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ABSTRACT

This document contains the proceedings of the Second International Conference on the Teaching of Mathematics at the Undergraduate Level. It provides a forum for bringing together faculty from countries with varied educational systems who are committed to introducing innovative teaching methods and new pedagogies. The conference presentations are centered around the following themes: (1) Educational Research: Results of Current Research in Mathematics Education and the Assessment of Student Learning; (2) Technology: Effective Integration of Computing Technology (Calculators, Computer Algebra Systems, WWW Resources) into the Undergraduate Curriculum; (3) Innovative Teaching Methods: Innovative Ways of Teaching Undergraduate Mathematics, such as Cooperative and Collaborative Teaching, Writing in Mathematics, and Laboratory Courses; (4) Curricula Innovations: Revisions of Specific Courses and Assessment of the Results, History of Mathematics, Innovative Applications, and Project Driven Curricula; (5) Preparation of Teachers: Trends in Teacher Education, and Changing Needs of Teachers; (6) Mathematics and Other Disciplines: The Effects of Changes in the Teaching of Mathematics on Other Fields, the Needs of Client Disciplines, and Interdisciplinary Courses; and (7) Distance Learning: Distance Learning Technologies (Networking, Tele-Education) for Teaching and Learning Mathematics, Current Hardware and Software Delivery Media, Educational Materials, and Visions for the Future. The proceedings includes 405 papers. (KHR)

Proceedings of the International Conference on the Teaching of Mathematics (at the Undergraduate Level) (2nd, Hersonissos, Crete, Greece, July 1-6, 2002)

Deborah Hughes Hallett and Constantinos Tzanakis, Editors

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C ONFERENCE THEMES

The conference presentations are centered around the following themes:

- **EDUCATIONAL RESEARCH**: Results of current research in mathematics education and the assessment of student learning. Access and equity.
- **TECHNOLOGY**: Effective integration of computing technology (Calculators, Computer Algebra Systems, WWW resources) into the undergraduate curriculum
- **INNOVATIVE TEACHING METHODS** : Innovative ways of teaching undergraduate mathematics, such as cooperative and collaborative teaching. Writing in mathematics; laboratory courses.
- **CURRICULA INNOVATIONS**: Revisions of specific courses and assessment of the results. History of mathematics; innovative applications; project driven curricula.
- **PREPARATION OF TEACHERS**: Trends in teacher education. Changing needs of teachers.
- **MATHEMATICS AND OTHER DISCIPLINES** : The effects of changes in the teaching of mathematics on other fields. The needs of client disciplines; interdisciplinary courses.
- **DISTANCE LEARNING**: Distance learning technologies (networking, tele-education) for teaching and learning mathematics. Current hardware and software delivery media; educational materials. Visions for the future.

From the Conference Organizers of ICTM2

Mathematics is central to our world. Mathematical ideas are essential for developments in science and engineering. Contributions from mathematicians have revolutionized finance and biology over the past decades. A mathematically literate citizenry is essential to a country's vitality. The teaching of mathematics is therefore a cornerstone of a country's educational health.

Yet most countries today are concerned about the level of mathematics their students learn, and concerned that interest in mathematics is falling at a time when the need for technical skills is rising. Many countries are wrestling with shortages of teachers, curricula that do not reflect modern needs, and teaching practices that do not always work for their students. Fortunately, recently there have also been significant advances in understanding how students learn and a surge of interest in the teaching of mathematics.

Following the success of the *First International Conference on the Teaching of Mathematics* (Samos, Greece, July 1998), the *Second International Conference on the Teaching of Mathematics* (ICTM2), provides a remarkable opportunity to bring together faculty from around the world who are committed to introducing innovative teaching methods. Mathematicians have traditionally not talked to each other much about teaching, nor have they talked to mathematics educators. Certainly, international communication between mathematicians is often more about research results in mathematics than about teaching strategies. This conference attempts to foster a conversation to fill this gap.

ICTM2 received about 420 proposals for presentations from over 65 countries—over one third of the world's nations. Their topics span educational research, technology, innovative teaching methods, curricula innovations, the preparation of teachers, connections of mathematics with other disciplines, and distance learning. Papers from ten distinguished plenary speakers, representing several continents, are also included in the proceedings. We hope that the published papers will lead the reader to a better understanding of the issues facing instructors of mathematics around the globe and that this understanding will lead to a higher level of international cooperation in the effort to improve the teaching of mathematics.

In addition to the papers, abstracts of the accepted oral and poster presentations are included. Abstracts were reviewed be members of the program committee and authors of accepted abstracts given the opportunity to submit a full paper. The papers were also reviewed by the International Program Committee.

We would like to express our immense gratitude to each and every member of the organizing committee, for his or her time, dedication, and invaluable comments in the refereeing process. We are also deeply indebted to the conference sponsors for making such an international event possible in beautiful surroundings on the island of Crete.

Special thanks to the University of Crete for hosting the Conference and to John Wiley & Sons Inc. for publishing the Proceedings.

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NEW CHALLENGES IN THE TEACHING OF MATHEMATICS

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ABSTRACT

Does the manifold, but discrete, presence of Mathematics in many objects or services around us impose new constraints to the teaching of Mathematics? If citizens need to be comfortable in various situations with a variety of mathematical tools, the learning of Mathematics requires that one starts with simple concepts. How can one face this dilemma?

"L'école doit enseigner à analyser et à discuter les paramètres sur lesquels se fondent nos affirmations passionnelles."

Umberto ECO,
Le Monde, La Repubblica, 10.10.2001

The content of this lecture grew out of discussions with teachers and scientists. In the title one could replace "Mathematics" by other fields, as the following quote from the French philosopher Alain ETCHEGOYEN shows: *"Si la... apprend aux élèves à analyser les concepts, à raisonner de façon démonstrative et à argumenter, elle est une des disciplines qui façonnent l'"honnête homme" de jadis, le "citoyen" d'aujourd'hui, les deux étant liés."* For him of course the dots were to be replaced by *"philosophie"*. You can find Mathematics in the title of my lecture in good part because I am a professional mathematician. In putting my arguments in writing, my only ambition is to contribute to a debate. School is at risk in many societies because, in my opinion, not enough attention has been given neither to the variety of types of knowledge to which students have to be exposed there, nor to new links existing between Science and Society, nor finally to the need to position Mathematics as a human activity in the course of History.

1. How to Link Technical and Generic Knowledge?

a) Doing Mathematics and learning about Mathematics

Mathematicians tend to agree that one cannot study their discipline without actually "doing Math". This is why we are so keen on giving problems to our students. In doing so we hope to fight the misleading conception that Mathematics could be a new scholastics, when many of its concepts were born while taking up challenges coming from fields outside Mathematics.

To succeed in this, students need a certain familiarity with basic mathematical concepts and/or objects, and they must learn to manipulate them while getting some idea about their universality and their relevance. This last point needs to be further clarified since, as will be explained later, behind it lies a potentially annoying hiatus.

This very seldom leads students to the perception that beyond the mathematical exercises they struggle with lies hidden a profession. As mathematicians, we all had to face the (hard) question coming from relatives and/or friends: *"What can you do in a domain where facts do not change and everything has been known for thousands of years?"* Our situation is certainly very different from that of musicians. For them it is obvious to a wide public that the good practice of playing music can be learned through strenuous routines, and that music gets enriched through the contributions of creative composers. If one considers the percentage of our students who, later, will become mathematicians, this may appear a minor issue. For me, to the contrary, getting Mathematics recognized as a living science lies at the heart of the matter. I will say more on this later.

So far I have only touched upon technical knowledge in schools, about which of course there are very diverse opinions concerning its content, how to get it across to students and how to measure its appropriation by them. There is a lot to say on this but the point on which I would like to focus my attention is quite a different one, namely that the use of scientific knowledge in modern societies requires much more than this familiarity with simple concepts and tools. This is what I tentatively call "generic" knowledge. Because of the scientific underpinnings of many aspects of modern life, getting some understanding on how complex systems rely on knowledge has become of paramount importance for citizens to form enlightened opinions and to make independent decisions. How can school training help with that? The dilemma there is: How can one give the proper perspective on scientific issues relevant to the daily functioning of our society in the very constrained school world? This covers one specific issue: How to get the proper balance between "simplicity" and "complexity" at school?

b) Mathematics entertains special relations to language and truth

In this paragraph I will only raise two points with respect to which Mathematics appears special, namely its peculiar relation to language and its special link to truth. These two points are in some sense obvious but I do not think that they have been looked at in the proper perspective as far as the teaching of Mathematics in schools is concerned. There are mentioned here because they appear to me as possible obstacles to address the global challenge mentioned before.

Let us begin with the relation of Mathematics to language. It is well known that the mathematical language must be precise, for the good reason that the ultimate purpose of a mathematical development is to "prove" a statement. This is even sometimes the basis of jokes at the expense of mathematicians. Any imprecision opens the door to a misconception, and even the smallest one can destroy the whole edifice. One should be careful though with one point, namely that after all in a mathematical explanation one is often using ordinary language in a special way. Most of the time, ordinary and strictly mathematical expressions are mixed, forcing students to live a sort of "double life". Mathematicians should be aware that this situation is not without consequences, and does create a sense of frustration for a number of students, because they feel that their ability to express themselves has been substantially limited. This can be the basis of strong bad feelings about Mathematics on the part of a number of students. This potential handicap can even get worse in more advanced courses where names given to many concepts are purely conventional. It is a fact that most of the names are well chosen, but some choices may exaggerate the feeling that Mathematics is cut off from real life because students realize that practicing Mathematics may even require to give up the free use of language.

For the purpose of this lecture, I would like to limit the relation of Mathematics to truth to the fact that a student who masters an argument can win against his or her teacher and/or his or her classmates. Such an experience can play a major role in the structuring of the personality. It also forces students to practice the dialectics between doubt and certitude, a very healthy exercise. Other structuring effects can also be hoped for in relation with the strength of good argumentation. Evariste Galois put it in an interesting way. He proposed *"faire du raisonnement une seconde mémoire"* as possible motto for the great benefit of the mathematical training. All this has very important consequences for teachers. One of them is that their worst mistake can be to impose their views against those of students who are actually right. Mathematics has a major role to play in the training towards critical thinking. As a result, there are several instances in History where Mathematics, and/or mathematicians, were considered subversive.

2. New Links between Mathematics and Society

a) Making the link evident

It is not clear, even to some mathematicians that a great many of the mathematical notions are at work in Society around us. Moreover, our times are special. Indeed, there has never been so many instances where this happens. Very often this is through the use of a *mathematical model.* At the same time, they are very few cases at school where the notion of a model is properly introduced, and students invited to make use of it.

For me the variety of situations where mathematical notions are in action in objects and services of daily use justifies the claim that we are entering a new age for Mathematics in terms of its relations with Society. They are several aspects for this, some connected with Mathematics itself, some with the development of high technologies. Let us list some causes for this strengthening:

- the extraordinary increase in the power of *computers* now makes many more questions amenable to calculations via models;
- we are living in a society where *communications* play a major (if not dominant) role, and dealing with large amounts of data requires to think of them in mathematical terms. Mathematics needed for that purpose is sometimes sophisticated and can be of recent development; in some cases even, problems originating from dealing with these data do represent new challenges to mathematicians;
- more and more often *images* become the main object under consideration, and need to be stored, compressed and securely transmitted; this is new type of objects to be manipulated systematically by mathematicians;
- *stochastic aspects* of some phenomena have today to be taken into consideration and properly analysed, thanks to the progress of Probability Theory and of Statistics.

Let us give some specific examples, many of them having to do with complex systems (in which one must be careful with the fact that, in the long run, often secondary effects dominate primary ones):

- *telecommunication systems* are incorporating many different mathematical components to code messages, to compress data, to design cellular phone networks; etc.;
- *data collecting and accessing* have invaded, and will invade even more, our lives; think of the generalized presence of bar codes (fundamental to manage inventories), of GPS (Global Positioning System) which involves sophisticated Mathematics when one would naively think that, thanks to its satellite network, the problem to be solved is a mere Euclidean geometry one; the medical scanner is a machine whose principle is based on a mathematical theorem, the Radon transform;
- *automated systems* are hidden in very many objects of frequent uses, such as transportation means (planes, trains, buses, cars, elevators, etc.), telecommunications, and soon intelligent buildings or houses;
- *shape optimization* can be motivated either by technical reasons (improving the aerodynamics of a car, or a plane wing) or aesthetic ones. Dealing with shapes is very cumbersome experimentally. One needs to manufacture prototypes that have to be one by one tested in wind tunnels, hence the introduction of "numerical wind tunnels", i.e. pieces of software and combination of mathematical operations adjusting several parameters at once in order to improve the design.

This new situation is exemplified by the fact that today there are *mathematical products*, as they are chemical products. As professional we must acknowledge this new dimension. Mathematicians rarely do so, maybe because we are still facing the unpleasant situation that no industrial sector considers itself as a "mathematical" sector, although the finance industry is getting close to being one.

A good reason for the non obviousness of the presence of Mathematics around us is the fact that one is often tempted to focus one's attention on a concrete object, when its actual social use involves it as part of a network, something that is most of the time hidden, and rather invisible. This is typical in airplanes for example.

b) Learning about limits and detecting the impossible

Many dimensions of social life involve understanding the meaning and therefore the limits of the information one can draw from a given situation, even if it has some stochastic aspect. A typical example of this can be found in the *proper use of statistical data*. They are present in many different areas, from opinion polls to insurance estimations, from risks to forecasting. They do play an important role if one is to take seriously the task of helping citizens assume their responsibilities. The purpose is not at all to expect that a sophisticated technical training in statistics can be achieved at school, but rather to make sure that all citizens be ready to challenge some claims on the basis that they realize why these claims are either self-contradictory or impossible.

This can be coined as *a scientific approach to doubt*, which should be one of the targets given to the mathematical training at school. It has a technical side but putting it at a too technical level can obscure the issue, which is to improve the contribution school training can make to citizenship.

More broadly, school is also challenged to help future citizens to get a better apprehension of the impact of basic scientific knowledge in society. Indeed in the last part of the XXth century one could witness a number of *short-circuits*, direct connections between discoveries or innovations in research laboratories and new industrial fields. After all this is exactly how internet got started, or how the telecommunication industry boomed. There was no preexisting market. This forces to rethink the relationship between research and development, and to challenge the claim that the search for a concrete application has to be the driving force of a programme, since it is a misleading oversimplification of the real mechanism. Enough room must be kept for free thinking besides targeted research. Again such a goal will be easier to achieve if a larger number of people see more clearly how this mechanism works.

As a result, providing teachers with resources to illustrate their courses through concrete situations where notions they teach, and exercises they propose to students are put to work, becomes a very serious issue that has not yet been addressed properly in many countries, in particular at the secondary level.

3. Mathematical Sciences as Human Activities

a) How does, and did, knowledge form?

The resistance to some changes that we have been advocating in the previous paragraphs is likely to find some of its roots in a misconception on how Mathematics actually develops, and developed. A temporality was even claimed by some of us as a natural companion of the universality of Mathematics. I deeply disagree with such a statement. The need for a historic perspective on any technical knowledge is obvious. It dictates the introduction of the proper dose of History of Science in any science course, probably not as a subject in itself but rather as a facilitator of the acquisition of a new notion.

In this respect, an important role must be given to breaks in past conceptions. Indeed, they show that knowledge is not the result of a linear accumulation and requires some painful rediscussions of the heritage from the past. Such an approach is likely to provide opportunities to link the presentation of Mathematics with other disciplines, scientific or not, and to make more meaningful comparisons between the different methods at work in these disciplines.

b) What is known, what is not known and what cannot be known

As I said earlier, one of key features to hope and generate a different attitude towards Mathematics among new generations of students, is to make it perceptible to them that there are questions which presently do not have answers. Progress on them can be of different types: either they can be considered as non interesting (a highly subjective judgment of course), and as such not worthy of further investigation, or impossible to answer (realizing that some important statements in Mathematics can be proved to be non provable was one of the major achievements in XXth century Mathematics due to Kurt Gödel), or just beyond reach of present methods and concepts.

Giving some idea that there are challenges around us, and making them perceptible, and at the same time meaningful, is a challenge in itself. Today, to my knowledge, not much thought has been put towards this goal, and this lack of investment becomes a handicap in our societies where the relation of students to schools has changed a lot because of the huge amount of information on a variety of subjects they have access to outside the school system.

If we are to have a chance of convincing a large portion of the school population that Mathematics is a living science, the minimum we must achieve is to prove it has a future. We cannot take this for granted, and we have to design tools to do that.

c) The role and place of abstraction

One of the points that, in my opinion, needs to be addressed has to do with the process of *abstraction*. It has focused a lot of criticism, in fact the archetype of criticisms against Mathematics, when the nature of our science lies for the most part in it. Henri Poincaré went as far as saying *"Faire des mathématiques, c'est donner le même nom à des choses différentes."* For me, the request to make Mathematics less abstract is self-contradictory. It may be true though that we did not discuss enough, or at least make it enough evident, how the abstraction process functions within Mathematics. It does have several aspects: from realizing that a common structure is at work in different situations to coming up with the minimum formulation for it, therefore establishing[1] an ideal object.

The previous point is not at all separate from a discussion of the *axiomatic method*. Its widespread use in the teaching of Mathematics, especially at more advanced levels, confuses the issue concerning it. It is quite clear to me that its introduction is one of the achievements in the History of Mathematics. It clearly marked the independence of mathematical concepts, and forced to make precise the role that mathematical developments have to play in modelling a situation. It also made possible the fantastic expansion in the training of Mathematics that one could witness after the Second World War. Nevertheless, even if one makes the pedagogical choice of introducing some notions in a purely axiomatic manner, one is not freed from the obligation of making a connection, at some stage of the learning process, with what prompted this notion to be singled out, together with the interest or limitations of variants of it.

[1]The choice I made of the word "establishing" in the previous sentence is deliberate in order not to take sides in the deep philosophical debate as to whether mathematical objects are "created" or "discovered", the long lasting dispute between Platonicists and Intuitionists. It is of course worthy of a thorough discussion, but to conduct it requires some technical philosophical tools that I do not want to introduce here. It could also divert us from the main points I want to discuss which are, I believe, independent from these philosophical stands.

4. A few Points as Conclusion

As said in the introduction, the main purpose of this address is to open a discussion. In my opinion, recent developments in Society require paying serious attention to new requests put to the teaching of Mathematics. Finding the best way to meet them will require many exchanges and attempts. Some of them will fail in certain circumstances and succeed in others. Understanding what makes this happen will probably force us to examine more thoroughly than we are used to the great diversity of pedagogical situations teachers face today.

In this conclusive paragraph I only offer some goals, which, I feel, have to be pursued a bit systematically. For none of them can I claim to have the right solution for achieving it.

a) Linking Mathematics to the rest of knowledge in schools

Isolating Mathematics from the rest of knowledge is for me the worst that can happen, especially in connection with other sciences. This does not mean that Mathematics does not have its own territory, specific methods, and peculiar requirements. Much to the contrary, it is in confronting the various approaches used in several areas that one has a chance of presenting Mathematics in the right perspective. Differences will stick out, and therefore an identity should emerge from this. Again, I cannot imagine that this will become a teaching in itself. It is by putting the right touches at the right moment that it is the more likely to be achieved.

At the same time, the worst would be that this link be made artificially or a necessary condition for the validation of a school work. There are indeed some topics worthy of attention at school that find their roots in Mathematics and whose development keeps you within the discipline. One must just make sure that the exposure of students to cross-disciplinary activities is big enough to make it perceptible to them that the various learning processes are indeed complementary. They all aim at understanding the world around us, and making it possible for them to put their knowledge to use in several different contexts.

b) Making sure that the knowledge relevant for all is properly integrated in curricula

Choosing material to be covered in curricula is a very delicate matter, but I feel sometimes too much attention is given to it at the expense of other aspects of the school environment that, in the long run, play an even more important role. At least this is the impression I got from participating in the elaboration of the curriculum for French high schools.

The need for coherent programmes compatible with the time allocated for the study is of course a big constraint. The introduction of new topics requires that teachers be trained, and proper documents be available. This should be thought in a much broader way as just having textbooks. One must also help teachers by providing them with documents for independent reading.

Nevertheless, efforts have to be made and competences gathered in order to make sure knowledge that has become pervasive in the understanding of how Society functions is taught at the right level. A typical example of this has to do with Statistics. Making it adequately connected to the traditional mathematical training requires some thought in the context of the present curricula in some countries. For these questions one should be careful in not taking a too technical approach, and be caught in a narrow pursuit of performance when what is at stakes is transmitting a basic, but very solid, understanding of the underpinnings and general ideas.

c) Working with teachers

None of this can be achieved if working and confident *contacts with teachers* are not established. It requires creating places where this working together can take place, forums where personal successful initiatives can be given the necessary resonance, and monographies and/or other media from which relevant information can be found.

Another issue, which can of course be the basis for a debate, is *the very purpose of the training in Mathematics in schools*. In my opinion it cannot be limited to giving the basis for future studies to those who will become professional mathematicians, when citizens, but also so many professionals, need more than ever to relate in confidence with Mathematics, even if their technical knowledge of it is limited. Having a good evaluation of what Mathematics does, and does not achieve has become very important.

This raises two questions about possible pedagogical methods. Involving students with personal projects, of a size appropriate to their level of sophistication, definitely gives them a chance to get a feeling of a more independent approach to work and, more important, to discover new connections by themselves. There is evidence that the learning effect of such experiences lasts longer than a more systematic and more technically oriented one but it can come only after a sufficient technical ability has been built. Again what is to be looked for is an optimal combination of the two. In this respect, it is sure that methods to *evaluate performances* at school have to be enriched and diversified. Much too often the teaching is completely geared by the *evaluation* schemes put in place. A political figure of the first half of last century in France, Edouard HERRIOT, is remembered for having said *"La culture, c'est ce qui reste quand on a tout oublié"*. I have the feeling that mathematicians have too often forgotten that building a mathematical culture is a responsibility that has been entrusted with them. It is indeed much broader than just training the new generation of people who are going to replace us as specialists. I am afraid that, at this moment, we, as a community, have not put enough thinking to our broad responsibilities.

'LIFE WASN'T MEANT TO BE EASY': SEPARATING WHEAT FROM CHAFF IN TECHNOLOGY AIDED LEARNING?

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ABSTRACT

The paper commences by reviewing some of the issues currently being raised with respect to the use of technology in undergraduate mathematics teaching and learning. Selected material from three research projects is used to address a series of questions. The questions relate to the use of symbolic manipulators in tertiary mathematics, to undergraduate student attitudes towards the use of computers in learning mathematics, and to outcomes of using technology in collaborative student activity in pre-university classrooms. Results suggest that teaching demands are increased rather than decreased by the use of technology, that attitudes to mathematics and to computers occupy different dimensions, and that students adopt different preferences in the way they utilise available resources. These outcomes are reflected back on the literature, and implications for teaching, learning, and research discussed.

KEYWORDS: undergraduate; mathematics; technology; Maple; graphical calculators; attitudes; collaborative learning.

1. Introduction

In this paper I want to reflect on outcomes from three research projects that span the interface from senior school to undergraduate programs. The common elements in the programs are mathematics, students, and technology. The purpose is to describe findings from the selected research foci, and relate them to matters raised in the wider literature, and to implications for theory and practice.

Papers addressing the use of technology in undergraduate mathematics make for interesting and varied reading. For example:

- The impact upon educational practice of powerful software like Mathematica has been less profound than optimists hoped or pessimists feared...tendency to begin by looking for electronic ways of doing the familiar jobs previously done by textbooks and lectures. (Ramsden, 1997).
- Of all the flaws in our mathematics training this seemed to us to be the most dangerous and insidious, for as we removed mathematics from our courses in response to 'student failings', the need for mathematics to do real science was in fact increasing...firstly there was the pious hope that a computer assisted approach would require less staff...problems arose from attempts to use Mathematica in two ways-which were incompatible. Was software an arena for exploration of mathematical ideas, or a channel for their transmission? (Templer et al, 1998)
- There is growing evidence (in the UK and elsewhere) of a general decline in the mathematics preparedness of science and engineering undergraduates...one response has been to simply reduce the mathematics content and to rely on computer-based tools to do much of the mathematical computation...difficult questions (emerge) at the intersection of cognitive and epistemological domains; to what extent must the structure of mathematics be understood in order for it to be used effectively as a tool? (Kent & Stevenson, 1999)

These excerpts canvass some of the challenging and problematic issues that are emerging in undergraduate mathematics education. The discussion that follows will raise issues associated with the use of symbolic manipulators as central agents for teaching and learning undergraduate mathematics; with affective characteristics of students using technology in undergraduate coursework; and with the use of technology in collaborative learning activity. The latter project has been implemented with pre-university school mathematics students as subjects. The qualities displayed by the students, and their approaches to learning have implications for the undergraduate programs in which they subsequently enrol.

2. Background

One fundamental component of any discussion of undergraduate learning is the composition and background of the student cohort. As noted above (Kent & Stevenson, 1999) the widening of secondary education, and curriculum decisions in relation to school mathematics, mean that the mathematical preparedness of entering undergraduates is perceived to be changing. Clearly this perception is impacting on course design and teaching approaches, in particular in the way that technology is utilised. However a nostalgic review of the past should not obscure the reality that there were really no "good old days". Studies addressing the (mis) understanding of basic concepts and procedures displayed by undergraduate mathematics students have been reported over a substantial period of time. Findings from these studies have a common theme viz. that the standard of performance of the 'current' student group is much lower than hoped for, given the investment of time and energy that has been directed towards the teaching and learning of mathematics over many years.

Characteristics of flawed performance have been historically consistent over a quarter of a century:

...After twelve years of schooling followed by two years of university, they had all but accepted the mindless mathematics that had been thrust upon them...Misconceptions, misguided and underdeveloped methods, unrefined intuition tend to remain assignments, corrections, solutions, tutorials, lectures and examinations notwithstanding. (Gray, 1975)

It appears that students have developed special purpose translation algorithms, which work for many text book problems, but which do not involve anything that could reasonably be called a semantic understanding of algebra. (Clement et al., 1980)

Weaker students suffered from the continued misinterpretation that algebra is a menagerie of disconnected rules to do with different contexts. (Tall & Razali, 1993)

In attending module after module, students tended to 'memory dump' rather than to retain and build a coherent knowledge structure...Their presumed examination strategy resulted in such a fragile understanding that reconstructing forgotten knowledge seemed alien to many taking part. (Anderson et al, 1998)

A common thread running through these studies is the powerful negative influence of fragmented learning, and the apparent absence of cognitive strategies to co-ordinate conceptual and procedural knowledge. The successive comments can be read as evidence supporting the constructivist paradigm, for students continue to carry mathematical 'baggage' and habits that inhibit the goals of instructors hoping to provide a fresh beginning in tertiary mathematics. Into the wake of this historical legacy, curriculum reforms and innovative teaching methods (often incorporating electronic technologies), have been injected as fountains of hope, at times accompanied by extravagant claims.

3. Focus A: Computer-Based Undergraduate Programs

The form of computer-based instruction varies widely, indicative of a range of beliefs among program designers and instructors - both about mathematics, and the nature of mathematics learning. Olsen (1999) discusses one of the most extensive examples of technology used to provide automated instruction. She describes (page 31) how politicians visiting Virginia Tech's Mathematics Emporium, a 58 000 square foot (1.5-acre) computer classroom:

see a model of institutional productivity; a vision of the future in which machines handle many kinds of undergraduate teaching duties-and universities pay fewer professors to lecture...On weekdays from 9 am to midnight dozens of tutors and helpers stroll along the hexagonal pods on which the computers are located. They are trying to spot the students who are stuck on a problem and need help.

This program appears to be openly driven by economic rationalism, and an assumption that mathematics is something primarily to be delivered and consumed. By contrast Shneiderman et al (1998) describe a model, in which electronic classroom infrastructure is extensive and expensive. Courses are scheduled into electronic classrooms, following a competitive proposal process, requiring full use of an interactive, collaborative, multi-media environment. Active engagement with a variety of learning tools is highly valued here.

In between the extremes occur a variety of models of instruction, concerned in varying degrees with factory production on the one hand, and student understanding and engagement on the other, and it is instructive to note comments from those describing the characteristics of such programs: here are some selections.

Templer et al (1998) noted problems accompanying efforts to provide meaningful learning that were perceived to arise as a direct result of a symbolic manipulator (*Mathematica*) environment. They noted that typically having mastered the rudiments, the majority of students:

"began to hurtle through the work, hell bent on finishing everything in the shortest possible time."

The following comment, or a close relative, was noted as occurring frequently among the students:

"I just don't understand what I'm learning here. I mean all I have to do is ask the machine to solve the problem and it's done. What have I learned?"

Kent & Stevenson (1998) in elaborating on their concerns about student quality (see Introduction), question whether mathematical procedures can be learned effectively without an appreciation of their place in the structure of mathematics. They argue that unless some kind of breakdown in the functionality of some concept or procedure (say integration) is provoked, students do not focus on the essential aspects of that concept or procedure. On the other hand they observed that the demands for formal precision that a programming environment places on its user, serves both to expose any fragility in understanding, and to support the building and conjecturing required in the re (construction) of concepts by learners. This comment interfaces with a debate about whether computer technology should be employed following prior understanding of mathematical concepts and procedures (Harris, 2000), or as a means integral to the development of such understanding (Roddick, 2001).

Interesting comment has been made also about specific issues relating to the introduction of technology into mathematics learning settings. Templer et al (1998) noted that the screen dominated the attention of most (although not all) students, and that some balance needs to be struck between directing students from paper to screen, and vice-versa. A lack of symmetry was evident in that some students are reluctant to move from screen to text, whereas the move the other way is more flexibly undertaken. An interesting slant on the 'how and when' debate is provided by the observation that mathematical 'tools' are forged through use, in contrast to conventional tools that are first made and then used. This then calls into question a sequence that seeks first to master a tool and then apply it. Specifically whether training in a manipulator such as *Mathematica, Derive, or Maple* requires prior time and effort, or whether a careful design can enable mathematics to be learned and applied contiguously with increasingly sophisticated manipulator use? Clearly this matter is not yet resolved.

3.1 Research Program

The teaching programs that form the background for this section of research took place at the University of Queensland during the period 1997-2000. As mainstream courses located between the extremes described above they represent models that may be located comfortably within present university structures and resources. The programs involve the use of *Maple* in first year undergraduate teaching, and issues associated with implementation connect with those of other researchers as sampled above. In keeping with Kent & Stevenson (1998) there is interest in the range of questions raised by students as they work with the software, as well as in their performance. With Templer et al (1998) there is concern with the links between computer-controlled processes and their mathematical underpinnings, noting the similarities and differences between the respective symbolism. This project had several aims, including the following:

1. *To classify the range of student-generated questions that emerge when learning of mathematical content interacts with a symbolic manipulator environment.*

2. *To identify structural properties associated with the Maple environment that can be identified as linking task demand and student success.*

The research was conducted within first-year undergraduate mathematics courses taken by students studying mainly within Science and Engineering degree programs. As taught in 1999 and 2000 the courses comprised a lecture series complemented by weekly workshops, in which approximately 40 students were timetabled into a laboratory containing networked computers equipped with *Maple* software. The lecture room was fitted with computer display facilities so *Maple* processing was an integral and continuing part of the lecture presentation. To support their workshop activity students were provided with a teaching manual (Pemberton, 1997), continually updated to contain explanations of all *Maple* commands used in the course, together with many illustrative examples. During laboratory workshops two tutors and frequently the lecturer also, were available to assist the students working on tasks structured through the provision of weekly worksheets. The students could consult with the lecturer during limited additional office hours, and unscheduled additional access to the laboratory was available for approximately 5 hours per week. The course was also available on the Web. Solutions to the weekly worksheets were provided subsequently.

The formal course assessment was constrained by departmental protocol and the availability of facilities. The major component comprised pen and paper exams at mid-semester and at end of semester (combined 80%). The balance consisted of *Maple* based assignments and a mark assigned on the basis of tutorial work (20%). To succeed students needed to transfer their learning and expertise substantially from software supported environment to written format, which means that they must be able to develop understanding through the medium with which they work, while simultaneously achieving independence from it. This involves the ability to learn and maintain procedures that a *Maple* environment does not enforce, so that attention is focused on the relationship between the mathematical demands of tasks, and their representation in a *Maple* learningscape.

3.2 Data sources

The data for addressing these questions come from two sources. Tutors assigned to the *Maple* workshops were provided with diaries in which they entered, on a weekly basis, examples indicative of the range of questions raised by students in the course of their workshop activity. The second source of data was a test given 7 weeks after the program started. This test was a voluntary exercise, and comprised a series of questions to be addressed with the assistance of *Maple* in its laboratory context. It provided formative feedback to the students on their performance, and ranged from simple school level manipulations to new material introduced in the tertiary program. Sample questions are included in the appendix, together with their *Maple* solutions. The test was directly relevant to preparing for the formal assessment at the end of semester, for the procedures required were ones that the students need to be proficient with, irrespective of software support. The tests were analysed and marked by two of the course tutors using criteria designed by the researchers. For this purpose the quality or indeed presence of a final interpretation of graphical output was not taken into account, so that the correct/incorrect dichotomy was on the basis of *Maple* operations only. On the basis of a review of the 250 (approx.) scripts submitted, it appeared that the first 16 questions had been attempted seriously by the whole group. For technical reasons two of these were deemed unsuitable for inclusion, so that responses to 14 questions formed the final data set.

3.3 Regression Analysis

Performance was analysed in terms of the influence of two categories labelled SYNTAX and FUNCTION respectively.

SYNTAX: refers to the general *Maple* definitions necessary for the successful execution of commands. These include the correct use of brackets in general expressions, and common symbols representing a specific syntax different from that normally used in scripting mathematical statements (such as *, ^, Pi, g:=).

FUNCTION: refers to the selection and specification of particular functions appropriate to the task at hand. Specific internal syntax required in specifying a function is regarded as part of the FUNCTION component, including brackets when used for this purpose. Complexity is represented by a simple count of the individual components required in successful operation. The way these definitions work is illustrated by applying them to the examples given in the appendix.

Q2. SYNTAX: Incidence of ^ [2] *plus* * [2]; total=4.
FUNCTION: General structural form of factor (argument); factor [1] *plus* ( ) [1] *plus* argument entry [1]; total=3.
Q8. SYNTAX: Incidence of ^ [1] *plus* *[2] *plus* ( ) [2] *plus* x1[1] *plus* := [1]; total=7.
FUNCTION: General structural form of plot (function, domain); plot [1] *plus* ( ) [1] *plus* , [1] *plus* function entry [1] *plus* domain entry [1] *plus* domain specification [1]; sub-total=6.
General structural form of fsolve (function, domain); sub-total [5] *plus* domain specification[1]; total =12.
Q14. SYNTAX: Incidence of*[2] *plus* ( ) [3]; *plus* y [1] *plus*: = [1]; total=7.
FUNCTION: General structural form of plot(function, domain); sub-total [5] *plus* domain specification [1];
General structural form of int(y, integ interval); sub-total [5] *plus* (subtraction) [1] *plus* integration interval specifications [2]; total=14.

Similar pairs were assigned to each of the 14 questions in the sample. Our diagnostic approach involves scoring on a correct/incorrect basis, as we are not (in this analysis) concerned with apportioning partial credit as would be necessary if grading student performance. The success rate on the questions is given by the fraction of students (N~ 250) obtaining the correct answer. We can regard these as providing a measure of the probability of success of a student from this group on the respective questions. For the questions in the Appendix the respective values are 0.89, 0.26, and 0.14. A linear regression analysis was performed using these probabilities as measures of the dependent variable (success), and SYNTAX and FUNCTION as input variables (Tables 1 & 2).

Table 1: Regression statistics

Multiple R	0.8710
R Square	0.7586
Adjusted R Square	0.7148
Standard Error	0.1419
Observations	14

Table2: Regression Statistics cont

	Coefficients	Standard Error	t Stat	P-value
Intercept	1.0947	0.0961	11.383	2E-07
SYNTAX	-0.0482	0.0168	-2.874	0.015*
FUNCTION	-0.0396	0.0122	-3.246	0.008**

According to this analysis both the SYNTAX ($p<.05$) and FUNCTION ($p<.01$) complexity measures contributed significantly to the task demand of the questions.

3.4 Student-generated questions (question 2)

A total of over 1300 questions indicative of the range of concerns displayed by students in the 2000 cohort when working mathematically in a *Maple* environment, was assembled from the tutor diaries. The categories were selected using a mix of empirical judgment, theoretical positioning, and the results of a pilot study in the previous year. The distribution is shown in Table 3.The number of questions per category varied from a maximum of 333 (24.6%) to a minimum of 29 (2.2%). The number of questions in which some aspect of *Maple* was unequivocally involved exceeded 80%.

Table 3: Student Question Types

Question Category	Percentage
1. Identify problem caused by a typo (TYPO)	8.4%
2. Resolve syntax error (SYN)	24.6%
3. Problem with function choice (FCHCE)	4.2%
4. Problem specifying function (FSPEC)	14.6 %
5. Stuck on mathematics (STMATH)	14.9 %
6. Procedurally stuck on Maple (STMAPLE)	19.5 %
7. Interpreting aspects of output (INTOUT)	11.6 %
8. General procedural (PROC)	2.2 %

The patterns evident in Table 3 confirm that when students interact with mathematics through technology, questions are generated rapidly and their scope is vastly increased. We can identify at least four types of inquiry from the responses. Those that are simply procedural (what to do next); those that are mathematical in the traditional sense; those that are software related (syntax and symbols); and those that are generated by the interaction of mathematics with software (function choice and specification). The intensity and scope of student questioning has escalated in comparison with traditional practice classes, with software the major contributor through properties of fast processing, scope for formatting and specification errors, just plain knowledge blocks in bringing the mathematics and software together, together with student initiative in exploring. In examining the analysis relevant to the first question, it can be observed that while achieving more rapid and efficient closure to algorithmic procedures the use of *Maple* has not reduced the need for the mathematical attributes of understanding and attention to detail. We note this in the significant impact of the variables SYNTAX and FUNCTION on success rate. SYNTAX errors penalise those who lack sufficient care in expressing their work symbolically, while the demands imposed by FUNCTION are proportional to the principles and sophistication of the associated mathematics. On the other hand, for those students who possess conceptual understanding and due regard for precision, the *Maple* environment has provided a means to progress rapidly and successfully at a greater rate than could otherwise be achieved. Our conclusion to this point is that there is no 'free lunch' (indeed laboratory tutors are lucky to get lunch at all). The propensity of students to alter their approach to reduce the learning potential available to them is apparent. Properties arising from the mutual interaction of students, mathematics, and technology can support approaches extending beyond the models that still seem to motivate some proponents of automated learning – models with goals of doing faster and more cheaply that which was done formerly with blackboard, chalk, and paper. These are limited goals indeed. The present research contributes to this broader endeavour, both in terms of identifying and classifying student responses to laboratory activities, and in linking mathematical demand to the complexity of manipulator operations and task success.

3. Focus B: Student attitudes to mathematics and technology

While there have been enthusiastic claims for the positive impact of technology on the teaching and learning of mathematics, systematic evaluations of impact have been harder to access. And while the study of *attitudes* in mathematics learning has a substantial history, the relationship between *attitude* and *performance* is not clear-cut although positive correlations have often been noted between these characteristics. Early claims that affective variables can predict achievement (e.g. Fennema & Sherman, 1978) have been balanced by later comments (e.g. Schoenfeld, 1989) indicating that research does not give a clear picture of the direction of causal relationships. Ma & Kishor (1997) set out to assess the directional relationship between attitude and achievement but their meta-study was essentially correlational, so that the Tartre & Fennema (1995) comment that described *confidence* as the affective variable most consistently related to mathematics achievement is probably a safe summary of the position.

More recent studies among tertiary students have continued to pose the direction of the relationship between *attitude* and *performance* as an open question. Thus while Tall & Razali (1993) argued that the best way to foster positive attitudes is to provide success, Hensel & Stephens (1997) concluded that "it is still not totally clear whether achievement influences attitude, or attitude influences achievement". Shaw & Shaw (1997) noted that among engineering undergraduates the top performing students (at entry) had a much more positive attitude to mathematics, and lower performing students a commensurately negative one – again leaving the direction of causality open.

The study of attitudes towards information technology (most frequently computers) has a shorter but more intensive history, probably because information technology, while newer, is pervasive in its permeation of curriculum areas. In considering attitudes to information technology among tertiary students it is useful to note that the disciplinary focus of target groups has tended to be in areas like Education, Psychology and Social Work. Reports involving mathematics students appear harder to come by, although some studies have included affective variables almost incidentally when evaluating general project outcomes (see below). It is this very breadth of discipline background, which has served to keep the investigation of attitudes to technology at a general level, appropriate to the majority who will not be called upon to use computers in the same technical sense as mathematics students working intensively with specialised software.

The relevance of studying attitudes to technology in conjunction with those relating to mathematics is emphasised and reinforced by the increasing use of technological devices in mathematics instruction. Several studies refer incidentally to attitudinal impacts as well as proficiency measures and Mackie (1992) in an evaluation of computer-assisted learning in a tertiary mathematics course indicated six positive learning outcomes, three of which were related to attitudinal factors. Park (1993) in comparing a Calculus course (utilising *Mathematica*) with a conventionally taught program, found some improvement in disposition towards mathematics and the computer in the experimental group. However Melin-Conjeros (1992), in comparing the performance of a group of Calculus students (equipped with limited access to *Derive*) with a control group, noted that the attitude of both groups decreased slightly. It has not been generally clear in the mathematically focused studies just which 'attitudes' have been affected by technology, as the reporting tends to be non-specific. By inference it appears that it is 'attitude' to mathematics that is referred to, and we are led to consider the implications of technology in impacting upon component attributes. The relationship between mathematics confidence and performance noted in the literature (whatever the direction of causality), means that the implications of a nexus between technology and mathematics needs specific research attention. The broad reporting of studies on the use of technology in mathematics instruction makes it difficult to disentangle whether reported affective outcomes are associated with changed attitudes to mathematics, or are linked directly to the technology. So theoretically we are moved to ask about the interpretation of outcomes if students possess high mathematics confidence and motivation, but low computer confidence and motivation, and vice versa. And beyond this, whether structural changes in attitudes will occur as technology becomes more and more a part of the students' life experience, past and present. The specific research purpose addressed here may be expressed as follows:

To investigate the stability of attitude scales for use in programs in which computer technology is directed towards assisting undergraduate mathematics learning.

4.1 The Attitude Scales

Given the purpose of developing scales for use in settings involving interaction between technology and mathematics learning, the positions articulated by Hart (1989), Mandler (1989), and McLeod (1989, 1994) have proved helpful in fashioning approaches to the definition of terms and hence instrumentation. The distinction between an *attitude* and a *belief* is tenuous to a degree – an *attitude* focus has been sought by wording items so that the respondent is personally involved:

e.g. I feel more confident of my answers with a computer to help me; rather than

Computers help people to be more confident in obtaining answers.

The students for whom the measures are designed are tertiary undergraduates in mathematics courses. They have made this a deliberate choice - whereby mathematics has been selected as both useful in pursuing career aspirations, and as a subject compatible with themselves as individuals. Hence while an overall monitoring interest in gender and usefulness has been maintained, these emphases, which have figured prominently in attitude studies among school students, (e.g. Fennema & Sherman, 1976), have not played a dominant role in the design. Two of the nine attributes (confidence and motivation) represented in the Fennema-Sherman formulation have been reflected in scale development, with appropriate items constructed for use by undergraduates. The choice of these attributes was influenced strongly by the total purpose of designing instruments for use when computer technology is used in the teaching/learning context. *Confidence* and *motivation* have been selected because of their extensive appearance in the literature for both mathematics and technology, and because of their potential for discriminating between attitudes when technology and mathematics interact. These four scales are designed to measure attitudes on both dimensions so that such differences can be identified and their implications noted. In particular the choice of *confidence* and *motivation* enables two circumstances of particular interest to be identified viz. situations where students hold strong positive feelings towards mathematics and negative feelings towards technology, and vice-versa.

A further scale measures the degree of interaction between mathematics and computers that students perceive they apply in learning situations. The interactive significance of the learning and instructional context has been emphasised in general (e.g. McLeod 1989). In a computer environment students may simply respond to the screen or be active in note making, summarising, and experimenting. Indeed they may choose not to utilise technology when it is available and relevant. The physical separation of the learning components; pen and paper, computer screen, and human brain adds a further dimension to the co-ordinating processes required for effective learning strategies. The computer-mathematics *interaction* scale assesses the extent to which students bring their mathematical thinking into active inter-play with the computer medium.

Within each scale the eight items were arranged randomly with half requiring the reversal of polarity at the coding stage. Students were asked for a measure of their agreement (or rejection) with respect to item wording on a Likert scale. The item groups were presented in such a way that the underlying constructs were unknown to the students. The scale items themselves were theoretically determined from the respective underlying constructs and from cognate literature. See (Galbraith & Haines, 1998,2000) and Galbraith, Haines & Pemberton (1999) for more details on developmental aspects of this work.

4.2 Administration & Outcomes

The instrument was given initially in October 1994 to 156 first year students on entry to courses in engineering, mathematics and actuarial science at City University, London, and subsequently to the corresponding cohorts in 1995 and 1996. At the University of Queensland, Australia the scales were administered to 170 entering engineering undergraduates in 1997, and to parallel groups in 1998 and 2000. For present purposes the 1994, 1997, and 2000 results have been selected to be representative across time and place.

The responses have in fact displayed similar patterns across both place and time. Polarities have been adjusted so that a higher score means more of the property described by the scale label. Included below for sample scales, are the positively worded item(s) attracting the strongest support, and the negatively worded item(s) invoking the strongest rejection (L=London, B=Brisbane). L942&B971&B001 means that the item was the second strongest choice of London '94 students, and the strongest choice of Brisbane '97 students and Brisbane '00 students etc.

mathematics confidence:	I can get good results in mathematics (L941& B971&B002) *No matter how much I study, math is always difficult for me (L941& B971&B001)
computer confidence:	I am confident I can master any computer procedure that is needed for my course (L941& B971&B001) *As a male/female (*cross out that which does not apply*) I feel disadvantaged in having to use computers (L941& B971&B001) * items whose polarities are reversed in calculating scale scores.

4.3 Scale reliabilities

These were obtained for each scale as shown in Table 4. London data first followed by Brisbane data in brackets (1997), [2000].

Table 4: Scale Reliabilities (Cronbach α)

mathematics confidence	0.77 (0.85)[0.81]	computer confidence	0.82 (0.88)[0.85]
mathematics motivation	0.80 (0.84)[0.82]	computer motivation	0.85 (0.86)[0.81]
		comp/math interaction	0.70 (0.70)[0.71]

The scales are coherent with reliabilities from strong to moderate. Internal scale statistics verify that all items contribute usefully to the respective constructs.

4.4 Scale validity

This rests primarily upon the theoretical base behind the construction of the scales. Additional structural evidence may be inferred from the sample items given above. For example the two items attracting the strongest responses for *mathematics confidence* (expecting good results, and rejecting that mathematics is difficult irrespective of effort), are both centrally to do with confidence. The coherence of the scale as indicated in the α value then supports the argument for validity without examining each additional item. Similar arguments apply to the other scales.

4.5 Differences in Attitude to Mathematics and Computing

A main purpose in this research has been to investigate the extent to which attitudes to computer use and to mathematics represent different inputs into technology based teaching contexts involving mathematics learning. In this section the student responses are analysed to address this issue further. London and Brisbane data indicated as in the previous table.

Table 5:Inter-scale correlations

	mconf	Mmotiv	cconf	cmotiv	cmint
mconf		.47(.68)[.51]	.29(.21)[.22]	.14(.19)[-.04]	.13(.16)[.04]
mmotiv			.25(.23)[-.07]	.29(.29)[.00]	.35(.26)[.15]
cconf				.71(.75)[.62]	.61(.58)[.56]
cmotiv					.68(.66)[.65]

Table 5 displays correlations between the five scales. The entries indicate that for all three cohorts the confidence and motivation scales are strongly associated within mathematics, and within computing respectively. However they are less strongly associated across the areas, as shown by the weak correlation, for example, between mathematics confidence and computer confidence. The computer-mathematics interaction scale is more strongly associated with computer confidence and computer motivation scales than with the mathematical scales, suggesting that computer attitudes are more influential than mathematical attitudes in determining the level of active engagement with computer related activities in mathematical learning. A Factor Analysis using the five scales as input variables with a two-factor solution (using oblimin rotation (SPSS) following a principal components analysis) yielded the loadings shown in Table 6. The two-factor solution confirms that the computer and mathematics related scales define different dimensions with computer properties dominant in the interaction scale.

Table 6: Factor Pattern Matrix

	Factor 1	Factor 2
mconf	.11(-.06)[.02]	**.55(.87)[.88]**
mmotiv	.14(.03)[-.02]	**.85(.89)[.87]**
cconf	**.89(.89)[.84]**	-03(-.03)[.05]
cmotiv	**.92(.90)[.89]**	-.05(.02)[-.11]
cmint	**.80(.83)[.85]**	.13(.02)[.06]

Percentage of variance 67.2(69.7)[75.3]

With respect to the research question we note the properties independently confirmed among students from different cohorts at different times and in different locations. Two further potentially significant inferences emerge from this stability and robustness. Firstly the confirmation that attitudes to mathematics and computing occupy different dimensions (the respective factors are almost orthogonal), with interaction loading with the computer scales. Secondly, at least an interim conjecture regarding the following question. Given that students' prior access to and experience with computers is continually increasing, will structural differences identified between mathematics and computer based affective responses diminish with time, or do they represent distinctive sets of characteristics with a permanent presence in computer-assisted mathematics learning? The data discussed here suggest the latter.

A final point of interest is associated with the data plotted in Figure 1 which shows an item-by-item plot of the differences between the means registered by females (F) and males (M) at the University of Queensland, using 2000 data.

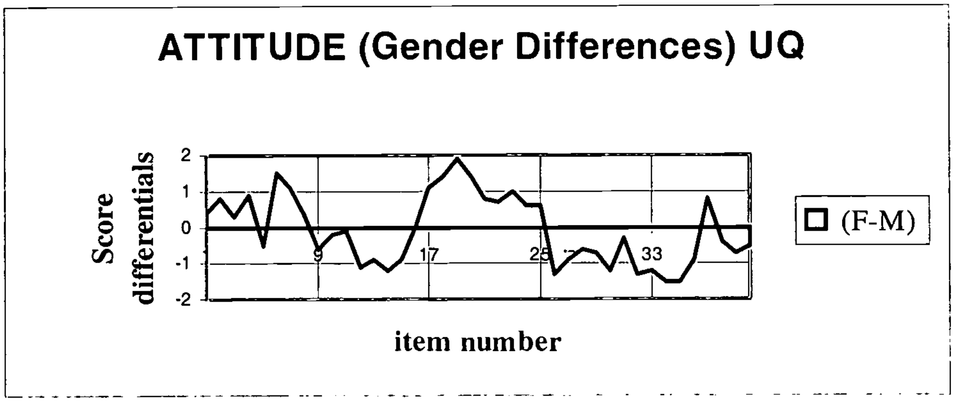

Figure 1. Gender differences on attitude scales (UQ 2000).

The vertical bars delineate the five 8 item scales, which, reading from left to right, are Mathematics confidence, Computer motivation, Mathematics motivation, Computer confidence, and Mathematics-Computer interaction. It is clear that females score more highly on the mathematics scales, and males more highly on the computer scales suggesting a systematic gender difference exists. A similar pattern occurs within other data. Both of these outcomes (robust scales and gender differences) suggest implications for the design and implementation of teaching programs that integrate computer-based activities into mathematics learning.

4. Focus C: Technology augmented Collaborative Learning

For this third focus the context is changed and the notion of technology broadened to include graphical calculators and also peripheral devices such as viewscreens. Different criteria apply when we allow the purpose of technology in mathematics teaching and learning to widen. If we are concerned purely with mathematical versatility and power, and features such as screen resolution then a symbolic manipulator may be a preferred choice. If we value portability, accessibility, and continuous access to a more restricted but still substantial range of mathematical functions then graphical calculators provide advantages. This is particularly so if the *learning environment* is a research interest. In a comprehensive review of research on graphical calculator use (in the decade to 1995), Penglase & Arnold (1996) noted a dearth of studies addressing learning environments and teaching approaches designed to maximise learning benefits. A subsequent review of research (Asp & McCrae, 2000) commented that this particular gap did not appear to have been seriously addressed, although substantial work on other aspects of graphical calculator use was noted. The teaching-learning environment remains an important context for examining alternative ways in which technologies, teachers, and students, combine in the pursuit of mathematical goals when these are not obscured by narrow definitions of desired outcomes.

Sociocultural perspectives on learning emphasise the socially and culturally situated nature of mathematical activity, and view learning as a collective process of enculturation into the practices of mathematical communities. The classroom as a community of mathematical practice supports a culture of sense making, where students learn by immersion in the practices of the discipline. Rather than relying on the teacher or textbook as an unquestioned external authority, students in such classrooms are expected to defend and critique ideas by proposing justifications, explanations and alternatives. Collaborative practices are called for, and in considering alternative models Brandon (1999) has usefully pointed out that the 'C' in Collaborative Learning has been used ambiguously to refer to both co-operative based learning (group members share the workload); and collaboration-based learning (group members develop shared meanings about their work). While interrelated there is a clear difference in the respective emphases. Collaborative activity in this latter sense, is characterised by equal partners working jointly towards an end (Anderson, Mayer, & Kibby, 1995), as a co-ordinated activity directed towards construction and maintenance of shared meaning and understanding (Rochelle & Teasley, 1995). A key element is *elaboration* (Webb & Palincsar, 1996), through which students: provide specific examples to illustrate concepts; use multiple representations (charts, diagrams etc) to explain concepts; create and evaluate analogies; translate terms; provide detailed descriptions of how to perform tasks or illustrate differences between concepts; provide detailed justifications for their problem solving; or use observations and evidence to support opinions or beliefs. These characteristics of collaborative learning, that emphasise the social construction of knowledge and shared conceptions of problem-based tasks, carry across as important elements in the design of computer based - supported collaborative learning (CSCL) as described by Brandon (1999). In generalising this property beyond computers to encompass technology in general we distance ourselves from models of 'Co-operative learning' wherein members of a group of peers are assigned individual roles (e.g. recorder, checker) prior to structured group activity. In this model role assignment may interfere with group processes by overemphasising organisational tasks at the expense of learning processes. Role assignment effectively restricts the opportunity of individuals to engage with problems freely, and to use their knowledge in the widest and most relevant way. This is in fundamental conflict with the goals that motivate a community of scholars.

A central tenet of sociocultural theory is that human action is mediated by cultural tools, and is fundamentally transformed in the process (Wertsch, 1985). The rapid development of computer and graphical calculator technology provides numerous examples of how such tools transform mathematical tasks and their cognitive requirements.

The approach then is predicated on three basic assumptions.

1. Human action is mediated by cultural tools, and is fundamentally transformed in the process.
2. The tools include technical and physical artefacts, but also concepts, reasoning, structures, symbol systems, modes of argumentation and representation.

3. Learning is achieved by appropriating and using effectively cultural tools that are themselves recognised and validated by the relevant community of practice.

The approach is informed by a Vygotskian framework, that has moved beyond the most widely known interpretation of the *Zone of Proximal Development* (ZPD) as the distance between what a learner can achieve alone and what can be achieved with the assistance of a more advanced partner or mentor. Two other representations are of particular relevance to our learning context. These are firstly the conceptualisation of the *ZPD in egalitarian partnerships*. This view of the ZPD, involving equal status relationships, argues that there is learning potential in peer groups, wherein students have incomplete but relatively equal expertise – each partner possessing some knowledge and skill but requiring the others' contribution in order to make progress. In the research context this feature becomes relevant through the collaborative activity of students in bringing technology to bear on mathematical tasks with varying levels of individual technological and mathematical expertise. One advantage of these groups is that, when the teacher withdraws, the students are provided with the opportunity to *own* the ideas they are constructing, and to experience themselves and their partners as active participants in creating and testing personal mathematical insights.

A second extension of the ZPD concept is created by the *challenge of participating in a classroom culture* constituted as a community of practice. Students as participants in a learning community are viewed as having partially overlapping ZPDs that provide a changing mix of levels of expertise that enables many different productive partnerships and activities to be orchestrated. (Brown et al., 1993; Brown & Campione, 1995) Through the establishment of a small number of repeated participation frameworks such as teacher-led lessons, peer tutoring, and individual and shared problem solving, students are challenged to move beyond their established competencies and adopt the language patterns, modes of inquiry, and values of the discipline. Such a classroom environment, representative of an active community of learners, is then augmented by the availability of technology as another agent in the search for powerful and meaningful mathematical learning and application.

To elaborate then, technology is viewed as one of several types of cultural tools - sign systems or material artefacts - that not only amplify, but also re-organise, cognitive processes through their integration into the social and discursive practices of a knowledge community (Resnick, Pontecorvo & Säljö, 1997). The amplification effect may be observed when technology simply supplements the range of tools already available in the mathematics classroom, for example, by speeding tedious calculations or verifying results obtained by hand. By contrast, cognitive re-organisation occurs when learners' interaction with technology as a new semiotic system qualitatively transforms their thinking; for example, use of spreadsheets and graphing software can alter the traditional privileging of algebraic over graphical or numerical reasoning. Accordingly, learning becomes a process of appropriating cultural tools that transform the relationships of individuals to tasks as well as to other members of their community (Doerr & Zangor, 2000).

This conceptualisation of technology usage in mathematics classrooms differs in its emphasis in that, in addition to its contribution in addressing mathematical concepts and processes, it encompasses also the sociocultural dimension: interactions between teachers and students, amongst students themselves, and between people and technology, in order to investigate how different participation patterns offer opportunities for students to engage constructively and critically with mathematical ideas. That is, while technology may be regarded as a mathematical tool (*amplifies capacity*), or as a transforming tool (*reorganises thinking*), it may also be regarded as a cultural tool (*changes relationships between people, and between people and tasks*).

5.1 Research procedures

A team of researchers, comprising a mix of academics and teachers, has been investigating the potential of collaborative learning in mathematics at pre-university level for a number of years. The student subjects are serious students of mathematics, many of whom enrol in undergraduate degrees in science and engineering in the year following their participation in the study. One particular study followed a group of students during their final two years of secondary education. On average a lesson was observed and videotaped every one to two weeks, with more frequent classroom visits scheduled if a technology intensive approach to a topic was planned. Each student had permanent access to a graphical calculator and spreadsheets were available as a normal classroom resource. Audiotaped interviews with individuals and groups of students were conducted at regular intervals to illuminate factors such as the extent to which technology was contributing to the students' understanding of mathematics, and how technology was changing the teacher's role in the classroom. This data triangulated information obtained from analysis of videotapes and questionnaires. At the beginning of the course and at the end of each year students completed a questionnaire on their attitudes towards technology, its role in learning mathematics, and its perceived impact on the life of the classroom.

The quality of mathematical exchanges is captured on the videotape record and is not reported in this paper. The interest here is in characteristics displayed as students work collaboratively, aided by technology, as a means towards collective and individual mathematical competence. While the most illuminating data are in the form of videotaped segments, featuring student and teacher discourse, (Goos et al., 2000) for present purposes we skip to a summary of some of the findings related to the learning characteristics identified. These have to do with the different ways in which students use technology, and see themselves in relation to it.

5.2 Metaphors for technology use

Observations have led to the development of a descriptive taxonomy of sophistication with which students work with graphical calculators. This is expressed in terms of metaphor.

Technology as Master. The student is subservient to the technology-a relationship induced by technological or mathematical dependence. If the complexity of usage is high, student activity will be confined to those limited operations over which they have competence. If mathematical understanding is absent, the student is reduced to blind consumption of whatever output is generated, irrespective of its accuracy or worth.

Technology as Servant. Here technology is used as a reliable timesaving replacement for mental, or pen and paper computations. The tasks of the mathematics classroom remain essentially the same—but now they are facilitated by a fast mechanical aid. The user 'instructs' the technology as an obedient but 'dumb' assistant in which s/he has confidence.

Technology as Partner. Here rapport has developed between the user and the technology, which is used creatively to increase the power that students have over their learning. Students often appear to interact directly with the technology (e.g. graphical calculator), treating it almost as a human partner that responds to their commands – for example, with error messages that demand investigation. The calculator acts as a surrogate partner as students verbalise their thinking in the process of locating and correcting such errors. Calculator or computer output also provides a stimulus for peer discussion as students cluster together to compare their screens, often holding up graphical calculators side by side or passing them back and forth to neighbours to emphasise a point or compare their working

Technology as an Extension of Self. The highest level of functioning, in which users incorporate technological expertise as an integral part of their mathematical repertoire. The partnership between student and technology merges to a single identity, so that rather than existing as a third party technology is used to support mathematical argumentation as naturally as intellectual resources.

Having constructed the taxonomy, through example and repeated observation the research team asked a group of students near the end of their course to reflect on its structure in relation to themselves as individuals. A selection of responses from the 2000 cohort is given below.

Master (M): because I often don't understand how to use every specific function of the technology, thereby limiting my use of such technology. I often don't know if I've used it correctly and as a consequence I can't be sure if my answer is correct or not.
I think I'm between *master* and *servant*. I tell the calculator what to do sometimes but only stick to what I know usually. I don't know exactly what it allows me to do, and if I haven't been taught, I won't look for it.
Servant (S): because I do not have enough knowledge of technology to be able to investigate new concepts. However I do regularly use it for *familiar* tasks purely as a time saver and to verify and check my answers.
Partner (P): Because my calculator has become my best friend. His name is Wilbur. Me and Wilbur go on fantastical adventures together through Maths land. I don't know what I'd do without him. I love you Wilbur.
Extension of Self (ES): Because my calculator is practically a part of myself. It's like my 3rd brain. I use it whenever it can help me do anything faster.

The student group had no problem reaching a personal decision and justifying it, and the 15 responses from the Year 12 students produced the following distribution. M (1), M-S (1), S (7), P (2), ES (4).

Following the earlier choice of metaphor to describe the taxonomy of sophistication with which students may work with technology, observation and discussion then suggested that a similar taxonomy may be useful in classifying instructional uses of technology.

Technology as Master

Here the teacher is subservient to the technology, and is able to employ only such features as are permitted either by limited individual knowledge, or force of circumstance. This seems clearly the case in large-scale transmissive programs where, as described by Olsen (1999), helpers are reduced to assistants responding to students on the basis of what the software has generated, and to marking computer generated quizzes. Here course organisation forces the relationship. However this circumstance may also apply in classrooms where teachers have individual autonomy. As described by Stuve (1997), pressure to be seen to implement technology following 'training', results in implementation dominated by whatever basic skill has been acquired, without consideration of impact beyond the present.

Technology as Servant

Here the user may be knowledgeable with respect to the technology, but uses it only in limited ways to support preferred teaching methods (Thorpe, 1997). That is the technology is not used in creative ways to change the nature of activities in which it is used. For example just as a calculator can be restricted to the purpose of producing fast reliable answers to routine exercises, a viewscreen may be limited to providing a medium for a teacher to demonstrate output to the class as an alternative to chalkboard, or a computer to crunching numbers faster.

Technology as Partner

Here the user has developed 'affinity' with both the class and the teaching resources available. Technology is used creatively in an endeavour to increase the power that students collectively exercise over their learning, rather than exercising it over them (Templer et al., 1998). This can occur both in the use of mathematically based technology (calculators and computers), for the purpose of enhancing individual prowess, and in the use of communications technology to enhance the quality of class learning through sharing, testing, and reworking mathematical understandings. For example, instead of functioning as a transmitter of teacher input, a viewscreen may be a vehicle for engendering otherwise non-existent student participation or act as a medium for the presentation and examination of alternative mathematical conjectures.

Technology as an Extension of Self

This is the highest level of functioning, in which powerful and creative use of both mathematical and communications technology forms as natural a part of a teacher's repertoire as fundamental pedagogical skills and mathematical knowledge. Writing courseware to support and enhance an integrated teaching program would be an example of operating at this level. Successful use of the rich electronic classroom (Shneiderman et al., 1998) would appear to demand this kind of expertise. However, ironically, too much sophisticated technology may exact a price! The sheer volume of technological choice can reduce opportunities to explore fully creative uses of individually productive items. It is noted that these levels of operating are neither necessarily tied to the level of mathematics taught, nor to the sophistication of technology available. Simple mathematics and basic technologies are sufficient to provide a context for highly creative teaching and learning. Conversely, powerful computers and expensive infrastructure can be associated with programs that are limited in what they are able to achieve, or indeed attempt.

6. Reflections

It seems almost fatuous to say that (without further qualification) the term 'technology assisted learning' is effectively meaningless. Much has been written that belongs to the genre of 'show and tell' rather than to information carefully collected and rigorously scrutinised. Almost anything can be argued to have enjoyed some success, in some form, with someone, at some time. Over a decade ago James Fey surveyed developments in the use of technology in mathematics education to that date. In noting that there was no lack of speculative writing on the promise of revolution that would follow from the application of various calculating and computing tools, he drew attention to the paucity of data available to back extravagant claims.

> It is very difficult to determine the real impact of those ideas and development projects in the daily life of mathematics classrooms, and there is very little solid research evidence validating the nearly boundless optimism of technophiles in our field. (Fey, 1989)

It is bemusing to reflect that this comment seems as relevant today as it was over a decade ago, even if the questions have become more refined. The literature confirms the existence of diverse factors that impact on the development and testing of theoretical frameworks, and on the conduct of practice. Such factors include not only inter-product competition (competing brands and genres) that extends also to users, but competing educational philosophies with respect to the teaching and learning of mathematics, and institutional politics.

It seems that one viewpoint of significance at all levels of debate, is whether technology is regarded primarily as a *learning tool* or a *power tool*. If we see calculators and computers as *power tools* then we use them as a high tech means of accomplishing mathematical tasks more quickly, or attacking problems that are intractable without the technology. Either way their use in these ways is enabled by the expert knowledge base of the user. Some of the most incisive discussion in the literature concerns the debate about whether students need to understand the mathematics independent of the technology, or whether it can be learned through technology. This raises the question of using technology as a *learning tool,* and what this means for educational practice. Those who treat mathematics as something to be transmitted and consumed, and see technology essentially as a means to this end, ignore both the message of history and the evidence accumulating from studies that pay attention to the learning context (e.g. Templer et al, 1998; Kent & Stevenson, 1999). Our work inhabits but a small corner of this domain: however consistent observations have indicated that access to technology impacts not only on task requirements, but on the culture of the learning approach, and on ways in which students reposition themselves with respect to the technology, the task, and each other. The fact that pages of output can be generated when operating with software packages gives a misleading measure of learning productivity, and creates even further need to subject such output to quality control and follow-up. Ironically this requires additional human resources at a time when institutional managers are looking to technology to reduce this very thing. The point has been underlined (Olsen, 1999) following her description of the 1.5 acre budget driven automated instruction initiative at Virginia Tech.

> Instructional software issues are unlikely to be resolved quickly... If we want the software to help at all... it's got to understand how students might misconceive what is presented to them--and to figure that out from the student's response. And right now, only people do that well. (p. 35)

The search for complexity measures for demands incurred in using *Maple* software, is an intended contribution to the 'replacement' debate - about the extent to which a student can adopt a black box mentality to software and focus on the purpose of a task. While results are preliminary they do not lend any support to the view that mathematics and technology are separable in the learning phase, and that technology essentially is a means to stronger mathematical capability among students. Put another way, it cannot be assumed that students use technology as experts use a *power tool* even when provided with sufficient enabling information. If learning is to be achieved then technology's role in initiating and consolidating understanding needs further intensive study and careful documentation. It is doubtful that enough of this is being done despite the plethora of projects using technology for instructional purposes. Studies such as Drijvers (2000) help to reinforce that obstacles arising when students work with computer algebra systems are generated by the interaction of mathematical and technological aspects. The idea then, of technology as simply a power tool to enable stronger mathematics, or as a replacement for transmissive models of teaching, is effectively rebutted by an increasing number of studies.

Work on attitudes has tended to be blurred by interactions between computers, calculators, and mathematics in programs involving technology-aided learning. Studies over many years have found that attitude and performance are related in school mathematics, although the direction of causality has been open to question. Several papers over the past five years have specifically made reference to attitude in relation to performance in undergraduate programs (e.g. Shaw & Shaw, 1997; Hensel & Stevens, 1997). Suspicion that in technology aided learning settings, confidence and motivation (in mathematics and technology respectively) may occupy different dimensions has been consistently confirmed in our research. Furthermore the results appear to be stable with no change apparent over a period of six years using students in different locations. An anticipated softening of the technology data due to increasing access and experience with calculators and computers has not eventuated. Gender differences in attitudes to mathematics and computers respectively, favouring females for mathematics and males for computers raise additional issues for course design, when technology and mathematics are brought together in undergraduate programs.

Studies on the impact of calculators and computers as cultural tools that change the nature of learning and relationships, as distinct from their agency as mathematical aids, promise to expand and challenge notions of what can be achieved in technology aided instruction. The emergence of different levels with which students see themselves using and interacting with calculators and computers also challenge approaches that see technology purely in terms of increasing mathematical power. Failure to recognise taxonomies of competence, preference, and confidence in using technologies increases the risk that inappropriate expectations and methods of instruction will drive course design and implementation. The risk that through unquestioned acceptance of a perceived authoritative source, a 'tyranny of the text' becomes replaced by a 'tyranny of technology' emphasises the role of the teacher as a custodian of mathematical values that must be continually articulated and embedded in instructional practices (Guin & Trouche, 1999). As the use of calculators and computers as cultural and mathematical tools in communities of practice approaches to learning become more prevalent in secondary education, there are implications for the design and implementation of undergraduate courses into which the students subsequently flow.

Finally, in order to make more systematic progress in evaluating quality and identifying problems we need to look at improving the relevance of research methods. It is probably fair to say that a substantial majority of us received research training within the scientific paradigm of the controlled experiment. Many have questioned its relevance in testing for outcomes of quality in educational settings-many more need to do so. What is valuable in knowing that approach A achieves statistically better results than approach B when both are terrible, and about 5% of variance is involved? Furthermore it is frequently not clear that the condition being 'tested' has been faithfully applied. Some unsuccessful attempts to replicate the success of Schoenfeld's (1985) problem solving program with College students provide cases in point. Johnson & Fishbach (1992) and Lester et al., (1989) reported studies that foundered in their attempts to replicate the success of the teaching approach advanced by Schoenfeld. While these studies specifically implemented elements of that teaching program (in terms of strategies), they did not nurture and sustain the culture of "mathematics community" that was of equal or greater importance. In the former study, the College students, used to other methods of mathematics teaching, were uncomfortable with the learning approaches and setting. On the other hand, their teachers were uncomfortable with the teaching style required of them, which was substantially different from that developed over many years. No positive change was achieved over a ten-week period. In the latter study, two classes of primary year 7 students showed little 'improvement' in metacognitive control behaviours over the seven weeks of the trial. These students had limited domain specific knowledge on which to draw, were reluctant to reflect on strengths and weaknesses, and inexperienced in the small group settings which formed a key part of the instructional program. Failure to establish a community of practice culture renders invalid attempts to evaluate the effectiveness of teaching strategies that necessarily draw from such a culture. Yet parallels to this failure, often compounded by inadequate reporting, torment study after study. This is quite apart from an increasing concern with ethical considerations that would question the integrity of studies that allocate a group of subjects to a 'treatment' believed to be inferior! The social context of the classroom is an inextricable component in the development of a community of practice. It becomes central therefore to locate identifiers by means of which the operation of such a community can be recognised, monitored and developed, and within which the achievements of teaching approaches can be assessed. Such methods involve establishing criteria against which to measure the quality of outcomes, for which purpose the use of videotapes, transcript analysis, and other methods of triangulation augment written data. Qualitative research methods and Grounded Theory approaches need to complement appropriate applications of quantitative methods more than they have so far managed to do. The development and implementation of rigorous research within a rich environment of outcomes is perhaps our greatest challenge in seeking to test and improve the effectiveness of instructional strategies involving technology.

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Appendix

Sample Questions

(Questions in italics: *Maple* commands in bold: *Maple* output in ordinary type)

Q2. Factorize $x^3 - 6x^2 + 11x - 6$

Maple Solution

> **factor(x^3-6*x^2+11*x-6);**

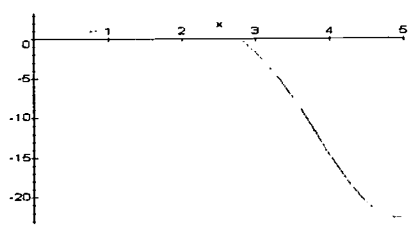
(x - 1) (x - 2) (x - 3)

Q8. Find where the graph of $x^2 \sin x + x \cos x$ *for* $0 \le x \le 5$ *is:*

(a) above the $x-axis$ *(b) below the* $x-axis$ *(c) cuts the* $x-axis$.

Maple Solution

> **plot(x^2*sin(x)+x*cos(x),x=0..5);**



> **x1:=fsolve(x^2*sin(x)+x*cos(x),x=2..3);**

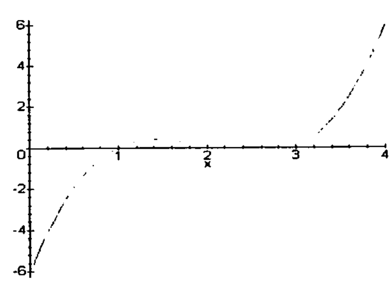
x1 :=2.798386046

****

Q14. Plot the graph of f(x) = (x-1)(x-2)(x-3) and use this to find the physical area under the graph from x= 1 to x= 3.

Maple Solution

> **y:=(x-1)*(x-2)*(x-3);**
> **plot(y,x=0..4);**



> **int(y,x=1..2)-int(y,x=2..3);**
> 1/2

THE ROLE OF VISUALIZATION
In the Teaching and Learning of Mathematical Analysis

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ABSTRACT

In this paper a brief introduction is presented to the nature and different types of mathematical visualization. Then we shall examine some of the influences visualization has had on the development of mathematics and its teaching, exploring in particular its current status. We then inspect the particular role it may have in what concerns mathematical analysis and the difficulties that surround the correct use of it, with or without the computer. Finally a sample of exercises in visualization in basic real analysis is presented in order to show with examples its possible role in the teaching and learning of this subject.

1 What is visualization in Mathematics?

The following story may convey the savor of visualization much better than many analyses. The protagonist here is the great Norbert Wiener, but I am sure that most mathematicians have been able to observe something similar happening to more than one of his or her teachers or colleagues. Wiener was giving one of his lectures at the MIT before a numerous audience.He was immersed in the intricate details of a complicated proof. The blackboard was almost full of formulas and he was marching on unblinkingly towards his goal. Suddenly he got stuck. One minute, two minutes,... To the students it seemed the end of the world... The great Wiener stuck...incredible! He was looking at the formulas, he was messing his hair, he was humming..., until he seemed to know what to do. He went with decision to one of the still empty corners of the blackboard and there he stayed for a little while drawing some mysterious pictures. He did not say a word, his great shoulders almost concealing from everybody what he was doing. Finally he sighted with relief, erased with care what he had drawn and went back to the point he had interrupted his proof and concluded it without any hesitation.

Mathematical concepts, ideas, methods, have a great richness of visual relationships that are intuitively representable in a variety of ways. The use of them is clearly very beneficial from the point of view of their presentation to others, their manipulation when solving problems and doing research.

The experts in a particular field own a variety of visual images, of intuitive ways to perceive and manipulate the most usual concepts and methods in the subject on which they work. By means of them they are capable of relating, in a versatile manner the constellations of facts and results of the theory that are frequently too complex to be handled in a more analytic and logic manner. In a direct way, similar to the one in which we recognize a familiar face, they are able to select, through what to others seems to be an intricate mess of facts, the most appropriate ways of attacking the most difficult problems of the subject.

The basic ideas of mathematical analysis, for instance order, distance, operations with numbers,... are born from very concrete and visualizable situations. Every expert is conscious of the usefulness to relate to such concrete aspects when he is handling the corresponding abstract objects. The same thing happens with other more abstract parts of mathematics. This way of acting with explicit attention to the possible concrete representations of the objects one is manipulating in order to have a more efficient approach to the abstract relationships one is handling is what we call mathematical visualization.

The fact that visualization is a very important aspect of mathematics is something quite natural if we have into account the meaning of the mathematical activity and the structure of the human mind. Through the mathematical activity man tries to explore many different structures of reality that are apt to be handled by the process we call mathematization in the following way. Initially we have the perception of certain similarities in the real objects that guide us to the abstraction from these perceptions of what is common and to submit it to a peculiar rational and symbolic elaboration that allow us to efficiently handle the structures which lie behind such perceptions.

Arithmetic, for example, arises with the intention to rationally dominate the multiplicity what is present in reality. Geometry tries to rationalize the properties of the form and extension in space. Algebra, in a second order abstraction process, explores the structures lying behind numbers and operations related to them. It deals with a sort of symbol of symbol. Mathematical analysis arose in order to deal with the structures of change of real things in time and in space,...

The mathematization process has proved to be extraordinarily useful in order to better understand and manipulate the common structures of many real things. Our human perception is very strongly visual and so it is not surprising at all that the continuous support on its visual aspect is so entrained in many of the tasks related to mathematization, not only in those that, like geometry, deal more directly and specifically with spatial aspects, but also in some others, like mathematical analysis, that arose in order to explore different kinds of changes occurring in material things.

Even in those mathematical activities in which abstraction seems to take us much beyond what is perceptible to our material vision, mathematicians very often use symbolic processes, visual diagrams, and many other forms of mental processes involving the imagination that accompany them in their work. They help them to acquire what we could call a certain intuition of the abstract, a set of mental reflexes, a special familiarity with the object at hand that affords them something like a holistic, unitary and relaxed vision of the relationships between the different objects of their contemplation. In this way they seem to know in advance how these different objects are going to react when they introduce some convenient changes in some part of the structure.

Visualization appears in this way like something absolutely natural not only in the birth of the mathematical thought but also in the discovery of new relations between mathematical objects and also, of course, in the transmission and communication processes which are proper to the mathematical activity.

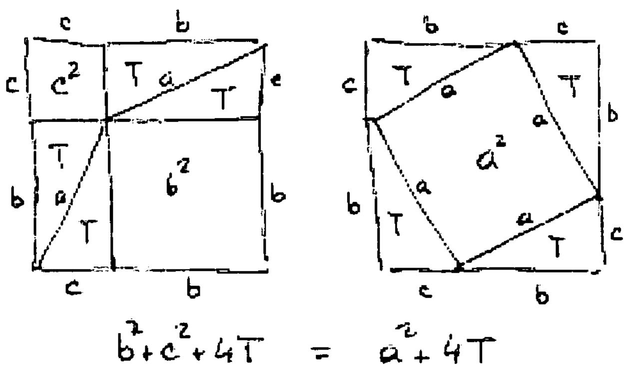
2 Different types of visualization

Our human visualization, even the apparently superficial phenomenon that we call "vision" in its more physiological sense, is not a process that merely involves the optical processes of our eyes. It is much more complex, since it entails in a quite important form, the activity of our brain. Perhaps in the newly born child the phenomenon that takes place is much more similar to the one occurring in a photographic camera, but the cerebral processes that immediately start taking place in his brain cause that, after a rather short time, after experimenting with the objects of the world outside the child transforms his vision into a true mental interpretation of what before was a simple physical optical phenomenon.

The visualization experiences with which we are going to deal here have a much more interpretation weight. In many of the forms of visualization we are going to experiment we have to follow a true process of codification and decodification in which intervene very crucially a whole world of personal and social interchanges, a good part of them firmly rooted in the history of the mathematical activity.

This makes the process of visualization largely based in the interaction with many person around us and in the immersion and enculturation in the historical and social context of mathematics. Visualization is therefore not an immediate vision of the relationships, but rather an interpretation of what is presented to our contemplation that we can only do when we have learned to appropriately read the type of communication it offers us. Here we have an example.

The following figure uses to be presented as a paradigm of a visualization in mathematics, a proof of Pythagoras' theorem. Probably the novice who looks with attention to this drawing arrives to see, with some luck, two equal squares that have been dissected in two different ways and perhaps will be able to understand, through the written indications, that the square over the hypotenuse of the rectangular triangle that arises, that seems to be copy of the other two that appear in different positions in the figure, have an area that is equal to the sum of the areas of the other two squares over the other two sides of the triangle.



But in order to arrive to the Pythagoras' theorem it will be necessary that he may prove that those triangles marked with T are of the same area, and that this same situation appears in any possible rectangular triangle, i.e. he needs to perceive that he is having before his eyes a generic situation.

The purported absolute immediacy of this dissection in order to show the general truth of Pythagoras' theorem is to a certain point deceiving, since it requires for such a purpose an involved work of decodification that is obvious to the expert, but far to be open to the novice. This consideration is one of the reasons why the introduction to visualization, for example in the teaching and learning of mathematics, is not an easy task that requires the clear conscience that the transparency of the process, perhaps real for the teacher because of the familiarity, acquired by the continued practice along many years, may be absent at all for the one who starts with this type of process.

But the presence of this type of decodification process in any visualization makes clear that mathematical visualization is not going to be a univocal term at all. According to the degree of correspondence between the mathematical situation and the concrete way of representation, that can be more or less close, natural, symbolic, even more or less personal and perhaps incommunicable... there are going to be many different types of visualization. In what follows I am going to try to distinguish several of them. At the light of some examples we can try to perceive the deep differences among them and some of the difficulties inherent to their practice.

Isomorphic visualization

The objects may have an "exact" correspondence with the representations we make of them. This means that, in principle, it would be possible to establish a set of rules to translate the elements of our visual representation and the mathematical relations of the objects they represent they represent. In this way the visual manipulations of the objects could be transformed, if we so desire, into abstract mathematical relationships. This kind of representation might be called an isomorphic visualization.

The modelization of a mathematical problem, which in many cases is possible, may be in many cases an isomorphic visualization. Its usefulness is rather manifest. The manipulation of the objects that we perceive with our senses or with our imagination is normally easier and more direct than the handling of abstract objects, that frequently may be rather complicated in its structure.

An example: Josephus problem.

In his book *De bello judaico*, Heggesipus tells about the siege by the Romans of the city of Jotapat. Josephus and other 40 Jewish men took refuge in a cave near the city and decided to kill themselves rather than surrendering. To Josephus and to a friend the idea was not making them very happy. They decided to take their measures. They suggested to do it in a certain order. All men should set themselves in a circle and, starting by an enthusiast who by all means wanted to be the first in killing himself, they would commit suicide by turn counting three. Josephus's idea, of course, was to place himself and his friend in such a way that they would be the two last ones in this order and so, being in absolute majority after the massacre of all the others, to decide to stop it. What places should Josephus and his friend take in order to accomplish their purpose?

The solution is rather easy. One takes 41 little stones, marks each one of them with a number 1, 2, 3,..., 41. One simulates the suicides and looks which two stones are left at the end.

The handling of the problem is clearly isomorphic and shows one of the shortcomings that can accompany visualization. We have been able to solve this particular problem, but the solution is going to vary when, for instance, there are 47 instead of 41 stones or when one counts five instead of three. Our visualization solves our particular problem, but the mathematician is interested in knowing what to do when there are m stones, one puts them in a circle and takes out the n-th one starting by a particular stone in a definite orientation.

We are confronted with a situation similar to the previous one concerning Pythagoras theorem. Will it happen in general what I observe in this particular triangle? There, after a rather simple conceptual elaboration one can arrive to the fact that the situation is in fact generic, independent of the rectangular triangle considered. Here, however, our manipulation only has solved our particular problem. Not a little achievement and besides, from such concrete manipulations very often arise very illuminating ideas which lead us to the general solution of our abstract problem.

A great part of our visualizations in mathematical analysis is of this isomorphic kind. They are probably the ones that mathematicians accept and use more profusely without objections. The visualization of the real numbers on the real line or that of the complex numbers by means of the points in the plane not only made its incursion in mathematics without resistance, but in the case of the complex numbers (Argand, Gauss), it was the means that made possible the general acceptance of this expansion of the number system against the resistance to admit complex or imaginary numbers as decent and honest mathematical objects.

In any case one has to be aware that our visualizations contain many aspects that have to do with tradition, tacit agreements, consensus and this makes them dependent in their use of a whole code to understand them that has to be transmitted, acquired and made sufficiently familiar to each one of their users. It is true that *"an image is worth a thousand words"*, but this presupposes an important condition, that the image comes to be correctly deciphered and understood. Otherwise an image is worth nothing.

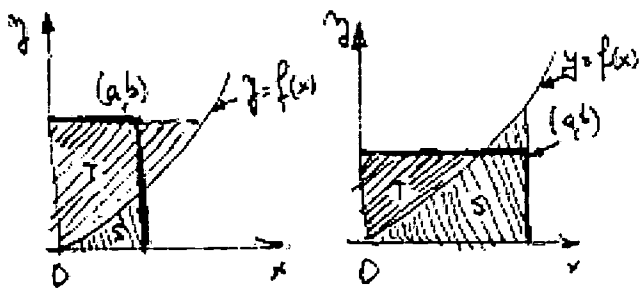
Another example of isomorphic visualization: Young's theorem.

Young's theorem, an inequality with plenty of important applications in analysis affirms the following.

Let y=f(x) be a real function defined on [0,inf) such that $f(0) = 0$, $f(x) > 0$ for each $x > 0$, f is continuous and strictly increasing on [0,inf) and $f(x)$ tends to infinity when x tends to infinity. Let y=g(x) the inverse function of f, i.e. for each x in [0,inf) we have g(f(x))=x. Then, for each pair of positive numbers a and b, one has

$$ab \leq \int_0^a f(x)dx + \int_0^b g(x)dx$$

The proof of this interesting result becomes obvious by merely inspecting the following figure



The inequality stated above simply affirms that the area of the rectangle with opposed vertices at the points (0,0) and (a,b) is less than or equal than the sums of the shaded areas S and T of the picture. The equality is exactly obtained when b=f(a), i.e. when the point (a,b) is a point of the graph of y=f(x). It would not be difficult at all to translate this into a completely formalized proof, if one has to content somebody with a especial desire of rigor.

Homeomorphic visualization

In this kind of visualization that I am calling "homeomorphic" some of the elements have certain mutual relations that imitate sufficiently well the relationships between the abstract objects and so they can provide us with support, sometimes very important, to guide our imagination in the mathematical processes of conjecturing, searching, proving,... Let us analyze an example that might be useful in order to make clear the nature of the homeomorphic visualization.

The Schröder-Bernstein theorem

Let A and B be two sets. Assume there exists an injective function f (i.e. a one-to-one mapping) from A to B and another injection g from B to A. Then there is a bijection h from A to B, i.e. an injection h such that h(A)=B. The following simple and elegant proof which appears in the classical and well-known textbook *Modern Algebra*, by Birkhoff and MacLane, is based on a convenient visualization of the sets and mappings of the statement. The presentation will be very succinct but, I hope, sufficiently clear.

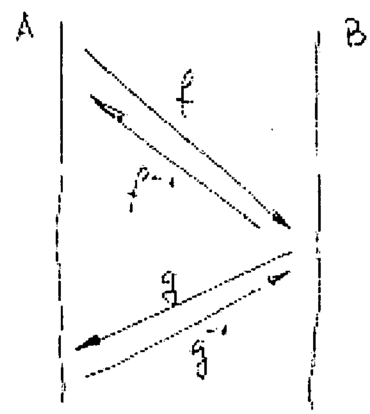
We start by representing the two sets A and B by the two straight lines of the figure above and the functions f and g by the descending arrows of the figure. We consider their inverse functions f^{-1} and g^{-1} and represent them by the corresponding ascending chains (we shall also consider as a chain a point in A or B that has no ascending arrow starting from it). We consider the ascending chains of linked arrows and classify them in the following way:

Class 1: ascending chains that end in A

Class 2: ascending chains that end in B

Class 3: chains that never end, i.e. chains that either are cyclic or pass through infinite points.

It is easy to see that this classification of the chains induces a classification of the points of A (and of B) into three disjoint sets according to the type of chain that goes through it.



And now we can easily define the bijection $h(x)$ we are looking for:

$$h(x) = \begin{cases} f^{-1}(x) & \text{if } x \text{ is of type 2} \\ g^{-1}(x) & \text{if } x \text{ is of type 1} \\ f(x) & \text{if } x \text{ is of type 3} \end{cases}$$

To check now that h is a bijection is an easy matter.

Here it is quite clear that our sets A and B may have nothing to do with straight lines, that our reference to the "ascending" and "descending" chains in the proof of the theorem is totally arbitrary, but they give us a very useful mental support for the key idea of "inverse image of a mapping" that is here the key for our proof.

And it is also quite clear that we could efface any visual connotation and write a completely formal proof that could astonish our reader who would keep wondering where our magnificent ideas could come from. Unfortunately this has been the prevalent fashion for quite a long time in papers, textbooks, lectures... inspired in such a style of mathematical miscommunication.

In this example it becomes manifest the power of this type of homeomorphic visualization that in many cases can become a quite personal and subjective process, perhaps often not easily communicable, but in any case the effort to hand it over to our students is worth doing.

Analogical visualization

Here we mentally substitute the objects we are working with by other that relate between themselves in an analogous way and whose behavior is better known or perhaps easier to handle, because it has been already explored.

This kind of visualization or analogical modelization was one of the usual discovery methods used by Archimedes, according to what he tells his friend Eratostenes in the famous letter which is known by *The Method.* There are many spectacular discoveries by Archimedes, for example his calculation of the volume of the sphere, which was first obtained by following this way of analogies and thought experiments of mechanical nature.

The following example, which arose in a workshop on solving problems with university students, can illustrate the way of proceeding.

The problem is the following: *we are given four segments of lengths a, b, c, d, with which one can form a convex quadrilateral in the plane of side lengths a, b, c, d, in this order. It is clear that if we can form one, then we can form many different convex quadrilaterals. Among them find the one enclosing the maximal area.*

The mechanical problem that can provide the adequate analogy leading to the solution is the following. We are given four thin rods forming an articulate plane and convex quadrilateral. We enclose it in a big soap film that contains the quadrilateral in its interior. We puncture the film at a point inside the quadrilateral. The equilibrium position of the rods will be such that that the tension of the soap film outside is minimal, i.e. the area of the quadrilateral is maximal.

Therefore our problem is reduced to find the equilibrium position of the rods in this situation. The forces acting on our system are reduced to four perpendicular forces to the sides applied at their midpoints, directed towards the exterior of the quadrilateral and each of magnitude proportional to the length of the corresponding side. It is easy to see that the equilibrium is obtained when the four perpendiculars at the midpoints of the sides concur, i.e. when the quadrilateral can be inscribed in a circle. This solves our original problem.

The use of the analogical method should not surprise any mathematician. It has been very often put to work in mathematics, not only by Archimedes but also, for instance, by Johann Bernoulli in his analogical solution to the brachistochrone problem proposed by him in the *Acta Eruditorum* "to the most acute mathematicians of the whole world". In this case an analogy with the behavior of the light rays was the guide towards his solution.

Even the most ingrained formalist should consider that the fields on which such analogies are based are capable of the most rigorous development, if this is what one should strive for.

Diagramatic visualization

In this kind of visualization our mental objects and their mutual relationships concerning the aspects which are of interest for us are merely represented by diagrams that constitute a useful help in our thinking processes. One could say that in many cases such diagrams are similar to mnemotechnic rules.

The tree diagram we use in combinatorial theory or in probability and many others each mathematician develops for his or her own use, of a very personal nature, are of this type. Such symbolizations and diagrams become in some cases of generalized use, but in many cases they are of a very personal, individual use, and cannot be easily shared with others.

But in many cases they could be communicated with little effort to many others that would find them extremely useful. However sometimes people think that such images, diagrams,... constitute a real obstacle for the development of the individual in mathematics, since what matters, they say, is only the formal justification of our arguments.

It is my opinion that the success that is experimented by the great teachers in mathematics is very often due to the efforts they make to transmit to others and to share with them not only the results of theirs and others researches, but also the processes by which somebody somewhere was able to obtain such results.

When one examines the mathematical writings of Euler, *the teacher of us all*, one perceives this expositive quality of one of the great geniuses of mathematics.

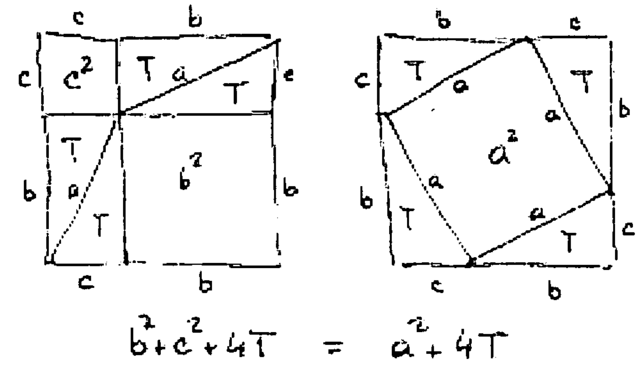
It is clear that the classification of the possible types of visualization we have seen here is neither exhaustive nor a clear cut one. There will be obviously many cases which cannot be enclosed in anyone of the types we have described here.

3 Visualization over the centuries

What has been the role of visualization along time? We shall briefly examine some of the most significative points.

The visualization at the origin of modern mathematics

The Greek word *theorein* means "to contemplate" and *theorema* is what is contemplated and not, as we now understand it, what is proved. In particular, among the early Pythagoreans who first cultivated mathematics in our modern sense, the study of the numbers and the relationships among them was performed by means of different configurations done by means of pebbles, small stones, *psefoi*, in Latin *calculi*. As a token here we can see above two of their most simple theorems.



For the Pythagoreans visualization was something connatural to the exercise of mathematics. In Plato the specific role of the image in the mathematical construction is more explicit and strongly emphasized. The image evokes the idea as the shadow evokes the reality. The drawn circle is not the reality. The real thing is the idea of the circle, but its image plays a very important role as evocative element of the idea. The way of knowledge he calls *dianoia* is very specific of the mathematical knowledge. The mathematician gets close to the intelligible through the reference to the sensitive.

The Elements of the mathematicians preceding Euclid probably contained, as Euclid's Elements do, many references that form an indispensable part of the text. But one can venture that it was probably in Euclid's lost *Book of Fallacies* where the references to geometrical paradoxes and fallacies had a especially important role. One could guess that this book rather than the *Book of Elements* could have been the one that was used by Euclid and his pupils in his learning practice.

As we have already seen, Archimedes used with advantage his analogical method as a very fundamental tool for his mathematical discoveries, although, one has to add, with a certain sense of embarrassment.

The modern classics

Descartes, in his *Regulae ad directionem ingenii*, has several rules that directly involve visualization processes. He strongly emphasizes the different roles of images and figures in the mathematical thinking.

Here one can see three of the most significative rules in this context:

REGULA XII.

Denique omnibus utendum est intellectus, imaginationis, sensus, et memoriae auxiliis, tum ad propositiones simplices distincte intuendas, tum ad quaesita cum cognitis rite comparanda ut agnoscantur, tum ad illa invenienda, quae ita inter se debeant conferri, ut nulla pars humanae industriae omittatur.

(Finally it is necessary to make use of all the resources of the intellect, of the imagination, of the senses and the memory: on the one hand in order to distinctly feel the simple propositions, on the other hand in order to compare that which we are looking for with what is already known, in order to recognize those; and also to discover those things that must be compared to each other in such a way that no element of the human ability is omitted).

REGULA XIV.

Eadem est ad extensionem realem corporum transferenda, et tota per nudas figuras imaginationi proponenda: ita enim longe distinctius ab intellectu percipietur.

(This rule must be applied to the real extension of the bodies. It all must be proposed to our imagination by means of pure figures. Since in this way it will much more distinctly perceived by the intellect).

REGULA XV.

Juvat etiam plerumque has figuras describere et sensibus exhibere externis, ut hac ratione facilius nostra cogitatio retineatur attenta.

(It is also useful in many occasions to describe these figures and to show them to our external senses, so that in this way our thought might maintain more easily its attention).

It seems also clear that the original idea driving Descartes to the development of the analytic geometry arose as an attempt to combine the geometric image of the ancient Greeks with the already at his time sufficiently well structured algebra.

The calculus of the seventeenth century is born with a very strong visual component and remains so in the first centuries of its development, in continual interaction with geometrical and physical problems. The following words of Sylvester may summarize and represent the feeling of some of the great classics of mathematics about visualization: *"Lagrange... has expressed emphatically his belief in the importance to the mathematician of the faculty of observation. Gauss has called mathematics a science of the eye..."* (The Collected Works of James Joseph Sylvester, Cambridge University Press, 1904-1912, quoted by Philip J.Davis (p.344) in *Visual Theorems*, Educational Studies 24 (1993) 333-344).

Visualization, as we see, has been a technique generally used by the most creative mathematicians of all times. One or other type of image accompanies their mathematical lucubrations, even the most abstract, although the nature of these images presents a difference from person to person much greater than we suspect.

Visualization, as we can see through these small samples extracted from the history of mathematics, has played a very important role in the development of mathematics. And so it had to be, given the peculiar structure of human knowledge, very strongly conditioned by visual, intuitive, symbolic, representative elements, and given the nature of mathematics and its purposes of obtaining an image, as accurate as possible, of the world around us.

The formalism of the 20th century and the visualization

In spite of the role played traditionally by visualization, the formalistic tendencies prevailing during a good part of the 20th century, as we shall see in a moment, had as a consequence a sort of demotion of visualization to an inferior position. Visualization was looked upon with mistrust and suspicion. It would take too long to analyze the reasons that may cause this situation, but I try to schematically pinpoint some of them.

The rational status of the Calculus in the 17th century was beset by doubts and confusion and it was not until the end of the 19th century, with the arithmetization of analysis, that became free of any doubt.

The non Euclidean geometry's in the middle of the 19th century lead many persons to be highly diffident of intuition in mathematics.

The initial polemic against Cantor's set theory at the end of the 19th century and the paradoxes around the foundations of mathematics drove many mathematicians to emphasize the formal aspect in the structure of mathematics, trying to achieve in them a solid basis for the mathematical edifice.

The results falsely or incompletely proved (for instance, of the four-color theorem or the Jordan closed curve theorem) based on a naive confidence in certain intuitive elements contributed to foster a more rigorous attitude towards the intuitive proofs, looking with distrust the merely intuitive arguments.

All these facts lead to create a trend towards the strict formalization, not only in what is related to the foundations of mathematics, what seemed to be amply justified, but also in what relates to the normal intercommunication among within the mathematical community and even, what is still much worse, in what attains the mathematical teaching and learning processes at every level. The consequences were very serious in what visualization concerns. The atmosphere of mistrust so created lead some mathematicians to aggressively advocate a more or less complete abandon it. The influence of formalism in the presentation of new results and theorems in the journals was the unavoidable norm. Even the structure of text books at the university level, and sometimes even at secondary and primary levels (*"modern mathematics"*) tended to conform to the same standards.

As a sample of such attitude one can read a couple of sentences in the introduction of a text book by Jean Dieudonné on linear algebra and elementary geometry: "I have decided to introduce not a single figure in the text... It is desirable to free the student as soon as possible of the straitjacket of the traditional "figures" mentioning them as scarcely as possible (excepting, of course, point, line, plane)..."

The model for the mathematical activity for long time was the formalist model, and even the teaching at the secondary level in many countries was contaminated by such tendencies.

One can find a clear testimony of such tendencies together with a brief attempt to explain it in the work *A Mathematician's Miscellany* by J.E. Littlewood, where he openly acknowledges the many benefits of visualization in his own research work.

"My pupils *will* not use pictures, even unofficially and when there is no question of expense. This practice is increasing; I have lately discovered that it has existed for 30 years or more, and also why. A heavy warning used to be given (*footnote*: To break with 'school mathematics') that pictures are not rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety. Some pictures, of course, are not rigorous, but I should say most are (and I use them whenever possible myself). An obvious legitimate case is to use a graph to define an awkward function (e.g. behaving differently in successive stretches): I recently had to plough through a definition quite comparable with the "bad" one above, where a graph would have told the story in a matter of seconds."

(In *Littlewood's miscellany*, edited by Béla Bollobás (Cambridge University Press, Cambridge, 1986), p.54)

Towards a return of visualization?

What is the present situation? It seems that in the last decade or so one can perceive a much more flexible attitude and a certain tendency toward a renewal of the influence of visualization in the mathematical activity, teaching, learning, doing research and publishing it. With decision, especially among many of those who do research in mathematics education. With many different attempts, not always very successful, among those who have tried to explore the possibilities of the computer for the mathematical tasks. And also with certain inertia, if not opposition, of a good part of the mathematical community.

4 The role of visualization in Mathematical Analysis

The image, as we have seen, has very important uses in many different types of mathematical activity. The image is frequently the matrix from which concepts and methods arise. It is a stimulating influence for the rise of interesting problems in different ways. It often suggests relationships between the different objects of the theory which are in a way somewhat difficult to detect by just logical means. It suggests in subtle ways the path to follow in order to solve the most intricate problems of the theory and even those connected with the development of the theory itself. The image is also a very powerful tool to grasp in a unitary and holistic way the different contexts constantly arise in the different task connected with the theory. It is also a rapid vehicle for the communication of ideas. It is also an auxiliary tool for the unconscious activity around the most obscure problems connected with it.

Visualization is therefore extraordinarily useful in the context of the initial process of mathematization as well as in that of the teaching and learning mathematics. All this makes very clear the convenience of training our own visual ability and to introduce to it those whom we are trying to introduce to mathematics. This applies not only to geometry, where all these considerations are quite obvious, but also to, for instance, mathematical analysis. The ideas, concepts, methods of analysis have a great richness in visual, intuitive, geometrical contents, that are constantly arising in the mental workings of the analyst. It was not in vain that mathematical analysis arose as a need to quantitatively mathematize in the first place the spatial relationships of the objects of our ordinary life. These visual aspects are present in all kinds of activities of the mathematician, in the presentation and handling of the most important theorems and results as well as in the task of problem solving. They seldom pass over to the written presentation, perhaps partly because of the difficulty inherent to this task, and in some other occasions because of the adherence to the most fashionable form of presentation *"the more formal, the better"*.

In fact the experts in a field of mathematical analysis have visual images, intuitive ways of approaching certain usual situations, imaginative ways of perceiving concepts and methods of great help for them and that would be of great value also for others in their own work. The experts, through the assistance of such visual tools, are able to relate, in a very versatile and flexible way, constellations, frequently very involved, of facts and results of the theory and through such relationships they are able to select in a co-natural way and without effort the most adequate strategies for solving the problems of the theory.

These images are able in many cases to offer all the necessary elements to build, if one so wish, the whole formal structure of the corresponding theoretical context or the problem. The expert knows, even without having done it so, that just by investing the necessary amount of time and by accepting to suffer the corresponding boredom inherent to the task, they, or any other, would be able to afford all the necessary ingredients to build up a proof capable to satisfy the most exacting appetite of rigor.

The following testimony of Hadamard on the role of visualization is quite representative of the influence of the image in the mathematical processes of an analyst:

"I have given a simplified proof of part (a) of Jordan's theorem [*that the continuous closed curve without double points divides the plane into two different regions*]. Of course, my proof is completely arithmetizable (otherwise it would be considered non-existent); but, investigating it, I never ceased thinking of the diagram (only thinking of a very twisted curve), and so do I still when remembering it. I cannot even say that I explicitly verified or verify every link of the argument as to its being arithmetizable (in other words, the arithmetized argument *does not* generally appear in my full consciousness). However, that each link can be arithmetizable is unquestionable as well for me as for any mathematician who will read the proof: I can give it instantly in its arithmetized form, which proves that that arithmetized form is present in my fringe-consciousness." (Jacques Hadamard, *The Psychology of invention in the mathematical field*, p.103, footnote).

My opinion is that one of the important tasks of the expert in analysis in his intention of introducing the young students to his or her field should be to try to transmit not only the formal and logical structure of the theorems in this particular area, but also, and probably with much more interest, to offer them these strategical and practical ways of the profession with which he or she has perhaps learned and become familiar with much effort through the passage of the years. They are probably much more difficult to make explicit and assimilable to the students, precisely because they are often located in the zones less conscious of the activity of the expert. It is quite clear that this task is going to present many aspects that are strongly subjective and that they are much more difficult to make explicit and assimilable for our students, precisely because of the fact that they are situated in the zones less conscious of the own activity of the expert.

By its own nature this task is going to involve many elements that are strongly subjective. The ways to visualize and to make more close and intuitive the ideas of mathematical analysis to make them work in certain concrete problems and situations are going to depend in an intense way of the mental structure of each one. The degree of help the visual support affords varies, with certainty, in a strong way, from individual to individual. What for one helps perhaps may be a hindrance to some other person. But these differences should not represent an obstacle in our attempts to offer with generosity to other those instruments that for us have resulted quite useful in our work to such a point that this work without them would be much more difficulty, abstruse and boring.

5 Difficulties around visualization

Obstacles and objections

There are many obstacles and objections that hinder a more decisive progress in order to put visualization in the right place it deserves in the job of communicating and transmitting mathematics at the educational level and also to restore its status in the tasks concerning research. Here we present some of them.

"Visualization leads to errors"

It is quite true that an incorrect use of visualization can lead us to errors in different ways. Sometimes because the figure we rely upon suggests a situation that in fact does not take place. This is the case of many geometrical fallacies like the ones to be found in the classical book by W.W.Rouse Ball *Mathematical Recreations and Essays,* Chapter III. An efficient way to get rid of such false arguments that seem to originate in an incorrect interpretation of the figure is to consider a figure similar to one proposed but in an extreme position of its elements. It often happens that our intuition leads us to a false conclusion because the figure in question approximates the one that in fact takes place. When we take a similar figure in a limit position, the truth shows up.

In some other cases the visual situation misleads us to accept certain relationships that appear so highly obvious that never comes to our mind the need or the convenience to justify them more rigorously. Euclid's axioms, for instance, with all its astonishing maturity, are not exempt from some very subtle gaps coming from this type of geometrical situations that had to be corrected by Hilbert in his *Grundlagen der Geometrie* (1902).

The *"proof"* built up by Arthur Kempe in 1879 of the *"four-color theorem"* was based in a geometric relationship that, although false, seemed so clear that it was accepted by the mathematical community of the moment until 11 years later when Heawood became aware of the fact that the proof was incomplete. By the way, it was Kempe's attempt the one that inspired the strategy, more than one century later, that lead Appel and Haken to a successful proof of the fact that four color suffice to appropriate color any particular map.

The *"proofs"* by Jordan (1893) and by other mathematicians of the visually obvious fact that a plane simple closed curve divides the plane into exactly two regions, the interior and the exterior, were not rigorous, since they contained assertions without rigorous justification based on intuitive relationships. Later on such assertions were established with a considerable effort. The first correct proof came 20 years after the statement of the theorem by Jordan in the work of O.Veblen (1913).

But the possibility that visualization can lead to error should not be a valid argument against its efficiency in the different processes of the mathematical activity, as well in the creative tasks it entails as in the processes of communication and transmission. Even the most formal techniques are open to errors, incomplete reasoning, fallacies,... And one should take this fact as something quite natural.

Mathematical thinking is not normally presented through a completely formalized exposition that could be automatically checked and controlled in each one of its steps. The communication style of mathematicians is at the moment rather far from that stage and it is probable that it will keep for some time. On the other hand it is debatable whether it would be convenient to adopt such a style of communication, if it becomes possible. The mathematical language is today a sort of mixture halfway between the natural language and the formalized language, a rather bizarre jargon consisting of elements of the natural language, some esoteric words, and logical and mathematical symbols. And in this curious mixture mathematicians are constantly alluding, in a more or less explicit way, to certain tacit agreements of the mathematical community of the time, which are loaded with intuitive, visual connotations, which each one presupposes to be known by the others.

In my opinion, it is not very surprising that such a language, especially in rather elaborate contexts, may be open to ambiguities, mistakes and obscurities. To illustrate this fact let us consider a rather recent example. The *"proof"* of Fermat's theorem solemnly presented in June 1993 by Andrew Wiles was able to convince the experts in the field for several months before they detected a rather serious gap. Some thought that to fill it could take another couple of centuries. The work of Andrew Wiles and Richard Taylor for a year was again successful. In 1995 the proof met with the approval of the experts and was published in the *Annals of Mathematics.*

"And now, please, give us a mathematical proof"

I imagine that a multitude the teachers share more or less the same experience. After having made a strong effort to make quite obvious to our students of a mathematical situation by means of a visual argument, we hear: "Now, please, give us a truly mathematical proof"

What is a proof? For the Pythagoreans working at the seashore with their pebbles it would be: *"Just look!"* For Littlewood: *"A proof is just a hint, a suggestion: look in this direction and convince yourself"*. For René Thom: *"A theorem is proved when the experts have nothing to object"*.

Should we say that an assertion is only proved when it comes at the end of a more or less lenghty chain of logical symbols? Maybe it is so in the paradise of the imagination of the formalists or logicians, but certainly not in the real world of the mathematician. He or she is already satisfied with a more reasonable degree of rigor. An isomorphic visualization, for instance, with well identified rules of codification and decodification that make it clear how to go from the image to the formal argument, is sufficient for the ordinary mathematician. It could be converted, with some effort in cases, in the most rigorous proof in order to content the most entrenched of the formalists.

Some other types of visualization, homeomorphic, diagramatic, are able to smooth out the path of other mathematicians, experts or students in order for them to explicitly construct a rigorous proof, if it is necessary, much more easily than with the terse, pedantic and often unintelligible kind of proof that the fashion has imposed for already too long time in our mathematical communication.

Of course the student that asks for a "real proof" after having been offered a faultless visual one has possibly in mind the bias, often transmitted by his teacher, that only that assertion which results after some logical quantifiers deserves the name of a proof. And this happens, it seems to me, because in our mathematical education we seldom have had into account the importance of the habit of correctly interpreting our visualizations, translating them, when it seems adequate, into a more formal language.

Visualization is difficult

Theodore Eisenberg and Tommy Dreyfuss have written an interesting paper with title *On the Reluctance to Visualize in Mathematics.* In it they try to analyze the different obstacles that one encounters in the visualization processes in mathematical education.

As I have said before, visualization is an intellection process which is direct and effortless, but only for the one who is sufficiently prepared to perform it in an efficient way. This preparation implies an immersion and familiarization with the task of decodification of the image. When such a preparation is absent, what for others might be an effortless and pleasant exercise can become a worrying and absolutely incomprehensible hieroglyph. It is true that *an image is worth a thousand words,* but one forgets to add the all-important condition that the image is understood. Otherwise it is worth nothing.

A road map, for instance, is not the reality of what is represented. It is just a set of symbols and codes one has to learn to interpret. The correct performance of a visualization requires a previous preparation, an education that not many mathematicians are able to transmit because they are not conscious of what it presupposes of convention, of tradition, of familiarity with certain codes nowhere explicitly written. And this is one of the aspects which our mathematical community, and especially our educational community, should emphasize.

On the other hand there are also difficulties which come from the low status that visualization has in our mathematical community. Our researchers make a continuous use of visualization, but their use is timid, half-hearted, something they seem to be ashamed of. No prestigious journal would admit for publication a paper in which the arguments and the proofs of the theorems would not be presented in the more or less formalized language in vogue, even if any other mathematician could recognize through them their validity. It is a question of observance to the prevailing norms. One often hears with scorn many people speaking about proofs presented "waving hands", when it is a fact that an adequate gesture can often open the minds of our audience.

Our students suffer of a certain distortion with respect to visualization and this is the origin of their attitude with respect to it and also of the following phenomenon rather frequent in our mathematical courses. We start by trying to explain for them the intuitive meaning of a theorem, what perhaps is for us the most important portion of our intervention in the hour. In the most favorable cases they will look at us with a certain attention, but without writing down a single word in their notebook. Just when we start writing down on the blackboard what is going to be a formal proof, i.e. what probably is already carefully written in their textbook, they start trying to get down in their notebooks "black over white" what seems to be for them the essential part of their work in class.

Visualization is also difficult for some other reasons of a practical nature and that become especially apparent at the level of the written, non-direct communication. Visualization is a dynamical process. The transmission means until now used in articles and in the textbooks that our students use is, basically, the written word, a statically vehicle that is not well adapted to the needs of the visualization processes. In the direct, oral presentation of a visualization its different elements start to appear little by little, rounding off an image that starts being rather simple and possibly finishes by appearing extremely complicated. In a book or article one presents usually the final image with all its elements and this becomes quite difficult to interpret. In order to show in the textbook something which would be near to the oral presentation of the same fact one would need perhaps six different figures. No editor would allow such a waste, insisting that the space is expensive and so everything has to be in a single figure.

Probably the communication means of the near future, especially for textbooks, will be something similar to the CD-ROM that allows one to mix in an interactive form text, dynamical images, computer programs that are

adequate for the field one is dealing with...

Some of the tasks ahead

I shall list some lines along which we could start working in order to put visualization in the place that corresponds to it according to its usefulness and to mathematical tradition.

Prevent possible deviations. We should try to explicitly teach to perform correctly the processes of visualization. We should pay special attention to the different types of visualization and to their especifi usefulness in the mathematical teaching and learning. We should try to be aware of the process of codification and decodification implied in the practice of visualization and trying to make them explicit for our students.

We should try to stress in our teaching the habits of visualization, trying to make very explicit their value in the practice of mathematics.

We should hold visualization in high esteem. We should insist in visualizing and, from time to time, we should transcribe our visualization into formal expressions in order to put it out of doubt that what we are doing is "real mathematics" and that what we explain by visualizing it can be also written in formal language.

We should appreciate the value of visualization not only in our frequent use of it but also in our evaluation of the uses our students and others make of it and of the different skills which visualization involves.

6 Visualization with and without the computer

The few examples we have proposed in the preceding pages have not needed at all the help of any sophisticated tool. A great part of the visualization that we advocate can be performed as it always has been done, by means of our imagination and representative ability, with the help of the normal tools at hand, paper and pencil, chalk and blackboard.... In general it is not even necessary to resort to straightedge and compass, since the main objective of our drawings is to help our intuition which is able to think correctly with the help of incorrect figures. The accuracy and precision of our drawings should be proportionate to what we expect from the type of representations we are using. It is of very little use to draw with straightedge and compass when a hand made figure is more that sufficient to suggest the relationships that are important for us. In most occasions the drawings are mere auxiliary tools of our imagination helping it to get a better grasp of the relations that help us towards the comprehension of subjects we are dealing with.

But it is quite clear that at this moment many technological tools are at hand that can help us in some circumstances when a simple hand made drawing is not satisfactory. The practice of visualization can be now importantly enhanced with the help of these tools in many different ways.

In what attains mathematical analysis one can say that the existence of symbolic calculus programs, such as MAPLE, MATHEMATICA, DERIVE, and many others, with their versatile representative abilities, with their capacity for interaction in every field of mathematics is already producing deep transformations in the new ways of doing research, teaching and learning mathematics. And this tendency seems to show no limitations.

Let us just consider a simple example. Some years ago, in order to represent a curve in the plane given by a not too simple equation $f(x,y)=0$, one used to advise the student to plot first a few elements of easy computation in order to get an initial feeling about the curve (intersections with the axes, possible horizontal and vertical asymptotes ....). Today almost any symbolic calculus program, even those incorporated into many pocket calculators, allow our students, given a rather sophisticated function, to obtain a graph of it and so to have an immediate grasp of many of its most important features. This already helps them to look in the right direction towards the solution of many problems that curve might offer. The student who is able to establish an intelligent dialogue with the machine through its representation capacities is in much better position to understand all the problems that might be proposed.

The new tools that are now in the hands of most of our students have opened quite new worlds to exploration that a few years before were closed to our view. To obtain 200 iterations of a simple function like $4x(1 -x)$ with 12 exact decimal digits starting with $x=0.7$, for example, was a gigantic task some years ago. Not so anymore. Now it may be made in a fraction of a second. Such capacities have opened new worlds for exploration on different topics such as dynamical systems, mathematical chaos, fractal geometry, and many others. On the other hand we have today many programs which are specifically destined to promote the visualization in different fields of mathematics, multidimensional analysis, geometry of different types.... All this is going to contribute to stimulate the current trend towards revitalization of the visual aspects of mathematics in many different areas.

In what follows I shall present a few examples that help to perceive how visualization may be of great help in the teaching and learning of some of the most basic aspects of mathematical analysis and later on I add also some other which are already a little more sophisticated. The images I introduce here are mostly handmade, in order to emphasize that the visual help one can obtain from such representations does not depend on the accuracy and precision of the pictures.

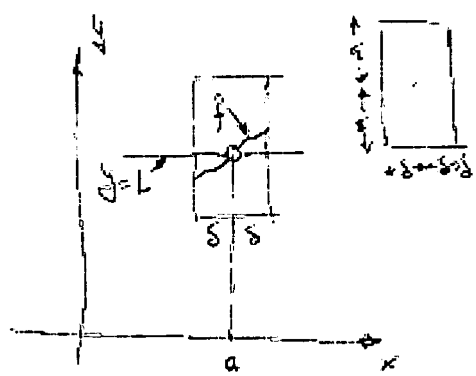
7 Samples of visualization in basic Real Analysis

One could easily present a whole course of introductory real analysis with the concrete goal of giving a visual slant to the most basic notions and results of the field. In my opinion it would contribute to balance the still prevailing bias toward explicit logical rigor and formalization.. I myself have written a small work with this orientation entitled *El rincón de la pizarra* (Pirámide, Madrid, 1996). But I think that in a normal situation it is healthier to make use in our teaching and learning of all possible recourses.

On the one hand the best advice to correctly choose our ways to confront a specific problem should come from the inspection of the features of the problem, and on the other hand a permanent bias in favor of the visualization processes could become also harmful for our students. Also one should take into account that each one of our students has its own peculiarities concerning the ways (logical, formal, intuitive,...) to attack a problem. In any case it seems very convenient to show the different possibilities that are available when one tries to introduce them to a particular field.

In what follows we shall explore the possibilities of a visual approach in order to get an adequate comprehension of the main concepts and results of introductory real analysis. We do it by offering some pictures accompanied by a few sentences in order to convey the meaning of them. The drawings are going to be handmade and rather rough, but, I hope, intelligible. I proceed so in order to make clear that the precision and accuracy of our pictures is not very important in order to achieve the goal we aim for.

Continuity of a function at a point

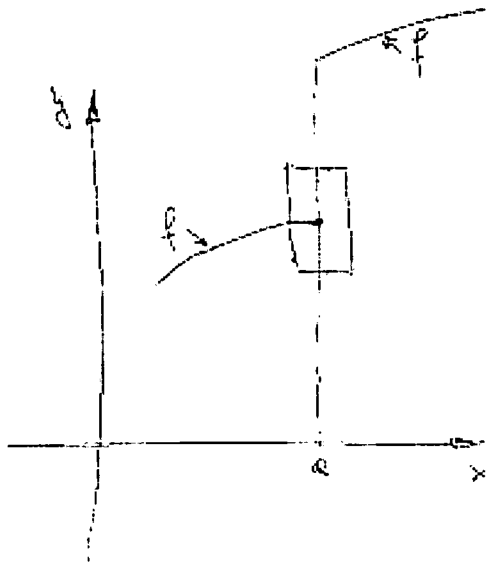


A function $f : \mathbb{R} \to \mathbb{R}$ can be isomorphically visualized by its graph.

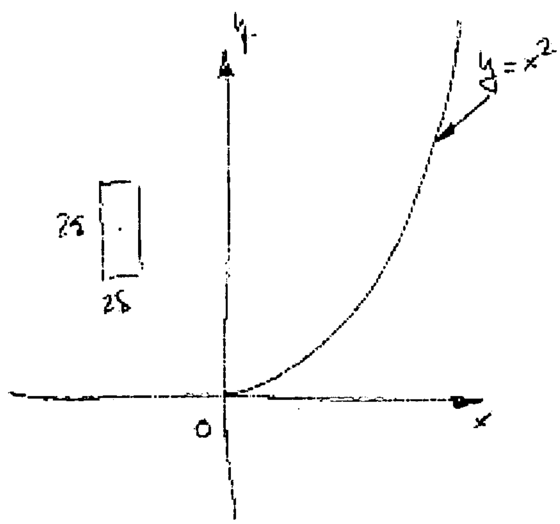
In order to deal with the notion of continuity of f we are going to introduce rectangular windows of height 2ε and width 2δ (of sides parallel to the axes Ox, Oy) centered at the points of (the graph of) f.

A function f is continuous at the point $a \in \mathbb{R}$ when the following happens: no matter how small we fix the height of a window centered at the point $(a, f(a))$ we can choose its width conveniently so that we can see the graph of f going from the left side of the window to its right side without going across its lintel (upper side) nor its threshold (lower side).

The function f of the picture in the next page is clearly not continuous at the point a.



It is clear that if we consider a different point, it may be possible that for the same height of the window we have to choose a different width in order to see the graph inside the window. An example follows



If we consider the function $y = x^2$, it is clear that we if take a point far to the right the slant gets more pronounced and the width which was adequate for points close to 0 is not any more valid.

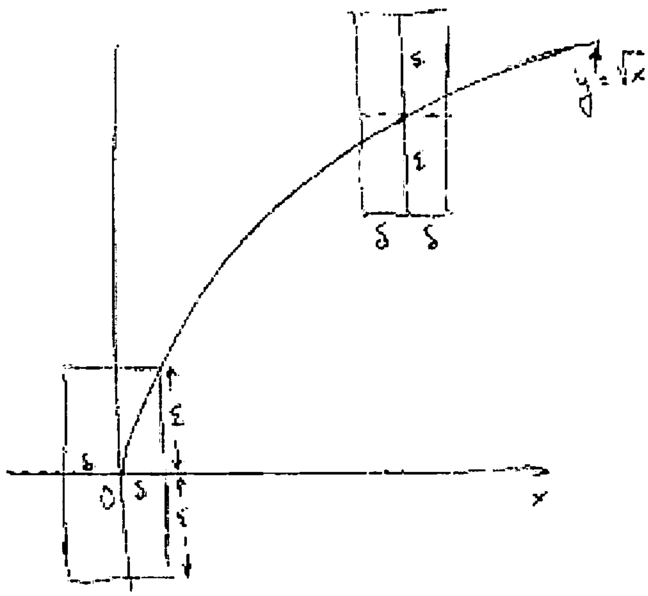
Uniform continuity

The motivation for this notion comes from the final remark of the preceding paragraph.

The function f will be said uniformly continuous on $\mathbb{R}$ when given a window height we can choose a window width such that this window centered at *any* point of the curve allows us to see the curve inside it.

An example follows

The function $f(x) = \sqrt{x}$ for $x \geq 0$ is uniformly continuous. The window which is adequate when

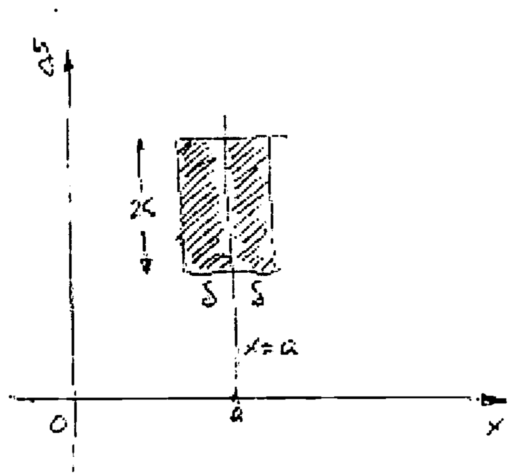


centered at $(0,0)$ is also good at any other point (one sees that the slope of the function decreases and this is what makes the window appropriate in this case).

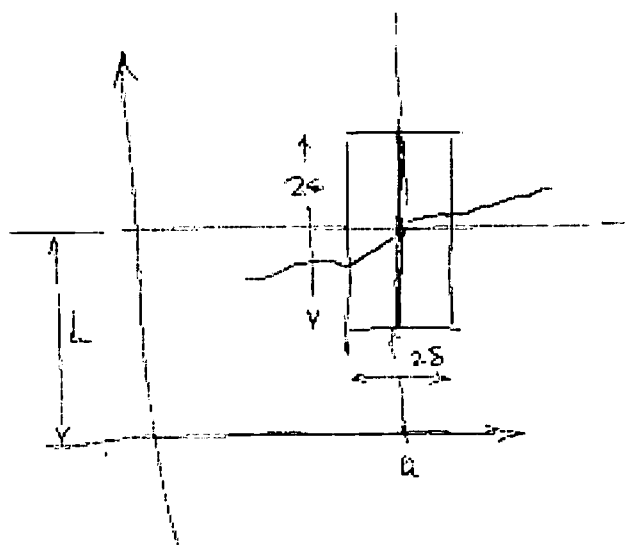
After considering this example one can easily conclude: *if the absolute value of the slope of the curve is always below a fixed finite value k, then the function is uniformly continuous*, since for each height 2ε we can choose a width $2\delta = 2\varepsilon/k$ so that when this window is centered at any point of the graph we can see the curve inside. More precisely: if $f : \mathbb{R} \to \mathbb{R}$ has a derivative at each point and $|f'(x)| \leq k$ at any x, then f is uniformly continuous on $\mathbb{R}$.

Limit of a function at a point

Now we consider windows as above, but we are going to disregard what happens along the vertical segment splitting it in two equal portions. To be more clear, we shall be interested in what happens in the shaded portion of the window in the figure in the next page (we shall call it *a split window*)



And now we can say that a function f has limit L at the point a , when for each window height there we can choose a width such that the corresponding split window centered at (a, L) lets us see the graph of the curve. The following figure will make it more clear.



Through it we try to suggest that what happens at the line $x = a$ does not matter.

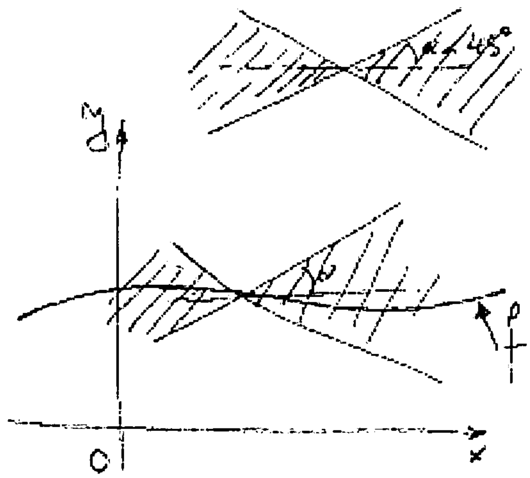
It is not my intention here to do so, but it would not be a difficult exercise to use the notions we have introduced in order to visually deduce the main properties of the real functions related to continuity and limits.

Contractive functions

The notion of contractive function on the real line is easily visualizable in a very interesting way. We are going to introduce now *"angular windows"*.

An angular window of angle $\alpha \in [0, \pi/2)$ centered at the point (a, b) is the portion of the plane enclosed by the two straight lines passing through (a, b) and forming angles with Ox of magnitudes α and $-\alpha$ containing the horizontal line through the point (a, b).

In the figure below the angular window is the shaded zone, corresponding to an angle $\alpha < 45$



A function $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function of constant $k \geq 0$ when the angular window of angle $\alpha = \arctan(k)$ centered at any point of f contains the graph of f.

The translation of this definition into analytical terms is, of course:

A function $f : R \to R$ is a Lipschitz function of constant $k \geq 0$ when for each $a, b \in R$, $|f(b) - f(a)| \leq k\,|b - a|$.

When the constant k is less than 1, i.e. when the angle is less than 45^o, then the function is called a contractive function.

From the visual definition it easily follows that any Lipschitz function is uniformly continuous (for any window height 2ε one chooses the width $2\delta = 2\varepsilon/k$ corresponding to the window with that height and whose diagonals have slope k and $-k$.)

The iterations of a function

As we shall see, it is often useful, given a function $f(x)$, $f : \mathbb{R} \to \mathbb{R}$, to consider the iterated values starting from $x = a$, i.e. the values, $f(a), f^2(a) = f(f(a)), f^3(a) = f(f(f(a))), ..., f^n(a), ...$ The visual determination of these values starting from the graph of the function is interesting:

from the point $(a, 0)$ one draws a vertical segment to the curve and one obtains $(a, f(a))$

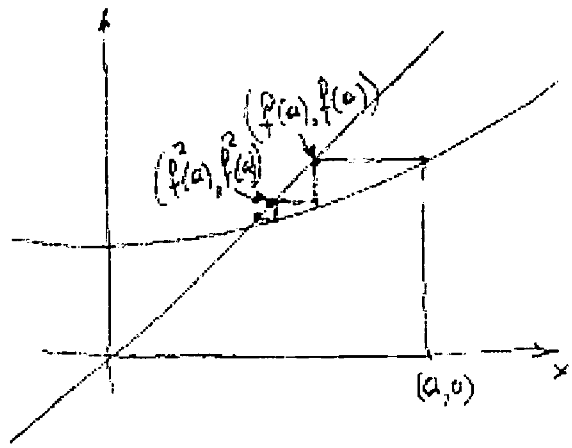
from $(a, f(a))$ one draws a horizontal segment that intersects the bisector $y = x$ at the point $(f(a), f(a))$

from this point one draws a vertical segment to the curve and one obtains $(f(a), f^2(a))$

....

In this way we obtain the different values $f(a), f^2(a), f^3(a), ...f^n(a), ...$

The following figure makes the process clear



It also suggests that when the function is contractive, the sequence of points on the bisector line are going to converge to a point which belongs both to this line and to the curve, i.e. it is a point $(p, f(p) = (p,p)$. This means that $f(p) = p$. We shall visually prove this property in detail.

Fixed points

Given a function $f : \mathbb{R} \to \mathbb{R}$ we say that $p \in \mathbb{R}$ is a fixed point for f when $f(p) = p$. The visual translation is: *a fixed point p for the function f is any of the abscissae of the intersections , if they exist, of the graph of f with the line $y = x$.*

Fixed points are extremely important in modern analysis and for this reason the following theorem is at the center of the theory.

A visual proof of the fixed point theorem for contractive functions

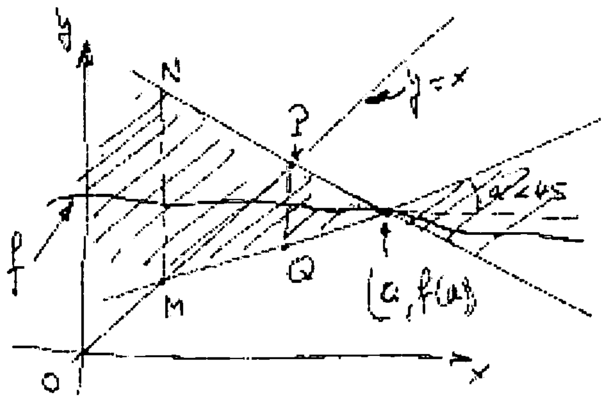
After the exploration we made above, when dealing with the iterations of a function, the following theorem should not be a surprise.

If $f : R \to R$ is a contractive function then there is a unique fixed point p for f that can be obtained by choosing any $a \in R$ and determining $p = \lim_{n\to\infty} f^n(a)$.

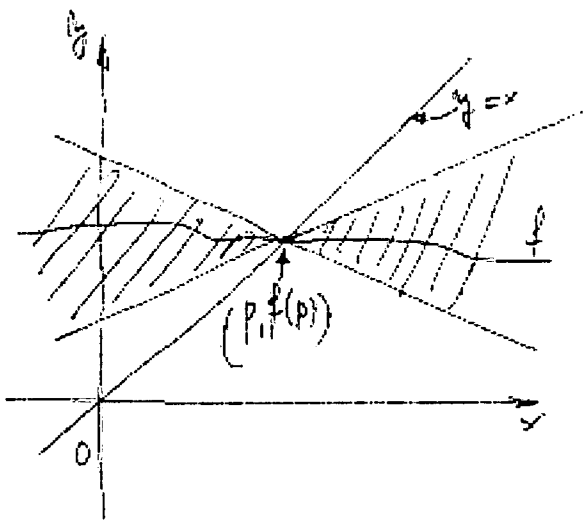
The existence of at least one fixed point is visually proved in the following way.

If we take any point $(a, f(a))$ of the graph of f and we center on it the corresponding angular window, as in the figure, it becomes clear that the sides of this window (since $\alpha < 45^o$) intersect the line $y = x$ at two points P, M (unless $a = f(a)$, but then we already have our fixed point a).

Since the graph of f is enclosed in the window we have drawn, it is obvious that it has a point on the segment PQ , below the line $y = x$ and another one on the segment MN , above the line $y = x$. Therefore the continuous curve f has at least one point of intersection with $y = x$, i.e f has at least one fixed point p.



The fact that this point is unique follows visually by just centering the angular window at the point $(p, f(p))$ as indicated in the figure below



Since $\alpha < 45^o$ and the graph of f is in this angular window, it cannot intersect again the line $y = x$. This means that f has a unique fixed point p.

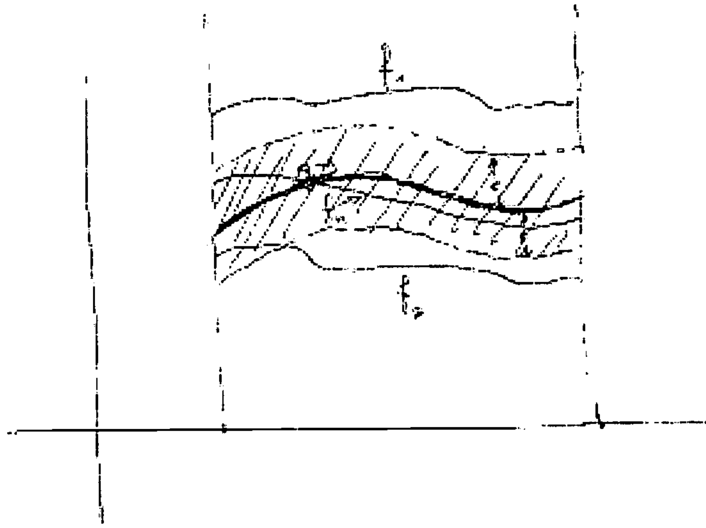
From the analytic characterization of the contractive function we have

$$|f^n(a) - p| = |f^n(a) - f^n(p)| \leq k\left|f^{n-1}(a) - f^{n-1}(p)\right| \leq k^n\,|p - a|$$

and, since $k < 1$, we obtain $p = \lim_{n\to\infty} f^n(a)$ and so the theorem is proved.

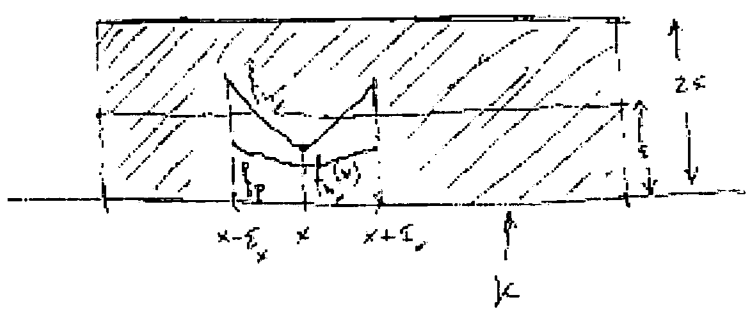
Sequences of functions and uniform convergence. Dini's theorem.

Let f_n and sequence of functions from $K \subset \mathbb{R}$ to K. That f_n converge uniformly on K to another function g means visually that for any plane strip of width $\varepsilon > 0$ around g we can choose a subindex m such that for each $n \geq m$ the function f_n is inside that strip as the figure below suggests. One can check that this is the exact translation of "for each $n \geq m$ and for each $x \in K$ one has $|f_n(x) - g(x)| \leq \varepsilon$.



Dini's theorem asserts that *if K is a compact set and if the sequence f_n of continuous functions converge montonically at each point $x \in K$ to $g(x)$, g being also a continuous function on K, then the convergence of f_n* to *g is uniform on K.*

The visual proof of this theorem is interesting.First one can assume that $f_n(x)$ decreases at each point and and one can reduce the theorem to the case where g is 0 on K by considering the functions $f_n - g$.



We fix a strip of width $\varepsilon > 0$ around the axis Ox. For each x in K there is an n_x such that for each $p \geq n_x$ one has $0 \leq f_p(x) \leq f_{n_x}(x) \leq \varepsilon$. Therefore for each $x \in K$ there is an open interval $(x - \varepsilon_x, x + \varepsilon_x)$ such that for each point $p \geq n_x$ and each t in the interval one has $0 \leq f_p(t) \leq f_{n_x}(t) \leq 2\varepsilon$.

Since K is compact we can choose a finite number of such intervals covering K. If N is the greatest n_x corresponding to these finite collection of intervals we see that for each $n \geq N$, f_n is in the 2ε -strip of the function g. This concludes the proof of Dini's theorem.

As an exercise I would like to suggest a visual proof of the following theorem related to the one by Dini: *if K is a compact set and if the sequence f_n of monotone continuous functions converge at each point $x \in K$ to $g(x)$, g being also a continuous function on K, then the convergence of f_n* to *g is uniform on K.*

I think the proof that results becomes significatively more transparent than the one usually offered.

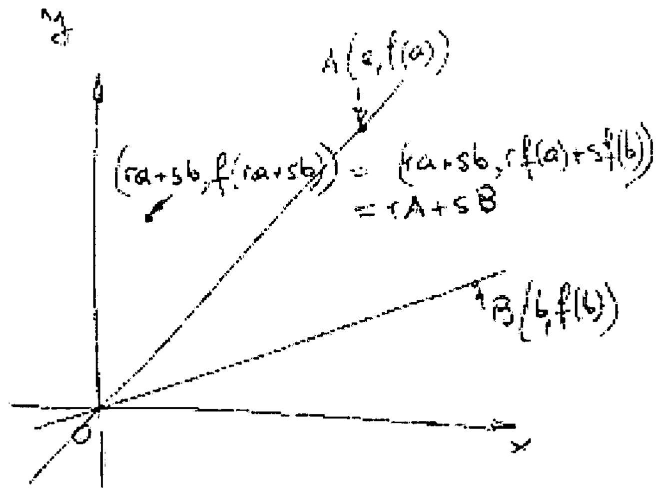
A theorem made simple by means of a visualization

An additive function $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(a+b) = f(a) + f(b)$ for each $a, b \in \mathbb{R}$.

It is then easy to see that $f(0) = 0$, that for each $m \in \mathbb{Z}$ we have $f(mx) = mf(x)$ and that for each $r, s \in \mathbb{Q}$ we have $f(ra + sb) = rf(a) + sf(b)$.

The following interesting fact has an immediate visual proof: *the graph of any additive function f is either a line through the origin or else is a set of points dense in the plane.*

Assume that the graph has two points $A(a, f(a))$ and $B(b, f(b))$ such that the straight line AB does not go through the origin.



The set of points $\{rA + sB : r, s \in \mathbb{Q}\}$ is obviously dense in the plane.

From this fact one can easily conclude that any function g which is additive and continuous, then it is of the form $g(x) = \lambda x$ for $\lambda = g(1)$. It is not difficult to see that any additive and measurable function has also to be of this same form. If we determine a function which is additive and not of this form we deduce the existence of non-measurable functions.

Such an additive function not of the form $g(x) = \lambda x$ is determined in the following way. Let us consider the vector space $\mathbb{R}$ over the field of rational numbers $\mathbb{Q}$. We determine a basis of this vector space by taking first the elements 1 and $\sqrt{2}$, which are clearly linearly independent over the rationals, and completing this set to a basis in an arbitrary way. Let this basis be $\{1, \sqrt{2}, e_3, e_4, ...\}$. Let us now define $g(1) = 1$, $g(\sqrt{2}) = 2$, and for any element $\alpha \in \mathbb{R}$, $\alpha = r_1 1 + r_2\sqrt{2} + r_3 e + r_4 e_4 + ...$ we set

$$g(\alpha) = g(r_1 1 + r_2\sqrt{2} + r_3 e + r_4 e_4 + ...) = r_1 g(1) + r_2 g(\sqrt{2}) + r_3 g(e_3) + ...$$

In this way g is additive and obviously the line passing through $(1, g(1))$ and $(\sqrt{2}, g(\sqrt{2}))$ does not go through the origin. The function we have so defined cannot be measurable.

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W.S. Peters, Introduction, 77-78

H. Kautschitsch, "Neue" Anschaulichkeit durch "neue" Medien, 79-82

S. Cunningham, Some strategies for using visualization in mathematics teaching, 83-86

R. Schaper, Computergraphik und Visualisierung am Beispiel zweier Themen aus der linearen Algebra, 86-92

- ZDM, Zentralblatt für Didaktik der Mathematik, Analyses: Visualization in mathematics and didactics of mathematics, Part 2, 26 (1994),109-132:

T. Eisenberg, On understanding the reluctance to visualize, 109-113

N.C. Presmeg, The role of visually mediated processes in classroom mathematics, 114-117

W.S. Peters, Geometrische Intuition, mathematische Konstruktion und einsichtige Argumentation, 118-127

W.S. Peters, Bibliographie zur Visualisierungsdiskussion, 128-132

CONCEPTUALIZING THE REALISTIC MATHEMATICS EDUCATION APPROACH IN THE TEACHING AND LEARNING OF ORDINARY DIFFERENTIAL EQUATIONS

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The undergraduate curriculum in differential equations has undergone important changes in favor of the visual and numerical aspects of the course primarily because of recent technological advances. Yet, research findings that have analyzed students' thinking and understanding in a reformed setting are still lacking. This paper discusses an ongoing developmental research effort to adapt the instructional design perspective of Realistic Mathematics Education (RME) to the teaching and learning of differential equations at Ewha Womans University. The RME theory based on the design heuristic using context problems and modeling was developed for primary school mathematics. However, the analysis of this study indicates that a RME design for a differential equations course can be successfully adapted to the university level.

Key Words: Differential equations; Realistic Mathematics Education (RME); College mathematics; Reform in mathematics education; Teaching practice

During the past decades, there has been a fundamental change in the objectives and nature of mathematics education, as well as a shift in research paradigms. The changes in mathematics education emphasize learning mathematics from realistic situations, students' invention or construction solution procedures, and interaction with other students or the teacher. This shifted perspective has many similarities with the theoretical perspective of Realistic Mathematics Education (RME) developed by Freudenthal (1973, 1991). The RME theory focuses on guided reinvention through mathematizing and takes into account students' informal solution strategies and interpretations through experientially real context problems. The heart of this reinvention process involves mathematizing activities in problem situations that are experientially real to students. It is important to note that reinvention is a collective, as well as individual activity, in which whole-class discussions centering on conjecture, explanation, and justification play a crucial role. In the reinvention approach, researchers build upon the work that has been done on symbolizing and modeling in primary-school mathematics (Treffers, 1991; Gravemeijer, 1994, 1999. Can the framework that was developed for primary school mathematics be adapted to teach differential equations in collegiate mathematics?

For three decades, international comparisons of mathematics achievement have favored primary and secondary students in Korea (Husen, 1967; McKnight, Travers, Crosswhite, & Swafford, 1985a and 1985b; Horvarth, 1987; U.S. Department of Education, 1997a, 1997b). For instance, Korean eighth grade students ranked second among 41 different nations on the Third International Mathematics and Science Study (TIMSS) (U.S. Department of Education, 1996). Superficially, it appears as if Korean students possess advance mathematical knowledge and skills when compare to other students of the same age in different countries. Lew (1999) and Kwon (2002) argued, however, that most Korean students seem quite unable to relate their well-developed manipulative skills to realistic context problems to the real-world situations, as secondary mathematics lessons in Korea put much emphasis on computation and algorithm skills. Korean students, however, are the only students who have difficulties adapting their mathematical knowledge to real-world situations. Lack of students' understandings of real-world situations and the characteristic of mindless, symbolic manipulation in differential equations has also been noted by a number of mathematicians (e.g., Boyce, 1994; Hubbard, 1994). The question then becomes how do instructors teach students differential equations in such a meaningful way as to foster students' mathematical growth. RME may give a perspective for conceptualizing this teaching of differential equations since realistic context problems play an essential role from the start and also the point of departure is that context problems can function as anchoring points for the reinvention of mathematics by students themselves (Gravemeijer & Doorman, 1999). Such a reinvention process in RME will be paved with realistic context problems that offer students opportunities for progressive mathematizing in differential equations. From the RME perspective, students should learn mathematizing subject matter from realistic situations in differential equations.

The overall purpose of this study is to examine the developmental research efforts to adapt the instructional design perspective of RME to the teaching and learning of differential equations in collegiate mathematics. A differential equations course, highlighting reinvention through progressive mathematization, didactical phenomenology and emergent models design heuristics, was developed. Informed by the instructional design theory of RME and capitalizing on the potential of technology to incorporate qualitative and numerical approaches, this paper offers an approach for conceptualizing the learning and teaching of differential equations that is different from the traditional approach.

Theoretical Orientation

Realistic Mathematics Education

RME is rooted in 'mathematics as a human activity,' and the underlying principles are guided reinvention, didactical phenomenology, and emergent models. These principles are based on Freudenthal's philosophy which emphasizes reinvention through progressive mathematization (Fredenthal, 1973, 1991). In RME, context problems are the basis for progressive mathematization, and through mathematizing, the students develop informal context-specific solution strategies from experientially realistic situations (Gravemeijer & Doorman, 1999). Thus, it is necessary for the researchers who adapt the instructional design perspective of RME to utilize contextual problems that allow for a wide variety of solution procedures, preferably those which considered together already indicate a possible learning route through a process of progressive mathematization.

Three guiding heuristics for RME instructional design should be considered (Gravemeijer, Cobb, Bowers, & Whitenack, 2000). The first of these heuristics is reinvention through progressive mathematization. According to the reinvention principle, the students should be given the opportunity to experience a process similar to the process by which the mathematics was invented. The reinvention principle suggests that instructional activities should provide students with experientially realistic situations, and by facilitating informal solution strategies, students should have an opportunity to invent more formal mathematical practices (Freudenthal, 1973). Thus, the developer can look at the history of mathematics as a source of inspiration and at informal solution strategies of students who are solving experientially real problems for which they do not know the standard solution procedures yet (Streefland, 1991; Gravemeijer, 1994) as starting points. Then the developer formulates a tentative learning sequence by a process of progressive mathematization.

The second heuristic is didactical phenomenology. Freudenthal (1973) defines didactical phenomenology as the study of the relation between the phenomena that the mathematical concept represents and the concept itself. In this phenomenology, the focus is on how mathematical interpretations make phenomena accessible for reasoning and calculation. The didactical phenomenology can be viewed as a design heuristic because it suggests ways of identifying possible instructional activities that might support individual activity and whole-class discussions in which the students engage in progressive mathematization (Gravemeijer, 1994). Thus the goal of the phenomenological investigation is to create settings in which students can collectively renegotiate increasingly sophisticated solutions to experientially real problems by individual activity and whole-class discussions (Gravemeijer, Cobb, Bowers & Whitenack, 2000). RME's third heuristic for instructional design focuses on the role which emergent models play in bridging the gap between informal knowledge and formal mathematics. The term model is understood in a dynamic, holistic sense. As a consequence, the symbolizations that are embedded in the process of modeling and that constitute the model can change over time. Thus, students first develop a model-of a situated activity, and this model later becomes a model-for more sophisticated mathematical reasoning (Gravemeijer & Doorman, 1999).

RME's heuristcs of reinvention, didactical phenomenology, and emergent models can serve to guide the development of hypothetical learning trajectories that can be investigated and revised while experimenting in the classroom. A fundamental issue that differentiates RME from an exploratory approach is the manner in which it takes account both of the collective mathematical development of the classroom community and of the mathematical learning of the individual students who participate in it. Thus, RME is aligned with recent theoretical developments in mathematics education that emphasize the socially and culturally situated nature of mathematical activity.

Traditional and Reform-Oriented Approaches in Differential Equations

Traditionally, students who take differential equations in collegiate mathematics are dependent on memorized procedures to solve problems, follow a similar pattern of learning in precalculus mathematics, and follow model procedures given in the textbook or by a teacher. Also, the search for analytic formulas of solution functions in first order differential equations is the typical starting point for developing the concepts and methods of differential equations. This traditional approach emphasizes finding exact solutions to differential equations in closed form, i.e., the dependent variable can be expressed explicitly or implicitly in terms of the independent variable. However, in reality, when modeling a physical or realistic problem with a differential equation, solutions are usually inexpressible in closed form. Therefore, as Hubbard (1994) pointed out, there is a dismaying discrepancy between the view of differential equations as the link between mathematics and science and the standard course on differential equations.

The teaching of differential equations has undergone a vast change over the last ten years because of the tremendous advances in computer technology and the "Reform Calculus" movement. One of the first textbook promoting this reform effort was published by Artigue and Gautheron (1983). More recently, a number of textbooks reflecting on this movement have been written (e.g., Blanchard, Devaney, & Hall, 1998; Borelli & Coleman, 1998; Kostelich & Armbruster, 1997; Hubbard & West, 1997). Primary features of these reform-oriented textbooks are content-driven changes made feasible with advances in computer technology. Thus, these textbooks have decreased emphasis on specialized techniques for finding exact solutions to differential equations and have increased the use of computer technology to incorporate graphical and numerical methods for approximating solutions to differential equations (West, 1994).

According to Boyce (1995), the primary benefit of incorporating computer technology in differential equations is the visualization of complex relationships that students frequently find too complicated to understand. For example, a typical differential equation, $u''+0.2u'+u=\cos wt$, $u(0)=1$, $u'(0)=0$, can be easily executed with technology, and students can understand the behavior of the system by using technology to draw a three-dimensional plot as a function of both w and t. The main reasons to use computers in a differential equations course are that geometric interpretations of solutions through the use of computer software help students to understand basic concepts such as initial value problems, integral curves, direction fields and flows for dynamical systems (Lu, 1995). In addition, many concepts including phase portrait, stability, stable and unstable manifold, bifurcation and chaos can better be understood by introducing a computer program for teaching and learning. However, the current reform movement in differential equations emphasizes a combination of analytic, graphical, and numerical approaches from the start. Although different from traditional approaches to differential equations, this movement is quite similar to traditional approaches in the way in which conventional graphical and numerical methods are used as the starting point for students' learning, as Rasmussen (1997, 1999) documented. Thus, as is the case with the traditional approach, students typically do not participate in the reinvention or creation of these mathematical ideas associated with graphical and numerical methods, the representation that conventionally accompany these ideas, and the methods themselves. The learning that occurred was characteristic of mindless graphical and numerical manipulation in the reform-oriented approach. In these respects, the learning demonstrates little improvement over traditional approaches where mindless symbolic manipulation was the prevalent mode of operation.

The current curriculum-oriented reform movement in differential equations has some content-based advantages. The approach being developed here seeks to build on and complement these positive aspects by adapting principled perspectives and approaches that have informed the re-thinking of mathematics learning and teaching at the elementary and secondary level to the re-thinking of mathematics learning and teaching of differential equations.

Guided by the RME instructional design theory, students may participate in the reinvention of mathematical idea and methods that comprise a differential equations course. The emphasis on reinvention by no means implies that the instructor is a bystander in the learning process. In fact, the instructor's role might even be more important in this approach than in the traditional dissemination approach to learning. For example, the instructor guides the construction of classroom social and sociomathematical norms (Yackel, Rasmussen, & King, 2000) that foster students' reinvention and sophisticated mathematical reasoning in differential equations. Initial work (Trigueros, 2000; Yackel et al., 2000; Zandieh & McDonald, 1999) suggests that this perspective demonstrates some promise to foster students' mathematics growth in differential equations.

Project Classroom & Preliminary Analysis

A classroom teaching experiment in an introductory course in differential equations was conducted during Fall 2001 at Ewha Womans University with a group of 43 students, most of whom were first-year undergraduate students majoring in mathematics education. Ewha Womans University has over 20,000 students and is one of the most prestigious schools in Korea. Ewha is also well-known for pre-service teachers education. Over 30% of newly employed in-service secondary mathematics teachers have graduated from Ewha Womans University.

Data based on a methodology for determining the emergence of classroom mathematical practices were collected (Cobb, Stephen, McClain, Gravemeijer, 2001). Data from the teaching experiment consisted of videotapes of each class session, including the small group work; field notes made by the observers and the instructor; records of instructional activities and decisions, and copies of students' work such as in-class work, homework assignments, weekly electronic journal entries and reflective portfolios. In addition, experimental curriculum materials as well as programs for the TI-92 calculator were developed. The materials were guided and informed by the RME instructional heuristic and were designed to help students to complete reinvention activities, which occur when students try to devise their own ways of working through a mathematical concept.

In the typical collaborative learning environment of this project, the instructor poses a task, students work in groups of two to four students, and after most groups obtain initial ideas about the task, the class engages in a discussion of students' approaches to the task. Whole-class discussions might continue for 10-15 minutes before another 5-10 minute segment of small group work took place. This cycle was typically repeated three to four times in a 75-minute class period. The nature of small group work was not for students to solve a specific problem but to analyze a question and develop reasons to support their thinking. Because of the continuous emphasis on reasoning, whole-class discussions resulted in the emergence of key concepts such as slope fields, phase lines, and bifurcation diagrams.

In this paper, one of the themes emerging during this teaching experiment is exemplified with a sample from the data and preliminary analysis. Holistic data analysis and its implications to undergraduate mathematics education from the RME perspectives will be discussed during the presentation.

Research on the design of primary school RME sequences has shown that the concept of emergent models can function as a powerful design heuristic (Gravemeijer, 1999). The following example illustrates the RME heuristic that refers to the role models can play in a shift from a model-of a situated activity to a model-for mathematical reasoning in the learning and teaching of differential equations.

Suppose a population of Nomads is modeled by the differential equation dN/dt =f(N).
The graph of dN/dt is shown below.

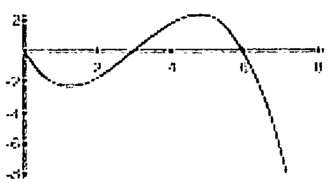

Figure 1. Graph of dN/dt.

For the following values of the initial population,
What is the long-term value of the population?
Be sure to briefly explain your reasoning.
(1)N(0)=2, (2)N(0)=3, (3)N(0)=4, (4)N(0)=7

The development from a model-of to a model-for can be illuminated by the four different levels of activity: situational, referential, general, and formal (Gravemeijer & Doorman, 1997; Gravemeijer, 1997). Each of these four different levels emerged during this teaching experiment.

At the situational level, students' interpretations and solutions depend on understanding how to act in the setting. For example, one participant named Jungsun was trying to figure out how to use the given differential equation to approximate the long-term value of the population for each initial population. This situation means that once she interpreted the differential equation as an experientially realistic context, she understood how to act in the setting. For this level, the TI-92 graphing and symbolic calculator can play an essential role by allowing the slope field to emerge as an initial record of students' reasoning and mathematical activities for their numerical approximations. Then it becomes a tool for fostering students' reasoning about solution functions to differential equations (Figure 2).

At the referential level, models-of is grounded in students' understanding of pragmatic, experientially real settings. Students' activities might be considered referential (that is, referring back to the discrete approximations) when they are initially acting with the slope field as if there is an indication of the differential equation at any conceivable point (Figure 3).

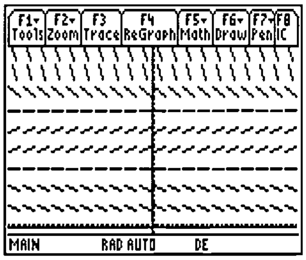

Figure 2. Slope field for dN/dt.

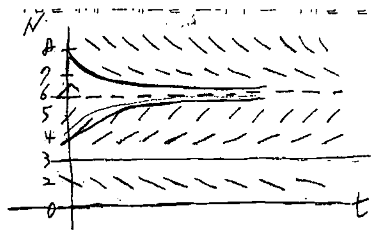

Figure 3. Jungsun's solution graph.

At the general level, models-for makes possible a focus on interpretations and solutions independent of situation-specific imagery. Students' interpretations and responses to solution functions are no longer referring back to discrete approximations or specific solutions. Their activities involve holistically interpreting rates of change and solution functions (Figure 4). That is, students' solutions involve simultaneous reasoning about individual solution functions, as well as collections of solution functions.

At the formal level, students' activities are often characterized by the formal use of conventional notation. This fact is a useful and important way to differentiate activity at the general level from activity at the formal level. For example, one student, Miju, uses a dynamic image of the phase line which differentiates activity at the general level from activity at the formal level, thus demonstrating that her reasoning regarding solution functions is at a higher level (Figure 5).

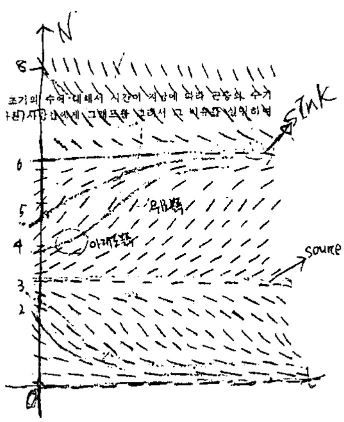

Figure 4. Rami's solution graphs.

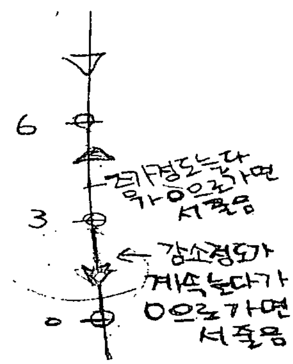

Figure 5. Miju's phase line.

Guided and informed by the RME instructional heuristic, students in the differential equations course first act in mathematical situations in progressively more formal ways where the model comes to the fore as a model-of a mathematical context. Then subsequently, the model changes so that it can begin to function as a model-for increasingly sophisticated ways of mathematical reasoning.

Concluding Remarks

The study of ordinary differential equations is essential for students in many areas of science and technology. Many useful and interesting phenomena in engineering and life sciences that continuously evolve in time can be modeled by ordinary differential equations. Therefore, it is very important for students to have a firm understanding of ordinary differential equations, their solutions, and the different kinds of qualitative behavior the systems of ordinary differential equations can exhibit. Several recent curriculum reform efforts in differential equations are decreasing the traditional emphasis on specialized techniques for finding exact solutions to differential equations and increasing the use of computing technology to incorporate qualitative and numerical methods of analysis. Yet, research findings (e.g., Habre, 2000; Rasmussen, 1997) on students' thinking and understanding of differential equations are still minimal.

Through conceptualizing RME perspectives to the learning and teaching of differential equations, this research illustrates that when students are engaged in instruction that supports reinventing conventional representations out of mathematizing experiences, slope fields and graphs of solution functions can and do emerge for their mathematical activities. Specifically, students in Korea might more readily adapt their well-developed manipulative skills to experientially real situations with the incorporation of the RME instructional design. Further this research demonstrates how emerging analyses of student thinking and symbol-use can be profitably coordinated to promote students' sophisticated ways of reasoning with mathematical concepts in differential equations. Thus this paper suggests that an RME design for a differential equations course offers an alternative perspective for conceptualizing the learning and teaching of differential equations, even in undergraduate mathematics. This research also implies that researchers should consider, investigate, and adapt principled approaches that have been useful for reform in K-12 mathematics when conceptualizing the reform of undergraduate mathematics.

Research in the teaching and learning of mathematics at the university level is a relatively recent and new phenomenon (Artigue, 1999); research in the teaching and learning of differential equations is even newer. The problems in undergraduate mathematics education are not easily solved by just writing or adopting new textbooks. The problems are related to the forms of students' work, the modes of interaction between university teachers and students, and the methods and content which students are assessed. The perspectives reported in this study can complement the growing research base in the teaching and learning of differential equations in both practical and theoretical aspects.

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ACCESSING KNOWLEDGE FOR PROBLEM SOLVING

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ABSTRACT

This paper studies the modes of thought that occur during the act of solving problems in mathematics. It examines the two main instantiations of mathematical knowledge, the conceptual and the structural, and their role in the afore said act. It claims that awareness of mathematical structure is the lever that educes mathematical knowledge existing in the mind in response to a problem-solving activity, even when the knowledge evoked is far from being evidently connected with the activity. For didactical purposes it proposes the consideration of mathematical techniques to facilitate the accessing of pertinent knowledge. All the assertions above are substantiated by close examination of some exemplars taken from various mathematical topics, and the presentation of some recent fieldwork results.

Introduction

Let us set the scene by immediately referring to a particular problem.

Example 1

Show that k! divides the product of any k consecutive positive integers.

The most efficient way to argue for this problem is the following. Consider for any positive integer n

$$\frac{n(n+1)(n+2)...(n+k)}{k!} \quad .$$

If we show that this fraction represents an integer we are done. However we notice that the above expression is in the form of a binomial coefficient, and so is guaranteed to be an integer.

In the approach above we have introduced and applied some knowledge that was not evidently relevant to the initial context of the question. The role of accessing appropriate knowledge here is decisive, as it is of course important generally in problem solving in Mathematics. It is essential for a solver to be able to transfer ideas from one context to another. To promote such an ability, there would seem to be two fronts that have to be nurtured. The first is to mentally organize mathematical knowledge as it is learned and develop it in a way that is conducive for application in problem solving. Indeed our example could even be regarded as a fact that could have been assimilated previously when learning about binomial coefficients. Such broader knowledge accumulated about a certain notion will be called a 'schema'. The second front is how the practitioner becomes skilled in making the connections she/he needs whilst working on non-routine mathematics. Are we simply reduced to say one just happens to notice something as in the example above, or can we analyze the process further? We shall consider awareness of mathematical structure as a possible way to achieve this.

The importance of accessing knowledge for solving activities and the creative challenge it demands means that it is natural to try to systematize the ways to cope with this mental action as far as possible. One way to effect this systemization is through techniques. (We shall specify exactly what we mean by a technique later in the paper.) In creating techniques we are often cementing interactions of different entities or systems, hence strengthening schemas. We feel, then, that students' acquisition of techniques is crucial for them to become efficient problem solvers. Some techniques are taught explicitly in the curriculum, but many others have to be garnered by the students themselves. Quite a few require only a slight shift in perspective in looking at acquired knowledge, but cognitively speaking we should not assume that such shifts would be easy for the student to accomplish on her/his own. Potentially anyway, yet further techniques would be gleaned from the students' experiences whilst occupied with their exercises, in drawing together similarities with previous work. However the required assimilation in order to process such perceived parallelisms into clear descriptions, as techniques would no doubt require certain maturity. There is a common saying in the professional community of mathematicians that "a trick met twice becomes a method". This disregards, though, the problems involved in identifying and extracting your method from the (possibly very different) contexts encountered.

This paper will address in more detail the issues raised above, and will discuss some pedagogical implications. In particular we will consider techniques that really only comprise simple reformulation of known material, as this class may be the most realistic to take in order to positively influence students' thinking patterns. In this context, I shall describe briefly some fieldwork that I conducted involving one such technique, employing bijections for the purposes of enumeration.

Knowledge Acquisition and Retention

It is plain that if we wish to access knowledge, we are first assuming that that knowledge is present. Hence knowledge acquisition and retention are relevant topics for our theme.

Here we shall be thinking only about mathematics content knowledge. (This excludes then knowledge of heuristics such as identified by Polya (1945) and metacognition as espoused by Schoenfeld in Schoenfeld (1992) for example.) There has been a tradition in mathematics education literature to compare 'conceptual knowledge' with 'procedural knowledge', see e. g. Hiebert & Lefevre (1986), however we shall add another category that we shall call 'structural knowledge'.

The conceptual, for us, concerns some sort of issue, circumstance or entity that can be modeled mathematically but may be also manipulated mentally to some degree independently of the mathematical model. Conceptual mathematics always in this way refers to a cognitive environment where the mind can process ideas that should be readily transferable to the mathematics. The part of the environment that supports these ideas is often referred as the concept image in the educational literature, see for example Tall & Vinner (1981). The concept image may take many forms, such as descriptive wording or use of diagrams. The concept image should be thought of as being much more than an informal representation; cognitively the concept image is more or less identified with the 'working' of the mathematics that it parallels. This strong identification between a mathematical system and a more intuitive realm means that a concept has the potential to convince the practitioner of the truth of some related proposals without having to make recourse to formal proof. Any known result that is at least partially understood via the concept will be termed 'conceptual knowledge'. It should be remembered that often the act, or we might say the art, of forming definitions must necessarily compromise the original concept image. [This is amply shown with Lakatos' work, as in Proofs and Refutations (1976)]. If the image is not adapted accordingly, there will be clashes between the image and the mathematical system leading to possible dysfunction in performance. In tertiary level mathematics, at least, images are not often induced within taught curricula, so this problem is usually never quite resolved completely. Even when they are 'officially' introduced, images may not capture every feature or special case involved in the mathematical system. [E.g. in Pinto & Tall (2001) it is remarked how a student could not reconcile the convergence of a constant sequence in the standard 'dynamic' graphical depiction for limiting properties of sequences often shown in text books.] The above suggests that conceptual knowledge may not be so easily assimilated or retained as one might have believed; and it is likely to be mentally processed inflexibly.

When mathematics education was still quite young as an autonomous discipline, Skemp (1978) emphasized the difference between 'to know how' (instrumental understanding) and 'to know not only how but also why' (relational understanding). Ever since the same concern has been voiced dressed in various guises and perspectives. Jones & Bush (1996) suggested that the notion of mathematical structure is a good medium to explain the state of 'comprehending the why'.

Following Rickard (1996), we describe a (mathematical) structure as a set of objects along with certain relations among those objects. Rickard's paper continues to further define structure abstractly (via the notion of isomorphism), but we shall not follow this here. As far as we are concerned, even though structure may be highly abstractly represented in axiomatic systems, it may also be identified locally within a given context. If the structure has to be analyzed, it must be to some extent extracted from the context, but this can be done in such a way where the contextual referents are always at hand. [As Mason (1989) points out abstraction involves 'drawing away', or 'divorcing', rather than just extraction.] Our perspective of structure, then, is to strip away all the intrinsic features and properties that are not relevant to a certain coherent means of manipulation of a system. In this kind of analysis, then, a sense of what is essential and what is not is built up, which surely contributes to an enhanced understanding of why approaches developed from the said means of manipulation should work.

Though we will not claim that conceptual knowledge is disjoint from structural knowledge (i.e. knowledge that is accrued from structural considerations), in essence the two are different in character. Structural knowledge is based on analysis or at least on reflection on connections and (inter-) relations (see Mamona-Downs & Downs, 2002), whereas conceptual knowledge depends on holistic mental images where structure should be implicitly represented but its presence not necessarily realized. However structural knowledge is meaningful; as a collorary, we contend that not everything that makes sense in mathematics is due to it being somehow 'conceptual'!

Structural knowledge is more flexible than it might at first seem. First, parallel structure may be identified in different contexts and so associations are made between diverse mathematical topics. If you do allow the notion of abstract structure, then these concrete manifestations may be regarded as the various **representations** of the structure (again following Rickard.) Second, new perspectives of structure or connections between non-parallel structures may be made by considering (for example) different approaches of solving the same question. An important facet here is that proofs often 'import' structure that is not explicitly present in the context of the proposition to be demonstrated. Taking these two notes together, we claim that thinking in structural terms is highly beneficial in forming schemas, which in turn contributes to the range, depth and linkage of the knowledge that is available for accessing.

Perhaps the role of representations deserves a little more explanation. Note now that we have both concept images and representations as some sort of description of a mathematical entity; how do these differ? Well, the difference is perhaps a matter of perspective, and may be best understood by contrasting the following two casual phrases: 'you can see it **as**' for the image and 'you see it **in**' for a representation. A representation then can have features that may be exploited that would not be available from a concept image. (For example, a graph as a concept image may be taken as a way of understanding functions, but as a representation it may introduce notions like slope, not integral to the abstract function definition.) In fact, because an image is identified with the entity, a more relevant issue seems to be whether cognitive images can have representations (rather than to ask how the two differ). Although we base representations on an abstract structural basis, we do not want to give the impression that it is not appropriate to talk about a representation of a concept image. But when we do refer to such a thing, we shall assume that the image is robustly consistent to the structure of the mathematics that models the concept (so, if need be, the representation may be put onto a structural footing).

Structural knowledge, being well suited to explain why things work, should be conducive for acquiring and retaining knowledge. However neither traditional methods of teaching nor indeed many reform or innovative pedagogical approaches put much emphasis on fostering structural appreciation, so this potential source of cementing knowledge is largely not available for the average student.

Because of the reasons given above, the typical student can have an impoverished stock of knowledge compared to what could be hoped for from the curricula. As much of the information received is not backed up with a sufficiently secure sense in meaning, either at the conceptual or structural level, students will not retain much of the mathematical content to which they are exposed to, and also much of the knowledge of certain powerful trains of thought needed in successfully working in mathematics. True, procedural knowledge quite often can be memorized through repeated use, but this knowledge is not valuable as a tool in problem solving unless some of its structural underpinnings are appreciated. (We characterize procedural knowledge as knowledge that is mentally held with little meaning or significance. Typically procedural knowledge is the result of rote memory or results from procedures that were not comprehended or appraised.) Hence often a student's mathematical knowledge is 'frail', a term used by Steiner (1990), and as such must be largely 'inert', as put by Whitehead (1929). The perspective of this paper will be to concentrate on how to make inert knowledge into a more 'active form'; we will not take into account the possibility that the relevant knowledge might not be registered in the individual's mind in any form. Largely we will employ examples where the knowledge 'prerequisites' are not demanding. However this activity should in itself enrich and reinforce the way that the underlying knowledge is understood, which, in turn, should strengthen its retention in the mind.

Knowledge Schema Building – An Example

In order to maximize possibilities for applications of some particular knowledge to be made available it is highly desirable to explore the knowledge from different perspectives and to seek for linkages with other bodies of information. Doing this we say that we are forming a schema centered around this knowledge. The notion of 'schema' has been given different interpretations in the cognitive and educational literature. We mention the following three exemplars: (a) in the Piagetian theory adaptation of knowledge occurs through the construction and modification of schemata that constitute sequential manifestations of knowledge at different levels of mental maturation, see for example Flavell (1963), (b) the schema based mathematical performance, as analyzed by Hinsley, Hayes and Simon (1977), where it is argued that the student deals with a problem by placing it in a broad category often from the statement (or parts of it) of the problem, (c) in the APOS (Action – Process – Object – Schema) framework the schema associated with a mathematical object encapsulates the building up and expresses the connections that relate actions, processes or different protogenic objects to this particular mathematical object, Dubinsky (1991). Analyses in these traditions tend to be either psychologically dominated, or if mathematical content is a focus (as in APOS) the schema tends to be fairly 'closed' (self-referential to a single conceptual source). An exception to this can be found in some strands of the epistemological tradition in mathematics education, epitomized by the work of Anna Sierpinska. Here care is taken to compare related concepts in order to enhance the structural appreciation of a core concept. A good exposition of this approach is to be found on Sierpinska's work on limits of sequences, Sierpinska (1990). In this section we are not concerned in analyzing schemas per se, but we will illustrate the kinds of dynamics of thought that may be involved in the actual process of building up some strands of a schema.

Example 2

Consider the number of ways of selecting r things out of n things ($r \leq n \in \mathbb{N}$). Denote this number by $C_{r,n}$. We shall call $C_{r,n}$, as r and n vary, as **choice numbers** (rather than binomial coefficients as not to anticipate the course of the exposition.) Cognitively, a choice number has been assigned a certain significance in meaning apart from the fact that it represents an integer. This meaning may be thought of as a conceptual counterpart of the following more formally stated problem: calculate the number of subsets of order r in a set of order n. The words "selecting r things out of n things" then qualify as a concept image. This image is stable enough to allow some mental manipulation. For example, apart from informally arguing to obtain its standard algebraic expression, we may further convincingly argue the identity

$C_{r-1,n-1} + C_{r,n-1} = C_{r,n}$. All that has to be done is to pick out one thing A, and consider two cases; in the first case we consider all choices of r things including A, in the second all choices excluding A. This partition yields the result. Of course to accomplish this train of reason needs a certain mental agility. Note that the reasoning involved is completely parallel with that which would have been used had we attacked the counterpart problem instead. In general, it has been noted often that different formulations of essentially the same problem can cause considerable change in solving performance. (One might recall the famous experiment made by Simom and his colleagues, Simon (1989) which showed that most people took significantly less time to 'solve' an Hanoi Tower problem compared to an exactly analogous task where the discs used in the Hanoi Tower puzzle were replaced by acrobats of varying size, jumping off and on each others shoulders.) In the case of comparing arguments afforded by the concept image with the parallel ones afforded by the corresponding mathematically defined system, perhaps it is not so much appropriate to say that the former will be the 'simpler'. Rather they will tend to be the more transparent, whereas the arguments from the formal system will be more concrete in the sense that one has the access to the structure that the system avails.

We proceed now to describe two further ways of obtaining the identity

$C_{r-1,n-1} + C_{r,n-1} = C_{r,n}$.

(a) If you expand out $(1+x)^n$, there are 2^n terms depending on whether you pick 1 or x in each of the factors $(1+x)$. For any one of these terms, if you have selected x in exactly r out of n factors, the term equals x^r. Collecting like terms, we obtain the result that the coefficient of x^r must equal the number of ways of choosing r things out of n, i.e. is $C_{r,n}$.

Consider now the reformulation below:

$(1+x)^n = (1+x)^{n-1}(1+x)$.

If a choice of r x's is made such that the choice for the isolated factor $(1+x)$ is 1 (x resp.), then a choice is induced of picking r (r-1 resp.) x' s out of n-1 for $(1+x)^{n-1}$. The identity

$C_{r-1,n-1} + C_{r,n-1} = C_{r,n}$ follows.

(b) Imagine that you have an $a \times b$ rectangular array of squares ($a, b \in \mathbb{N}$) and denote the extreme bottom left square by L and the extreme top right square by R. Placed on L is an object that can be moved around the array only by making successive moves either going some spaces to the right along a row or going up some spaces along a column. The number of routes that the object can take to arrive at R is $C_{a-1,a+b-2}$. This is because necessarily each route must involve exactly a+b-2 'crossings' from one square to another; each crossing can be done either vertically or horizontally, but in total for the route to end at R we must have exactly a-1 of the crossings done vertically. By setting

$a = r+1$ and $b = n-r+1$, we may identify $C_{r,n}$ with the number of paths as described above. Now all paths ending at R must either pass through the square immediately to its left or the square immediately below. Knowing the number of routes going to these two squares as $C_{r,n-1}$ and $C_{r-1,n-1}$ respectively, we obtain our identity again.

Notice that both (a) and (b) constitute representations of the basic concept of enumerating ways of choosing r things out of n. Apart from varying terminology due to contextual differences, the argument to justify the identity $C_{r-1,n-1} + C_{r,n-1} = C_{r,n}$ is exactly the same in (a) and (b) as for our initial concept image processing. As the identity on its own is clearly sufficient to calculate $C_{r,n}$ for any r, n by assuming appropriate initial values (basically the Pascal triangle represents the identity), any relationship involving the choice numbers $C_{r,n}$ that can be shown in one situation may be shown analogously in the other two. However this misses some important points; the context that the representations provide can either contribute to providing cognitive tools or can actually afford techniques that otherwise would not be available as we are now going to illustrate.

The choice numbers $C_{r,n}$ are well known to provide a very rich system of formulae. (See e.g. Anderson (1989), Chapter 2.) We will take the representations (a) and (b) and illustrate how having experience with them could help a practitioner to procure some of these relationships. We shall start with (b). This representation has the special feature of being able to be treated visually, and it is the availability of diagrams that lead us quite naturally to obtain some results. We shall give just one example. Any route from L to R must necessarily enter the top row at some column; once the route has reached the top row, its path is determined. Hence the number of routes must equal the sum (over all squares S in the second top row) of the routes starting from L and ending at S. This yields the identity: $C_{r-1,r-1} + C_{r-1,r} + C_{r-1,r+1} ...+ C_{r-1,n-1} = C_{r,n}$.

Now this result simply could be attained by recursive use of $C_{r-1,n-1} + C_{r,n-1} = C_{r,n}$, but by the time that we start long summations it is difficult to maintain the original concept image in terms of numbers of selecting things. Routes on arrays provide us with an alternative way to describe the numbers $C_{r,n}$, where now interpretations of sums and products can be made. Because of this new representation, certain identities become particularly significant and natural to extract (some of which are not so obvious as the one that we have employed).

However, although (b) may guide or inspire direction for posing and solving, essentially it does not offer any new methodology. The representation (a) is very different in this matter. The fact that we are now imbedding the set of choice numbers into the system of polynomials brings in a much more elaborate structure available for exploitation. For example, differentiation is now a device we can utilize. The following formula is just an expression of the representation (a):

$$C_{0,n} + C_{1,n}\, x + C_{2,n}\, x^2 + ...+ C_{n,n}\, x^n = (1 + x)^n$$

By substituting $x = 1$ does not give us anything that is not already conceptually clear. However by the simple action of differentiating both sides and then substituting $x = 1$ we obtain a relationship which is far from being intuitive:

$$C_{1,n} + 2\,C_{2,n} + \ldots + n\,C_{n,n} = n\,2^{n-1}.$$

In this example, we have introduced a couple of representations of the set of choice numbers that allow certain relationships between the choice numbers to be more easily and naturally formed. However, in a way this situation can also be reversed. Many combinatorial problems yield answers in terms of choice numbers, and it is important to simplify the resultant expressions if possible. In this activity, we might well want to make use of the cognitive or structural tools that our representations can offer. Hence the representations do not act only as platforms to inspire problems and results, but they can also be invoked in a solving activity.

In general, representations, as well as weaker associations, provide the kind of net of connections that would form a schema of a type likely to promote the application of knowledge to problem solving. Representations are particularly potent components in the fabric of a schema because of the closely drawn structural ties they have with the central concept image. What is even more important is that representations often offer quite powerful and novel ways of thinking about a theme, as we have illustrated in this section. If we can instill within the student an appreciation of 'neat' arguments, a representation that contributed to forming one is likely to be remembered. A schema critically depends on memory enforced by structural awareness.

Forming broad schemas will significantly increase the chance that problem solving triggers links with knowledge resources. However to take advantage of this fully the student must actively seek out potential applications; this will be explained further in the next section.

The Process of Identifying Applications of Knowledge in Problem Solving

To start with, we will consider a problem that needs little demand on the knowledge base. The style of writing dealing with its solution is meant to highlight the role of the educational research notion of control in problem solving, as explained in the book of Schoenfeld (1985). Indeed the problem we use is taken from this book (p.94).

Example 3

Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be given sets of real numbers. Determine necessary and sufficient conditions on $\{a_i\}$ and $\{b_i\}$ such that there are real constants A and B with the property that

$$(a_1 x + b_1)^2 + (a_2 x + b_2)^2 + \ldots + (a_n x + b_n)^2 = (A x + B)^2$$

for all values of x.

We will write down a solution, but not as you would expect it to be presented in a text, but to reflect a plausible line of thought which could guide you to obtain an answer. You might start off to see whether you can gain some conceptual image for this expression. For this problem, obtaining such an image is unlikely; and it would be an act of control to come to this realization. Hence it would seem a structural approach is needed. A first perusal of the situation might draw your attention to the variable x, and lead you to an assessment that the expression having to hold for all x is a strong condition. Because of this it would seem to be a good idea to try out some particular values of x. Are there 'special' values that would be especially useful to employ? With this question in mind you review the expression again. You note the special feature of the expression that the left hand side is a sum of squared terms; if we drive this sum to zero then each term must also become zero. This can be done by setting $x = - B/A$ on the right hand side of the expression. This breaks the back of the problem; necessarily all the quotients $b_i : a_i$ must be equal, and then it is straightforward to show that this condition is also sufficient.

Although the argument is not difficult, none of the students involved in the relevant fieldwork in Schoenfeld (1985) were able to solve the task. The problem may be due to the very structural level in which the strategy lies. When you are operating structurally, the main things concerning you are to regard the variable x as a degree of freedom or choice, and to access basic knowledge of which the most sophisticated is that a square is always non-negative. The combination of these two things then might in fact be challenging for students to achieve. How can we help them? Well the core of the strategy of our solution is to force the system into a special state of an often-used form; if the sum of terms squared equal 0 then all the terms are zero (in |R). If this becomes a part of the students' knowledge together with a habit to recall this knowledge whenever s/he meets a sum of squares, the student would be in a much better position to answer. True this kind of 'cueing' of knowledge does not represent the most creative thought, but we believe that it does play a very important role in doing mathematics at any level. We shall resume this theme by discussing techniques in the next section.

Example 4

Is there a partial sum of the harmonic series that is an integer, apart from 1? *

A sketch solution: Rewrite the n^{th} partial sum as

$$\frac{\sum_{k=1}^{n} 1 \times 2 \times \ldots \times (k-1) \times (k+1) \times \ldots \times n}{n!} \qquad (A)$$

For $n \geq 2$ an analysis of the numerator would reveal that the highest power of 2 dividing the numerator is less than the highest power of 2 dividing $n!$. Hence none of the partial sums for $n \geq 2$ is integral.

We shall flesh out this solution, but (as in the previous example) in such a way to represent some of the 'background' thoughts that would enable the forming of the argument. The philosophy in doing this is to point out possible difficulties that students might have in obtaining this approach by themselves, not necessarily to present a typical way that an expert would tackle the problem cognitively. We proceed to consider some different stages of the solving process.

(1) Would students necessarily rewrite the n^{th} partial sum as done above? If they want to 'size up' the problem to start off with, they might first be wanting to link up the issue raised in the question with their knowledge of the harmonic series. The basic information they are likely to bear

* We were introduced to this problem by S. J. Hegedus

on the issue is the fact that there always is a partial sum of the series which surpasses any particular integer, and that the terms of the series tend to zero. This knowledge would seem inviting to accommodate in a real number line image. The focus would then be naturally drawn to how successive partial sums 'jump over' integers and how long these jumps are. If the students believe that the answer to the question in the task is no, the natural strategy would be to try to construct around each integer an interval such that no partial sum could be contained in the interval. Clearly the lengths of these intervals would have to tend to zero. If, on the other hand, the students believe that the answer to the question is yes, then they might be tempted to try to justify this existentially on the following basis. As the closest partial sum to an integer gets arbitrarily close to the integer as the integer becomes arbitrarily large, the expectation would be for the two to coincide eventually. The first argument is not plausible, the second is an instance of a common misunderstanding that students show for sequences and series, see for example Mamona-Downs (2002).

Hence the knowledge that would seem the most pertinent to the problem because of the setting of the question does not seem to help us much. If students had started thinking in the ways described above, they would likely to have to abandon it soon. It would then be an act of self-regulation to decide to seek for alternative ways of approaching the question. Conceptually there doesn't seem to be much else to hold on to, but...

(2) there is a natural algebraic maneuver to make, the one taken in our sketch solution. It is motivated more by a practice (i.e. if you have a sum of fractions what you 'normally do' is reformulate it into a single fraction) rather than a conscious shift in strategy. The act performed here is quite modest, but what is impressive is how this small move has opened up a very different realm for the mind to explore compared to the one offered in (1) above. Students are now presented with a quotient of two integers with the issue whether that quotient can represent an integer. Now connections should be coming through from a completely different source of knowledge, including fractions in lowest terms, highest common factors, the Euclidean algorithm, and prime decomposition. Because of the algebraic form of the quotient, it is not likely to be able to carry out the steps of the Euclidean algorithm. What seems to be the most propitious tool available is prime decomposition. Up to now what has been employed is a global viewpoint of the question that did not prove fruitful; prime decomposition offers a way to look at the present state of the solving by local analysis (i.e. to consider prime power divisors for any prime independently from other primes) and hence promises to be flexible. The processing of the knowledge of the unique factorization theorem to suit the issue would be to check whether the greatest prime power divisor of the numerator is greater or equal to the greatest prime power divisor of the denominator for each prime. If for any $n \geq 2$ this is so, our question will be answered in the affirmative; if none of $n \geq 2$ satisfy it, our answer will be negative. The issue is now set into a particular milieu.

(3) Now it is a good time to pause and take stock of the new issue and perspective. Some structural reflection would reveal that for any given prime the greatest prime power divisor (GPPD) of n! may be feasibly found; similarly for the separate terms in the summation of the numerator (this information is not crucial anyway). What really should be of a concern is how to tackle the additions in the numerator. For this we might step away from the context and consider this local issue in the theoretical milieu, i.e. to work within the schema of the unique factorization theorem. What readily available information is there about GPPDs over addition? Suppose that a and b are positive integers, p is a prime and $p^r \| a$ means that r is the greatest power of p dividing a. Then

$p^r || a$ and $p^s || b$ with $r<s \Rightarrow p^r || (a+b)$ (B)

This result is straightforward; however no such universal results will be available in the case when

$r = s$. The only elementary fact that can be deduced is:

$p^r || a$ and $p^s || b$ with $r = s \Rightarrow p^r | (a+b)$ (C)

Hence in the first case (i. e. $r \neq s$) there is perfect control of the GPPD, whereas in the second ($r = s$) only little. What significance do these results have for the problem?

(4) As the main considerations about knowledge access for the problem are now covered, the exposition we give shall be briefer from now on. We return to the present solving situation with the attention on applying the knowledge given in (3) above in an efficient way, i.e. loosely speaking to arrange things such that case (B) is used rather case (C) as far as possible. To prove that the answer of the question is 'no', there are two working variables at hand; a particular prime p for basing the GPPDs, and the order in which the summation of the terms of the numerator of (A) are to be taken.

It happens that if we choose $p = 2$ (for whatever partial sum considered) and take the natural order of summation as suggested by the algebraic form of (A), then whenever we add the next term to the aggregate presently considered we always are encountering case (B) rather than case (C). (We leave the reader to explore this situation to understand why this happens.) This means that the GPPD of the numerator for 2 equals the lowest GPPD of any term of the numerator for 2. If $n>1$, the second term of the numerator is $n!/2$, which has a lower GPPD for 2 than does the denominator $n!$. Thus it is established that there are no partial sums of the harmonic series that equal an integer greater than 1.

Comments on educational issues concerning application of knowledge in example 4.

Despite the tools employed in this task are elementary, we feel that most mathematicians would agree with us in saying that this approach would be understandably difficult for students to create on their own. This can be partially explained by some of the classic themes espoused in the problem-solving tradition. For example, self-regulation to decide when to change tactics or focus, usage of explorative work, identifying patterns and extracting the right structure to construct proofs are all likely to have their roles for anyone adopting the approach. All these types of activities require skills involving flexible and individual thought. But on top of these there are further demands on the students, in accessing knowledge. It is on this facet we will concentrate on.

The first thing to note is that the context of the question could lead a student to follow an unpromising direction. The explicit mention of the harmonic series rather than just writing the mere algebraic expression $(1+ 1/2 + \ldots + 1/n)$ would in itself encourage dynamic imagery related to limiting properties. Even if the terminology was avoided in the presentation of the question, the student is quite likely to make the association with the harmonic series anyway. What this illustrates is that automated triggering in recalling knowledge may be misleading unless it is accompanied with a sense of criticism. Given the observation that if students fail to succeed in obtaining a solution using one argument they tend to give up rather than trying to find another, quite a few students would be frustrated in this question because they happened to follow this line.

When we gather all the fractional terms of the partial sums into one fraction, we are entering a mode of algebraic manipulation. Once students are in such a mode it seems very difficult for many of them to get out of it again. They might have some insight how to handle symbolism to guide it into some desired form, but it seems a rather foreign practice to impute meaning or intuitive significance whilst working algebraically. Without extracting meaning or significance, we are unlikely to link our work with our (long-term) knowledge. The seemingly simple act of mentally processing (A) as a quotient of two integers may well not be a natural one for students to perform. Students' behavior in this way could be enhanced in regular problem-solving courses.

The task in forming the connection between the situation of when a quotient of two integers yields another and prime decomposition was underplayed in (2) above. Really would this connection occur to a student? In general, the difficulty about accessing knowledge when it is not triggered automatically is that the application of the knowledge has to be anticipated at the same time as it is being accessed. In our case we might have to have an inkling how the fundamental theorem of arithmetic will help before being motivated to recall it. This kind of impasse perhaps might be avoided in our particular problem more than in others; triggering attention to prime decomposition likely may be achieved by deliberately seeking for hints how to proceed. A reflection that the present processing of the question is just an issue involving integers, and a recollection that an important tool in analyzing integers are GPPDs would seem enough to make the connection open for consideration. However how many students would make both the reflection and the recollection would be debatable.

Another feature of our approach is how it illustrates how knowledge interacts with problem solving. In (3) an issue raised on the level of specifically working on a particular problem was 'lifted' to the environment of the knowledge supporting that issue. Doing this helps to deliberate the issue in its full generality, and having done this the resultant expanded knowledge is pumped back into the solving environment to guide further strategy. Hence, in a sense the knowledge is responsive to the working as well as vice-versa. This though represents another switch of mode in thinking, and so comprises yet another challenge to the student.

Techniques

We regard a (mathematical) technique as a (mathematical) method with the following characteristics:

(I) There is a recognizable structural cue that suggests that the technique may be applied.

(II) There are one or more standardized steps or sub-goals to achieve, but typically there may be substantial problem solving involved in attaining these goals.

(III) The final step will yield some information of an identifiable type.

Perhaps any method might satisfy the above traits to a degree, but we think of a technique of being quite tightly constrained by them. In general, we consider methods to be less explicit and more flexible than techniques.

Techniques are associated with certain structural references, and as such are very different from heuristics, which tend to act as general advice in setting up strategy in problem solving. However there are some similarities between the two, in particular in the way that both can be rather speculative ways of working. (The problem solving aspect of a technique means that we are not assured to be able to carry out the technique even if it is suitably applied.) Because of this it is quite useful to think of a technique as, loosely speaking, lying between algorithms and heuristics as suggested by Schoenfeld (personal communication).

It is the feature of the cue that makes techniques highly significant in the process of accessing knowledge in problem solving. This feature means that whenever the relevant structural pattern is recognized, the student should be triggered to think about the technique. The technique itself comprises a rather specialized processing of knowledge. Hence the technique automates the (otherwise cognitively difficult) act of retrieving pertinent knowledge.

When the structural cue is strongly associated with some particular conceptual imagery then the application of the technique usually becomes habitual after some experience. For example students soon familiarize themselves with the standard techniques of optimization of (smooth) real functions using calculus tools. (Note that such techniques are not algorithmic, as finding roots is not necessarily easy.) However when the structure implicit in the cue is not identified with a single specific mathematical context then we find that techniques are usually not taught nor consciously held in the mind of the students. As a consequence, the tendency is that the broader a technique is, the less it is appreciated.

One broad technique that definitely is usually taught though is induction. Note that although the technique itself can be supported by fairly evocative imagery (e.g. a line of dominoes placed in a line in such a way that the knocking down of the first will cause all the others to fall in sequence), the description of the cue must be very general and may not seem very concrete. Perhaps it could be characterized by the identification of a family of objects indexed by the positive integers together with an explicit hypothesis about a property of the objects. This encompasses a much more extensive vista of applications of induction than those typically 'registered' by the student that might only stretch to proving algebraic identities. (And even in this case students may only use induction when directed to do so.) Another facet that further restricts students' vision about induction is that usually their experience with the technique is limited to situations where the 'hypothesis' to use is more or less given to them. A more creative situation (and one that would be more true to research work) would be for the students to provide the hypothesis themselves. One way of attempting to do this would be to do some experimental work by examining the property for some specific members of the family of objects and to try to discern a pattern as a basis to forming a hypothesis. In this way we have added a new constructive first step to our original technique (i.e. to develop a hypothesis), and as a result the cue widens even more. We shall call such extensions as **constructively widened techniques**.

Another hugely important technique that also admits a constructively widened technique is the use of 1:1 correspondences for enumeration purposes. In the basic form of the technique the cue is the situation of having two (finite) sets, A and B say, for one of which (say B) we know the order (i.e. the number of elements) and for the other (A) we wish to find the order (or a bound to it). The task involved in the technique is to construct a 1:1 correspondence from a certain set of subsets of B into or onto A, and then deduce some information about $|A|$. In the constructively widened form the cue becomes simply a set A about whose order we want some information. The first stage of the technique now is to identify or construct a second set B for which it would seem propitious to form a 1:1 correspondence with A.

Even though the knowledge on which this technique is based on is both elementary and fundamental (i.e. a bijection preserves set order), students might well not be able to utilize it as suggested in the technique above. It is important that the students have processed the knowledge exactly into the context of the cue. Then whenever instances of the cue are recognized, there should be awareness on the part of the students that the technique is available. (Of course they might choose not to pursue it because they can foresee difficulties or an alternative approach that they prefer.) There are two ways of instigating such awareness; first to explicitly introduce a description of the technique and its cue in class, and second to give the students a sequence of relevant tasks, starting with those yielding the most transparent applications. An important technique deserves some focused pedagogical attention.

To illustrate the points I have just made, I shall briefly describe some fieldwork that I have recently conducted with the collaboration of M. Downs on the technique of employing 1:1 correspondences for enumeration. The participants were volunteers from a 'proof' course that is mainly directed towards students contemplating to take a major in Mathematics. They all had similar tertiary-level mathematical background; each had passed a couple of courses in calculus and one in linear algebra. The institution involved is the University of California, Berkeley. The fieldwork comprised two stages. In the first the six participants worked on a problem sheet on their own. The second was a teaching experiment; it consisted of an open discussion between the participants (four students) about the same problems, with the researchers sometimes prompting its direction. The tasks were designed so that each could be solved by constructing a suitable bijection; however some afforded alternative approaches, but these would always be tedious and more 'messy' in comparison. In their proof course, the students had just been exposed to a short formal treatment of bijections.

The motivation behind the 'written' stage was to see how well the students were already equipped for applying the technique. The responses indicated that on the main the students did not exploit the bijections that were fairly natural to invoke. In one problem one student did give a correct answer by an informal bijective argument, but it transpired that that student had met the question and the approach before. Otherwise the students either did not progress, or opted for the more tedious methods available or worked experimentally. These results would strongly suggest that this population was not able to apply the technique. But what was interesting is that at several places the students wrote notes as asides to their main argument that expressed the basic idea that would have supported the construction of a bijection had the technique been followed. The students seemed not to have the means or confidence to develop the ideas. The mere awareness of the technique as a mentally registered entity probably would have been sufficient to allow the students to utilize these ideas to promote complete arguments.

We attempted to test the validity of this conjecture in the 'teaching experiment' part of the fieldwork. Once we had introduced a background of employing bijections into the session, we wanted to see how easily the students would construct the appropriate correspondences and to observe any ways that they seemed not to be at ease. We illustrate the results by summarizing what happened with one task considered in the session.

Fieldwork Question

Let C be a circle, and suppose that $p_1, \ldots, p_n$ are n points on C. Construct all chords of C connecting 2 points from $p_1, \ldots, p_n$. A crossing is a point strictly inside C that is an intersection point of the constructed chords. What is the maximum number of crossings? (That is find the number of crossings with the assumption that each crossing lies on only two chords.)

In order to start the discussion for this question, the researchers drew two simple diagrams on the blackboard, both showing a circle. Picture 1 further indicated a single chord, picture 2 one crossing and the two chords that intersect there. Picture 1 was meant to act as a prompt towards an analysis via considering the number of crossings on a chord. As this number is not constant, this approach is involved though still viable. Picture 2 was meant to hint a neat way of solving the problem using a bijection in the spirit of our technique; correspond any crossing with the set of four boundary points formed by the end points of the two chords passing through it. We may then deduce that the number of crossings is $C_{4,n}$.

After agreeing early on that the number of crossings on a chord is not constant, the students' attention was solely caught on figure 2 rather than figure 1. Almost immediately one student put forward the bijective argument that allows you to equate the number of crossings with the number of subsets of order 4 of the set $\{p_1, \ldots, p_n\}$. For this student, though, this action had merely transposed the original problem to a new one, because he was not familiar with choice numbers. Another student who had studied combinations before helped out, so the participants could at least understand that the new form of the problem was now a standard one. However this is rather a side issue in respect to the application of the enumeration technique. Two out of the four students showed themselves very comfortable with the bijective argument; even though it was understood on the intuitive level, it proved quite robust when these two students were asked to justify why the relation is 1:1 and onto. The other two students though obviously had misgivings. One of these students consistently showed a dislike or mistrust of the technique in general. She preferred alternative approaches such as breaking the problem down into stages or cases, or employed experimental examination. These procedures seemed a lot more secure and concrete to her than the highly constructive aspects of the technique. The remaining student though had shown himself receptive to the technique in other tasks, his qualms were more local to this particular question. He seemed to appreciate the bijective argument but he appeared not to believe that the simple local structure (as suggested in figure 2) can possibly represent the complicated looking structure of the whole system. In a way, his wish to reconcile the local structure with the global is to be applauded, but it put him to some disadvantage compared to the students who did not feel the cognitive need to attempt such assimilation.

Let us now try to draw together our thoughts about how techniques affect the process of accessing knowledge for problem solving purposes. Techniques that are intimately tied with a certain closed content domain should largely become part of the schema centered around the relevant concept image; the linkage of problem solving in this case would likely first pass through the image and then (if appropriate) to the technique. (For example problem solving might reveal an issue on optimization that leads you to use the standard calculus techniques.) This kind of circumstance is not so significant to our theme. However the situation where the technique is broader and can be applied in many mathematical contexts is different. Indeed we do not suppose that these contexts have been identified or listed. Whenever the problem solving activity happens to wander into any one of these contexts and the present state of the system reveals (or brings up an issue concerning) the particular type of structure as described in the cue, then the technique is available for application. This needs both an awareness of the technique and a general alertness in 'spotting' the cue. However what is provided by the technique is a standardized way of channeling a common structural feature or issue into knowledge processed in a particular way likely to advance a solution. We believe that this role of techniques is vitally important in rendering some of the more creative aspects in mathematics somewhat more routine and accessible. The most general techniques, such as employing correspondences, could truly be considered as very potent universal lines of thought in doing mathematics. However we should not forget about techniques of more modest significance; these are often under-employed because they have been given dominant associations, so that the technique tends to be used in limited contexts. (For example the technique of partial fractions is likely to be used only for solving integrals.)

Our fieldwork on enumeration via bijections suggests several educational issues. Firstly, because broad techniques do not usually have an identity for students, even if students have an intuition about a relevant relationship they may lack the framework to develop it. Hence it seems important to teach students some of the most consequential techniques, just as induction is taught. Doing and discussing a sequence of tasks pinpointing applications of the technique seems an effective way to achieve this. In our fieldwork, three out of four of the students seemed to come out of the discussion stage with a fair appreciation of the enumerative technique; one student at the end of the session said: "*I learned a lot and had never thought of bijections in this way before*". However there are caveats. The fourth student did not seem to get on with the technique at all. From the constructivist perspective of mathematics education we might be criticized in trying to impose methodology. However we feel that this may be countered by the argument that basic techniques form such vital ways of thinking that we cannot afford to let students believe that they can bypass them by inventing their own methods each time. The student would risk lacking the possession of essential problem-solving tools.

A second consideration is that although a technique has its problem-solving aspects, it also has procedural aspects. The latter means that an application of a technique may not elucidate its role within the global structure. Hence a reliance on a technique may represent an undue restriction in thinking about a system. This problem, though, is really a question concerning self-regulation.

Epilogue

The main pioneer of problem solving as a discipline in mathematics is generally considered as being Polya. His work on heuristics, especially the book "How to solve it" (1945), on the main received a good reception from mathematicians. However subsequent fieldwork based on his philosophies did not live up to expectations. Later, educational researchers such as Schoenfeld attempted to find the cause of these disappointments. What was decided was that Polya had succeeded to lay down a tactical base for problem solving, but had left out a managerial aspect. This led to mathematics educators to adopt the psychological notion of metacognition (roughly speaking, self-consciousness of your own cognitive processes). This is split into four main categories: resources, control, belief systems and classroom community influences (see Schoenfeld, 1985). It is in the category of resources that knowledge is treated; Schoenfeld summarizes it thus, p.44 ibid:

Resources are the body of knowledge that an individual is capable of bringing to bear in a particular mathematical situation. They are the factual, procedural, and propositional knowledge possessed by the individual. The key phrase here is "capable to bear"; one needs to know what an individual might have been able to do, in order to understand what the individual did do.

Clearly the topic of resources pertains a lot to cognitive science, as it is the human brain that is storing and processing information. However scientists in this field have only been able to model mental operations relevant to mathematical knowledge where linkages occur spontaneously and are "nearly automatic". [For a comprehensive account of this work see Silver (1987).] Mathematics educators in problem solving have noted these limitations, but without the backing of cognitive theory for more sophisticated channels of accessing knowledge they have preferred not to expand so much on the knowledge base, but to concentrate on control with which "... *solvers can make the most of their resources*" (Schoenfeld, 1985). From this standpoint problem solving then seems to depend on triggering associations with the available resources. The perspective of this paper is how to make these triggering processes more effective, and to stress that the act of knowledge accessing for problem-solving purposes can be far from being mechanized in contrast to what the psychological literature seems to suggest.

In this regard, we are guided by a naive metaphor where we imagine knowledge providing 'hooks' and problem solving situations as providing 'loops'. By increasing the number and size of the hooks and loops we increase the chance that a pair will clasp. Augmenting the size of a hook involves securing and enhancing a reliable concept image, and processing it in a convenient way for its application. Creating new hooks, in the context of a fixed body of definitional knowledge, is done through making connections and forming schemata. By enlarging a loop we mean that we become more aware of the structural aspects of the present state of the working system. Finally we may guide our system into another state, perhaps motivated by a realization that a linkage with some knowledge is imminent, to make further 'loops'.

Poincaré in his essay Mathematical Creation (Poincaré, 1913) made a similar metaphor, for knowledge interactions in the context of unconscious incubation preceding a sudden inspiration; *"the future elements of our combinations are something like the hooked atoms of Epicurus"*. However our use pertains to a different circumstance; we are consciously attempting to let knowledge bear on our solving activities. But very often in order to do this we have to simultaneously anticipate what knowledge is required and consider how to manipulate the system into a state that affords an application of that knowledge. Cognitively this is a difficult demand on students, and the situation is worsened by the fact that the issues overcome in such situations are completely lost in standard style presentation. In general, we advise that some account of the 'thought behind' a solution appears in its exposition, in the same sort of spirit Leron (1983) recommended that the rationale of the constructions made in a proof be informally explained.

However what was said above might suggest a picture that for every individual problem-solving scenario considered we are creating essentially a novel set of ideas, connections and strategies. This clearly is misleading. Mathematical arguments in detail have a bewildering variety, but in outline there seems to be a relatively few types of central features that support them. Taking advantage of these common characteristics is of huge importance; it allows mathematicians to identify types of arguments that can be treated in a similar way. This factor has to be accounted for in our thesis. In this paper we restricted ourselves to what might be the most tangible form of unifying argumentation, that is through techniques. Our description of techniques is such that some very fundamental ways of thinking in mathematics are represented. These require on the face of it only slight re-processing of basic knowledge, but a fieldwork we conducted suggested that students were not alert to the particular technique involved. We propose that some important broad techniques should be explicitly taught. Here we acknowledge the difficulty of already crowded syllabi. However we believe if we were able to accustom students not only to interiorize arguments together with the mathematical facts they provide, but also to take in some of the structural features of the arguments divorced from the facts then the exercise would be justified. For in this circumstance students will start to get in the habit of developing working techniques for themselves.

We note that techniques help in the problem of 'transfer' often referred to in the educational literature. The problem of transfer is about the common phenomenon that students (at all ages) often behave as if they do not recognize analogous problems set in different contexts. Silver's research, Silver (1979), showed that this was mainly due to students not being aware of the underlying mathematical structure. Sierpinska (1995) elaborated this theme, claiming that a present trend in mathematics education at school level suggesting that task contexts should emulate as far as possible 'real-life' situations is detrimental to the transferal of problems. The main message in her paper is that school tasks should be concerned about 'applications', without worrying too much about the applications' status of being either abstract or authentic to reality. She states:

*We need 'contexts', but only in the sense of problems that give meaning and sense to what students learn: **knowledge is always an answer to a question.***

Silver's and Sierpinska's position for school mathematics is somewhat similar to ours for tertiary level problem solving. What we can expect of more mature students is to develop a sense of structure. We believe that **meaning** in mathematics has both conceptual and structural aspects. Conceptual thinking can be both limited and unreliable without accompanying structural appreciation. Structural considerations do not have to be regarded as being abstract, but we conjecture that it is mostly at the level of recognition of parallel structure that the transfer problem is to be resolved. The parallel structure 'connects' with the same knowledge basis, which then is 'applied'.

The theme of reflecting on structure is a recurring one in our paper. The word structure is one that is commonly employed in mathematics education literature but it rarely forms a focus for analysis. Picking up from Sierpinska's assertion that "knowledge is always an answer to a question", we note that this does not seem to represent well how most students retain their knowledge. In truth, the typical student is not often engaged in pondering about mathematical issues but usually is immersed in tackling a mass of exercises. For this reason the student's appreciation of her/ his mathematical knowledge will usually be superficial, and not very effective for use in problem solving. To learn through issues typically requires the unraveling of a rich mixture of motivations both at the structural and the conceptual level. Much the same combination is required for problem solving itself; sometimes informal arguments based on the conceptual image suffices, sometimes arguments are made completely from structural considerations, but most problem-solving tasks involve a blending of the two. But thinking conceptually and thinking structurally seem to form disparate modes. On the metacognitive level we feel that it is important for students to be aware of these two modes of thought. This would represent an important aspect in control; that is, taking the decision about which mode would be the more profitable to assume at any particular time in a solving path.

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MATHEMATICS AND OTHER DISCIPLINES
The Impact of Modern Mathematics in other Disciplines

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ABSTRACT

The impact of modern mathematics and its application in other disciplines is presented from the 20th century historical perspective. In the period 1930's to 1970's mathematics became more inward looking, and the distinction between pure and applied mathematics became much more pronounced. In the 1970s, there was a return to more classical topics but on a new level and this resulted in a new convergence between mathematics and physics. The 20th century approach to mathematics resulted in a more developed mathematical language, new powerful mathematical tools, and inspired new application areas that have resulted in tremendous discoveries in other applied sciences. Towards the end of the 20th Century, mathematicians were making a re-think on the need to bridge the division lines within mathematics, to open up more for other disciplines and to foster the line of inter-discipline research. The current cry is that this interaction will be further strengthened in the 21st Century.

1. Introduction

Mathematics has been vital to the development of civilization. From ancient to modern times mathematics has been fundamental to advances in science, engineering, and philosophy. Developments in modern mathematics have been driven by a number of motivations that can be categorised into the solution of a difficult problem and the creation of new theory enlarging the fields of applications of mathematics. Very often the solution of a concrete difficult problem is based on the creation of a new mathematical theory. While on the other hand creation of a new mathematical theory may lead to the solution of an old classical problem, (Monastyrsky, 2001). This paper is discussing the current role of mathematics in other disciplines.

The presentation is in four parts. Section 2 is dealing with trends of application areas of mathematics at the wake of the twentieth century, Section 3 looks at the changes in mathematics application as a result of the modern approach to mathematics and discoveries in other scientific fields, section 4 addresses the current (21^{st} century) thinking of collaborative and inter discipline mathematics and the section 5 gives some examples of application areas where mathematics is emerging as a vital component with great opportunities for inter discipline research.

2. Trends of Applications in the 20^{th} Century

The 20^{th} century made a rethink on the foundations of mathematics, it was characterised by a new approach to mathematics, fuelled by David Hilbert's (1862-1943) famous set of "mathematical problems" in the 1990 International Congress of Mathematicians. Hilbert's vision was to analyse axioms of each subject and state results in their full generality. This vision became concrete in the 1930's through the development of the axiomatic approach to algebra, pioneered by E. Artin and Edith Noether. Parallel trends took place in functional analysis with Banach Spaces. This spread rapidly to algebraic topology, harmonic analysis and partial differential equations. In addition to this axiomatic approach, the Bourbaki group introduced the idea that there was one universal set of definitions, which once learnt, would be the foundations of everything more specialised (Mumford, 1998). In the drive to seek generality, 20^{th} century mathematics became more diverse, more structured and more complex.

2.1 Divergence of Mathematics from Physics

In the 18^{th} and 19^{th} century mathematical language was vague and did not allow much interaction among mathematicians of different fields. In the period 1950's to 1970's Mathematicians concentrated around problems of algebraic topology, algebraic geometry and complex analysis and they developed new concepts and new methods. New powerful mathematical tools were developed and the language of mathematics became highly developed and very powerful. This has had great impact on diverse fields such as number theory, set theory, geometry, topology and partial differential equations. This new approach to mathematics resulted in greater abstraction. Mathematicians spent years of apprenticeship in a full set of abstraction before doing their own thinking. When the basics were clear enough there was a search for powerful tools that allowed for development and expansion of the geometric intuition into new domains. Examples are topology, homological algebra and algebraic geometry. These new developments made it possible for great breakthroughs in solving several difficult problems that were stuck. For example the Deligne's proof of Weil conjectures, Faltings' proof of Mordell conjecture and Wiles' proof of Fermat's theorem could not have been done in the 19th century just because mathematics was not developed enough. Mathematics of the 20th century has started the path for harmonising and unifying diverse fields. The unification of mathematics started with a common language that has greatly simplified the interaction between mathematicians. This language became the basis for development of new technical tools for the solution of old problems and the formulation of research programmes.

As a consequence of the new approach to mathematics, pure mathematicians drifted away from applications and saw no need to collaborate with other scientists, even their traditional neighbours, and the physicists. On the other hand, application of the highly abstract modern mathematics could not be easily visualised by the traditional users of mathematics. The period 1930's to 1970's saw a divergence within mathematics itself and between mathematics and other applied sciences. Mathematics became more inward looking, and the distinction between pure and applied mathematics became much more pronounced. The diversification of mathematics was first of all connected with external social phenomenon, the rapid growth of the scientific community and the breaking discoveries in physics.

The traditional area of application of mathematics is physics. Within this area the deepest mathematics and success stories have been achieved. For example, Einstein's general theory of relativity was based on classical differential geometry of Riemannian spaces, the Hilbert spaces, the theory of linear operators, and spectral theory. In the 1930's the connection of mathematics and other sciences, especially physics was broken. Physicists got interested in solving more concrete problems that could be solved without the application of sophisticated and abstract modern mathematics. The developments of pure mathematics in the post World War II period became weakly connected with applied sciences especially physics. Mathematicians' could not view how physics could assist modern mathematics while physicist could not imagine the application of new abstract mathematical concepts such as sheaf, cohomolgy, J- functor and the like in their fields (Monastyrsky, 2001).

2.2 Re-Convergence of Mathematics with Physics

From the beginning of 1970s, there was a return to more classical topics but on a new level. These developments resulted in the new convergence between mathematics and physics. Some modern mathematicians (e.g. S. Novikon, S.T. Yau, A. Connes, S. Donaldson and E. Witten) quickly saw new opportunities and challenges hidden in the new physics. Examples of mathematical results that got inspired by physical ideas include Donaldson's proof of the existence of different differential structures on simply connected 4-dimensional manifolds. This has very deep consequences for quantum gravity and the gauge theory on strong and weak interactions and resulted in the revisit of the Yang-Mills equations of elementary particles, which had been developed by physicists C. N. Yang and R. Mills almost twenty years earlier in 1954. The Yang-Mills equations had been considered non-physical and had attracted very little attention of physicists. Structures in the elementary particles are described by highly nonlinear equations with deep topological properties. Donaldson's proof inspired physicists to do a deeper study of the Yang-Mills equations. In the 1970's information flow between mathematicians and physicists resumed and led to new and deeper connections between modern mathematicians and physicists. Basing on this new union, theoretical physicists have made substantial progress in uncovering the principles governing particle interaction. The new conservation laws developed in the last part of the 20^{th} Century are believed to be the most fundamental in physics. Most success stories of application of pure, most abstract mathematics are in physics. The application of modern abstract mathematics in physics has resulted in astounding discoveries of the 20^{th} Century in the physical sciences, the life sciences and technology.

The new approach to mathematics resulted in a more developed mathematical language, new powerful mathematical tools, and inspired new application areas that have resulted in tremendous discoveries in other applied sciences including computer science and computer technology. The new mathematical tools and the developments in computer technology, the development of algorithms, mathematical modelling and scientific computing have led to remarkable new discoveries is physics, technology, economics and other sciences in the last half of the 20^{th} century. This has also enabled mathematicians to use modern mathematical tools to solve deep classical problems left by the previous generation of mathematicians.

3. New Application Areas

The branch of mathematics traditionally used in the applications in physics is analysis and differential geometry. Most of the advances in pure mathematics were propelled by problems in physics. In the last part of the 20^{th} century researchers in many other sciences have come to a point where they need serious mathematical tools. The tools of mathematical analysis and differential geometry were no longer adequate. For example a biologist trying to understand the genetic code will need tools of graph theory than differential equations because the genetic code is discrete. Issues of information content, redundancy or stability of the code are more likely to find tools of theoretical computer science useful than those of classical mathematics are. Even in physics discrete systems such as elementary particles need use of combinatorial tools and statistical mechanics need tools of graph theory and probability theory. Traditionally economics is a heavy user of applied mathematics toolbox. Now economics utilises sophisticated mathematics in operations research such as linear programming, integer programming and other combinatorial optimisation models, (Lovasz, Laszlo, (1998)).

3.1 Bridging the Division Lines

Developments in computer technology have re-activated some areas in the fields of discrete mathematics, formal logic and probability that were otherwise dormant for a long time. Examples include the vast and rapid developments in the areas of algorithms, databases, formal languages, as well as cryptography and computer security. Just about 25 years ago questions in number theory that seemed to belong to the purest, most classical and completely in applicable mathematics now belong to the core of mathematical cryptology and computer security.

Towards the end of the 20^{th} Century, mathematicians were making a re-think on the need to bridge the division lines within mathematics, to open up more for other disciplines and to foster the line of inter-discipline research. The current cry is that this interaction will be further strengthened in the 21st Century. Many believe it is better to view pure and applied mathematics as a continuum rather than as two competing and hostile camps.

Efforts being undertaken in other scientific communities will bring the full range of mathematical techniques to bear on the great scientific challenges of our time. It is quite obvious that in this century, the need for mathematics to enrich other scientific disciplines, and vice versa, is most urgent. Currently there is a sense of readiness among mathematicians to interact with the world around them. Currently there is a sense of readiness among mathematicians to interact with the world around them. This is in addition to continuing the pursuit of mathematics for internal motivations such as revealing its inherent beauty and understanding its coherent symmetries.

Being the language of sciences, mathematics has a great potential to make tremendous contributions to the other sciences. The current move is to breakdown barriers that still exist between mathematicians and other scientists. For example, there is still a large gap in the knowledge of physics. The two main pillars of 20^{th} century physics, quantum theory and Einstein's general theory of relativity are mutually incompatible. It is speculated whether string theory and other most abstract mathematics areas will provide the solution. Mathematicians and theoretical physicists are busy working to bridge this gap.

3.2 Potential Contribution to Other Fields

As evidenced by the discoveries of the last half of the 20^{th} century, mathematics can enrich not only physics and the other physical sciences, but also medicine and the biomedical sciences and engineering. It can also play a role in such practical matters as how to speed the flow of traffic on the Internet or sharpen the transmission of digitised images, how to better understand and possibly predict patterns in the stock market, how to gain insights into human behaviour, and even how to enrich the entertainment world through contributions to digital technology.

Through mathematical modelling, numerical experiments, analytical studies and other mathematical techniques, mathematics can make enormous contributions to many fields. Mathematics has to do with human genes, the world of finance and geometric motions. For example, science now has a huge body of genetic information, and researchers need mathematical methods and algorithms to search the data as well as clustering methods and computer models (among others) to interpret the data. Finance is very mathematical; it has to do with derivatives, risk management, portfolio management and stock options. All these are modelled mathematically, and consequently mathematicians are having a real impact on how those businesses are evolving. Motion driven by the geometry of interfaces is omnipresent in many areas of science from growing crystals for manufacturing semiconductors to tracking tumours in biomedical images. The convergence of mathematics and the life sciences, which was not foreseen a generation ago, is a tremendous opportunity for application.

4. Inter-Discipline Mathematics

Currently, efforts are being undertaken to facilitate collaborative research across traditional academic fields and to help train a new generation of interdisciplinary mathematicians and scientists. Also similar efforts are slowly being introduced in undergraduate and postgraduate mathematics curricula and pedagogy. Disciplines that hitherto hardly used mathematics in their curricula are now demanding substantial doses of knowledge of and skills in mathematics. For example the pre-requisites for mathematical knowledge and skills for entry in into biological and other life sciences as well as the mathematics content in the university curricula of these programmes is becoming quite substantial. Curricula for the social sciences programmes now include sophisticated mathematics over and above the traditional descriptive statistics. Curricula of some universities in the developed countries have inter-disciplinary programmes where mathematics students and students from other sciences (including social sciences) work jointly on projects. The aim is to prepare graduates for the new approaches and practices in their fields and careers.

4.1 Examples of Inter-Discipline in Research

Complexity theory is an example of inter-discipline and is the new focus on research in mathematics (Hoyningen-Huene, *et al* 1999). Certain essential details of complexity have been known for quite some time. At the end of the 19^{th} century, the first source of a general idea of complex systems was research in dynamical systems, in the context of classical mechanics. It is an interdisciplinary approach fuelled by sophisticated mathematics, mathematical modelling and computer simulation, inspired by observations made on complex systems in the most diverse fields including meteorology, climate research, ecology, economics, physics, embryology, computer networks and many more. Examples are systems that adapt to changes in their environment in an extremely surprising way. They include Economics (economy of a country), Biodiversity (ecosystem of a pond), Biology (the immune system of an organism) and Artificial Intelligence (Computer Networks).

Probability theory seems to bridge most of the division lines within mathematics. The importance of probabilistic methods in almost all areas of mathematics is exploding. Probability theory is one illustration of the unity of mathematics that goes deeper than just using tools from other branches of mathematics. With probability theory, many basic questions can be modelled as discrete or as continuous problems.

4.2 Illustration of Current Needs Of Mathematics in University Curricula

The role of mathematics in other disciplines has become clearer. I will illustrate this by making quotations from a public reaction to a decision by the Rochester University to reduce the size of mathematics faculty.

Below are quotations from an article titled "Demotion of mathematics meets groundswell of protest" by Arthur Jaffe, Harvard University, President-elect, American Mathematical Society (AMS), Salah Baouendi, University of California at San Diego, Past Chair, AMS Committee on the Profession and Joseph Lipman, Purdue University, Chair, AMS Committee on the Profession presents the statements from different people. The article dated February 1, 1996, is available on the Internet http://www.ams.org/committee/profession/rochester.html and it appeared in Notices of the American Mathematical Society. "In 1996, the University of Rochester planed to downgrade its mathematics program by reducing faculty size and closing down some postgraduate programmes. University of Rochester's plan met with outright protest not only from mathematicians but also from well-known scientists both in universities and in business. Strong protest statements were made by at least six Nobel laureates, by dozens of members of the

National Academy of Sciences, as well as by other leaders in science and industry. The outpouring came from many fields, including biology, chemistry, computer science, economics, geology, mathematics, philosophy, physics, and sociology".

Below are verbatim quotations of some of the statements:

31 professors in the Harvard physics department (including 3 Nobel laureates) wrote: "Recent history confirms the interaction between fundamental mathematical concepts and advances in science and technology. We believe that it is impossible to have a leading university in science and technology without a leading department of mathematics".

Norman Ramsey, Nobel laureate in physics, remarked: "If you had only one science department at a university, it would be mathematics, and you build from there".

All members of the Harvard chemistry department, including one Nobel laureate wrote: "For centuries, mathematics has rightly been termed "the queen of the sciences," and this is just as apt today. In particular, chemistry has benefited more and more from mathematical developments and concepts. A university that aims to have a worthy program in science and technology simply must have a genuine department of mathematics pursuing original research"

Steven Weinberg, University of Texas, Nobel laureate in physics stated the following: "I am not a mathematician, but I regard mathematics as the core of any research program in the physical sciences. If you do not have a graduate program in mathematics, then eventually you will have no research mathematicians, which will make Rochester far less attractive to theoretical physicists. Experimental physicists may not feel the loss of the mathematics program directly, but with fewer first-rate theoretical physicists you will begin to lose your best experimentalists as well. You will also be weakened in your ability to compete for good students; both graduate and advanced undergraduate physics students need to take advanced courses in mathematics, which can only be taught well by active research mathematicians. I imagine that similar effects will eventually be felt in your chemistry and optics departments. I would not advise any prospective undergraduate or graduate student who wishes to concentrate on the physical sciences to go to a university that did not have a graduate program in mathematics".

Joel Moses, a computer scientist and provost at MIT, wrote: "I for one cannot imagine operating a school of engineering in the absence of a strong and research-oriented mathematics department. The same can be said for a school of science. I am also dismayed at the prospect of covering a substantial portion of the teaching load in mathematics with adjunct faculty".

George Backus, research professor of geophysics at the University of California at San Diego and a member of the National Academy of Sciences, wrote: "At UCSD, the Institute of Geophysics and the Scripps Institute of Oceanography often recommend that our Ph.D. students take graduate courses in the UCSD Department of Mathematics. Modern theoretical geophysics and physical oceanography simply cannot be done without sophisticated modern mathematics. To teach these [advanced mathematical subjects] with sophistication and insight requires people for whom they are the primary research interest".

Neil A. Frankel, manager, Advanced Components Laboratory at the Xerox Corporation expressed the following industrial point of view: 'It is evident that neither [Kodak nor Xerox] is well served by the elimination of two technology-related [graduate] departments [chemical engineering and mathematics]. To stay ahead of the very significant competition from Japan and elsewhere, [Kodak] will need all the quality engineering talent it can find. The availability of a quality university in Rochester enhances our ability to attract the very best people to our company. If graduate mathematics is eliminated, I really don't see how UR can support first-rate programs in the sciences and in engineering, and I fear that all of these will decline".

Professor Sir Michael Atiyah, director of the Newton Institute in Cambridge, England; also the past president of the Royal Society wrote: "Increasingly the complex problems that scientists now face require more sophisticated mathematical understanding. The advent of more powerful computers has in no way decreased the fundamental relevance of mathematics. I can illustrate the scope of mathematical interaction with other fields by listing just a few of the inter-disciplinary programmes that we have run at the Newton Institute in the past few years: computer vision, epidemics, geometry and physics, cryptology, financial mathematics, and meteorology".

Edward Dougherty, editor of the Journal of Electronic Imaging, wrote: " While at first this might appear to most people as simply one major research university deciding to restructure itself into a not-so-major university, for those of us in the imaging community there is much more at stake. Because it is home to both Kodak and Xerox, Rochester is one of the major imaging centers in the world, and therefore the future of imaging is closely tied to significant imaging events in Rochester. Suspension of graduate research and teaching in two key foundational imaging disciplines is not insignificant. Chemical engineering plays a role in imaging materials, toners, and numerous other staples of digital imaging. The case for mathematics is even more compelling when it comes to digital imaging. Simply put, there is no scientific phenomenology without mathematics. The kind of mathematics graduate courses necessary for contemporary research in image processing might simply cease to exist in the city of Kodak and Xerox".

Marvin L. Goldberger, dean of the Division of Natural Sciences in the University of California at San Diego wrote: "Not only is mathematics an exciting and vital intellectual endeavour, but from a number of standpoints, plays an exceptional educational role at both the undergraduate and graduate levels. Advanced mathematics is essential in all areas of applied science; economics; technological risk analysis; to an increasing extent in fundamental and applied biology (e.g., drug design); in national security issues involving communication, cryptanalysis, satellite reconnaissance--the list is endless, but one more example is particularly relevant: in recent years topology has played a central role in elementary particle physics where string theory is a candidate for "Theory of Everything." This is another case of the remarkable and mysterious relationship between mathematics and the physical world. Topology is one of the strengths of the Rochester Mathematics Department".

These public reactions illustrate the ever-expanding interrelationship between mathematics and other disciplines, today and in the immediate future.

5. Examples of Key fields where Mathematics is emerging vital

Friedman, A., 1998, presented three examples of key fields in science and technology to the 1998 Berlin International Congress of Mathematicians. The examples are from the disciplines of materials sciences, the life sciences, and digital technology. Also recently, Hu, J.J and Wang, H. 2001, presented to a conference a brief outline of a perspective from the USA army research office on trends in army funding for mathematics research. Below are summaries of the four examples:

5.1 Mathematics in Materials Sciences

Materials sciences is concerned with the synthesis and manufacture of new materials, the modification of materials, the understanding and prediction of material properties, and the evolution and control of these properties over a time period. Until recently, materials science was primarily an empirical study in metallurgy, ceramics, and plastics. Today it is a vast growing body of knowledge based on physical sciences, engineering, and mathematics.

For example, mathematical models are emerging quite reliable in the synthesis and manufacture of polymers. Some of these models are based on statistics or statistical mechanics and others are based on a diffusion equation in finite or infinite dimensional spaces. Simpler but more phenomenological models of polymers are based on Continuum Mechanics with added terms to account for 'memory.' Stability and singularity of solutions are important issues for materials scientists. The mathematics is still lacking even for these simpler models.

Another example is the study of composites. Motor companies, for example, are working with composites of aluminium and silicon-carbon grains, which provide lightweight alternative to steel. Fluid with magnetic particles or electrically charged particles will enhance the effects of brake fluid and shock absorbers in the car. Over the last decade, mathematicians have developed new tools in functional analysis, PDE, and numerical analysis, by which they have been able to estimate or compute the effective properties of composites. But the list of new composites is ever increasing and new materials are constantly being developed. These will continue to need mathematical input.

Another example is the study of the formation of cracks in materials. When a uniform elastic body is subjected to high pressure, cracks will form. Where and how the cracks initiate, how they evolve, and when they branch out into several cracks are questions that are still being researched.

5.2 Mathematics in Biology

Mathematical models are also emerging in the biological and medical sciences. For example in physiology, consider the kidney. One million tiny tubes around the kidney, called nephrons, have the task of absorbing salt from the blood into the kidney. They do it through contact with blood vessels by a transport process in which osmotic pressure and filtration play a role. Biologists have identified the body tissues and substances, which are involved in this process, but the precise rules of the process are only barely understood. A simple mathematical model of the renal process, shed some light on the formation of urine and on decisions made by the kidney on whether, for example, to excrete a large volume of diluted urine or a small volume of concentrated urine. A more complete model may include PDE, stochastic equations, fluid dynamics, elasticity theory, filtering theory, and control theory, and perhaps other tools.

Other topics in physiology where recent mathematical studies have already made some progress include heart dynamics, calcium dynamics, the auditory process, cell adhesion and motility (vital for physiological processes such as inflammation and wound healing) and bio-fluids. Other areas where mathematics is poised to make important progress include the growth process in general and embryology in particular, cell signalling, immunology, emerging and re-emerging infectious diseases, and ecological issues such as global phenomena in vegetation, modelling animal grouping and the human brain.

5.3 Mathematics in Digital Technology

The mathematics of multimedia encompasses a wide range of research areas, which include computer vision, image processing, speech recognition and language understanding, computer aided design, and new modes of networking. The mathematical tools in multimedia may include stochastic processes, Markov fields, statistical patterns, decision theory, PDE, numerical analysis, graph theory, graphic algorithms, image analysis and wavelets, and many others. Computer aided design is becoming a powerful tool in many industries. This technology is a potential area for research mathematicians. The future of the World Wide Web (www) will depend on the development of many new mathematical ideas and algorithms, and mathematicians will have to develop ever more secure cryptographic schemes and thus new developments from number theory, discrete mathematics, algebraic geometry, and dynamical systems, as well as other fields.

5.4 Mathematics in the Army

Recent trends in mathematics research in the USA Army have been influenced by lessons learnt during combat in Bosnia. The USA army could not bring heavy tanks in time and helicopters were not used to avoid casualty. Also there is need for lighter systems with same or improved requirement as before. Breakthroughs are urgently needed and mathematics research is being funded with a hope to get the urgently needed systems. These future automated systems are complex and nonlinear, they will likely be multiple units, small in size, light in weight, very efficient in energy utilisation and extremely fast in speed and will likely be self organised and self coordinated to perform special tasks.

Research areas are many and exciting. They include: (i) Mathematics for materials (Materials by design - Optimisation on microstructures; Energy Source - compact power, Energy efficiency; Nonlinear Dynamics and Optimal Control). (ii) Security issues (needs in critical infrastructure protection, mathematics for Information and Communication, Mathematics for sensors, i.e. information/ data mining and fusion, information on the move i.e. mobile communication as well as network security and protection). (iii) Demands in software reliability where mathematics is needed for computer language, architecture, etc. (iv) Requirements for automated decision making (probability, stochastic analysis, mathematics of sensing, pattern analysis, and spectral analysis) and (v) Future systems (lighter vehicles, smaller satellites, ICBM Interceptors, Hit before being Hit, secured wireless communication systems, super efficient energy/ power sources, modelling and simulations, robotics and automation.

During the last 50 years, developments in mathematics, in computing and communication technologies have made it possible for most of the breath taking discoveries in basic sciences, for the tremendous innovations and inventions in engineering sciences and technology and for the great achievements and breakthroughs in economics and life sciences. These have led to the emergency of many new areas of mathematics and enabled areas that were dormant to explode. Now every branch of mathematics has a potential for applicability in other fields of mathematics and other disciplines. All these, have posed a big challenge on the mathematics curricula at all levels of the education systems, teacher preparation and pedagogy. The 21st Century mathematics thinking is to further strengthen efforts to bridge the division lines within mathematics, to open up more for other disciplines and to foster the line of inter-discipline research.

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LOOKING FOR LEVERAGE: ISSUES OF CLASSROOM RESEARCH ON "ALGEBRA FOR ALL"

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ABSTRACT

In the United States, African Americans, Latinos, and Native Americans have lower success rates and higher drop-out rates in mathematics than other racial or ethnic groups. Given that quantitative competency serves increasingly as a vehicle for economic enfranchisement, these differential success rates make mathematics achievement a civil rights issue. Failure and dropouts start early. Moreover, "algebra" is becoming a major stumbling block: many states require students to pass algebra tests in order to graduate high school. This social/mathematical problem is becoming increasingly urgent.

This paper describes the American context and suggests its relevance world-wide. It then explores the following issue. Suppose one wants to do classroom-based research on Algebra for All: one will observe what takes place in middle school mathematics classrooms where there are diverse populations of students. What kinds of data should one gather in order to determine which practices support the learning of mathematics by diverse groups of students and how they work? What theoretical frame will provide the best purchase on these issues?

Issues addressed include: whether mathematics is "culture-free" and what the implications for instruction might be, even if it is; the institutional support necessary for high quality instruction; the differential treatment of student groups; pedagogical practices that enfranchise a wide range of students; the roles of language and discourse in learning and classroom communities; individual agency; and what it means to engage meaningfully with mathematics. The challenge is to conduct classroom research that helps to explain, at a level of mechanism, how classroom interactions can be structured to help students who vary widely in terms of cultural backgrounds and prior mathematical success to *all* learn some very solid mathematics.

First.
You have to *understand* the problem.
– George Pólya, *How to Solve It*

Introduction

This paper differs from those I am accustomed to writing in one fundamental way. Typically, researchers spend a fair amount of time working on a problem. Then, after significant progress has been made, they write up the results. The purpose of writing such a paper is to share understandings with others. I will do some of that here. But my goal is also to problematize a research arena – to grapple with the question of how one can productively study classroom attempts to help middle school students with widely divergent cultural and socio-economic backgrounds learn the mathematics that leads to and includes the study of algebra.

Here is why the topic matters. Issues of "algebra for all" are absolutely central in the current America context. In the United States, poor children and under-represented minorities (African Americans, Native Americans, and Latinos) tend to earn lower grades and to stop taking mathematics courses much earlier than others; access to and treatment in mathematics classes also differs by gender. Broadly speaking, a lack of mathematical competence and credentials constitutes a barrier to full participation in the economic mainstream. Hence differential participation and success rates in mathematics become an issue of social justice. Moreover, the stakes are about to be raised. California and other states have instituted standardized examinations as a prerequisite for high school graduation. The mathematical content focus of the examinations is on algebra. Students who do not succeed at learning algebra will be denied a high school diploma – and thus seriously marginalized.

A team of researchers from three universities (The University of Wisconsin at Madison, the University of California at Berkeley, and the University of California at Los Angeles) has received funding from the U. S. National Science Foundation to address these issues. Our project, "Diversity in Mathematics Education" (DiME), covers a lot of territory. Project goals include preparing a new generation of researchers to work on issues of diversity and mathematics education, working in partnership with bcal school districts to create enhanced models of teacher preparation and professional development, and creating a set of resources that can be used by teachers and school districts to address these issues. Central to such resources is developing a deep sense of what happens in classrooms as students grapple with the ideas of algebra.

There is always uncertainty in research; that is the nature of the process. As an established researcher, I have of course developed my own *modus operandi* and a substantial level of comfort for dealing with uncertainty. Typically I approach a problem with some sense of what is likely to be important, in both theoretical and pragmatic terms. I identify phenomena of interest, gather relevant data (which might include videotapes and various artifacts), labor over the data until they begin to make sense, draw some tentative conclusions, and look for more data or perspectives that will yield triangulation. The results of that work may be some or all of the following: theory refinement, methods development, or a deeper understanding of a particular problem. (For me, problems tend to be of the type "how does something work"; answers are usually at a detailed level, describing the way things fit together.) I am accustomed to starting with rough ideas of problem, theory, and method – with some notions of what things are important and what will help me make sense of them – and then living with the phenomena until a reasonably clear picture emerges. Indeed, much of the pleasure of being a researcher is in figuring out how to turn one's intuitions into new methods, perspectives, and findings. When my intuitions feel solid, they often pay off – not necessarily in the ways I expect, but often in ways that are close.

As I begin this project, I do not feel comfortably equipped to address classroom issues at the heart of DiME's "diversity and algebra[1]" agenda. Despite having spent many years of thinking about issues of mathematical thinking, teaching and learning; despite having spent one morning every week in local public school mathematics classrooms for the past decade; and despite having read widely and thought hard about issues of "mathematics for all," I am not at all confident that I have an appropriate framing of the issues or that the methods I know are appropriate for grappling with them.[2] This paper represents an attempt to think through some of those issues – to lay out some of what is known and seems to be relevant, and to see if I can elaborate some of the conceptual and methodological problems that need to be confronted.

The paper proceeds as follows. In the next section I start with a bit of international context, to show the relevance of the issues discussed here to non-American readers. Then I focus on the American context, providing a bit of historical background – how high school mathematics moved from a subject to be studied only by the elite to a subject to be studied by all. I proceed to discuss plausible goals for mathematics instruction, and the reason that learning a solid core of mathematics is an important and plausible goal for all students. This is followed by a brief discussion of demographic data. These data on the mathematical performance of diverse groups indicate clearly that in the United States, mathematics education is an issue of social justice.

Having established context for DiME's agenda, I move on to review some of what is known about making mathematics accessible to a wide range of students. That section of the paper is where I try to untangle the issue of classroom research on algebra for all. As I work through various dimensions of what is known, I point to issues that still strike me as problematic.

Before moving to my announced agenda in the next section, I want to conclude this introductory section by posing and reflecting on some questions about the nature of mathematics and mathematics instruction. These questions have provoked me, through the years, to think about issues of diversity and mathematics. I begin with a question that haunted me for a long time as a mathematician, then move to ones concerned with pedagogy and research.

- Isn't mathematics "culture-free" or "culture-independent?"

At international mathematics conferences, for example, it's astounding how people who have never met each other and may share only a few words in a common language can communicate

[1] In what follows I shall say a fair amount about diversity and rather little about algebra. That is because issues concerning algebra are somewhat more straightforward, and do not cry for elaboration here: see, e.g., NCTM's (2000) *Principles and Standards* and the U.C. ACCORD Mathematics working group's (2000) report *Pathways to algebra for all of California's children.*
[2] This sense of discomfort is, of course, intimately tied up with my sense of what counts as understanding or explanation. My goal as a researcher is to understand how and why things work, so I'm not satisfied personally until I have a sense of how things fit together.

meaningfully about deep mathematical ideas. While it may or may not be the case that "a rose is a rose is a rose," there is no doubt that from the typical mathematician's point of view a Banach space is a Banach space is a Banach space: once the definition is made the properties are established, and anyone who plays by the rules can determine those properties. At a more elementary level, a square is a square is a square: once one says that a quadrilateral in the Euclidean plane is a square, then (for example) its diagonals must be perpendicular and must bisect each other. The point from the mathematician's perspective is that the properties follow from the definition, no matter who does the proving. At an even more elementary level, it doesn't matter who counts a finite set of objects, or what culture that person comes from– the answer will always be the same.

An affirmative answer to the first question leads to a corollary question:

- If mathematics is culture-free, then how does it make sense to speak of "teaching mathematics to students of different cultures"? That is, if mathematics is culture-free, shouldn't mathematical pedagogy be culture-free?

How one answers this question depends, of course, on how one conceptualizes teaching and learning.

One view, which predominated when I began to teach mathematics and is still, I suspect, rather common at the university and perhaps secondary levels, is that the responsibility of the mathematics teacher is to present lucid explanations of the mathematical ideas at hand. In this view, the truly competent teacher is the one who has three of four (maybe more) different ways of explaining a topic or concept, so that students who don't "get" the first may find the second more accessible, or perhaps the third, or fourth.

It is important to recognize possible concomitants of this view. When the teacher has presented mathematically clear explanations at the right level, he or she has met his or her pedagogical obligations. Thus this approach creates a clear division of responsibilities. The faculty's job is to make the material accessible to students; the students' job is to learn that which has been clearly presented. In consequence, this perspective allows the faculty to abdicate responsibility for some student learning: if the presentation has been clear, then it's the student's fault if he or she didn't learn the material. It also supports "deficit" models of instruction, with the assumption that students from particular backgrounds have particular deficits. (Students for whom English was a second language might, for example, be taken out of mathematics classes until their English was deemed adequate for full participation in the mathematics classes. The net result was that those students got further behind in mathematics.)

When it is presented in such stark terms, the "lucid explanation" perspective described in the previous paragraphs might well be rejected by a fair percentage of today's teachers. It harks back to the "old days," when teachers lectured and students took notes. In the United States today's mathematics classes are much more interactive; students engage in a wide range of mathematical activities. A more contemporary view might be that the responsibility of the mathematics teacher is to provide students with a range of activities (possibly including lecture, individual or small group work, whole class activities, the use of manipulative materials, and more) that allow students to engage with the mathematical ideas at hand, and to learn as a result.

This does indeed sound contemporary. The point to recognize, however, is that everything that I said about the "lucid explanations" perspective applies to this more contemporary view as well. Here the master teacher might be viewed as the teacher with a large bag of tricks, including a large range of activities that support multiple approaches to the mathematics. This certainly covers more territory than the first perspective. But, like the other, it creates a clear division of responsibilities. The teacher now has a larger set of responsibilities – the pedagogical tool kit is expected to be much larger. But here too, faculty are given a warrant for abdicating responsibility for some student learning: if classroom activities have been field tested and are thought to be of high quality, then it's the students' fault if they don't leam the material.

A third view is that effective teaching (defined as "things the teacher does that lead to successful learning") is teaching that helps students to negotiate the terrain between what they bring to the learning environment and what one wants them to learn. Of necessity, this kind of teaching calls for understanding and building upon what the students bring – predispositions and understandings, habits of mind, patterns of engagement, patterns of communication (including norms of social interaction and linguistic patterns), and more. It should be obvious that many of these are shaped by the student's experiences outside classroom boundaries – that is, they are shaped culturally. From this perspective, then, effective teaching must be responsive to what the students bring with them to the classroom – in Ladson-Billing's (1994) words, pedagogy must be "culturally responsive."

If one accepts the notion that one has to "meet students where they are," the next set of questions to address concerns how to understand what the students bring to the classroom, and how to foster productive interactions between students and mathematics. As will be elaborated below, there is reason for optimism about what can be achieved. Indeed, there are some suggestions of the kinds of conditions that might, in concert, sustain positive change. These will be reviewed, albeit briefly. But even given these, I find myself confronted with a series of questions about the kind of research I would like to produce.

The question I would like to address is this:

- Suppose one wants to do classroom-based research – that is, one's work will be grounded in observations of what takes place in middle school mathematics classrooms in which there are diverse populations of students. What kinds of data should one gather in order to determine which practices support the learning of mathematics by diverse groups of students, which do not, and how they work? What theoretical frame will provide the best purchase on these issues?

As simple as these questions may seem, the answers are anything but simple.

Context

Why this might matter to people outside the United States.

The United States has often gone its own way in curricular matters. For example, the traditional U. S. mathematics curriculum consists of a year's study of elementary algebra in 9^{th} grade, Euclidean geometry in 10^{th} grade, and a return to more advanced algebra and trigonometry in 11^{th} grade. In the traditional curriculum, geometric problems are not dealt with in the algebra courses, and vice-versa; applications are few and far between. This course configuration, along with the nature of topic coverage in the U.S. ("a mile wide and an inch deep"), are somewhat anomalous internationally (see, e.g., Schmidt, McKnight, and Raizen, 1997). Given the atypical nature of the curriculum, and the somewhat atypical history of race relations in American society, why might the study of issues of diversity and mathematics education in the U.S. be relevant anywhere else?

I shall answer by assertion – but someone else's rather than mine. In a paper written for the International Commission on Mathematics Instruction, Robyn Zevenbergen writes the following:

The international phenomenon of expansion of the higher education sector has resulted in greater diversity in the intake of students. No longer is higher education the domain of the elite, but now more students can access it than in any previous times. . . . Students who, in earlier times would not have gained access to (or even considered enrolling in) tertiary mathematics, are now coming to classes. These students have very different needs and expectations of their study and are likely to encounter difficulties... (Zevenbergen, 2001, p. 13).

In short, the democratization of higher education worldwide will result in more diverse groups of students in tertiary mathematics classes, and a concomitant set of pedagogical issues. And such issues will not appear for the first time at the post-secondary level; they will appear in the mathematics "pipeline," as students are being prepared for the further study of mathematics.

100 years of American curricular history in a few paragraphs

The 20th century can be seen as a century of mathematical "democratization" in the United States. As the century began, mathematics was the province of the elite. As it ended, arguments were being made that all citizens need to be quantitatively literate in order to participate fully in the American democracy.

In 1890 only 6.7% of the 14 year-olds in the United States attended high school, and only 3.5% of the 17 year-olds graduated (Stanic, 1987). The purpose of schooling was to provide the vast majority of students with workplace skills and little else. Schooling for the masses focused on what were called the three R's: Readin', Ritin', and Rithmetic." Education for the elite was reserved for high school and beyond.

Over the course of the 20th century there were continuing pressures for additional schooling. By mid-century almost three-fourths of the children of age 14 to 17 attended high school, and 49% of the 17 year-olds graduated. (Stanic, 1987, p. 150). These enrollment changes resulted in the pressures identified above by Zevenbergen: courses once designed for a select group of students were being studied by increasing numbers of students. These demographic trends continued through the end of the century. A part of the American ethos is that education is a pathway to social and financial advancement: the "G.I. Bill," for example, provided soldiers returning from World War II with incentives to take courses at the post-secondary level. General social goals included high school graduation and access to further study for all students. By the end of the century, more than half of the high school graduates in the U.S. had enrolled in some form of post-secondary education.

Outside the classroom the world had changed in significant ways. Inside the classroom, however, the mathematics curriculum was largely unchanged: for most students grades 1-8 consisted of the study of arithmetic. In grade 9 they studied algebra. Half the students stopped taking mathematics at that point, and half went on to geometry in grade 10. Half the students stopped taking mathematics at that point, and half went on to "advanced algebra/trigonometry" in grade 11. The attrition rate from the mathematics pipeline continued at 50% per year as students proceeded through pre-calculus and then calculus, either in their senior year in high school or in their first year of post-secondary education.

1989 and beyond: New curricular goals

In 1989 the U. S. National Council of Teachers of Mathematics issued the *Curriculum and Evaluation Standards for School Mathematics*, a volume that proposed significant changes in mathematics teaching. This was followed in 1991 by the *Professional Standards for Teaching Mathematics* and in 1995 by the *Assessment Standards for School Mathematics*. I shall refer to these three volumes collectively as the *Standards*, while noting that the first volume, published in 1989, is the one that had the greatest influence. Part of the reason for creation of the *Standards* and the changes they suggested was dissatisfaction with the then-current curriculum, including the huge attrition rate from the mathematics pipeline described in the previous paragraph. But equally important was a reconceptualization of the underlying goals and purposes of mathematics instruction. The curriculum had been inherited from a time when mass education was for limited purposes of general literacy, and advanced education was for the elite. The *Standards* specified new instructional goals for *all* students: "New societal goals for education include (1) mathematically literate workers, (2) lifelong learning, (3) opportunity for all, and (4) an informed electorate" (NCTM, 1989, p. 3).

The publication of the *Standards* catalyzed a large (and not uncontroversial) change in mathematics instruction, which came to be known as "reform." Desired reforms (which were grounded in contemporary research, but had not been empirically tested on a large scale) included the following:

"We need to shift –

- toward classrooms as mathematical communities—away from classrooms as simply a collection of individuals;
- toward logic and mathematical evidence as verification—away from teacher as the sole authority for right answers;
- toward mathematical reasoning—away from merely memorizing procedures;
- toward conjecturing, inventing, and problem solving—away from an emphasis on mechanistic answer-finding;
- toward connecting mathematics, its ideas, and its applications—away from treating mathematics as a body of isolated concepts and procedures." (NCTM, 1991, p. 3)

The *Standards* emphasized mathematical processes as well as content. Specifically, there was a focus at all grade levels on problem solving; on reasoning; on connections within mathematics and from mathematics to ideas outside mathematics; and on communicating using mathematical ideas. In the years from 1989 to the present, there has been some slow implementation of reform, along with a fair amount of experimentation[3]. After the publication of the *Standards*, some groups (sometimes with funding from the U. S. National Science Foundation) began the development of

[3] The *Standards* did not specify curricula, but rather a set of learning goals for students. Thus it was possible to develop very different approaches to instruction that were "in the spirit of the standards."

curricula aligned with (their authors' interpretation of) its goals. These curricula became available in the mid-to-late 1990s. Reliable data on their use, discussed later in this section, is just beginning to accumulate.

Toward the end of the 20th century, NCTM realized that it needed to re-examine the contents of the *Standards*. Part of the reason for this reconsideration was political: the original document had been interpreted in so many different ways that some clarification was in order. More importantly, a lot had been learned in the years since the *Standards* had been issued. Ideas that had been speculative (that is, research-based but not extensively field-tested) had since been examined in practice, and methods, ideas, and materials had been significantly refined over the ensuing decade. Equally important, there had been important changes in the world outside of school. When the *Standards* were written, its authors took a bold stance, arguing that all high school students should have access to (and use) graphing calculators. Just a few years later, computers and the World Wide Web became accessible resources. Numbers no longer had to be "nice"; machines could do number crunching. Large data sets were available on the web, meaning that students didn't have to work with "faked" data. Graphing packages were available, as were various modeling tools. With such tools and data available, the nature of the mathematics that could be done in classrooms changed considerably. And, the threshold of mathematical competence for full participation in America's participatory democracy kept rising.

All of these reasons led NCTM to issue *Principles and Standards for School Mathematics* in April 2000. (Full disclosure: I was a member of NCTM's Commission on the Future of the Standards, which decided that a new vision was needed, and a member of the writing team that produced *Principles and Standards*.) *Principles and Standards* represents an evolutionary change from its antecedent, in that it is informed by a decade's experience working toward the content and process goals of the *Standards*. But there are ways in which *Principles and Standards* is itself revolutionary. Just as the original *Standards* represented a vision statement – a set of goals for the future – so do *Principles and Standards.* Perhaps one of the strongest positions in the document is that all students should study a basic core set of mathematics courses each and every year that they are enrolled in secondary school. The expectation is that this common core will prepare all students for quantitatively literate citizenship, entry into the workplace upon graduation, and the pursuit of mathematics at the university level if they desire.

This expectation flies in the face of 100 years' curricular tradition in the United States. It is also a bold (and some would say impossible) cry for social justice, given the data that I shall soon describe.

Part of the rationale for the recommendation is as follows. There are basically two audiences to consider: those who (for the time being at least) see themselves as having no mathematical needs beyond those required for a good job and literate citizenship, and those who will pursue the further study of mathematics. A good case can be made that the needs of these two groups are converging. The threshold for quantitative literacy has been rising. Today one expects people to be able to model and understand real-world phenomena using quantitative tools, to analyze and understand (and even make) complex logical arguments; to make decisions about social issues; to use technological tools appropriately when necessary; and to communicate effectively orally and in writing. Such skills are required for decision-making in one's personal life (e.g., when choosing mortgages or telephone plans), for interpreting information in newspapers (which is increasingly given in graphical or tabular form), for making informed choices regarding public policy (just how dangerous is a pesticide suspected of causing damage, or living near power lines?), and on the job (e.g., making predictions using spreadsheets and other software, defending one's choices or line of argument in a memo).

Many of these skills were given scant attention in the traditional curriculum. They can be seen not only as part of the foundation for quantitatively literate citizenship, but also as part of the foundation for mathematical and scientific careers. Let me describe my own background. My Ph.D. is in mathematics. Through secondary school and well into my collegiate career I studied no statistics and learned nothing about analyzing data. (I first studied statistics when I had to teach it.) I never did any "real world" modeling, or had practice at representing real world phenomena in mathematical terms. With the exception of a rather stilted form of writing up proofs in 10th grade geometry, I was not asked to make mathematical arguments of any sort until I was asked to *reproduce* proofs in calculus, then write them in a linear algebra course. I was rarely if ever asked to communicate using the language of mathematics; more often than not, producing a string of symbols and the right number at the end of my computations sufficed to get full credit for working a problem. In sum, my preparation as a mathematician-to-be would have been far richer had I been asked to develop the skills that are now relevant for all citizens. A common core can serve both groups (with the mathematically inclined studying additional mathematics if they wish.

That being the goal, what is the reality?

The data speak: Diversity and equity must be major concerns with regard to mathematics education.

As a mathematician, I value mathematics for myriad reasons: its beauty, its clarity and coherence, its power as a way of thinking, its role as the "language of science," its contributions to our intellectual heritage, and more. As an educator, I realize that access to high quality mathematics instruction – the kind of instruction that will enable students to develop mathematical competency – is a matter of social justice.

Everybody Counts, a 1989 report from the U. S. National Research Council, made the case this way:

More than any other subject, mathematics filters students out of programs leading to scientific and professional careers. . . . Mathematics is the worst curricular villain in driving students to failure in school. When mathematics acts as a filter, it not only filters students out of careers, but frequently out of school itself. . . .

Low expectations and limited opportunity to learn have helped drive dropout rates among Blacks and Hispanics *much* higher -- unacceptably high for a society committed to equality of opportunity. It is vitally important for society that *all* citizens benefit equally from high quality mathematics education. (National Research Council, 1989, p. 7)

This last statement situates mathematics instruction firmly as an equity issue. The "gender gap" in mathematics performance and the role of mathematics as a "critical filter" for women have been documented for some time (see, e.g., Sells, 1975, 1978). Similar data exist for under-represented minorities (specifically African Americans, Latinos, and Native Americans). In 1990, the U. S. National Research Council published *A Challenge of Numbers,* which synthesized a great deal of data regarding the mathematical trajectories of various sub-populations of the United States. Here in tabular form are data regarding the percentage of students enrolled at various levels in mathematics in the late 1980s.

	8th Grade	12th Grade	B.S. in math	M.S. in Math	Ph.D. in Math
Asians	2	2	6	8	8
White Male	40	41	45	55	70
White Female	39	39	40	33	17
Black	12	11	5	2	2
Hispanic	7	6	2	2	2

Percentage of students at various points in the mathematics pipeline.
Data drawn from Figure 4.2 of NRC, 1990.
(Rounding results in some column sums not being 100)

Reading each row from left to right provides documentation of increasing or decreasing participation in mathematics, from eighth grade on. Since schooling is essentially universal at eighth grade, the first column represents the approximate proportion of each demographic group in the U.S. population. One sees a substantial percentage increase in mathematics participation among Asians and White males, and a substantial decrease among White females, Blacks, and Hispanics. These data represent just the tip of the iceberg, for they fail to capture the "performance gap" between various demographics groups (in terms of scores on various standardized exams) at all levels of the educational system. A synthesis of current performance and demographic data has just been published in the *Educational Researcher* by Jaekyung Lee. Lee's (2002) findings are not encouraging. They suggest that the progress toward narrowing racial and ethnic achievement gaps in the 1970s and 1980s (as reflected by scores on a range of standardized tests) may have slowed or reversed in the 1990s. In what follows, NAEP refers to the U.S. National Assessment of Educational Progress, a federally funded national sampling of student performance in core subject areas. The SAT is a "high stakes" examination taken by a large percentage of students applying for post-secondary study. Among Lee's findings were the following.

- Black-White average score gaps on the NAEP mathematics tests tended to diminish from 1971 through 1990, but then stabilized or increased through 1999. In 1999 these differences were between 25 and 35 points at all grade levels. (NAEP defines five "performance levels" of mathematical proficiency corresponding to of 150, 200, 250, 300, and 350. The average differences of 25 points represent a very large and significant difference.)
- Hispanic-White average score gaps on the NAEP mathematics tests showed a similar trend, in that they tended to diminish from 1971 through 1990, but then stabilized or increased through 1999. In 1999 these differences were between 20 and 30 points at all grade levels.

- Black-White average score gaps on the SAT mathematics exams followed a similar pattern over the period from 1977 to 2000, with a steady decrease in the score gap from 123 in 1977 to a low of 91 in 1990, but then very slow increases to a difference of 94 in 2000. (SAT scores are on a 200-800 scale, with a mean of 500 and a standard deviation of about 110. These are very large and significant differences.)
- Hispanic-White gap trends on the SAT mathematics exams were similar, although the magnitude of the gaps has been a bit smaller (as it was on NAEP). There was a steady decrease in the average score gap from 80 points 1978 to a low of 57 points in 1989, but then a steady increases in differences from then on, to an average difference of 69 points in 2000. These too are very large and significant differences, with the trend moving away from equality.

Lee also offers comparative data on trends of selected measures of socioeconomic, cultural, and educational conditions among Blacks, Whites, and Hispanics from 1970 through 1998. These data offer few reasons for cheer, other than the fact that, generally speaking, things do tend to be better now than they were thirty years ago. Here are some of the relevant data. Data are given in terms of ratios of proportions of the populations being compared. In 1998,

- The likelihood of a Black family living in poverty was 2.5 times that of a White family; the likelihood of a Hispanic family living in poverty was 2.3 times that of a White family.
- The likelihood of a Black family being headed by a single parent was 2.5 times that of a White family; the likelihood of a Hispanic family being headed by a single parent was 1.3 times that of a White family.
- The high school dropout rate for Blacks was 1.8 times that for Whites, and the high school dropout rate for Hispanics was 3.8 times that for Whites.

These statistics are troubling – and, of course, data summaries capture the realities in rather dry ways. Kozol's (1992) *Savage Inequalities* brings them to life in dramatic (and much more disturbing) fashion.

It should be noted that while the data portray some of the harmful realities that need to be addressed at both the social and school levels, they do not at all paint a clear picture of precisely how they are related. Indeed, some trends such as high school dropout rates differed substantially for Blacks and Hispanics, while many of the trends regarding socioeconomic and cultural conditions looked remarkably similar. Lee summarizes his presentation of the data with the following comment: "In brief, this analysis of schooling conditions and practices shows that none of the conventional indicators examined above fully accounts for the bifurcated racial and ethnic achievement gaps trends that I have described" (Lee, 2002, p. 10).

Despite the absence of a clear causal (or in some cases, correlational) mechanism, aspects of the problem are clear. There are huge performance gaps in mathematics. There is differential access to mathematical resources, with poor and underrepresented minority students less likely than others to have access to high quality instruction. (See Kozol, 1992, for graphic descriptions of educational inequities in the United States; see Secada, 1992, for a broad characterization of racial, ethnic, and class issues in mathematics education.) The legal term for guaranteed access to educational opportunities is "opportunity to learn (OTL)." OTL has become a major civil rights issue in the U.S.

Generally speaking, a lack of credentials or poor performance in mathematics is likely to lead to decreased opportunities. Assuring high quality instruction, and moving toward a high level of performance for all students, is an issue of social justice.

This point has been highlighted by Robert Moses, civil rights leader and founder of the Algebra Project (a project intended to help provide disenfranchised minority students access to mathematics). Moses notes that algebra has come to take on a powerful filtering role in school curricula: those who will "make it" do so by passing algebra, while the rest will have severely limited opportunities. In *Radical equations: Math literacy and civil rights*, Moses writes:

> Today ... the most urgent social issue affecting poor people and people of color is economic access. In today's world, economic access and full citizenship depend crucially on math and science literacy. I believe that the absence of math literacy in urban and rural communities throughout this country is an issue as urgent as the lack of Black voters in Mississippi was in 1961. (Moses, 2001, p. 5)

Focusing in on the classroom: Some of what we know

Let me begin this section by reiterating two points. The first is my emphasis on examining classroom instruction – albeit with the recognition that factors outside the classroom walls obviously play a powerful role shaping what can and does take place inside them. The second is my notion of teaching as a set of actions that help students negotiate the terrain between what they bring to the learning environment and what one wants them to learn. The question for me in thinking about focusing on the classroom is deceptively simple: What can we know, and how can we know it?

There is a clear policy context, which I shall summarize in brief. And there are suggestions (both in terms of findings and methods) from research on gender; on language; on attempts to teach "mathematics for all"; on individual agency; on classroom community; and in fine-grained analyses of learning.

Policy Assumptions

As noted above, there have been some dramatic changes in American mathematics curricula since the issuance of the NCTM *Standards* in 1989. These changes have not been uncontroversial. Curricula constructed in line with the *Standards* tended to emphasis "process" to a significant degree: the first four standards at each grade level concerned problem solving, mathematical reasoning, making connections, and communicating mathematically. There has been a concomitant de-emphasis on practicing basic skills and on the mastery of procedural algorithms (e.g., the procedures for long division and multiplication of multi-digit numbers). This raised for some the concern that students would lose foundational mathematical skills, without which they would be seriously handicapped. For some years the controversies lay primarily in the political arena, since there were no hard data to make the case one way or another. The first volume of *Standards* was published in 1989, and "standards-based" curricula were developed in the mid-1990s. They were first implemented on a large scale in the late 1990s, and data concerning their implementation have only begun to be available over the past year or two. Those data suggest the following:

The alignment of curriculum, student assessment, and professional development (enhancing the capacity of teachers to implement curricula as intended) is essential. When a standards-based curriculum is implemented in a stable context and when assessment and professional development are consistent with that curriculum, there can be significant improvements in student learning. Those improvements include:

- scores on measures of skills that meet or exceed the scores of students who study traditional (U.S.) mathematics curricula. (In other words, fears that less direct attention to basic skills would result in an absence of those skills are not warranted.)
- tremendously enhanced performance on measures of concepts and problem solving, in comparison with the test scores of students who study traditional curricula. (This, of course, should come as no surprise; traditional curricula give much less attention to concepts and problem solving than do reform curricula.)
- a significant decrease in racial "performance gaps." In one well-documented case, Black/White racial differences essentially vanished on measures of skills; they dropped substantially on measures of concepts and problem solving.

Data supporting these assertions may be found in Schoenfeld (2002). These data provide a warrant for looking at contexts where students are encouraged to engage with meaningful mathematics – that is, with mathematics curricula consistent with NCTM's *Principles and Standards* or the earlier *Standards*. The data also point to the fact that such engagement is much more likely to be successful in the right "policy surround" – one in which teachers are supported in their efforts to make the mathematics accessible to students, both by means of assessment policies and by professional development.

Issues of Context

Though they are not the focus of the classroom analyses I propose to discuss here, one must keep in mind the variety of contextual factors that shape the opportunities made available to students. These include differential opportunities due to unequal distribution of resources and tracking or "curriculum differentiation." Secada (1992) documents relationships between various contextual factors (race, ethnicity, social class, and language) and mathematics achievement (typically measured on standardized achievement tests); Lee (2002) updates some of these. As noted above, Kozol (1992) portrays the stark realities that lie behind some of those data. Oakes, Gamoran, and Page (1992) describe the effects of tracking:

> "Curriculum differentiation works against the success of academically deficient students: By the end of the year, they tend to fall even further behind. Even in the best of cases, in which ability grouping benefited low-ability as well as high-ability students in certain elementary school studies, high-group students tended to gain more, so that the gaps still widened." (Oakes, Gamoran, & Page, 1992, pp. 599-600)

Putting aside for the time being the problematic nature of constructs such as "high ability" and "low ability" students[4], this does suggest some issues that could be examined in classrooms, e.g., the uses of grouping and the consequences thereof. Of particular interest to me is explanation at the level of mechanism. Such studies exist in reading, for example: "At the elementary level, low reading groups spend relatively more time on decoding activities, whereas more emphasis is placed on the meanings of stories in high groups" (Oakes et al., 1992, p. 583). This serves as an explanation of why the rich get richer, in that the more advanced students are presented more opportunities to learn the things that all students need to learn. Similarly in high school mathematics, teachers of "low ability" classes tended to emphasize mathematical procedures, while teachers of "high ability" classes gave much greater emphasis to inquiry skills, problem solving, and the preparation for further study (Oakes et al., 1992, p. 584).

Issues of Differential Treatment

The previous section focused on differential treatment at the group level. Classroom analyses have also focused on differential treatment at the individual level (aggregating the individual data). Some studies with the best potential for the detailed examination of classroom practices regarding differential treatment were gender studies, which have a tradition that goes back some 30 years. After examining patterns of classroom interactions, for example, Good, Sikes, and Brophy (1973) concluded that "male and female students are not treated the same way" (p. 85; quoted in Koehler, 1990). Typical studies examined the frequency of the questions teachers asked boys and girls, and their nature – whether questions were at high or low content levels, how often they were focused on disciplinary issues, and how often teachers' comments focused on substantive content issues or superficial aspects of work such as neatness.

In early work on classroom practices, in the 1970s, achievement scores were not examined. As a result, systematic patterns of interactions could not be related (even statistically) to outcomes. Also, the scope of processes covered was rather narrow. Hence it is not clear what would correlate with what (or even if the right variables had been chosen), even if outcome measures had been used.

A next generation of studies in the late 1970s and 1980s, called differential effectiveness studies, employed the "process/product" paradigm, which attempted to link differential teacher and student behaviors to differential performance outcomes. Such studies rapidly revealed unexpected complexities. First, correlational patterns were not what one might naï vely expect. Differential patterns of engagement did not consistently produce differential scores, raising hypotheses that some teacher behaviors might be appropriate for some students, and not others. (In the language of the time, there might be an "aptitude-treatment interaction" that confounded the relation between teacher actions and student outcomes.)

Leder (1992) reviews a broad spectrum of gender studies in mathematics. A jaundiced summary of Leder's summary might be "there are lots of interesting things to look at, but very few if any clear-cut conclusions that one can draw." Environmental variables listed by Leder

[4] Such classifications are often made on the basis of standardized tests, which tend not to make accommodations for linguistic skills. The use of such tests can thus lead to the classification of a mathematically talented student who is taking the test in his or her second (or third)language as being "low ability."

included school variables, teacher variables, the peer group, the wider society, and parents. Learner-related variables included intelligence, spatial abilities, confidence, fear of success, attributions, and persistence.

The process/product paradigm died pretty much a natural death, and for good reason. There were two main difficulties regarding such studies. The first is that the work was correlational – and as indicated above, the correlations did not provide much by way of insight. The second is that outcome measures were almost all mathematically superficial. Standardized tests were typically employed. These gave little attention to the complex processes of mathematical thinking and learning that are now central to educational discourse. Thus, while such studies suggest interesting things to look for in patterns of teacher-student interactions, a new (and much more fine-grained) perspective is required. Such a perspective would attend much more to the mathematical richness of the interactions, and would try to link the character of the interactions more directly to student performance.

Looking more closely at teacher practices

One lens through which one might examine teacher practices is that of "culturally relevant pedagogy," as described by Gloria Ladson-Billings (1994). Ladson-Billings (1997) abstracts some principles of productive pedagogies for *all* students as follows:

- Students treated as competent are likely to demonstrate competence.
- Providing instructional scaffolding for students allows students to move from what they know to what they do not know.
- The major focus of the classroom must be instructional.
- Real education is about extending students' thinking and abilities beyond what they already know.
- Effective pedagogical practice involves in-depth knowledge of students as well as subject matter.

Ladson-Billings goes on to note that researchers face serious theoretical (and methodological) challenges in trying to frame productive "next steps" in research – the job being to confront the necessary complexity of classroom interactions and characterize it in ways that allow for building productively on what students know. That is indeed the challenge.

It is worth noting that culturally relevant pedagogy need not be "culturally specific." Some programs, such as the Algebra Project (Moses, 2001; Moses, Kamii, Swap, & Howard, 1989) and the Jaime Escalante Math Program (Escalante & Dirmann, 1990) are designed to address the perceived needs of specific groups of students. Other programs, such as Cognitively Guided Instruction, or CGI (Carey, Fennema, Carpenter, & Franke, 1995) and QUASAR (Silver, Smith, & Nelson, 1995), or many of the standards-based curricula, were not designed for implementation with specific populations of students. The key desideratum is that they were designed to meet students "where they are."

Additional factors to consider follow.

Issues of Language and Discourse

In recent years there has been a significant change in perspective regarding the mathematics instruction of "English language learners" and/or those students whose cultural backgrounds are from other than mainstream U.S. culture. Older studies tended to look upon mathematics learning as the acquisition of vocabulary and of skills; English language learners were often thought of as having language (and other) "deficits" and instructed narrowly in terms of vocabulary. Today it is understood that engaging in mathematics involves a form of sense-making that far transcends the acquisition of a technical vocabulary; also that deficit models are not a productive way to address the educational needs of students with non-mainstream backgrounds. Echoing the comments of Ladson-Billings summarized above, for example, Garcia and Gonzales (1995) note the following characteristics of teachers considered successful with linguistic and cultural minority students: high expectations for all students; a rejection of models of their students as intellectually disadvantaged; commitment to students' academic success; commitment to student-home communication; and willingness to modify curriculum and instruction to meet the specific needs of their students.

The new emphases in standards-based curricula on mathematical processes – on problem solving, reasoning, connections, and communication – call for a much higher level of mathematical discourse.

> "Research in mathematics education documents a variety of perspectives regarding what it means to learn mathematics. Learning mathematics can be seen as learning to carry out procedures, develop hierarchical skills, solve mathematical problems, or mathematize situations. Recent theoretical perspectives have focused increasingly on mathematics learning as a process that intrinsically involves the use of language. Such notions include descriptions of mathematics learning as sense-making (Lampert, 1990; Schoenfeld, 1992), as participation in communities of practice (Lave & Wenger, 1991; Brown, Collins, & Duguid, 1989), as developing socio-mathematical norms for participating in the discourse of mathematics classrooms (Cobb, Wood, & Yackel, 1993), and in general as learning to participate in mathematical discourse practices such as modeling and argumentation (Brenner, 1994; Forman, McCormick, & Donato, 1998; Greeno, 1994)." (U.C. ACCORD Mathematics working group, October 2000, p. 10).

As Brenner (1994) observes, *Standards*-based curricula typically call for discussing and analyzing problem situations, choosing the relevant analytical and representational tools, solving problems, and communicating the results. In comparison with traditional curricula, this requires the increased use of language in the service of mathematical sense making. Hence classrooms in which these curricula are employed run the risk of placing English language learners at risk – unless their teachers can find ways of taking advantage of the first language resources the students bring with them to instruction. This will call for mediating between the linguistic resources that the students come with – typically everyday language in their first language and some mastery of English – and the specialized use of the "mathematics register" (Halliday, 1978), a precise technical form of expression using mathematical terms that has its own specialized syntax and meanings (see, e.g., Khisty, 1995; Moschkovich, 1999, 2000; Pimm, 1987; Warren & Rosebery, 1995). More generally, an argument can be made that teachers (and researchers on teaching) need to be familiar with a range of issues pertaining to language, language development, and language acquisition (See Fillmore and Snow, 2000). In terms of classroom research, this will call for fine-grained analyses to see how interactions among students and between the students and the teacher work to support or inhibit students' meaningful engagement with the rich conceptual aspects of mathematics.

To make this discussion concrete, let me give some examples of how an inappropriately high linguistic threshold can impede English language learners' participation in mathematics and other subjects, and paint a distorted picture of the students' competencies. Lily Wong Fillmore has investigated the language demands in "high stakes" contexts such as high school exit examinations in various states. Fillmore (2002) points out that the tests examine not only subject matter mastery, but students' command of academic English. Here is a sample problem from the Arizona exit exam.

> If x is always positive and y is always negative, then xy is always negative. Based on the given information, which of the following conjectures is valid?
>
> A. $x^n y^n$, where n is an odd natural number will always be negative.
>
> B. $x^n y^n$, where n is an even natural number, will always be negative.
>
> C. $x^n y^m$, where n and m are distinct odd natural numbers, will always be positive.
>
> D. $x^n y^m$, where n and m are distinct even natural numbers, will always be negative.

Fillmore writes:

"What's difficult about it? Nothing, really, if you know about, can interpret and use—

- exponents and multiplying signed numbers;
- the language of logical reasoning;
- the structure of conditional sentences;
- technical terms such as *negative, positive, natural, odd,* and *even* for talking about numbers.
- ordinary language words and phrases such as *if, always, then, where, based on, given information, the following, conjecture, distinct,* and *valid.*" (Fillmore, 2002, p. 3.)"

Fillmore continues with sample questions from the tenth-grade Massachusetts Comprehensive Assessment System (MCAS).

1. *Which of the points below is not collinear with the others?*
   M (3, -2) N (-5, 6) S (-9, 10) T (10, -21)

   A. N only
   B. S only
   C. T only
   D. They are all collinear.

2. *The amplitude, frequency, and shape of an electrical signal can be displayed and measured using*

   A. a signal generator.
   B. a multimeter scope.

C. an oscilloscope.

D. an odometer.

3. *The Petition of Right that the English Parliament forced King Charles I to sign in 1628 included the principle of habeas corpus, which means that*

A. only a legislative body can collect taxes in time of peace.

B. civil law cannot apply to the clergy.

C. martial law can only be applied by the head of the government.

D. no one can be imprisoned unless charged with a specific crime within a reasonable time.

These examples are clearly problematic, if one takes seriously the idea that assessment should help reveal "what students know and can do." These examples, taken from formal assessments, also highlight potential linguistic issues in the acquisition of mathematical understandings. It will not be terribly difficult, I suspect, to find evidence of unhelpful discourse practices in mathematics classrooms. The question is, how does one document what are likely to be productive practices, and provide meaningful evidence of the relationship between the practices and their impact?

Issues of Participation and Agency

Active engagement (of a mathematically appropriate and productive kind) is likely to be a major factor contributing to students' mathematical success. There are various ways one can look at issues of engagement, at both the collective and individual levels. One can examine participation structures, both whole class and small group. Are all students "invited" to participate fully? Are there moves by teacher and/or students that enfranchise various students, or that disenfranchise them? Analyses of this type, combined with analyses of the kinds of comments made by individual students, can paint a good picture of local engagement – of what students are doing and how they engage with the material. But then there are at least three other issues that need to be considered, if one is to have a chance of seeing the "big picture."

First, there is the issue of linking participation and engagement to outcomes. In the past, some of my explanatory work has been at the aggregate level. For example, I was able to argue on the basis of classroom observations that particular practices in high school geometry classrooms led to the development of particular student beliefs regarding the nature of the mathematical enterprise. It is not clear to me whether the study of aggregate or individual trajectories is more promising for linking participatory experiences with student perceptions and behavior.

Second, at what point in students' mathematical histories is it most profitable to start looking at interactive and engagement patterns? To give a specific example: a few weeks ago my research group viewed a videotape of a group of students working together on an applied problem. The interactions were nothing short of wonderful; the three students (two girls and one boy, all of different ethnicities) all contributed in substantive ways to the solution of the problem they were addressing. In terms of the methods discussed above, it would be straightforward to do a discourse analysis indicating how each was enfranchised by the others, what their contributions were, and so on. And that's essential. But looking at this tape raised more questions than it answered. How in the world did the students learn to interact like that? How typical were the interactions? How far back do you have to go to trace the ways these students learned to interact with each other, to describe the role of the teacher in shaping the group's interactions? A comprehensive data analysis would take a huge amount of time. What strategies are there for targeting the "right" things for the "right" kinds of analysis?

Third, it must be recognized that in-class interactions are shaped in myriad ways by events that take place outside of class. To name one essential feature of the interactions, consider the issue of students' mathematical agency and mathematical identities. Whether students will engage mathematically and how they will do so is a function of how they see themselves, how they see the instrumentality of the mathematics they are studying, and how they see themselves fitting in with their environment (Eckert, 1989; Martin, 2000). Martin (2000), for example, describes interviews with African Americans who felt that, now matter how well they did mathematically, they would never be given job opportunities that would use such skills – so why bother? Other interviews reveal that parents, by underestimating the specific mathematical prerequisites for progressing through the educational system, can limit their children's opportunities. How far back in time, and how far outside the classroom, must one go to trace such things appropriately? Another issue has to do with beliefs. For example, the typical American belief that one is either born good or bad at mathematics (in contrast to the typical Japanese belief that one's performance in mathematics is directly related to the amount of work one puts into studying) clearly shapes how students engage mathematically.

Issues of meaningful mathematics (in and out of the classroom)

The question here is: what is meaningful to students, in what ways; what unexpected territory might one enter when trying to introduce students to rich mathematical terrain? This is, in a broad sense, a curricular issue. (I take "curriculum" to mean both the materials that students study and the ways in which they are brought together to study them.) Horn (2002; in preparation) has studied the concept of "group-worthy" activities used by one reform-oriented mathematics department. These are mathematical problems and activities that can be accessed from multiple starting points and that can engage students with diverse mathematical backgrounds. Group-worthy activities provide affordances for classroom interactions that can enfranchise and support a wide range of students. Teasing out the interaction of such curricular materials with the kinds of interactions that can and do take place in the classroom adds yet another level of complexity to the task of seeing "what counts."

Curricular choices intended to "reach the students where they are" can raise issues that are not encountered when one teaches more traditional mathematics. Silver, Smith, & Nelson (1995) describe one such example. Teachers in the QUASAR program had administered the following open-ended task to students:

> Yvonne is trying to decide whether she should buy a weekly bus pass. On Monday, Wednesday, and Friday she rides the bus to and from work. On Tuesday and Thursday, she rides the bus to work, but gets a ride home with her friends. Should Yvonne buy a weekly bus pass? Explain your answer.
>
> *Busy Bus Company Fares*

One Way	$1.00
Weekly Pass	$9.00

Teachers were surprised by the number of students who responded that the weekly pass was a better buy, given that the one-way fares described in the problem statement added up to only $8.00 per week. When they discussed their answers with students, "many students argued that purchasing the weekly pass was a much better decision because the pass could allow many members of a family to use it (e.g., after work and in the evenings) and it could also be used by a family member on weekends." (Silver, Smith, & Nelson, 1995, p. 41) This makes good sense – it's a real-world solution to a "real world" problem. It points to the complexities one faces in designing and implementing curricula that try to bridge meaningfully to children's lives, and to the subtleties that one faces in assessing issues such as student thinking and what it means for a curriculum to enfranchise students.

Concluding Comments

I have argued for some years (see, e.g., Schoenfeld, 1999) that the state of the art is such that educational researchers can now conduct research in contexts that really matter. For me, that means mathematics classrooms. I also have my own personal standards for what constitute well-warranted claims in education. Those have to do with explanation at a level of mechanism, where one is obligated to explain how things fit together and why things happen. My research on problem solving and on teaching has typically been at a very fine-grained level of analysis: a typical claim has been that the student or teacher behaves in particular ways because he or she has very specific knowledge, goals and beliefs. Looking for causality has often caused me to expand the scope of inquiry, and to expand the theory within which the empirical work that characterized the behavior was situated. For example, my analysis of student problem-solving protocols revealed that students routinely made conjectures in contradiction to things they "knew" (and had proved just a short time before). This led to studies of beliefs – e.g., the idea that some students "believe" that proof-related knowledge is not relevant or useful when working "discovery" problems of a particular type. That raised questions about the origins of such beliefs – which turned out to be the mathematical practices in which the students had engaged, over time, in their mathematics classes. The chain of causality for "simple" behavior in a twenty-minute problem solving session in the laboratory reached back to formative experiences, over a period of years, in mathematics classrooms.

The challenge of the problem solving research pales in comparison to the challenges of developing a coherent frame within which to examine issues of diversity and mathematics learning. It should be clear that the brief summary of some of what is known about issues of "algebra for all" given in this paper raises far more questions than answers. Each of the arenas addressed – context, differential treatment, teacher practices, language and discourse, participation and agency, and meaningful mathematics in and out of the classroom – is itself complex and not well understood. Interactions among them are that much more complex. Painting the "big picture" while maintaining a focus on detail and a predilection for explanation at a level of mechanism will be an interesting challenge.

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"ALGORITHMIC MATHEMATICS" AND "DIALECTIC MATHEMATICS": The "Yin" and "Yang" in Mathematics Education

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ABSTRACT

Peter Henrici coined the term "algorithmic mathematics" and "dialectic mathematics" in a 1973 talk. I will borrow these two terms and attempt to synthesize the two aspects from a pedagogical viewpoint with illustrative examples gleaned from mathematical developments in Eastern and Western cultures throughout history. Some examples from my teaching experience in the classroom will also be given.

1 Introduction

At the 1973 Joint AMS-MAA (American Mathematical Society – Mathematical Association of America) Conference on the Influence of Computing on Mathematical Research and Education Peter HENRICI of Eidgenössische Technische Hochschule coined the terms "algorithmic mathematics" and "dialectic mathematics" and discussed the desirable equilibrium of these two polarities [8; see also 4, Chapter 4]. In this talk I will borrow these two terms and attempt to synthesize the two aspects from a pedagogical viewpoint with illustrative examples gleaned from mathematical developments in Eastern and Western cultures throughout history. This paper is to be looked upon as a preliminary version of the text of my talk, which will surely suffer from the lack of the much needed reflection which usually arises *after* the talk and the much desired stimulation which is brought about by the audience *during* the talk.

Maybe at the outset I should beseech readers to bear with a more liberal usage of the word "algorithm" in this talk, viz any well-defined sequence of operations to be performed in solving a problem, *not* necessarily involving branching upon decision or looping with iteration. In particular, this talk does *not* aim at probing the difference and similarity between the way of thinking of a mathematician and a computer scientist. (The latter question certainly deserves attention. Interested readers may wish to consult the text of a 1979 talk by Donald KNUTH [9].) Hopefully, the meaning I attach to the terms "algorithmic mathematics" and "dialectic mathematics" will become clearer as we proceed. Let me quote several excerpts from the aforementioned paper of Henrici to convey a general flavour before we start on some examples:

> "*Dialectic mathematics* is a rigorously logical science, where statements are either true or false, and where objects with specified properties either do or do not exist. *Algorithmic mathematics* is a tool for solving problems. Here we are concerned not only with the existence of a mathematical object, but also with the credentials of its existence. *Dialectic* mathematics is an intellectual game played according to rules about which there is a high degree of consensus. The rules of the game of *algorithmic* mathematics may vary according to the urgency of the problem on hand. ⋯ *Dialectic* mathematics invites contemplation. *Algorithmic* mathematics invites action. *Dialectic* mathematics generates insight. *Algorithmic* mathematics generates results." [8]

2 Examples of "algorithmic mathematics" and "dialectic mathematics"

My first example is a very ancient artifact dating from the 18th century B.C. (now catalogued as the Yale Babylonian Collection 7289), a clay tablet on which was inscribed a square and its two diagonals with numbers (in cuneiform expressed in the sexagesimal system) 30 on one side and 1.4142129... and 42.426388... on one diagonal (see Figure 1).

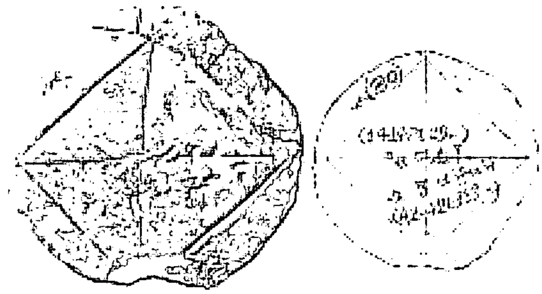

Figure 1

There is no mistaking its meaning, viz the calculation of the square root of 2 and hence the length of the diagonal of a square with side of length 30. The historians of mathematics Otto NEUGEBAUER and Abraham SACHS believe that the ancient Babylonians worked out the square root of 2 by a rather natural algorithm based on the following principle. Suppose x is a guess which is too small (respectively too large), then $2/x$ will be a guess which is too large (respectively too small). Hence, their average $\frac{1}{2}(x + 2/x)$ is a better guess. We can phrase this procedure as a piece of "algorithmic mathematics" in solving the equation $X^2 - 2 = 0$:

Set $x_1 = 1$ and $x_{n+1} = \dfrac{1}{2}(x_n + 2/x_n)$ for $n \geq 1$.
Stop when x_n achieves a specified degree of accuracy .

It is instructive to draw a picture (see Figure 2) to see what is happening. The picture embodies a piece of "dialectic mathematics" which justifies the procedure:

ξ is a root of $X = f(X)$ and ξ is in $I = [a, b]$.
Let f and f' be continuous on I and $|f'(x)| \leq K < 1$ for all x in I. If x_1 is in I and $x_{n+1} = f(x_n)$ for $n \geq 1$, then $\lim_{n\to\infty} x_n = \xi$.

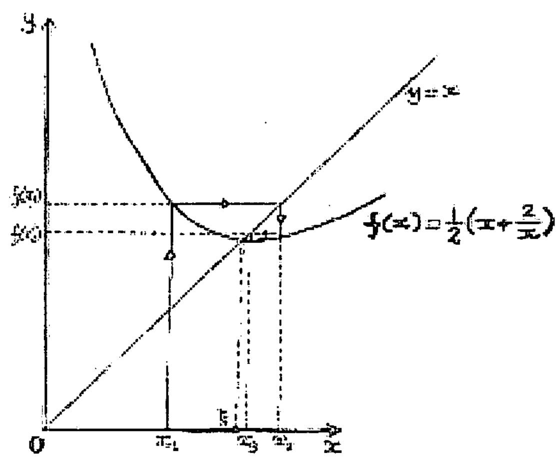

Figure 2

"Algorithmic mathematics" abounds in the ancient mathematical literature. Let us continue to focus on the extraction of square root. In the Chinese mathematical classics *Jiuzhang Suanshu* [*Nine Chapters On the Mathematical Art*] compiled between 100B.C. and 100A.D. there is this Problem 12 in Chapter 4:

> "Now given an area 55225 [square] *bu*. Tell: what is the side of the square?
> ... The Rule of Extracting the Square Root: Lay down the given area as *shi*. Borrow a counting rod to determine the digit place. Set it under the unit place of the *shi*. Advance [to the left] every two digit places as one step. Estimate the first digit of the root. ..." (translation in [3])

The algorithm is what I learnt in my primary school days. It yields in this case the digit 2, then 3, then 5 making up the answer $\sqrt{55225} = 235$. Commentaries by LIU Hiu in the mid 3rd century gave a geometric explanation (see Figure 3) in which integers $a \in \{0, 100, 200, \ldots, 900\}$, $b \in \{0, 10, 20, \ldots, 90\}$, $c \in \{0, 1, 2, \ldots, 9\}$ are found such that $(a + b + c)^2 = 55225$.

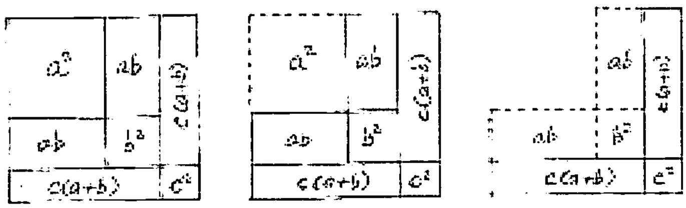

Figure 3

A suitable modification of the algorithm for extracting square root gives rise to an algorithm for solving a quadratic equation. One typical example is Problem 20 in Chapter 9 of *Jiuzhang Suanshu*:

> "Now given a square city of unknown side, with gates opening in the middle. 20 *bu* from the north gate there is a tree, which is visible when one goes 14 *bu* from the south gate and then 1775 *bu* westward. Tell: what is the length of each side?" (translation in [3])

Letting x be the length of each side, we see that the equation in question is $X^2 + 34X = 71000$. A slight modification of the picture in Figure 3 (see Figure 4) will yield a modified algorithm.

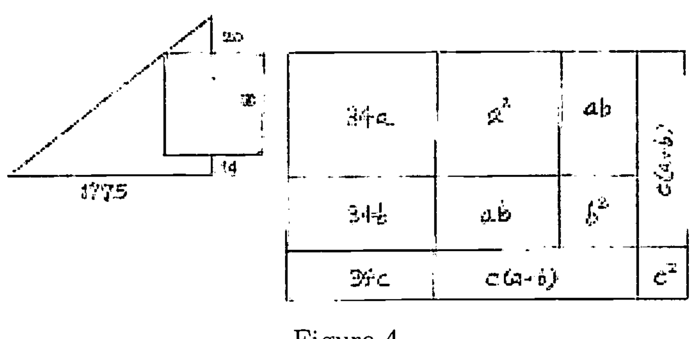

Figure 4

The same type of quadratic equations was studied by the Islamic mathematician Muhammad ibn Mūsā AL-KHWARIZMI in his famous treatise *Al-kitāb al-muhtasar fī* *hisab al-jabr wa-l-muqābala* [*The Condensed Book On the Calculation of Restoration And Reduction*] around 825A.D. The algorithm exhibits a different flavour from the Chinese method in that a closed formula is given. Expressed in modern day language, the formula for a root x of $X^2+bX=c$ is $x=\sqrt{(b/2)^2+c}-(b/2)$. Just as in the Chinese literature, the "algorithmic mathematics" is accompanied by "dialectic mathematics" in the form of a geometric argument (see Figure 5).

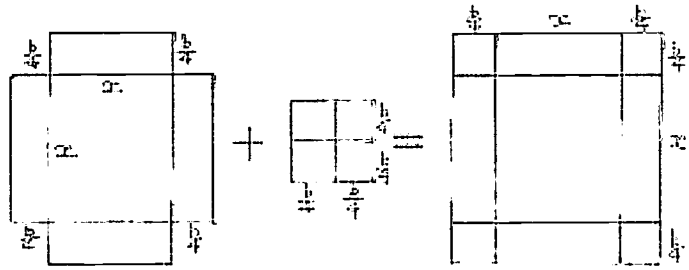

Figure 5

The author concluded by saying, "We have now explained these things concisely by geometry in order that what is necessary for an understanding of this branch of study might be made easier. The things which with some difficulty are conceived by the eye of the mind are made clear by geometric figures."

3 Intertwining of "algorithmic mathematics" and "dialectic mathematics"

Let us come back to the equation $X^2-2=0$. On the algorithmic side we have exhibited a constructive process through the iteration $x_{n+1}=\frac{1}{2}(x_n+2/x_n)$ which enables us to get a solution within a demanded accuracy. On the dialectic side we can guarantee the existence of a solution based on the Intermediate Value Theorem applied to the continuous function $f(x)=x^2-2$ on the closed interval $[1,2]$. The two strands intertwine to produce further results in different areas of mathematics, be they computational results in numerical analysis or theoretic results in algebra, analysis or geometry. At the same time the problem is generalized to algebraic equations of higher degree. On the algorithmic side there is the work of QIN Jiushao who solved equations up to the tenth degree in his 1247 treatise, which is equivalent to the algorithm devised by William George HORNER in 1819. On the dialectic side there is the Fundamental Theorem of Algebra and the search of a closed formula for the roots, the latter problem leading to group theory and field theory in abstract algebra. In recent decades, there has been much research on the constructive aspect of the Fundamental Theorem of Algebra, which is a swing back to the algorithmic side. A classic example to illustrate this back-and-forth movement between "algorithmic mathematics" and "dialectic mathematics" is the work of Paul GORDAN and David HILBERT in the theory of invariants at the end of the 19th century. Gordan was hailed as the "King of the Invariants" and in 1868 established the existence of a finite basis for the binary forms through hard and long calculations covering page after page. The work was so laborious already for the binary forms that people could not push forth the argument for forms of higher degree. Hilbert came along in 1888 to give an elegant short existence proof of a finite basis for forms of any degree. It is frequently reported that Gordan commented, upon learning of the proof by Hilbert, "This is not mathematics. This is theology." What is less frequently mentioned is that Hilbert worked hard to find a constructive proof of his theorem on basis. He succeeded in 1892, finding a constructive proof through knowledge of the existence proof. Upon learning of this constructive proof, Gordan was reported to say, "I have convinced myself that theology also has its merits." [12, Chapter V]

Thus we see that it is not necessary and is actually harmful to the development of mathematics to separate strictly "algorithmic mathematics" and "dialectic mathematics". Traditionally it is held that Western mathematics, developed from that of the ancient Greeks, is dialectic, while Eastern mathematics, developed from that of the ancient Egyptians, Babylonians, Chinese and Indians, is algorithmic. As a statement in broad strokes this thesis has an element of truth in it, but under more refined examination it is an over-simplification. Let me illustrate with a second example. This example may sound familiar to readers, viz the Chinese Remainder Theorem. The source of the result, and thence its name, is a problem in *Sunzi Suanjing* [*Master Sun's Mathematical Manual*] compiled in the 4th century that reads:

> "Now there are an unknown number of things. If we count by threes, there is a remainder 2; if we count by fives, there is a remainder 3; if we count by sevens, there is a remainder 2. Find the number of things." (translation in [10])

To solve this problem, which can be written in modern terminology as a system of simultaneous linear congruence equations

$$x \equiv 2 \pmod 3, \qquad x \equiv 3 \pmod 5, \qquad x \equiv 2 \pmod 7 \ ,$$

the text offers three magic numbers 70, 21, 15 which are combined in a proper way to yield the least positive solution

$$2 \times 70 + 3 \times 21 + 2 \times 15 - 105 \times 2 = 23 \ .$$

In his treatise *Suanfa Tongzong* [*Systematic Treatise on Arithmetic*] of 1592 CHENG Dawei even embellished this solution as a poem which reads:

" 'Tis rare to find one man
Of seventy out of three,
There are twenty one branches
On five plum blossom trees.
When seven disciples reunite
It is in the middle of the month,
Discarding one hundred and five
You have the problem done."

It is interesting to note (but I am no qualified historian of mathematics to trace the transmission of knowledge) that the same problem with its solution also appears in *Liber Abaci* of 1202 by Leonardo of Pisa, better known as FIBONNACI. It reads:

> "Let a contrived number be divided by 3, also by 5, also by 7; and ask each time what remains from each division. For each unity that remains from the division by 3, retain 70; for each unity that remains from the division by 5, retain 21; and for each unity that

remains from the division by 7, retain 15. And as much as the number surpasses 105, subtract from it 105; and what remains to you is the contrived number." [4, p.188]

In ancient China the problem was handed down from generation to generation, gradually attaining a glamour which was attached to events as disparate as a legendary enumeration of the size of his army by the great general HAN Xin in the late 3rd century B.C. to a parlour trick of guessing the number of a collection of objects. (The story about Han Xin may explain a common confusion some people make in identifying the author of *Sunji Suanjing* with another Sun Ji who flourished seven centuries earlier and who was famous for his treatise on military art.) This much is a familiar story told and re-told. We will turn to look at the problem from an angle not as commonly adopted by popular accounts.

The first time I myself encountered the name of the Chinese Remainder Theorem (CRT) explicitly mentioned was when I, as a student, read Chapter V of *Commutative Algebra* by Oscar ZARISKI and Pierre SAMUEL [14]. The name is given to Theorem 17 about a property of a Dedekind domain, with a footnote that reads:

> "A rule for the solution of simultaneous linear congruences, essentially equivalent with Theorem 17 in the case of the ring J of integers, was found by Chinese calendar makers between the fourth and the seventh centuries A.D. It was used for finding the common periods to several cycles of astronomical phenomena."

In many textbooks on abstract algebra the CRT is phrased in the ring of integers $\mathbb{Z}$ as an isomorphism between the quotient ring $\mathbb{Z}/M_1 \dots M_n\mathbb{Z}$ and the product $\mathbb{Z}/M_1\mathbb{Z} \times \cdots \times \mathbb{Z}/M_n\mathbb{Z}$ where M_i, M_j are relatively prime integers for distinct i, j. A more general version in the context of a commutative ring with unity R guarantees an isomorphism between $R/I_1 \cap \cdots \cap I_n$ and $R/I_1 \times \cdots \times R/I_n$ where $I_1, \dots, I_n$ are ideals with $I_i + I_j = R$ for distinct i, j. Readers will readily provide their own "dialectic" proof of the CRT.

For many years I have been curious as to how the abstract CRT develops from the concrete problem in *Sunzi Suanjing*. One mostly cited (but not quite accurate) account appears in Volume II of *History of the Theory of Numbers* by Leonard Eugene DICKSON which says:

> "Sun-Tsŭ, in a Chinese work Suan-ching (arithmetic), about the first century A.D., gave in the form of an obscure verse a rule called t'ai-yen (great generalization) to determine a number having the remainders 2, 3, 2, when divided by 3, 5, 7, respectively. ..." [5, Chapter II]

This account probably originated from a series of articles published in the Shanghai newspaper *North-China Herald* titled "Jottings on the science of the Chinese" written by the British missionary Alexander WYLIE of the London Missionary Society. Wylie was one of the most prominent pioneers in the study of Chinese Science after Antoine GAUBIL of the first half of the 18th century and Edouard BIOT of the first half of the 19th century. In No. 116 (October 1852) of the *North-China Herald* he wrote:

> "The general principles of the *Ta-yen* are probably given in their simplest form, in the above rudimentary problem of Sun Tsze;

> Subsequent authors enlarging on the idea, applied it with much effect to that complex system of cycles and epicycles which form such a prominent feature in the middle-age astronomy of the Chinese. The reputed originator of this theory as applied to astronomy is the priest Yih Hing who had scarely finished the rough draft of his work *Ta-yen leih sháo*, when he died A.D. 717. But it is in the "Nine Sections of the art of numbers" by Tsin Keu chaou that we have the most full and explicit details on this subject. ..."

The account of Wylie was subsequently translated into German by K.L. BIERNATZKI in 1856, elaborated by L. MATTHIESSEN in 1874/76, who pointed out that the Chinese result is same as that expounded by Carl Friedrich GAUSS in Section II of his *Disquisitiones Arithmeticae* of 1801 [6]. (Kurt MAHLER clarified this mistaken point in a short paper published in *Mathematische Nachrichten* in 1958 [11].)

The author of the 1247 treatise *Shushu Jiuzhang* [*Mathematical Treatise in Nine Sections*] referred to in Wylie's account was one of the most famous Chinese mathematicians of the 13th century by the name of QIN Jiushao (Tsin Keu chaou). From the first two problems in Book I we can discern the source of the problem as well as the naming of the technique he introduced, viz "Da Yan (Great Extension) art of searching for unity". Problem 1 states:

> "In the *Yi Jing* [*Book of Changes*] it is said, "The Great Extension number is 50, and the Use number is 49." Again it is said, "It is divided into 2 [parts], to represent the spheres; 1 is suspended to represent the 3 powers; they are drawn out by 4, to represent the 4 seasons; three changes complete a symbol, and eighteen changes perfect the diagrams." What is the rule for the Extension and what are the several numbers?" (translation in Wylie's article)

This is a problem about the art of fortune telling by combination of blades of shi grass. It provides an exercise about residue classes of congruence. Problem 2 states:

> "Let the solar year be equal to $365\frac{1}{4}$ days, the moon's revolution, $29\frac{499}{940}$ days, and the Jia Zi, 60 days. Suppose in the year A.D. 1246, the 53rd day of the Jia Zi is the Winter solstice or 1st day of the Solar year; and the 1st day of the Jia Zi is the 9th day of the month. Required the time between two conjunctions of the commencement of these three cycles; also, the time that has already elapsed, and how much as yet to run." (translation in Wylie's article)

This is a problem about the reckoning of calendar where the number of days was counted from a beginning point called the Shang Yuan, that being the coinciding moment of the winter solstice, the first day of the lunar month and also the first day of the cycle of sixty.

Let us phrase the "Da Yan art of searching for unity" in modern terminology to illustrate the algorithmic thinking embodied therein. The system of simultaneous congruence equation is

$$x \equiv A_1 \pmod{M_1}, \qquad x \equiv A_2 \pmod{M_2},\ \ldots,\ x \equiv A_n \pmod{M_n}\ .$$

Qin's work includes the general case when $M_1, \ldots, M_n$ are not necessarily mutually relatively prime. His method amounts to arranging to have $m_i | M_i$ with $m_1, \ldots, m_n$ mutually relatively prime and $LCM(m_1, \ldots, m_n) = LCM(M_1, \ldots, M_n)$. An equivalent problem is to solve $x \equiv A_i \pmod{m_i}$ for $i \in \{1, \ldots, n\}$, which is solvable if and only if $GCD(M_i, M_j)$ divides $A_i - A_j$ for all $i \neq j$. The next step in Qin's work reduces the system (in the case $M_1, \ldots, M_n$ are mutually relatively prime) to solving separately a single congruence equation of the form $k_i b_i \equiv 1 \pmod{M_i}$. Finally, in order to solve the single equation $kb \equiv 1 \pmod{m}$ Qin uses reciprocal subtraction, equivalent to the famous euclidean algorithm, to the equation until 1 (unity) is obtained.

Writing out the algorithm in full, we have

$$m = bq_1 + r_1, b = r_1 q_2 + r_2, r_1 = r_2 q_3 + r_3, \text{ etc. with } m > b > r_1 > r_2 > \cdots$$

so that ultimately r_i becomes 1. Set $k_1 = q_1$, then $k_1 b \equiv q_1 b \equiv -r_1$ (all congruences refer to modulo m). Set $k_2 = k_1 q_2 + 1$, then $k_2 b \equiv k_1 q_2 b + b \equiv -r_1 q_2 + b \equiv r_2$. Set $k_3 = k_2 q_3 + k_1$, then $k_3 b \equiv k_2 q_3 b + k_1 b \equiv r_2 q_3 - r_1 \equiv -r_3$. Set $k_4 = k_3 q_4 + k_2$, then $k_4 b \equiv k_3 q_4 b + k_2 b \equiv -r_3 q_4 + r_2 \equiv r_4$, etc. In general, we have $k_i b \equiv (-1)^i r_i \pmod{m}$. This algorithm provides a method for solving $kb \equiv 1 \pmod{m}$ as well as a proof that what is calculated is a solution. The method is to start with $(1, b)$ and change (k_i, r_i) to (k_{i+1}, r_{i+1}), stopping when $r_i = 1$ and i is even. Then k_i is a solution. For example, to solve $14k \equiv 1 \pmod{19}$ we start with $(1, 14)$, which is changed to $(1, 5)$, then to $(3, 4)$, then to $(4, 1)$, then to $(15, 1)$. Hence 15 is a solution. When the calculation is performed by manipulating counting rods on a board as in ancient times, the procedure is rather streamlined. Within this algorithmic thinking we can discern two points of dialectic interest. The first is how one can combine information on each separate component to obtain a global solution. This feature is particularly prominent when the result is formulated in the CTR in abstract algebra. The second is the use of linear combination which affords a tool for other applications such as for curve fitting or the Strong Approximation Theorem in valuation theory.

It is not surprising that the euclidean algorithm is used in Qin's work. The principle was familiar to the ancient Chinese who explained it in Chapter 1 of *Jiuzhang Suanshu* as:

> "Rule for reduction of fractions: If [the denominator and numerator] can be halved, halve them. If not, lay down the denominator and numerator, subtract the smaller number from the greater. Repeat the process to obtain the *dengsu* (greatest common divisor). Reduce them by the *dengsu*." (translation in [3])

It is called the euclidean algorithm in the Western world because it is contained in the first two propositions of Book VII of *Elements* compiled by EUCLID in about 300 B.C. If we read these two propositions we would be struck by its strong algorithmic flavour. Proposition 1 states:

> "Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another." (translation in [7])

This is followed by Proposition 2 which says:

> "Given two numbers not prime to one another, to find their greatest common measure." (translation in [7])

A reading of the proofs of these two propositions will offer the reader a more balanced view of the style of the book *Elements*. The kind of mathematics developed in *Elements* is traditionally seen as an archetype of "dialectic mathematics". This more balanced view betrays the over-simplified belief that Eastern-Western mathematics is synonymous with algorithmic-dialectic mathematics. Furthermore, some people even stress above all only the formal and rigorous aspect of "dialectic mathematics". I will now follow the reasoning put forth by S.D. AGASHE [1] to reveal the (somewhat algorithmic) background and motives of the mathematics contained in the first two books of *Elements*. Proposition 14 in Book II addresses the construction of a square equal (in area) to a given rectilinear figure. It seems the problem of interest is to compare two rectilinear figures, whose one-dimensional analogue of comparing two line segments is easy. For two line segments we can put one onto the other and see which one lies completely inside the other (or is equal to the other). Actually this is what Proposition 3 of Book I sets out to do:

> "Given two unequal straight lines, to cut off from the greater a straight line equal to the less." (translation in [7])

To justify this result we have to rely on Postulate 1, Postulate 2 and Postulate 3. Unfortunately, for rectilinear figures the problem is no longer as straightforward, except for the case of two squares when we can reduce the investigation to the sides of each square by putting one onto the other so that one square lies completely inside the other (or is equal to the other). Incidentally we need Postulate 4 to guarantee that. Hence we have found a way to compare two rectilinear figures, viz we try to reduce a rectilinear figure to a square, which is the content of Proposition 14 in Book II:

> "Construct a square equal to a given rectilineal figure." (translation in [7])

Let us first try to reduce a rectangle to a square. A rectangle can be readily converted to an L-shaped gnomon which is the difference between two squares. Actually that is the content of Proposition 5 in Book II (see Figure 6).

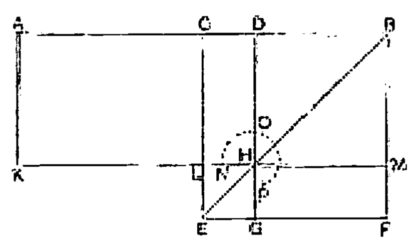

Figure 6

To make the difference of two squares a square we can ask a reversed question about the sum of two square being equal to a square. The latter question is answered by the famous Pythagoras' Theorem which is Proposition 47 in Book I! To complete the picture we must construct a rectangle equal to a rectilinear figure. By decomposing a rectilinear figure into triangles and by contructing a rectangle (or more generally a parallelogram with one angle given) equal to each triangle, the problem will be solved. The construction of a parallelogram (with one angle given) equal to a triangle is the content of Proposition 42, Proposition 44 and Proposition 45 in Book I, whose proofs all rely on Postulate 5 about parallelism. Viewed in this way, the axiomatic approach exemplified in *Elements* gains a richer meaning.

4 Pedagogical viewpoint

I now come to the pedagogical viewpoint. In the first part of my talk I tried to show how the two aspects — "algorithmic mathematics" and "dialectic mathematics" — intertwine with each other. It reminds me of the "yin" and "yang" in Chinese philosophy in which the two aspects complement and supplement each other with one containing some part of the other. (To go even further than that I would even borrow a metaphor probably from the biologist and popular science writer Stephen Jay GOULD: Is a zebra a white animal with black stripes or a black animal with white stripes?) If that is the case, then in the teaching of mathematics we should not just emphasize one at the expense of the other. When we learn something new we need first to get acquainted with the new thing and to acquire sufficient feeling for it. A procedural approach helps us to prepare more solid ground to build up subsequent conceptual understanding. In turn, when we understand the concept better we will be able to handle the algorithm with more facility. In the mathematics education community there has been a long-running debate on procedural vs conceptual knowledge, or process vs object in learning theory, or computer vs no-computer learning environment. In a more general context these are all related to a debate on algorithmic vs dialectic mathematics, which are actually not two opposing forces but can be joined to provide an integrated way of learning and teaching. I will now give five examples on learning and teaching, with the last two having more to do with research. I apologize for the obvious lopsided emphasis on algebra in these five examples. My excuse is that they all have to do with my own classroom experience.

(1) Solving a system of linear equations by reduction to echelon form is clearly algorithmic in nature. (By the way, the algorithm was explicitly recorded and explained in Chapter 8 of *Jiuzhang Suanshu*. The title of the chapter itself is telling — *Fangcheng*, which means literally "the procedure of calculation by a rectangular array".) However, a clear understanding of this working does much to help us understand the more abstract and theoretical part of linear algebra and see why many of the concepts and definitions make sense. I will not therefore regard an exercise in manipulating a system of linear equations as a routine exercise for those who are less apt at coping with abstract theory, but as a preparation for it. Suitably dressed up, even a routine exercise can become a useful lead into interesting and useful theory. As an example, we can ask:

> "Let W_1 be the subspace in $\mathbb{R}^3$ spanned by $(1,1,2)$, $(3,0,-1)$, $(1,-2,-5)$ and let W_2 be the subspace in $\mathbb{R}^3$ spanned by $(4,1,1)$, $(1,4,-1)$, $(2,-7,3)$. Calculate the intersection of W_1 and W_2. Describe the geometry of it."

An ad hoc calculation in this concrete case supported by a clear geometric picture, with

(4, 1, 1) lying on the line of intersection of the two hyperplanes W_1 and W_2, leads to a more theoretical discussion in a general situation.

(2) As a pupil I came across in school algebra many homework problems which ask for writing expressions like $p^3q + pq^3$ or $5p^2 - 3pq + 5q^2$ or $p^4 + q^4$ or ... in terms of a, b, c where p, q are the roots of $aX^2 + bX + c = 0$. Each time I could arrive at an answer, maybe sometimes after long calculation. I used to query why an answer must come up for such so-called "symmetric" expressions. It was only many years later that I came to understand this in the form of the Fundamental Theorem on Symmetric Polynomial. There are different proofs for the result and it can be formulated in a rather general context of polynomials over a commutative ring with unity. But I still find it helpful to work out one example in an algorithmic fashion to get a flavour of the dialectic proof. For instance let us try to express the polynomial

$$X_1^3X_2^2 + X_2^3X_3^2 + X_3^3X_1^2 + X_1^2X_2^3 + X_2^2X_3^3 + X_3^2X_1^3$$

in terms of $\sigma_1 = X_1+X_2+X_3$, $\sigma_2 = X_1X_2+X_2X_3+X_3X_1$, $\sigma_3 = X_1X_2X_3$. Naturally we can write the polynomial in X_1, X_2, X_3 as a polynomial in X_3 with coefficients involving X_1, X_2, i.e.

$$f(X_1, X_2, X_3) = (X_1^3X_2^2 + X_1^2X_2^3) + (X_1^3 + X_2^3)X_3^2 + (X_1^2 + X_2^2)X_3^3 \ .$$

Applying our knowledge of polynomials in X_1, X_2 (after so much working in school algebra), we arrive at

$$f(X_1, X_2, X_3) = \tau_1\tau_2^2 + (\tau_1^3 - 3\tau_1\tau_2)X_3^2 + (\tau_1^2 - 2\tau_2)X_3^3$$

where $\tau_1 = X_1 + X_2$, $\tau_2 = X_1X_2$. Now, write $\sigma_1 = \tau_1 + X_3$, $\sigma_2 = \tau_2 + \tau_1X_3$, $\sigma_3 = \tau_2X_3$. From the first two relationships we can express τ_1, τ_2 in terms of σ_1, σ_2 and X_3, i.e. $\tau_1 = \sigma_1 - X_3$, $\tau_2 = \sigma_2 - \sigma_1X_3 + X_3^2$. Substituting τ_2 back to the third relationship we can express $X_3^3 = \sigma_3 - \sigma_2X_3 + \sigma_1X_3^2$. Hence we can express the coefficients $\tau_1\tau_2^2$, $\tau_1^3 - 3\tau_1\tau_2$, $\tau_1^2 - 2\tau_2$ in terms of $\sigma_1, \sigma_2, \sigma_3$ and X_3 up to the second power. Substituting back to $f(X_1, X_2, X_3)$ we obtain, after some rather tedious (but worthwhile!) work,

$$f(X_1, X_2, X_3) = \sigma_1\sigma_2^2 - 2\sigma_1^2\sigma_3 - \sigma_2\sigma_3 \ .$$

Note that suddenly all terms involving X_3 vanish and that is the answer we want! Coincidence in mathematics is rare. If there is any coincidence, it usually begs for an explanation. The explanation we seek in this case will lead us to one proof of the Fundamental Theorem on Symmetric Polynomial.

(3) The simplest type of extension field discussed in a basic course on abstract algebra is the adjunction of a single element algebraic over the ground field, say $\mathbb{Q}$. The element α, say in $\mathbb{C}$, is said to be algebraic over $\mathbb{Q}$ if α is the zero of some polynomial with coefficients in $\mathbb{Q}$. The dialectic aspect involves the "finiteness" of the extension field $\mathbb{Q}(\alpha)$ viewed as a finite-dimensional vector space over $\mathbb{Q}$. It is helpful to go through some algorithmic calculation to experience the "finiteness". For instance, take $\alpha = \sqrt{2}$. It is easy to see that a typical element in $\mathbb{Q}(\alpha)$ (by knowing what $\mathbb{Q}(\alpha)$ stands for) is of the form $(a + b\alpha)/(c + d\alpha)$ where a, b, c, d are in $\mathbb{Q}$, because any term involving a higher power of α can be ground down to a linear combination (over $\mathbb{Q}$) of 1 and α.

The procedure on conjugation learnt in school allows us to revert the denominator as part of the numerator, i.e.

$$\begin{aligned} 1/(c+d\alpha) &= (c-d\alpha)/(c+d\alpha)(c-d\alpha) = (c-d\alpha)/(c^2-2d^2) \\ &= [(c/(c^2-2d^2)] + [(-d)/(c^2-2d^2)]\alpha \, . \end{aligned}$$

Hence, a typical element in $\mathbb{Q}(\alpha)$ is of the form $a+b\alpha$ where a,b are in $\mathbb{Q}$. It is more instructive to follow with a slightly more complicated example such as $\alpha = \sqrt{1+\sqrt{3}}$. It is not much harder to see that we can confine attention to linear combinations of $1, \alpha, \alpha^2, \alpha^3$, but this time it is much more messy to revert the denominator as part of the numerator. This will motivate a more elegant dialectic proof modelled after the algorithmic calculation for $\alpha = \sqrt{2}$. Another useful piece of knowledge about algebraic elements is: If a and b (say in $\mathbb{C}$) are algebraic over $\mathbb{Q}$, then $a+b$ is algebraic over $\mathbb{Q}$. The dialectic aspect involves the notion of "finiteness" by viewing $\mathbb{Q}(a,b)$ as a finite-dimensional vector space over $\mathbb{Q}$. Going through an algorithmic calculation may help to consolidate understanding. For instance, take $\sqrt{2}$, which is algebraic over $\mathbb{Q}$ as a zero of X^2-2, and take $\sqrt[3]{3}$, which is algebraic over $\mathbb{Q}$ as a zero of X_3^3-3. Try to find a polynomial with coefficients in $\mathbb{Q}$ such that $\sqrt{2}+\sqrt[3]{3}$ is a zero of it. We can follow an algorithm which expresses $X^2-2 = (X-\sqrt{2})(X+\sqrt{2})$ and $(X^3-3) = (X-\alpha)(X-\alpha\omega)(X-\alpha\omega^2)$ where $\alpha (\in \mathbb{R})$ is such that $\alpha^3 = 3$ and $\omega = \frac{1}{2}(\sqrt{3}i - 1)$, then consider the polynomial

$$g(X) = (X-\sqrt{2}-\alpha)(X+\sqrt{2}-\alpha)(X-\sqrt{2}-\alpha\omega)(X+\sqrt{2}-\alpha\omega)(X-\sqrt{2}-\alpha\omega^2)(X+\sqrt{2}-\alpha\omega^2)$$

which reduces after some calculation to $X^6+6X^4-6X^3+12X^2-36X+1$ (noting that $\alpha^3 = 3$ and $1+\omega+\omega^2 = 0$). It is certainly not incidental that ultimately no coefficient involves $\sqrt{2}$ or α or ω ! Further enquiry will suggest a constructive proof of the general result by making use of symmetric polynomials.

(4) To begin with a simple example, let z be a (complex) root other than 1 of the equation $X^5-1=0$, so $z^4+z^3+z^2+z+1=0$, or $(z^1+z^4)+(z^2+z^3)=0$. Write $\eta_0 = z^1+z^4$ and $\eta_1 = z^2+z^3$ and note that $\eta_0+\eta_1 = -1$ and $\eta_0\eta_1 = \eta_0+\eta_1 = -1$. Hence, η_0, η_1 are roots of $Y^2+Y-1=0$, say

$$\eta_0 = \frac{-1+\sqrt{5}}{2}\,,\ \eta_1 = \frac{-1-\sqrt{5}}{2}\,.$$

From $\eta_0 = z+\frac{1}{z}$ we obtain $z^2-\eta_0 z+1=0$ so that one value for z is $z = \frac{1}{2}(\eta_0+\sqrt{\eta_0^2-4}) = \frac{1}{4}[-1+\sqrt{5}+\sqrt{-10-2\sqrt{5}}]$. This calculation is the basic idea Carl Friederich GAUSS applied to solve the equation $X^N-1=0$ where N is a prime number. (I have a slight suspicion that Gauss was inspired by the work of Alexandre-Théophile VANDERMONDE who solved that equation in a brilliant 1774 paper titled "Memoire sur la résolution des équations" [13, Chapter 11 and Chapter 12].) The calculation will go through in general if at each stage we can break up the sum of powers of z into two halves, which is the case when N is of the form $2^{2^s}+1$, i.e. N is a Fermat prime. This is the theory of cyclotomy developed by GAUSS in Section VII of his *Disquisitiones Arithmeticae* of 1801 in connection with his celebrated discovery in 1796 of the constructibility of a regular seventeen-sided polygon by straight-edge and compasses [6].

We now go tangentially off the work of Gauss but take with us one crucial point: express $\eta_0\eta_1$ in the form $a\eta_0 + b\eta_1 + c$ for some integers a, b, c. Let p be an odd prime of the form $2f+1$ and g is a primitive root of p. Let $C_0 = \{g^{2s} | s \in \{0, 1, 2, \dots, f-1\}\}$ and $C_1 = \{g^{2s+1} | s \in \{0, 1, 2, \dots, f-1\}\}$, then $\{1, 2, \dots, p-1\}$ is decomposed into the disjoint union $C_0 \cup C_1$. We call C_0, C_1 cyclotomic classes and $(i,j) = |(C_i+1) \cap C_j|$ (with $i, j \in \{0, 1\}$) cyclotomic numbers. If $\eta_0 = \sum_{t \in C_0} z^t$ and $\eta_1 = \sum_{t \in C_1} z^t$, then it turns out that $\eta_0 + \eta_1 = -1$ and $\eta_0\eta_1 = (1,0)\eta_0 + (1,1)\eta_1 + c$ where c is the number of 0 in $C_0 + C_1$ (repetition counted). More generally, let p be a prime number and $q = p^\alpha = ef+1$ and g is a generator of the multiplicative group of the finite field $GF(q)$, which is decomposed into a disjoint union $C_0 \cup C_1 \cup \cdots \cup C_{e-1}$ where $C_i = \{g^{es+i} | S \in \{0, 1, 2, \dots, f-1\}\}$ (cyclotomic class). We call $(i,j) = |(C_i+1) \cap C_j|$ (with $i, j \in \{0, 1, \dots, e-1\}$) cyclotomic numbers. The fascinating property which comes out of the calculation is that, when and only when $(i, 0) = (f-1)/e$ for all $i \in \{0, 1, \dots, e-1\}$, then C_0 is a difference set in $GF(q)$, i.e. each nonzero element in $GF(q)$ is the difference $x - y$ of the same number of pairs of elements (x, y) in $C_0 \times C_0$. For instance, this is true for $q = 11$ so that $C_0 = \{1, 3, 4, 5, 9\}$, the set of quadratic residues modulo 11, is a difference set. If you look at all the differences (modulo 11) $x - y$ of pairs (x, y) of numbers in C_0, you will find each nonzero number appearing exactly twice. Research on difference sets is a nice mixture of "algorithmic mathematics" and "dialectic mathematics".

(5) The last example is a personal anecdote about a piece of research work. Let me first describe the problem. Let F be the finite field with $q = p^s$ elements, i.e. $F = GF(q)$. A function $f : F \to \mathbb{C}$ is called a nontrivial multiplicative character of F if $f(0) = 0$, $f(1) = 1$ but $f \not\equiv 1$ on $F^* = F \setminus \{0\}$, and $f(b_1 b_2) = f(b_1) f(b_2)$ for all b_1, b_2 in F. In this case, it is well-know that

$$\sum_{b \in F} f(b)\overline{f(b+a)} = \begin{cases} q-1 & \text{if } a = 0\,; \\ -1 & \text{if } a \neq 0\,. \end{cases} \qquad (\#)$$

Harvey COHN asks whether the converse is true: If $f : F \to \mathbb{C}$ is such that $f(0) = 0$, $f(1) = 1$, $|f(a)| = 1$ for all a in F^* and (#) holds, must f be a nontrivial multiplicative character of F? In the summer of 1996 I could settle the real case (so that $f(a)$ is either 1 or -1 for nonzero a) with an affirmative answer when F is a prime field. That much is "dialectic mathematics". I failed to extend the argument to the case when F is not necessarily a prime field. Hence the work was put aside until my interest was resurrected in the spring of 1999 when a young colleague, Stephen CHOI, gave a seminar on the same problem arising in a different context, attacked by a different approach. Naturally we joined forces to look at the general case. We noted that (#) involves only the addition in F but not the multiplication in F. If we compose a specific injective multiplicative character $\chi : F \to \mathbb{C}$ of F with an additive bijection $\varphi : F \to F$, then $f = \chi \circ \varphi$ satisfies (#) since χ satisfies (#). It remains to see if there exists any additive bijection φ which is not multiplicative. I turned to "algorithmic mathematics" by actually doing the calculation using a representation of F as the quotient ring of $GF(p)[X]$ modulo the ideal generated by an irreducible polynomial of degree s. One day upon re-checking the calculation of some concrete cases, I found an error, which I corrected. But in either case — the original incorrect version and the correct version — (#) was satisfied. To my dismay more errors in the calculation were detected, but each time, with correction or no correction, (#) was still satisfied. That made me become aware that more often than not, φ is not multiplicative. Finally we could prove this and give a negative answer to the problem in the case of non-prime fields [2].

5 Epilogue

To conclude I would like to share with readers a Zen saying from the Tang monk Qingyuan Weixin:

> "Before I had studied Zen for thirty years, I saw mountains as mountains, and waters as waters. When I arrived at a more intimate knowledge, I come to the point where I saw the mountains are not mountains, and waters are not waters. But now that I have got its very substance I am at rest. For it is just that I see mountains once again as mountains, and waters once again as waters."

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HOW PEOPLE LEARN ... MATHEMATICS

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ABSTRACT

I address four of the seven themes of the 2nd International Conference on the Teaching of Mathematics – research, technology, pedagogical innovation, and curricular innovation – from the point of view that learning mathematics is, first of all, *learning*. Research from a variety of fields – education, neurobiology, cognitive psychology – provides a consistent set of messages about what learning is, how learning takes place, and how teachers can facilitate learning.

I offer necessarily brief surveys of research on the main themes, and then I describe how my understanding of this research has led to the design of a learning environment (a combination of an interactive classroom, an online delivery system, a rich set of tools, demanding course requirements, innovative course materials, effective in-class and assessment practices, and intangibles) that is radically different from my practice of, say, 20 years ago. I also provide an example of a research-based design for a single lesson.

My conclusions touch on the need for continuous curriculum renewal, for effective strategies to stimulate deep learning, for goal-directed assessments, for addressing the needs of a would-be mathematically literate public, and for preservice and inservice professional development.

1. Introduction

The 2nd International Conference on the Teaching of Mathematics intends to address new ways of teaching undergraduate mathematics. The first four of seven conference themes (slightly abridged) are

- **EDUCATIONAL RESEARCH**: Results of current research in mathematics education and the assessment of student learning. ...
- **TECHNOLOGY**: Effective integration of computing technology...into the undergraduate curriculum
- **INNOVATIVE TEACHING METHODS**: ... cooperative and collaborative teaching, writing in mathematics, laboratory courses.
- **CURRICULA INNOVATIONS**: Revisions of specific courses and assessment of the results ... innovative applications, project driven curricula.

This paper cuts across all four of these themes – and has some implications for the other three as well – professional development, relationships to other disciplines, and distance learning technologies.

I write from the perspective of a 40-year teaching career at Duke and other universities, including many attempts at innovative curriculum development and incorporation of technology into the learning process. To be candid, for the first half of my career I mostly failed to have any significant impact on my students, at least in the sense of stimulating sound knowledge and understanding of mathematics. My truly successful students were few enough in number that I still remember their names – and I have always suspected that they would have succeeded just as well without me.

I'm obviously a slow learner, but frustration is a powerful motivator. A series of opportunities in the 1980's and since has permitted me to learn a good deal about my profession that I should have learned much earlier, and to put that learning to use as a teacher and curriculum developer. At first my learning was experiential (that is to say, *ad hoc*), trying things in the classroom, rejecting what did not work, and reinforcing what did. One might describe this as "natural selection" in the evolutionary sense. Later I began to study the research literature – not just in mathematics education, but also in cognitive psychology and neurobiology – to find reasons for my successes and failures. It probably would have been more efficient to proceed in the other order – as I said, I'm a slow learner. In this paper I share some of what I have learned, along with connections to the conference themes.

2. Research

The first part of my title comes from the book *How People Learn: Brain, Mind, Experience, and School*, a (U.S.) National Research Council study (Bransford, *et al.*, 1999) that summarizes the very substantial body of research on learning, especially that of the past 30 years. Here is the start of the Executive Summary (p. xi):

> "Learning is a basic, adaptive function of humans. More than any other species, people are designed to be flexible learners and active agents of acquiring knowledge and skills. Much of what people learn occurs without formal instruction, but highly systematic and organized information systems – reading, mathematics, the sciences, literature, and the history of a society – require formal training, usually in schools. Over time, science,

> mathematics, and history have posed new problems for learning because of their growing volume and increasing complexity. The value of the knowledge taught in school also began to be examined for its applicability to situations outside school.
>
> "Science now offers new conceptions of the learning process and the development of competent performance. Recent research provides a deep understanding of complex reasoning and performance on problem-solving tasks and how skill and understanding in key subjects are acquired. ... "

My point in citing this and other works on learning research is that learning mathematics is, first and foremost, *learning*. Our subject is not exempt from what others have learned about learning, and indeed our curricula and pedagogy, to be successful, must be informed by research on learning. Readers of this paper will probably not be surprised by any of the findings in the NRC study – but may be surprised to learn the strength of the research base underlying the strategies we have come to associate with the words "reform" and "renewal."

The 1990's have been described as "The Decade of the Brain," a period in which the study of live, functioning, normal brains has come into its own through non-invasive technologies, such as positron emission tomography (PET) and functional magnetic resonance imaging (fMRI). This research will continue for many decades, of course. As the NRC study states (p. xv), "What is new, and therefore important for a new science of learning, is the *convergence* of evidence from a number of scientific fields." (Emphasis in the original.) That is, the messages from neuroscience are entirely consistent with and supportive of what we have learned from developmental psychology, cognitive psychology, and other areas of research.

There is one sense in which learning mathematics is different from learning many other things, such as speaking our native language, remembering visual and aural images of familiar people and places, and driving a car. The first and most fundamental biological fact about our brains is that they have not evolved significantly from the brains of our hunter-gatherer ancestors. Thus, we are superbly adapted – or would be if it were not for environmental influences – for fight-or-flight decisions and other survival tactics. As Dehaene (1997) has so beautifully documented in *The Number Sense*, this means that humans (and other species as well) are practically hard-wired to do arithmetic with small integers – but everything else in mathematics is *hard*, because it doesn't come to us instinctively. On the other hand, we learn many things that are not instinctive in an evolutionary sense, such as history, philosophy, foreign languages (beyond infancy), music, and neurobiology. One might say the Education is about *learning the things that hard to learn* – of which mathematics is just one example. [Exercise for the reader: Why is "driving a car" – clearly not an evolutionary adaptation – a relatively easy task for adolescents and adults in a developed society?]

We summarize here some of the key findings from the NRC study (Bransford, *et al.*, 1999, pp. xii-xviii) that are relevant to collegiate education, in particular, to undergraduate mathematics.

- **Collateral Development of Mind and Brain**
  - "Learning changes the physical structure of the brain."
  - "Structural changes alter the functional organization of the brain, [i.e.], learning organizes and reorganizes the brain."
  - "Different parts of the brain may be ready to learn at different times."
- **Durability of Learning and Ability to Transfer to New Situations**
  - "Skills and knowledge must be extended beyond the narrow contexts in which they are first learned."

- "...a learner [must] develop a sense of *when* what has been learned can be used .... Failure to transfer is often due to ... lack of ... conditional knowledge."
- "Learning must be guided by general principles .... Knowledge learned at the level of rote memory rarely transfers ...."
- "Learners are helped in their independent learning attempts if they have conceptual knowledge. ..."
- "Learners are most successful if they are mindful of themselves as learners and thinkers. ... self-awareness and appraisal strategies keep learning on target ... . this is how human beings become life-long learners."

♦ **Expert vs. Novice Performance**

- "Experts notice ... patterns ... that are not noticed by novices."
- "Experts have ... [organized] content knowledge ..., and their organization ... reflects a deep understanding of the subject matter."
- "Experts' knowledge cannot be reduced to sets of isolated facts ... but, instead, reflects contexts of applicability ...."
- "Experts have varying levels of flexibility in their approaches to new situations."
- "Though experts know their disciplines thoroughly, this does not guarantee that they are able to instruct others ...."

♦ **Designs for Learning Environments**

- "*Learner-centered environments* ... Effective instruction begins with what learners bring to the setting ... learners use their current knowledge to construct new knowledge ... what they know and believe at the moment affects how they interpret new information ... Sometimes learners' current knowledge supports new learning; sometimes it hampers learning."
- "*Knowledge-centered environments* The ability to think and solve problems requires knowledge that is accessible and applied appropriately. ... Curricula that are a 'mile wide and an inch deep' run the risk of developing disconnected rather than connected knowledge."
- "*Assessment to support learning* ... Assessments must reflect the learning goals .... If the goal is to enhance understanding and applicability of knowledge, it is not sufficient to provide assessments that focus primarily on memory for facts and formulas."
- "*Community-centered environments* [An] important perspective on learning environments is the degree to which they promote a sense of community. ..."

♦ **Effective Teaching**

- "Effective teachers need 'pedagogical content knowledge' – knowledge about how to teach in [the] particular [discipline], which is different from knowledge of general teaching methods."
- "Expert teachers know the structure of their disciplines and [have] cognitive roadmaps that guide the assignments they give ..., the assessments they use ..., and the questions they ask in the ... classroom ...."

♦ **New Technologies**

- "Because many new technologies are interactive, it is now easier to create environments in which students can learn by doing, receive feedback, and continually refine their understanding and build new knowledge."
- "Technologies can help people visualize difficult-to-understand concepts ...."

- "New technologies provide access to a vast array of information, including digital libraries, real-world data for analysis, and connections to other people who provide information, feedback, and inspiration, all of which can enhance the learning of teachers and administrators as well as students."
-

3. Technology

There has been a great deal of controversy over the past two decades about the presumed effects, good and bad, of using technological tools (calculators and computers) in teaching and learning mathematics. The debate is beginning to be informed by a substantial and growing body of research, which one hopes in time will replace strident assertions of deeply held opinions. The NRC report cited in the preceding section highlights the positive features, particularly of interactive technologies, for learning in general. A forthcoming volume (Heid and Blume, to appear) surveys research on the role of technology in teaching and learning mathematics at all levels. As a co-author of one of the chapters in that volume (Tall, *et al.*, to appear), I have had an opportunity to learn more about this research as it relates to college-level mathematics. Our paper includes an analysis of a large number of recent research papers and Ph.D. theses in mathematics education that focus on technology in calculus and related subjects. In simplified form, the key messages are

1. *Technology used inappropriately makes no significant difference.* In particular, adding calculators and/or computers to a traditionally taught and assessed mathematics course may make it marginally better or worse, but there won't be much change. "Better" is likely to be associated with students finding ways to use the technology that are not necessarily planned by the instructor. "Worse" is likely to be associated with time and effort devoted to yet another task, particularly if it is seen as disconnected from all the others.
2. *Technology integrated intelligently with curriculum and pedagogy produces measurable learning gains.* It may be impossible to tease out whether the gains are the direct result of the technology or of the rethought curriculum and pedagogy. (Do it matter?)
3. *There is little evidence that one technology is "better" than another.* What matters is how the technology is used.
4. *There is substantial evidence that using computer algebra systems for conceptual exploration and for learning how to instruct the software to carry out symbolic calculations leads to conceptual gains in solving problems that can transfer to later courses.* In comparison, students in traditional courses tend to use more procedural solution processes that do not easily transfer to new situations.
5. *Technology enables some types of learning activities* (e.g., discovery learning) *and facilitates some others* (e.g., cooperative learning) *that are harder or impossible to achieve without technology.*

These results are completely consistent with what is known about learning in general – which reinforces my point that learning mathematics is, first of all, *learning*, and only secondarily about mathematics.

One of the more interesting points in the research on technology in mathematics courses is the role of the teacher in influencing the outcome. Keller and Hirsch (1998) found that students' preferences for numeric, graphic, or symbolic representations reflect in part the teacher's preference. Kendal and Stacey (1999) studied three teachers who taught the same calculus syllabus using TI-92 symbolic calculators. Teacher A enthusiastically used the computer algebra system at every opportunity, while Teacher B was more reserved and underpinned the work with paper-and-pencil calculations. Teacher C was enthusiastic about the graphing abilities of the calculator and used it more often for graphical insight than for symbolic calculation. The three teachers also had different predictions about their students' algebraic competence, geometric competence, and likelihood of success while using the technology. Mean scores on the common end-of-course assessment were essentially the same for the three sections, but students in each of the sections were successful on different questions, more or less in accord with their teacher's expectations and privileging of specific uses of the technology.

4. Curriculum

What do we really want to teach, and why do we want to teach it? Are the important topics in mathematics essentially unchanged over time, or should the curriculum be viewed as something like a living organism – perhaps as a species of organisms, with births, deaths, evolution?

Whenever I think about these questions, I am reminded of our sister sciences, for which the answers are much more obvious. For example, when I was a student, continental drift was considered a heretical theory – not just wrong but wrong-headed, not worth serious scientific discussion. One could easily list several dozen significant paradigm shifts in science over the past 50 years, most of which have been reflected in science curricula at some level.

Over the same period of time, mathematical knowledge has literally exploded, both in its pure sense and in its relationship to science and technology. And yet we tend to think of the academic content of our discipline (at least K-14) as essentially static. We know better, of course. When I was a student, the list of important skills (necessarily paper-and-pencil skills, except for occasional use of a slide rule) included calculation of square roots, interpolating in trig and log tables, and polar and logarithmic graphing, along with others that subsequently disappeared from the "standard" curriculum. It is very rare now to encounter a student who has ever calculated a nontrivial square root by hand or who has ever seen a log table or a slide rule (never mind knowing what to do with them). The non-Cartesian graphing techniques disappeared because the presumed benefits were not commensurate with the intellectual demands of learning how to do them (not to mention the cost of special graphing paper). But now those techniques are back in our curricula (or should be), because they have important conceptual content and modeling significance, and because our modern technology makes them easy, cheap, and accessible to all.

So why do some of our colleagues continue to insist on advanced factoring techniques as a prerequisite skill for calculus, when the original reason they were in the curriculum was to be able to solve carefully contrived max/min problems? And why do we assume that essentially all of single-variable calculus is a prerequisite for differential equations – or that the really important techniques in differential equations are the purely symbolic ones? Any problem that has been reduced to a button on an omnipresent calculator – such as square root, log function, max/min, or graphical-numerical solution of a differential equation – can no longer be considered a difficult or inaccessible problem. Now that many of these formerly difficult problems have been rendered easy, we have to confront the fact that solving the problems does not imply understanding of the conceptual content.

Much of our profession continues to resist research-based calls for curricular (and other) changes, such as the NCTM *Principles and Standards* (NCTM, 2000). The current *Standards* are themselves the product of extensive debate, development of curricular materials, trial, research, and revision since publication of the predecessor document in 1989. And yet many academic mathematicians cannot conceive of a successful secondary curriculum that is not organized by presumed precursor topics for calculus, organized into courses titled Algebra I, Geometry, Algebra II, Trigonometry (perhaps in combination with, say, Analytic Geometry), and Precalculus.

The calculus reform initiative in the U.S. (see Roberts, 1996, Ganter, 2000) has more or less coincided with NCTM efforts to reform school curricula and has been the driving force in reform of collegiate curricula at all levels. Successes and failures of this initiative have to be viewed against the backdrop of an established system in which the table of contents of a textbook was seen as a complete description of a course. Thus, among the early "reformers" were some who saw their task as grafting technology onto an unchanged (unchangeable?) syllabus. (We have already noted in the preceding section the failure of these efforts to produce significant learning gains.) Others saw their task as creating the next best-selling calculus textbook – or, in some cases, grudgingly accepted commercial publication of a textbook as the primary means of dissemination of their good ideas for reform. Only a relative handful of these curricular efforts ever made it to commercial publication, and, for a number of reasons, only one (Hughes Hallett, *et al.*, 2001) was ever a true commercial success. Each subsequent edition of this work looks more "traditional" but still retains the creative problems and other tasks that set it apart from a traditional text. Meanwhile, the commercially successful traditional calculus books are taking on a more "reformed" appearance without a significant change in real content or approach.

Over the next few years, and perhaps beyond, we will see growing use of the World Wide Web for dissemination of innovative curricular materials, both commercial and free (or grant-supported), bypassing the traditional publishers and enabling direct access to interactive materials that cannot reasonably be reduced to print. One example of this is the Web publisher Math Everywhere, Inc. (http://matheverywhere.com/), an enterprise created by Bill Davis and colleagues to market interactive courseware, including *Calculus & Mathematica®* (1994), one of the most successful products of the calculus reform initiative. By "successful," I do not mean in the commercial sense – it's not clear to an outside observer that Addison-Wesley's marketing attempts were ever successful. On the other hand, a number of the research studies cited by Tall, *et al.* (to appear) compared *C&M* to traditional courses and found significant learning gains for the *C&M* students. In addition to the "classic" *C&M*, the MEI Web site now offers a range of similar courses, in various stages of maturity, addressing much of the lower-division college curriculum.

The *Connected Curriculum Project* (http://www.math.duke.edu/education/ccp/), in which I am a principal, is an example of free distribution (supported by a National Science Foundation grant) of materials that grew out of an earlier calculus reform project (Smith and Moore, 1996), another commercial failure for which the research studies generally showed significant learning gains. The *CCP* materials are not entire courses – rather they are modular, interactive units that lead students through important concepts and applications throughout the lower-division curriculum.

There are a number of free Web sites offering peer-reviewed college-level curriculum materials in a variety of disciplines, including mathematics. Among these are the *Mathematical Sciences Digital Library* (MathDL, http://www.mathdl.org/), *MERLOT* (http://www.merlot.org/), and *iLumina* (www.ilumina-dlib.org/). I am affiliated with the first of these – an NSF-funded project of the Mathematical Association of America – as Editor of the *Journal of Online Mathematics and its Applications* (JOMA, http://www.joma.org/). JOMA is a peer-reviewed academic journal that includes, among other things, high-quality, innovative, and class-tested curricular materials, as well as user and research articles about these materials.

5. Pedagogy

The NRC study (Bransford, *et al.*, 1999), while extensive, does not encompass all of the important research threads in the study of higher education. For example, researchers in Scotland, Australia, and Sweden (Entwistle and Ramsden, 1983; Entwistle, 1987; Ramsden, 1992; Bowden and Marton, 1998) have studied student *approaches* to learning, with a focus on approaches that lead to deep vs. surface learning. (See also Rhem, 1995.) Deep learning approaches are quite different from surface learning approaches, and a given student – whatever his or her "learning style" – may exhibit different approaches simultaneously in different courses. These student-selected "coping strategies" are often influenced by expectations set by the instructor, consciously or unconsciously.

In particular, surface learning is encouraged by

- excessive amounts of material to be covered,
- lack of opportunity to pursue subjects in depth,
- lack of choice over subjects and/or method of study, and
- a threatening assessment system.

On the other hand, deep learning – the organized and conceptual learning described in the NRC study – is encouraged by

- interaction with peers, especially working in groups,
- a well-structured knowledge base with connections of new concepts to prior experience and knowledge,
- a strong motivational context, with a choice of control and a sense of ownership, and
- learner activity followed by faculty connecting the activity to the abstract concept.

These are especially important aspects of pedagogy for those of us whose goals include teaching mathematics to a much broader audience than just those who intend to replace us as mathematicians. Notice in particular, the similarity of the "surface" list to the way many mathematics courses are taught in many colleges and universities – with results that are almost universally considered unacceptable. And notice also that the "deep" list comprises principles that have been incorporated into all of the major "reform" efforts of the past 15 years or so.

Much of the reform has been carried out with scant or no knowledge of research – in some cases, even as the relevant research was under way. However, it is no accident that the strategies we found empirically to be effective are the same as those that have been shown by research to be effective. Perhaps the most significant aspect of the reform efforts has been the near-universal realization that revision of curricula is not enough, that decisions about topics are not enough, that inclusion of technology is not enough – that none of this matters unless our pedagogical strategies are also effective.

6. Putting it All Together: Research, Technology, Curriculum, Pedagogy

In a recent paper (Smith, 2001) I wrote about the Web-supported classroom environment in which I have taught for the past three years. The courses I teach now are the product of what I have learned over the past two decades about research on learning (in neurobiology, cognitive psychology, and empirical educational studies), supported by modern computer technology, carefully designed curricular and assessment materials, and active-learning strategies in and out of the classroom. My students and I benefit from Duke University's commitment to quality education in the form of an Interactive Computer Classroom, Web delivery support via Blackboard 5.5, an extensive array of site-licensed software, and excellent staff support. Unfortunately, one of the disadvantages of committing a classroom or course description to paper is that it quickly goes out of date, expecially if Web resources are involved. There is an online version of my 2001 paper at http://www.math.duke.edu/~das/essays/classroom/ in which I have kept the links to classroom and course resources current.

Key features of my courses include

- articulated goals and assessments directed toward achieving the goals;
- a goal-setting exercise at the start of each term to give students a sense of common purpose and joint ownership;
- weekly plans that spell out the objectives, activities, readings, and problem assignments;
- a carefully cultivated sense of *community* in which students see each other and me as partners in their learning enterprise, not as competitors or adversaries;
- an online discussion board, plus easy access to e-mail for all course participants, to facilitate the sense of community;
- a mix of in-class activities – lecture supported by online interactive "notes" in a computer algebra file, informal group activities in teams of two to four (with or without use of a computer), structured lab activities using *Connected Curriculum Project* materials, and online use of resources from remote sites;
- challenging take-home open-book tests with all resources available;
- regular homework graded assignments on a weekly cycle, with a requirement that all submitted solutions be accompanied by a check and/or a correctness argument;
- campus-wide access to a computer algebra system (currently *Maple*® 7);
- use of every learning task as an assessment (formal or informal) for which feedback is given, and conversely, use of every assessment as a learning opportunity;
- a non-threatening distributed grading system among a range of different activities, roughly half with group grades and half with individual grades;
- a weekly electronic journal submission with a paragraph or two of reflection on the week's work;
- team projects with classroom presentation and multiple-submission papers;
- Web delivery of all important course documents and online submission of most student work;
- emphasis on realistic or real-world problems that are meaningful to students on their own terms and that serve as motivators and scaffolding for the mathematical concepts

Without my belaboring the point, the reader should find many points of contact between this list of strategies and the research findings cited earlier.

To illustrate the construction and use of research-based materials, I will give one example of a module (Moore, *et al.*, 2001) that I use early in a multivariable calculus course. This module could be used with any students who have had some exposure to polar coordinates, parametric representations, logarithmic graphing, and the relationship between tangent lines and derivatives.

The module, which may be seen at the URL given in the References, starts with a background page on spirals in nature, in particular, the spiral shell of the chambered nautilus (*N. pompilius*). This page is linked to other sites for information about Aristotle, who studied gnomonic growth, and D'Arcy Thompson, author of the 1917 classic *On Growth and Form*, from which some of the content of the module is taken. There are also links to other sites with information about spirals in nature (seed patterns, nebulae, etc.) or related mathematical topics (Fibonacci numbers, evolutes of curves, etc.). My observation has been that students seldom follow any of these links – that they may do no more with the background page than look at the pictures, because it doesn't appear to contribute anything to completion of their assignment. However, part of the richness of the Web is that one can provide alternate learning paths for those who choose to take them – and without interfering with those who want to follow a straight line toward a specific goal.

The "business" of the module starts on the next page, where students are shown an enlarged cross-section of the nautilus shell superimposed on a polar grid and are challenged to reproduce the spiral shape. Their first step is to make a list of radial measurements (with a ruler), either on the screen or on a printed version of the picture. Thus we start with a tactile activity that leads to student ownership of the data from which the model will be derived. Students then test their data by logarithmic plotting for an exponential growth pattern, from which they can then derive a polar formula, $r = f(\theta) = Ae^{k\theta}$, and immediately test their model to see if the polar graph fits the data. They don't have to ask anyone "Is this right?" – they see immediately if they have made a mistake, and they have to get the formula right before they can move on.

On the next page, students link polar plotting to parametric plotting via the polar-to-Cartesian change-of-coordinate formulas and plot their spiral again in rectangular coordinates. They also use this representation to zoom in at the origin and discover the self-similarity of the exponential spiral – a rather different result from the local linearity they usually associate with "zooming in."

Finally, students use the power of the computer algebra system (CAS) to explain the name "equiangular" – that is, to show that the angle between radius vector and tangent line is constant. This calculation involves calculus and algebra steps that only a few students would complete successfully with pencil and paper. With the CAS, almost everyone can complete the calculation and at the same time keep their focus on the mathematical concepts involved.

At the end of the lab activity, each student team completes their CAS-based report by answering the following summary questions:

1. Describe in general terms the process of finding a polar formula from the radial measurements on a seashell picture.
2. What happens when you zoom in at the center of an equiangular spiral? The behavior you observed is called **self-similarity**. Explain the name.
3. What remains constant as $\boldsymbol{r}$ grows in an equiangular spiral?
4. Describe in geometric terms why the equiangular spiral has the name it has.
5. What is the shape of an equiangular spiral with $\boldsymbol{\beta = \pi/2}$? How is this reflected in the formula for $\boldsymbol{r}$ as a function of $\boldsymbol{\theta}$? How is it reflected in the relationship between $\boldsymbol{\beta}$ and $\boldsymbol{k}$?

The last question asks about a case not previously encountered in the module – that in which the "equiangle" β is a right angle and the "spiral" is a circle. Since the relationship they have found is $\tan \beta = 1 / k$, they have make sense of this formula when the left-hand side is ∞.

This module illustrates design that takes students through at least one complete Kolb learning cycle (see Wolfe and Kolb, 1984):

- Concrete experience: input to the sensory cortex of the brain in the form of seeing, touching, moving – e.g., taking measurements;
- Reflection and observation: mainly right-brain activity, reinforced by use of previous learning – e.g., logarithmic plotting);
- Abstract conceptualization: left-brain activity – e.g., finding the polar exponential growth formula;

- Active experimentation: often involves the motor brain, sometimes the sensory cortex as well – e.g., testing the conceptual model against the reality of the data.

If the testing phase does not show complete success, the cycle may start over with the same problem, now being viewed from a slightly enhanced knowledge base – at least with the knowledge that something they thought would work in fact did not. When students achieve success at one experimentation point, they are ready to move on to the next learning cycle.

This example links Kolb's research on experiential learning to the neurobiological evidence that *deep learning is whole-brain activity* (see e.g., Rhem, 1995, Zull, 1998).

7. Conclusions

Research studies on learning in general and on learning mathematics in particular (with or without technology), together with my teaching and development experiences of the last two decades, lead me to several conclusions:

1. Curricula need to be rethought periodically from the ground up, taking into consideration the tools that are available. It is not enough to think of clever ways to present mathematics as the content was understood in the mid-20th century, when the available tool set was quite different, as was the intended audience.
2. Much of the effort that goes into curriculum design can be squandered if one does not also rethink pedagogical strategies in the light of research showing the effectiveness of active-learning strategies and distinguishing between good and bad ways to stimulate deep learning approaches. It is not enough to adopt (or write) a new book or even a new book-plus-software package.
3. Our tools for assessing student learning – whether for purposes of assigning grades or for evaluating effectiveness of our curricula – need to be consistent with stated goals for each course and with the learning environments in which we expect students to function. It is not enough to continue giving timed, memory-based, multiple-choice, no-tech examinations.
4. If we are serious about mathematical understanding for everyone with a "need to know" – not just the potential replacements for the mathematics faculty – then we must plan our curricula, pedagogy, and assessments for effective learning of the skill sets and mental disciplines that will be needed by a mathematically and technologically literate public in the 21st century. It is not enough to keep using ourselves as "model learners."
5. Revision of curricula, pedagogy, assessment tools, and technology tools will accomplish little without concurrent professional development to keep faculty up to date with the required skills, knowledge, attitudes, and beliefs. It is not enough to continue acting as though an advanced degree in mathematics is evidence of adequate preparation to teach.

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A HUMAN ACHIEVEMENT: MATHEMATICS WITHOUT BOUNDARIES

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Keywords: Mathematical concepts, history of mathematics

After suffering a series of defeats in battles against Napoleonic armies, Prussia decided to reform, amongst other things, its system of education. This work was entrusted to Wilhelm von Humboldt, who was appointed as the head of department of culture and education of the kingdom. During the eighteen months at this post, von Humboldt completely reorganized the school system of Prussia and wrote the charter of a new university. This university was called Berlin University: It enrolled its first students in 1810 and operated in a royal mansion donated by the King of Prussia.

The charter of Berlin University was revolutionary and it was based on three fundamental principles dictated by von Humboldt. The first was the inseparable unity of education and research. According to von Humboldt, research activity was what distinguished a university from other institutions of education. In Berlin University all subjects were present from philosophy to natural sciences, from medicine and engineering to arts and religious studies. University professors and students were constantly engaged in research, accepting no theory or idea as given, without subjecting it to critical reasoning.

The second fundamental principle concerned academic freedom. Berlin University was to be an arena of intellectual freedom. Activities of the university were to be conducted without any influence or interference of external sources of authority. This principle was summarized in the German motto "Lehrfreiheit und Lernfreiheit".

Students in Berlin University were obliged to have a fundamental education in natural sciences, philosophy and humanities in their first years before specializing in their degree areas. This was the third fundamental principle of von Humboldt. Wilhelm von Humboldt himself was a philosopher and a linguist. He knew thirty languages. It was von Humboldt's hope that the graduates of Berlin University would be first and foremost universal intellectuals and propagators of enlightenment. This was in direct contrast to the new French institutions of higher education whose mission was to educate expert professionals who were also good citizens of France.

The new model of Berlin University was received enthusiastically by other German Universities. Although universities in other European countries did not altogether take Berlin University as a blueprint, the fundamental principles set forth by von Humboldt were acclaimed by many. It was in the United States that von Humboldt's principles were widely adapted as the basic philosophy of higher education. Von Humboldt had hoped that in the new universities modelled on his fundamental principles, a unified grand theory of knowledge would develop by time, transcending all national or geographical borders. With a common culture based on similar general courses taken in the first years of their universities, the age of enlightenment would produce a new generation of professionals, who would also be intellectuals equipped with all tools necessary for critical thinking, refuting all dogma and bigotry.

However this did not happen - history took a different turn: Europe entered a phase of rapid industrialization and formation of strong nation states, creating new rivalries. The industry needed workers willing to do the same simple manual work for long hours at low wages. The state needed loyal and obedient citizens, who would heed a call to arms without hesitation whenever this was considered necessary by the government.

There was definitely a need for experts -engineers, doctors, and so on- but there was almost no "*Lebensraum*" for independently minded intellectuals who would not automatically hate the designated enemy of the state.

At the beginning of the new millenium, we are somewhat caught between two main currents of historical events. Or rather, there is one main current, that of globalization and some strong reactions to globalization which can form a strong coalition of opposition. There are also those who think that mankind cannot do without the devil, which has to be invented if there is none readily available.

Yet the alteration of geographic borders, fear of clash of civilizations, globalization, anti-globalization may well be temporary trends here today and gone tomorrow, belonging to the world that we see on the surface, the world where ideas are limited by boundaries of the widest variety. To the erring person who imagines the true world to be just a reflection of what he sees, everything is bound to appear like a seemingly endless, unproductive tug-of-war.

Yet below this surface is another world, the world of the infinite, where progress is always in a steady forward direction. In this world there can be no notion of "the shortness of the human life span" or even "time"; definitely no notion of material gains, for each idea is a drop that will expand within the never-ending flow that endures beyond centuries and milleniums. This may be why we mathematicians are perhaps among those people who can sense the true meaning of the word "infinite" in the most acute way.

Mathematics is a precious human achievement. It transcends boundaries of all kinds - geographical, historical, national, philosophical or linguistic. Mathematics is accumulative and ageless. Whenever I give the proof of Euclides that there are infinitely many primes, I ask my students to conduct a survey of the physics or astronomy of that period in history, and to compare it with what we know now. The proof attributed to Euclides is still valid today. Furthermore, I hope that it gives at least to some of my students as it does to me, a sense of aesthetic pleasure, whereas the model of the universe by Ptolemy, although at its time of formulation considered a masterpiece, is actually not only false but also quite naive.

Mathematics is full of true masterpieces. It is through the use of accumulation of the masterpieces of mathematics that scientists understand nature much better today than even just a century ago. We have developed means of harnessing the forces of nature for the benefit of humanity. What we describe as "high technology" has its roots in some field of mathematics. Today we use mathematics more widely than ever. Mathematics is and has always been a part of our common heritage, a part of the common wealth we share. We mathematicians do not patent our theorems, but publish them so that everyone can use them, criticize them or even prove them false.

To teach mathematics in the general context of humanities, I propose a course or a series of courses highlighting some concepts of mathematics, interplay between the concrete and the abstract and between heuristic arguments and formal proofs. Let me try to illustrate by means of some sketchy examples: Assuming that our students know basic arithmetic, one could define prime numbers and prove the prime factorization theorem that there are an infinite number of primes. We can then continue to discuss the twin prime problem and the Goldbach conjecture. For a more advanced class one can describe some of the futile attempts to obtain a formula giving all primes, and even include a discussion of some of the heuristic arguments making the conjecture that there are an infinite number of twin primes plausible. A discussion of the use of big primes in cryptography would bring us to today, from our starting point which was around 300 BC.

Another line of advance could start from utilitarian geometry and how it was formalized in the *Elements*. This masterpiece deserves certainly some attention, especially as the first example of the axiomatic approach and rigorous proofs. The fifth postulate could be discussed at some depth. One could also deal with the systematic approach of Appolonius to the conic sections and jump to Kepler's laws and maybe mention Newton's discovery of the gravitational force. Another path could take us to different geometries motivated certainly by the fifth postulate. In this discussion of geometry one could display how the *Elements* survived until the modern times, transmitted from one civilization to another through translations from one language to another, written on papyrus, parchment, "in palimpsest" and on paper.

A more ambitious project would be to take up the abstract notion of a group and illustrate the wide range of applications that is hidden in this simple algebraic structure. Even if briefly, one could touch upon Galois groups and how one can prove the impossibility of the trisection of an angle using compass and ruler only. Symmetry and ornaments can also be discussed in this context. A short discussion of Klein's Erlangen program would demonstrate the link between algebra and geometry.

These are just the initial thoughts that spring to my mind within the framework of what I know as a 20^{th} century mathematician – within time and the fertility of the human imagination, naturally new projects will be produced, existing projects will change form. However, we know from the history of mankind that in the land of the infinite, no idea or project –however incomplete it may be- goes wasted, if it is of any value: Sooner or later, it is bound to sparkle someone else's imagination -be it in another geography or another century- and in the end, turn into a sturdy brick contributing to the beauty of the magnificent joint product of mankind of all ages -immortal and transcending all worldly matters.

If education is to make a significant contribution to our future, I believe it must stress much more the achievements of humanity, not only in technology, health or natural sciences, but also in humanities in general.

We should strive to increase the awareness of our young people, that throughout history we have created a tremendous amount of human wealth – in music, literature, architecture, philosophy and in mathematics. These human values, when taught properly, will infuse a new sense of pride and confidence in ourselves, a new hope for a better and peaceful life on our planet. We should revise the unfulfilled dream of von Humboldt and try to make it come true.

Panel "On the role of the history of mathematics in mathematics education"

Fulvia FURINGHETTI (Coordinator)
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ABSTRACT

In recent years, important works on the relationship between history and mathematics education have appeared:

(a) The Proceedings of the "European Summer University on History and Epistemology in Mathematics Education" (Montpellier, France, 1993, Braga, Portugal, 1996, and Leuven/Louvain-la-Neuve, Belgium, 1999),

(b) Two books based on the elaboration of papers which were presented during the satellite meetings of HPM (History and Pedagogy of Mathematics, one of the ICMI affiliated international groups), the first edited by R. Calinger (MAA 1996), and the second edited by V. Katz (MAA 2000),

(c) The ICMI Study book on "History in Mathematics Education", edited by J. Fauvel and J. van Maanen.

(d) Journals for Mathematics Teachers and/or Mathematics Education Researchers have published special issues on the History of Mathematics in Mathematics Teaching (e.g. *For the Learning of Mathematics* in 1991, *Mathematics in school* in 1998 and *Mathematics teacher* in 2000). The re-born newsletter of HPM (International Study Group on the Relations between History and Pedagogy of Mathematics) is becoming (we hope) a forum where piece of information and ideas are shared.

These material and the experiments carried out all over the world make further discussion on the *role of the History of Mathematics in Mathematics Teaching* both possible and necessary. In recent discussions the expression "integration of History in Mathematics Teaching" appears frequently. Which ideas are behind this expression? The main idea is that of using History as a mediator to pursue the objectives of Mathematics Education. This means that, these objectives, together with the study of the historical evolution of concepts should be analysed. This work has to be carried out by educators and historians in a collaborative way. Among the benefits, which are expected to result from this work, is the new perspective offered by History to consider students' difficulties in learning Mathematics. To make teachers active actors in this process we need to give a convenient place to the History of Mathematics in pre-service and in-service teacher education.

Members of the Panel:

-***Abraham Arcavi***, Department of Science Teaching, Weizmann Institute of Science, Israel.
-***Evelyne Barbin***, IUFM de Créteil, France
-***Fulvia Furinghetti***, Department of Mathematics, University of Genoa, Italy.
-***Man-Keung Siu***, Department of Mathematics, University of Hong Kong, Hong Kong SAR, China.
-***Constantinos Tzanakis***, Department of Education, University of Crete, Greece.

ON THE ROLE OF THE HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION

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ABSTRACT

In recent years important works on the relationship between history and mathematics education have appeared. Some of them, such as the proceedings of the European Summer Universities in History and Pedagogy in Mathematics education, the HPM satellite meetings of ICMI Conferences, the French publications of IREM, are evidence of rather regular activities in the field. The re-born newsletter of HPM (International Study Group on the Relations between History and Pedagogy of Mathematics) is becoming (we hope) a forum where piece of information and ideas are shared.

These materials and the experiments carried out all over the world make it possible to go further in the discussion about the role of the history of mathematics in mathematics teaching. In the recent discussions a word is appearing frequently: integration [of history in mathematics teaching]. What behind this word? The main idea is that of using history as a mediator to pursue the objectives of mathematics education. This means to develop an analysis of these objectives together with the study of the development of concepts in history. This work has to be carried out by educators and historians in a collaborative way. In the present paper we show how the preceding ideas may be applied in introducing a concept of infinitesimal analysis.

1. Introduction

In the recent years important works on the relationship between history and mathematics education have appeared. Often they are the results of initiatives particularly addressed to teachers, such as the proceedings of the European Summer University (held in 1993, 1996, and 1999). Other times they are the output of meetings among researchers (historians, mathematicians, educators), such as the two books originated by the HPM satellite meetings of ICMI conferences (1996 editor R. Calinger, and 2000 editor V. Katz), the ICMI Study book edited by J. Fauvel and J. van Maanen (2000), the book *Learning from the masters!* edited by F. Swetz, J., Fauvel, O., Bekken, B., Johansson, & V. Katz (1995), the proceedings of the Brazilian meetings *Encontro Luso-Brasileiro de história da matemática & Seminario Nacional de história matemática*, the book *History of mathematics and education: ideas and experiences* edited by H. N. Jahnke, N., Knoche and M. Otte (1996),

Journals for mathematics teachers have published special issues on the history of mathematics in mathematics teaching (e.g. *For the learning of mathematics* in 1991 and 1997, *Mathematics in school* in 1998 and *Mathematics teacher* in 2000).

Particularly impressing is the net of publications (mainly in French) edited by the French University Institutes for teacher education (IREM): they constitute a kind of common thread in the development of the subject "The history of mathematics in mathematics education".

The *Newsletter of HPM* (International Study Group on the Relations between History and Pedagogy of Mathematics, affiliated to ICMI) informs three times a year the readers about a range of initiatives (conferences, meetings, exhibitions) and publications concerning the history of mathematics in mathematics education.

Eventually I like to point out the importance of the new information and communication technology in establishing a new relationship with history, especially for those people as teachers, who had difficulties in finding the suitable materials. As illustrated in (Barrow-Green, 1998), the access to historical sources, to biographical information and references is now more available than in the past to everybody.

In the publications that I have mentioned we may find attempts of answering the central question "What is the role of the history of mathematics in mathematics education?". This question may be split into more focused sub-questions:

- which educational benefits are introduced by the history of mathematics?
- which teaching strategies are to be applied?
- how mathematics teachers are prepared to this introduction?

In theory, these issues are the same as those faced by researchers in mathematics education or curriculum developers when introducing innovations in mathematics teaching. I am thinking, in particular, at the introduction of information and communication technology. I may explain this similarity reminding the view of historiography as a "literary artifact" expressed by Hayden White, as reported by Eco (1994, p.161). Extending this concept we may say that as the technology, history too is an artifact which intervenes in teaching. As an artifact it may play the role of mediator in the process of teaching/learning. Of course in these similarities there are differences specific to the specific object of study, but at a first level we may take the plan in Fig.1 as the common path when using different mediators (e.g. history, technology).

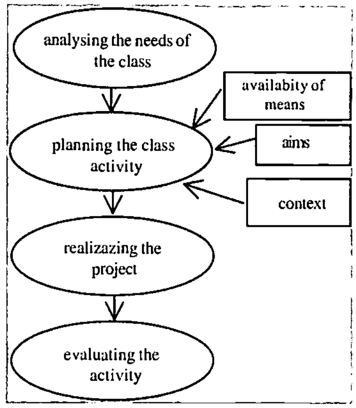

Figure1. Plan for the implementation of a teaching sequence

In the case of history the striped zone has to be specified according to the plan of Fig.2.

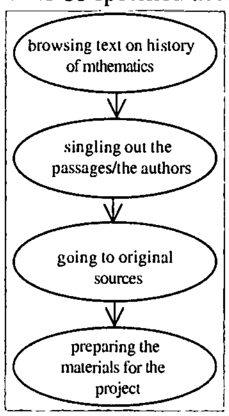

Figure 2. Plan for introducing the history of mathematics in a teaching sequence

Of course, there are variations to this plan such as going directly to the sources, if one has a suitable knowledge of the history of mathematics. The point is that the choice of passages/authors (striped zone of Fig.2) has to be carried out in the light of the educational needs.

2. An example

I give an example of this way of working by outlining the features of a project on which I have worked myself with two secondary teachers. The subject was the introduction of derivative. Our main concern was the poor concept images held by undergraduate students. To focus on the elements that may intervene in the formation of this concept image we designed a questionnaire addressed to students. The questionnaire consisted of 14 questions related to the derivative, each question containing four options plus an option allowing comments. The full work is reported in (Boggiano, Furinghetti & Somaglia, submitted). The questionnaire was given to the students of the scientific lycei of Genoa (big town) and two little towns near Genoa. All together we analyzed 434 questionnaires. The findings show that the students answer in a satisfying manner when they resort to prototypes, but fail in facing new situations. Moreover the questions containing graphics bring to light the weakness of concept images held by students, since graphics require an *active* and *aware* construction of mathematical objects. Also it emerges the students' weakness in passing from the algebraic to the geometric domain and vice versa. We may say that derivative is one of the mathematical object to which students connect manipulation of formulas, but not mathematical meaning.

Thus the problem is to recover the mathematical meaning. The plan illustrated in Fig.3, taken from (Furinghetti & Somaglia, 1998), shows the steps we use to bridge the gap between informal and formal mathematics.

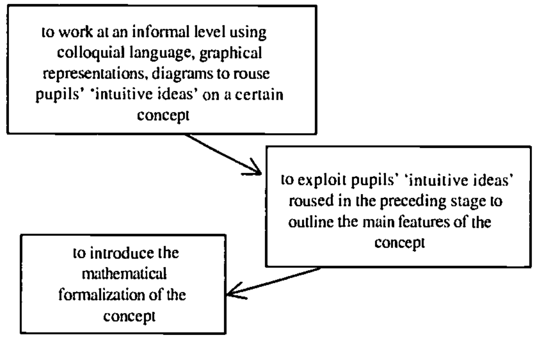

Figure 3. Steps from informal to formal mathematics

To make students work at an informal level before tackling a given topic formally allows the reification of concepts. Sfard (1994) ascribes a central role to the birth of metaphors, as explained in the following passage

> If the meaning of abstract concepts is created through the construction of appropriate metaphors, then metaphors, or figurative projections from the tangible world onto the universe of ideas, are the basis of understanding. [...] the leading type of sense-rendering metaphor in mathematics is the metaphor of an ontological object. (p.5)

For us to work at the informal level means to work in a world which is close to the students' experience, i.e. the "tangible world" mentioned by Sfard. My position is in line with Freudenthal's ideas on the efficacy of context problems as an opportunity to let formal mathematics emerge. As explained in the paper (Gravemeijer & Doorman, 1999), context problems have to be intended in a broad sense as "problems on which the problem situation is experientially real to the student" (p.111).

In this framework to use history may reveal itself fruitful and sense-carrier. In our project it was considered the pioneering period at the beginning of calculus, where the roots of the mathematical entities in the world of material objects are more visible. The tangent line to a curve was taken as the first step in the construction of the derivative. Other authors have tried this way, see, for example, (Grégoire, 2000; Villareal, 1997).

Passages from original sources were proposed to the classroom. One was taken from *Observations sur la composition des mouvements et sur le moyen de trouver les touchantes des lignes courbes* by Gilles Personne Roberval (1602-1675). Already in 1644 Marin Mersenne informed the scientific community about a method by Roberval to find tangents based on kinematics. The manuscript containing the method was written by a pupil of Roberval (Du Verdus) and presented to the *Académie des Sciences* by Roberval only in 1668.

The method holds if the kinematic generation of the curve is known, and thus only particular curves may be treated with this method. The author assumes that the direction of the movement of a point on a curve is the tangent to the curve in any position of this point. The parallelogram law for the addition of constant velocity vectors was already known. Roberval applied this law to instantaneous velocity vectors. From the specific properties that define the curve Roberval finds the components of the movement and afterwards the tangent as the composition of them, see Fig.4. A discussion of the Roberval's method may be found in many texts, see, for example, (Edwards, 1979).

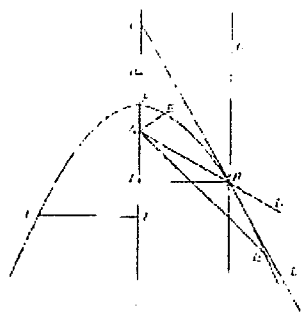

Figure 4. Kinematic construction of tangents (Roberval)

There are two aspects of the chosen extract that make it close to the ideas about the use of history that I explained before: geometry and movement. Both these aspects are part of students' experience: geometry mainly belongs to school experience, movement to everyday experience. The construction may be applied to other curves. We have chosen the second order parabola, since we wish that students work on a well known curve, applying its definition in an operative way.

The passage by Roberval was available in Italian in a reliable translation taken from one of the few readers published in Italian (Bottazzini, Freguglia, Toti Rigatelli, 1992). Thus we bypassed the problem of translation, which is one of the main problems in using the history of mathematics in teaching. Jahnke et al. (2000) distinguish at least two types of translation:

> translation into modern mathematical language, and translation from one language into another. While the former serves in particular to reconstruct a mathematical argument, the latter has promising educational advantages insofar as it initiates students and trainees into mastering a language and to conceptual analysis. (p.316)

Usually to have to deal with a foreign or dead language (Latin, Greek) is a great difficulty which takes teachers away from using history.

3. Conclusions

I have outlined the basic ideas that I see behind the use of history in mathematics teaching. To simplify my discussion I made the choice to skip the big problem of teacher education in history. I'm aware that this problem exists: it is not by chance that a full chapter of the ICMI Study on the use of the history in mathematics teaching is devoted to this subject (Fauvel & van Maanen, 2000). As far as I know the related problem of teachers' attitude is less investigated, see (Philippou & Christou, 1998). I think that the discussion on my model may be a starting point both to encourage teachers to approach history as a mediator in their work and to make plans for teacher training (pre-service and in-service).

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*Panel **ICMI Study on The Teaching and Learning of Mathematics at Undergraduate Level***

Derek HOLTON (Coordinator)
University of Otago, Dunedin, New Zealand

ABSTRACT

A short history of the Study will be given to set the background for a deeper discussion of three of the main areas of the Study.

Educational Research: One of the goals of the Study was to determine what educational research carried out at this level of formal education had to offer; to evaluate the researches potential to help us understand better the observed problems and to offer strategies for tackling these; and to identify the current limitations of research and suggest orientations for its future.

Practice: Recent changes in undergraduate mathematics teaching have been in response to external factors that impinge on the teaching of the discipline, as well as a result of different epistemological views of mathematical learning. Several innovative teaching approaches were highlighted in the Study. These include new approaches to teaching topics of a traditional curriculum, as well as attempts to redefine the nature of undergraduate mathematics teaching and learning.

Technology: Innovations in this area affect both curriculum and pedagogy. Much of the Technology area of the Study centred on the use of technological tools for supporting students learning, particularly via visualisation, computation, and programming both during and after formal lecture time. Consideration was given to technologies potential to foster more active learning, to motivate explanations of surprise feedback, to foster co-operative work and to open a window on students thought processes.

Members of the Panel:

- ***Michèle Artigue***, Université Paris 7, Paris, France
- ***Derek Holton***, University of Otago, Dunedin, New Zealand
- ***Joel Hillel***, Concordia University, Montreal, Canada
- ***Alan Schoenfeld***, University of Berkeley, California, USA

Panel: ICMI Study on The Teaching and Learning of Mathematics at Undergraduate Level

Michèle ARTIGUE, Université Paris 7, Paris, France
Joel HILLEL, Concordia University, Montreal, Canada
Derek HOLTON, University of Otago, Dunedin, New Zealand
Alan SCHOENFELD, University of Berkeley, California, USA

1. Introduction

The Study began in 1997 with the first meeting of the International Programme Committee. Their Discussion Document appeared in the ICMI Bulletin of December 1997. Somewhat surprisingly, we completed on time, all the Study goals outlined on the timeline and, in addition, produced an extra publication (marked with an asterisk below). The main items on the timeline were

- December 1998: Study Conference, Singapore;
- Special issue of the International Journal of Mathematical Education in Science and Technology, Volume 31, No. 1, 2000*;
- Presentation of main findings 2000, ICME-9, Makuhari, Japan;
- Study Volume, The Teaching and Learning of Mathematics at Undergraduate Level, Kluwer Academic Publishers, Dordrecht, 2001.

We list below some of the main questions raised in the Discussion Document.

- What research methods are employed in mathematics education? What are the major research findings of mathematics education?
- Are the educational theories that are relevant at school level, relevant at university level as well?
- What do we know about the learning and teaching of specific topics such as calculus and linear algebra?
- What alternative forms of assessment exist? How can assessment be used to promote better learning and understanding?
- What are the effects of the use of technology in the teaching and learning of mathematics?
- To what extent do potential teachers of school mathematics, scientists, engineers, etc., need specially designed courses?
- What changes are, or should be, taking place in the curriculum?

Most of the questions raised were discussed in the two publications that have arisen form the Study. We take up issues related to mathematical research, practice and technology for this panel.

2. Educational Research

Some of the goals of the Study were to determine what educational research carried out at this level of formal education had to offer; to evaluate the research's potential to help us understand better the observed problems and to offer strategies for tackling these; and to identify the current limitations of research and suggest orientations for its future.

Research in mathematics education carried out at the university level helps us better understand the learning difficulties our students have to face, the surprising resistance of some, and the limitations and dysfunction of some of our teaching practices. Moreover, in various cases, research has led to the production of teaching designs that have been proved to be effective, at least in experimental environments. It has also been the source of specific theoretical frames. This is well evidenced by the section 3 of the ICMI Study Book and elsewhere. But the Study also shows that the research carried out up to now has been restricted in its cover. For instance, efforts have been concentrated on a few areas of the subject and on the training of future mathematicians or teachers. The Study also shows that, up to now, the influence of research on university teaching remains quite limited. This phenomenon cannot only be explained by the limitations of current research noted above and the Study allows us to better understand this limited impact. For instance, it shows us to that we are unlikely to get substantial gains without more engagement and expertise from teachers and significant changes in practices. One essential reason is that what has to be reorganised is not only the content of teaching but more global issues such as the forms of students' work, the modes of interaction between teachers and students, and the form and content of assessment. This is not easy to achieve and is not just a matter of personal good will. Another crucial point is the complexity of the systems in which learning and teaching take place. Because of this complexity, the knowledge that we can infer from educational research is necessarily partial. The models research can elaborate are necessarily simplistic ones. We can learn a lot even from simplistic models but we cannot expect that they will give us the means to really control didactic systems. As evidenced by the Study, the current links between research and practice do not allow research to play the role it could play. Improving these links is a necessity but has not to be considered as the sole responsibility of researchers. It is the common task of the whole mathematical community.

3. Practice

Recent changes in undergraduate mathematics teaching have been in response to external factors that impinge on the teaching of the discipline, as well as a result of different epistemological views of mathematical learning. Several innovative teaching approaches were highlighted in the Study. These include new approaches to teaching topics of a 'traditional' curriculum, as well as attempts to redefine the nature of undergraduate mathematics teaching and learning.

A fairly accurate picture of current undergraduate mathematics is that, by and large, it is still dominated by the 'chalk-and-talk' paradigm, a carefully selected linear ordering of course content, and assessment which is heavily based on a final examination. Even the highly publicised 'computer revolution' has not really made a sweeping impact on mathematics. The agenda is still basically defined by pure mathematics and one can reasonably claim that as long as the primary goal of mathematics education is conceived in terms of preparing future professional mathematicians, existing curricula function optimally if they just keep abreast of new developments within mathematics. Nevertheless, there are many calls from the general scientific community and professional associations of mathematicians and users of mathematics, to overhaul undergraduate mathematics education. This overhaul might include: goals, epistemology, learning styles, motivational issues, technology, and breadth of training.

In practice, it turns out that actual trends tend to be more modest and depend very much on the contexts and goals of the institutions involved. Changes are most discernible in departments that consider the goal of training future mathematicians as being too narrow, too expensive, or simply unrealistic in terms of who is actually enrolled in their programmes. Rather, they see their goals nowadays as being both academic and vocational. Certain trends however, can be seen. These include:

- Some departments are becoming more explicit about their aims and objectives for courses and for programmes as well as in describing a desired 'profile' of a student completing each of their programmes.
- There is a general trend towards reducing the mathematical content of courses, both for programme and client students.
- There is also an increased emphasis on applications and computer simulations both in main-stream mathematics courses and in courses targeted for client students.
- The transition problem from secondary to tertiary level has led to the appearance of bridging courses aiming to facilitate students' entry into university mathematics.
- The one-maths-course-for-all model is giving way to customised courses for different clientele.
- Though assessment is still dominated by the end-of-year exams there is a move towards a more varied assessment based on projects, weekly tests, essays, report writing, and seminar presentation, and group projects.
- Joint degrees, traditionally in mathematics and physics, have now given way to degrees such as mathematics and finance, mathematics and ecology, mathematics and information technology.

4. Technology

Innovations in this area affect both curriculum and pedagogy. Much of the Technology area of the Study centred on the use of technological tools for supporting students' learning, particularly via visualisation, computation, and programming both during and after formal 'lecture' time. Consideration was given to technology's potential to foster more active learning, to motivate explanations of 'surprise' feedback, to foster co-operative work and to open a window on students' thought processes.

A range of questions was raised by the working group on technology. Some of these questions are listed below. They were discussed to various degrees in the Study volume.

- How can you use technology to teach theoretical concepts?
- Does current literature make convincing arguments for using technology?
- How should the curriculum be reorganised to make effective use of technology?
- How does technology change mathematics (what is considered mathematics, how it is done)?
- How do we characterise teacher-student interactions with technology (the Internet, calculators, computers)?
- Should we focus on the current curriculum and how to integrate technology into it or should we consider what the mathematics curriculum could be now we have technology?
- How do we manage computers and calculators efficiently in the classroom?
- What strategies (e.g., starting with a black box and exploring) do we have for using technology to teach mathematics?
- How do we design technology and build it into the curriculum?

Panel Why School Mathematics matter: A Cross-Country (TIMSS) Examination of Curriculum and Learning

William SCHMIDT (Coordinator)
College of Education,
Michigan State University USA

ABSTRACT

We will present TIMSS data examining the relationship of curriculum to mathematics learning at the eighth grade. Data from 31 countries will be used to explore through formal statistical modelling the relationship among the three aspects of curriculum and learning. The four aspects of curriculum include measures of a country's content standards, textbook emphases, emphasis on the more complex cognitive demands of materials and the time allocations of the teachers. The dependent variables in the analyses are the gain scores in twenty specific topic areas such as congruence and similarity; functions; and 3-D geometry. By using gain scores the analyses focus on the mathematics that was learned during eighth grade, which then is related to the measures of the eighth grade curriculum. The patterns and relationships that emerge are discussed from a mathematics point of view. A panel of mathematicians from several countries will then discuss the implications of theses results both generally and in terms of the perspective of their own countries.

Members of the Panel:

- ***Johann Engelbrecht***, University of Pretoria, South Africa
- ***Curtis McKnight***, University of Oklahoma, USA
- ***Oh Nam Kwon***, Ewha Women's University, Korea
- ***William Schmidt***, Michigan State University USA
- ***Tosun Terzioglu***, Sabanci University, Turkey

Panel "Mathematics Is For All"

Coordinator: William Yslas Velez, Professor of Mathematics and University Distinguished Professor, Department of Mathematics, University of Arizona, Tucson, Arizona, USA

ABSTRACT

As mathematicians we believe that mathematics is useful, beautiful, and necessary in order to address the scientific problems that society confronts. We would all like to have a citizenry that is mathematically literate. Yet, many of us complain about the small number of students who choose to study mathematics in college or to choose mathematics for their major. Interestingly, there have been considerable efforts at increasing these small numbers and these efforts have been directed at sections of the population that have not historically participated in the mathematical enterprise. The purpose of this panel is to learn about these efforts and how to integrate these efforts into the culture of a university mathematics department.

Every country has "minority" populations that do not participate fully in the mathematical enterprise in that country. Minority populations oftentimes have to overcome more barriers than the majority population, barriers that stand in the way of the full expression of latent mathematical ability. These barriers take on many forms. Preparatory schools may not fully prepare students for the rigors of a university curriculum. The lack of financial resources is a common impediment. Social structures may prohibit the consideration of a mathematical career. The lack of knowledge about mathematical careers certainly plays a factor. Perhaps even the organizational structure of the university should factor in. One of the goals of this panel is to explore these impediments.

Concern for these under-represented groups sometimes results in special efforts or programs to address this inequity. These special efforts and programs are designed to encourage minority populations to gain access to mathematical careers. In many instances, minority mathematicians have led the efforts and have devoted a considerable portion of their careers in an effort to provide better access to the under-served. The mathematical community can learn a great deal about increasing access to mathematics by looking at minority programs. Efforts aimed at improving access for minority populations can also increase access for all students, and that is another goal of this panel.

A common dictum in the United States is that "Mathematics is for all". It is the goal of many pre-college programs in the U.S. to have all students complete a solid program of study in mathematics, one that will prepare them to pursue a mathematically based career in college. When we look at the professorate in mathematics departments at our research universities in the U.S., it is abundantly clear that the professorate is not representative of the U.S. population. The phrase, "mathematics is for all", does not appear to apply at the level of university professor of mathematics. The percentage of women is nowhere near equity. Historically, there were three main minority groups in the U.S., African-Americans, Mexican-Americans and Native Americans. These minority populations are almost invisible among the professorate at research universities in the U.S.

This panel will provide the opportunity to learn about these special efforts to increase the participation of minority populations in mathematics. Panelists will be invited to provide examples of the work that they have done to increase the accessibility, for minority groups in their countries, of mathematics and mathematics-based careers. Examples will be chosen that will give full evidence that these efforts have a broader appeal and, when incorporated into the way a mathematics department functions, will serve to increase the interest in mathematics in more students, not just minority students.

Members of the Panel[1]:

- ***Megan Clark***, Centre for Mathematics and Science Education School of Mathematical and Computing Sciences Victoria University, Wellington, New Zealand
- ***Cyril Julie***, School of Science and Mathematics Education, University of the Westem Cape, South Africa
- ***William Yslas Velez***, Department of Mathematics University of Arizona, Tucson, Arizona, USA

[1] Members of the panel at the time of publication (April 2002).

CONVEX SETS AND HEXAGONS

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ABSTRACT

Euclid presented his fundamental results about 300 B.C., but Euclidean Geometry is still alive today. We studied the new properties of convex sets and its inscribed hexagons in a two dimensional Euclidean space. As an application, these results solved a question in Geometry of Banach Spaces. From my teaching experience at Community College of Philadelphia, I think the material is reasonable and suitable to be added to the Linear Algebra course and/or Functional Analysis course. It may encourage others to know that the tools we give our students remain useful in modern research.

1 Introduction

In [1], we used elementary geometry to discuss the properties of the rhombi inscribed in the unit circle C of a two dimensional normed vector space, and proved that the well-known property from Euclidean geometry, namely that every rhombus inscribed in unit circle C has sides of C-length $\sqrt{2}$, does not characterize the Euclidean space. The result is that if the curve C of unit vectors is invariant under rotation by 45^o, then every rhombus inscribed in C has sides of C-length $\sqrt{2}$. In the first part of this paper we still use elementary geometry to discuss the properties of so-called normal hexagons inscribed in the unit circle C of a two dimensional normed vector space, and we consider another well-known property from Euclidean geometry, namely that every normal hexagon inscribed in an unit circle C has side-medians of C-length $\frac{\sqrt{3}}{2}$. However, we also prove that this property does not characterize the Euclidean space either. By using the term side-median for a polygon inscribed in the unit circle C of a normed vector space, we mean the median of the triangle with the origin as a vertex and a side of the polygon as base. In the second part of this paper, which is an appendix, we present more properties of rhombi inscribed in the unit circle C we discussed in [1].

2 Inscribed Hexagons

As we have already shown in [1]: we can use any bounded convex set which is symmetric with respect to the origin and contains the origin as an interior point in a two dimensional Euclidean space to define a new norm. On the other hand, the unit disk of any normed vector space is a bounded convex set which is symmetric with respect to the origin and contains the origin as an interior point.

Definition: A hexagon in a normed vector space with unit circle C is called a normal hexagon if it has six sides of same C-length, and each pair of opposite sides are parallel. The normal hexagon is called a unit normal hexagon if it has six sides of C-length 1.

The unit circle of the standard Euclidean space E^2 is a standard circle, and there is unique regular hexagon inscribed in the standard circle with a given point on the standard circle as the one of its vertices. From [2], for any invertible matrix A we can define an inner product on E^2 by $< x, y >= Ax \cdot Ay$, and every inner product arises in this way. Under the linear isometry $x \longrightarrow A^{-1}x$, the image of the standard Euclidean unit circle is the unit circle C of unit vectors with respect to the inner product, which is an ellipse, and the image of any regular hexagon inscribed in the Euclidean circle is a normal hexagon inscribed in this ellipse C. Since the unique regular hexagon in the standard Euclidean circle has sides of Euclidean length 1, and six side-medians of Euclidean length $\frac{\sqrt{3}}{2}$, it follows that the unique normal hexagon inscribed in an ellipse C with a given point as one of its vertices has sides of C-length 1, and side-medians of C-length $\frac{\sqrt{3}}{2}$.

The question is: does the property above characterize the Euclidean space? That is, if a normed vector space has the property that every normal hexagon inscribed in C of unit vectors has side-medians of C-length $\frac{\sqrt{3}}{2}$, does the norm arise by an inner product?

Observe that the two dimensional standard Euclidean space E^2 and a two dimensional normed vector space with C as its unit circle are set up in the same plane. In the following, for a given vector x in the plane we use $|x|$ to denote the general Euclidean length (in the Euclidean space) and $\|x\|_C$ to denote the C-length (in the normed vector space). Let K and $C = \partial K$ be the unit disk and unit circle of the two dimensional normed vector space respectively, then both K and C are symmetric with respect to the origin, in addition K is a convex set with the origin as an interior point. So, geometrically the question above is equivalent to the following question: if a convex set K, which is symmetric with respect to the origin and contains the origin as an interior point (and therefore $C = \partial K$ could be the unit sphere of some normed vector space), has the property that every normal hexagon inscribed in $C = \partial K$ has side-medians of C-length $\frac{\sqrt{3}}{2}$, must C be an ellipse in E^2 (Therefore $C = \partial K$ should be the unit sphere of an Euclidean space)?

To answer this question we need the following results.

Let T be a tangent line of K, then $T \cap K = T \cap C$ is either a single point or a line segment with $||T \cap K||_C = ||T \cap C||_C \leq 2$.

Lemma 1: Let $x \in C$, T be the tangent line parallel to the vector x, and L be a line parallel to x too. Then when L moves parallel from the position passing through the origin towards T, the $||L \cap K||_C$ is non-increasing from 2 to $||T \cap K||_C = ||T \cap C||_C$. Furthermore, for any a, where $||T \cap K||_C = ||T \cap C||_C \leq a < 2$, there is unique $u \in C$ and corresponding $v \in C$ such that vector $u - v$ is parallel to x, and $||u - v||_C = a$.

Proof: Let L_1 moves parallel to L_2 towards T, and $u_1, v_1 \in L_1 \cap C, u_2, v_2 \in L_2 \cap C$ (see Figure 1). If $||u_2 - v_2||_C > ||u_1 - v_1||_C$, or $||u_2 - v_2||_C \geq ||u_1 - v_1||_C < 2$, then at least one of u_1, v_1 falls inside the trapezoid with vertices $-x, x, u_2$, and v_2.

This contradicts the convexity of K. Therefore $||u_2 - v_2||_C \leq ||u_1 - v_1||_C$, or when $||u_1 - v_1||_C < 2, ||u_2 - v_2||_C < ||u_1 - v_1||_C$.

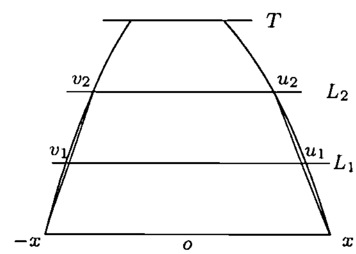

Figure 1:

Lemma 2: Let $x \in C$, then there exists at least one normal hexagon inscribed in C with x as one of vertices.

Proof: Let T be the tangent line to C, and parallel to x. If $||T \cap K||_C = ||T \cap C||_C \leq 1$ (see Figure 2), from lemma 1 we can take $u, v \in C$, such that $u - v$ is parallel to x, and $||u - v||_C = 1$. From parallelograms with vertices u, v, o, and x, and vertices $u, v, -x$, and o, we have $||u - x||_C = ||v||_C = 1$, and $||v - (-x)||_C = ||u||_C = 1$. So, the hexagon with vertices $x, u, v, -x, -u$, and $-v$ is an inscribed normal hexagon.

¿From lemma 1 again there is unique u and corresponding $v \in C$ such that $u - v$ is parallel to x, and $||u - v||_C = 1$. So, in this case the inscribed hexagon with x as one of vertices is unique.

If $||T \cap K||_C = ||T \cap C||_C > 1$ (see Figure 3), we can take infinite many pairs of $u, v \in T \cap K = T \cap C$ such that $u - v$ is parallel to x, and $||u - v||_C = 1$. So, in this

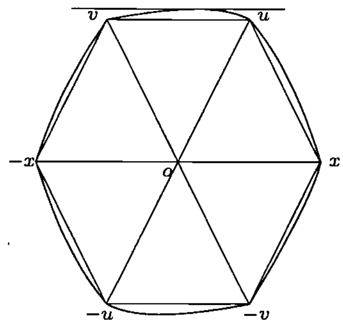

Figure 2:

case there are infinite many normal hexagons inscribed in C.

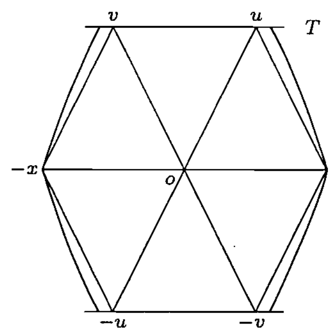

Figure 3:

Consider a normed vector space in a plane with a normal hexagon as the unit circle C, then the normal hexagon itself is an inscribed normal hexagon in C. It has side-medians of C-length 1, but it is not a inner product space.

Lemma 3: Let x be a vector in C, and x, u, v, and $-x$ in C are counterclockwise located, then $||v-x||_C \geq ||u-x||_C$, and $||v-(-x)||_C \leq ||u-(-x)||_C$.

Proof: Let u' and v' be the normalizations of $u-x$, and $v-x$ respectively. Then u' and $v' \in C$. If $u' = v'$, then u, v, and x are colinear. So $||v-x||_C \geq ||u-x||_C$. Otherwise x, u', v', and $-x$ are counterclockwise located too (see Figure 4).

Case 1: If the line L_v passing through v and v' intersects the line L_x through $-x$ and x at a point Q, and Q is on left side of $-x$, then $||v-x||_C > 1$ (see Figure 5). The L_u passing through u and u' is either parallel to the line L_x (in this case $||u-x||_C = 1$, therefore $||v-x||_C \geq ||u-x||_C = 1$), or intersects L_x at a point P. If P is on the right side of x, then $||u-x||_C < 1$ (therefore $||v-x||_C \geq ||u-x||_C$). If P is on the left side of $-x$, then from the convexity of K, P must be on the left side of Q. By considering similar triangles with vertices u, x, P and vertices u', o, P, we have $||u-x||_C = \frac{|Px|}{|Po|}$. Similarly from similar triangles with vertices v, x, Q and vertices v', o, Q, we have $||v-x||_C = \frac{|Qx|}{|Qo|}$. Since $\frac{|Px|}{|Po|} \leq \frac{|Qx|}{|Qo|}$, we have $||v-x||_C \geq ||u-x||_C$.

Case 2: If line L_v is parallel to line L_x , then $||v-x||_C = 1$ (see Figure 6). ¿From convexity of K, the line L_u either intersects line L_x on the right side of x, or L_u is

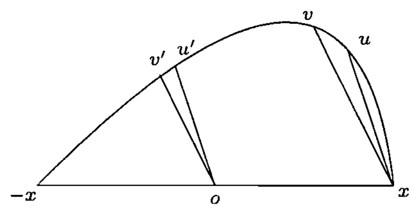

Figure 4:

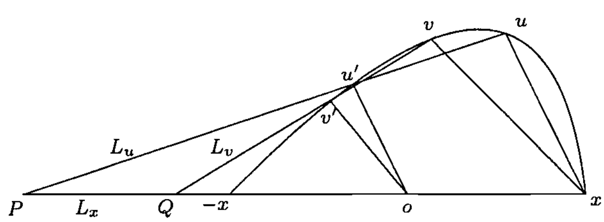

Figure 5:

parallel to L_x so $||u-x||_C \leq 1$. We still have $||v-x||_C \geq ||u-x||_C$.

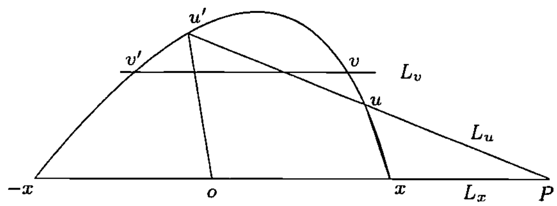

Figure 6:

Case 3: If the point Q, the intersection of line L_v and line L_x is on the right side of x, then $||v-x||_C \leq 1$ (see Figure 7). From convexity of K again, the point P, the intersection of L_u and line L_x, is either on the left side of Q, or coincides with Q. Similar to case 1, by considering the similar triangles we have $||v-x||_C = \frac{|Qx|}{|Qo|} \geq \frac{|Px|}{|Po|} = ||u-x||_C$.

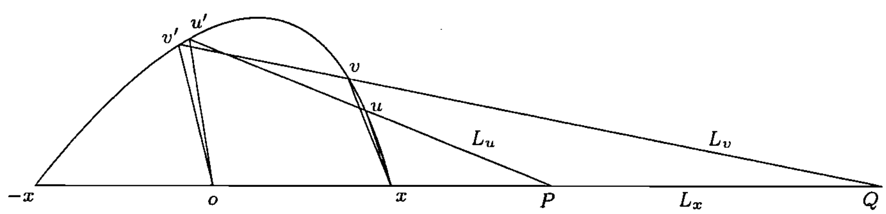

Figure 7:

Similarly, we can prove $||v-(-x)||_C \leq ||u-(-x)||_C$. The proof of lemma 3 is completed.

Lemma 4: If the curve C of unit vectors is invariant under rotation by 30^o, then C does not contain any line segment with C-length greater then or equal to 1.

Proof: Suppose $u, v' \in C$ such that $||v'-u||_C \geq 1$, and the line segment L connecting u and $v' \subseteq C$. If $\angle v'ou < 30^o$, take a vector v such that $\angle vou = 30^o$, and $|ov| = |ou|$, then $v \in C$, and by lemma 3 $||v-u||_C \geq ||v'-u||_C \geq 1$. If $\angle v'ou \geq 30^o$, take $v \in L$ such that $\angle vou = 30^o$. From the hypothesis, $|ov| = |ou|$, and the line segment $[u, v]$ connecting u and v coincides with L. So we have $v' = v$ and therefore $||v-u||_C \geq 1$. Let $w = \frac{u+v}{2}$, then $w \in K$, and $||w||_C \leq 1$ (see Figure 8). Let $t = v-u$, then $||t||_C = ||v-u||_C \geq 1, |t| = |u-v| = 2|u|\sin 15^o$, and the angle between t and w is 90^o. Let s be the image of rotating w couterclockwise by 90^o, then $||s||_C \leq 1$, and $|s| = |w| = |u|\cos 15^o$. Since $\cos 15^o > 2\sin 15^o$, we have $|s| > |t|$. But $||t||_C \geq ||s||_C$. This is a contradiction. The proof is completed.

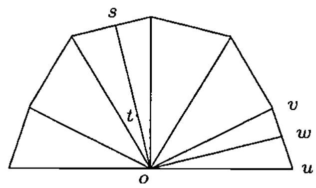

Figure 8:

Theorem 1: If the curve C of unit vectors is invariant under rotation by 30^o, then every normal hexagon inscribed in C has side-medians of C-length $\frac{\sqrt{3}}{2}$.

Proof: Let u be a vector in C. Since C is invariant under rotation by 30^o, C does not contain any line segment with C-length greater than or equal to 1. Therefore the normal hexagon inscribed in C with u as one of its vertices is unique (lemma 2). Let u_1, u_2, u_3, and u_4 be the vectors obtained by turning u counterclockwise by successive steps of 30^o (see Figure 9). Then the hexagon with vertices $u, u_2, u_4, -u, -u_2$, and $-u_4$ is the unique normal hexagon inscribed in C.

We have $||\frac{u+u_2}{2}||_C = \frac{|\frac{u+u_2}{2}|}{|u_1|} = \frac{|\frac{u+u_2}{2}|}{|u|} = \frac{\sqrt{3}}{2}$. Similarly, we have $||\frac{u_2+u_4}{2}||_C = ||\frac{u_4+(-u)}{2}||_C = ||\frac{(-u)+(-u_2)}{2}||_C = ||\frac{(-u_2+(-u_4)}{2}||_C = ||\frac{(-u_4)+u}{2}||_C = \frac{\sqrt{3}}{2}$. The proof is completed.

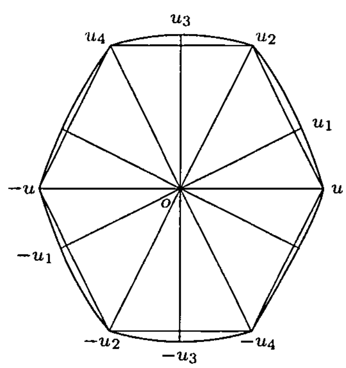

Figure 9:

So ellipses are not only curves C with the property that every inscribed normal hexagon in C has side-medians of $\frac{\sqrt{3}}{2}$. A regular polygon with 12n sides in particular a regular twelvegon will satisfy the condition. Therefore the image of any regular polygon with 12n sides under any invertible linear map has this property too. Equivalently we have proved that the property that every normal hexagon inscribed in C of unit vectors has side-medians of C-length $\frac{\sqrt{3}}{2}$ does not characterize the Euclidean space.

3 Appendix

In the second part of this paper we study more properties of inscribed normal parallelograms, rhombi, in the unit circle C.

We have already proved the uniqueness of rhombus inscribed in the curve C of unit vectors with a given point of C as a vertex in [1]. Now we prove the existence of this kind of rhombus.

Theorem 2: There is a rhombus inscribed in C of unit vectors with a given point of C as a vertex.

Proof: Let $x \in C$, from lemma 3 when u moves from x to $-x$ counterclockwise, $||u - x||_C$ continuously increases from 0 to 2, and $||u - (-x)||_C$ continuously decreases from 2 to 0. So there exists $y \in C$, such that $||y-x||_C = ||y-(-x)||_C$. The Parallelogram with vertices $x, y, -x$, and $-y$ is a rhombus inscribed in C, with a given point x as a vertex.

Finally, by combining theorem 1 of [1] and the theorem 2 above, we have the following theorem.

Theorem: There is one and only one rhombus inscribed in C, with a given point in C as a vertex.

4 Discussion

In this paper, the question we posed: a conjecture about the characteristic of Euclidean spaces belongs to the subject of the Geometric Functional Analysis. All figures which appeared: hexagons, circles, ellipses, symmetric convex sets belong to Elementary Geometry, the course students studied at high schools and/or in a freshman level at colleges. The concepts and methods which we need to prove the lemmas and main theorem: linear vector spaces, norm and normed vector spaces, Euclidean spaces, and linear transformations belong to Linear Algebra, the course we are teaching. Based on the knowledge in Elementary Geometry, all the concepts and methods about linear spaces and linear transformations, which make one of the most important parts of the Linear Algebra course are needed to prove the lemmas and main theorem. After my lectures students learned that the basic figures in Elementary Geometry have meaning in the Geometry of Banach spaces they never imagined: different Ellipses are unit spheres of different Euclidean spaces, and different symmetric convex sets are unit spheres of different normed spaces and so on. And students also learned that the concepts, methods and results in Linear Algebra course are useful and powerful in proving results in more advanced mathematical courses. The students told me that they understood better and deeper what the definitions of the abstract spaces really mean, relations among topics in the different chapters of the course, and learned how to think mathematically, and how to use their knowledge in practice. They also told me that they were inspired by my lectures to do research, and they recognized the tools they acquired in the classroom remain useful in modern research.

So lectures on this subject in my Linear Algebra course help students to review the Elementary Geometry, to enhance the understanding of the Linear Algebra course, and encourage them to study Real and Functional Analysis in the future. I think the material of this paper is suitable and reasonable to be added to current Linear Algebra course and/or Functional Analysis course.

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USING MATHEMATICA IN TEACHING ROMBERG INTEGRATION

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ABSTRACT

High order approximations of an integral can be obtained by taking the linear combination of lower degree approximations in a systematic way. One of these approaches for 1-d integrals is known as Romberg Integration and is based upon the composite trapezoidal rule approximations and the well-known Euler-Maclaurin expansion of the error. Because of its theoretical nature, students in a classical Numerical Analysis course usually find it difficult to follow. In order to overcome the difficulty, Mathematica software is utilized to illustrate the method, and the underlying theory. A Mathematica program and a set of experiments are designed to explain the method and it's intricacies in a stepwise manner. The program is expected to help the student to learn and apply the method to 1-d finite integrals. However, with minor modifications, it is possible to extend the method to multi-dimensional integrals.

1. Introduction

The Romberg integration is the problem of approximating the integral below using the linear combinations of well-known trapezoidal sums T_i^1's in a systematic way in order to achieve higher orders in an effective manner.

$$I = \int_a^b f(x)\,dx\,, \qquad a,b \in \Re,\ \ f \in C^k[a,b]$$

The method is based on the Euler-Maclaurin asymptotic error expansion formula and the Richardson extrapolation to the limit (Joyce 1971). Romberg, a German mathematician, (Romberg 1955) has been the first to organize the Richardson's method in a systematic way suitable for automatic calculations on the computer in 1955.

Geometrically speaking, the value of I is the area under the curve of $y=f(x)$ bounded by the x-axis, and the lines x=a, and x=b. T_1^1 is the area of the trapezium and approximates the value of I as shown in Figure 1 below.

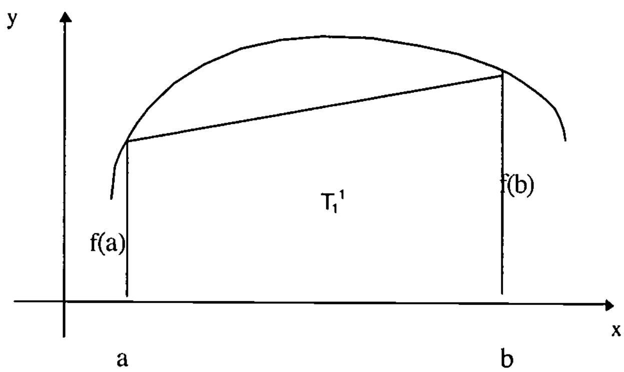

Figure 1 Basic trapezoidal computation T_1^1 over [a,b].

Each trapezoidal sum is defined as

$$T_i^1 = \frac{b-a}{2^i}[f[a] + f[b] + 2\sum_{j=1}^{2^{i-1}-1} f[x_j]]$$

for i=1,2,...,n (n ≡ maximum level of subdivision), $x_j = x_0+jh$, j=1,2,...,i and h = $(b-a)/2^{i-1}$. Note that for the ith subdivision of the interval x_0 = a, and x_i = b. The computation starts with T_1^1 on the interval [a,b], and T_2^1, T_3^1, and so on are computed by successively halving the interval and applying the basic rule T_1^1 to each subinterval formed. In this subdivision process the Romberg sequence {1,2,4,8,16,...} is utilized. Other subdivision sequences are also possible (Yazýcý 1990).

For example, after the computation of T_1^1 as shown above, the interval of integration is bisected and the second composite approximation T_2^1 to I is formed as shown below. Obviously, as the number of subintervals increases a better, although same order of, approximations to I are obtained.

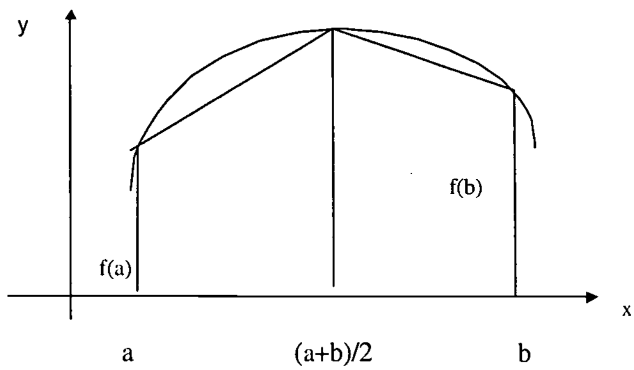

Figure 2 Composite trapezoidal sum T_2^1 over 2 subintervals.

Once the composite Trapezoidal sums are available, so-called Romberg table can be formed.

$$\begin{array}{cccccc}
T_1^1 & & & & & \\
T_2^1 & T_2^2 & & & & \\
T_3^1 & T_3^2 & T_3^3 & & & \\
T_4^1 & T_4^2 & T_4^3 & T_4^4 & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
T_n^1 & T_n^2 & T_n^3 & T_n^4 & \cdots & T_n^n
\end{array}$$

via

$$T_i^j = \frac{4^{j-1}T_i^{j-1} - T_{i-1}^{j-1}}{4^{j-1}-1}, \quad i = 2,3,\cdots,n \quad and \quad j = 2,3,\cdots,i .$$

It is known that the entries in the second column of the table are composite Simpson's approximations to the same integral (Burden & Faires 1985). The third column entries are also composite approximations based on the Newton interpolatory formulae. The consecutive columns have no resemblance to any known method based on interpolation. The trapezoidal rule is of polynomial order one. That is, trapezoidal sums are exact whenever the integrand *f(x)* is a first-degree polynomial in x. Provided that the 1st column entries converge to I, all diagonal sequences over the table converge to I as well (Kelch 1993). Moreover, if column k is of order p then the column k+1 entries are of order p+2. This could easily be justified using the asymptotic error expansion formula:

$$I - T_i^1 = c_1h^2 + c_2h^4 + \cdots + c_kh^{2k} + O(h^{2k+2}) \quad , \quad h = \frac{b-a}{2^{i-1}} \ .$$

The c_i's are constants (based on the Bernoulli numbers) independent of h. This is an even expansion in powers of h. The linear combinations formed by the Romberg procedure causes the c_i's vanish one by one. Obviously, h approaching to zero (application of the composite rule for smaller and sma ller values of h) suggests that T_j^1 converges to I. For singular integrals this expansion is not valid and it takes different forms depending on the nature of singularity (Lyness & Mc Hugh 1970).

In order to show the way c_i's vanish when the linear combination of the composite values are formed, the expansion formula above is applied with two different step sizes h_1=b-a (original interval size), and h_2=(b-a)/2, (interval size after the first bisection) to obtain

$$I - T_1^1 = c_1h_1^{\,2} + c_2h_1^{\,4} + \cdots + c_kh_1^{\,2k} + O(h_1^{\,2k+2}) \quad , \quad h_1 = b-a$$
$$I - T_2^1 = c_1h_2^{\,2} + c_2h_2^{\,4} + \cdots + c_kh_2^{\,2k} + O(h_2^{\,2k+2}) \quad , \quad h_2 = \frac{b-a}{2}$$

Multiplying both sides of the latter by 4 and subtracting from the first, and rearranging the resulting equation, one gets

$$I - \frac{4T_2^1 - T_1^1}{3} = -\frac{1}{4}c_2h_1^{\,4} + O(h_1^{\,6})$$

which shows that the first error term of the expansion vanishes and the linear combination of T_1^1 ,and T_2^1 , (T_1^2 = [4T_2^1 - T_1^1]/3), produces a higher order approximation to I.

2. Romberg Integration with Mathematica

It is the feeling of the authors that, in learning the Romberg integration, students face some difficulties in understanding the rational behind the method. The discussion over the asymptotic error expansion and Euler-Maclaurin series and convergence makes the presentation more complicated. Working out the details of the derivations and combinations of the composite rules and the formation of the Romberg table is time consuming, if not boring. Instead, a simple symbolic program could be quite beneficiary to show all the details and derivations. Such an approach will give the student a chance to play around with the formulas and observe easily the relation between the composite sums, order of an approximation and the high orders achievable by forming the simple linear combinations.

A text-based Mathematica (Burbulla & Dodson 1992) is used to develop the program below:

romberg/: ***romberg[f_,{a_,b_},n_]:=*** (

1. Define h and initialize other variables

   h = b - a;

2. Generate array t for composite sums (to maximum level 10)

   Array[t,10];

3. Apply the basic trapezoidal rule to f

```
t[1] = h/2 (f[a] + f[b])
```

4. Create array x to hold abscissas of the points generated as a result of subdivision. The newly generated nodes (x, •, and +) utilized by t[2], t[3], and t[4] as depicted by the Romberg subdivision sequence are illustrated in Figure 3.

```
Array[x,512];
```

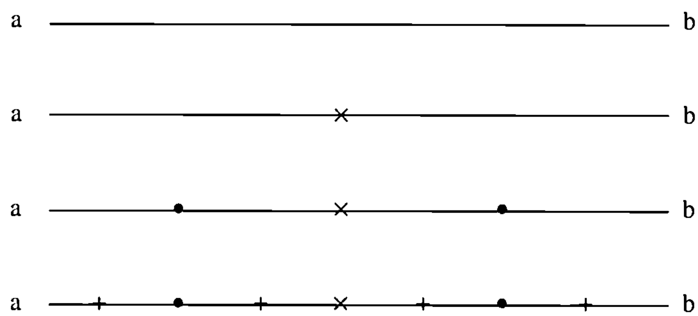

Figure 3 Nodes of subdivision at levels 1,2,3, and 4

5. Compute composite trapezoidal sums

```
For[m = 2, m <= 10, m++,
  k = m - 1;
  Do[ x[j] = a + (j-1)h/2^k , {j , 1 ,  2^k+1}];
  t[m] = h/2^m ( f[a] + f[b] +2 Sum[f[x[j]],{j , 2 , 2^k}] )
```

6. Define the Romberg Extrapolation table r (10x10 matrix) and initialize its first column to t

```
Array[ r , {10,10}]
For[m = 1 , m <= 10 , m++ , r[m,1] = t[m]]
```

7. Form the Romberg table using the first column entries

```
For[ i = 2 , i <= 10 , i++ ,
  For[ j = 2 , j <= i  , j++ ,
    r[i,j] = (4^(j-1) r[i,j-1] - r[i-1,j-1] )/(4^(j-1)-1)]]
```

Once, this program is made available to the student, the method can be investigated for a symbolic function f over [a,b] in an effective manner by calling the subprogram **romberg** with f for, say, 10 levels of subdivision as

```
f[x_] := g[x]
romberg[f,{a,b}, 10]
```

Romberg integration uses the so-called Romberg sequence $R = \{1,2,4,8,16,..., 2^k,...\}$ to subdivide the interval. Other subdivision sequences are also possible and may reduce the number of function evaluations for the same accuracy. However, Romberg sequence provides full overlapping of the nodes of integration, i.e., all the nodes at level k of subdivision are included in level k+1. This idea is incorporated in Step 5 above by replacing t[m] by a recursive definition as follows:

```
For[m = 2, m <= 10, m++,
```

```
k = m - 1;
For[j = 1, j <= 2^(k-1), j++,
Do[x[j] = a + (2j-1)h/2^k , {j, 1, 2^k+1}];
t[m] = t[m-1]/2+ h/2^k(Sum[ f[x[j]], {j, 1, 2^(k-1)}])
```

3. Experiments

A sample Mathematica session is set up to demonstrate the power of the Romberg integration for a general function *f*. The following instructions are to be carried out after setting up the definitions above.
Experiment 1: Set up the first trapezoidal approximation t[1] to I over [a,b].

In[1] := t[1]

$$\mathbf{Out[1]} = \frac{(-a+b)(g[a] + g[b])}{2}$$

Experiment 2: Set up the composite trapezoidal rule t[2] over 2 sub-intervals.

In[2] := t[2]

$$\mathbf{Out[2]} = \frac{(-a+b)\,(g[a] + g[b] + 2\,g[a + \frac{-a+b}{2}])}{4}$$

Experiment 3: Set up the composite trapezoidal rule t[3] over 4 sub-intervals.

In[3] := t[3]

$$\mathbf{Out[3]} = (\,(-a+b)\,(g[a] + g[b] + 2\,(g[a + \frac{-a+b}{4}] + g[a + \frac{-a+b}{2}] + g[a + \frac{3(-a+b)}{4}])))/8$$

Experiment 4: Simplify the expression

In[4] := Simplify[%]

$$\mathbf{Out[4]} = \frac{(-a+b)\,(g[a] + g[b] + 2\,g[\frac{a+b}{2}] + 2\,g[\frac{3a+b}{4}] + 2\,g[\frac{a+3b}{4}])}{8}$$

Experiment 5: Set up the first Romberg value as a linear combination of t[1] and t[2] and observe that this is identical to Simpson's approximation over [a,b].

In[5] := Simplify[(4 t[2] - t[1])/3]

$$\textbf{Out[5]} = \frac{(-a+b)\,(g[a] + g[b] + 4\, g[\frac{a+b}{2}])}{6}$$

Experiment 6: The Romberg table is generated and stored in the two-dimensional array r. Compare Out[5] with the value of r[2,2].

In[6] := Simplify[r[2,2]]

$$\textbf{Out[6]} = \frac{(-a+b)\,(g[a] + g[b] + 4\, g[\frac{a+b}{2}])}{6}$$

Experiment 7: Display the value of r[3,2] (Simpson's rule applied to 2 sub-intervals)

In[7] := Simplify[r[3,2]]

$$\textbf{Out[7]} = \frac{(-a+b)\,(g[a] + g[b] + 2\, g[\frac{a+b}{2}] + 4\, g[\frac{3a+b}{4}] + 4\, g[\frac{a+3b}{4}])}{12}$$

Experiment 8: Display the value of r[3,3] (First entry in the third column of the Romberg table). Observe that this is also an approximation based on the Newton interpolatory formula. The subsequent columns have no resemblance to any known formulae based on interpolation.
In[8] := Simplify[r[3,3]]

$$\textbf{Out[8]} = \frac{(-a+b)\,(7\, g[a] + 7\, g[b] + 12\, g[\frac{a+b}{2}] + 32\, g[\frac{3a+b}{4}] + 32\, g[\frac{a+3b}{4}])}{90}$$

Experiment 9: Compute the integral below numerically by displaying the value of r[6,6]. Compare the result with that of Mathematica's build-in function **Integrate**.

$$\int_0^\pi Sin[x]\,dx = 2$$

(* DEFINE f *)

In[9] := f[x_]:= Sin[x]

(* DEFINE END POINTS OF INTEGRATION *)

In[10] := a =0

In[11] := b = Pi

(* DISPLAY SEVERAL ROMBERG TABLE VALUES *)

In[12] := r[2,2]//N

Out[12] = 2.0944

In[13] := r[4,4]//N

Out[13] = 2.0001

(* COMPUTE ACTUAL VALUE AND DISPLAY ERROR *)

In[14] := actual = Integrate[Sin[x],{x,0,Pi}]

Out[14] = 2

In[15] := err = Abs[actual - r[6,6]] // N

Out[15] = 1.32072 10^{-12}

The values of the Romberg table, r[i,j]'s, computed by the program, are as follows:

0					
1.5708	2.0944				
1.89612	2.00456	1.99857			
1.97423	2.00027	1.99998	2.00001		
1.99357	2.00002	2.	2.	2.	
1.99839	2.	2.	2.	2.	2.

Experiment 10: As discussed earlier, the basic Trapezoidal rule is linear and therefore integrates first-degree polynomials exactly, and each Romberg column doubles the order of approximation. To investigate this let f be x^7, over [0,1/2], and observe that r[4,j] is exact (1/2048 = 0.000488281).

In[16] := f[x_] := x^7

In[17] := a = 0

In[18] := b = 1/2

(* CALL ROMBERG WITH F OVER [A,B] *)

In[19] := romberg[f, {a,b}, 4]

In[20] := r[4,4] // N

Out[20] := 0.000488281 (exact!)

The Romberg table produced by the execution of the subprogram is as follows:

0.00195312			
0.000991821	0.00671387		
0.00626326	0.00504494	0.000493368	
0.00523608	0.000489369	0.000488361	<u>0.000488281</u>

4. Justification of the Method

Romberg extrapolation method is based upon the existence of the asymptotic error expansion discussed in section 1. Mathematica can be used to illustrate how and why the method works by assuming such an expansion and symbolically deriving expressions that correspond to the entries of the Romberg table. For this purpose, let

In[21] := Array[c,4]

In[22] := e[h_] := Sum[c[i] h^(2i), {i,1,4}]

In[23] := x = (4 e[h/2] - e[h]) / 3

In[24] := y = (4 e[h/4] - e[h/2]) / 3

In[25] := Expand[Simplify[x]]

Out[25] = $\frac{-(h^{4}\ c[2])}{4} - \frac{5\ h^{6}\ c[3]}{16} - \frac{21\ h^{8}\ c[4]}{64}$

In[26] := Expand[Simplify[(16 y- x) / 15]]

Out[26] = $\frac{16\ c[3]\ h^{6}}{64} + \frac{21\ h^{8}\ c[4])}{1024}$

The last two results illustrate that the values in the second column of the Romberg table are $O(h^4)$ and the third column entries are of $O(h^6)$.

5. About the Error Term of the Trapezoidal Rule

Mathematica function **Series** can be used to verify the error term of the Trapezoidal rule given by

$$E = -\frac{h^2}{12}[f'(b) - f'(a)] + \frac{(b-a)h^4}{720} f^{(4)}(\mu) \quad , \quad \mu \in [a,b]$$

For this purpose, we investigate the error in the basic rule for the integral

$$s = \int_a^{a+h} Sin[x]\, dx = -Cos[a+h] + Cos[a]$$

```
(* DEFINE F AND CALL ROMBERG OVER [a,a+h] *)

In[27] := f[x_] := Sin[x]

In[28] := romberg[f,{a,a+h},10]

In[29] := t[1]

          h (Sin[a] + Sin[a+h])
Out[29] = ---------------------
                    2
In[30] := s = Integrate[Sin[x],{x,a,a+h}]

Out[30] = Cos[a] - Cos[a+h]


(* FIND THE ERROR IN t[1] *)

In[31] := e = Series[s - t[1], {h,0,3}]


                  3
           Sin[a] h        4
Out[31] = ----------  + O[h ]
              12

(* USING THE DEFINITION ABOVE FOR ERROR IN TRAP. RULE *)

In[32] := terror = -h^2/12 (Cos[a+h] - Cos[a])

                   2
          -(h  (-Cos[a] + Cos[a+h]))
Out[32] = --------------------------
                       12

In[33] := Series[terror,{h,0,3}]

                  3
           Sin[a] h        4
```

Out[34] = ------------ + O[h]
12

The values of Out[34] and Out[31] are shown to be identical verifying the dominant term of the error formula.

6. Computational Complexity of Romberg Integration

The complexity of any numerical integration algorithm based upon interpolation is mainly depicted by the number of integrand function evaluations at the nodes of integration of the numerical rule. Romberg extrapolation described in this study is no exception. An additional cost is incurred in this case in the formation of the Romberg table, which is negligible.

The Mathematica program discussed earlier in Section 2 is a static implementation of the algorithm, i.e., for a fixed subdivision level, say, **maxlevel**, all of composite Trapezoidal sums are computed first and then the Romberg table is formed. In this case, considering the overlapping of the nodes in bisecting the interval, each level n introduces 2^n additional integrand evaluations. In higher dimensions, this may result in too many function evaluations, and hence the method may not be computationally efficient. This could be avoided by forming the rows of the Romberg table dynamically. That is, at each level, rows of the table are completed by the Romberg formula and an error test is performed to check the accuracy of the diagonal value *r[n,n]*. Whenever, the error criteria is satisfied, the algorithm terminates avoiding further unnecessary subdivisions and function evaluations. Otherwise, next composite sum is to be formed by bisecting the interval one more time.

This idea can be easily incorporated into the Mathematica code given in this work. The dynamic implementation is given below.

```
dynamic_romberg/: dromberg[f_,{a_,b_},n_, tol_]:=
(h=b-a;
 Array[t,n];
 t[1]= N[h/2(f[a]+f[b])];
 Array[x,512]; Array[r,{n,n}];
 For[m=2,m<=n,m++,
   k=m-1;
   Do[x[j]= N[a+(j-1)h/2^k],{j,1,2^k+1}];
     t[m]= N[h/2^m(f[a]+f[b]+2 Sum[f[x[j]],{j,2,2^k}])]];
     For[m=1,m<=n,m++, r[m,1]=t[m]];
       reler = 1.;
       i=2;
       While[reler >= tol && i<n,
         For[j=2,j<=i,j++,
```

```
        r[i,j]=(4^(j-1)r[i,j-1]-r[i-1,j-1])/(4^(j-1)-1)];
      reler = Abs[(r[i,i]-r[i-1,i-1])/r[i-1,i-1]];
      Print["i=",i,",",r[i,i],
            "computed relative error=", reler];
      i++
   ];
)
```

A sample run and its output is given for the approximation of *f[x]=Sin[x]*, over [0,Pi/2].

```
f[x_]:=Sin[x]
dromberg[f,{0,Pi/2},10,0.00001]
```

i= 2 , 1.002280 computed relative error =0.276142

i= 3 , 0.999992 computed relative error =0.00228311

i= 4 , 1.000000 computed relative error =$8.44274\ 10^{-6}$

7. Comparisons and Conclusions

In this article, Romberg extrapolation technique is illustrated using the symbolic computing facility as provided by Mathematica. Main objective of this article is to facilitate symbolic computations in order to present a highly technical method in a simplified manner. Because of the nature of the work done, numerical calculations are mostly avoided. A brief comparison of different approaches to numerical integration is outlined below.

Romberg method is built on the trapezoidal rule that is based on the linear interpolation over the two points on the interval. Higher order interpolatory rules (Newton-Cotes type formulae) can be used for high order approximations. However, the coefficients of such rules alternate in sign causing loss of accuracy. Another class of integration rules are Gaussian type that uses coefficients based on the roots of certain orthogonal polynomials over the domain of integration. Gaussian type rules provide higher degrees of accuracy compared to Newton-Cotes formulae, however, amount of work done increases dramatically because of lack of overlapping during the subdivision of the interval to obtain composite sums. Monte Carlo methods involve generating random numbers over the domain of integration, and then computing the expected value (approximation to I) by simply averaging the function values at the randomly generated points. Monte Carlo methods are suitable for N-dimensional integration for its low cost compared to the rules mentioned before. For a detailed comparison of these methods the reader is referred to, for example, (Davis & Rabinowitz 1975).

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SERVICE WITH A SMILE

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ABSTRACT

In this paper I will discuss some aspects of *specialised service teaching*, by which I mean the teaching of mathematics to an identifiable group of students with a shared primary interest which is not mathematics. I will first argue for the vital importance of service teaching in general, not because of its budgetary implications for mathematics departments, but because of its role in ensuring the overall health of mathematics as a discipline. I will then examine two key issues concerning the teaching of specialised service courses, namely whether mathematicians should teach such courses, and if they do, how they should approach this task.

1. Introduction

In this paper I will be mainly concerned with what I will call *specialised service teaching*, by which I mean the teaching of mathematics to an identifiable group of students with a shared primary interest, which is not mathematics. A course in complex variable for engineering students would qualify, as would an introductory course in calculus for biology majors, but not a general introductory course in calculus for students including mathematics majors, or for all students other than mathematics majors. First, though, I want to say something about service teaching in its broadest sense, that is, the teaching of mathematics to students whose primary interest is not mathematics.

The issues I want to discuss fall under three headings:

- Why is service teaching important?
- Who should teach specialised service courses?
- How should mathematicians approach the teaching of specialised service courses?

2. Why is Service Teaching Important?

> *"Of all the resources which the human spirit possesses ... none is so momentous and so inseparable from our inner nature as the concept of number. ... Every thinking person ... is a number-person, an arithmetician".*
>
> J.W.R. Dedekind, undated manuscript

Some years ago I took part in a study of the mathematical needs of school-leavers in New Zealand. The particular aspect that I was involved in was the investigation of the mathematical needs of everyday life. The detailed conclusions that we reached are not relevant here. But the study brought home forcibly to me the fact that virtually everybody does mathematics frequently in the course of their daily lives, in many different contexts: shopping, completing tax returns, working out household budgets, calculating quantities for home decorating, playing games ... — the list is endless. The fact that everyone *does* mathematics makes it almost unique among academic disciplines; people may take an interest in history or geography, but they do not *do* it inescapably in their daily lives.

There is nothing new here, of course, we all know this. But the point I want to emphasize is that this is why society regards mathematics as deserving of a special place in the school curriculum — not because of the aspects of mathematics that we mathematicians regard as important. By most ordinary standards of importance, it is arithmetic and elementary geometry that are the most important parts of mathematics, not functional analysis or group theory.

Much the same can be said at the level of tertiary education. What gives mathematics a special place in tertiary education is the fact that it is needed by scientists, engineers, economists, sociologists ... — and again the list is endless. If it were not so, mathematics departments would be small groups teaching small classes of a few devotees.

Once again there is nothing new here. We all know that large service classes are a budgetary necessity for most mathematics departments, so of course service teaching is important! But that is not the point I want to make. If, as I argue, almost everyone does mathematics at least some of the time, then service teaching is important simply because it is the way almost all of those who do mathematics learn the subject. It is vital for the health of the discipline that it should be done well. If most of the people who do mathematics do so unwillingly, inexpertly and with feelings of dislike if not actual nausea, then mathematics is in a bad way. If on the other hand they do mathematics with a sense of enjoyment and view it as a friend rather than a foe, then we as teachers have done well and our subject will flourish at all levels.

The teaching of mathematics majors is of course essential for the continuation of the subject, but we do not need encouragement to be attentive to that aspect of our educational task. Service teaching, on the other hand, often risks being neglected because it is seen as a tiresome necessity, a digression from our main task of educating mathematicians. I believe that for service teaching to receive the attention it deserves, it needs to be seen for what it is — one of the most important things that we as teachers of mathematics do.

3. Who should teach specialized service courses?

It may not be so everywhere, but certainly in the university systems that I have worked in, the question of who should teach specialized service courses is a perpetual source of tension. Because of its budgetary implications, the question is all too often seen as a purely political one, but here I want to focus on the academic question. Who are the best people to teach such courses — the mathematics subject specialists or the specialists in the students' primary interest subject? We might like to say that mathematicians are the best qualified people to teach such courses, but what reasons can we advance to justify this?

The most obvious reason is that mathematicians are the experts in mathematics, and university students should be taught by experts. When it comes to teaching mathematics majors, this argument is conclusive. In the case of service courses, it remains valid, but the acknowledged expertise of mathematicians does bring disadvantages as well. As mathematicians, we see the subject from a particular viewpoint, which is not the same as the viewpoint of students in service courses. For example, a mathematician would probably see Fourier series as a special case of the general phenomenon of the representability of elements of a Hilbert space in terms of orthonormal bases. But if the students are electrical engineering students, they will see the subject in terms of signal processing and spectral analysis. Unless their mathematician teacher takes this into account, the students may feel (perhaps rightly) that they are being taught by someone who does not understand their needs. Again, mathematicians tend to be excited by singular cases and exceptions, which help to sharpen our understanding of the conditions under which various results hold good. But students in other disciplines care much less about such things since they seldom or never arise in practice. We need to keep a sense of proportion when teaching service courses and not get too carried away by "interesting" special cases, which are really of interest only to ourselves.

The second reason that might be advanced is that we are the experts on the teaching of mathematics. Here again it can safely be said that we are the experts on teaching mathematics to budding mathematicians (though even so we are not always conspicuously successful). We tend to take it for granted that this expertise will easily transfer to service courses, and are unimpressed by the doubts sometimes expressed by our colleagues from other disciplines. But when teaching service courses we are not teaching people like ourselves (or even people with ambitions to be like us). We need to keep reminding ourselves that while we may be teaching mathematics, we are not teaching mathematicians. Making our teaching acceptable to students who do not necessarily share our interest in mathematics is not easy. It may require us to take an interest in things non-mathematical, rather than assuming that the students have an interest in things mathematical.

The contrary case for leaving the teaching of specialized service courses to specialists in the discipline being served is of course made by turning the negative features of teaching by mathematicians into positive arguments for the contrary. The positive features of teaching by mathematicians will naturally then become arguments against the contrary! But the case against teaching by mathematicians is not without strengths and we certainly cannot simply dismiss it as ill-conceived.

In short, I do not think we can or should expect others to take it for granted that we are the people best fitted to teach service courses in mathematics. To prove our case we need to take such teaching very seriously and put in the effort required to overcome some of the handicaps that I have mentioned. This brings me to my last section.

4. How should mathematicians approach the teaching of specialized service courses?

I think one of the biggest problems facing mathematicians teaching specialized service courses is that the students tend to see both the teacher and the subject as alien. Advanced students in, say, engineering or ecology usually form a coherent group, attending many classes and laboratories as a group and getting to know the teachers in their chosen fields very well. By contrast the mathematician appears for a few hours each week and may well seem like a being from another world, particularly if the mathematics is obviously being taught from the point of view of a mathematician rather than an engineer or a biologist. Terminology and notation that is different from what the student sees in other subjects can increase the feeling that mathematics is an alien subject. To take a very simple example in connection with the teaching of engineering students: mathematicians (and textbooks on engineering mathematics written by mathematicians) invariably denote the solutions of $x^2 + 1 = 0$ by $\pm i$, while engineers (and textbooks on engineering mathematics written by engineers) denote them by $\pm j$. So students are immediately conscious of a distinction between the two worlds, yet there is really no reason why a mathematician teaching engineers should not adopt their notation.

You can probably guess what my proposed solution is: as far as possible, mathematicians teaching specialized service courses should try to see the subject from the point of view of the discipline being served. Now you may say: "But I am a mathematician, not an engineer or ecologist or whatever. How can I not see the subject from the point of view of a mathematician?" — and of course there is some truth in that. But as professional mathematicians we are often confronted by problems brought to us by people outside mathematics, and in order to help them we have to understand their points of view and interpret our mathematical solutions in their terms. On the whole, I think we are pretty good at this, and their is no reason why we cannot do the same in our teaching. It does require some extra effort though: it is important to talk to practitioners of the other discipline and to read the textbooks that the students will use in their other subjects. Just using some of the terminology and notation that these textbooks use can make a big difference. And perhaps most important of all, the teacher should have or be willing to develop a genuine, even if only amateur interest in the other discipline. A service mathematics teacher who really has no interest in the discipline being served is not likely to be successful.

Let me give a few examples of what I mean:

(i) Textbooks on calculus for economics generally define concepts such as marginal cost, elasticity of demand and so on in terms of derivatives, give a brief explanation of their significance and then plunge into examples and exercises involving the calculation of these quantities for specific, often quite arbitrary, functions. This has its place, of course, but textbooks on introductory microeconomics do very little of this. The focus is much more on qualitative questions involving the interpretation of these quantities and effects of changes in them. Often the material in the mathematics text is of little direct help in understanding these matters. Yet it would not be difficult to incorporate such ideas into the mathematics course and thereby make it much more relevant to the students' real needs.

(ii) Functions of a complex variable are very important for engineers in connection with control theory. Textbooks on mathematics for engineers typically focus on residue theory, leading towards applications such as the evaluation of certain definite integrals. This is a mathematically beautiful theory, but it is of only marginal relevance to control engineers. Certainly they need to know about poles, but their interest is in the location of poles in connection with the stability and behaviour of control systems. There is plenty of interesting mathematics here, but it needs to be dug out of texts on control theory, not mathematics texts, and it tends to use its own specialized language. Time spend on finding out these things and incorporating them into a service course is well rewarded by having a much more motivated class.

(iii) Mathematicians may find themselves teaching a course to ecology students on the mathematical modeling of populations using differential equations. It is very easy to get carried away by the mathematical tidiness of the models involved and forget that real populations do not always behave in the tidy way predicted by our models. A look at texts and journal articles on ecology will provide plenty of material for a more critical look at the relevance and applicability of our models, surely just as important as training our students in the mathematical techniques, and probably more interesting for most of the students, since only a few will go on to become specialists in mathematical modeling.

To sum up: I have argued that service teaching is of the highest importance for the health of mathematics. I believe that we as mathematicians are the best people to do it provided we are prepared to make the effort to meet the students halfway. My experience is that if we do this, service courses can be immensely satisfying and enriching for both teacher and students.

CLASSIFYING STUDENTS' MISTAKES IN CALCULUS

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ABSTRACT

This article analyses some structural errors in calculus problems from first year mathematics undergraduates. They arise for reasons related to generalisation, intuition, inadequacy of concepts, instrumental understanding, problems of language and symbol manipulation. The lack of metacognitive control is also an important factor.

1. Introduction

There are many accounts of mathematical errors, which have an underlying logical explanation. In a pioneering study Brown and Burton (1978) catalogued many such errors in the domain of arithmetic. There have been similar studies since, for example Van Lehn (1980), and Maurer (1987). In the *Concepts in Secondary Mathematics and Science* project, reported in Hart (1981) misconceptions in other areas of school mathematics were investigated.

In many situations what appears to happen is that a procedure is learned instrumentally (Skemp 1976) in a way which does not reflect the underlying mathematical structure but which gives the correct answers in a particular set of examples. It is then extrapolated, but gives incorrect results, because of the structural mismatch, which the instrumental learning cannot adapt to. A common undergraduate example is

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \text{ (correct)} \quad \rightarrow \quad \frac{\partial y}{\partial x} = \frac{1}{\frac{\partial x}{\partial y}} \text{ (incorrect).}$$

Maurer's (1987) article, which discusses mainly subtraction, considers generalisation, and makes the point that this seems to happen purely syntactically, ignoring semantic considerations. Norman & Pritchard (1994) relate errors to Krutetskii's (1980) ideas about generalisation.

In undergraduate mathematics Orton (1983) discussed errors in basic calculus. Following Donaldson (1963) he focused on Structural Errors (as distinct from mistakes in calculation (Executive Errors) which nevertheless sometimes have a structural explanation). Orton's work concentrated on basic concepts and calculations in one-variable calculus. He explored things like limits, the meaning of dy and dx and the differential quotient, rates of change and turning points. In dealing with integration he looked at the integral as the limit of a sum, and at area and volumes of revolution. The examples he reported use simple polynomials, concentrating on basic conceptions. In this paper we explore structural errors occurring in first year university calculus arising as a result of procedural extrapolation as described above. We have chosen examples which emphasise algorithmic procedures, and which are some way beyond the basic ideas which Orton discussed.

Cipra (1989) gives examples of student errors in his book on mistakes in calculus, and suggests methods of checking and monitoring. This relates to the ideas of Schoenfeld (1985) concerning metacognitive *control*. Cipra does not analyse individual student errors to categorise them structurally, as will be discussed in this article. He gives some hypothetical explanations, for example for Fractional Inversion (p.61).

All the examples below were encountered in students' written work, or during problem classes. In the former case the written work was followed by discussions, where students explained their (erroneous) procedures. In problem classes one was able to interrogate students' thinking as they worked on problems. In the examples we give a condensed version of the students' solutions, and a résumé of their explanations, using their own language, to clarify the observations.

It is important to realise that these are not isolated errors. All the examples here were encountered during a one semester first year university calculus course. They are a small but representative sample, not only of that course but of many of the structural errors one has observed teaching calculus over many years.

The examples below are split into three categories: procedural extrapolation, pseudolinearity and equation balancing. These are not designed to be a definitive taxonomy, but to indicate that one can observe common features among the student errors one encounters.

2. Procedural extrapolation

We give three examples involving differentiation, and then two on integration, where the second has several integrals giving rise to similar errors.

Example 1: Find the first five derivatives of $f(x) = \exp(x + x^2)$.

Solution: $f'(x) = \exp(x + x^2); f''(x) = \exp(x + x^2);\ldots$ and so on: they are all the same.

Explanation: *Well, the derivative of the exponential function is always the same.*

Comment: The student has used the fact that the derivative of the exponential function is the exponential function. This has however been used as if it were a universal procedure. One can observe this particular extrapolation in many similar contexts. The students appear to be operating on the (exponential) function as an object, having lost sight of its process or action attributes (Thompson 1994, pp. 26-7). However, as Thompson points out

> " it is easy to be fooled - to think that students are reasoning about functions as objects when it is actually the function's literal representation (i.e. marks on paper) that are the objects of their reasoning."

In fact one might also refer to *oral* representation since they say *the exponential function* in their explanations. The kind of error in this example is encountered both when the function is written as e^{x+x^2} and also as $\exp(x + x^2)$, emphasising that the students associate a name (the exponential function) with what they see on paper, and then operate with the name (verbal symbol) and the properties they associate with that. Evidence from students' written work in the context of this error suggests that they also operate internally in this way. Subsequent discussions confirm that their internal verbalisation of their procedures follows the same pattern as that which they offer when they work "out loud".

Example 2: Find the Maclaurin expansion of $f(x) = \ln(1 + 2x)$.

Solution and explanation:

You need to work out the derivatives and then put x = 0. The first one is $f'(x) = \dfrac{1}{1+2x}$.

This is a fraction so you have to use the quotient rule. First you square the denominator. On top you have two terms. The first is the denominator times the derivative of the numerator, and there are no x terms so the derivative of that is 1. The second term is minus the numerator times the derivative of the denominator, which is 2. So you get

$$f''(x) = \frac{(1+2x)-2}{(1+2x)^2} = \frac{2x-1}{(1+2x)^2}.$$

You do the next one the same way, by the quotient rule

$$f'''(x) = \frac{(1+2x)^2 \cdot 2 - (2x-1).2(1+2x)\cdot 2}{(1+2x)^4}.$$

This is getting too complicated. It must be easier but I can't find a mistake.

Comment: This example was observed in a tutorial class with the student being asked to think out loud. The student had seen so many applications of the rule that he could not imagine the possibility that the denominator term should be multiplied by zero in applying the quotient rule. He was easily able to follow the alternative solution, writing the first derivative in the form $f'(x) = (1+2x)^{-1}$ and continuing by using the chain rule successively, but he could still not find his mistake. When it was pointed out that the derivative of his original numerator is zero and not 1 he was genuinely surprised, responding

So the quotient rule doesn't always have two terms on top?

He took a lot of convincing that he did have two terms, but one of them was zero.

But nought is the same as nothing. So there is really only <u>one</u> term is there?

Here the form of the result of the procedure as well as its description is being extrapolated.

Example 3: Find the first and second partial derivatives of $f(x, y) = \exp(x^2 y^2)$.

Solution: $\dfrac{\partial f}{\partial x} = 2xy^2 \exp(x^2 y^2)$; $\dfrac{\partial^2 f}{\partial x^2} = (2xy^2)^2 \exp(x^2 y^2)$. [The other second order partial derivatives were subject to the same error.]

Explanation: *When you use the function of a function rule the derivative of the exponential function is the same again. Then you differentiate what is in the brackets and so you multiply by* $2xy^2$. *You do the same again to work out the second partial derivative, so you multiply by another* $2xy^2$.

Comment: The first partial derivative has been calculated correctly. The problem seems to be that this procedure has been formulated in the instrumental form

multiply by $2xy^2$.

The first step does not involve the product rule and so the student performs the following steps by extrapolating the procedure used at step 1, namely

multiply by $2xy^2$, *and the exponential function is unchanged.*

Example 4: Evaluate the indefinite integral $\int x \cos x dx$.

Solution A: $\int x \cos x dx = \frac{1}{2}x^2(-\sin x)$.

Explanation: *Integration by parts is the reverse of the product rule for differentiation. In the product rule you differentiate both functions, so for integration by parts you must integrate both functions.*

Comment: What is interesting is that in this example many of the students making the error were able to apply the product rule for differentiation correctly during the course of discussion. What appears to be extrapolated here is not so much the procedure but an informal verbal description of the procedure. The linguistic register (Pimm, 1987, Chapter 4) has been shifted from that of mathematical English to everyday English, where the fuzziness of ordinary discourse is a factor.

Other students arrived at this kind of error by asserting that

the integral of a product is the product of the integrals.

Solution B: $\int x \cos x dx = \frac{1}{2}x^2 \cos x + x(-\sin x)$.

Explanation: This student also said that

integration by parts is the reverse of the product rule for differentiation.

She continued

For differentiation $(fg)' = gf' + fg'$, *so for integration*

$\int fg = g \times \int f + f \times \int g$.

Comment: As in example 2, preservation of form appears to be a factor here.

Example 5: Evaluate the following indefinite integrals:

$$\int\frac{1}{1+3x}dx;\int\frac{1}{1+x^2}dx;\int\cos(x^3)dx;\int(t^2+1)^{\frac{1}{2}}dt.$$

Solution:

$$\int\frac{1}{1+3x}dx=\frac{\ln(1+3x)}{3};\int\frac{1}{1+x^2}dx=\frac{\ln(1+x^2)}{2x};$$

$$\int\cos(x^3)dx=\frac{\sin(x^3)}{3x^2};\int(t^2+1)^{\frac{1}{2}}dt=\frac{(t^2+1)^{\frac{3}{2}}}{\frac{3}{2}\cdot 2t}.$$

Explanation: *If you differentiate ln(1+3x) you get the function we are supposed to be integrating, except for an extra 3, so we have to divide by the 3 to get the answer. The others are similar. In the second one "1 over" gives you a log again. This time if you differentiate ln(1+x²) you get an extra 2x, so you have to divide by it like we did in the first one. In the next one the integral of cos is sin, and this time if you differentiate you get an extra 3x², which you divide by. In the last one, if you integrate x^n you get x^{n+1} over n+1. But it isn't x, it's t²+1, so again if you differentiate you get an extra 2t, so this gets divided as well as the* $\frac{3}{2}$.

Comment: The students appear to have formulated the instrumental procedure "divide by the derivative of what is in the brackets". This works in the case when that derivative is a constant, where the "inner function " is *linear*. It is being extrapolated to situations where the inner function is non-linear. This is a very commonly encountered error. Many students display it, and their explanations usually follow similar lines to the one reported here. One can speculate as to how this extrapolation might be a consequence of instruction as follows. Students are urged always to check their integration by differentiating the result. When they first encounter differentiation of composite functions they are given examples like the first one, where the inner function is linear, in order to keep the calculations straightforward initially. For example the textbook Adams (1995) presents the Chain Rule in §2.5 (p.121), but prior to that in §2.3 there is a separate explanation "A Special Case of the Chain Rule" for derivatives of functions of the form $g(x)=f(ax+b)$. In the context of integrals like the first one in this example what they observe is that when they differentiate the inner function they always obtain what is in the denominator. The resulting cancellation gives the correct result. We have encountered students who have consciously extrapolated this aspect of the procedure when checking other results. For example in the second integral, when checking by differentiating $\frac{\ln(1+x^2)}{2x}$ they do not use the quotient rule. Instead they follow a sequence of steps which imitates what happens with $\frac{\ln(1+3x)}{3}$, so that the procedure

> *differentiate ln(1+3x) and you get 1 over (1+3x) times 3, which cancels with the 3 in the denominator*

is transformed to

> *differentiate ln(1+x²) and you get 1 over (1+x²) times 2x, which cancels with the 2x in the denominator.*

When the error is pointed out, it is not uncommon for students to query the first (correct) integration. Having been shown that a procedure they have used gives an incorrect result in one case, they feel that it must be wrong in all cases (a further extrapolation), as their own comments indicate.

As well as using written work, where students are asked for explanation some time later, situations like this have been observed directly in a tutorial setting. Students give verbal explanations along the lines reported here, and when asked "is that how you thought about it" respond affirmatively. In many cases the students provide observable evidence that their subvocal (self-talk) explanations (Pimm, 1987, p. 24) are very close to what they say out loud.

3. Pseudolinearity

A well known class of extrapolations is the erroneous use of linearity in such examples as
$\ln(x+y) = \ln x + \ln y$; $e^{x+y} = e^x + e^y$; $\tan(x+y) = \tan x + \tan y$;
$(a+b)^2 = a^2 + b^2$; $(\sin x + 1)^2 = \sin^2 x + 1$;
$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$; $\sqrt{4t^2+4} = 2t + 2$; $\sqrt{1+\frac{1}{x}} = 1 + \frac{1}{\sqrt{x}}$.

It is clear from discussions with students that they do not consciously think of the various functions involved (square root, tangent, etc.) as linear. Errors such as these occur before they encounter linearity in an overt, systematic manner as linear operators in differentiation and integration, linear transformations in linear algebra etc. One of the underlying possibilities is extrapolation of the distributive rule, and Norman & Pritchard (1994) label such examples unequivocally as generalised distributivity, as does Maurer (1987). One does in fact come across students who say

log times x plus y is log times x plus log times y.

One also encounters, in connection with the square root error for example, explanations like

well, when you do something to a + b you get the same as doing it to a and doing it to b. It's the same as with a times b.

Norman & Pritchard formulate this as $F(a*b) = F(a)*F(b)$, where $*$ is some binary operation (extrapolated from situations such as $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$, where it *is* true). In practice the situation is more complicated than just the single category of *distributivity* would imply. We find the use of the generalised rule $f(a*b) = f(a) \circ f(b)$, with different binary operations on each side, for example $\ln(a+b) = \ln(a) \times \ln(b)$, which is an erroneous extrapolation from the two (correct) relationships $\ln(a \times b) = \ln a + \ln b$ and $\exp(a+b) = \exp(a) \times \exp(b)$. The focus of attention seems to be the a and the b. These seem to be the primary objects, with the binary operations not being regarded as objects to the same extent. What is clear is that the binary operation plays a lesser role than the algebraic variable. With many of these errors students will spontaneously correct them when challenged. They often put it down to memory,

Oh! I never remember whether it's plus or times,

rather than the structural considerations above. This is not surprising, because they do not have the language to describe these things in structural terms.

As well as the rule $F(a*b) = F(a)*F(b)$ being applied when F is a real function, it is also applied when F is an operator such as differentiation. A typical example is the assertion that the derivative of a product is the product of derivatives.

Example 6

$$\frac{d}{dx}\ln\left(\frac{\sin x}{x}\right) = \frac{x}{\sin x}\left(x^{-2}\sin x\right)\left(x^{-1}\cos x\right)$$

Explanation:

Well because it's ln you have one over what's in the brackets. Then you have to differentiate the bracket, so it's easier to write it as $x^{-2}\sin x$. *So you have to differentiate the first and the second and multiply by the one over bit.*

Comment:

This student performed well on basic differentiation exercises using the product rule, and did not use pseudolinearity. However in previous exercises on the chain rule the "inner function" did not involve products or quotients, but simple polynomials or trigonometric or exponential functions. The expectation was therefore established that the answer would always be in the form of a product of expressions. In extrapolating this expectation we see that pseudolinearity comes to the surface again. The earlier exercises on differentiating products (and quotients) had not eradicated this deep-seated structural misconception.

It is sometimes unclear whether examples in this category are structural (e.g. application of linearity), or arbitrary (randomly mis-placing of the binary operations), in the sense of Orton and Donaldson. This would benefit from further research.

4. Equation balancing

How often do we emphasise in elementary algebra the principle "you do the same thing to both sides of an equation and they are still equal"? (Pimm, 1987, p.20)

Well here are some situations where the students' explanations involve this principle. What may be significant is that on many occasions in their comments the students replace the phrase "*to* both sides" with "*on* both sides".

Example 7:

$$\int \frac{1}{(1+u)^3}\,du = \ln|1+u|^3; \quad \int \frac{1}{x^2+4x+7}\,dx = \ln\left|x^2+4x+7\right|$$

Explanations:

$\int \frac{1}{(1+u)}\,du = \ln|1+u|$, *and you cube on both sides.*

$\int \frac{1}{x}\,dx = \ln|x|$, *only it's* x^2+4x+7 *on both sides instead.*

Comment: The students are not applying a general rule of the form $\int \frac{1}{f} = \ln|f|$, for if they are given an example where f is a trigonometric function they do not respond in this way. One does not find errors like $\int \cos\left(\frac{1}{x}\right)dx = \cos(\ln x)$, or $\int \frac{1}{\cos x}\,dx = \ln(\cos x)$ with anything like the frequency with which this type of error appears when simple polynomials are involved as in these two examples. So there are some limits to the extent to which procedural extrapolation occurs. (It is tempting to talk about a "Zone of Proximal *Extrapolation*", *à la* Vygotsky.)

Finally we have an example from the examination paper on the course from which all the errors in this article come.

Example 8: Find the Maclaurin expansion of $f(x) = \frac{1}{(1+2x)^{\frac{1}{4}}}$, by any method.

Solution:

$$\frac{1}{1+2x} = 1 + 2x + (2x)^2 + (2x)^3 + \ldots, \text{ and so}$$

$$\frac{1}{(1+2x)^{\frac{1}{4}}} = 1 + 2x^{\frac{1}{4}} + 4x^{\frac{2}{4}} + 8x^{\frac{3}{4}} + \ldots$$

Comment: In this case there wasn't a verbal explanation because it was an examination question, but the results of the application of the principles of equation balancing and pseudo-linearity can be clearly seen.

5. Conclusions

Many studies concerning students' mistakes analyse elementary mathematics (Brown and Burton (1978), Van Lehn (1980), Hart (1981), Maurer (1987)) or basic concepts of more advanced topics - Orton (1983).

In this study we have discussed mistakes relating to algorithmic processes in one variable calculus, lying beyond the basic principles. The study demonstrates that mistakes occurring here reflect structural errors, which Donaldson (1963) found in elementary mathematics. These involve confusion between object, action and process (Thompson (1994)), mis-application of language (Pimm (1987)), generalisation (Krutetskii (1980), Maurer (1987)), confusion between syntax and semantics (Norman & Pritchard (1994)), and inadequate metacognitive control procedures (Schoenfeld (1985)). This provides evidence that the types of error present in elementary mathematics continue into more advanced mathematics. This confirms the suggestions of Maurer (1987) and Norman & Pritchard (1994) that such structural errors cannot be avoided. In teaching mathematics we emphasise qualities such as flexibility, reversibility, generalisation and intuition, and so paradoxically it seemes that these very qualities can give rise to structural errors. From a constructivist viewpoint they will happen in the course of learners constructing their own meanings.

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SYMBOLIC MATH GUIDE : AN INNOVATIVE WAY OF TEACHING AND LEARNING ALGEBRA USING TI-89 AND TI-92+ GRAPHING CALCULATORS

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ABSTRACT

During the past decade, handheld graphers have fundamentally changed the teaching and learning of many school mathematics concepts – particularly those dealing with graphical representation and visualization (Demana and Waits, 1992). Graphing calculators have enabled many students to experience mathematics as a more dynamic, interactive, and visually-appealing area of study. Yet, because graphers have historically lacked symbol manipulation capabilities - relying on numerical approximations to calculate - their impact on the teaching and learning of *equation solving* and *symbolic manipulation* has been minimal. While many secondary school teachers and students use calculators to study graphs, they continue to examine algebraic manipulation using pencil-and-paper or chalkboard-based activities.

However, a powerful new generation of graphing calculators equipped with symbolic manipulation capabilities is likely to change this situation. Handheld *Computer Algebra Systems* (CAS) – including the Casio FX 2.0 and Hewlett Packard 49g – will likely prompt instructional changes that mirror those precipitated by handheld graphers a generation ago.

In the following article, the authors discuss features of *Symbolic Math Guide (SMG)*, a CAS designed for use with Texas Instruments TI-89 and TI-92+ graphing calculators. Unlike earlier CAS, *SMG* was developed primarily as a pedagogical teaching and learning tool for high school mathematics students – not a research tool for university faculty. In the first sections of this document, the authors present research findings suggesting a need for such pedagogically-oriented CAS. In subsequent sections, the authors provide sample calculator exercises that highlight *SMG*'s ability to simplify algebraic expressions, exploring differences between pedagogical and traditional CAS (e.g. *SMG* and TI-92 CAS). The calculator exercises are provided as an introduction to *SMG* for both teachers and researchers.

Keywords: Educational Technology, Graphing Calculators, Computer Algebra Systems, Algebra

1. Introduction

As educators, we must prepare our students and ourselves for new and exciting forms of technology that take the best of what we have to offer as teachers and apply it to our subject matter.

(Diem, 1992, p.109)

Today, we live in a world significantly different from that of only a generation ago. Over the past two decades, technology's influence on everyday life has been pervasive and powerful - challenging our notions of human interaction, communication, and learning. Incorporating previously unthinkable tasks into daily routine, technologies such as word processors, electronic mail and the internet have made life richer, more convenient, and more productive. In a similar way, handheld graphers have profoundly transformed many aspects of school mathematics. Graphing calculators have enabled students to experience mathematics as a more dynamic, interactive, and visually-appealing area of study. Graphing tools have heightened the importance of graphical representation and visualization in mathematics classrooms (Demana and Waits, 1992).

Despite the revolutionary role that graphing calculators have played in the past, their impact on the teaching and learning of *equation solving* and *symbolic manipulation* has been minimal. Because graphers have historically lacked symbol manipulation capabilities, many teachers have used the devices to study *graphical concepts* - while continuing to examine algebraic manipulation using more traditional *pencil-and-paper* activities. The introduction of a powerful new generation of graphing calculators (e.g. Texas Instruments TI-89 and TI-92+, Casio FX 2.0, Hewlett Packard 49g) promises to change this situation. Equipped with symbolic manipulation capabilities, these handheld Computer Algebra Systems (CAS) challenge popular notions of algebraic manipulation in school mathematics. While providing students with powerful means of investigating the richness of mathematical symbolism in more dynamic and interactive ways, they call into question the continued role of pencil-and-paper in school algebra instruction.

Although studies of CAS with secondary school students have existed since the early 1990's (Aldon, 1996; Hirlimann, 1996; Klinger, 1994), early investigations have typically taken place in school computer labs using CAS on desktop computers. Important distinctions exist among CAS studies using calculators and computers.

- CAS-equipped graphing calculators may be used in traditional classroom settings on an "as-needed" basis. Unlike school computer labs, the use of CAS-equipped calculators requires no interruption in classroom instruction and no special trips to a remote lab.
- Calculators are more portable and more convenient. Students can use handheld CAS tools in other classes or to do homework without installing additional computer software or hardware.
- CAS-equipped calculators integrate symbolic manipulation functionality within an environment with which many students are already familiar – that of graphing calculators.

Portable CAS-equipped devices have only recently begun to appear in school classrooms. Therefore, research studies involving the use of *CAS-equipped calculators* in school settings are not commonplace. Preliminary research involving the use of CAS-equipped calculators with secondary school students has indicated that the tools are useful as "conjecture building" devices (Edwards, 2001). However, research also indicates that CAS-equipped devices have a tendency to perform "too many steps" for novice algebra students, while employing symbolism that is unfamiliar or even contradictory to that found in school textbooks (Edwards, 2001). The findings of Edwards (2001) have findings suggested a need for CAS tools designed primarily as pedagogical teaching and learning tools – not as a tool for researchers and mathematicians. Texas Instruments *Symbolic Math Guide* (*SMG*) was developed to address issues such as these.

2. The Need for *Symbolic Math Guide*

During a year-long study of CAS use with secondary school students, Edwards (2001) found that CAS students were dissatisfied with emphasis on calculator-based methods when solving manipulation-intensive problems. CAS student dissatisfaction appeared to be related to the calculator's tendency to complete large portions of problems for students.

> I think most knowledge about math is learned through hand-written work. Hand-written work gives the student a visible and mental track of what work was done and how the problem is solved. Calculators don't always show the individual steps to solving equations (Mike Fine, second-year algebra student).

The screenshots highlighted in Figure 1 illustrates the results of entering the equation $\frac{x^3-x}{x+1}=\frac{x^3}{x}$ on the home screen of a TI-92.

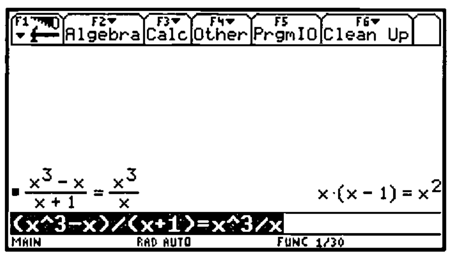


Figure 1: Steps automatically performed upon entering equation into TI-92 CAS

As Figure 1 suggests, the TI-92 homescreen CAS automatically performs the following tasks:

1. re-expresses x^3-x as $x\cdot\left(x^2-1\right)$
2. re-expresses $\left(x^2-1\right)$ as $\left(x+1\right)\cdot\left(x-1\right)$
3. re-expresses $\frac{\left(x+1\right)}{\left(x+1\right)}$ as 1
4. re-expresses $\frac{x^3}{x}$ as x^2

After a student decides to subtract x^2 from each side of the equation, the calculator automatically performs several more steps. These are highlighted in Figure 2.

1. Expands $x \cdot (x-1)$ as $x^2 - x$
2. Simplifies $(x^2 - x) - x^2$ as $-x$

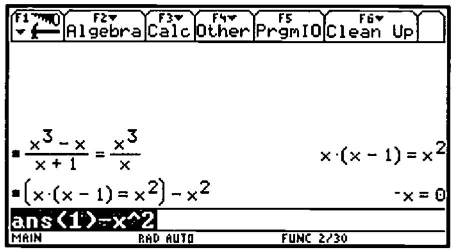


Figure 2: More calculations automatically performed by TI-92 CAS

In addition, Edwards' students complained that calculator notation differed significantly from notation typically found in school textbooks. Several major differences are highlighted in Figures 3 and 4.

For instance, unlike conventional mathematical text, in which algebraic steps are written one below the next, TI-92 output is read from left to right, then from top to bottom (like sentences in a book). This is shown in Figure 3.

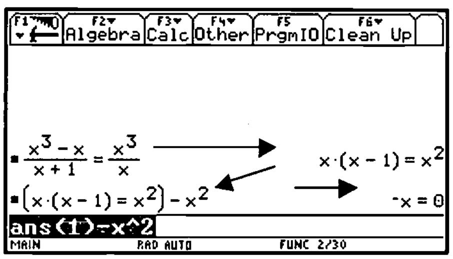


Figure 3: Algebraic output is read like "sentences in a book" on the TI-92 home screen

Additionally, the manner in which the TI-92 homescreen CAS simplifies expressions suggests to students that transformations are applied to *entire equations* (rather than to each side of an equation). This tendency caused confusion with novice algebra students. An example is provided in Figure 4.

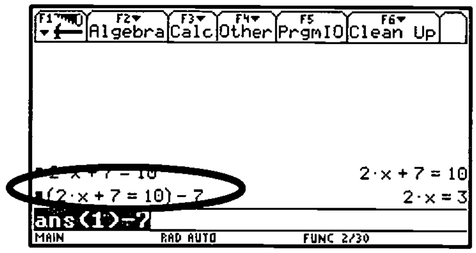


Figure 4: The TI-92 CAS applies a single transformation to an entire equation rather than performing separate transformations to each side of an equation

Edwards concluded the following at the end of his study:

> CAS based equation solving does not appear to support conceptual understanding to the same extent as traditional by-hand equation solving. The awkwardness of the TI-92 output as well as the calculator's tendency to perform "too many steps" automatically may have contributed to students' preference for by-hand methods. (Edwards, 2001, p. 299)

Traditional CAS were designed as tools for researchers – not as learning tools for young students. Thus, they tend to perform algebraic steps automatically – with little explanation provided to the user. In addition, CAS often display algebraic information in non-standard formats. Although these tendencies may suffice for university researchers who need fast answers and *already know significant mathematics*, they render CAS unsatisfactory as a learning tool for beginning algebra students. As we note in the following section, tools such as *SMG* provide students with access to the computational power of CAS, while at the same time providing an environment explicitly designed to *teach*, not confuse.

3. Features of Symbolic Math Guide

A primary purpose of *Symbolic Math Guide* (*SMG*) is to help students develop a deeper understanding of various algorithms used to solve algebraic manipulation-style problems. Unlike the raw symbolic manipulation utilities studied by Edwards (2001), *SMG* is more faithful to the mathematics and mathematical notation found in school textbooks. *Symbolic Math Guide* was built first and foremost as a pedagogical teaching tool - not an answer generator. The program encourages teachers and students to solve problems in a step-by-step fashion in a manner similar to traditional pencil-and-paper methods. Several features of *SMG* are listed below.

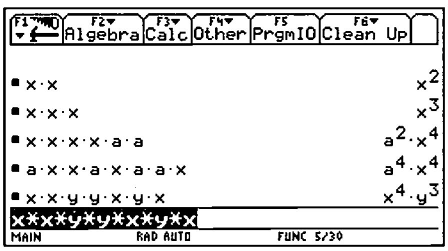
- Student exercises are organized by problem sets. Teachers, publishers, and students are able to create problem sets for particular lessons or activities. The sets may be easily shared online or in class.
- As they select algebraic steps from menus and dialog boxes, students solve algebraic problems in an interactive manner.
- While considering the results of students' most recent calculations, *SMG* generates intelligent problem-solving options that focus student attention on new material being learned.

- Because *SMG* simplifies arithmetical expressions automatically, students may focus more attention on theoretical aspects equation-solving. Student work is not unduly hampered by arithmetic and lower-level algebra mistakes.
- While using *SMG*, students are encouraged to consider algebraic expressions and equations as mathematical objects. *SMG* encourages students consider appropriate transformations to apply to these mathematical objects to solve problems.

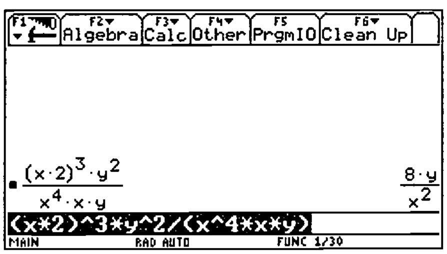
SMG is a self-paced learning tool to help students in learning symbolic manipulation. It offers a source of extra problems for students who haven't mastered a certain symbolic manipulation skill and can be used as a quick review for exams or a quick review of previously learned symbolic manipulation skill. The authors of this document have informally used *SMG* with students when introducing new classes of problems.

4. Simplifying Expressions With Powers with TI-92 CAS and *SMG*

4.1 Exploring Powers with Traditional CAS

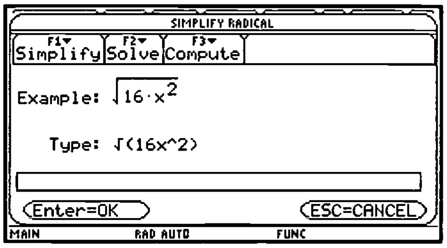


CAS allows students to discover rules about simplifying powers. By typing in several related examples into the TI-92 home screen, students form conjectures regarding algebraic rules. The examples to the left suggest a well-known "exponent multiplication" rule.

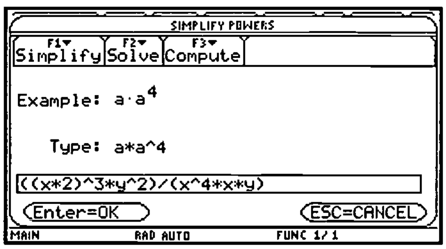


Unfortunately, the TI-92 CAS has a tendency to simplify more complicated expressions in one or two steps. This tendency creates confusion for inexperienced students, impeding their understanding of algebraic equivalence.

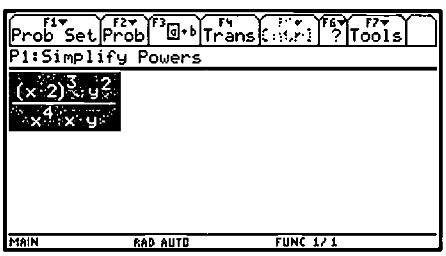
4.2 Exploring Powers with *Symbolic Math Guide*



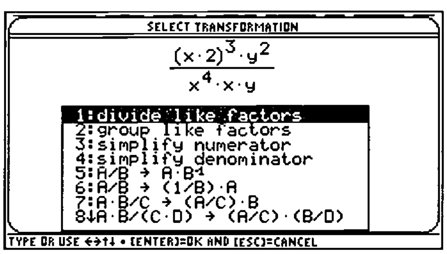
After starting *SMG* and upon selecting the `New Problem` option, *SMG* prompts the user to select a problem type. For instance, if a student wants to simplify an algebraic expression, he or she should press F1. Equation solving options appear under F2. Computational options appear under F3.



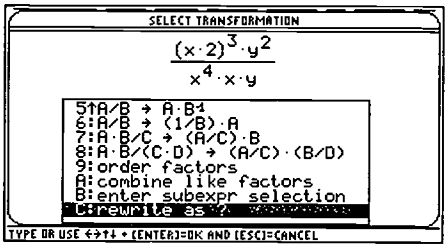
Inside the data entry line (at the bottom of the screen), type in the expression `((x*2)^3*y^2)/(x^4*x*y)` then press enter. The problem is now entered into the *SMG* main work screen.



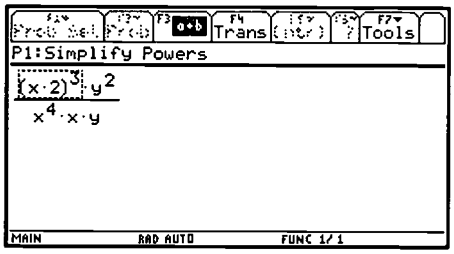
Several tools are available to the user at this point. In particular, the F3 menu option allows the user to select subexpression. The F4 menu option provides the user with different algebraic transformations that may be applied to selected expressions.



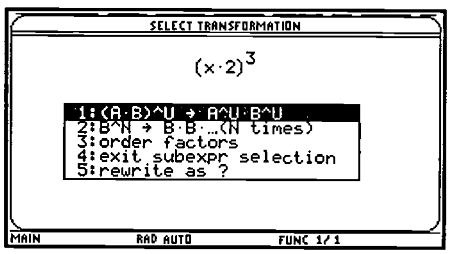
When the problem is entered into *SMG*, twelve legal choices are provided for the user. The student can choose any of them - although some selections lead to more efficient solutions than others. By offering legal steps, the *SMG* strengthens student understanding of rules used in simplifying powers.



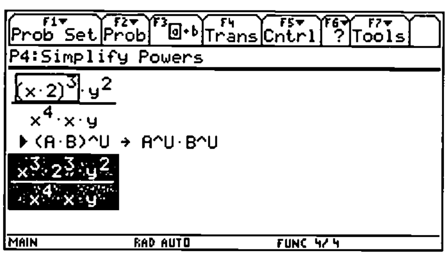
Since the twelve legal choices do not include a "power of a power" rule, students are encouraged to look at subexpressions within the problem. Students may use the subexpression feature of *SMG* to choose a smaller portion of the problem to simplify first.



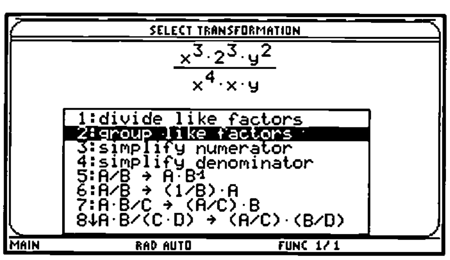
The screenshot to the left shows the selection of the subexpression $(x\cdot 2)^3$. Sub-selection is accomplished by pressing F3 and highlighting an expression with the calculator's keypad.



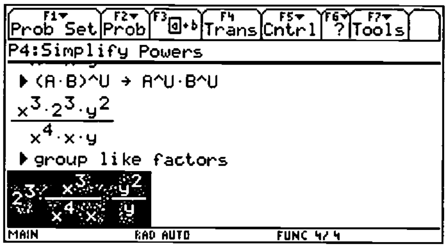
When the subexpression $(x\cdot 2)^3$ is selected and F4 is pressed, a different list of algebraic options is made available to the user.



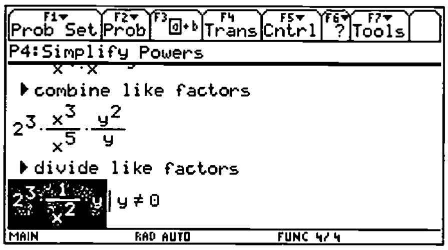
The first option - `(A•B)^U    A^U•B^U` - distributes an exponent across factors within parentheses.



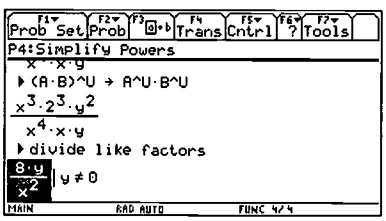
After $(x\cdot 2)^3$ is re-expressed as $x^3\cdot 2^3$, a new listing of algebraic options is once again provided to the user. The `group like factors` and `divide like factors` options are both reasonable selections.



By selecting the `group like factors` option, one is able to look at different variables combined.



The application of the `combine like factors` and `divide like factors` options makes it easier for many students to understand what is meant by "cancelling out."



oHowever, if students are already familiar with "canceling," one step cancelation is accomplished by omitting the application of `combine like factors`.

4.3 Anecdotal Evidence regarding *SMG*

Using *Symbolic Math Guide* informally with second-year algebra students, Edwards notes that the tools do offer some benefits over traditional CAS. Specifically, *SMG* does not seem to skip algebraic steps. Students have the ability to think more critically about algebraic transformation when simplifying algebraic expressions with *SMG*. In addition, the application offers classroom teachers an instructional partner in the classroom. Students needn't wait for teacher approval when checking the correctness of various algebraic manipulations. As one of Edwards' students notes:

> I like the math guide (*SMG*). It doesn't do as much work for you as other calculators, so you still have to think about the algebra. That's a good thing for future classes. Plus, I like the fact that the program (*SMG*) lets us find our own solutions. I think it makes algebra a little more interesting because we can experiment. The teacher doesn't have to lecture to us so much (Zak Stevens, second-year algebra student).

Nevertheless, the application isn't a perfect learning tool, and it certainly isn't as flexible as a seasoned classroom teacher. For instance, when using *SMG* in classroom situations, Edwards noted the following problems related to *SMG*:

- Selecting "subexpressions" within a term (e.g. highlighting $(x \cdot 2)^3$ within the expression $\frac{(x \cdot 2)^3 y^2}{x^4 xy}$) requires manual dexterity not required with pencil and paper. Some students become frustrated with the "subexpression" selection features of *SMG*.
- After selecting a specific expression to simplify, menu options do not always contain desired transformations. Students are left wondering "what to do next?"
- Inconsistencies exist with regard to domain restrictions. For instance, when simplifying the algebraic expression $\frac{(x \cdot 2)^3 y^2}{x^4 xy}$, SMG notes that $y \neq 0$ but no such restrictions are generated for x (see last screenshot).
- Functionality does not exist for roots other than square roots.

5. Discussion

SMG has an important role in helping students to give meanings to the algebraic transformations they frequently employ in secondary mathematics classes. In addition, *SMG* provides students with a more interactive method for learning concepts of symbolic manipulation than possible with pencil and paper. While using *SMG*, students are less preoccupied with calculations – spending more time considering algebraic transformations and concepts of equation solving. The authors of this document have found that learning is maximized when students are encouraged to anticipate the result of each transformation they select before pressing the ENTER button within *SMG*. By using *SMG*'s 'Press ENTER' mode, the application provides students with extra time to write down predictions – showing results to students only after ENTER is pressed again. The physical act of writing down each step with pencil and paper appears to help some students as they learn appropriate manipulation steps. While less effective when reviewing previously learned material, we've found the 'Press ENTER' mode to be quite useful when teaching material to students for the first time.

Because *SMG* allows students to select a variety of legal algebraic steps that are not necessarily the "best" or "most efficient" steps, students often construct different methods to solve individual problems. By comparing different solution strategies, students begin to appreciate the richness of algebraic problem solving (a subject which many students see as having "only one right way of doing things"). On the other hand, if legal algebraic steps do not lead to a solution, *SMG* makes it easy for students to go back to any previous step and try different transformations. To accomplish this task, students press the "up cursor" to get back to the step they wish to change. Then they choose a new transformation from a variety of menu options. In addition, SMG allows students to select subexpressions and replace them with an equivalent expressions from the keyboard. For instance, if a student knows that x+x is equivalent to 2x, the student can highlight "x+x" and choose a "replace with equivalent expression" menu option. *SMG* tests for equivalence of original and the user-defined expressions.

We always discuss that it is necessary to connect mathematics with real life situations. However, algebraic manipulation is one of the areas in the secondary mathematics curriculum that can be very abstract and very monotonous for students. Because students' minds and attention are always busy with calculation details, it is all too easy for them to lose sight of general equation solving techniques - particularly those involving algebraic transformation. *SMG* attempts to address this problem by offering teachers and students a novel approach to learning algebra. The CAS makes it possible for students to focus on the transformations in a visual, interactive and technology-rich environment.

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CASE STUDIES IN THE SHORTCOMINGS OF MAPLE IN TEACHING UNDERGRADUATE MATHEMATICS

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ABSTRACT

In this paper, we will investigate some of the algorithmic inadequacies and limitations of Maple as well as the common misuses of the software when used as a tool in teaching undergraduate mathematics. We will present examples for which Maple produces misleading or inaccurate results. We will also refer to situations where Maple gives accurate, but incomplete, results which are misused or misinterpreted by novice users of the software, specifically the undergraduate students. The authors have over ten years of experience in using Maple as a teaching tool and some examples presented here are based on those classroom experiences. Other cases have been reported by our students, by our colleagues and in various newsgroups devoted to discussions on Computer Algebra Systems (CASs). Many of the previously reported software bugs, observed in the earlier versions of Maple, are now corrected in the most recent release of the software. So, although we have occasionally referred to the older versions, we have presented the actual output only from the latest version of Maple, namely Maple7, in this paper. For the sake of brevity, we have limited our discussions to the topics which are ordinarily covered in the first two years of a typical undergraduate mathematics curriculum such as limits, single and multivariable integration, series, and floating point arithmetic. We have also tried to limit our case studies to the most common features of Maple, specifically those features that are widely used by the undergraduate students who are new to Maple.

Keywords: Maple; technology; shortcomings; bugs; undergraduate mathematics

Introduction

Computer Algebra Systems (CASs) have become increasingly popular tools in teaching mathematics in the past decade. The use of CAS has caused drastic changes in teaching undergraduate mathematics courses, particularly pre-calculus and calculus courses. According to the CBMS survey [1], 18% of Calculus I and II courses involved computer assignments in 1995, up from 9% in 1990. Assuming that the same trend has continued throughout the 90's, one could speculate that CAS has now become a major component of teaching in approximately 50% of Calculus I and II classes. The extensive mathematical assistance, symbolic manipulations, computational power and graphical abilities of CAS can greatly help students to explore mathematical topics and experiment with ideas without labouring through cumbersome calculations. The educators in mathematics community have hoped that CAS would enable students to develop an investigative attitude toward mathematics. A multitude of textbooks, workbooks and project manuals have been published to encourage and help the students toward this goal. Unfortunately, most of the literature is focused on the power of CAS, use of the commands, and to a lesser extent the programming aspects of CAS. Few of these books discuss the limitations and inadequacies of the software and the potential for misuse of CAS. As a result, the novice users such as beginning undergraduate students, who lack mathematical maturity, often mistakenly, assume that the "black box" software can solve any mathematics problem completely and accurately. This paper is written to demonstrate some of the shortcomings of one of the most popular CASs, namely Maple. We'll present examples from a typical pre-calculus and calculus course where Maple produces incomplete, inaccurate, or misleading results. We'll start each section with an example or two where the earlier versions of Maple produced inaccurate results and later these algorithmic bugs were corrected in the most recent version of Maple (version 7), and conclude the section with examples and actual output of Maple7 where the software still has difficulty to produce an accurate result. The examples are taken from a variety of topics. Although we have many examples in our disposal, we have limited our presentation to those examples which best demonstrate the shortcomings of the software. In section 1, we'll discuss solving scalar equations, section 2 is devoted to limits, and section 3 deals with sums and series. Single and multivariable integrals are discussed in section 4. Some of the examples presented here are based on the authors' classroom experience and our students and colleagues have reported some examples to us. However, most of our information is based on the *Maple User Group archives* and the internet discussion groups devoted to CASs, specifically: *sci.math.symbolic*, and *comp.soft-sys.math.Maple.*It is important to note that the authors have no intention of downplaying or downgrading the importance of CASs in general, and Maple in particular. CASs have revolutionalized the teaching of mathematics and we wholeheartedly endorse the CAS-based mathematics instruction. The pitfalls of earlier versions of Maple (which many have been corrected in Maple7) have not diminished our interest in the use of the software in our classes. We have used Maple in our classrooms for over a decade and we'll continue to do so enthusiastically in the future.

1. Solving scalar equations:

It was reported in [2] that *fsolve(x^5-5^x,x,x=3..5)*, using MapleV3 gives an output of $x=4$ which is clearly incorrect. The solutions are $x=1.76$ *and* $x=5$. Apparently there was a bug in the Newton's algorithm. The algorithm is corrected in Maple7. In [3], it is reported that MapleV5 command of *solve(x^2=Pi*tan(1)+sin(1),x)* which should have two obvious and trivial solutions produces no answer. The following two examples are the actual output of Maple7 which demonstrates some of the inadequacies of the software. The first example is using the command *allvalues*, which should return all of the solutions of the polynomials. It appears that for the first problem *x^3*(x-1)*, Maple produces the expected four roots (three roots of 0 and one equal to 1), but in the second example we get only one root of 5 (instead of 4 equal roots of 5):

```
> #Example1.1- find all roots of x^3(x-1)
> allvalues(RootOf(x^3*(x-1)));
```

$$1, 0, 0, 0$$

```
> #Example1.2- find all roots of (x-5)^4.
> allvalues(RootOf((x-5)^4));
```

$$5$$

The *solve* command of Maple sometimes has difficulty with equations that contain floating-point numbers, particularly when the expression involves exponents. The following example and solution taken from [4], demonstrates such a case and offers a remedy. Specifically, it suggests that we replace the value of the exponent by a symbolic parameter, then solve the equation in terms of the parameter and substitute the value of the parameter at the end of the procedure.

```
> #Example1.3-solve the given equation using the floating-point values.
> solve(1.03*x^0.67=67,x);
Warning, computation interrupted

> # solve appears unable to get the solution.Abort the computations and
use rational representations.
> evalf(solve(103/100*x^(67/100)=67,x));
```

$$508.5395605, 506.3050286+47.62040174I, 499.6210698+94.82231377I,$$
$$488.5464231+141.1909244I, 473.1784128+186.3187436I,$$
$$453.6520937+229.8091870I, 430.1390637+271.2800590I,$$
$$402.8459560+310.3669124I, 372.0126237+346.7262499I,$$
$$337.9100311+380.0385447I, 300.8378729+410.0110472I,$$

We get the solution we want (508.5395605) and a lot of complex solutions, which are omitted for the sake of brevity, so we'll try another approach [4]:

```
> # solve by replacing the exponent with a symbolic parameter.
> solve(1.03*x^p=67,x);
```

$$e^{\left(4.175133817\,\frac{1}{p}\right)}$$

```
> eval(subs(p=0.67,%));
```

$$508.5395595$$

2. Limits

It appears that if the command *limit* is used to determine limit of unassigned functions *f* and *g*, all versions of Maple, including Maple7, return *f(0)g(0)* which is clearly incorrect. Example 2.2, taken from [5], demonstrates another strange behaviour of the command *limit*. The limit in both cases should return unevaluated. Consider the Maple 7 output:

```
> #Example2.1- find limit of f(x)*g(x) as x approaches 0
> limit(f(x)*g(x),x=0);
```

$$f(0)\,g(0)$$

```
>#Example2.2- find limit of f(x)*exp(-x) as x approaches infinity
>limit(f(x)*exp(-x),x=infinity);
```

$$0$$

3. Sum and Series

The earlier versions of Maple (V3 and V5) had an algorithmic bug in summing infinite terms of divergent series. Apparently, Maple did not check for convergence first, rather it used various sum formulas, which are only valid outside the range of convergence of the series. Here are a few examples: it is reported in [6] that the command *sum((-1)^(n+1),n=1..infinity)* produces a sum of 1/2, which is clearly incorrect, since the series is a well- known divergent series. The command *sum(n!,n=0..infinity)* produces a surprising (complex) result of $0.69717488-1.1557273i$ [7]. Most of these bugs have been corrected in Maple7. However there are still a few left. Following is an actual output of Maple7 for a limit/series problem. Note that generally Maple looks at the leading term of a series for finding limits. In the following example [8], Maple clearly fails to see that the sum of the two trigonometric terms is zero and mistakenly returns zero (instead of x) as the limit of the expression.

```
>#Example3.1- find limit of the given expression
>g:=x+(-cos(9/50*Pi)+sin(8/25*Pi))/h;
```

$$g := x + \frac{-\cos\left(\frac{9}{50}\pi\right) + \sin\left(\frac{8}{25}\pi\right)}{h}$$

```
>limit(g,h=0);
```

$$0$$

Here is another example from Maple7 that perhaps has more to do with the floating-point arithmetic [9] than series. Note that a simple change of exponent from an integer "1"to a floating-point representation "1."creates a totally different and incorrect result.

```
>#Example3.2-Comapre series expansion of 1/(1-x)^1 and 1/(1-x)^1.
>series(1/(1-x)^1,x);
```

$$1 + x + x^2 + x^3 + x^4 + x^5 + O(x^6)$$

```
>series(1/(1-x)^1.,x);
```

$$1$$

4. Integration

There is a multitude of problems in single-variable integration that Maple, specifically the earlier versions of Maple, fail to produce correct results. In fact, the majority of reported software bugs to Maple-related Internet sites were (and continue to be) about antiderivatives and definite integrals. The primary reason behind many of the inaccurate or incomplete results appears to be the issue of multivalued functions in the complex plane. If the path of integration crosses the branch cut then the definite integral often returns an inaccurate result. We suspect that there are also problems with the implementation of Risch's algorithm. Here are a few examples from MapleV which since have been corrected in the most recent versions of Maple (versions 6 and 7): it is reported in [2] that both MapleV3 and V4 fail to produce an accurate result for the simple antiderivative problem of *int(sqrt(x)*sqrt(1+1/x),x)* .In another example [10], *int(log(sin(t)),t=0..Pi)* returns 0 which is incorrect, while *int(log(sin(x),x=0..Pi)* returns *–Pi*ln(2)* which is correct. Following is the actual output produced by Maple7, which demonstrates some of the persisting bugs in the software. In the first example [11], Maple gives a complex result to a definite integral, which clearly has a real value. However, If we use the inert command for integration (*Int* instead of *int*) we'll get the correct result. The reason appears to be that by using Int, Maple avoids finding antidrivatives and employs a numerical approach to find the result of the integral. Whereas, if we use the int command, Maple first finds the antiderivative, and then uses the Fundamental Theorem of Calculus to calculate the integral and somewhere in that process Maple7 commits an error. In the second example, the *int* command again produces a complex result to a positive integrand evaluated over a real interval. Although the integral is not a trivial one, but one expects that it be either returned unevaluated or some kind of message is given about the non-existence of an elementary antiderivative. The numerical integration using *Int* produces the correct result.

```
> #Example4.1-evaluate the integral using int and Int
> evalf(int(log(5+cos(x)),x=0..1));
```

$$1.764697796 + .88\ 10^{-9}\ I$$

```
> evalf(Int(log(5+cos(x)),x=0..1));
```

$$1.764697791$$

```
> # Let's increase the digits to see if int does better
> Digits:=15;
```

$$Digits := 15$$

```
> evalf(int(log(5+cos(x)),x=0..1));
```

$$1.76469779083464 + .136\ 10^{-13}\ I$$

```
> #Example4.2-evaluate the given integral using int and Int
> evalf(int(1/sqrt(2+x^4),x=0..1));
```

$$.4790759386 - .4790759386 I$$

```
> evalf(Int(1/sqrt(2+x^4),x=0..1));
```

$$.6775156893$$

We close this section with an example on double integrals. The example is taken from [12] and involves a trivial double integral over a rectangular region. It appears that Maple7 produces different results when the order of integration changes. The correct answer is 3.066667. The error first reported in 1996 and it appears that it has not yet been corrected. Here is the actual Maple7 output:

```
> #Example4.3-evaluate the double integral over the rectangular region.
> evalf(int(int(abs(y-x^2),x=-1..1),y=0..2));
```

$$3.216988933$$

```
> evalf(int(int(abs(y-x^2),y=0..2),x=-1..1));
```

$$3.066666667$$

As a final note, it is worth mentioning that the users of Maple or any other CAS sometimes use the words "pitfall", "bug" or "error" improperly. The user of the software, occasionally, makes an assumption (presumption?) about a command, which simply is not shared by Maple. In the following example [13], the user is surprised at the fact that Maple7 can not simplify ln(exp(f)) which is expected to be f. However, as it is explained in [13], Maple does not know that the parameters, t, C and R represent time, capacitance and resistance which are real numbers. Therefore, one has to inform Maple7 explicitly that all the parameters are real. Maple7, then returns the simplified expression. The actual output of Maple7 is presented here.

```
> # simplify the given expression
> result:=ln(exp(t/R*C));
```

$$result := \ln\left(e^{\left(\frac{t\,C}{R}\right)}\right)$$

```
> simplify(result);
```

$$\ln\left(e^{\left(\frac{t\,C}{R}\right)}\right)$$

```
> # simplify does not work since Maple needs more information about t, C
and R
> simplify(result,assume=real);
```

$$\frac{t\,C}{R}$$

Concluding remarks and acknowledgements

In this paper, we examined some of the shortcomings of Maple through examples. We presented examples from older versions of the software, which are now corrected in the latest version of Maple. We also presented examples to demonstrate some of the bugs, which still exit, even in the newest version of Maple. Some of the examples presented in this paper are taken from the posted problems and solutions in various newsgroups, most notably *sci.math.symbolic*, *comp.soft-sys.math.Maple* and *the Maple user Group archives*. We are very thankful to all of our colleagues in the mathematics community, who were, and continue to be, involved in these discussions, particularly those who utilize CASs in teaching of undergraduate mathematics. Our main objective in writing this paper is to encourage a conversation among the mathematics educators on the practical aspects of using CASs. We hope that our work will be of some use to the educators in the mathematics community who are involved in CAS-based mathematics instruction. We also would like to thank the Texas Lutheran University, which partially supported this research through the TLU Research and Development fund.

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EXAMS AND COMPUTER ALGEBRA SYSTEMS

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ABSTRACT

In the French system of CPGE (undergraduate level), CAS (Computer Algebra System) is used as a mathematical aid. The ability of students to use CAS as a tool in a real mathematical activity is relatively easy to test if you are tutoring them in the context of their research projects. It is not the same in exams. On these occasions, the examiner chooses the question and the examinee has just an hour to tackle it.

In our presentation, using examples from French exams in Maple, we will endeavour to show the various pitfalls to avoid and how an examiner can become able to assess the ability of students to use CAS as a mathematical tool.

EXAMS AND COMPUTER ALGEBRA SYSTEMS

In order to show how we are tackling the problem of using Computer Algebra Systems (CAS) at exams, we have to describe the system where we are teaching (CPGE), what are the goals of using CAS in it and what kind of exams we are talking about.

THE CPGE

The French "Grandes Ecoles" system was the result of the French Revolution (1789-1799). Before that time, the University was mainly concerned with Theology. For some reasons that are not relevant here, it appears that the only way to change this predominance was to create a new kind of university. These were called "Grandes Ecoles". As they were cut from the clerical tradition of the official university, they became more interested in real life. Two centuries later, they remain closer to industry than the university. The main usual criticism of the latter is its remoteness from the real world. The CPGE are the undergraduate level of the "Grandes Ecoles". Nowadays, this represents 40 thousand students and 1200 professors of mathematics (of a population of 60 millions) working on the same national curriculum. Since the exams are national, it is difficult to change that aspect. Nevertheless, despite this centralised side, they are dispersed in large and small "lycées" throughout country ("lycées" also deal with the secondary education of pupils aged 15 to 18). Their size depends on the population of the city where they are, the larger ones being in the largest cities. Some of them can have just about 40 students, while others such as Janson de Sailly (Paris) have one thousand or more. After two or three years in CPGE, the students usually pass their exams (roughly 85%). Then they can choose to enter one or other "Grande Ecole" depending on their exam results, those with the highest ones can choose whatever they want (usually the "Ecole Polytechnique" or the "Ecole Normale Supérieure"), the other ones with the lowest score results taking whatever places are left available. Since the Revolution, the practical success of our system has turned it into a rather selective one. In fact, most of the best science students avoid the university to try to enter one of the "Grandes Ecoles" so they begin their studies in the CPGE even if they finish elsewhere.

COMPUTER ALGEBRA SYSTEMS AND THE CPGE

In CPGE, the official idea is that CAS should be used as a problem-solving tool. This means that the students have to deal with abstract ideas and CAS will perform the calculations after that. It is quite easy to check the ability of students to use CAS in this way during the year. You have just to give them usual problems (at home in one week) and exercises (at the blackboard in one hour), wait and see what they have done. The final interview is sufficient to understand how and why they use (or do not use) CAS.

The two CAS in use in CPGE are Maple and Mathematica but we mainly use Maple V (Release 5). Programming is taught but without recursion (see [1] and [2] for an example of this teaching). To be more precise, we use a kind of micro-Maple limited to a list of operators and functions (see table 1) even if the students have the right to use every function they know. So, in this paper, we will focus on it. Nevertheless, our remarks remain relevant for other versions of this software as well as for Mathematica and others.

COMPUTER ALGEBRA SYSTEMS AT EXAMS

At exams, the written part takes place at the same time for thousands of students nationally. So, on the one hand, it is difficult to provide them access to computers, on the other hand, we cannot allow them to use their own ones (whatever their size are) because it may be a source of fraud. Thus, we have decided to test their abilities to use CAS at the oral part of the exams.

So, during this part, a computer with Maple and Mathematica is available in the examination room. The examiner provides technical assistance. For example, he can answer to questions as: "what is the instruction for computing an integral?" or "how can I re-initialise Maple?" He is not supposed to judge the examinee through the questions he asks. The fact that a computer is in the room does not mean students have to use it. Using CAS or not using it is their choice. So, the first difficulty is to recognise that CAS may be helpful in a particular question. Anyway, the students will be judged on their ability to do mathematics, not on their knowledge of software. Thus, we have to ask them to tackle usual exercises where CAS can be used but is neither necessary nor sufficient.

We have chosen to discuss our criteria on examples (see [3] for others) because it is difficult to consider this question in the abstract. Our philosophy concerning the use of CAS at exams is to use classical exercises but to keep only those where CAS may help without doing everything. Nevertheless, without applying it on examples, this simple idea does not take its full sense.

COMPUTER ALGEBRA SYSTEMS AS VIDEO GAMES

In a lot of cases where real programming is not needed, students can produce results without understanding a thing on the matter. For example in the two following exercises, a basic technical expertise is enough to find the right answer:

Exercise 1: Compute $\lim\limits_{x\to 0}\dfrac{2\tan x-\tan 2x}{x(1-\cos 3x)}$.

Exercise 2: Compute $\displaystyle\int_0^{+\infty}\frac{\mathrm{d}x}{1+x^4}$.

Testing student's ability to find the right results ($-\dfrac{4}{9}$ and $\dfrac{\pi\sqrt{2}}{4}$) in such cases is not far from testing their ability to play a video game. In fact, the only thing to know is how to encode the mathematical expressions written above in the Maple language. So, you cannot tell much about the student's mathematical knowledge through their way of doing such exercises.

COMPUTER ALGEBRA SYSTEMS AS PROGRAMMING LANGUAGES

A solution to avoid the use of CAS as a kind of video game is to ask students to program them on mathematical examples. So, the following exercises concern programming. They have been chosen to show the boundary between fair and unfair exercises of this kind at the mathematical part of exams.

Exercise 3: Two distinct natural numbers are called amicable if the sum of the proper divisors of one number equals the other. Write an algorithm finding all couple of amicable numbers smaller than 1500.

Exercise 4: Write a function f returning the sum of the cubes of the digits of an integer n in decimal expansion. Find the n such that $f(n) = n$.

Exercise 5: Write a function returning the index of the maximum value of a sequence of real numbers $(u_1, u_2, \ldots, u_n)$.

Exercise 6: Find a method to compute x^{10} with just four multiplications. How many multiplications are necessary to compute x^{55}? Generalise and write the corresponding function.

Exercise 3 is very simple if you know that there is a function computing the sum of the proper divisors of a natural number in the number theory package of Maple (it is called sigma). It is rather difficult if you are limited to micro-Maple. As we cannot forbid the use of the functions available in full Maple, such an exercise must be avoided. A consequence is that, for every exercise, we have to check if it cannot be solved by the use of just one magical function out of our micro-Maple (result: 220, 284 and 1184, 1210).

A good solution of the first part of exercise 4 involves recursion (using the mathematical property: $f(0) = 0$ and if $n = 10q + r$ then $f(n) = f(q) + r^3$). To solve the second part, you have to realise that a solution has at most four digits and then to try all of them which is easy (result: 0, 1, 153, 370, 371 and 407). Thus, this exercise is not a bad one to test mathematical ability at undergraduate level but it involves a real knowledge of programming. So, as it is not the main goal of our teaching, this kind of exercises must be avoided or kept for a second question to apparently good students in this domain, just to check how far they can go.

Exercise 5 and a lot of the same kind (as sorting and searching for example) must be avoided as they are just programming exercises. They must be reserved for the computer science exam. Exercise 6 is at the frontier of this kind. May be it is better to avoid it too.

Anyway, in this paragraph, we see that to avoid the testing of the ability to play video games, we test the students on their knowledge of computer science. The goal is missed. If this kind of exercises can be used for those who are obviously good, they must be kept to them.

COMPUTER ALGEBRA SYSTEMS AS TRAPS

Another way of testing whether students really know what they are doing when using CAS may be to give them examples where CAS results are wrong. Such cases are not difficult to find, especially in computing integrals:

Exercise 7: Compute $\int_0^{+\infty} \frac{\mathrm{d}x}{x^2 - a^2}$ if it exists.

Exercise 8: Compute $\int_{-1}^{1} \frac{\sin a \ \mathrm{d}x}{1 - 2x\cos a + x^2}$.

Exercise 9: Compute $\int_{-\infty}^{+\infty} \frac{e^{ix}}{1 + ix} \mathrm{d}x$.

In exercise 7, Maple gives a clearly impossible answer ($\frac{i\pi}{2a}$). It is really easy to tell that this result is wrong because it is not real. So, the reaction of a student on this kind of output may be interesting. In fact, a lot of them have a bad tendency to believe everything they see on a computer screen.

In exercise 8, Maple gives the correct but amazing result:

$$\frac{\sin a\left(-\arctan\frac{-1+\cos a}{\sqrt{1-\cos^2 a}} + \arctan\frac{1+\cos a}{\sqrt{1-\cos^2 a}}\right)}{\sqrt{1-\cos^2 a}}$$

To simplify it properly in order to find the right answer ($\frac{\pi}{2}$ if $\sin a > 0$, $-\frac{\pi}{2}$ if $\sin a < 0$ and 0 if $\sin a = 0$), you must have some knowledge on the inverse tangent function.

In exercise 9, Maple gives 0 as the answer (and Mathematica $\frac{\pi}{e}$). Both are wrong but there is no obvious reason for students to doubt any of these results (the right one is $2\frac{\pi}{e}$).

It is easy to imagine a number of traps of this kind. If there are really strong reasons to doubt the result of CAS, they can provide an opportunity to test the understanding of students but in other cases, they must be forbidden. Thus, at exams, we prefer to avoid such exercises even if some of them are very interesting. An implication of that is that the examiners have to really use CAS to solve their exercises before the examination just to check they are not mining the road to the solution without realising that.

COMPUTER ALGEBRA SYSTEMS HELPING INTUITION

The best way of finding good exercises is to take a look at our use of CAS in real mathematical life. One of its uses is to help our intuition. Here are some examples of exercises where they can be used in this way.

Exercise 10: Find the complex numbers z such that $\frac{z^2}{2z+3i}$ is imaginary. Represent them in the complex plane.

Exercise 11: Let Γ be the curve given in parametric form by: $M(t)\begin{cases} x = 3t^2 \\ y = 2t^3 \end{cases}$. Find the locus C of the point from where the tangents to Γ meet at right angle. What kind of curve is C? Determine the position of C relatively to Γ.

Exercise 12: Let $A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$. Compute A^n.

Exercise 13: Let $f_n(x) = \frac{\sin nx}{n\sqrt{x}}$, $x \in (0, +\infty)$. Does the sequence (f_n) converge uniformly?

Exercise 14: Consider the sequence (u_n) defined by: $u_0 = a$ and $u_{n+1} = \frac{3}{2u_n^2+1}$ for all $n \geq 0$, $a \in$ □.

What can you say about $\lim_{n \to +\infty} u_n$?

If students understand the exercise 10 properly, they can plot the solution directly (see figure 1). Of course, more can be said on the curve on figure 1. The examiner will judge students on their ability to recognise and to prove that the curve is composed of a straight line and a circle.

In exercise 11, CAS is useless to find a parametric representation of C ($\begin{cases} x = t^2 - 1 + \dfrac{1}{t^2} \\ y = \dfrac{1}{t} - t \end{cases}$) but very useful to plot it (see figure 2). With a minimum knowledge, students can suspect that this curve is a parabola. In the same way, they can see that the two curves are tangent. Then they have to prove these two properties.

In exercise 12, through some computations done with the help of CAS, it is quite easy to see that $A^n = \begin{pmatrix} a_n+1 & a_n & a_n & a_n & a_n \\ a_n & a_n+1 & a_n & a_n & a_n \\ a_n & a_n & a_n+1 & a_n & a_n \\ a_n & a_n & a_n & a_n+1 & a_n \\ a_n & a_n & a_n & a_n & a_n+1 \end{pmatrix}$ for some a_n. Then, through multiplying this matrix by A, CAS help to find the law $a_{n+1} = 6\, a_n + 1$, the result follow ($\dfrac{6^n - 1}{5}$) but it needs a minimum of mathematical knowledge.

Without visual aids, most of the students do not see that the sequence of exercise 13 converges uniformly. With CAS, they generally see that but it is more difficult to prove it. For that purpose, CAS is useless. This exercise does not miss the goal but is rather difficult and must be left for further testing of an apparently good student.

In exercise 14, a lot of students see incorrectly that $\lim_{n \to +\infty} u_n = 1$. The reason lies in their interpretation of the drawing of the graph of the function f defined by $f(x) = \dfrac{3}{2x^2 + 1}$ (see figure 3). Starting from any number a, they felt that the sequence approached 1 in absolute value and oscillated from one side of 0 to the other. At this step, it is not too bad but what can be very upsetting is when they are able to prove it! With a better understanding, they went to examine the graph of $f \circ f$ (see figure 4). Generally, not only they see the correct result (two limit points $\dfrac{3 \pm \sqrt{7}}{2}$ if $a \neq \pm 1$, the limit 1 occurs only if $a = \pm 1$) but they are able to isolate the properties to be proved in order to prove it (CAS can be helpful for this purpose). In this example, we note that "seeing" requires a lot of knowledge in mathematics. As the previous exercise, this one must be left for a further testing.

In conclusion, all these exercises (10 to 14) are rather good to test mathematics ability even if they are not all of the same level. In all of them, CAS is used to "see" the correct result and to see requires a lot of knowledge in mathematics.

COMPUTER ALGEBRA SYSTEMS TAKING CHARGE OF CALCULATIONS

In a lot of cases, CAS can be used to take charge of calculations. In this kind of use, it is important to choose exercises where the students have either to analyse the results or to find the right calculation to do.

Exercise 15: Compare $2+2\sqrt{2}$ and $\sqrt{5+2\sqrt{6}}+\sqrt{9-2\sqrt{6}-4\sqrt{5-2\sqrt{6}}}$.

Exercise 16: Find the zeros of the polynomial $P = x^4 - 2x^3 + x^2 - 2x + 1$. Represent P as a product of irreducible polynomials over the real domain.

Exercise 17: Let $M = \begin{pmatrix} a & 0 & 0 & 0 & b \\ 0 & a & 0 & b & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & b & 0 & a & 0 \\ b & 0 & 0 & 0 & a \end{pmatrix}$ where a and b are complex numbers. On which condition is M diagonalisable?

Exercise 18: Let a, b and c be three real numbers and $M = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$. Find the matrices M such that $M^2 = \mathrm{I}$.

Exercise 19: Solve the following differential equation: $xy'+(1-x)y = \dfrac{xe^x}{x^2+1}$. Plot some integral curves. Is there any continuous solution on □?

Exercise 20: Let C is the curve given by $M(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$. Show that the osculator planes to the curve at three different points and their plane have a common point.

In exercise 15, an approximation of the difference gives a probable answer (the numbers are equal). To prove it with CAS, you have to use a special function not in our micro-Maple (radnormal) thus this kind of exercise is a trap because the students have no simple reason to realise that $\sqrt{5\pm 2\sqrt{6}} = \sqrt{3}\pm\sqrt{2}$ even if this equality is really easy to prove. This kind of example shows that we have really to check if an exercise can be done with the use of our micro-Maple only. As a matter of fact, knowing the property, it is very easy to believe that CAS will be a good help but it is not always the case.

In exercise 16, CAS find the zeros of P ($\dfrac{1\pm\sqrt{2}}{2} \mp \dfrac{1}{2}\sqrt{-1\pm 2\sqrt{2}}$) easily (with allvalues, solve is not enough). The only mathematical problem is to assemble them in polynomials over the real field (result $x^2+\left(\sqrt{2}-1\right)x+1$, $x-\dfrac{1+\sqrt{2}}{2}\pm\dfrac{\sqrt{2\sqrt{2}-1}}{2}$).

In exercise 17, students can think that CAS gives the answer directly but, in fact, they have to check that the eigenvectors given are linearly independent. Computing the determinant of the proposed vectors, we find that it is not the case if $a + b = 2$.

In exercise 18, CAS gives M^2 and help to transform the matrix equation in a system of three equations ($\begin{cases} a^2+2bc=1 \\ b^2+2ca=0 \\ c^2+2ab=0 \end{cases}$) which can be solved with CAS (result: $\pm\mathrm{I}$, $\pm\dfrac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$).

In exercise 19, it is easy to plot a lot of integral curves (see figure 5) but the continuous solution is not easy to spot among them without a theoretical study. For that, the students have to write the general solution ($y(x) = \frac{e^{x}\left(\frac{1}{2}\ln\left(x^2+1\right)+A\right)}{x}$) and to realise that the only point of discontinuity is 0 and to compute its limit at 0 (see figure 6).

In exercise 20, with CAS, it is possible to write the equation of the osculator plane at $M(t)$ and of the plane passing through points with parameters a, b and c ($f(t) = 3t^2 - 3ty + z - t^3$, $g(x,y,z) = (ab + bc + ca)\,x - (a+b+c)\,y + z - abc$). Then, they can solve the system of equations $\begin{cases} f(a) = 0 \\ f(b) = 0 \\ f(c) = 0 \end{cases}$ and substitute the solution into g.

In conclusion, most of these exercises (16 to 20) are rather good for testing the ability of doing mathematics with CAS. Exercise 15 is not a good choice because it is not of the same level whether you know a magical function or not. This kind of cases must be avoided, as they are unfair

CONCLUSION

Through these examples and counter examples, we can see that the exams in CPGE test the official way of using CAS taught in CPGE (see [4]). The exact opposite of this way of thinking can be found in [5]. Our main problem is that we cannot use it in the written part for the time being. CAS should be available all the time to allow students to become completely accustomed to using CAS but there are two problems firstly the high price of small portable computers and secondly the possibility of fraud. Thus the testing is done only at the oral part of the exams.

Our general idea to choose exercises is to use classical ones but to keep only those where our micro-Maple (see table 1) may help without doing everything. A good exercise must not give a decisive advantage to those who have a knowledge of functions out of our micro-Maple or of recursion because it will be unfair to the others. Thus, exercises involving too much programming must be reserved as further tests for those who are obviously good. For the same kind of reasons, traps as cases where Maple gives a wrong answer must be avoided or reserved for further testing. An implication of that is that the examiners have to really use CAS to solve their exercises before the examination. To finish with the question, we will stress two points. Firstly, experiments where the students have to see something are rather good to test mathematical ability because "seeing" requires a lot of mathematical knowledge. Secondly, exercises involving parameters are often good because CAS do not discuss particular cases. So, in this kind of exercises, the students have to understand and to interpret the results of CAS.

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TABLE AND FIGURES

table 1

^	$	allvalues	display	for	local	Pi	series	while
*	&*	assume	do	I	map	plot	simplify	with
–	( )	coeffs	dsolve	If	map2	plots	sin	
+	–>	conjugate	eigenvals	Im	matrix	pointplot	solve	
=	{ }	convert	eigenvects	infinity	max	polynom	sqrt	
<	[ ]	cos	eval	int	multiply	print	subs	
>	< >	D	evalc	inverse	nops	proc	sum	
,	<=	denom	evalf	iquo	numer	product	taylor	
;	>=	det	evalm	irem	numeric	Re	trace	
"	' '	diag	exp	limit	op	restart	transpose	
/	:=	diff	expand	linalg	Order	RETURN	unapply	
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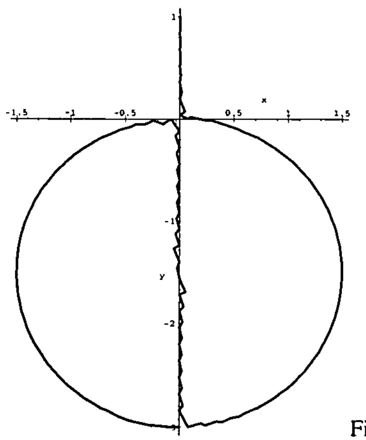

Figure 1

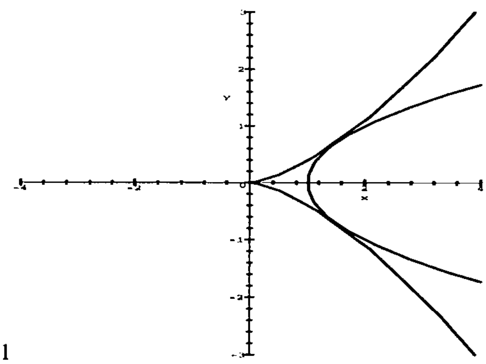

Figure 2

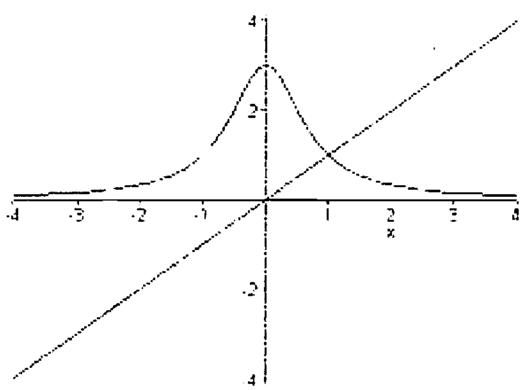

Figure 3

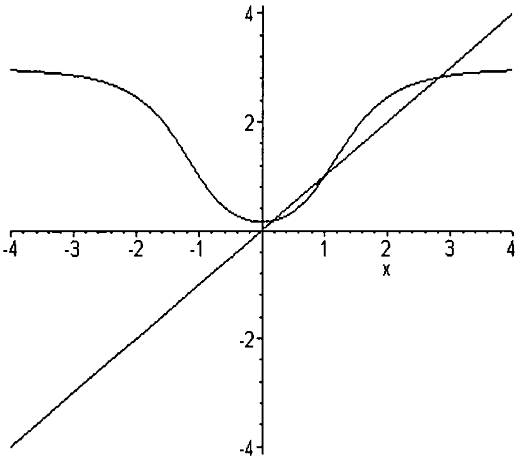

Figure 4

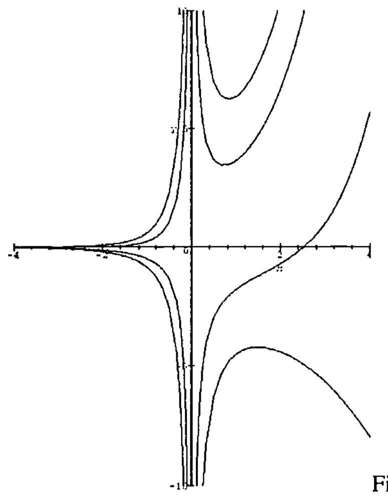

Figure 5

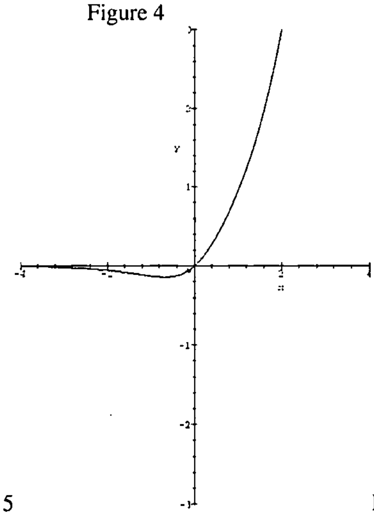

Figure 6

PREPARING TECHERS-A DIAGNOSTIC MATHEMATICS COURSE

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ABSTRACT

A Diagnostic Mathematics Clinic serves students who are having difficulty with mathematics. In the clinical setting, University preservice mathematics education students work on a one-to-one basis with a student. The university students administer and evaluate diagnostic tests; conduct parent, student, and teacher interviews; and analyze measurement and screening data provided by the school. Based on these data, the clinician and university director develop an achievement plan for each student. This article describes the effects of the clinical experience on undergraduate students who are pursuing certification to teach mathematics.

Preparing teachers-a diagnostic mathematics course

The University of Houston-Clear Lake (UHCL) Diagnostic Mathematics Clinic serves second through eighth grade students who are having difficulty with mathematics. University preservice mathematics education students (clinicians) work on a one-to-one basis with a student. They administer and evaluate diagnostic tests; conduct parent, student, and teacher interviews; and analyze measurement and screening data provided by the school. Based on these data, the clinician and university director develop an achievement plan for each student. This article describes the impact of the clinical experience on university preservice students. In order to better understand this impact, it is necessary to describe the operations of the clinic.

Background Information

The National Council of Teachers of Mathematics (2000) states: "Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well." The National Commission on Teaching and America's Future (1996) also believes that effective teachers need to understand and be committed to their students as learners of mathematics. Working with students in a one-to-one format in a math clinic provides a unique opportunity for preservice teachers to focus instruction based on the needs of individual students.

According to Engelhardt (1985) mathematics clinics have different purposes: teacher education, public service, or research. One focus of a clinic is to educate teachers to cope with students who have difficulty learning mathematics. In this setting clinicians typically attend a seminar sequence and practica. In the seminar, theoretical and practical topics are explored; while in the practica these ideas are implemented with students. Dockweiler (1993) believes that the *Curriculum and Evaluation Standards* published by the National Council of Teachers of Mathematics should guide the establishment of any mathematics clinic. According to Dockweiler, a clinic should serve three roles: providing service to the community, training teachers in diagnosing and remediating the difficulties of students, and research. The Diagnostic Mathematics Clinic was established using the teacher education model described by Engelhardt and encompasses the three roles described by Dockweiler. It is staffed by undergraduate students working toward elementary or secondary mathematics certification.

Sheila Tobias (1999) suggests finding ways to integrate the needs of future teachers into standard undergraduate mathematics courses is difficult. UHCL has addressed this concern by creating math courses specifically designed for preservice teachers. One such course, the Diagnostic Mathematics Course, is offered each fall, and enrollment varies from 15 to 24 undergraduate mathematics education students. This course has been offered for more than 16 years at UHCL. Children are typically referred to the clinic by teachers or their parents. Information about the clinic appears in local newspapers, and flyers describing the clinic are sent to area mathematics supervisors, elementary schools, and middle schools. There is a registration fee for the clinic, and the children meet at the university one and a half hours each week for ten weeks. Questionnaires about the child and mathematics are completed by the parents and the child's mathematics teacher.

Palmer (1994) stresses the importance of obtaining a solid, reliable picture of the student's current understanding of mathematics before beginning instruction. Eaves (1992) believes the key to successful instruction is beginning with the known and working towards the unknown. Since the level and accuracy of prior knowledge varies with each child, he describes diagnostic testing as a positive action to determine each student's knowledge level. The Diagnostic Mathematics Clinic utilizes the KeyMath Revised Diagnostic Test (KMR; Connolly, 1988). The KMR has three major areas with the following subtests: Basic Concepts - numeration, rational numbers, and geometry; Operations - addition, subtraction, multiplication, division, and mental computation; Applications - measurement, time and money, estimation, interpreting data, and problem solving. In establishing these content areas, Dr. Connolly reviewed mathematical curricula, mathematics programs, basal mathematics text books, research articles and other publications, especially those of the National Council of Teachers of Mathematics. According to Nicholson (1988) KMR is well constructed with excellent directions for interpretation and comparison of scores, both within the KMR and other instruments. If the area of mathematics is the only problem area delineated for a student, Davis (1989) recommends the KMR as the best measure for assessing the student. The KMR was also favorably reviewed by Bachor (1989-1990), Huebner (1989), and Finley (1992). According to Beck (1992):

> In the galaxy of educational test, KeyMath-R can only be described as a brightly shining star. From all aspects-content development, technical and normative underpinnings, and presentation of materials-the test is an outstanding example of the test-maker's craft.

Once the KMR has been administered and scored, the clinician develops an achievement plan for the student. This plan is based on previous test data; information from parents, teachers, the student; and the KMR data. The objectives described in the achievement plan form the basis for the remaining eight weeks of the clinic.

The clinic sessions are scheduled in viewing rooms and are under the supervision of the clinic director. Clinicians submit lesson plans prior to the clinic session, along with reflections of previous lessons. The clinic director observes sessions and provides clinicians with feedback.

Data Collection

Preservice students were surveyed at the end of the fall semester, 2000, and asked to respond to the following question: Will your experience working with one child in a diagnostic setting impact your classroom teaching? If so, in what way?"

Focus on the Child

Almost all clinicians noted the importance of fostering the child's self-esteem and positive attitude toward mathematics. They found that by focusing on the child's strengths during the session, the child became more confident in his mathematics skills. The clinicians found that a lack of self-confidence could be a hindrance to succeeding in mathematics. They indicated that knowing this will encourage them to keep a positive, "You can-do-it" attitude as they teach.

Clinicians learned the value of establishing a supportive environment for students. One clinician noted she thought her student understood everything because he did not ask questions. In reality, the student was very shy and was afraid to ask questions. The clinician realized the importance of establishing an atmosphere of trust to encourage questions from students. This clinician plans to use a journal in her classroom and have students write down what they did not understand in class. These problems can be addressed the following day, without the student feeling uncomfortable.

According to Kennedy (1998) how a subject is taught tells students whether the subject is interesting or boring, clear or fuzzy, applied or theoretical, relevant or irrelevant, and challenging or routine. Clinicians found the importance of making instruction relevant to their students. Using the individual child's interests as a learning tool was found to be effective in providing meaningful instruction. For example, one child enjoyed hunting, fishing and working with animals. Whenever possible, these interests were included in instruction. Another student's interest in cats was used in a shopping game in which everything sold had to do with cats. In her classroom, this clinician is going to survey her students about hobbies and interests and use this information in creating mathematics problems.

Clinicians found it was important to consider a student's attention span. A clinician noted that lecturing to her student resulted in his becoming distracted immediately. In order not to "lose" him, she had to completely involve him in her teaching. For example, if she were teaching a lesson on fractions, she would give him fraction pieces to use as she taught the lesson. In her classroom teaching, she plans to actively involve students.

The ability to learn and process knowledge at an average rate of speed were attributes one preservice teacher had always taken for granted. After working with a special needs student, he no longer takes this ability for granted. He never knew with certainty what his student would retain and be able to do in subsequent tutoring sessions. This clinician decided to begin each tutoring session and each class session with a review of the material covered the previous day. He learned that when working with students similar to this student, it will be important to vary instructional activities and to provide as much structure as possible when working mathematics problems.

In working with one child who was experiencing difficulties in mathematics, clinicians felt they would be more aware of special needs students in their classrooms and have a better idea of how to fulfill their needs. "I have the tools to be able to determine the ways in which a student with a learning problem might learn best."

Instruction

All clinicians administered the KeyMath diagnostic test and developed and administered their own diagnostic test. They noted the importance of the diagnostic test in identifying each child's individual needs. In their own classroom, many clinicians indicated they would administer a diagnostic test prior to teaching a new unit or chapter to identify students' weaknesses as well as prior knowledge. They felt, in a classroom, diagnosing problems quickly can prevent wasted time and "grasping at straws." According to clinicians, "Whether it be discussing the results of a diagnostic test administered by a diagnostician or developing, administering, and evaluating my own diagnostic test, I feel prepared to discover the weaknesses of a particular student or of the class as a whole. I now realize the importance of diagnosis in my class, and I am going to use techniques for a quick diagnosis during the monitoring and adjusting phase of my teaching."

University students also noted the importance of minimizing or managing frustration. One clinician noted his student seemed to have forgotten some important concepts that had been discussed in the previous tutoring session. Throughout the previous session, the student seemed to understand the concept. Yet, when she tried applying the concept during the following session, she could not. The clinician was frustrated for a number of reasons. The clinician began to wonder if he had done a good job; had he spent enough time with his student on the subject matter; had she already forgotten what she had learned last week; or was something else preventing the student from working with the concept. The clinician's comment, "If there is a frustration level in working with one child, there must be a twenty-fold frustration level in working with twenty children." This clinician believes that by realizing that some children may have low retention, he can "turn it around" and use it as a challenge or opportunity. "When I teach I will be constantly asking myself, 'What can I do to maximize retention?' This is where opportunity knocks on my door, and I have to be ready to answer it."

Success of students, according to the clinicians, is very dependent on mastery of concepts at lower levels. If, for example, a child has trouble with rational numbers, it would be easy to assume the problem lies with rational numbers. Yet, with proper diagnosis, the problem may be with earlier concepts such as numeration, addition or multiplication. Clinicians believe working in the diagnostic setting has given them tools and knowledge to work with students to determine where the actual "breakdown" of knowledge occurs.

Organization and being prepared are critical attributes noted by clinicians. Each tutoring session required a lesson plan, which incorporated manipulatives and a variety of activities. Obtaining the manipulatives, organizing each lesson, and adapting the lesson to students' abilities are required by classroom teachers every day. Clinicians found the tutoring sessions were more successful when they were better prepared and more organized.

Clinicians also reported the importance of flexibility. One ADHD student would come to the tutoring sessions in various moods. One day he would come to the clinic eager to work, while the next week he would complain about being tired and choose not to do any work. The clinician found letting the student rest or simply talk about his problems helpful. After the student was able to rest or vent his frustrations, an assignment or activity could be performed successfully. In his regular classroom, this teacher said he would develop a "time out" format where students will be given short breaks away from the regular activity. Upon completion of the break, students would return to the activity and complete the assigned task without penalty. Clinicians learned that even with the best preparations and the best intentions, sometimes a lesson does not work the way you thought it would. They learned to take a deep breath, back up, and try again.

Perseverance was also recognized as a valuable asset. If something does not work, do not give up; try something different. A clinician noted you "must be patient enough to take the time to find the method that will work with each child." Another clinician noted she had the time to look at a variety of different manipulatives, to try them out, and discover which ones were the most effective for her student. When working with an entire classroom of students, this clinician feels that if a certain method or tool does not help a particular student, she will be able to use another one that might better meet the student's needs.

Clinicians have learned to never underestimate a student, to always have more material and activities planned than they think they will be able to do. This was a surprise for preservice teachers. According to one clinician, she had planned her first lesson plan, establishing reasonable time goals to accomplish each objective. However, at the end of the lesson she had extra time and no additional activities planned. The clinician had not considered what she would do if this happened and stated "After this experience I will forever have more materials and activities than necessary."

One preservice teacher discovered the use of manipulatives and games to enhance children's learning. Based on the success of using manipulatives and games with one child, she plans to incorporate manipulatives and games in her own classroom. Another clinician found his student learned mathematical concepts by first using manipulatives and then applying the concept. He realized everybody learns in different ways and he will have to be prepared to teach twenty or so students in several different ways. "I will be prepared with two or more manipulatives for each mathematical concept I plan to teach."

A preservice teacher found that students can effectively learn mathematics without a lot of worksheets. Her student enjoyed the games and manipulatives and based on test scores, the student's mathematics skills improved. This clinician is going to incorporate games and manipulatives in her classroom, and she is also going to recommend that her parents use games and manipulatives with their children at home.

Additional Insights

Clinicians realized there were factors outside of school that as a teacher they will have no control over, and they must stay focused on what they can do to help their students, including asking for help from other teachers and administrators. Scheer and Henniger (1982) describe the clinic as an ideal setting for parental involvement in the educational process. One of the requirements of this program was that the clinician interview the parents and teacher of his or her student. A clinician indicated she has learned how to discuss a student's mathematical weaknesses with parents and teachers. She also learned questions to ask that lead to a greater insight into the student's problems. Clinicians learned the importance of input from the student's parents. They were able to provide background information that helped the clinician determine the best strategies in working with the student. Clinicians indicated that communication with the parents of the students in a teacher's classroom will be equally important.

Assessment of Clinical Experiences

A preservice teacher noted:

A teacher's teaching abilities are always work-in-progress. She can always improve, if not a new technique for teaching, perhaps a new understanding of learning - a diagnostic setting provides that opportunity.

According to another preservice teacher:

The diagnostic clinic has given me time to work, talk, and enjoy a student in a way that would be difficult to do when there are twenty-five students in a classroom. This experience will be a valuable memory to remind me to take time to enjoy and get to know my students so that I can provide for them, in a personal way, learning that is exciting and fun.

Another clinician noted:

I can use the knowledge I gained from teaching in the diagnostic clinic to become a better teacher. I will remember my work in the clinic and consider often if different activities or a different approach might help the learning process. I will also remember to praise students often when they are successful and try to be flexible with my teaching methods when I see frustration from my students. Most of all, I will try to be available to the students for the one-on-one contact that is often lost in the large classroom.

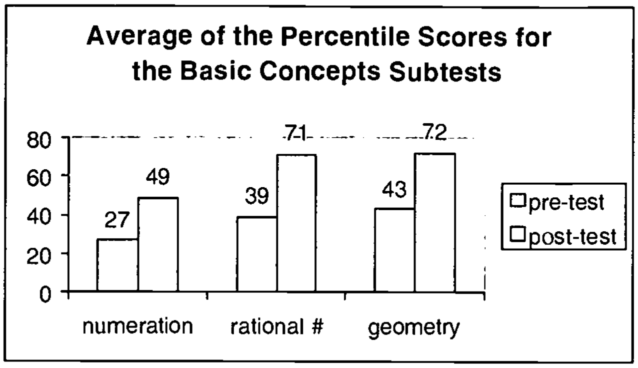
A preservice teacher reported:

The most important lesson I learned from working with Kristen in the mathematics clinic is accomplishments, accomplishments by Kristen and accomplishments by me. When Kristen felt good about correctly answering the problems assigned to her, a little grin would appear on her face; she would have this little smile that I interpreted as "I'm good." This had to be one of the most warming experiences I have ever felt. I knew I had done well. Not only had Kristen accomplished the task of learning, but I had accomplished the task of teaching. Diagnostic teaching is an attitude that cares very much about each student's learning. I will carry this attitude with me into the classroom.

Impact on Public School Students

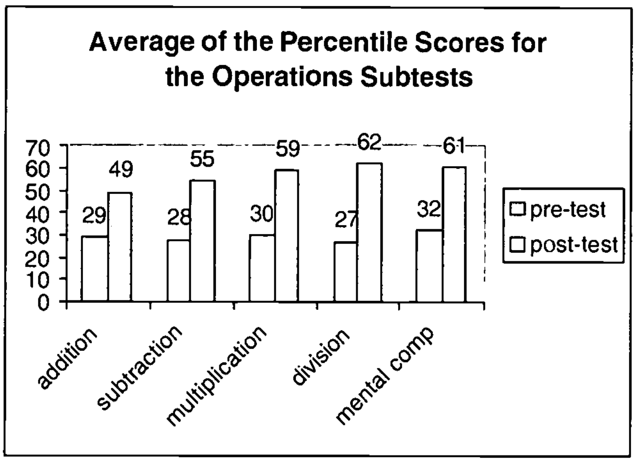
Public school students who participated in the Diagnostic Mathematics Clinic from Fall 1991 to Fall 1995 were surveyed. Fifty-five students who completed the program with both pre and post scores were included in this study. On the first day of the clinic, the university student administered the KMR to his/her student. On the tenth and final day of the clinic, the alternate form of the KMR was administered. A total of 45 students received tutoring in the Basic Concepts area of the KMR. Table I presents the mean scores for the Basic Concepts subtests.

Table 1



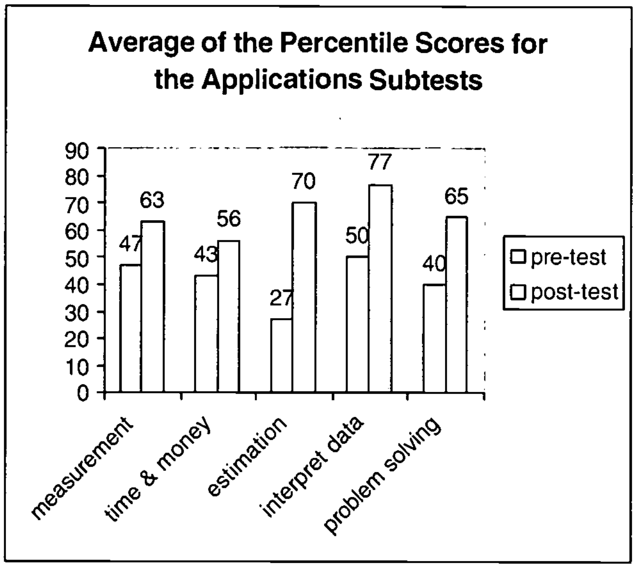
The second area of the KeyMath-R test is the area of Operations. Table 2 shows the mean scores for the Operations subtests. Forty-eight students received tutoring in the Operations subtests with more students receiving tutoring in subtraction than in any of the other subtests.

Table 2



The final area of the KeyMath-R diagnostic test is Applications. Fifty-four students received tutoring in this area. Table 3 presents the mean scores for the Applications subtests.

Table 3



Conclusion

The results of this study indicate the positive impact of clinical mathematical experiences on preservice mathematics teachers. Novice teachers have detailed how the clinical experiences have assisted them in focusing on both the student and instruction as they teach and plan to teach mathematics. The clinic gives university students the opportunity to practice mathematics instructional techniques with a student on a one-to-one basis and the confidence to try various manipulatives with students. In addition, the clinic provides teachers with practice in writing lesson plans, diagnosing students' problems, and reflecting on lessons taught. The impact of the clinical experiences of public school students is also significant. These students improved in their understanding of mathematics and informal assessment indicated a change in students' attitudes about themselves and mathematics.

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TEACHING AND LEARNING MATHEMATICS WITH VIRTUAL WORLDS

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ABSTRACT

Much of mathematics and statistics is taught using textbook examples and exercises. It is difficult through these to give students a feel for the broad issues involved. Ideally students could carry out experiments in the real world to then model and analyse mathematically. However, this is not practical for large classes and the logistics may even detract from the learning. We propose using virtual worlds instead, allowing students to manipulate the parameters of a simulated experiment and record the results. These simulations should be messy, requiring the students to think about measure ment issues and noise and how these impact on the mathematics. The results can then be used as a starting point for teaching, in place of the traditional exercise settings. We give experiences and feedback from several virtual worlds for statistics and dis cuss current work on virtual worlds for calculus.

1. Introduction

Teaching large service courses in calculus and statistics is a difficult challenge, requiring the motivation of students whose main interest is typically not mathematics and whose mathematical backgrounds can vary greatly. On the other side, student learning is affected by similar issues. Additionally a current concern for many students is time pressure and the need for efficiency. Lectures and laboratories based on **virtual worlds** aim to address these issues. They motivate theories and methods by providing a concrete setting, which relates to nonmathematical interests. They emphasise qualitative aspects of the course, allowing students with weaker mathematical backgrounds to gain confidence. And they achieve these aims without adding to student workloads, an important efficiency for large classes. The use of virtual worlds is also suitable for smaller and more specialised settings, and for secondary schools.

In this paper we describe some virtual world activities used in teaching an introductory statistics course. Section 2 describes an exercise using virtual rats, a simple enrichment of a textbook exercise. This is extended in Section 3 to a virtual world involving a more complex interface, with animated plants growing under various conditions, in which students have a more active role. Section 4 then looks at similar settings under development for teaching a comparable service course in calculus. We conclude with general discussion in Section 5 on the effectiveness of the approach.

The focus of the paper is on student experiences with the virtual worlds. Technological approaches to teaching such as this are often exciting for the developer but are ultimately worthless unless students enjoy them and, more importantly, can see that they will help their learning. At the end of the final laboratory session the students were surveyed to obtain feedback on their use of the various virtual worlds. The survey questions were very general, such as "Please comment on the use of virtual rats in Practical 2". Of particular interest is whether the students would identify the motivating principles behind the use of the virtual worlds. The comments received, both positive and negative, are given along with descriptions below.

2. Virtual Rats

Most introductory statistics textbooks are rich with real data sets, allowing students to relate the results of their explorations and analyses back to real scenarios. This is certainly desirable; Cobb and Moore (1997) suggest that "statistics requires a different *kind* of thinking [to mathematics], because data are not just numbers, they are numbers with a context." However, it is still a somewhat passive experience because the students have to take the context for granted. They have not been involved in obtaining the data and so lack ownership of the setting. Mackisack (1994) gives an overview of the other benefits of experimental work. For instance, the students also get an appreciation of the practical issues involved in carrying out experiments and collecting data, an outcome encouraged by Higgins (1999). The first aim of using virtual worlds is to engage the students in thinking about the design of the experiment and the origin of the variability in the data, while not allowing this to be so time consuming that the rest of their learning suffers. Additional emphasis on the practical issues mentioned above is provided by the virtual plants in Section 3.

As an example of a virtual experiment, consider a setting described by Moore and McCabe (1998) of a two-way analysis of variance, which involves an experiment for exploring the effects of calcium and magnesium on blood pressure. Three levels of each mineral in the diet of rats were considered, giving nine possible treatments to try. A standard textbook exercise would give the resulting data, or simply the summary statistics from the nine treatment groups, and then use questions to have the students visualise the data or test for main effects and interactions using ANOVA.

A simple interface for a virtual version of this experiment is given in Figure 1. Here the user can specify the calcium and magnesium levels (each Low, Medium, or High) for a particular rat and then click the measure button to find out its blood pressure after the treatment. Clicking again returns the blood pressure for another rat. An appreciation for the effect of the treatments and of the variability of results can be obtained easily in this simulation; carrying out this actual experiment would not be practical in a statistics course, especially with a large first-level class. A student commented that "it was an interesting way of collecting the data rather than simply being given data to work with." Another noted that "it was a good way to investigate the effects on rats without using actual rats", suggesting that they really identified with the virtual setting, despite the simplicity of the interface. Along similar lines, a further student appreciated the difference between the virtual world and the real world by saying that it was "better than using live ones as it cuts outside variables."

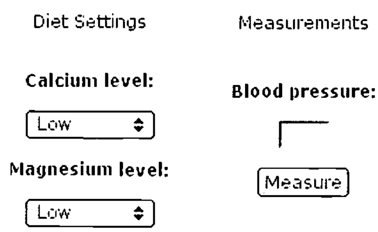

Figure 1. Blood pressure experiment

Using this in a computer laboratory exercise, the students are not told to try all nine possible treatments. Instead they are told that they have a budget of 30 rats and have to use these to explore the effects of the two minerals on blood pressure. They have to think about how best to do this, including such standard questions as determining the number of possible treatments. Indeed some will just look at the four treatments using the Low and High levels of each mineral, giving more observations for each treatment, while some will work with the full nine. One mineral can also be left fixed, allowing exercises in one-way ANOVA or in two-sample comparisons. One student complained that "the number of combinations we had to do was a bit tedious". However, this is the whole point of the exercise, to make the activity of data collection more concrete. In fact most students highlighted the speed of the process: "very easy to use, and being able to repeat the treatment so quickly makes it very efficient and time saving."

The virtual world is set up so that there is an identifiable interaction between the two minerals. This exercise is used at the beginning of the course, introducing main effects plots and interaction plots from which the students can detect and understand the interaction present. The lecture course itself does not say much about two-way analysis of variance, yet this simple exercise allows students to appreciate the ideas involved and the types of effects that can result.

3. Virtual Plants

The virtual rats have proved successful in getting students to think about statistical issues beyond the mechanical techniques that are typically emphasised. However, they leave out one important step and that is the measurement process. The way the measurements are carried out, with possible errors and biases, can have profound effects on the statistical analyses that are carried out. It is not clear, for example, whether the variability that the students observe in their virtual blood pressures is coming from differences in the rats or from errors in the measurements.

Simulating the measuring of blood pressure on rats is a difficult activity to capture in a concrete manner. An alternative setting was created using virtual plants (Bulmer, 2001). The interface, shown in Figure 2, is similar to that for the rats with two factors that can be controlled. Four levels of nitrogren fertilizer can be specified (None, Low, Moderate, and High) along with three levels of irrigation (None, Some, and Lots). However, rather than being able to obtain a series of measurements by clicking a button, the user instead gets a movie which shows the growth of 12 plants, 6 with one treatment and 6 with another. Figure 3 shows the last frame of one such movie; all plants received some water, but the plants at the rear had moderate fertilizer while the plants at the front had low fertilizer. A student noted that this was "better than [the rats practical] – could actually see the results in picture form which gave a better idea of what happened." The virtual plants were generated using L-systems (Prusinkiewicz and Lindenmayer, 1990), which has the side benefit of introducing curious students to some contemporary mathematics.

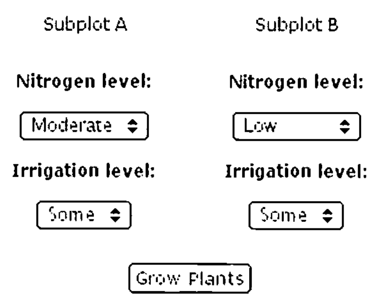

Figure 2. Plant growth experiment

Figure 3. Final frame of plant growth movie

Instead of being given a measure of growth for each plant, the students now have to deal with task. Students need to start by thinking about why they might be measuring plant growth. They could measure the heights of plants if they wanted to see which treatment gave taller plants, or they could count branches, leaves, or flowers if they wanted to see which gave higher yields.

Measurements of height can be made from the screen using a plastic ruler. It is quite a satisfying experience to walk into a computer laboratory and see a room full of students with rulers up to the screens. They are physically engaging with the setting, rather than passively taking a set of mysterious numbers from a textbook exercise, or even from a real study.

Measuring height with a ruler is difficult because it is hard to know where the top of a plant is; they branch outwards after an initial vertical growth. Counting flowers or leaves is difficult because it is hard to know you've seen them all, just as in real life. The measurement process should be difficult and students should have think about what simplifications or estimates they are making. One student complained that "it would have been more helpful and easier to use if some kind of variable was presented with the movie as a result (rather than making student measure it off the screen)" but most saw the purpose of the exercise in that it "made it more interesting than copying information out of a data set."

The main complaint was that the graphics were not clear enough to make accurate measurements, such as counting the number of leaves. Again this is partly useful, keeping the measurement process difficult, but it does not reflect reality very well. If students were working from a photograph of the field then they would not have the pixelation problem. A new set of virtual plants has been developed for 2002 using ray traced graphics to give much clearer images of the plants. One disadvantage of this is that it uses a perspective projection, rather than the parallel projection seen in Figure 3, which will make measurements of height more difficult. This may be offset by the inclusion of shadows and other visual cues.

4. Calculus and Dynamics

Service teaching in statistics is not the only area where students need to appreciate the relationship between the mathematical ideas and the underlying reality that it models. Much of calculus teaching is directed towards students in other disciplines, such as engineering and physical and biological sciences, for whom similar motivational issues are present.

Exercises are currently being developed based around such settings as a virtual pendulum and a virtual planetary system. The pendulum world is simply a series of videos of real pendulums with different string lengths and weights. This is a simple setting that could in fact be done quite easily by students, but again it may be more efficient to have the virtual world preprepared on the computer. Different students can collaborate on their analysis of the same physical setting. As with the plants, no measurements are given to the students. It is up to them to make measurements of the string length and the weight (which is an interesting visual problem) and then of periods and other dynamic quantities. The planetary system is computer generated, showing a fictitious system and allowing students to make measurements about radii (a non-trivial task for elliptical orbits) and motions.

These settings aim to work on two levels. Firstly, the students can graphically explore the relationships between the quantities they measure, such as the period and string length of the pendulums. This is appropriate for students at the secondary level, and leads on to the idea of summarising relationships using function curves. Secondly, students at the upper secondary and tertiary levels can look at mathematical models for these systems, use their measurements or external information to estimate the parameters in these models, and then compare their models with the observations. They can then look at discussing why there might be discrepancies between the model and the measurements.

5. Discussion

The virtual rats experiment was very simplistic, lacking any graphical output to immerse the user, but in the past four semesters it has consistently received positive feedback. Almost any statistical textbook exercise could be converted into such an experiment by first modelling the data and then using that model and random number generation to recreate the data "live". It gives students ownership of their work by putting the data into a context. It also means that students each have different data sets, which helps encourage cooperative learning (Magel, 1998) and broadens their learning experiences.

For the course in question, the students are also given projects in which they are asked to design and carry out their own experiments, followed by statistical analysis and the writing of a mock journal article. This is a rich form of assessment, but in a class of 500 it is difficult to have one-on-one discussions about each student's intended experiment and the issues they may face. The use of the virtual worlds in the laboratory setting allows students to discuss experimentation aspects in an immediate way, using the simulation to highlight important points.

In all the use of virtual worlds has been very effective, both in terms of time and in terms of engaging student interest. "It was efficient (non time consuming) and allowed for an understanding (better) through experiencing it (working it out)", as one student wrote. The virtual plants emphasise the "nonmathematical statistics" proposed by Higgins (1999) and along with other visual models, like the pendulum, are thus suitable for a range of mathematical abilities, particularly in secondary schools and in service courses. Fearnley-Sander (2001) has suggested a similar approach for teaching and learning algebra, and it is likely that virtual worlds have many other applications in mathematics education.

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AN APPROACH OF LINEAR ALGEBRA THROUGH EXAMPLES AND APPLICATIONS

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ABSTRACT

Linear algebra is a *language* which is used in all sciences (and beyond). For a class consisting of students in mathematics, computer science, physics, engineering, microtechnics, chemistry, we use a multidisciplinary approach to this field by example and application. Starting with linear systems, we extract the general features from three motivating examples.

In the first one, we show that it is impossible to cover a sphere with (curvilinear) hexagons only. In any subdivision using hexagons and pentagons, a fixed number of twelve pentagons is needed. This is shown by row operations on a system of 4 equations in 5 variables. Here, the *surprise* is that although the system is under-determined, one variable has a fixed value. Several natural examples may illustrate this necessity: Football ball, buckminsterfullerene C_{60}, architecture, protozoa... From the dodecahedron we get a special solution having no hexagons. All others are derived from this one by addition of a solution of the associated homogeneous system.

In the second example, we consider a chemical reaction (composition of the atmosphere, according to Lord RAYLEIGH), in which the coefficients have to be determined. The superposition principle for homogeneous systems appears quite naturally in this context.

Finally, to exhibit the power of the general principles, we consider a huge system obtained by digitalization of a potential on a grid. If the values are given on the boundary, then there is one and only one solution for which the value at each interior point is the mean value of the four neighboring points. It is indeed easy to show that the associated homogeneous system has only the trivial solution.

In our opinion, these motivating examples are accessible to undergraduate students. Linear equations may be amplified and added; thus linear combinations appear. They can be dependent, whence the interest in giving a maximal number of independent ones; here is the rank. Linear equations thus furnish an ideal approach for the language of vector spaces and their dimension.

Introduction

Linear algebra is a cornerstone in undergraduate mathematical education. It develops a general language used by all scientists and is interdisciplinary in essence. It hence evolves naturally towards abstraction. For most students, it is a first contact with modern mathematics. I propose to approach it by concrete examples. In this way, its power and relevance is immediately realized.

Let me only sketch here a possible start with linear systems, already furnishing a meaningful and valuable part of linear algebra. Two by two (and three by three?) systems may have been solved in high school. But now it is important to consider more general ones, and choose examples creating surprise, leading to questions, general methods..., with cultural relevance, aesthetic sense, or having as many of these qualities as possible!

1 First Example: Covering a Sphere with Hexagons and Pentagons

Question: Is it possible to cover the surface of a sphere with (curved) hexagons only?

Answer: From a bee "it is difficult!"; from EULER: "*It is impossible*!"

To prove the impossibility, we consider a generalization. Let us try to cover a sphere with hexagons and pentagons only. We know that this is possible.[1] The dodecahedron yields such a covering with 12 pentagons (and no hexagon). By convention, we juxtapose two polygons along a common edge, three polygons having a common vertex. It is easy to find a few equations, linking the unknown numbers of such polygons. More precisely, let us introduce

$$x : \text{number of pentagons}, \quad y : \text{number of hexagons},$$
$$e : \text{number of edges}, \quad f : \text{number of faces}, \quad v : \text{number of vertices}.$$

The number of faces is equal to the sum of the numbers of pentagons and hexagons, hence a first obvious relation: $f = x + y$. Since each pentagon has five edges, and each hexagon has six, the expression $5x + 6y$ counts twice the number of edges (edges belong to exactly two polygons). Hence a second relation $5x + 6y = 2e$. Our convention shows that the sum $5x + 6y$ also counts vertices three times and we get $5x + 6y = 3v$. From this follows $2e = 3v$, but this relation adds nothing new since it is a consequence of the previous ones. Another, more subtle relation was discovered by EULER, namely[2] $f + v = e + 2$. We have obtained a system consisting of four equations linking the five variables x, y, e, f, and v:

$$\begin{cases} x + y &= f, \\ 5x + 6y &= 2e, \\ 5x + 6y &= 3v, \\ f + v &= e + 2. \end{cases}$$

[1] Such configurations occur in architecture, sport, chemistry...

[2] It is valid for any decomposition of the sphere into polygons, with no restriction on the number of incidences at the vertices.

Grouping the variables in the left-hand side in the order e, f, v, x, y, these equations are

$$\begin{cases} f - x - y &= 0, \\ 2e - 5x - 6y &= 0, \\ 3s - 5x - 6y &= 0, \\ e - f - v &= -2. \end{cases}$$

To save space—this savings has enormous benefits—we replace an equation by the sequence of its coefficients, not forgetting to include a 0 in the place of a variable that does not appear explicitly. For example, the equation $f - x - y = 0$ stands for

$$0e + 1f + 0v - 1x - 1y = 0 \quad \text{abbreviated by the row } (0\ 1\ 0\ -1\ -1 \vdots 0),$$

separating the left- and right-hand sides by vertical dots. The whole system is thus

$$\begin{pmatrix} 0 & 1 & 0 & -1 & -1 & \vdots & 0 \\ 2 & 0 & 0 & -5 & -6 & \vdots & 0 \\ 0 & 0 & 3 & -5 & -6 & \vdots & 0 \\ 1 & -1 & -1 & 0 & 0 & \vdots & -2 \end{pmatrix}.$$

The big parentheses have the sole purpose of isolating the system from the context(!). It is advisable to start the enumeration by an equation containing the first variable, so we permute the first and last equations and obtain an equivalent system... As is explained in any linear algebra textbook, *row operations* may be used to bring the system into a staircase form

$$\begin{pmatrix} 1 & -1 & -1 & 0 & 0 & \vdots & -2 \\ 0 & 1 & 0 & -1 & -1 & \vdots & 0 \\ 0 & 0 & 1 & -3/2 & -2 & \vdots & 2 \\ 0 & 0 & 0 & -1/2 & 0 & \vdots & -6 \end{pmatrix}.$$

The last equation of this equivalent system is $-x/2 = -6$ implying $x = 12$.

Here comes a *surprise*: Although the system is *under-determined* (only four equations linking five variables), the number of pentagons in any subdivision of the sphere (into hexagons and pentagons only) is fixed and equal to 12. Isn't this remarkable! On the other hand, the number of hexagons is not fixed. Several natural examples illustrate this. (Recall that the audience is not necessarily interested in pure mathematics, so why not spend a few minutes to show the importance and ubiquity of the result found; a few slides may help.)

(a) We already mentioned that a partition of the sphere is easily obtained with twelve pentagons and no hexagon: $x = 12$ and $y = 0$ (simply project a regular dodecahedron onto the surface of a sphere).

(b) Another solution with $y = 20$ (and $x = 12$) is obtained as follows. Start with a regular icosahedron (12 vertices and 20 faces formed by equilateral triangles). Cut the vertices, replacing them by pentagonal faces (thus replacing the triangular faces by hexagonal ones). The polyhedron thus obtained has 60 vertices representing the positions of the carbon atoms in the buckminsterfullerene C_{60}.

(c) One can construct a geometrical solution with $y = 2$. Start with six pentagons attached to one hexagon. This roughly covers a hemisphere. Two such hemispheres—placed symmetrically—will cover the sphere.

General solution. Mathematically speaking, one can take for y any value—say $y = t$—and then

$$x = 12, \quad y = t, \quad e = 3t + 30, \quad f = t + 12, \quad v = 2t + 20.$$

provides the algebraic solution of the proposed linear system. [3]

General Principle. The general solution is the sum of the particular solution coming from the dodecahedron and the *general solution of the associated homogeneous system*, here depending on the choice of a parameter t (there is one free variable).

Further themes. (1) Construct infinitely many geometrical solutions with two groups of 6 pentagons (Hint: Consider two types of tubes). (2) What happens if the sphere is replaced by the surface of a torus? (The associated homogeneous system appears.)

2 Second Example: A Chemical Reaction

The first example has shown that homogeneous systems are both important and simpler to study. Let us turn to one of them. When Lord RAYLEIGH started his investigations on the composition of the atmosphere around 1894, he blew ammoniac and air on a red-hot copper wire and analysed the result. Let us imitate him, and consider a typical reaction of the form[4]

$$x\, NH_3 + y\, O_2 + z\, H_2 \rightarrow u\, H_2O + v\, N_2,$$

where the proportions $x, \ldots, v$ have to be found. Equilibrium of N-atoms requires $x = 2v$. Similarly, equilibrium of hydrogen atoms requires $3x + 2z = 2u$ and finally, for oxygen, we get $2y = u$. Proceeding systematically, we have to choose an order for the variables. We adopt their order of occurrence in the chemical reaction: x, y, z, u, and v, hence write the system in the form

$$\left\{ \begin{array}{lllllr} x & & & & -2v & = 0, \\ 3x & & +2z & -2u & & = 0, \\ & 2y & & -u & & = 0. \end{array} \right.$$

Now, observing that the right-hand sides are all zero, it is superfluous to include the last coefficient 0 common to all equations. Thus we simply replace the first equation by the row $(1\ 0\ 0\ 0\ -2)$, so that the system is represented by the array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 3 & 0 & 2 & -2 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{pmatrix}.$$

[3]Notice that many algebraic solutions have no geometric realization. For example, one may take $y = \frac{1}{2}$ $(x = 12)$ and adapt correspondingly $e = 31.5$, $f = 12.5$, $v = 21$. Similarly, one can take $y = -1$ together with $e = 27$, $f = 11$, $v = 18$. A necessary condition is that y should be a nonnegative integer! But this condition is not sufficient. There is no covering of the sphere consisting of twelve pentagons and just *one* hexagon.

[4]We add hydrogen for mathematical interest, but be careful of the explosive character!

From the second row (or equation), subtract three times the first one, and then, permute the second and third equations. This leads to the staircase shape system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 2 & -2 & 6 \end{pmatrix}.$$

The last equation is

$$2z - 2u + 6v = 0 \quad \text{or simply} \quad z - u + 3v = 0.$$

If we choose arbitrarily u and v—say $u = a$ and $v = b$—we have to take

$$z = a - 3b.$$

The second equation then leads to $2y = a$, and the first one furnishes $x = 2b$. Thus, for each choice of a pair of values for u and v, there is one and only one solution set[5]

$$\begin{cases} x = & 2b \\ y = & \frac{1}{2}a \\ z = & a - 3b \\ u = & a \\ v = & b \end{cases} \quad \text{or equivalently} \quad \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} = \begin{pmatrix} 2b \\ \frac{1}{2}a \\ a - 3b \\ a \\ b \end{pmatrix}.$$

Observations. This problem concerns proportions. We can deal with numbers of atoms, or numbers of moles.[6] If a solution is found, any *multiple* will also be one. We may also *add* or *combine* multiples of solutions to obtain new ones. A first case is given by the choice $u = 2$, $v = 0$, hence $x = 0$ (no ammoniac); it corresponds to the elementary reaction

$$O_2 + 2H_2 \rightarrow 2H_2O,$$

namely the synthesis of water. Another one—in which Lord RAYLEIGH was interested—is given by $u = 6$, $v = 2$, hence $z = 0$ (no danger of explosion!) which corresponds to the elementary reaction

$$4\,NH_3 + 3\,O_2 \rightarrow 6\,H_2O + 2\,N_2.$$

Any solution is a combination of these two basic solutions. The general solution of the system depends on *two arbitrary parameters*. It is easy to generalize.

Results. Any homogeneous system having more variables than relations has a nonzero solution. The solutions of a homogeneous system exhibit the following structure

- ◇ Any multiple of a solution is again a solution,
- ◇ The sum of two solutions is also a solution.

Linear equations (rows of a certain type) may be amplified and added; solutions (vertical lists) may similarly be combined. The language of vector spaces emerges in a relatively general context.

[5] A solution set is a list of solutions, written *vertically*.

[6] Each mole contains approximately $0.60221367 \times 10^{24}$ atoms. This is the *Avogadro number*, namely the number of atoms in 12g. of carbon, or the number of oxygen molecules O_2 in 32g. of oxygen, etc.

3 Third Example: Potentials on a Grid

It is important to realize that systems containing several hundred or even thousands of equations and variables occur frequently. These systems are often incompatible, or under-determined, and it is highly desirable to have efficient algorithms to discuss them. In particular, it is impossible to use tricks or guess work to solve them! This is why a systematic discussion has to be carried out. The first problem is that the alphabet is too poor to code so many variables and we have to number them, thereby ordering them:

$$x_1, x_2, x_3, \ldots, x_n.$$

As before, instead of the equation $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$, we simply write the row of its coefficients: $(a_1\ a_2\ a_3\ \ldots\ a_n \vdots b)$. At this point, one should explain the following

Basic Principle. A linear system having as many equations as variables can always be solved in a unique way if the rank of the associated homogeneous system is maximal. (Indeed, the reduced staircase system exhibits no compatibility condition, and there are no free variables.)

Consider now in the plane $\mathbf{R}^2$, a certain bounded domain D (e.g. a disc, the interior of an ellipse, or a rectangle, etc.). We are looking for a potential inside D, taking prescribed values on the boundary. To approach this physical problem, we introduce a square mesh in the plane, and only keep the squares having a nonempty intersection with D. We are left with a certain set of vertices P_i, edges and square faces. Here is an example

```
  + +         + +
  + + +     + + +
  + + + + + + + + +
  + + + + + + + + +
+ + + + + + + + + +
+ + + +     + + + +
+ + +         + + +
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+ + + +     + + +
+ + + + + + + + +
+ + + + + + + + +
  + + + + + + + +
    + + + + +
```

Replacing the boundary points by a bullet, we get

$$
\begin{array}{cccccccccc}
 & \circ & \bullet & & & & & \bullet & \circ & \\
 & \bullet & + & \bullet & & & \bullet & + & \bullet & \\
 & \bullet & + & + & \bullet & \bullet & + & + & + & \bullet \\
 & \bullet & + & + & + & + & + & + & + & \bullet \\
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\bullet & + & + & + & + & + & + & + & \bullet & \\
 & \bullet & + & + & + & + & + & \bullet & \circ & \\
 & & \bullet & \bullet & \bullet & \bullet & \bullet & & &
\end{array}
$$

The vertices which are not boundary ones have four neighbors, conveniently called North, East, South, and West. We are looking for a function (potential) defined at all interior points having the mean value property. Starting from known values at the boundary points,[7] we introduce variables x_i for the unknown values at the interior points P_i. If the four neighbors of an interior point P_i are P_p, P_q, P_r, and P_s, there is a corresponding equation

$$x_p + x_q + x_r + x_s = 4x_i.$$

Here, $p = N(i)$ is the index of the northern neighbor of P_i, etc. It may happen that all x_j are unknown, in which case we get a homogeneous equation

$$x_p + x_q + x_r + x_s - 4x_i = 0.$$

Or it may happen that certain values are prescribed, because the corresponding point lies on the boundary. For instance, we may encounter an equation of the form

$$x_p + x_q + x_r - 4x_i = -b_s$$

where b_s is the given value for the potential at the boundary point P_s. In any case, we group the unknown variables in the left-hand sides, while the known ones are gathered in the right-hand sides. Thus we get a linear system (S) for the variables x_i. *We are going to show that this linear system is compatible, and has a unique solution for each data on the boundary.*

If there are N interior points P_i, the system contains N variables x_i and also N equations: To prove that (S) has maximal rank $r = N$, we consider the associated homogeneous system (HS), simply obtained by requiring zero values on the boundary. In this case, it is enough to show that there is only one solution to the problem, namely the trivial one $x_i = 0$ for all indices i (corresponding to interior points P_i). Here is the crucial observation. For any solution set (x_i), select a variable x_j taking the maximal value (in a finite list, there is always a maximum!). Since this value x_j is the average of the four values at neighboring points, the only possibility is that these four values

[7] Certain boundary values may be irrelevant: Here, they are denoted by a $\circ$ instead of a $\bullet$.

are equal, and equal to the maximal value. Iterating this observation on neighboring points, we eventually reach a boundary point, where the value is 0. Hence the maximal value is itself 0. By symmetry, the minimal value is 0. Finally, we see that all $x_i = 0$, which proves the claim.[8]

4 Notes on a Teaching Approach

From my experience, the last example is much more difficult to grasp than the two preceding ones. But even if it is not possible to convey its significance, it will serve a purpose, namely to show that linear algebra is not a trivial matter. Linear equations in a large number of variables are used in extremely sophisticated situations, like weather forecasting, devising profiles for wings of supersonic planes, etc. This is not apparent on 4×5 examples, and is only suggested by the last example.

The preceding examples lead to the systematic elimination theory based on row operations. Each of them can easily fit in a one hour (or 45 min.) presentation, possibly followed by a discussion. In parallel exercise sessions (is it necessary to repeat that exercises constitute a must in the learning procedure?), one may try to lead the students to the *question* of the invariance of the rank. As soon as the vocabulary of independence, generation, and dimension is acquired, it is possible to give a positive answer.

It is widely recognized now that a *first part* of linear algebra should be devoted to linear systems, rank/dimension theory, linear maps and their kernels, eigenvectors (geometrical theory, incl. diagonalization). A *second part* should introduce inner product spaces with metric relations, orthogonality (Pythagoras theorem), best approximation (mean squares method). This is the "bilinear" part of linear algebra. Symmetric operators can be treated in this part (with their diagonalization). Finally, in a *third part*, the determinant is presented as a generalized volume, or volume amplification factor. Having some experience from bilinear algebra, the students may now grasp multilinearity. Applications abound with the characteristic polynomial. Spectral values for orthogonal, antisymmetric (and more generally normal) operators can be discussed.

It is important to me that a student able to follow only a first section of the course, can already apply it in his field. I hope that this type of introduction yields a valuable primer in linear algebra, complementing the classical approach by vectors in the usual 2- and 3-dimensional spaces.

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[8] The same reasoning shows more generally that any solution will take its values between the minimum and maximum on the boundary. Any solution attains both a maximum and a minimum at a boundary point.

ENHANCING MATHEMATICS TEACHER PROGRAMS AND RESPONDING TO THE SHORTAGE OF MATHEMATICS TEACHERS

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ABSTRACT

Through the Department of Mathematics the author has spearheaded many innovative courses and programs to improve the mathematics education of future teachers at all levels. This work has been recognized by a joint appointment to the Brock Faculty of Education. As co-chair of the Mathematics Education Forum of the Fields Institute for Research in the Mathematical Sciences, he has motivated strategies to address the shortage of mathematics teachers in Ontario. This presentation will consider the following: Too many middle school teachers in Ontario show a lack of understanding of and enthusiasm for mathematics. In 1990 the Mathematics Department,! with the collaboration of other Science Departments and the Faculty of Education, instituted a unique program for middle school teachers. To teach at the secondary level in Ontario an individual must present two subjects, a first teachable (a minimum of six university courses) and a second teachable (minimum of three university courses). Half of the teachers in Ontario teach mathematics with a second teachable qualification and with mathematical experiences gained in Service Courses. The Department of Mathematics has reviewed its programs and opened appropriate courses to students wanting mathematics as a second teachable. Teacher education in Ontario is principally consecutive, namely, teacher candidates apply to a Faculty of Education after a first degree. There are no mathematics requirements to qualify for elementary school teaching in Ontario. The author has instituted a mathematics course for future elementary teachers who did not complete their high school mathematics. This course is now required by the Brock Faculty of Education.

Ontario is facing a shortage of mathematics teachers. For three years, the Mathematics Education Forum of the Fields Institute has been developing strategies to address this concern. It is hoped that the sharing of these developments will help others to implement changes within their own educational systems.

Introduction

In Canada education is a provincial responsibility and in Ontario teacher education follows a consecutive model, where future teachers first complete a university degree and then apply to a Faculty of Education. Therefore the normal pattern is a three or four year undergraduate degree followed by one year in a Faculty of Education after which one is certified to teach in the Province of Ontario. Admission into Faculties of Education is based on a number of criteria including marks achieved in the undergraduate program, a portfolio, and undergraduate discipline requirements. For most programs in these Faculties there are far more applicants than positions and students must present an average of at least 75% in their undergraduate program. A portfolio outlines experiences with children, in schools, in camps, in tutoring situations, etc., and this can account for as much as 40% of the admission mark. Undergraduate discipline requirements depend on the school level certification. For the purpose of this presentation we shall summarize and simplify these requirements into elementary, middle, and high school certification. There are no subject specific requirements for elementary school certification and teachers are home-room teachers responsible for most disciplines. A minimum of three courses[1] in one subject taken from a list of 'teachable' subjects is required at the middle school level. These 'teachable' subjects include those that one would normally expect. Teachers at this level also teach across most disciplines. At the high school level candidates must present a minimum of six courses in one of a list of 'teachable' subjects and at least three courses taken from another subject from that list.

The consecutive teacher education model carries with it a number of implications for university mathematics departments and for groups interested in mathematics education. At the elementary level, mathematics departments need to be pro-active and offer a specially designed mathematics course, otherwise the present situation will continue where the great majority of elementary teachers enter their teaching career with very little understanding of mathematics, and how to teach it as a living discipline. Mathematics departments should be even more concerned about the mathematics background of teachers at the middle school level. Unfortunately very little has been done. In middle school students start to make the transition from arithmetic to algebra, in geometry they move from the visual/observational to the descriptive/analytical/relational, and they start their experiences in probability and data analysis. Middle school teachers need understanding of mathematics beyond an ability to perform a set of algorithms. At first sight undergraduate mathematics programs for secondary school teachers appear to be less problematic. But are they? Are mathematics teachers taking appropriate mathematics courses for their future career? Are they getting a breadth of experience in mathematics? What about future teachers who have a major in another discipline and have a minor of three courses in mathematics? Now that Ontario is experiencing a shortage of graduating teachers of mathematics, future teachers with mathematics as a minor will surely end up in a mathematics classroom. Are these future teachers selecting courses that provide a breadth of experience in mathematics and that present mathematics as a living discipline? Or is their mathematics a compendium of techniques? Do they understand what mathematics is and what

[1] A course in this context is a full year course.

mathematicians do? In this paper we present some of the initiatives that the Department of Mathematics at Brock University has implemented to address these many concerns.

Ontario has always had a shortage of middle school teachers who have any undergraduate background in mathematics. Recent data shows that Ontario is starting to experience a shortage of mathematics teachers at the high school level. Over the next five to ten years, and at the present rates of graduation, the number of new teachers is projected to meet only forty four percent of the demand for new mathematics teachers. This has implications for the government, for faculties of education and for departments of mathematics. The Mathematics Education Forum of the Fields Institute for Research in the Mathematical Sciences has undertaken a number of initiatives to address this concern.

Initiatives at Brock University

Brock University is a publicly funded university with just over twelve thousand undergraduate students. The Department of Mathematics plays a fundamental service role to many disciplines in the university and has an Honours program which attracts between twenty and thirty first year students each year. It also has joint programs with other disciplines and plays an active role in teacher education. In all courses and programs, students and faculty make extensive use of technology. Maple is used starting in the first year. In Statistics, Minitab and SAS are used. "Journey Through Calculus" and Geometer's Sketchpad provide learning tools in appropriate courses. The Honours Program is called MICA – Mathematics Integrating Computers and Applications. Within this Program students may select concentrations in Pure Mathematics, or Statistics, or Teacher Education, or others.

In the late eighties the author turned his attention to teacher education, especially to the education of future middle school mathematics teachers. Middle school mathematics plays a pivotal role in the the development of individual's understanding and progress in mathematics. Students start their transition from arithmetic to algebra, in geometry they move from the visual/observational to the descriptive/analytical/relational, and they begin experiences in probability and data analysis. To enhance the education of future middle school teachers in mathematics a Concurrent Program was developed on collaboration with members of the Faculty of Education and other members of the Faculty of Mathematics and Science.

In 1990 thirty students were admitted to this special program where they would do mathematics, science, and education concurrently. From the point of view of attracting students the timing was perfect. Mathematics and science graduates were finding it difficult to get places in Faculties of Education, because these Faculties had reduced the weighting on undergraduate program marks and had increased the weighting on the portfolio – evidence that applicants have worked with children or peers. In the Ontario context a Concurrent Program is attractive to students who aim to become teachers because the program guarantees them a place in Brock's Faculty of Education provided they continue to meet certain conditions involving marks, course selection, and so on. As expected the Concurrent Program continues to attract very good applicants, students who are interested and motivated in mathematics and science, and students who have a real desire to become teachers. Admission is done on the basis of marks and a letter that outlines the applicant's interest in teaching as demonstrated by activities with children or peers. The Program is highly structured and is demanding in its diversity of emphases. Students' have access to a Program director and a Program coordinator. The formation of peer groups is, for some students, the major reason for their success in the Program.

Professors report that concurrent education students form a real identifiable community, not only because they know each other and take most of their courses together, but also because they are proud to be in the Program. Members of faculty enjoy the dynamics that these students generate in their mathematics classes. They are eager to share their knowledge and are ready to ask questions. The Program consists of six (full year) courses in mathematics, three in different sciences, a number in education, one in child and youth studies, one in psychology, and one selected from the humanities. It aims to provide a breadth of experience while it retains a concentration in mathematics. Students must maintain a 75% average. In mathematics the students are exposed to different areas of mathematics, that include calculus, linear algebra, discrete mathematics, combinatorics, probability, statistics, geometry, applied abstract algebra, geometry, history of mathematics and teaching/learning mathematics at the middle school level.

There are many enrichment possibilities. Students can instruct in the annual Brock University residential mathematics and science camps organized for over 2000 middle school students in May and June. They can instruct in an annual camp for Aboriginal students, and in a camp for top Ontario grade 9 and 10 mathematics students. They can help in local and regional Science Fairs, and can participate in a government-sponsored program called "Tutors in the Classroom". Parents from the region can draw for assistance from the list of mathematics tutors maintained by the Department.

Over eight hundred students have graduated from this Concurrent Program and school boards are approaching the University specifically for these graduates. Because these students have completed enough mathematics and science courses to qualify for high school teaching a small number upgrade their teaching certificates. In general however most of them are teaching at the middle school level and are rapidly taking leadership roles with other teachers in their schools.

I believe that this Program is an example where a small but consistently implemented change can produce quite an effect in the educational system as a whole. I have tended to shy away from innovations that will not be sustained by the Department of Mathematics. When I started introducing technology in the mathematics courses in the mid-eighties most of my time was spent getting other faculty on board. There are too many examples of innovative courses and programs in departments of mathematics that have collapsed when the sustaining faculty member has moved out of them.

The Concurrent Program for future mathematics teachers at the middle school level suggested that the Department should play a more important role in the preparation of future mathematics teachers at the high school level. The Department extensively advertised the shortage of mathematics teachers and developed appropriate packages of mathematics courses for them. Finding an appropriate set of courses for majors was not difficult. What was a challenge was the selection of appropriate courses for those students who would be majoring in another discipline and would be seeking to complete three mathematics courses. The looming shortage of mathematics teachers would make it certain that these graduates would be placed in a mathematics classroom. While doing this, the Department of Mathematics also identified three appropriate courses for future middle school teachers, not in the Concurrent Program, who would be selecting mathematics as their 'teachable' subject. Whereas for high school teacher preparation it would make sense to require calculus and linear algebra, for the middle school level it would not be appropriate to allocate one and a half courses out of three to these two areas. The prerequisite structure of upper year mathematics courses made this a real challenge. Students without Calculus and Linear Algebra would not have access to courses in the history of mathematics nor to courses in geometry, two essential areas of mathematics for future teachers at the middle school level.

Three years ago the Department of Mathematics decided to completely review and restructure its curriculum. Although technology was the main, opening up courses to more students was another reason. The review had three objectives. It analyzed the impact that the availability of technology in every course had on curriculum and sequencing of mathematical concepts. It seriously explored what it meant to teach mathematics in this new environment, and it made every effort to open up the prerequisite structure of courses. The impact for future teachers was the splitting of both the geometry and the history of mathematics course with their first half not requiring calculus and linear algebra. The history of mathematics course at Brock is particularly useful for future teachers as it is sequenced historically and students do mathematics within the mathematical constraints of the time. Although one can always improve the content and approach in courses if they were only for future teachers, the Department of Mathematics believes that it has done the best it can with the resources it has.

The focus on future teachers at the elementary level is very much the author's interest and is informed by his cross appointment to the Faculty of Education. For the past two years I have been teaching a mathematics course for students who have not completed their high school mathematics but are hoping to teach at the elementary level. The prerequisite for this course is failure or an incomplete program in mathematics at the high school level. The course runs as a set of workshops using hands on materials and using Mason (1) type problems that the class works on until everyone is able to explain to a peer how they have completed the activity and understood the mathematics. The students are encouraged and coaxed to ask questions, to make hypotheses and not to get emotionally attached to them, to look for generalizations, to explore the nature of mathematics, to do simple mathematics in different ways, to consider how mathematics at the elementary level empowers students to do mathematics at higher levels, and to do explorations in a non-threatening environment. I get a lot of satisfaction from the noticeable progress of the majority of these students. By the end of the course most of them are able to work on substantial mathematical problems and they are capable to translate their understanding of mathematics as a human endeavour to the mathematics they will be teaching.

Initiatives by the Fields Mathematics Education Forum

The Fields Institute for Research in the Mathematical Sciences has mathematics education as one of its mandates. It achieves this responsibility through a Mathematics Education Forum that brings together individuals from universities (both from departments of mathematics and from faculties of education), from colleges, schools, industry and from business. The Forum is Chaired by the author and it has developed and completed a number of mathematics education initiatives both at the Provincial and National levels. One of the initiatives was to address, through the work of a Task Force, the looming shortage of mathematics teachers in Ontario. Because the Forum has a wide representation, it has a certain standing among communities that can impact the problem. The Task Force identified a number of aims. The first was to make Faculties of Education aware of the problem so that they may increase the intake of students who present a concentration of mathematics courses in their undergraduate degree. The second aim was to encourage departments of mathematics to reflect on their responsibilities for the education of future teachers. The Ontario consecutive model of teacher education will be most effective when departments of mathematics, within their programs, provide opportunities for future teachers to reflect on their learning of mathematics and when they offer environments that model good teaching practice. Future teachers benefit from a diversity of mathematical experiences that arise in courses from a variety of mathematical areas. They also benefit from the experience of different assessment practices. If these undergraduates have opportunities to tutor, to work in groups, and to assist teachers in schools, they will develop a better understanding of what teaching is all about. The third aim of the Task Force was to develop an advertising campaign directed at students in schools, colleges and universities. For this a Website (2) was developed and a poster advertising this site was sent to every Ontario high school, college and university.

Conclusions

The consecutive model of teacher education in Ontario provides opportunities for university departments of mathematics to influence and improve the mathematics preparation of future teachers. However to do so departments have to be proactive and have to consider what mathematics courses are most appropriate for future teachers at different school levels. Unfortunately few university departments of mathematics have been proactive and the number of applications to faculties of education by students who have mathematics in their undergraduate degree is insufficient to meet the demand for teachers of mathematics. The Department of Mathematics at Brock University has developed some innovative courses and programs for future teachers. The Mathematics Education Forum of the Fields Institute for Research in the Mathematical Sciences has and continues to address these concerns.

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POLYNOMIALS IN THE CONTEXT OF LINEAR ALGEBRA: EXPRESSIONS? SEQUENCES? FUNCTIONS? VECTORS?

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ABSTRACT

Problem: Textbooks offer different definitions for polynomials. Examples:

- Expressions over a ring;
- Infinite sequences;
- Functions from the ring of coefficients into itself.

Mathematical and epistemological implications of the different interpretations will be discussed.
Methodology: In a one semester Linear Algebra course polynomials were defined as functions but the coefficient-criterion for equivalence was assumed. Vectors were defined as elements of a Vector Space (systemic definition). After the course the students were interviewed about polynomials and their role as vectors.

Findings:

- Two of the above interpretations of polynomials were present in the students' responses: Expressions and functions.
- Students evoked images that were never introduced in class, such as a curve for a polynomial, and a floating oriented segment for a vector.
- Students experienced difficulties in consolidating their contradicting prototypes of vectors and polynomials.
- The coefficient-criterion for polynomial-equality was rarely applied.
- Only one student used the systemic definition of a vector.

What is a polynomial? Textbooks often offer the following definitions:

Polynomials as expressions over a ring: In this interpretations polynomials are defined as expressions of the form $\mathbf{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}$ where $\mathbf{x}$ is a symbol which has no particular meaning and a_j are elements of the ring (Dubinsky & al., 1994). As such, two polynomials are considered equal if equal powers have equal coefficients.

Polynomials as infinite sequences with elements in a ring, of which all but a finite number equal zero. While in the previous definition $\mathbf{x^i}$ had *no particular meaning*, here it is defined as

$$\mathbf{x^i} := \{ 0, 0, \ldots, \underset{i}{1}, 0 \ldots \}$$

Together with the following definitions: $\alpha := \{ \alpha, 0, 0, \ldots \}$

$f+g := \{ \alpha_0+\beta_0, \alpha_1+\beta_1, \ldots \}$

and $f \bullet g := \{ \Sigma_{i+j=0}\alpha_i\beta_j, \Sigma_{i+j=1}\alpha_i\beta_j, \ldots \}$

We then have that $f = \{ \alpha_0, \alpha_1, \ldots \alpha_r, 0, 0, \ldots \}$ can be expressed uniquely in the form $f = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_r x^r$.

Here again two polynomials are defined to be equal if and only if $\alpha_0 = \beta_0$, $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, ... (Curtis, 1974). We call this definition of equality via the coefficients "the coefficient criterion for equality".

Polynomials as functions p(x) from the ring or field of their coefficients into itself. Consider the following definition:

> Let $f = \sum \alpha_i x^i$, where all but a finite number of α_i equal zero. In order that $f \in F[x]$, we define $f(\xi)$ as follows: Let $\xi \in F$. We define an element $f(\xi) \in F$ by $f(\xi) = \sum \alpha_i \xi^i$, and call $f(\xi)$ the *value of the polynomial f when ξ is substituted for x.*

(Compare, for example to Curtis, 1974, p. 168.)

The equality of polynomial functions is taken to be that of functions:

$$f = g \text{ if and only if } f(\xi) = g(\xi) \text{ for all } \xi \in F.$$

A **theorem** follows:

Two polynomial functions f(x) and g(x) are equal if and only if $\alpha_i = \beta_i$ for all i.

Proof:

$\alpha_i = \beta_i$ for all i $\Rightarrow f = g$ is obvious.

Not so obvious, although seldom treated with students, is the other direction:

$f = g \Rightarrow \alpha_i = \beta_i$ for all i.

It is easily proven in C[x] and F[x] , relying on the differentiability of polynomial functions over C and F: For any i , differentiate the equal polynomials i times, substitute in the ith derivative 0 for **x**, and you get $\alpha_i = \beta_i$.

What about other fields? In fact, the Coefficient Criterion for Equality does not hold for Polynomial Functions over any Zp with P prime. In Zp[x], x^p and x, polynomials of different coefficients, are equal functions. This follows from **Fermat's Little Theorem**:

Let p be a prime which does not divide the integer a, then a(p-1) = 1 (mod p).

Sometimes Fermat's Little Theorem is presented in the following form:

Corollary:

Let p be a prime and a any integer, then $a^p \equiv a \pmod p$.

Research literature

Rauff (1994) deals with students' difficulties when operating on polynomials as expressions, without referring to them as functions. Harel (2000) claims that difficulties with vector spaces of functions in general (and of polynomials in particular) arise from the fact that the students have not formed the concept of a function as a mathematical object. To use his words: "as entities which they can treat as inputs for other operations" (there, p. 181). Using APOS terminology (Asiala & al., 1996) one might describe the need for the concept of function to have developed from action via process into object, in order for the student to be able to treat polynomials as members of a vector-space, and hence as vectors (The systemic definition of vectors, Syrpinska, 2000).

Dorier & al.(2000) treated vector spaces of polynomial functions and examined students' operating with specific values of the functions and their derivatives. So did Rogalsky (2000).

I did not find research on the flexibility required of students for shifting from one definition (interpretation) of the concept to the other, or the intuition students might or might not have regarding the coefficient criterion for equality.

Methodology of the reported research. Fifteen students took a one-semester course in linear algebra at a college for prospective high-school teachers of technological subjects. This was a first semester in the first year of their college training with no preparatory course in mathematics.

Teaching of this course tried to follow principles derived from the theoretical perspective APOS (Asiala et. Al 1996). Those teaching according to this perspective often use the ISETL software in undergraduate mathematics courses, but due to technical problems ISETL was used only partially in this course – only at its opening phase. Some ISETL activities were dedicated to the amelioration of the concept *function* in the minds of the students, bringing it closer to the level of object. It is accepted by APOS-oriented researchers that a significant development of the concept of function is a pre-requisite to the student's ability to construct adequate linear algebra concepts. For example: The construction of linear-combinations as functions with input scalars and vectors and output a single (new) vector ([1]). Similarly, the ability to treat polynomial functions as objects, to operate upon them the vector-space operations, and consider them members of a vector-space, also depends upon the student's previous development of a polynomial function as object.

A general characteristic of ISETL is that some of the more effective activities it enables can only be carried out on finite sets. Hence using ISETL in a linear algebra course naturally deals with finite fields Zp and vector-spaces over them. Hence a distinction between the different interpretations of polynomials arises in such a course.

<u>Polynomials and vectors in the course</u>

- The term Vector was first introduced with tuples, then broadened to other examples, and finally to the general (systemic) definition: A vector is an element of a vector space (See Fischbein, 1995, Sierpinska, 2000, on systemic thinking).
- Polynomials were defined as functions, and dealt with over R only.
- Vector spaces of polynomials over R were dealt with throughout the course;
- The coefficient-criterion for equivalence was presented and taken for granted (no proof).

[1]) RUMEC - Research in Undergraduate Mathematics Education Community (2001). *Initial genetic decompositions for topics in linear algebra.* Unpublished report.

The interviews

Of the 15 students, 12 agreed to be interviewed after the course. The interviews consisted of a structured questionnaire, were conducted individually, with each interview lasting about 45 minutes, and were video-recorded. Question no. 8 was constructed to examine the concept of polynomial:

Question no. 8:

One. What is this: $x^4 + x^3 + 7$.

Students who did not identify this expression as a polynomial were reminded of this term. If the question "what is a polynomial" was not brought up spontaneously, then the interviewer asked it.

Two. How does one check whether $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ and $b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0$ are equal? (In some cases, when the student could not relate to the general expression of polynomials, the question was repeated with explicit examples, such as $4x^5 + 3x + 1$ and $4x^5 - 2x + 1$).

Three. Is $x^4 + x^3 + 7$ a vector?

Four. What is a vector?

Responses to this question were organized according to the following aspects:

- What is a polynomial?
- Equality of polynomials.
- What is a vector?
- The confrontation of contradicting interpretations of vector and polynomial.

I will start with a lengthy analysis of a single student's interview.

Harve - What is a polynomial?

Int.: *What is this*? [Points at the written polynomial].

Har.: *A polynomial.*

Int.: *What is a polynomial*?

Har.: *A polynomial is a... an equation. Wait, actually it is not an equation, a polynomial is un... addition of..., with un,...and, well,... I want to say, no, at the beginning I wanted to say equation, but I don't have any equality-sign here, so I dropped it.*

Int.: *Well.*

Har.: *Eh, it's an expression, with, eh, of the deg, of the deg, of some particular degree. From degree two up it can be a polynomial.*

Harve – Equality of polynomials – A.

Int.: *O.K., just give me a minute* [writes: $4x^5 + 3x + 1 \qquad 4x^5 - 2x + 1$]. *Here are two polynomials.*

Harv.: *Ehem.*

Int.: *Are they equal*?

Harv.: [Thinks.]

Int.: *Equal or unequal*?

Harv.: *Maybe they are equal, when x equals zero. Only then they will be equal. Yes, then I..., Actually no, I don't understand...Ah, the x, yes, the x must be zero, for them to be equal. Any other digit...*

Harve – Equality of polynomials – B.

Int.: *But the polynomial, the polynomial itself.*

Harv.: *Polynomial to polynomial*?

Int.: *Yes.*

Harv.: *What does it mean, polynomial to polynomial? And what, you need...*

Int.: *What do you need to do in order to check, whether they are equal. That is the question.*

Harv.: *This means to start what, to reduce between them, to make equal* [writes an equality sign = between the two polynomials: $4x^5 + 3x + 1 \quad = \quad 4x^5 - 2x + 1$].

...

To find the x or something like that? No? ...To find what is the x itself, which maybe they will be equal.

Harve's relation to the polynomials brings into mind the *Action* conception of function, described within the APOS theoretic perspective:

> *An **action** is a repeatable mental or physical manipulation of objects. Such a conception of function would involve, for example, the ability to plug numbers into an algebraic expression and calculate. (Dubinsky and Harel, 1992)*

Within this perspective we also have a detailed description of the limitations of the action conception of function. Applied to polynomials, we can anticipate that a student whose conception of polynomial is limited to action conception of function, would probably be able to calculate (component per component) a linear combination of two polynomials, but will not be able to discuss and investigate characteristics of operations such as polynomial addition or multiplication by scalar. Hence he or she will find it difficult to consider whether a given set of polynomials is or is not a vector space.

Now we can sum up what we know about Harve's conception of a polynomial:

- A polynomial is not an *Equation*;
- It is an *Expression*;
- His function concept is at the *Action* level of development – far from the required level of *object*. This is especially evident when he asks*: What does it mean, polynomial to polynomial*? As if for him, polynomials are not comparable objects.
- Equality: Point-wise Equality, and for but some substitutions.

Let us look at Harve's responses in relation to the other aspects.

<u>Harve – Is a polynomial a vector?</u>

Int.: *Is the expression that we had here*, [reads and points at $x^4 + x^3 + 7$], *is it a vector*?

Harv.: [Thinks]...*A vector needs to have a size and a direction.* [Thinks] *And here...* [sighs], *I can't even turn it into a vector, what should I do, x to the power of 4 and x to the power of 3* [writes: $\begin{pmatrix} x^4 \\ x^3 \\ 7 \end{pmatrix}$] *and sev not, it does not seem right.*

<u>Harve – What is a vector?</u>

Int.*: O.K. So for you a vector is,... So what is a vector*?

Harv.: *A vector is a number that has both a size, its size, and the direction. That means, to which direction does it move…*

So for Harve, a vector is either a mathematical *thing* that has size and direction, or else, maybe a tuple.

Having analyzed meticulously the responses of one particular student, I will present other students' responses accumulated according to the suggested categories.

What is a polynomial.

The students related to three different interpretations of polynomials:

- Polynomials as equations
- Polynomials as meaningless expressions.
- Polynomials as functions.

Polynomials as equations.

Some students thought that polynomials were equations. Here are some such answers to the question *What is this* [x^4+x^3+7]?

Kid: *An equation.*

Guil: *Does it not have to be equal to zero?*

Jul: *This is an equation.*

Others, after considering this interpretation, rejected it. They concluded that x^4+x^3+7 was *not* and equation:

Harve: *A polynomial is a… an equation. Wait, actually it is not an equation*

And then:

…at the beginning I wanted to say equation, but I don't have any equality-sign here, so I dropped it.

Ala: *It is not an equation, as if, it is not equal to anything*

Mad: *It is not an equation.*

Polynomials as meaningless expressions:

Hersch: *A polynomial is…, a set of elements*

Ala: *…It is some exercise.*

And then:

Are these powers? 3 and 4 [Points at them]?

Int.: *Yes, it is* [reads] x^4+x^3+7.

Ala.: *Then it is an exercise*

Michel: *x to the power of 4 and …a number, two variables and a number.*

Exercise was considered in this category as I think that by this term students referred to some combination of mathematical symbols to be manipulated according to some syntactic rules.

Polynomials as functions

1st. Function as an input-output mechanism (*action* level in the development of the concept):

Harv. About $4x^5+3x+1$ and $4x^5-2x+1$:
Ah, the x, yes, the x must be zero, for them to be equal. Any other digit…

Mad said About x^4+x^3+7 :

If you substitute any number, it gives a result. ... Let's say 7, gives us 7. Zero, seven. ... ...one is nine.

2nd. Function as a graph

Joel: *When I have, when I have a polynomial* [says and writes:] $x^2 + x +$ *it looks like this* [draws:]

Joel was the only student to present a graphic interpretation of function.

The coefficient criterion for equality of polynomials

A. Knowledge

Lin is one example of a student who can be considered to actually know this criterion:

Int.: *Here are written letters, but suppose you had numbers, how would you check?*

Lin: *I should have, I would, would compare, ah, greatest power to greatest power.*

Int.: *Even if the greatest power here was 5 and the greatest power here was 3, you would...*

Lin: *No, no, no, the meaning is if the power here 5, as I said, and here 5, then I...*

Int.: *So what would you compare?*

Lin: *The..., if the powers were equal then I'd compare the numbers.*

Int.: *They are called coefficients.*

Lin: *Coefficients. And they are equal then they are equal, If they are not equal...*

Int.: *Wait, wait, if they are equal with the high powers then you can stop checking?*

Lin.: *No. no. I am speaking about, for example here it was 5 and 5* [points at both given polynomials, in letters]. *I'd look, 5, 5. And I'd look and see that the numbers are equal, that the coefficients are identical, then I'd move to the lower power, to 4 or 3, depends what, what was there.*

Why do I categorize this response under knowledge? Here I am using a description of knowledge used often by researchers who work within the theoretic perspective APOS:

> *A person's mathematical knowledge is her or his tendency to respond to certain kinds of perceived problem situations by constructing, reconstructing and organizing mental processes and objects to use with the situation.* (Dubinsky, E., 1989).

We might say that Lin did reconstruct and describe an action-scheme which uses the coefficient criterion for the check of the equality of two polynomials.

5 (out of 12) students explained a proper version of the coefficient criterion, and could be categorized as knowers.

The coefficient criterion for equality of polynomials

B. An enlarged criterion

Mad was another student who expressed knowledge (proper use) of the Coefficient Criterion, but he went further to enlarge this criterion into an invention of a kind of "order relation" between polynomials:

Int.: *Suppose these are two polynomials and we wrote here letters instead of the coefficients. Yes? If theses were numbers, how would you check if these two polynomials were equal?*

Mad: *You subtract this from that, if it equals zero, they are equal.*

Int.: *How subtract?*

Mad: *Here is degree 4, and here 4* [points at the appropriate components] *you take the coefficient of this minus the coefficient of that, the coefficients, if it is negative then this is smaller than that. If it is zero than they are equal, if it is positive then this is larger.*

What is a vector?

In order to analyse the student's ideas about polynomials and vectors, I will first present their ideas of what a vector was. First some concept images (Vinner, 1983)

1st. *Size and direction*

Harve: *A vector needs to have a size and direction.*

Hersch: *O.K., we said that the definition of vector is* **not** *something that has direction and size, which is what it usually is, so if not, every number, every mathematical operation, any part is a vector,... I don't have a definition.*

That is an example of a student's awareness to the conflict between his concept image and concept definition (Vinner, 1983).

B. *Comes out of zero*

Mad: *Something that comes out of zero and goes up to some point.*

Tania: *It is an axis, that comes out of the origin.*

C. *Joins two points*

Joel: *A vector..., it joins two points. ... And it has a direction.*

D. A tuple?

Harv. [about $\mathbf{x}^4 + \mathbf{x}^3 + 7$, after saying that *A vector needs to have a size and a direction.*] *And here...* [sighs], *I can't even turn it into a vector, what should I do, x to the power of 4 and x to the power of 3* [writes]:

$$\begin{bmatrix} x^4 \\ x^3 \\ 7 \end{bmatrix}$$

and seven. It's not, it does not seem right.

E. A element of a vector-space

Lin

Int.: *Is this polynomial* [points at $\mathbf{x}^4 + \mathbf{x}^3 + 7$]...*a vector?*

Lin.: *Is this polynomial a vector?* [Thinks.]

Int.: *How do you know if something is a vector?*

Lin.: *If, eh, a vector is actually a sub, it needs to be a vector-space. If it fulfills all the rules of a vector-space, and if it is a vector-space, which I do not...*

Int.: *Ah, do you hesitate because you do not remember whether it is a vector-space?*

Lin.: *No, no, I, yes, I do not remember if it is a vector-space.*

Int.: *O.K.*

Lin.: *If it does...*

Int.: *If it were a vector-space?*

Lin.: *Then yes, it is a vector.*

This is an example of the systemic thinking, discussed by Sierpinska (2000) and Fischbein et al. (1995). It means that a student is able to analyze a mathematical concept from the point of view of a system comprised of elements of its own kind, and their transformations. For Piaget, such ability indicates the organization of the concept *vector* into an operation (Piaget, 1975, 1976, Piaget and Inhelder, 1971). In terms of APOS it also means that the concept *polynomial* has developed, in the student's mind, into an *object.*

Confrontations between concepts

The last of response categories I present deals with confrontations between the student's concept of *polynomial* and his concept of *vector.* Two examples:

Mad

Int.: *This, the polynomial we have started with* [points at $\mathbf{x}^4 + \mathbf{x}^3 + 7$ *]. Is it a vector?*

Mad: *If you substitute any number, it gives a result.*

Int.: *And that's why it is a vector?*

Mad: *Let's say 7, gives us 7. Zero, seven.*

Int.: *Ehem.*

Mad: *One is nine.*

Int.: *And that's why it is a vector?*

Mad: *Yes.*

Int.: *What is a vector?*

Mad: *It's, let's say, something that comes out of zero and goes up to some point.*

Int.: *And this* [points at the polynomial] *comes out of zero and goes up to some point?*

Mad: *Not in every case, only in the substitution of zero.*

Int.: *Ehem.*

Mad: *If you substitute zero gives us zero seven.*

Int.: *Ehem.*

Mad: *And this is not a vector.*

So we can see that for Mad, a vector is something that *comes out of 0*, while for polynomial he has an action concept of function (substitution). In the confrontation between the two he first thinks that *Yes, it is a vector,* because of (0,7), but finally he concludes that *this is not a vector*, perhaps because *Not in every case, only in the substitution of zero.*

Joel contributes our second example of confrontation. His is a confrontation between two graphical concept images, that of a polynomial as a graph of the function, and that of a vector as an arrow that joins two points. We have quoted him before in relation to each of these concepts separately. Here is his full discussion of $\mathbf{x}^4 + \mathbf{x}^3 + 7$ both as vector and polynomial:

Int.: *Is this polynomial* [points at $x^4 + x^3 + 7$] a vector?

Joel.: *Is it a vector? Ah, yes, it is a vector, yes.*

...

Int.: *What is called "vector"?*

...

Joel.: *A vector..., it joins two points.*

Int.: *A polynomial joins two points?* [Points at the polynomial.]

Joel.: *Ah?*

Int.: *Does a polynomial join two points?*

Joel.: *When I have, when I have a polynomial* [says and writes:] $x^2 + x +$ *it looks like this* [draws:]

Int.: *Ehem*

Joel.: *But this is not a vector* [adds a cord with an arrow:]

But the vector is between two points.

Int.: *So wait, wait, the arrow is a vector because it joins two points.*

Joel.: *And it has a direction.*

Int.: *O.K. So why is the polynomial a vector?*

Joel.: [Thinks] *Why is the polynomial a vector? Good question. I need to think of it.*

Conclusions

In a subject matter as difficult as linear algebra, even the "simplest" objects, polynomials, which are supposed to serve as "familiar" examples of the more abstract ideas, turn out to be interpreted in many different ways, both in the mathematics and in the students' minds. The consolidation of these interpretations poses problems for both teacher and student. How could research help us here?

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WRITING IN A REFORMED DIFFERENTIAL EQUATIONS CLASS

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ABSTRACT

In an attempt to promote the development of understanding over rote memorization, writing in mathematics has received increased attention in recent years. In Calculus, the Rule of Three (based on communicating ideas through *algebraic, graphical* and *numerical* means) has been replaced by the Rule of Four in which *writing* plays a central role. Educators agree that the benefits of writing include the promotion of understanding, and the initiation of the posing of questions. Writing also helps generate meaning, and helps in the retentio n of content. In this paper, I evaluate the use of writing for analyzing a problem and its solution. The setting is a reformed differential equations class offered at the Lebanese American University. Unlike a traditional ode course where students are provided with a cookbook of methods for solving differential equations, the emphasis in a reformed ode course is placed on the geometry of the solutions and on an analysis of the outcomes. In many instances, students are asked to solve a differential equation by plotting its solution curves without identifying them analytically, and the sketch is to be supplemented by an argument justifying it. In addition, various real life problems are modeled and essay questions are asked to analyze the graphs describing these models. Results show that students first reject the idea, but later rate writing as essential. Furthermore, an improvement in the style and content of the writing exercises is usually noticeable at the end of each semester.

Keywords: Reformed differential equations curriculum; writing in mathematics.

1. Introduction

In recent years, the curriculum of ordinary differential equations has undergone fundamental changes in favor of the visual aspect of the field. Traditionally, differential equations were taught in a very mechanical way: Equations are usually classified, and for each class a method of solution is presented. Since differential equations are widely used in engineering and the physical sciences, this mechanical approach has defeated the purpose of the course as an aid to understanding real life problems (such as the harmonic oscillator, predator-prey models, competing species models, and others.) The traditional approach to teaching differential equations has its roots in the way Calculus has been taught throughout the past centuries. Even though the ideas of Calculus were inspired by problems in astronomy, and even though Calculus later showed to be very useful for answering questions in various sciences, this mathematical field has been taught traditionally as a set of rules and procedures with very little reference to its uses in the real world. More than a decade ago, educators and researchers begun questioning this approach for teaching Calculus, and many discovered that teachers and students alike are "losing sight of both the mathematics and of its practical value" (Hughes-Hallett, Gleason, 1998, p. v). Following the first program announcement for Calculus reform of the National Science Foundation in the United States, many math instructors begun re-designing their classes, and many of them emphasize now the algebraic, the visual, and the numerical aspects of the field (the Rule of Three). Clearly, the development of advanced graphing calculators and of dynamical computer programs was a contributing factor to the adoption of this approach. More recently, *writing* was added to the Rule of Three. According to Hallett, Gleason, et al., students need to learn "to reason with the intuitive ideas and explain the reasoning clearly in plain English" (p. vi). In general, researchers agree that the benefits of writing include the promotion of understanding, and the initiation of the posing of questions; writing also helps generate meaning, and helps in the retention of content (Rose, 1989, 1990).

Differential equations are a beautiful application of the ideas and techniques of calculus to solve various real life problems. Consequently, the new approach for teaching calculus lead to a similar approach for teaching differential equations. In the article "Teaching Differential Equations with a Dynamical Systems Viewpoint", P. Blanchard (1994, p. 385) suggests that teachers do not give any more equations for which explicit solutions exist, but rather use computers and graphing calculators to graph the approximate solutions of a differential equation and require students to interpret and justify what they see. In the book Differential Equations by Blanchard, Devaney and Hall (1998), the authors write (p. v), " the traditional emphasis on specialized tricks and techniques for solving differential equations is no longer appropriate given the technology that is readily available.... Many of the most important differential equations are nonlinear, and numerical and qualitative techniques are more effective than analytic techniques in this setting." Addressing the students, the authors add that many exercises of the book ask to analyze models and to explain verbally the conclusions. Thus, in the new ode curriculum, writing is as essential as the solu tion process itself.

Research on writing in mathematics is not very extensive yet. In the literature, some papers and books have emphasized the skills required to write a good mathematical proof. (e.g. MAA Notes 14 (1989)); others, such as J. Meier & T. Rishel (1998), M. Porter & O. Joanna (1995), A. Schurle (1991), have discussed the effects of writing on the learning itself. In particular, Schurle discusses whether writing helps students learn about differential equations. However, the curriculum adopted by the author is the traditional one. In this paper, I will assess primarily writing as a tool for analyzing and understanding results obtained mostly geometrically in a reformed ode course. Writing to understand concepts is in addition evaluated.

2. The New ODE Curriculum

An ordinary differential equation of order n is an equation of the form:

$$\frac{d^n y}{dt^n} = f(t, y, \frac{dy}{dt}, \frac{d^2 y}{dt^2}, \ldots, \frac{d^{n-1} y}{dt^{n-1}})$$

Finding a solution to this equation means finding a function $y(t)$ satisfying that equation. Analytically, this requires expressing $y(t)$ implicitly or explicitly in terms of t. In a traditional differential equations course, analytical methods of solution are described for very specific types of equations. In a reformed course however, more emphasis is placed on the geometry of the solutions. In many instances, solutions are drawn without a slight knowledge of their analytic representations, and students are expected to read information from these graphs. For instance, in studying the logistic population model $\frac{dP}{dt} = kP(1-\frac{P}{N})$ (a first order differential equation), students are expected to read from the slope field the growth of the population given any initial condition (See Figure 1). In studying harmonic oscillators, second-order equations $\frac{d^2 y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = 0$ are transformed into systems of the form $\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -\frac{k}{m}y - \frac{b}{m}v \end{cases}$, and students are expected to read from its vector field the change in the position as well as in the velocity of the motion of a mass attached to a spring (see Figure 2). Clearly, this qualitative approach for solving differential equations gives a new dimension to the field of differential equations since in the traditional setting, rarely were students asked to interpret solutions that were obtained analytically.

3. The Setting

The course, Ordinary Differential Equations, as is offered at the Lebanese American University in Beirut, is a 3credit course aimed at engineering students who have taken prior to it the calculus sequence. Before enrolling in a school of engineering at any university, students of Lebanon have to pass the official baccalaureate exam (mathematics section) offered at the end of their secondary school years. Teaching in Lebanon is still traditional. Only in few private schools are graphing calculators and computers in use. Yet, the teaching of 3[d] and 4[h] semester calculus at the Lebanese American University incorporates the use of *Mathematica* in the form of projects combining the geometric and the analytic sides of mathematics. The class meets three times a week (50-minute sessions) in a regular classroom. The book adopted for the past three years has been Differential Equations by P. Blanchard, R. Devaney & G. Hall, a reformed text that emphasizes the geometric approach and analyses of outcomes. Furthermore, two computer software programs are used regularly: ODE Architect, a multimedia tool with enormous visual capabilities and generally used for classroom presentations; and

Interactive Differential Equations (IDE), a collection of labs designed to build a complete understanding of a particular concept. Computer homework are usually assigned from IDE and they generally require a great deal of visual observations that can only be communicated though writing.

4. Sample Writing Exercises and the Students' Reactions

As mentioned above, the book of Blanchard, Devaney and Hall emphasizes the geometrical approach to differential equations and requires analyses of outcomes. The authors for instance introduce the idea of a differential equation by modeling a population growth problem. According to them, how the differential equation is written is not of much importance; the importance lies in "what the equation tells us about the situation being modeled" (p5). Throughout the section, various models are solved geometrically and discussed primarily in a verbal manner. Exercises fall also in the same line of thought. For instance, in one exercise (p. 15), learning is modeled by the differential equation $\frac{dL}{dt} = 2(1 - L)$, where $0 \leq L(t) \leq 1$ is the fraction of a list learned at time t. One question asks students to analyze whether a person who starts up knowing none of the list can ever catch up with another who starts up knowing half of the list. Similarly, many problems that I give on exams always require some verbal discussions. Some questions for instance ask to analyze results obtained geometrically such as: Given a slope field of a first order differential equation, draw a representative collection of solutions and describe verbally the main similarities and differences of solutions to various initial value problems; or: Associate differential equations with slope fields and justify the answer with a short paragraph; or: Identify systems as being Predator-Prey or Competing Species systems, and write a small paragraph justifying the identification; in particular discuss what happens when one of the species is extinct.

Other exam questions require writing essays to examine the level of theoretical understanding. For instance, one might ask students to discuss the existence and uniqueness of solutions to initial value problems. Another question that I add frequently to my tests is about the general linear system:

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy.$$

Students are asked to discuss in an essay the condition(s) that a, b, c, and d have to satisfy in order to obtain for instance two distinct real eigenvalues. Then they are asked to discuss the different kinds of phase portraits that can occur in this case

Assignments from the workbook Interactive Differential Equations also encourage students to explore mathematical concepts through writing. In one favorite exercise, the love affair between Romeo and Juliet is modeled by a linear system of differential equations (see system above). The values of a, b, c, and d are changed to reflect new factors affecting the relationship. Questions posed require in most cases an analysis of feelings. Here is a sample: What are Juliet's feelings for Romeo when he is most attracted to her? What do you expect to happen to the relationship? Suppose the two lovers had exactly the same emotional profile in terms of their response to each other and their responses to their feelings ($\frac{dx}{dt} = ax + by, \frac{dy}{dt} = bx + ay$), investigate some situations and write a small paragraph.

Do students accept the idea of writing essays in mathematics? And how do they react to questions of this sort?

In the beginning, most students reject somehow the idea of writing in mathematics. As one student puts it: This is a Math, not an English class! Students have been trained in schools to solve any mathematical problem in a mechanical way; a discussion of the problem and a justification of its outcome have rarely been considered important. Consequently, the idea of writing in mathematics is alien to them. In fact, students of a reformed ode class have to adapt first to the idea of solving a mathematical problem geometrically rather than analytically. In Habre (2000), I investigated strategies for solving a differential equation as adopted by students of a reformed ode course offered in the United States. Results showed for instance that most students think primarily of analytic solution techniques; only few showed approval of the qualitative approach, while all the others had serious reservations about it. Since writing is a consequence of the geometric approach, it is not surprising therefore that most students initially reject this idea. For instance in the exercise modeling learning, only 36% of the students investigated discussed it verbally. It was unfortunate that by the time it was due, the analytic technique for solving a separable differential equation had been discussed in class. Consequently, 43% of the students tried to solve the problem analytically. The remaining students combined both approaches perhaps in an attempt to justify analytically what they had discussed verbally. By the time the first exam is usually given, students are in general more adapted to the idea of writing, yet their problem lies in not knowing how much writing is required. In many cases, their verbal discussions become lengthier as time goes on. As for the content, many discussions include the right amount of information needed and some may be even lengthy; but there are always students who do not write enough and students who cannot accept the idea of writing in Mathematics. Figures 3, 4, and 5 show a sample of students' answers to the Predator-Prey essay question:

$$\frac{dx}{dt} = 10x(1 - \frac{x}{5}) - 8xy$$
$$\frac{dy}{dt} = -4y + \frac{1}{10}xy$$

In Figure 3, the analysis of the student is almost complete with a detailed description of the behavior of the predator and prey population. This student writes: " [This] is a predator –prey system. When $x > 0, \frac{1}{10}xy > 0$ this has a positive effect on $\frac{dy}{dt}$ which makes the predator y grows because they are eating preys. When $y > 0, -8xy < 0$ this has a negative effect on $\frac{dx}{dt}$ which make the prey x decay because they are eaten by the predators. When x = 0, preys are extinct; $\frac{dy}{dt} = -4y$ [then] the predators will decay because they don't have food to eat $(-4y < 0)$. When y = 0, [then] predators are extinct, [then], the preys will grow according to a logistic model since there are no predators to eat them." Figure 4 on the other hand is the work of a student who seems to have understood the system but did not write enough. The student writes: " Here we have a predator-prey model because if the predator decreases or decays, [then] the prey will grow in logistic model; however if the prey decays, the predator will also decay exponentially…" In this writing exercise, this student was not specific as to which variable represents the prey and which represents the predator. The student also did not discuss explicitly the effect of the positive and negative signs of the mixed terms. In my opinion, this student may be one of those who do not know how much writing is required in problems of this kind. As for Figure 5, it shows the work of a student who simply writes very little. For many students however, writing is seen at the end of the semester as an essential component in the learning process. In one questionnaire distributed at the end of one semester, students were asked to answer to the following question: In many instances, writing was essential to communicate an idea/concept. What is your opinion on writing in mathematics (differential equations in particular)? Your opinion should be independent of your English capabilities. All the students who responded to this question agreed that writing was essential in the course. Some reasoned (rightfully) that writing complements the geometrical approach adopted in the solution process:

> "Since the course stresses on the geometrical way of solving DE's, this makes the writing very essential for the student to be able to express and tell the way or the steps followed in solving and drawing the solution."
>
> "Writing is very important in this course especially when no analytical solution is attainable. It is useful to describe the behavior of solutions where we have geometrical approach. Even when we have a quantitative solution, we need to explain it to let others understand what the equations we have written express."

Others argued that writing was also necessary for enhancing the learning and for showing that concepts have indeed been understood:

> "I found absolutely no problem in the "essay questions" on exams and in homework. In fact, I think that they were very useful because they clarify concepts in our mind. Once we write to explain an idea in our own words (often with the aid of sketches), we make sure we fully understand it."
>
> "Generally, writing in mathematics is very important. Personally I think solving an equation by only using the mathematical symbols without explaining the procedures followed isn't that good because it may become a procedure done by heart, while with writing and explaining the professors can make sure if the students understand the material."
>
> "Writing is an essential and useful process in mathematics.... For example, when solving a system analytically, a student may solve it either by chance or by cheating in some instances! So writing provides the instructor about the student's understanding of the subject taken."

In conclusion, the eventual positive reaction of the students concerning writing is extremely encouraging. It is comforting to know that students do consider writing as a tool to enhance learning as well as a tool to clarify ideas presented geometrically. However, I think that students will always ask questions such as: How much is enough? How detailed should I be? Or: Is this what you want? It has proven difficult to answer these questions and consequently to grade essay questions. However, time and practice will certainly improve the style in which essay questions are asked. This in turn should help students know what exactly to write, and help instructors in the grading process.

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Figure 1. The slope field of the population model & some solutions.

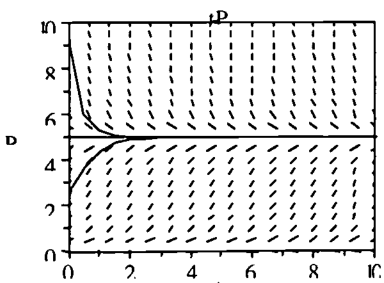

Figure 2. The vector filed of a simple harmonic oscillator, one solution curve, and its tx and ty time series.

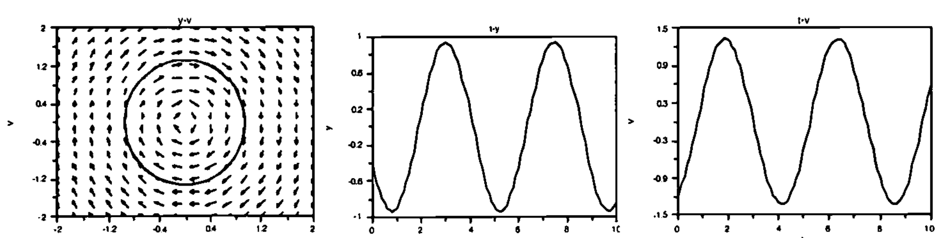

Figure 3. An almost complete analysis of the predator-prey system.

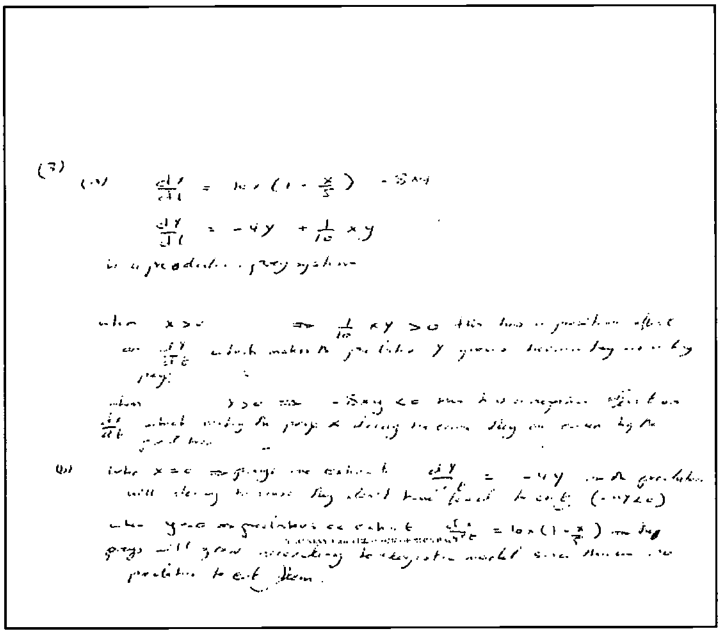

Figure 4. A student who does not seem to know how much and what to write in analyzing the predator-prey system.

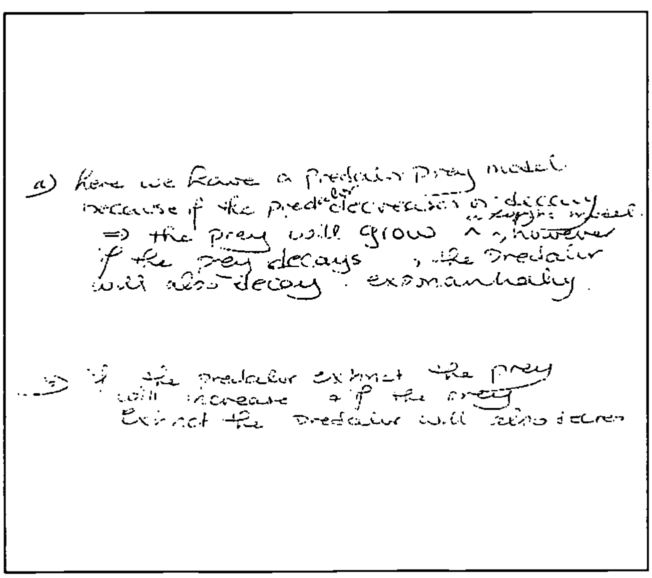

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Figure 5. A student who writes very little.

a) If $y = 0$ · $\frac{dx}{dt} = 10x - 2x^2$ — x = prey, y = predator

the prey obey a logistic model.

If $x = 0$: $\frac{dy}{dt} = -4y$.

the predator population obey an exponential decay model.

b) the population becomes extinct if $x = 0$ (for example).

$\frac{dx}{dt} = 0$ — if there is no prey

$\frac{dy}{dt} = -4y$ — the predator population will decrease exponentially

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CALCULATION OF AREAS: The discussion of a mathematical-historical problem that exposes students' conceptions of Proofs

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KEYWORDS: Proof, refutation, truth, example, counter-example, certainty.

ABSTRACT

The present study constitutes an attempt to check students' conceptions about the nature and the significance of mathematical proofs. The setting of this study was a mathematical-historical discussion within the framework of a course dealing with the development of mathematics. The students - elementary school pre-service mathematics teachers - were exposed to some problems taken from the Egyptian mathematics. After the lesson – that included the presentation of a formal proof of the main statement discussed - the students were asked to answer individually and in writing questions concerning the Egyptian method to calculate the area of a quadrilateral. The analysis of their answers reinforces the conception that pre-service teachers may know how to perform the "ceremony" of proof but in general, they do not appropriately conceive its meaning or its role establishing truth in mathematics.

1. Introduction

Already in the eighties, researches have shown that students do not quite understand the essence and significance of mathematical proof although they are generally capable of performing the "ceremony" of proof. This result fits Arthur Eddington's image, that as far as they are concerned "*Proof is the idol before whom the pure mathematician tortures himself*".

In the study conducted by Fishbein & Kedem (1992) students received a proof of a mathematics statement and then were asked to state whether further concrete examples were required in order to establish the truth of the same statement. Its main finding showed that, although most students claimed that they had understood the proof, they felt that they should examine further examples in order to consider whether it is true or not.

The present study constitutes another attempt to check students' conceptions about the nature and the significance of mathematical proofs.

2. The Study

Following Hanna (1996), I believe that "... proof deserves a prominent place in the curriculum because it continues to be a central feature of mathematics itself, as the preferred method of verification, and because it is a valuable tool for promoting mathematical understanding." (Hanna, 1996, p.22). And although mathematics teachers in elementary schools do not generally deal directly with proofs of mathematical statements, they must know and understand the legitimate mathematical methods to establish the validity of a statement. Moreover, as mathematics teachers they must teach their students to justify their assertions and how to present these justifications in a manner that would convince the others that their claims are valid. It is therefore important to examine mathematics students' and teachers' conceptions of *proof* and to what extent they are aware of the various functions of proof as a mathematical activity.

The setting of this study was a mathematical-historical discussion within the framework of a course dealing with the development of mathematics. The students - elementary school pre-service mathematics teachers - were exposed to some problems taken from the Egyptian mathematics. The population included 25 students majoring in mathematics teaching for elementary schools. This population consisted of 18 females and 7 males. The students belonged to the Jewish sector (n_1=13) and to the Arab sector (n_2=12). During the meeting the students communicated in Hebrew.

In general, the framework of this college course on the development of mathematics enables:

a. The review of contents with which the students are familiar – i.e.: calculation of areas and the review of contents they will teach (area, quadrilaterals and their properties);

b. The exposure of students to the need for more substantial tools in mathematics than measurement, observation and experimentation;

c. The exposure of students to the idea of proof, to different kinds of proofs and to discuss "what giving a proof means";

d. The exploration of students' conceptions of what constitutes evidence in mathematics and what the roles of proof are.

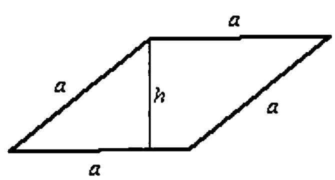
In a two-hours meeting they were shown a way to calculate the area of a quadrilateral as it appears in the Rhind Papyrus (Eves, 1982, p. 14)[1]. The meeting was designed as follows:

a. Calculation of the area of a *square* when the length of the sides is known.

b. Calculation of the area of *rectangle* when the length of each side is known.

c. For each one of these quadrilaterals, the teacher presented the calculation of its area according to the Egyptian method.

d. An attempt to calculate the area of a *rhombus* where the length of its sides is known and the teacher's presentation of the calculation according to the Egyptian method. (See Figure 1)

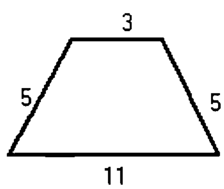


$$S_E = \frac{(a+a)}{2} \cdot \frac{(a+a)}{2} = a^2$$

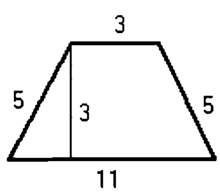
$$S = a \cdot h \leq a^2$$

Figure 1

e. Calculation of the area of an *isosceles trapezoid* where the length of its sides is known and the teacher's presentation of the calculation according to the Egyptian method (figure 2).



$$S_E = \frac{(3+11)}{2} \cdot \frac{(5+5)}{2} = 30$$



$$S = \frac{(3+11) \cdot 3}{2} = 21$$

Figure 2

f. Short discussion on the differences between the two values obtained in section (v)

g. The teacher asked the participants to build a quadrilateral – assisted by a suitable software – where the area calculated according to the Egyptian method (S_E) is smaller than its area calculated according to "our" method (S).

[1] The Egyptian method for finding the area of the general quadrilateral is to take the product of the arithmetic means of the opposite sides.

h. After such an example was not found, some students suspected that such a quadrilateral does not exist. In other words, the following conjecture was formulated: *In every quadrilateral, the number obtained from the Egyptian method (S_E) is bigger or equal to the area of the quadrilateral (S).*

i. The teacher presented a proof for that statement and the proof was discussed in class (see Appendix).

Immediately after the lesson, the students were asked to answer individually and in writing the following questions:

1. Describe the method according to which the ancient Egyptians calculated the area of a quadrilateral.
2. Is their method correct? Explain.
3. Have we the right to judge the method's correctness? Can't <u>we</u> be mistaken? Explain.

The *first* question was asked in order to make sure that the students understood the method to calculate the area of a quadrilateral according to the length of its sides. It also enables to find different ways of formulating this method.

The *second* question reveals to what extent the presentation of a proof in class influences the students' consideration of the validity of the statement proved.

The *third* question is the main one in this study, and it enables to disclose the degree of students' understanding that any result contrary to a proved one must be false.

3. Findings

All the students were able to describe correctly the Egyptian method of calculating the area of a quadrilateral. They established that with regard to rectangles, the method was precise, while in other cases it resulted in findings, which differed, from those obtained by "our" methods.

The distribution of students' answers to the other two questions is presented in Figure 3.

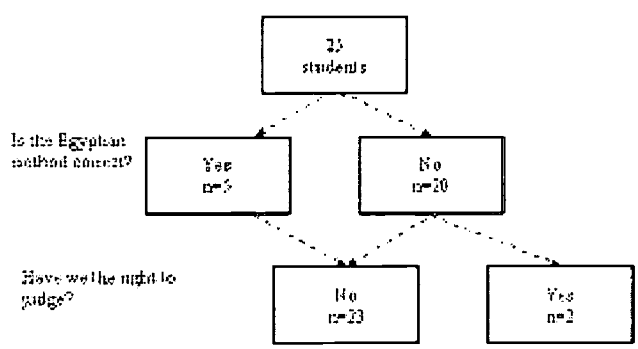


Figure 3

When designing the meeting, the rhombus and the trapezoid were chosen having in mind that these quadrilaterals are known to the students - i.e. they know how to calculate their area. The rhombus was introduced in a *generic* way by presenting the length of its sides by means of the parameter a and without giving any information about its angles. Students were supposed to notice that in this case, the number provided by the Egyptian method constitutes in fact the area of the square with side a and all the other rhombuses of the family have smaller area. According to Peled and Zaslavsky (1997), this should have been a generic counterexample, but there is no evidence that the students grasped it.

The trapezoid was introduced as a *specific* quadrilateral and it was chosen purposefully to constitute an example that leads to the conclusion that the Egyptian method is wrong since there is at least one quadrilateral for which the area given by the Egyptian method is different from the area given by "our" methods.

Five students maintained that the Egyptian method is correct. Although the results obtained from the calculation of the area of the trapezoid described in Figure 2 were still on the blackboard, these students were not aware of the fact that at least one of these results must be wrong since they contradict each other. These students recognized that the results are different but did not recognize the contradiction between them. One of them - Orly - was extremely skeptical:

"Who says *they* were wrong and *we* are right? The results are different, I agree. But maybe our method is the wrong one."

It seems that Orly forgot that "our" method to calculate the area of a trapezoid -for example- can be proved, making it true. From her comments, I learn that it is important to explicitly stress the fact that any result different from a proved one, must be false. This was not clear to the five students that contended that the Egyptian method is correct.

The eighteen other students claimed that the Egyptian calculation is indeed wrong. They related the error to the following categories:

The Egyptian method is incorrect because

i. "it is not proved" (8 students);

ii. "it is not clear how did they get it" (4 students);

iii. "we found a counterexample" (3 students);

iv. "we proved that their result is, in general, larger than the real area" (3 students).

The eight responses in category (a) illustrate another aspect of what De Villiers referred to as a "fundamental axiom" upon which mathematics is assumed to be based: "Something is true if and only if it can be (deductively) proved" (De Villiers, 1997, p.20). In their case, the statement is formulated in a variant that logically follows from the axiom: "If something is not proved, then is false." It may be interesting to investigate further this view since it may be interpreted at least in two ways: "If something *cannot* be proved, then it is false" (a statement that may be refuted if we think about statements like axioms or definitions) or "If *I* did not prove something, then I cannot accept it as true" (a statement that is not appropriate to elementary school teachers for whom the deductive nature of the mathematical contents they teach is not always very clear).

Orly's response is an illustrative one from this category:

"I think that they [the Egyptians] *were wrong, because they did not prove* that their way is the right one, and did not demonstrate how they arrived to these methods and why. I consider it possible that we are wrong, but that someone must refute our methods and prove that our calculation methods are wrong."

Category (b) includes responses of students that seem to refer to an aspect of the role of *explanation* played proofs: in general, the main role of proof is considered to be the *verification* that a mathematics statement is true but not always we are able to provide proofs that show *why* this statement is true. Although they were exposed to an *explanatory* proof that the Egyptian method is correct if and only if the quadrilateral is a rectangle, the students were not exposed to the reasoning that led the Egyptians to the formulation of their method of calculation. From this lack of information they concluded that – as Rinat expressed it – "the Egyptians were not rigorous enough, hence they were mistaken."

Out of all the students (n=25), only two of them – Daniel and Gabriel - pointed out that we do have the right to claim that the Egyptian method is wrong.

While Daniel argued:

"The proof at the blackboard *tells us* that our claim is true and it gives us the right to say *we are right*... The moment I saw the example of the trapezoid, I *knew* that the Egyptians were wrong... ",

Gabriel reasoned as follows:

"I think that everything is relative: The error is relative, because when one says 'error' one must add 'error with regard to...' or 'error in the framework of...". The Geometry with which we are familiar is Euclidean, but I won't judge according to it. According to the Euclidean Geometry, the Egyptians were wrong for the following reasons: 1) Our calculations today prove to be more precise than the Egyptians. 2) *Our calculations are based on proof*, but the Egyptians calculated according to estimations or maybe as a result of trial and error considerations. Maybe one day someone will say that we were wrong, because all our calculations were built on the Euclidean Geometry, which was based on axioms formulated by Euclid and these axioms, are irrelevant. The terms 'right' or 'wrong' are therefore relative and depend on the rules they were subjected to..."

These two students exposed two important aspects of proof that, in my opinion, deserve more attention among teachers educators: a) the role of examples and counter-examples while proving or disproving a conjecture - while a million of examples are not enough to establish the truth of an universal statement, one sole counter-example is enough to disprove it; b) the appreciation of the deductive methods used in geometry, specially the fundamental role played by axioms and definitions. Out of 25 students, only two were able to identify the notion of proof with the notion of certainty, meaning that - under the conditions the statement is proved – one can be sure that no counter-example exists and nobody will ever be able to construct or find such a counter-example. Moreover, Gabriel reminds us of the fact that no theorems or formal proofs are known in Egyptian mathematics and that there is no clear distinction between calculations that are exact and those that are only approximations.

The other 23 students advocated that we have no right to claim that the Egyptian calculation is wrong. For example, Ariel, Bruria and Chris claimed:

Ariel:

"We have no right to judge because it is a different culture and the knowledge depends on the culture. The achievements of every culture should be respected."

Bruria:

"We have no right to judge and claim that they were wrong, because their time was different from ours. They developed their methods according to the ways and means they had available ... We cannot say that their calculations were wrong as long as we do not have a proof of the error. *It is important to keep in mind, that everything is right until you prove the opposite.* This also applies to the Egyptians: In their time their calculation was correct."

Chris:

"I think, that the Egyptians arrived at their formula from a more ancient one. They proved it in their own way and the Egyptians accepted this formula at that time, until our formula was established. *I don't think that we have the right to claim that the Egyptians were wrong, because our formula is based on theirs*. The formula used by us is an improvement and development of the ancient one and *does not contradict it*. It is possible that in the future the calculations of the area of a general quadrilateral will be investigated and a better and more correct formula than ours will be discovered. Then our formula will not be good and right, or even not precise enough."

These three excerpts illustrate the students' lack of understanding of what the meaning of giving a proof really is. They were indeed able to recognize that the Egyptian method is easy to use since it only involves elementary arithmetical operations and that it was used for *every* quadrilateral. On the other hand, they admitted that nowadays they do not know of a method to calculate the area of a general quadrilateral but a special method for each kind of quadrilateral. These advantages of the Egyptian method are mere illusions if we consider the fact that the Egyptian method *was proved* not to be accurate. I agree with these students that it is important to respect the achievements of every culture but this respect does not imply that we cannot compare and point out that some results are contradictory and some methods are not accurate. Pre-service teachers need to be exposed to the developmental aspect of mathematics, to different paradigms of proof, to the meaning of truth in mathematics and to the ways truth is achieved in mathematics. This exposure may foster their conceptual understanding of proof beyond their algorithmic knowledge of how to prove.

4. Concluding Remark

These preliminary results ask for further analysis but from the exposed above it appears that Hanna's recommendation is relevant more than ever:

"With today's stress on teaching *meaningful* mathematics, teachers are being encouraged to focus on the explanation of mathematical concepts and students are being asked to justify their findings and assertions. This would seem to be the right climate to make the most of proof as an explanatory tool, as well as to exercise it in its role as the ultimate form of mathematical justification. But for this to succeed, students must be made familiar with the standards of mathematical argumentation; in other words, *they must be taught proof*" (Hanna, 1996, p.33).

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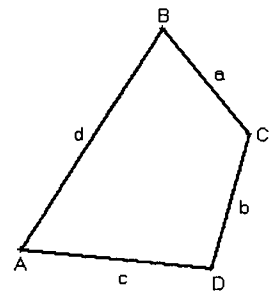
APPENDIX

Given:
Quadrilateral *ABCD*
BC=a, CD=b, DA=c, AB=d

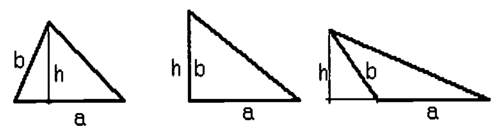


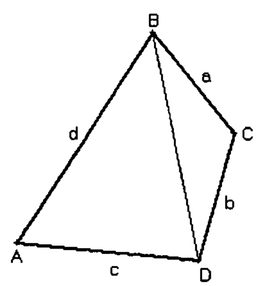
Prove that:

$$S(ABCD) \le \left(\frac{a+c}{2}\right)\cdot\left(\frac{b+d}{2}\right)$$

Proof:

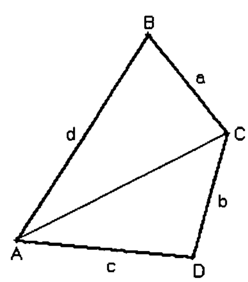
Lemma The area of a triangle with sides a and b, is no larger than $\frac{ab}{2}$.





In the quadrilateral *ABCD*, built diagonal *BD*. Then, according to the Lemma above:

$$S(ABCD) = S(\Delta BAD) + S(\Delta BCD) \le \frac{cd}{2} + \frac{ab}{2}$$



In the quadrilateral *ABCD*, built diagonal *AC*. Then, according to the Lemma above,

$$S(ABCD) = S(\Delta ABC) + S(\Delta ADC) \le \frac{ad}{2} + \frac{bc}{2}$$

Adding, we get that

$$2\cdot S(ABCD) \le \left(\frac{cd}{2} + \frac{ab}{2}\right) + \left(\frac{ad}{2} + \frac{bc}{2}\right) = \frac{d(c+a) + b(a+c)}{2} = \frac{(a+c)\cdot(b+d)}{2}$$

Therefore, $S(ABCD) \le \frac{(a+c)\cdot(b+d)}{4}$ in every quadrilateral *ABCD*.

The equality holds *if* and *only if* the quadrilateral is a rectangle.

GEOMETRICAL AND FIGURAL MODELS IN LINEAR ALGEBRA

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ABSTRACT

According to many university teachers, "geometrical intuition" can help students in their learning and understanding of linear algebra. Fischbein's theory about intuition and intuitive models provided us with a framework that confers a precise meaning to "geometrical intuition", and permits to examine its possible effects on students practices in linear algebra. We study especially geometrical models, stemming from a geometry, and figural models, whose elements are drawings. We describe here some aspects of the use of these models by teachers and students in linear algebra.

1 Introduction

University teachers in France often declare that "geometrical intuition" might help students in their learning and understanding of linear algebra. Several educational researches mention possible interactions between geometry and linear algebra (Dorier 2000) but none of them tries to clarify the specific problem of intuition. In order to design a theoretical frame allowing us to tell and answer research questions about geometrical intuition in linear algebra, we used Fischbein's theory about intuition in mathematics, and models as intuition factors. We will first briefly present that theory, and the questions it raises in the case we study. Then we will expose elements of our work about the use of geometrical and figural models by teachers and students in linear algebra.

2 Using models in mathematics : the theory of Fischbein

We will set out here the elements of Fischbein's work which are relevant for the present study.

Intuition and the use of models

According to Fischbein, every human being needs to act in accordance with a credible reality. Even within a conceptual structure, the reasoning endeavor needs a form of certitude. The role of intuition is to provide that kind of certitude. Intuition is for Fischbein a type of cognition characterized by self evidence, immediacy and certitude ; it always exceeds the given facts. Models are a central factor of intuition in mathematics ; Fischbein defines a model as follows :

> "A system B represents a model of system A if, on the basis of a certain isomorphism, a description or a solution produced in terms of A may be reflected consistently in terms of B and vice versa"(Fischbein 1987 p.121)

Fischbein distinguishes several kinds of models. The ones we use in our study are intuitive models. An intuitive model can be perceived like a concrete reality ; it can stemm from a mathematical theory, if it stays connected with a certain reality (the opposite is a theoretical model, i.e. a mathematical modelisation of a physical reality). There are also several kinds of intuitive models, in particular :

- *Analogical and paradigmatic models*

  An analogical model must be independent of the original ; in that case, the model and the original belong to two distinct conceptual systems. On the opposite, a paradigmatic model is a subclass of objects, used as a model. It is not a mere example, but a particular examplar, representative for the whole class.

- *Intramathematical and extramathematical analogies*

Fischbein also distinguishes different sorts of analogical models in mathematics. The main distinction is beetween intra and extramathematical models. In the case of an intramathematical analogy, the original and the model are both mathematical theories. On the opposite, extramathematical analogies occur with extramathematical models. In our study, it will be the case when the model is a material representation (we use in that case the term "drawing", or "picture", referring to (Laborde and Capponi 1992)). We will refer to such models as "figural models".

We define geometrical intuition in linear algebra as the use of geometrical or figural models.

Geometrical and figural models in linear algebra

We define here a geometry as a mathematical theory whose main objective is to provide a theoretical model for physical space (it is notably restricted to dimension 3). A geometrical model is a model stemming from a geometry ; it is an intuitive model, because the geometry is connected with physical space. It is an intramathematical model ; it can be either paradigmatic, or analogical, depending on the corresponding geometry (that geometry can be indeed a subclass of linear algebra, or can be independent of it). It is always associated with a figural model. The geometrical model can thus smuggle uncontrolled elements in the reasoning process. For example, when studying the general notion of quadratic form, students encounter in some cases vectors orthogonal to themselves. That property cannot be associated with a result in two-dimensional Euclidean geometry ; it is opposed to the drawing usually used to represent two orthogonal vectors in the plane. Thus in that case, the reference to a geometrical model stemming from Euclidean geometry might prevent the understanding of the general theory.
We also study the use of figural models in linear algebra for themselves, independantly of any geometry.

In the following study, we will rather refer to the use of models than to the general expression "geometrical intuition". The questions we study can then be formulated as follows :

- What are the possibilities and the limits of the use of geometrical and figural models in linear algebra ?
- What are the effective uses of models, by teachers and students ?

The results we present in the two following sections are partial answers to the second question.

3 Teacher's choices

We addressed a questionnaire to university teachers, who were used to teach linear algebra (in France). It included several parts, related to various aspects of the use of geometrical and figural models in linear algebra. We will give here details about their use of figural models, and the conclusions of the whole questionnaire.

3.1 Teacher's use of figural models in linear algebra

In our questionnaire, two tables were proposed for the teachers to fill in : one with three drawings, that are sometimes used in linear algebra (according to a previous textbooks study) ; the teachers were asked to say if they use them, and what they illustrate with them ; and an empty table (with five lines), where the teachers were asked to present other drawings they use.
The drawings of the first table, and examples of drawings and interpretations proposed by the teachers, are presented in Annex 1.
Analysis of the answers led to the following conclusions.

Little use of a figural model

A first global statement is that teachers do not use many drawings in their linear algebra courses. Only 16 of the 28 teachers who answered that question proposed drawings in the second part of the question, i.e. other drawings they might use in their courses. And the average number of drawings proposed by these 16 teachers is 2.25 ; this is very low, considering the fact that there were five lines to be filled in the table figuring in the questionnaire. The average number of drawing per teacher, for both parts of the question, is only 3.2.

No specific figural model ?

Moreover, most of the drawings are used to illustrate situations in dimension ≤ 3 , in fact situations occuring in $\mathbb{R}^2$ and $\mathbb{R}^3$. Only 43% of the teachers propose more interpretations referring to an abstract vector space than to $\mathbb{R}^2$ and $\mathbb{R}^3$. For example, for the first drawing (see Annex 1), nine teachers propose the interpretation : "Basis of the space[1]", and three "Orthogonal basis of the space", while only three of them quote the general notion of "Orthogonal basis", and only one the general notion of basis.
For the second drawing, eight teachers mention an intersection of planes, and only five an intersection of subspaces.
The drawings proposed by the teachers are not very different from what we proposed in the questionnaire : except for two quadric surfaces, they are mostly combinations of parallelograms, lines (plain or dotted) and arrows. Only five drawings represent a 2-space ; the thirty-one others are perspective drawings, evoking the 3-space, even if they are used to illustrate situations in a general vector space ; 3-space seems probably more representative than the plane, a better candidate for a paradigmatic model.
The notions illustrated by at least two teachers are projections, orthogonal projections, symmetries, rotations, supplementary subspaces, coordinates of a vector.

In fact, most of the notions and properties quoted by the teachers have already been encountered at secondary school in France, in the geometry course : lines, planes, symmetries, projections (it is not the case for supplementary subspaces and rotations around an axis).

[1]The term "space" refers here directly to geometry. In French indeed, "space" used on his own means "geometrical 3-space".

So the teachers do not seem to develop a specific figurative model, independant of a geometrical model, in linear algebra. For some teachers, drawings intervene only when they mention an affine geometry in their linear algebra course. Some others (a minority) use drawings in linear algebra, but only for $\mathbb{R}^2$ and $\mathbb{R}^3$. In that case, linear algebra in $\mathbb{R}^2$ and $\mathbb{R}^3$ could then be used as an intuitive, paradigmatic model for the whole theory. But there is is no evidence that the students will be able to use that model, especially if the teachers use no drawings in general vector spaces (we will not study that question here).

3.2 Conclusions of the teacher's questionnaire

Considering the answers to the whole questionnaire leads to distinguish two main tendencies among the teachers.
Some of them praise a structural approach to linear algebra, with almost no figural model associated. Geometry will then be presented as a mere application of the general theory.
On the opposite, the others choose to present an affine geometry, with an associated figural model, before introducing linear algebra.
This is a clear symptom of the influence, still very strong, of the discussions held before and during the reform of modern mathematics in France (1960-1970).
Only a minority of teachers propose a figural model especially elaborated for linear algebra. It might have negative consequences on the students practices : if some students need a figural model to help their reasoning in linear algebra, they will probably use a model associated with affine geometry, unsuitable in a vectorial space (they can for example mention "parallel" subspaces, when asked for their possible relative positions).

4 Use of models by students

4.1 Presentation of the test

Description of the activity

We have chosen to submit to first year university students an unusual linear algebra task : for two given sets of vectors of the plane, represented by two drawings, they were asked to say if there exists a linear application sending the first onto the second. Six couples of drawings were proposed ; the first was a parallelogram (except in the sixth case, where it was a segment) ; two basis vectors were drawn on the sides of the parallelogram. The second was either a parallelogram, or a circle, or a triangle, or a segment (see Annex 2).
The students were also asked to provide a justification for their positive or negative answer, but no proof, because we only wanted to observe the elements used to base their reasoning process.

Possible uses of models, and related difficulties

Several models can be used by students in that context ; we will briefly describe them here.

- *Geometrical models*

  - *Usual geometrical transformations*
    Students can use the model of the "usual" applications of the plane : rotations, projections, symmetries, dilations. That model can stemm from linear algebra in a two-space, but also from secondary school geometry. The main problem here is that students may answer negatively if they do not identify a usual geometrical transformation sending the first set of vectors onto the second. That problem has been pointed out by Sierpinska (Sierpinska 2000), in a research work about the learning of linear applications. She calls that kind of phenomenon "thinking of mathematical concepts in terms of prototypical examples".

  - *Preservation of spatial properties*
    Students can associate with linearity, or at least with linear applications of $\mathbb{R}^2$, some preservation properties. For example : "A linear application preserves alignment", or "A linear application preserves parallelograms". It can lead to wrong answers if only alignment is taken into account ; in that case, some students can declare that a parallelogram can be transformed into a triangle.

- *Linear algebra properties associated with a figural model*

Students can use figural models, associated with different aspects of linearity, and different properties of linear applications.

  - *Stability properties*
    The stability properties, for the sum and the scalar multiplication, can be associated with drawings. For example, the drawing of a parallelogram can illustrate the sum of two vectors, and the coresponding stability.

  - *Transformation of the basis vectors*
    The students we asked know that a linear application of the plane is characterized by the images of two basis vectors. So they can draw on the second picture two arrows representing these images, as a justification for the existence of a convenient linear application. The problem that can arise here is that students only care for the two vectors, forgetting the rest of the figure. In that case they can even answer that a parallelogram can be transformed into a circle.

  - *Other properties*
    Figural models associated with various other properties of linear applications can intervene.
    "A linear application sends a subspace on another subspace" ; "A linear application sends the nul vector on itself"... Some of these properties involve the notion of dimension : "the dimension of the image of a subspace F is less or equal than the dimension of F", for example. There is in that case a special difficulty, stemming from a figural model associated with the notion of dimension. The given drawings can be misinterpreted ; in particular, a confusion between "dimension" and "direction" can occur.

4.2 Answers analysis

The test was proposed during the second semester of the first university year ; 43 students answered it. They already had linear algebra during the first semester, with different teachers (the tutorial groups are reorganized between the two semesters).
Uses of the models mentioned above clearly appear in their answers, with the associated difficulties. Several models can intervene in the same answer. In fact, three main types emerge, corresponding to the following use of models :

- *Usual transformations and dimensional properties (13 students, labelled "U")*
  These students use the two models together ; they propose for example a usual transformation to justify their positive answers, and use a dimensional argument in a negative case.
  The association of these two models is surprising at first sight, because they are of different natures. But they both correspond to an attempt of students to elaborate a figural model that can help them in their task. For that purpose, they use familiar objects ; but these objects are unsufficient to provide here an appropriate model.

- *Transformations of the basis vectors (14 students, labelled "B")*
  Only one model intervenes in these answers : the characterization of a linear application of the plane by the images of two basis vectors. These students reduce to a minimum their use of a figural model. Their reasoning is based on a theoretical property ; they draw two vectors on the second picture, because they are asked to do so. But most of them neglict to condider the whole drawing ; they claim, for example, that a parallelogram can be transformed into a triangle, because they can represent two "image vectors" on the sides of the triangle.

- *Preservation of spatial properties and stability properties (10 students, labelled "P")*
  The two models used in these answers are in fact closely related. The properties : "A linear application preserves parallelograms", and : "A linear application preserves sum and scalar multiplication" can indeed be associated with the same drawing ; the first can be used as an intuitive model for the second. The associated figural model is well adaptated for the task we proposed here.

(37 answers are gathered in these types ; for the 6 remaining answers, there is no evidence of the used models, but all of them are wrong).

The following crosstable shows the distribution of the students answers in the three types, together with their success or failure to the test.

	Correct answer	Incomplete answer	Wrong answer	Total
U	1	1	11	13
B	2	2	10	14
P	4	2	4	10

A correct answer means here that the choice of a positive or negative answer was right in the six cases ; in a wrong answer, there is at least one mistake. Only seven

students are right in the six cases ; it is indeed a difficult task, where drawings play a major part, on the opposite of the students habits.
Despite the low number of students in each box, it appears clearly that the ones belonging to the type labelled "P" are more likely to succeed than the others.

5 Conclusion

We presented here very local results ; but they point out general phenomena, confirmed by the rest of our work (Gueudet-Chartier 2000).
Some students need a figural model to help their reasoning in linear algebra (in the experiment we presented, they were obliged to deal with drawings ; that statement comes from other parts of our work).
But most teachers do not propose in their linear algebra courses a suitable, specific figural model. What are the consequences for the students practices ? According to our observations (the test presented above provides an example of it), three main types stand out :

- Some students do not seem to use any figural model. A minority of these students proves nevertheless a good understanding of linear algebra.
- Others try to construct by themselves such a model, using for example secondary school geometry ; but it turns out to be inadaptated for linear algebra.
- Some students elaborate a suitable figural model ; moreover, they are quite successfull in various linear algebra tasks. However, it is difficult to decide if that model is a factor, or on the contrary an evidence, of their understanding of linear algebra.

Studying further the students uses of figural models in linear algebra would require the organization of a teaching experiment, allowing us to know exactly wich models have been proposed, and to observe their influence on the students practices.

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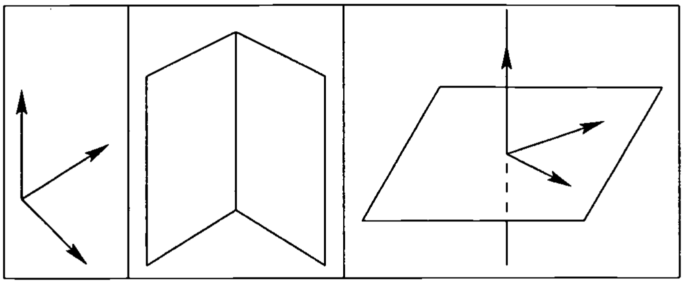
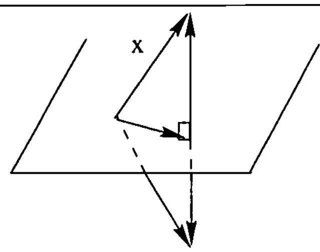
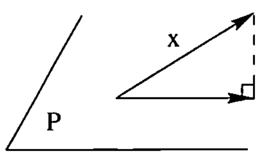
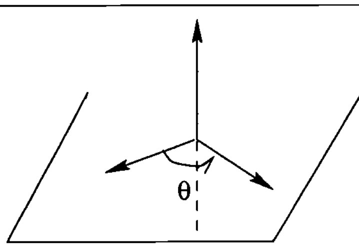
ANNEX 1

Drawings proposed to the teachers :

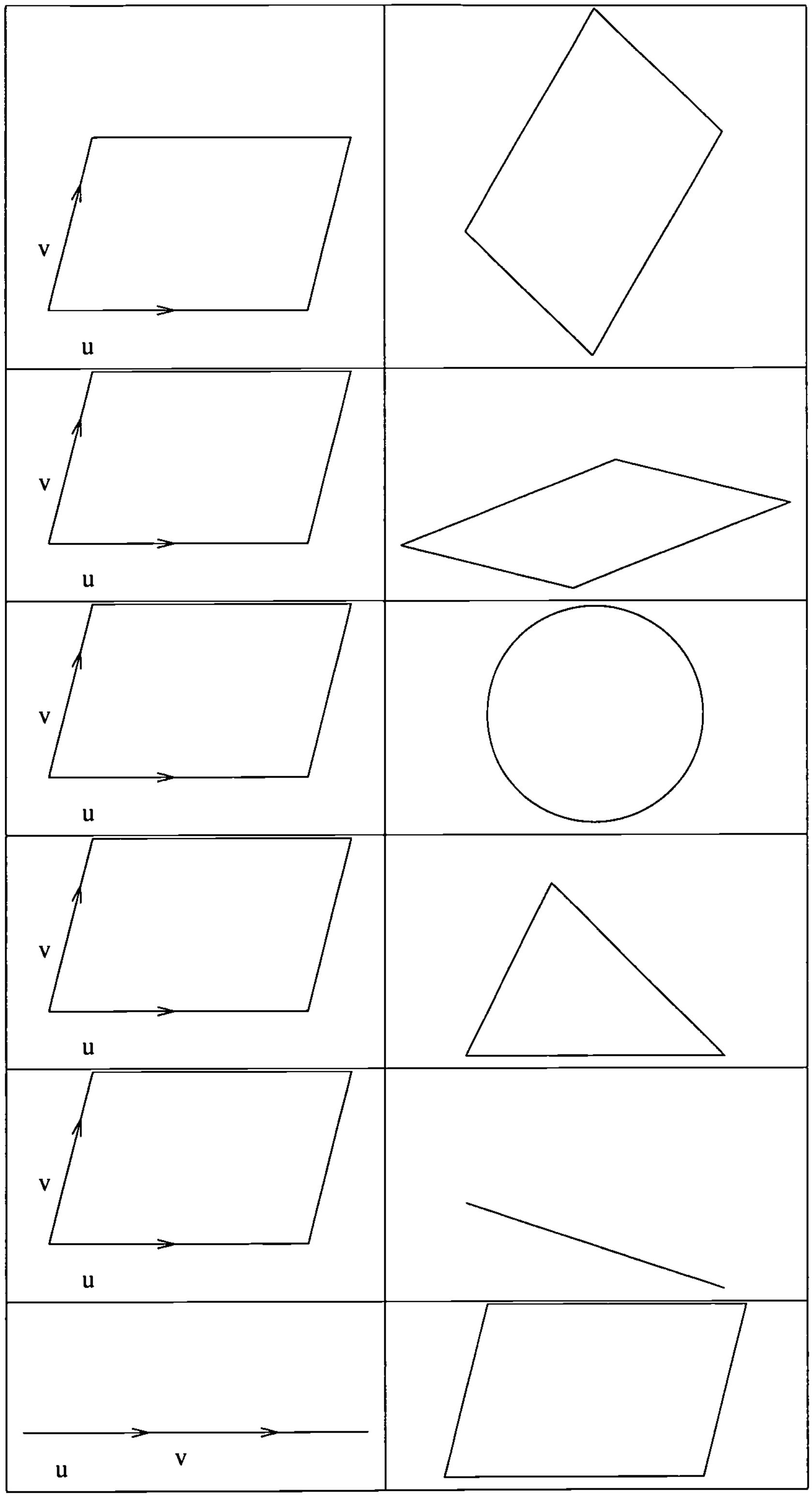


Examples of drawings proposed by the teachers :

Drawing	That drawing illustrates
	Orthogonal projection on a plane Orthogonal symmetry
	x and its orthogonal projection on P
	Rotation about an axis

ANNEX 2

Drawings proposed in the students test :



OPTIMAM PARTEM ELEGIT[1]

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ABSTRACT

This paper discusses some issues in numerical optimisation. It illustrates graphically the rationale behind some optimisation techniques. It shows the perils that await the unwary when extrapolating using functions whose parameters have been specified by choosing the values, which minimize a sum of squares of errors.

[1] Choose the better part. (Luke 10:42)

Introduction

The wisdom of the command: ' choose the best part', should be obvious to all. Optimisation is the branch of mathematics which deals with the techniques for locating the maximum or the minimum of a function, i.e. ' the best part''.

There is the common misconception that to determine the location of the minimum of a function of several variables, $f(x_1, x_2, \ldots, x_n)$, one simply needs to solve the system of non-linear equations formed by setting to zero the partial derivatives of f:

$$F_i(x_1, x_2, \ldots, x_n) = \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_i} = 0, i = 1,2,\ldots n$$

However, to solve such a system, usually, one needs to use a numerical procedure. Efficient numerical methods to do this are based on finding the minimum of

$$S = \sum_{i=1}^{n} F_i(x_1, x_2, \ldots, x_n)^2$$

Thus, numerical optimisation is required for solving systems of non-linear equations and not the other way around.

The computational methods for solving optimization problems are generally known as hill-climbing techniques that is because they mimic the strategy that a climber may use in trying to reach the summit of a mountain. Different strategies are open to the climber to reach the summit and we shall illustrate the rationale behind some of them.

Optimisation is frequently used to fit models to data with the intention of summarizing, interpolating or extrapolating from the observations. Extrapolation carries the implication that the estimated parameters are physically meaningful. However, it is very possible that parameters which produce a very good fit to the data lead to disastrously unsuitable extrapolations. Then, when is it safe to extrapolate? The paper discusses, through examples, the issues involved.

Finding the best part

Let us consider the simplest strategy for locating the optimum of a non-linear function using a hill climbing technique. Consider that a climber is trying to reach the summit (maximisation) of a hill, or the bottom of the hill (minimisation), without a map and in dense fog. The climber can rely on an altimeter to measure altitude and a compass, which allows him to maintain a fixed direction. Measuring is time consuming, but movement itself is easy. The climber wishes to move as fast as possible. What is the best strategy?

It seems that the simplest approach would be to move along an arbitrary direction, such as the north-south line making regular measurements of the altitude until the highest point on the line is reached. Starting from this new point the same operation can be carried out along the east-west direction. This process of alternating searching along fixed directions ultimately will take the climber to the summit.

The algorithmic implementation of such simple procedure is known as the univariate search. To illustrate it we consider a problem presented by Box et al [1]. We wish to specify a function that relates the concentration η of a chemical substance with time. The function is of the form:

$$\eta = \frac{\beta_1}{(\beta_1 - \beta_2)}(e^{-\beta_2 t} - e^{-\beta_1 t})$$

where, β_1 and β_2 are parameters which need to be estimated. Given a set of observed values for η and t, a common procedure is to estimate the β s by the method of least squares. That is: minimise the sum of the squared differences between the observed values and the predicted ones. That is, we want the location of the minimum of

$$f(x_1, x_2) = \sum_{i=1} (y_i - \eta(x_1, x_2))^2$$

where x_1 and x_2 stand for the possible values that we can, respectively, assign to β_1 and β_2 ; y_i correspond to the observed concentration at time t_i. A set of observations is listed in Table 1. Let us consider finding values for the betas by minimizing f using only the first six pairs of values of the data set.

Table 1. Observed concentration values y_i at times t_i.

t_i	0.0625	0.125	0.25	0.50	1.00	2.00	4.00	5.00	6.00	7.00
y_i	0.01	0.02	0.08	0.15	0.22	0.51	0.48	0.29	0.20	0.12

The shape of the function f is illustrated by its contours, shown in Figure 1(a). The picture also gives the path to the minimum using the univariate strategy. It is obvious from the graph that the path to the optimum requires a large number of short steps. However, the short steps could be used to define a general direction and a more efficient method would be to move along such a direction. The Davey, Swann and Campey (DSC) [2] algorithm does this. In contrast to the univariate search the DSC algorithm takes advantage of the accumulating information about the function. Starting at the point $x^{(0)}$ one cycle of the univariate search determines the point $x^{(1)}$. The next search is along the line joining $x^{(0)}$ and $x^{(1)}$ which determines the point $x^{(2)}$, and then we search at right angles to the previous search direction to determine $x^{(3)}$. The next search direction is along the line joining $x^{(2)}$ and $x^{(3)}$, and so on. Figure 1(b) shows the iterations using the DSC algorithm. In this case far fewer steps and function evaluations are required.

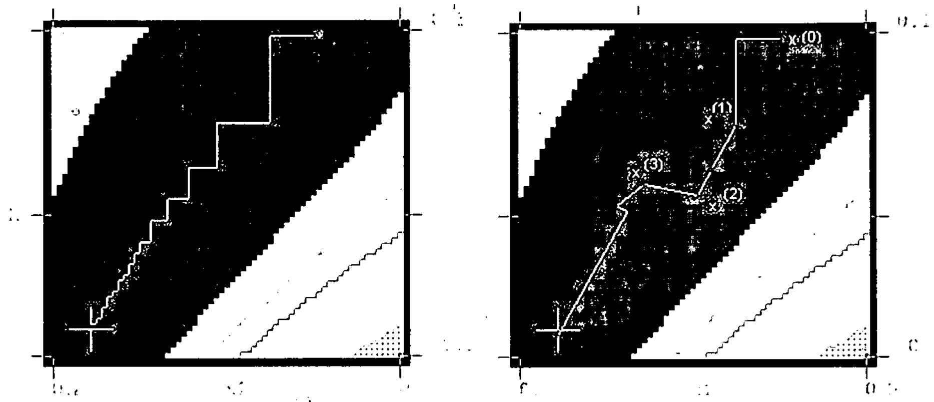


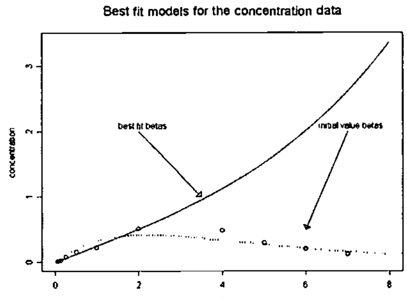
Figure 1. (a) The univariate search, locates the optimum, using 581 function evaluations, at (0.2442, -0.2402), with $f = 0.002$.

(b) The DSC algorithm uses 142 function evaluations to fin d the optimum at (0.2433, -0.2431). Both methods start at the point (0.5,0.39).

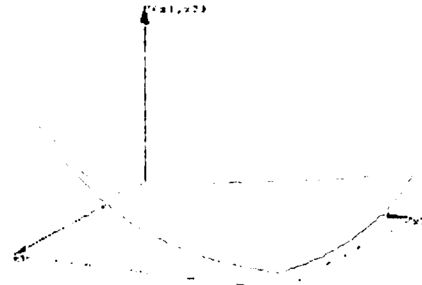
However, if our intrepid climber was allowed also to carry a spirit level, then he could use it to measure the lay of the land, and this extra information might lead him to choose his direction of search to be along the steepest descent. He might well find that such a strategy might produce a succession of large number of short steps similar to those of the univariate search. But being a smart climber he would realise that information about the gradients could be used, as in the DSC method, to determine a more efficient direction. This will lead him, no doubt, to discover the conjugate gradients method. Furthermore, having information about the gradients, he might consider gathering information about the curvature of the land, and using it might well develop Newton's type methods. It may well be that the terrain over which he is moving is very rocky - a noisy function - and therefore he may decide that he is much better off using the DSC strategy than the more elaborate methods which involve misleading gradient measurements.

All these strategies for numerical optimisation can readily be illustrated using graphs like those in Figure 1 and generalize to problems in more dimensions because the principles on which the methods are based are the same for two as for higher dimensions. The illustrations can easily be done using the software from McKeown et al. [2].

The function, specified with values for β_1 and β_2 which minimise f, fits the first six points of the data very well. There may be the temptation of assigning physical meaning to the estimated betas. However, when the rest of the observations are viewed, the fitted function is in complete disagreement with them. Any extrapolation using the fitted function, or a physical interpretation given to the parameters would have been unwise. On the other hand, it is simple to see that a set of values for x_1 and x_2 contained in the lowest contour of the sum of squares function are possible candidates for selection as values for the betas. For such a set there is not much change in the value of f. In particular, the pair of values at the start of the iteration fit the data almost as well as the ones that optimise f, and they happen to specify a function that gives reliable predictions for the extra data points. Figure 2 illustrates this.



(a)



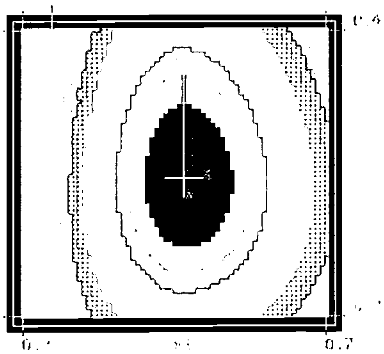
(b)

Figure 2. (a) Fitted function. (b) 3D Plot of $f(x_1, x_2)$.

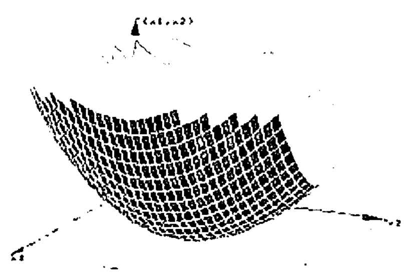
So, what is going on here?

The answer to the question lies in the fact that the function we are minimizing is insensitive to changes in x_1 and x_2. This is particularly visible in Figure 2 (b), which gives a 3D plot of f. The plot shows that f is practically constant along the line joining the initial and optimal values of x_1 and x_2. Though we found a local minimum, its location is insensitive to changes along the ridge of f shown in the figure. The problem is said to be ill-conditioned, and in such cases the fitted curve is only suitable for interpolation and no physical significance should be assigned to the estimated parameters. The data has forced us into a curve fitting problem and not a parameter extraction one.

By contrast when using the last six observed values to estimate the parameters we get the optimal values at $x_1 = 0.5153$, $x_2 = 0.3475$ and $f = 0.0363$. The contours of the new least squares function are given in Figure 3(a), they show that changes around the minimum lead to significant changes in f. The corresponding 3-D picture confirms that in this case there is no ill-conditioning.



(a)



(b)

Figure 3. (a) Contours of the sum of squares function for the last six data points. The steps of the univariate search are also illustrated from the starting point (0.5,0.39).

(b) The 3-D picture of the sum of squares function.

The plot of concentration against time in Figure 4 (a) shows that extrapolation is a lot less problematic when there is no ill-conditioning. Furthermore, a well-conditioned problem makes for a faster path to the optimum as illustrated in Figure 3 (a), showing the sequence of steps to the optimum when using the univariate search.

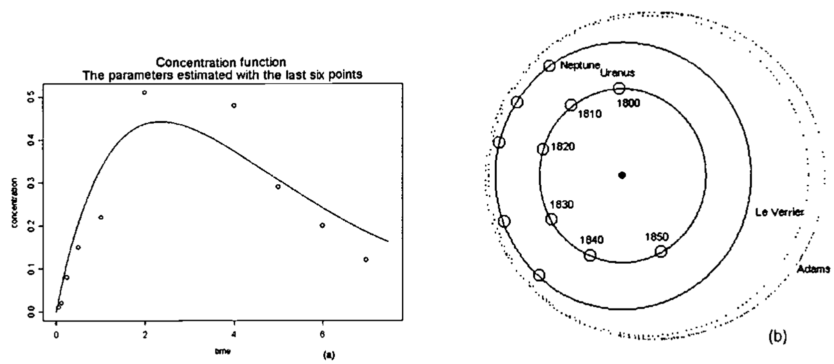


Figure 4. (a) Concentration against time fitted using the last six points.
(b) Orbits for Neptune - calculated and actual. The numbers on Uranus correspond to the year when its location was used to determine Neptune's orbit.

A classical story of ill-conditioning

Recently the fascinating story of the discovery of the planet Neptune was published in a popularized form [3]. The story in the book contains a fair dose of human drama. It is exciting also because it is an example of a successful theoretical astronomical prediction. Using the discrepancies observed in the orbit of Uranus two mathematicians working independently, one French, Urbain Jean-Joseph Le Verrier, the other English, John Couch Adams, accounted for the discrepancies by predicting the existence of a new planet - Neptune–

These two mathematicians were breaking new ground. Newton's theory of gravitation had been used to calculate the effects of bodies on one another, but this was the first time that the theory was used to predict the position of a body from observations of the effects of its gravity on other bodies. However, not everyone was using the new planet explanation to try to account for the problems in Uranus' orbit. The Astronomer Royal George Airy supported the hypothesis that Newton's inverse square law did not apply over large distances. The perseverance of the two young mathematicians on the validity of their assumptions, against the pressures from a famous and established scientist are only part of the intricate drama that led to the discovery of Neptune. Their work not only helped in the discovery but it confirmed the universality of the gravitation law, and produced a model of work for the interaction between mathematicians and experimentalists.

Adams and Le Verrier were able to point out were in the sky to look for the planet. The astronomers duly found it in 1846. However, it is interesting that both mathematicians failed in determining with any accuracy the orbit of the planet for the region where there were no observations. Figure 4 (b) shows the theoretically proposed orbits and the actual one. Note that the maximum error in the predicted orbits is about half the radius of Uranus' orbit. This is interesting to us, because it is an example of the consequences of ill-conditioning. To specify the orbit the mathematicians used the observations on the discrepancies observed in Uranus's orbit occurring during the first half of the 19th century. They were used to determine both, the position and the mass of Neptune. The mathematicians obtained a good fit to the data by overestimating the mass of the planet and the radius of the orbit. The errors compensated to give a fit acceptable in the region were the data was available but the calculated orbits were not suitable for extrapolation. The calculated orbits diverged more and more from Neptune's. Had the search for the planet taken place a few years earlier or later it would not have been found anywhere near the predicted location.

Optimisation and mathematical education

Optimization is a decision-making problem: how to maximize or minimize the value of some quantity. In many cases this amounts to assigning values to certain quantities called the decision variables. We showed that optimization problems are common in science and engineering and that they usually cannot be solved by analytical methods and that computational methods must be used. There are two educational issues here, the first one is how to present a rationale for the numerical procedures for optimization. The second issue is to identify the applicability of the results of the optimization.

The analogy of 'hill-climbing' can be used as a powerful teaching tool to illuminate the ideas behind many of the numerical optimization methods. This is so because the algorithms for optimization can be illustrated with two-dimensional functions. We looked in particular at the idea behind the Davies Swann and Campey algorithm. From a simple description of the idea, the specification of the method – for any number of dimensions – seems a trivial generalization of the 'hill climbing' analogy. For example, we can state the DSC procedure for optimising a function of n variables as:

1 Set $k = 1$. Select an arbitrary starting point $\mathbf{x}^{(0)}$
2 Carry out one cycle of the univariate search algorithm to produce $\mathbf{x}^{(k)}$
3 Select $\mathbf{q} = \mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}$ as a new search direction.
4 Generate $n - 1$ orthogonal directions and orthogonal to $\mathbf{q}$.
5 Search along $\mathbf{q}$ and each of the other $n - 1$ orthogonal directions to determine the new point $\mathbf{x}^{(k+1)}$. Each search begins at the end of the previous one.
6 If stopping criteria satisfied stop, else set $k = k+1$ and repeat from 3.

We used bold face to denote an n-dimensional vector. The algorithm above is a straightforward generalisation, to n dimensional functions, of the basic idea illustrated in Figure1 (b).

Further exploitation of the hill-climbing analogy might lead us to question the efficiency of obtaining exact determinations for the $\mathbf{x}^{(k)}$ s. It may be better not to find the optimum along a search direction but simply a better point from which to continue the search along a different direction. This policy may take more cycles, but overall, may require less use of the altimeter, and as changing direction involves no effort, a method with inexact line searches might be a more efficient one. The educational possibilities when using sensible, imaginative ideas derived from the hill-climber analogy are boundless.

Optimisation is also taught as a procedure to fit equations to data. The objective, of course is to model a physical situation. However, the applicability of the fitted model is highly dependent on the conditioning of the problem. We illustrated that for two-dimensional problems ill conditioning implies a flatness, about the optimum, of the function we wish to optimize. Thus, the effect of ill-conditioning is to provide many possible, near optimal, but possibly dramatically different solutions. When this occurs, the only sensible use for the fitted model is for interpolation, which is not an unimportant outcome as the history of the location of Neptune testifies.

Though a mathematical treatment of ill-conditioning is an advance topic, the ideas and consequences of ill-conditioned problems can and should, as we have shown, be presented in more elementary courses in data analysis and optimization.

Finally, we feel that the teaching of numerical optimization should not be constrained by the use of 'analogies'. Their value is simply to provide another point of view, which might help to make the topic more interesting. We do not think that there is a unique solution to the teaching of the subject. It may well be that the problem of optimizing the teaching of mathematics is ill-conditioned, in the sense that there are many equally satisfactory solutions, and hence one should be careful to extrapolate from any of them.

Concluding remarks

The analogy of hill-climbing has been shown to be useful for providing a motivation for numerical optimisation methods. The fundamental problem of using models, which are fitted to data, has been discussed. In particular we concentrated on the important distinction between data fitting and parameter extraction. We showed that when the problem is ill-conditioned, 'choosing the best part' can only be used for summarising the data and that no physical meaning should be associated to the parameters of the model. The discovery of the planet Neptune, during the middle of the 19th century, and the failure to specify its orbit was offered as an example of the effects of ill-conditioning. It would be an exciting project to investigate the conditioning of the problem using formal methods of analysis. There are, of course, such formal methods, McKeown and Sprevak [4] show how to use them in an application. It is not, however, the objective of this paper to deal with such formal methods but to offer a pictorial representation of ill-conditioning and of its consequences. We believe that everybody could profit by being aware that when fitting models to data, using optimization methods, the usefulness of the fitted model depends greatly on the conditioning of the problem. The moral of the lesson is: 'Optimam partem elegit', but be aware of its limitations.

Acknowledgments : We wish to thank Dr. J. J. McKeown for many interesting debates at the Thursday Seminars.

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STUDENT CENTRED LEARNING IN A MATRIX ALGEBRA COURSE

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ABSTRACT

A new approach was tried in presenting a matrix algebra course to students with differing abilities and diverse needs. The course had previously been presented using fairly traditional methods where the emphasis was on the transmission of knowledge from the lecturer to the students (using partially completed notes which students filled out during lectures). While exam results were reasonable, based on well practised examples, the course was fairly narrow and prescribed.

A change to a more student centred approach was effected using the following mix.

(1) The lectures paralleled a text book (Linear Algebra with Applications by David Lay), which closely followed the Linear Algebra Curriculum Study Group recommendations for an appropriate core syllabus responsive to client needs, and using a matrix-oriented problem solving approach. The resulting lectures were reasonably informal, encouraged students to read the text for themselves, attempted interaction and incorporated some technology.

(2) There was a weekly computer laboratory session using Matlab, where students were encouraged to work cooperatively in pairs on problems from the databank of Matlab exercises available with the text, and on other projects that emphasised understanding and demanded written interpretation.

(3) There was a weekly individual exercise, which provided a variety of question types from routine computations and standard algorithms to short proofs that required an understanding of key concepts.

The paper considers such questions as how successful the course was from the viewpoint of lecturer and student, how Matlab was used not only for calculating more realistic examples but to aid understanding, and how well the group approach worked.

Background

Admission to university in New Zealand is fairly open. School leavers require a modest performance in their final national school exams, while those aged 20 or over are granted automatic entry. As in other countries, there has been a large increase in the numbers of students attending university in recent years, with Otago's role increasing by some 50% in the past 10 - 15 years. This means the mathematical background and ability of entry students is very mixed. As well, students now pay sizeable fees (from some 2000 Euro for science to more than 5000 Euro for health science), so although there is a Government loan scheme, there is considerable pressure to have reasonable pass rates in courses.

The matrix algebra course is a 200 level one semester paper usually required by mathematics majors. Its prerequisite is two semesters of 100 level mathematics, which combine algebra and calculus and are fairly traditionally taught large classes using little or no technology. Topics taught in the algebra section include introductory material on 3-D vector algebra, matrices, determinants, 2-D linear transformations and eigenvectors.

The University enrols students in arts, science, business, health sciences and other disciplines (but has no engineering school), and the matrix algebra course caters to students from all of these areas, although predominantly science. For about half the students it is their only mathematics paper. But there are also some honours students from the likes of mathematics, statistics, physics and computer science.

The previous course used "outline notes", that is a bound set of partially completed notes in which students copied down examples, diagrams and arguments during lectures. Some problems using Matlab were set on the exercise sheets (from Kolman 1997). On the face of it students learned well, but I became concerned that the course was too prescribed and the students too "spoon-fed". When all mathematics papers were changed in 2000, I decided to try a more "student centred approach", that is making the students (where possible in collaboration with others) more responsible for their learning, and changing the lecturing role to more that of a facilitator, as advocated in Berry et al 1999. At the same time I introduced technology as an integral part of the course.

The Course

In designing any mathematics course at Otago, there is a certain tension between providing a challenging course but being aware of competing courses which often make fewer time demands. Even the other 200 level mathematics courses have only a one hour (come if you need help type) tutorial. After due consideration the following mix was chosen.

(1) The lectures (32) paralleled a text book (Lay 2000), the author being a member of the Linear Algebra Curriculum Study Group, who have put out a recommended curriculum for a first course in linear algebra (Carlson et al 1993) which this text follows. This syllabus, widely debated and generally acclaimed (see Dubinsky 1997 for a contrary view), worked well with the diverse students in my course.

Because of the sort of conceptual difficulties discussed in Dorier & Sierpinska 2001, the chapter on general vector spaces was avoided in favour of the more concrete approach of treating all vectors as n-tuples in R^n. The emphasis placed by Lay on considering the columns of a matrix aided understanding. For example the definition of Ax as a linear combination of the columns of A

with weights from the vector x eased the initial problems students have in reformulating between vector equations and matrix equations. The early introduction of key concepts such as linear independence and spanning (together with the computer work) also seemed to give students a much better and more confident grasp of these often troublesome ideas. The geometrical focus helped students grapple with concepts, from the more abstract notion of a subspace (which still takes time) to the more concrete ideas involving least squares.

The students received in advance, chapter by chapter, a one page summary of each section we covered in the book, giving key definitions, results and Matlab commands needed. So in theory, if not in practice, students could read ahead and prepare for lectures. The lectures themselves were reasonably informal, highlighting the principal issues in each section and leaving the students to work through much of the detail, although I still probably explained more than I should. The first year I gave no notes at all but just talked, and tried with varied success to get class participation (somewhat difficult with 90 students). The second time I gave some informal notes (due to feedback from the first year, possibly influenced by comparison with the companion calculus paper where all notes were given by hand), such as quick summaries, problems to watch for, or perhaps considering part of a proof.

Lay's text comes with a database of all the book's exercises in whatever CAS is used (we used Matlab). By typing the simple command cisj, Matlab prompts which question you want from chapter i, section j. This made it easy to consider any question from the text in lectures.

(2) Students attended a weekly two hour Matlab session, using problems from the text to practice standard algorithms and computations (before doing their individual exercises) and mini-projects from various sources, such as the CD or Matlab manual (Day 2000) accompanying Lay, or suggested in the MAA notes (Carlson et al 1997).

Matlab was selected because the command structure is straightforward (no programming was required). In the first lab before lectures began, students were given a handout and asked to get to grips with entering matrices and doing standard matrix operations. From then on students were asked to work in groups (most chose pairs because of the lab layout), jointly writing up their work and sharing the mark, which counted towards their internal assessment. Because of their algebra knowledge from 100 level mathematics, students started the course applying new technology to old mathematics (following the rule of thumb suggested by Berry et al 1999) and coped easily, perhaps occasionally needing a reminder about syntax (which was usually supplied by another student!).

There is evidence that using technology can develop understanding (see Mayes 1996). The following sample of examples we considered convinced me of this.

(a) Realistic examples of linear equations.

After looking at smaller examples, linear traffic flow problems or temperature grids involving perhaps 20 variables can be easily tackled. Homogeneous systems such as balancing the chemical reaction

$$MnS + As_2Cr_{10}O_{35} + H_2SO_4 \rightarrow HMnO_4 + AsH_3 + CrS_3O_{12} + H_2O$$

also give concrete examples of what a vector n-tuple might represent.

(b) Visualizing linear transformations.

The M-file planelt (which comes in a package with Kolman 1997) nicely demonstrates the effects of 2-D transformations. I ask students to take a shape and observe (sketch) how it changes under a succession of transformations and then find a single matrix with the same net effect (and check that it works – they quickly learn to compose matrices in the right order!) A similar M-file drawpoly (from Day 2000) can be used to illustrate affine transformations in the plane in the section on computer graphics, using homogeneous coordinates and 3 x 3 matrices. For example to rotate a figure about an arbitrary point.

(c) Standard algorithms.

The computer allows students to concentrate on the algorithm rather than the arithmetic, and is useful for practising routine calculations such as LU factorizations or diagonalizing matrices. Further, they can easily check their result works which may iron out any misconceptions, for example with regard to the order in which the eigenvalues are placed in the diagonal matrix.

(d) Computer insights.

Counting the number of operations (flops) used in a task is instructive, for example in calculating $(A*A)*\mathbf{x}$ or $A*(A*\mathbf{x})$, which brings the associative rule to life. Considering rounding errors also gives insights. For example students found by chance a simple integral matrix for which $\det(A) = 0$ but $\mathrm{rref}(A) = I$ (row equivalent to I). Then using the rrefmovie command they could observe how a pivot could be small but not small enough to be set to 0, unless the default value for the tolerance (numbers smaller than this value are set to 0) was increased. The effect of partial pivoting can also be explored. For example, students could use this technique to produce the same LU factorization calculated by Matlab (which differs from the usual hand calculation because of partial pivoting).

(e) Experiment and prove situations.

The computer can be quickly used to show patterns leading to conjecture and (possibly) proof. For example, the behaviour of triangular matrices (used in LU decompositions and other applications) is explored in many texts. Students get bored pretty quickly using random matrices to observe that like triangular matrices are closed under products (although a picture proof is instructive), but guessing the form of the inverse of a 3 x 3 (or 4 x 4) triangular matrix with integral entries (using rational format) usually requires a good number of repetitions to spot the pattern and the consequent conjecture can then be proved.

(f) Iterative processes.

Finding the steady state vector of a stochastic matrix or iterative solutions to linear systems using the Gauss or Jacobi method are ideal for CAS.

(g) Eigenvectors.

Lay introduces eigenvectors by considering a dynamical system involving the three life stages (junior, subadult, adult) of an owl population. Matlab enables these populations to be quickly modelled and graphed simultaneously. Students can then get a real feel for how the populations behave as the survival parameter (junior to subadult) varies. Later (after eigenvector bases have been explored) they can explore how the eigenvalues of the associated matrices change with this parameter. Matlab can also be used to plot the iterates of a point under the action of a matrix (so for example the trajectory of a dynamical system) and observe how these vary according to the eigenvalues.

(3) There was a weekly exercise that students worked on individually, with a variety of questions (mostly from Lay) such as routine calculations, standard algorithms or short proofs. The well designed questions from Lay require little computation, largely avoided being repetitive (often asking the same underlying question in different ways), and tested knowledge in quick but searching ways.

Conclusions

Delegating more responsibility to students for their learning, provided more time in lectures to stress the main results (using lots of transparencies), to give informal and intuitive meanings to concepts and to discuss pedagogical issues. However there was still a conflict between covering the material (for those who might use the linear algebra) and spending more time understanding it (for those for which the course was a vehicle for learning some mathematics). Using Matlab helped break up the lectures and generate discussion, but further class activities would help.

A fairly detailed student survey of the course was conducted and some of the responses are recorded in table 1. As can be seen, generally the students found the pace appropriate and the book easy to follow. Surprisingly (see table), they also preferred the lecture approach taken to the more secure outline notes which they had used in 100 level courses and which they rated favourably there.

	Strongly agree	agree	neutral	disagree	strongly disagree
Pace of material appropriate	30%	45%	22%	3%	0%
Book easy to follow	28%	55%	11%	6%	0%
New approach better than outline notes	19%	48%	26%	6%	0%
Matlab enhanced understanding	13%	57%	9%	21%	0%
Group work enhanced understanding	30%	33%	23%	10%	3%

Table 1

From my perspective the group activity in tutorials worked really well and students generally agreed with this. They also found Matlab helped their understanding although not all agreed (see table). There was a much better participation rate in tutorials and a more vibrant (nosier!) atmosphere. Various problems sometimes associated with group activities, such as inactive members or subdividing material, were largely avoided since the students worked mainly in pairs sharing one computer (largely dictated by the lab layout). Usually the longer the pairs worked together the better the collaboration, but I didn't force this and for various reasons there were realignments or the occasional person who wished to work alone. In most partnerships, even when one was mathematically weaker than the other, both were able to contribute in various ways (perhaps one might be more computer savvy or a better recorder than the other). There was the odd instance of a very lopsided liaison in which the weaker student was considerably helped by their partner.

I tried to encourage the groups to record a clear description of the object and outcome of each exercise as well as the mathematics involved, but success here was mixed (sometimes lots of numbers were recorded but not the big picture). The marking scheme tried to reward good explanations, but because not all parts were able to be marked the scores awarded were not always very discriminating and were generally quite high, but this had a good attitudinal spin off. I would probably design a lab sheet in future to encourage better explanations. The University now regularly conducts surveys of student opinion in all courses and mathematics performs well in areas such as "developing problem solving skills" but poorly in "written communication skills" and "ability to work as a team member". The computer labs should help address these concerns.

The individual exercise was more challenging and discriminating. Answers to odd numbered questions are given in Lay, so students could try an adjoining exercise, which although different might have some similarities. Of course more routine hand calculations were often checked by students using Matlab. Student's lack of experience in tackling proof type questions was obvious and even with lots of help (in office hours) success was mixed.

In general I was happy with the course, and felt it was a step in the right direction away from a passive lecturing style to a more active student involvement.

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IMPLEMENTING A 'EUROPEAN' APPROACH TO MATHEMATICS EDUCATION IN INDONESIA THROUGH TEACHER EDUCATION

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ABSTRACT

This paper reports on the results of a four-year study called CASCADE-IMEI that is a learning environment (LE) in the form of a face-to-face course and a web site (www.clix.to/zulkardi) which aims to introduce Realistic Mathematics Education (RME), Dutch approach to mathematics education, as an innovative teaching methods in Indonesia trough prospective mathematics teachers in initial teacher education. It also presents the background of mathematics reform in Indonesia by adapting RME as a promising approach. Then, the paper describes the process of a development research approach in which three prototypes of the LE have been developed and evaluated both by prospective mathematics teachers in Indonesian Educational University in Bandung and several experts in the Netherlands. Finally, it will discuss the changes on the prospective mathematics teachers after they followed the LE program with a more detailed on their teaching performance in junior secondary mathematics classroom.

Key words: mathematics learning environment, www, RME, development research

Introduction

Up to now, the teaching process in mathematics classrooms in Indonesia is still conducted mainly with a traditional (or mechanistic) approach. Teachers actively explain the material, provide examples and exercises, whereas the students act like machines, they listen, write and perform the tasks initiated by the teacher. Group or whole class discussions are seldom present and interaction as well as communication is often missing. Likewise, mathematical goals and curriculum materials used in the classrooms are still based on 'mathematician' mathematics not on student mathematics with a focus on real life application (de Lange, 2001). This is in contrary to the needs of the information society in which mathematics literacy is an important goal. In summary, it is clear that goals, content and teaching and learning approaches in the mathematics classroom need to be reformed.

Since the last three years, the CASCADE-IMEI study is tied to the current reform of mathematics education in Indonesia. In an attempt to combat the low achievement in mathematics of students on national exams, the Indonesian government has attempted to identify probable reasons for this problem. Research cites various causes, including inaccurate learning materials, inadequate mechanistic teaching methods, poor forms of assessment and the anxiety of students to mathematics. One of the promising approaches toward the teaching and learning of mathematics that is thought to address these problems is realistic mathematics education (RME). RME is a theory of teaching and learning mathematics that has been developed in the Netherlands since the early 70's (cf. de Lange, 1987; Freudenthal, 1991; Gravemeijer, 1994). Contrary to the current mathematics education in Indonesia, RME uses realistic and interdisciplinary materials as a source as well as a starting point for mathematics teaching.

This study aims to introduce RME to (prospective) mathematics teachers in teacher education in Bandung, Indonesia, by developing a learning environment in the form of a face-to-face RME course and web site support. In this learning environment (prospective) teachers are encouraged to build up their background knowledge as well as to develop knowledge regarding (Selter, 2001): the mathematical component (overview of RME theory, doing mathematics); the didactical component (how to design and teach RME lessons); the practical component (how to manage RME classroom during classroom practice); and the psychological part of RME (how do pupils in the school learn and understand RME lessons).

This paper will focus on the impact of the learning environment on (prospective) mathematics teachers and teacher educators as well as on pupils in the classrooms with regard to RME as an innovation in mathematics education in Indonesia.

Theoretical Framework

The learning environment, including both the course and the web site, is based on the RME philosophy and principles. The philosophy of RME is mostly determined by Freudenthal's view on mathematics (cf. Freudenthal, 1991). Two of his important points of view are: (1) mathematics must be connected to reality and (2) mathematics should be seen as a human activity. First, in order to start from reality that deals with phenomena that are familiar to the students, *Freudenthal's didactical phenomenology*, i.e. the view of learning as starting contextual experience is used. Second, by the *guided reinvention* principle through *progressive mathematizations*, students are guided didactically and efficiently from one to another level of thinking. These two principles and the concept of *self-developed models* (Gravemeijer, 1994) are used as design principles both in developing the course materials and the web site. Furthermore, these principles are operationalized into five basic characteristics of realistic mathematics education or five tenets of RME (Gravemeijer, 1994). In short those are:

(1) *Use of contextual problems* (contextual problems figure as applications and as starting points from which the intended mathematics can come out).

(2) *Use of models or bridging by vertical instruments* (broad attention is paid to development models, schemas and symbolization rather than just offering the rule or formal mathematics right away).

(3) *Use of students' contribution* (large contributions to the course are coming from student's own constructions, which lead them from their own informal to the more standard formal methods).

(4) *Interactivity* (the social live in the classroom including explicit negotiation, intervention, discussion, cooperation and evaluation among pupils and teachers are essential elements in a constructive learning process in which the student's informal strategies are used as a lever to attain the formal ones).

(5) *Intertwining of learning strands* (the holistic approach implies that learning strands can not be dealt with as separate entities; instead, an intertwining of learning strands is exploited in problem solving).

Research Methodology

This study uses a development research approach (van den Akker, 1999). With this method, the learning environment is developed and evaluated in three main phases: preliminary study, prototyping phase and assessment phase. In this paper the focus is on the research process up to the prototyping phase in which the three prototypes of the learning environment were designed and evaluated in the Netherlands and in Indonesia. In the Netherlands, eight experts from four different expertises (curriculum development, professional development, RME and web site development) were involved as evaluators of the learning environment. After being revised and adapted to the Indonesian context, these prototypes were evaluated and implemented to the target group in teacher education in Bandung.

Development phase

This section provides a brief description on both components of the learning environment (see also Zulkardi & Nieveen, 2001): the course and the web site as illustrated in Figure 1.

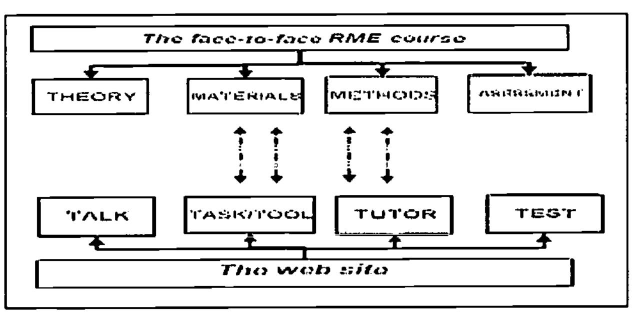

Figure 1. *The components of the learning environment*

The course

The RME course is a part of the learning environment that is developed in order to make (prospective) mathematics teachers understand what RME is and how to implement RME in the classroom. The main contents of this course include: (1) overview of the RME theory; (2) learning what are RME materials and how to redesign them; (3) learning how to teach using the RME approach in the classroom; and (4) learning how to assess the pupils in the RME classroom.

The web site

The web site, www.clix.to/zulkardi, is developed in order to support the course participants in a sustainable way. In order to do so, the following options are available:

(1) *Online Info-base or task.* The online info-base is the main component of the web site and consists of exemplary RME materials such as student materials and teacher guide; student productions from RME classes, applet programs and mathematical games, links to web sites that have relationship with mathematics education in general and RME.

(2) *Online Tutor.* In order to inspire (prospective) mathematics teachers before they conduct teaching practice in the school, an online tutor was designed. At his moment, the online tutor consists of theory on how to use RME materials in the classroom. In the future, a number of video clips that illustrate critical moments of teaching using RME materials in the classroom will be made available. For example, how to start the lesson, how to organize and to manage discussions.

(3) *Online Talk.* In order to provide (prospective) mathematics teachers with opportunities to discuss their problems and experiences, the web site provides an online talk element including e-mail facilities, a message board and a mailing list.

(4) *Online Test.* In order to facilitate (prospective) mathematics teachers with a number of RME problems, an online test called *problem of the month* was developed. It contains not only an example of RME problems but also a guide on how and when to use them in the classroom practice.

Research phase and questions

The results of the implementation and evaluation of the learning environment are discussed in the remainder of this paper based on the following questions:

- *What changes in (prospective) mathematics teachers as well as in pupils in schools in Bandung are reflected in their attitude towards RME?*
- *What changes in (prospective) mathematics teachers in Bandung are reflected in their knowledge of RME as the content of the learning environment?*
- *What are the effects of the learning environment with respect to the mathematics education society in Indonesia?*

Participants

In Indonesia, the main participants of the formative evaluation cycles of the learning environment (held in the period September 1999 to January 2000, May 2000 to August 2000, January 2001 to May 2001 and September 2001 to November 2001) were 27 (prospective) mathematics teachers at the Department of Mathematics Education, the Indonesian Educational University in Bandung. All of them had no teaching experiences except four of them, who were in-service teachers. About 480 pupils participated from 15 secondary school classrooms. In addition, six teacher educators were involved as supervisors of their students, and 15 school mathematics teachers were involved as observers in the classroom.

Instruments

The instruments that were used during the evaluation of the course are an entry and a final questionnaire, an end of unit test and a guideline for interviewing participants. The instruments that were used in the school are a final questionnaire, end-of-unit test and observation form. Furthermore, the instruments that were used in evaluating the web site are an observation form, a logbook and e-mails for collecting data from the (prospective) mathematics teachers.

Procedure

The course was implemented in the teacher education institute within a time frame of three to five blocks of four-hours. The course started by giving the participants information about the basic principles of RME and its characteristics. Then some examples of realistic mathematics problems were given and discussed in groups to get the idea of each characteristic of RME. Next, the participants were given a number of RME problems in various topics (such as linear equation system, symmetry, side seeing, statistics and matrices). After they solved the problems, they were guided in discussing the various strategies and in several cases they were invited to present their answers in front of the class. Finally, at the end of the course they were tested to see their performance in solving the problems. They were followed when they implemented the RME lessons in school classrooms. These activities took the longer time of the research period. They developed the lesson materials in collaboration with the researcher, who also observed their lessons.

The web site was evaluated using a cooperative evaluation, during which the (prospective) mathematics teachers performed as users and were asked to work aloud. Moreover, during the whole program, they discussed and reflected on their experiences using e-mails and a mailing list.

Results and Discussion

We present the results and discuss them based on the basis of the questions that were stated in the research methodology part.

- *What changes in (prospective) mathematics teachers as well as in pupils in schools in Bandung are reflected in their attitude towards RME?*

The sample reactions of participants that were gathered by a similar questionnaire are summarized in table 1.

Table 1 The results of final questionnaire of 29 student teachers in teacher education (TE), 36 senior high school students (SMUN) and 24 junior high school students (SMPN) after they followed the RME instruction process.

Items	TE			SMUN			SMPN		
Reactions overall	+	+/-	-	+	+/-	-	+	+/-	-
Learning process of RME is interesting	27	2	0	32	1	3	24	0	0
RME materials are interesting	28	1	0	32	2	2	23	0	1
Interactivity make me easy to understand	29	0	0	36	0	0	23	0	1
The role of teacher is helpful for me	28	0	1	35	0	1	24	0	0
Assessment materials challenge me	26	0	3	33	3	0	24	0	0
Motivates me to learn mathematics	28	1	0	33	1	2	24	0	0
Learning other's strategies is new for me	28	1	0	36	0	0	24	0	0

Note for reactions: '+': yes, '+/-': neutral, and '-': no

In general, the results in table 1 illustrate that the participants are very interested in the RME teaching approach both in the teacher education institute and in the schools. These positive reactions from participants are a necessary prerequisite to higher-level evaluation results. The items refer to the characteristics of RME. For instance, from the results on the second item 'RME materials are interesting', it can be concluded that the materials that were used are real to their situation (the first characteristic of RME) and integrated to other strands or subjects (the fifth tenet of RME). Further, they found 'a nuance of democracy' in learning mathematics such as the interactivity and a chance to learn other's strategies during the discussion (the fourth characteristic of RME).

- *What changes in (prospective) mathematics teachers in Bandung reflected in their knowledge of RME as the content of learning environment?*

In order to answer this question three kinds of results are used. The first kind of result consists of (prospective) mathematics teachers solutions on a test at the end of the course. Here, their understanding of RME either theoretically or mathematically were assessed. Overall, the results show that the participants were able to write down the philosophy and the characteristics of RME and solve RME typical problems in the sense of mathematization. However, the results are not discussed here because this falls somewhat beyond the theme of this conference. Second, the knowledge of (prospective) mathematics teachers in developing lessons based on the RME tenets was taken into account when answering the question. All of participants developed their lessons based on the RME materials, which were provided by the researcher. As a result of this, all of them were able to develop their own lessons in collaboration with the researcher. Of course, the results are not as good as truly RME materials. Nevertheless, as (prospective) mathematics teachers they have got a valuable experience in designing the lessons. Finally, the researcher observed the teaching skills of (prospective) mathematics teachers. An overall impression was that they were able to teach realistic materials in an interactive manner. They used their knowledge from teacher education such as how to start the lesson, how to make groups of students and how to guide group and class discussions. However, they also encountered some problems such as how to motivate the students to get involved in the discussion and how to conclude the lesson.

- *What are effects of the learning environment (the web site) to the mathematics education society in Indonesia?*

As the web site of the CASCADE-IMEI has been online since last three years, thousands of users, most of them from the mathematics education society from many countries (dominantly from Indonesia), have accessed the web site. On the basis of data that were gathered from user's feedback either through forms, e-mails, or a mailing list it can be concluded that this first mathematics web site in Indonesia has positive effects in:

- providing information, learning opportunities and communication facilities concerning mathematics education to not only mathematics education people but also parents and policy makers; and
- functioning as a dissemination tool of RME to other (prospective) mathematics teachers all over Indonesia.

Conclusion

Based on the results in the previous section we can concluded that:

- Changes in (prospective) mathematics teachers as well as pupils in schools in Bandung reflected in their attitude to RME have shown that they are interested in RME as an innovation as well as 'a nuance of democracy' in mathematics classroom (such as the interactivity and a chance to learn other's strategies during the discussion) has been accessed not only in the undergraduate (teacher education institution) level but also in the secondary school level.
- Changes in (prospective) mathematics teachers in Bandung reflected in their knowledge of RME theory have shown that they could perform better as RME teachers in classroom practice.
- The learning environment (the web site part), as the first web site of mathematics education in Indonesia has positive effects in supporting a traditional course in teacher education. Hopefully, the web site will be a nice dissemination tool for an innovation to mathematics society in Indonesia.

Nevertheless, these tentative changes have only been found mainly in the research locations of the CASCADE-IMEI study in Bandung. As Indonesia is a big country with about 225 million people, of course, the issues of scaling up and dissemination become of paramount importance. In this process we need to learn from experiences of mathematics education in Indonesia and in other regions all over the world.

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THE TEACHING OF CREATIVE MATHEMATICAL MODELING VIA AN EDUCATIONAL TOOLKIT FOR DESIGN OPTIMIZATION (TDO)

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ABSTRACT

The solution of a real-life problem via mathematical modeling often leads to the posing of a mathematical optimization problem. Even if the modeling exercise is relatively simple, the solution of the associated optimization problem represents a non-trivial and time-consuming process. In the teaching of mathematical modeling, this fact often inhibits the student from carrying out the repetitive but essential evaluation of various alternative models in order to arrive at an acceptable solution. To overcome this difficulty, the Toolkit for Design Optimization (TDO) was recently developed (Snyman et al. 2001). This system allows the student to easily solve his or her formulated optimization problem on a computer, through the interactive use of a graphical user interface (GUI), without doing any formal programming. This paper briefly describes the system, and presents some experiences of the authors in using TDO in teaching a course in creative modeling to a group of senior undergraduate engineering students. With very little formal knowledge of mathematical optimization algorithms, the students were capable of solving a wide range of modeling problems. Of particular importance is the finding that the system not only enables the students to be creative in solving non-trivial design problems, but also allows them to have fun in doing so.

Keywords: computing technology, mathematical modeling, optimization algorithms

1. Introduction

The attempt at solving a real-life problem via mathematical modeling requires the cyclic performance of the four steps depicted in Figure 1. The main steps are: 1) the observation and study of the real-world situation associated with a practical problem, 2) the abstraction of the problem by the construction of a mathematical model that is described in terms of preliminarily fixed model parameters **p**, and variables **x** that have to be determined such that model performs in an acceptable manner, 3) the solution of a resulting purely mathematical problem that requires an analytical or numerical solution $\mathbf{x}^*(\mathbf{p})$, and 4) the evaluation of the solution $\mathbf{x}^*(\mathbf{p})$ and its practical implications. After step 4) it may be necessary to adjust the parameters and to refine the model, resulting in a new mathematical problem to be solved with an associated new solution to be evaluated. It may be required to perform the modeling cycle a number of times before an acceptable solution is obtained. More often than not, the mathematical problem to be solved in 3) is a mathematical optimization problem requiring a numerical solution. In many cases, even if the modeling exercise is relatively simple, the solution of the formulated optimization problem represents a non-trivial and time-consuming process. In the teaching of mathematical modeling, this fact often inhibits the student from carrying out the repetitive but essential evaluation of various alternative models in order to arrive at a practical solution. The Toolkit for Design Optimization (TDO) (Snyman et al. 2001) allows the student to easily solve his or her formulated constrained or unconstrained optimization problem on a computer, through the interactive use of a graphical user interface (GUI) without doing any formal programming.

TDO employs gradient-based optimization algorithms and depending on the type of problem being solved the student has the option of experimenting with different algorithms. TDO can be used to select an analytical objective function to be optimized as well as additional analytical equality and inequality constraint functions if constrained problems are considered. Allowance is also made for the use of approximations in specifying the objective and constraint functions.

In this paper the use of the toolkit is illustrated through its application to two sample mathematical modeling problems, typical of those that may be posed in the classroom. The first problem is the determination of the minimum cost design of a beer can of prescribed volume. The objective of the second example is to find the equilibrium configuration of a cable of negligible weight subjected to concentrated loads. Experiences of the authors with TDO in teaching a course in creative modeling to a group of senior undergraduate engineering students are also discussed. Of particular importance is the finding that the system not only enables the students to be creative in solving non-trivial design problems, but also ensures that they have fun in doing so.

2. Statement and Numerical Solution of an Optimization Problem

A *mathematical optimization problem* can be stated as follows:

Find $\mathbf{x}=(x_1,x_2,\ldots,x_n)\in R^n$, that minimizes $f(\mathbf{x})$ subject to the constraints

$$g_j(\mathbf{x})\leq 0,\quad j=1,2,\ldots,m \text{ and} \qquad (2.1)$$
$$h_j(\mathbf{x})=0,\quad j=1,2,\ldots r$$

where $f(\mathbf{x})$, $g_j(\mathbf{x})$ and $h_j(\mathbf{x})$ are scalar functions of the *variables* **x**. The function f is called the *objective function* and g_j and h_j are respectively the *inequality* and *equality constraint functions*. A *local optimum* solution is denoted by $\mathbf{x}^*$.

TDO uses gradient-based optimization methods developed at the University of Pretoria to solve the above general problem. These methods have the common and unique property that no explicit line searches are required. The individual algorithms that may be selected by the student are LFOP (Snyman 1982 and 1983), ETOP (Snyman 1985), and SQSD (Snyman and Hay 2000a) for unconstrained optimization, and LFOPC (Snyman 2000), ETOPC (Snyman 1998) and DYNAMIC-Q (Snyman et al. 1994 and Snyman and Hay 2000b) for constrained problems. With ETOP(C) both a Fletcher-Reeves or a Polak-Ribiére implementation is available. DYNAMIC-Q allows for the solution of (2.1) through the solution of a sequence of simple quadratic approximate sub-problems, constructed from the sampling of the function values and gradient values of the objective and constraint functions at successive approximate solution points.

3. Mathematical Modeling

The *formulation* of a mathematical modeling problem as an optimization problem involves transcribing a verbal description of the problem into a well defined mathematical statement by performing the following three steps (Arora 1989) (i) In addition to fixed parameters **p**, identify a set of design variables to describe the system, i.e., the n-dimensional vector $\mathbf{x}=(x_1,x_2,\ldots, x_n)$. (ii) Determine a criterion that is needed to judge whether or a given model, corresponding to a given **x** is better than another. This criterion is called the objective function f and is of course influenced by the variables, i.e., $f=f(\mathbf{x})$. (iii) Specify the set of constraints within which the system must perform. Again the specified constraints are influenced by the design variables of the system. If the design satisfies all the constraints we have a *feasible* (workable) system or model.

The following two examples are typical of simple modeling problems that may be posed in the classroom. They will later be used as vehicles to illustrate the implementation of TDO.

3.1. ***Beer Can Problem*** (Arora 1989) The verbal statement of the design problem is as follows. Design a can that will hold at least a specified amount of beer and meet other design requirements. The cans will be produced in billions, so that it is desirable to minimize the cost of manufacturing them. Since the cost can be related directly to the surface area of the sheet metal used, it is reasonable to minimize the sheet metal required to fabricate the can. Fabrication, handling and aesthetics and shipping considerations impose the following restrictions on the size of the can: 1) the diameter should not be more than 8 cm and not less than 3.5 cm; 2) the height of the can should be no more than 18 cm and no less than 8 cm; and 3) the can is required to hold at least a specified volume, V_{spec} ml, of fluid (e.g., V_{spec}=400 ml =400 cm^3). The mathematical formulation is now obtained by performing the following steps. (i) The design variables are identified as x_1=D= diameter of the can (cm) and x_2=H= height of the can (cm). (ii) The objective function to be minimized is the total surface area of the can: area = $\pi DH+\frac{1}{2}\pi D^2$. This gives $f(\mathbf{x})=\pi x_1 x_2+\frac{1}{2}\pi x_1^2$. (iii) From the statement of the problem the following inequality constraints are identified. The volume = $\frac{1}{4}\pi D^2 H \geq V_{spec}$, i.e. , $g(\mathbf{x})= V_{spec} -\frac{1}{4}\pi x_1^2 x_2 \leq 0$. Constraints on the size can imply: $3.5\leq D = x_1 \leq 8$ and $8 \leq H = x_2 \leq 18$. The final formal mathematical statement of the design optimization problem is therefore:

minimize $f(\mathbf{x})=\pi x_1 x_2+\frac{1}{2}\pi x_1^2$

such that $g_1(\mathbf{x})= V_{spec}-\frac{1}{4}\pi x_1^2 x_2\leq 0$ with *side constraints* (3.1)

$g_2(\mathbf{x}) = 3.5- x_1 \leq 0; g_3(\mathbf{x}) = x_1 - 8 \leq 0; g_4(\mathbf{x}) = 8 - x_2 \leq 0;\ g_5(\mathbf{x}) = x_2 -18 \leq 0$

3.2. ***Cable Configuration Problem*** Consider the *symmetrical* system of three masses supported by an inextensible cable of negligible weight as shown in Figure 2. The problem is to find the equilibrium configurations of the cable for different choices of masses m_1 and m_2, and connecting lengths ℓ_1 and ℓ_2. This problem may also be formulated as an optimization problem by performing the following three necessary steps. (i) Identify the relevant variables as (x_3, x_1), the Cartesian coordinates of mass m_1 and x_2 the vertical position of mass m_2. (ii) Recognize, from elementary energy considerations, that for any given choice of the masses and the connecting lengths, the equilibrium configuration corresponds to that of minimum potential energy, i.e., choose the objective function as $f(\mathbf{x})= 2m_1gx_1+m_2gx_2$ or more simply, $f(\mathbf{x})=2m_1x_1+m_2x_2$ since the acceleration due to gravity g is constant. (iii) As the cable is inextensible, specify the associated constraints $x_1^2 + x_3^2 \le \ell_1^2$ and $(x_1 - x_2)^2 + (1 - x_3)^2 \le \ell_2^2$.

The final formal mathematical statement of the cable design optimization problem is therefore: minimize $f(\mathbf{x})=2m_1x_1+m_2x_2$

such that (3.2)

$$g_1(\mathbf{x}) = x_1^2 + x_3^2 - \ell_1^2 \le 0;\ g_2(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_3 + 1 - \ell_2^2 \le 0$$

3.3 Modeling-Optimization Interaction In the modeling process the student would normally like to quickly evaluate different options and strategies to arrive at an acceptable practical solution. This would normally require the changing of the different *parameters* of the specific problem, and then solving the correspondingly modified optimization problem to evaluate various alternatives. In the beer can problem (3.1) the typical parameter is V_{spec}, and in the cable problem (3.2) the parameters are the masses m_1 and m_2, and connecting lengths ℓ_1 and ℓ_2. Although the modification of the model through parameter variation is simple, the solution to the resulting reformulated optimization problem may be non-trivial by comparison. The latter exercise may also be time-consuming and distracting. Therefore, if the emphasis in the classroom is to be on the modeling aspects, i.e., on the formulation and evaluation of different models, then the availability of a computational device that may easily and quickly be used to solve the different formulated optimization problems, would clearly be an invaluable aid. The TDO graphical user interface is such a computational tool.

4. Graphical User Interface

4.1. Main Window The Toolkit for Design Optimization (TDO) is a graphical user interface (GUI) operating in the Windows 95/98/NT environment that allows the student to obtain solutions to optimization problems of the form (2.1). It was developed using Visual C++. The main window of TDO is shown in Figure 3. This Main window is used to control the whole optimization process, which includes the specification of the objective function (analytical or approximated), the specification of the design variable names, initial values and/or bounds, the specification of the constraints, and the optimization algorithm settings. After each item has been set or selected, control returns to this main window, from where the solution of the optimization problem is launched. This central control location allows the user to easily compare different algorithms, and to determine the influence of different settings, e.g., bounds and move limits. The current version of TDO is limited to five design variables, and the specification of three equality and three inequality analytical constraint functions. The approximation of the objective function and/or one constraint function is allowed for.

4.2 Specification of Analytical Functions TDO allows the user to specify analytical functions in terms of design variables. Several built-in analytical functions are provided. These are mainly selected through the specification of the coefficients of polynomials and reciprocal terms. The following general analytical objective function is included in the current version of TDO:

$$
\begin{aligned}
f(\mathbf{x}) = {} & a_{00} + a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4 + a_{51}x_5 \\
& + a_{12}x_1^2 + a_{22}x_2^2 + a_{32}x_3^2 + a_{42}x_4^2 + a_{52}x_5^2 \\
& + a_{13}x_1^3 + a_{23}x_2^3 + a_{33}x_3^3 + a_{43}x_4^3 + a_{53}x_5^3 \\
& + a_{14}x_1^4 + a_{24}x_2^4 + a_{34}x_3^4 + a_{44}x_4^4 + a_{54}x_5^4 \\
& + ac_{12}x_1x_2 + ac_{13}x_1x_3 + ac_{14}x_1x_4 + ac_{15}x_1x_5 + ac_{23}x_2x_3 + ac_{24}x_2x_4 + ac_{25}x_2x_5 \\
& + ac_{34}x_3x_4 + ac_{35}x_3x_5 + ac_{45}x_4x_5 \\
& + \frac{ar_1}{x_1} + \frac{ar_2}{x_2} + \frac{ar_3}{x_3} + \frac{ar_4}{x_4} + \frac{ar_5}{x_5} \\
& + as_1x_1^2x_2 + as_2x_1x_2x_3 \qquad (4.1)
\end{aligned}
$$

By specifying the $a_{i,j}$, $ac_{i,j}$, ar_i and as_i coefficients, the user can select any specific function from the set defined by (4.1). Of interest to the student is that transcendental and hyperbolic functions can also be approximated by polynomial functions, and can thus also be specified using (4.1). Refer to Figure 4 for the objective function dialog. The 'Approximated' setting is discussed in Section 4.4.

The settings of the coefficients of the analytical objective function, as selected by the 'Set Coefficients' button, are defined in the dialog contained in Figure 5. Note that the settings can be reset when different problems are run in succession. For constrained optimization problems, selecting 'Constraints' in the main window, allows for the specification of inequality constraint functions $g(\mathbf{x})$, and the equality constraint functions $h(\mathbf{x})$ in an identical manner to that shown for $f(\mathbf{x})$ in (4.1) and Figure 5.

4.3. Specification of Design Variables and Optimization Settings Returning to the main window, the selection of the 'Design Variables' setting calls the Design Variable window which enables the user to name variables and to select their initial values as shown in Figure 6. Any variable name can be given in the appropriate edit boxes. Constraints in the form of bounds can be placed on the variables through the selection of the appropriate check boxes and the specification of the relevant minimum and maximum values.

Selecting 'Algorithm Settings' in the main window enables the selection of a suitable optimization algorithm, ETOP, SQSD or LFOP for unconstrained problems, LFOPC or ETOPC for constrained problems and DYNAMIC-Q if approximations are to be employed. The window that allows for the appropriate selections is depicted in Figure 7. For each of these methods, the user can specify the convergence parameters (Design Variable Tolerance and Objective Function Gradient Norm) as well as a step size limit linked to the dimension of the design variable vector and range of the design variables. Default values of the algorithm control parameters are displayed in the dialog. These values are used if not modified. The maximum number of iterations and the print frequency of the results can also be adjusted in this dialog.

4.4. Optimization using approximations TDO uses successive spherical quadratic approximations (Snyman et al. 1994 and Snyamn and Hay 2000b) of the objective and one constraint function for cases where these functions, evaluated externally to TDO, are expensive to evaluate. The construction of the approximations at a local design point $\mathbf{x}^{(k)}$ requires the function value and its gradient at this current design point. The gradient of the function is obtained by first-order forward finite dfferences. If the 'Approximated' setting is selected TDO requires that the user enter the value of the relevant function at $\mathbf{x}^{(k)}$ as well as each of the values at the respective perturbation points $(\mathbf{x}^{(k)} + \mathbf{\Delta x}_i)$, i=1,n. Refer to Figure 8 for the dialog for the Approximated

Subproblem Setup. In the case of the first iteration, the checkbox for the first iteration is checked and the initial curvature can be specified (positive for a convex and negative for a concave approximation). A default value of 0.0 is used for the curvature (i.e. a linear initial approximation) if not changed. The solution of successive sub-problems is controlled from this dialog. For the first iteration, the initial design (starting values of the design variables) as well as the perturbations on the design variables, and move limits on design variable modification, are first specified. The objective and/or constraint function values, as obtained from an external numerical (or experimental) simulation, are entered in the Current Iteration fields. After a sub problem is solved, the previous objective function and design variables' values are automatically written in the Previous Iteration fields, and the design suggested by the optimizer automatically becomes the new Starting values of the design variables for the 'new' current iteration. The user then reruns the external simulation with the new design and the cycle is repeated.

4.5. Results of optimization problem The summary results of the direct solution of an analytical optimization problem, or of each sub problem using approximations, are given in the Results dialog shown in Figure 9. This window gives the results for the cable configuration problem with $m_1=m_2=1$kg and $\ell_1 = \ell_2 = 1$m. (As a matter of interest the computed solution for the beer can problem with V_{spec}=400 cm^3 is x_1 =7.97885 cm and x_2= 8.00000 cm). The detailed results are written to a file that can be imported into a spreadsheet program (Microsoft Excel) for graphical output by clicking on the 'View history in Microsoft Excel' button. A macro in Excel reads the data and plots the history of the objective function, design variables and constraints. The numerical data values are also given in spreadsheet format for further processing. Alternatively, the user can click on the 'Graphical Display' button to view the results in a plot inside TDO. This view is shown in Figure 10. The objective function, constraints and design variables are shown on the same axis, and are normalized. The normalization factors are given in the dialog for all the functions and variables.

5. Implementation of TDO in a Design Course

Over the past few years TDO has successfully been used in the teaching of optimization techniques to relatively large groups and to individual students. In particular, it was recently employed in a senior design course for engineering students where the following assignment was set:

"Assignment: Introductory mathematical modeling and optimization exercise using TDO.

This assignment represents a challenge to your creativity. Construct an original model of a real-world problem situation, simple enough (with respect to the forms of the objective and constraint functions and the number of variables) to be solved by TDO. In the formulation identify the parameters ***p*** *of the model and the design variables* ***x****. Obtain a realistic solution to the problem by executing the modeling-optimization loop* (Figure 1) *as many times as necessary."*

With very little formal knowledge of mathematical optimization algorithms, the students were capable of solving a wide range of realistic modeling problems. The problems ranged from the optimal design of amplifiers, filters and antennas of importance to electrical engineers, to design problems relating to combustion chambers, centrifuges, cycle chains and formula 1 GP racers of specific interest to mechanical engineers. Many other problems were also successfully solved. Some of those worthy of further mentioning include the design of a solid rocket fuel projectile, optimizing the flow in a continuous casting process, the shape optimization of a soap bar for longer life and the design of a feeding trough for animals. Most of the problems tackled involved three to five variables, with many side constraints and relatively complicated inequality and equality constraints. In solving these non-trivial design problems, the students had to consider many different possible models (by varying, for example, the set of parameters **p**), and solving for each model the associated optimization problem. These tasks the students accomplished with remarkable ease, mainly due to the availability of TDO's user-friendly GUI, through which the models could easily be modified and optimized.

The above teaching experience has shown that, by giving the student assistance in the detailed, laborious and repetitive optimization task, enables him or her not only to solve non-trivial design problems, but also to have fun in doing so. As summarized by one of the students: *"The TDO program is a fun and useful tool in learning design optimization practice."*

Future possible improvements in TDO include the automatic linking of TDO to other simulation software to evaluate the objective and constraint functions. Other considerations are the extension to a considerably larger number of design variables; the approximation of multiple inequality and equality constraints; and the availability of a wider class of built-in analytical objective and constraint functions.

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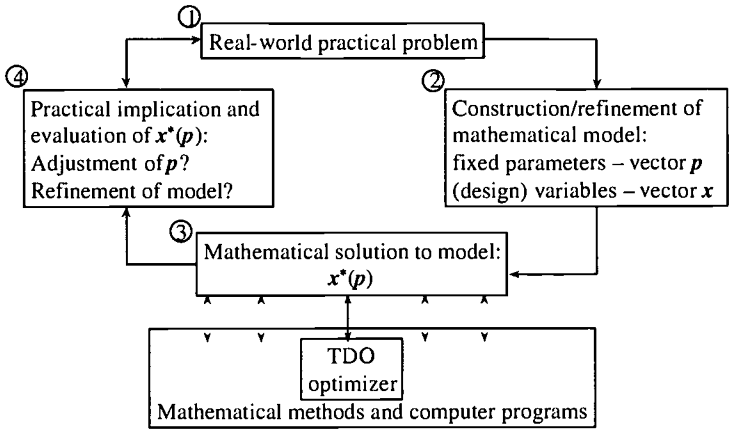
Figure 1. The mathematical modeling process.



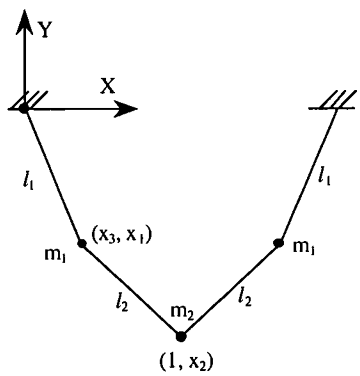


Figure 2. Cable configuration problem.

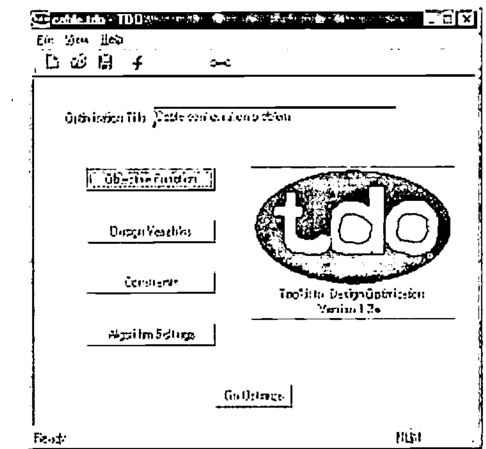


Figure 3. Main window of TDO.

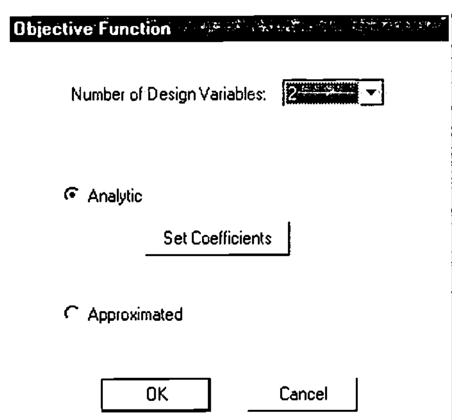


Figure 4. Objective Function dialog for beer can problem.

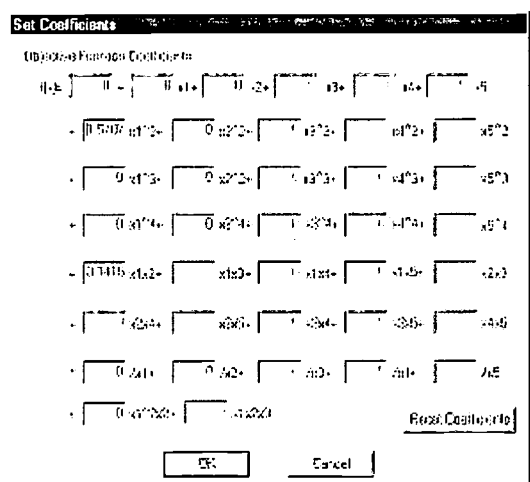


Figure 5. Analytic Objective Function Coefficient

settings dialog for beer can problem.

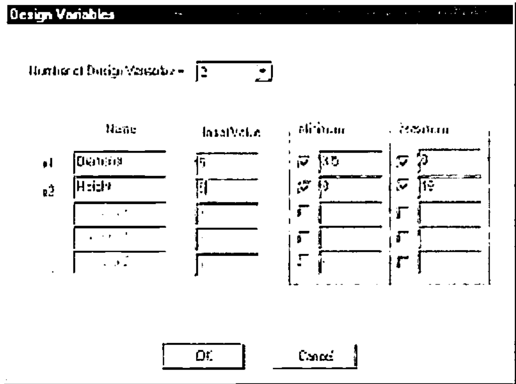


Figure 6. Design Variables dialog for beer can problem.

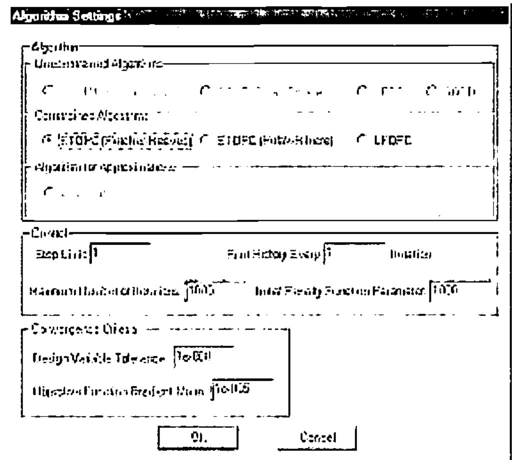


Figure 7. Optimization settings dialog for beer can problem.

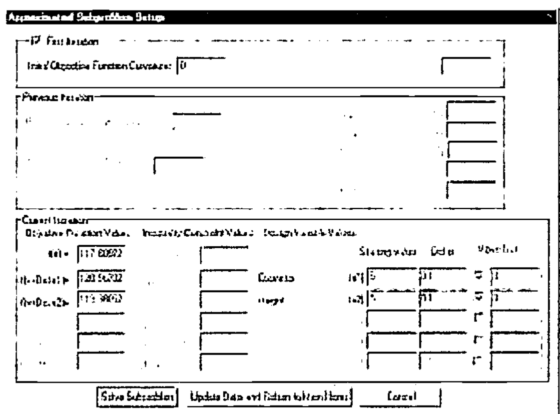


Figure 8. Approximations dialog.

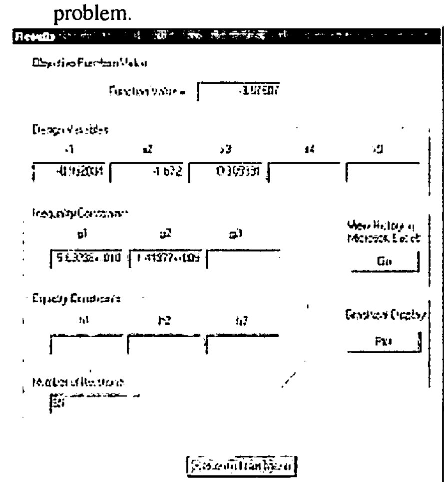


Figure 9. Results dialog for cable configuration problem.

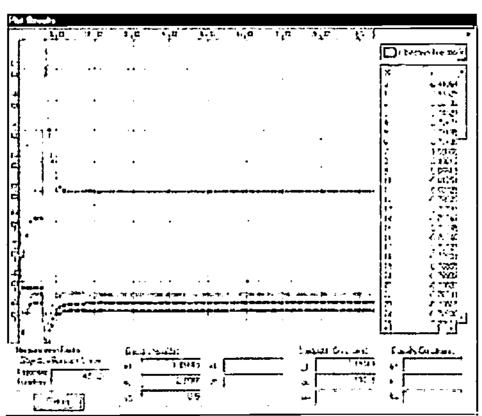


Figure 10. Graphical convergence histories of the values of the objective and constraint functions, and of the values of the design variables.

DEVELOPING OPEN AND FLEXIBLE COMPUTING ENVIRONMENTS FOR TEACHING MATHEMATICS AND SCIENCE

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ABSTRACT

Most software packages for the teaching of mathematics contain only a limited set of tools, utilities, and procedures. Therefore we often have to use more than one computer program to teach different subjects. This situation makes teaching very inefficient: We need to teach our students how to use each tool, we need to teach different, environment-dependent strategies for solving problems, and sometimes we even need to adapt to completely different computing philosophies. An ideal teaching package, on the other hand, would allow teachers and students to customize the program by modifying the resources within the program, adding their own procedures, functions and operations, and might even allow them to build their own mathematical libraries. In this paper, we show that *MuPAD*, a computer algebra system from SciFace Software and the University of Paderborn in Germany, is gradually becoming such an ideal electronic teaching environment in this sense since it already meets several of the mentioned requirements. We will show how teachers can build their own libraries, add and integrate them with *MuPAD* resources, and use them in their teaching with both a standalone and an online version of *MuPAD*. Finally, we will discuss some of the advantages of this open and flexible environment.

Desiderata

Let us begin by quoting from a letter by Carlos Fleitas, a teacher of mathematics from Spain, who mentions that he uses *Cabri* and *CarbiWeb* to make interactive geometry, uses *Derive* to study functions and graphs, tries to present the elementary ideas of probability with the help of *Excel*, and has recently begun to investigate *MuPAD* as a tool for generating L-systems. We know of other teachers of mathematics who are using even more expanded sets of tools in their classroom. Such approaches are difficult to follow.

In all of these cases, both the teachers and the students need to master several computing tools in addition to having to learn the mathematical concepts involved. There is a good reason for using such nonintegrated sets of tools in the classroom. Not one of these packages can be used to teach all or almost all topics in undergraduate mathematics. However, recent advances in the development of the *MuPAD* computer algebra system, for example, are giving us hope that the situation might change in the near future. Before we begin to analyze some of the promising aspect of *MuPAD*, we will identify some of the features we expect a computer package to have to be suitable for the teaching of undergraduate mathematics.

The package should have a broad and easily expandable mathematical base, be easy to learn and use, and fit naturally into most modern teaching and learning environments. By this we mean the following:

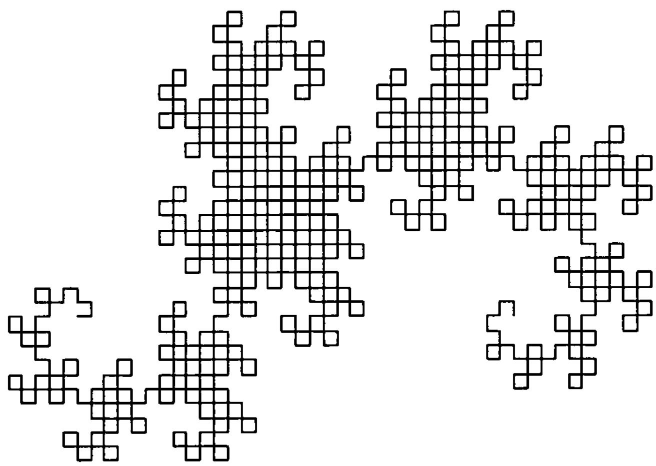
1. The package should provide an environment for the teaching of the widest possible range of topics in undergraduate mathematics: abstract algebra, linear algebra, geometry, calculus, differential equations, probability and statistics, as well other standard topics.
2. The package should provide means for teachers to enrich and expand the functionality of the package through customized libraries and software extensions.
3. The package should be easy to maintain and update by teachers and their assistants, even in schools with modest computer facilities, and should be portable across computer platforms such as the Windows, Linux and the Macintosh environments.
4. The package should be easy to use. In particular, it should be compatible with mainstream electronic course management systems such as WebCT. It should have a natural and easily learned interface and help facility. Moreover, it should be easy to learn by average students with relatively little supervision.
5. The package should provide for both command-line programming as well as for the menu-driven manipulation of mathematical objects, especially graphs and surfaces. Its programming language should be easy to learn and have as natural a syntax as possible.

This list can certainly be expanded and does not encompass all relevant features. Some of them were discussed in detail in our earlier paper (see [2]). Here we will concentrate mainly on the problem of developing *MuPAD* customized libraries and using them as teaching tools.

The Role of Libraries in Computer Algebra Systems

Libraries for computer algebra systems are sets of mathematical procedures usually grouped by topic. For example, a library may consist of procedures for teaching calculus, number theory, probability and statistics, and so on. The procedures defined in a library may range from simple tools for solving quadratic equations or calculating greatest common divisors to sophisticated methods for finding shortest paths in oriented graphs.

Computer algebra systems that allow teachers to build their own libraries are invaluable tools for advancing the teaching of mathematics. Using such systems, teachers and students can collaboratively build a wide range of toolboxes for their courses, modify them as needed, and improve and expand them over time. This provides a live environment for experimenting with mathematics. Teachers can exchange libraries with their colleagues and build extensive educational systems. Moreover, the ability to customize libraries may inspire enterprising students to solve mathematical problems that are often beyond of their school curriculum. The paper on L-systems mentioned below (see [1]), is one such example of a student project that went considerably beyond the standard undergraduate curriculum. Its author, *Michelle Raimbert*, began with an undergraduate paper on L-systems, written under the supervision of the second author, an introduction to L-systems by the first author, and created a *MuPAD* notebook on L-systems containing a beautiful collection of programs for generating fractals and other branching structures. The main objective of this paper was to investigate the suitability of MuPAD for the representation of such algorithms and techniques and to illustrate this suitability by creating some of the most famous fractal curves. One of them is a well-known Harter-Heightway Dragon curve,

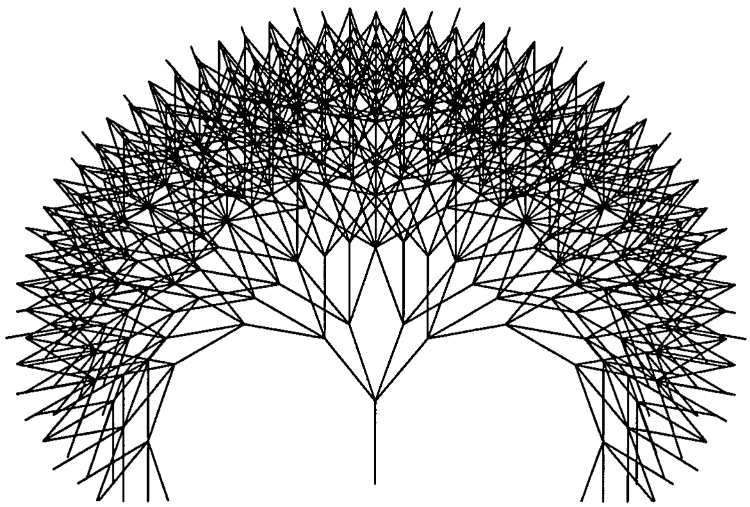


generated by the MuPAD code

```
L:=plot::Lsys(90, "BL",
   "L" = "L+R+",
   "R" = "-L-R",
   "L" = Line, "R" = Line,
   "B" = RGB::Black
):
L::generations:=11:
plot(L,Axes=None)
```

It is pedagogically valuable that the rule to generate the Harter-Heightway Dragon curve is conceptually quite simple. Start with a single segment of the length L. In each of the subsequent steps, replace any obtained segment by a semi-triangle, i.e. the figure that contains two equal segments separated by the right angle. This construction can be mimicked physically to some extent, by folding a piece of paper. However, for larger numbers of steps the use of computer is necessary. In most of the known cases, the creation of the Harter-Heightway Dragon curve requires a good knowledge of a programming language and programs creating this curve are usually not simple. With MuPAD, however, we can create this curve by typing eight lines of simple code without losing sight of the algorithm involved.

Another interesting example of a fractal structure from the same student's project is a beautiful tree like shape where she experimented with multiple colors, here for printing purposes presented as a black-and-white picture,



generated by the code

```
• L:=plot::Lsys (20, "L",
      "L"="BR[++YL][+OL][--YL][-OL]+",
      "R"=Line, "L"=Line,
      "Y"=RGB::OrangeRed,
      "O"=RGB::Pink,
      "B"=RGB::CadmiumRedDeep
  ):
  L::generations:=6:
plot(L,Axes=None)
```

Again, the creation of this type of fractal shape requires a good knowledge of a programming language and recursive programming techniques. With MuPAD nine lines of code suffice and only two lines require a bit more explanation.

The two above examples show that computer algebra systems provide us with entirely new tools for visualization of intriguing mathematical objects: elaborate curves and surfaces, and representations of biological phenomena such as the growth of algae, the veins in leaves, and the bronchi in the lungs. As such, they help us to instill a new physical meaning into mathematical objects.

As the above code shows, many of these objects are generated easily and intuitively. The *MuPAD* project on L-systems [1], for example, provides an enjoyable starting point in one such direction. We know of many other examples where students have discovered interesting mathematical facts by experimenting with mathematical concept by using a computer program.

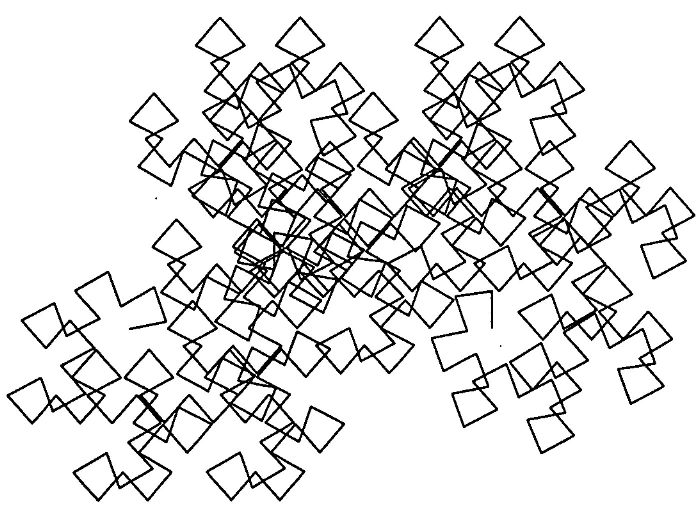
From an experimental student's project or classroom works there is one step to organizing the most interesting pieces of work and saving them later as reusable library of procedures.

For example, with a very basic knowledge of MuPAD programming student or teacher can convert the code presented in [1] into a library of new L-system procedures. Here we show how it can be done for the mentioned above dragon curve and the tree like shape.

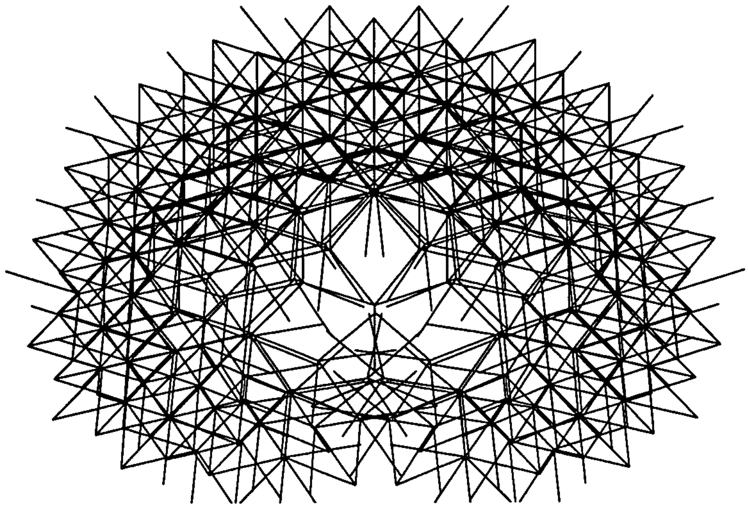
```
dragon := proc(angle, steps)
begin
   L:=plot::Lsys(angle, "BL",
      "L" = "L+R+",
      "R" = "-L-R",
      "L" = Line, "R" = Line,
      "B" = RGB::Black
   ):
   L::generations:=steps:
   plot(L,Axes=None);
end;

tree := proc(angle, steps)
begin
   L:=plot::Lsys (angle, "L",
      "L"="BR[++YL][+OL][--YL][-OL]+",
      "R"=Line, "L"=Line,
      "Y"=RGB::OrangeRed,
      "O"=RGB::Pink,
      "B"=RGB::CadmiumRedDeep
   ):
   L::generations:=steps:
   plot(L,Axes=None)
end
```

Later such procedures can be used in the classroom to experiment with dragon curves using various input parameters. For example,

- `dragon(100,10)`



- `tree(35,6)`



Developing Libraries for *MuPAD*

As our simple examples in this paper show, MuPAD is a command-line-based computer algebra system. It has a programming language similar in many aspects to Pascal. The package is highly universal. It can be used for almost any undergraduate mathematical topic, from logic to sophisticated problems in linear and abstract algebra. We would now like to discuss how the openness and flexibility of *MuPAD* make this system a promising environment for the teaching of mathematics.

There are many ways of developing a customized library for *MuPAD*. Teachers can easily write a set of procedures, test them on appropriate input data, and save them as reusable files on their computers. Here is a simple example that illustrates this process. Suppose we would like to create a library for basic statistical routines. We could start by writing a procedure for calculating the average of n numbers and save this procedure in a new library. The following steps accomplish this task.

```
average:=proc()
local n, i, result;
begin
   n:=args(0); result:=0;
   for i from 1 to n do result:=result+args(i) end;
   result:=result/n;
   return (result)
end:

WRITEPATH := "userlib";
write("mylibrary.mb", average)
```

We could then assign our students the task of extending this library[1] by writing simple procedures for other familiar statistical routines. Files such as **mylibrary.mb** can be saved in folders such as **userlib** in the *MuPAD* directory. To use this library, all a student has to do is to load the file into *MuPAD* when required. This can be as simple as invoking the following two commands:

```
READPATH := "userlib";
read("mylibrary.mb")
```

The maintenance of a computer lab in a school can be a time-consuming process. By specifying a location on the school network for *MuPAD* libraries, we can reduce this task to the maintenance of a single folder on a network server. This can be done by adding to each local installation of *MuPAD* a small configuration file with both **READPATH** and **WRITEPATH** commands that point to a network folder. From that moment on, all a teacher needs to do is to update and maintain a single folder on a network server. Recent innovations in the design of *MuPAD* lead us to believe that in the future, the development of customized libraries will be even simpler than in the given example. For more information on using and developing *MuPAD* libraries, we refer to chapter 5 of [2], where the benefits of these features of *MuPAD* are discussed in detail.

MuPAD Computing on the Web

In addition to making it easy to create customized libraries, the *MuPAD* Computing Server is a feature that may completely change how we use computer algebra systems in schools in the future. The *MuPAD* Computing Server is a special server application consisting of the *MuPAD* computing engine and *MuPAD* libraries. Users access the *MuPAD* Computing Server through a web page using standard browsers and perform calculations and solve problems directly on the Web. From the perspective of students and teachers, this tool is completely platform-independent. It can be accessed from a Macintosh, Windows-based PC, or even from a Linux-driven computer. Since all calculations are performed remotely on the server, the needs for specific hardware configurations at the user end are now redundant. Anyone connected to the Internet with any browser will be able to work with *MuPAD*. In order to maintain the *MuPAD* Computing Server, all teachers need to do is to update and load their libraries on the server. In addition, they can post on web pages all management aspects of working with their libraries: a listing of available topics and procedures, instructions on how to use them, tutorials, quizzes, glossaries, and so on. Students can use such *MuPAD* installations in the classroom as well as at home. They do not need to have *MuPAD* installed on their computers.

Conclusion

We have illustrated with simple examples how teachers and students can use the intuitive programming language of *MuPAD* to build mathematical libraries and use them in their teaching and research by integrating them with existing *MuPAD* resources. In our own teaching, we have incorporated such libraries in *WebCT*, where students can explore and experiment on the Web using several computing platform. At this point, our libraries encompass various fields of

[1] The statistical procedures in the basic MuPAD library are quite well developed. However, we will undoubtedly always need new additional routines for specific applications.

mathematics and some applications to other sciences. We have also shown how student projects can be used to expand our repertoire of libraries. Competing environments, based on other computer algebra systems with other features, are being developed elsewhere. What distinguishes the *MuPAD* system from most, if not all, of the other systems, is that it is open, flexible, user-friendly, portable across most computing platforms, and very affordable. We therefore already have here a tool that meets many of our listed needs. The era of device-independent, Web-based teaching and learning of mathematics and science has begun. Let us embrace it and use it for the betterment of global education.

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THE ROLE OF INSTRUMENTAL AND RELATIONAL UNDERSTANDING IN PROOFS ABOUT GROUP ISOMORPHISMS

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ABSTRACT

The ability to construct proofs is a crucial skill in advanced mathematics that most students lack. To investigate the causes of students' difficulty, we observed a small group of undergraduates and doctoral students constructing proofs about group isomorphisms. Undergraduates were able to construct very few proofs, despite having an understanding of mathematical logic and often possessing the instrumental knowledge needed to prove the propositions in our study. Doctoral students proved every proposition in our study. Our analysis reveals that doctoral students regularly used their relational understanding of group isomorphisms to guide their proof attempts, while undergraduates seldom did. We conclude that what one can prove solely using instrumental understanding is often limited, and using a relational understanding may be necessary to be an effective proof constructor.

1. Introduction

The ability to construct proofs about mathematical concepts is a crucial skill for any student of mathematics. Unfortunately, most college have serious difficulties constructing proofs (e.g. Moore, 1994). As students have difficulty with this crucial skill, it is natural to try to locate the cause of their difficulty. There has been considerable research on this topic, most of which has focussed on the logical aspect of proof construction. For instance, Harel and Sowder (1998) observed most students do not have an accurate conception of what constitutes a mathematical proof and Selden and Selden (1987) give examples of common invalid student proofs. While this research has produced rich data that is clearly important, there is a large and significant class of proofs that it cannot explain. Often, students fail to construct proofs because they do not know how to begin, spend all their time pursuing dead-ends, or reach an impasse where they simply cannot decide how to proceed (e.g. Moore, 1994; Schoenfeld, 1985). In these situations, the students' shortcomings are not logical in nature. Why students fail to construct proofs in these situations is poorly understood.

2. Instrumental and relational proofs

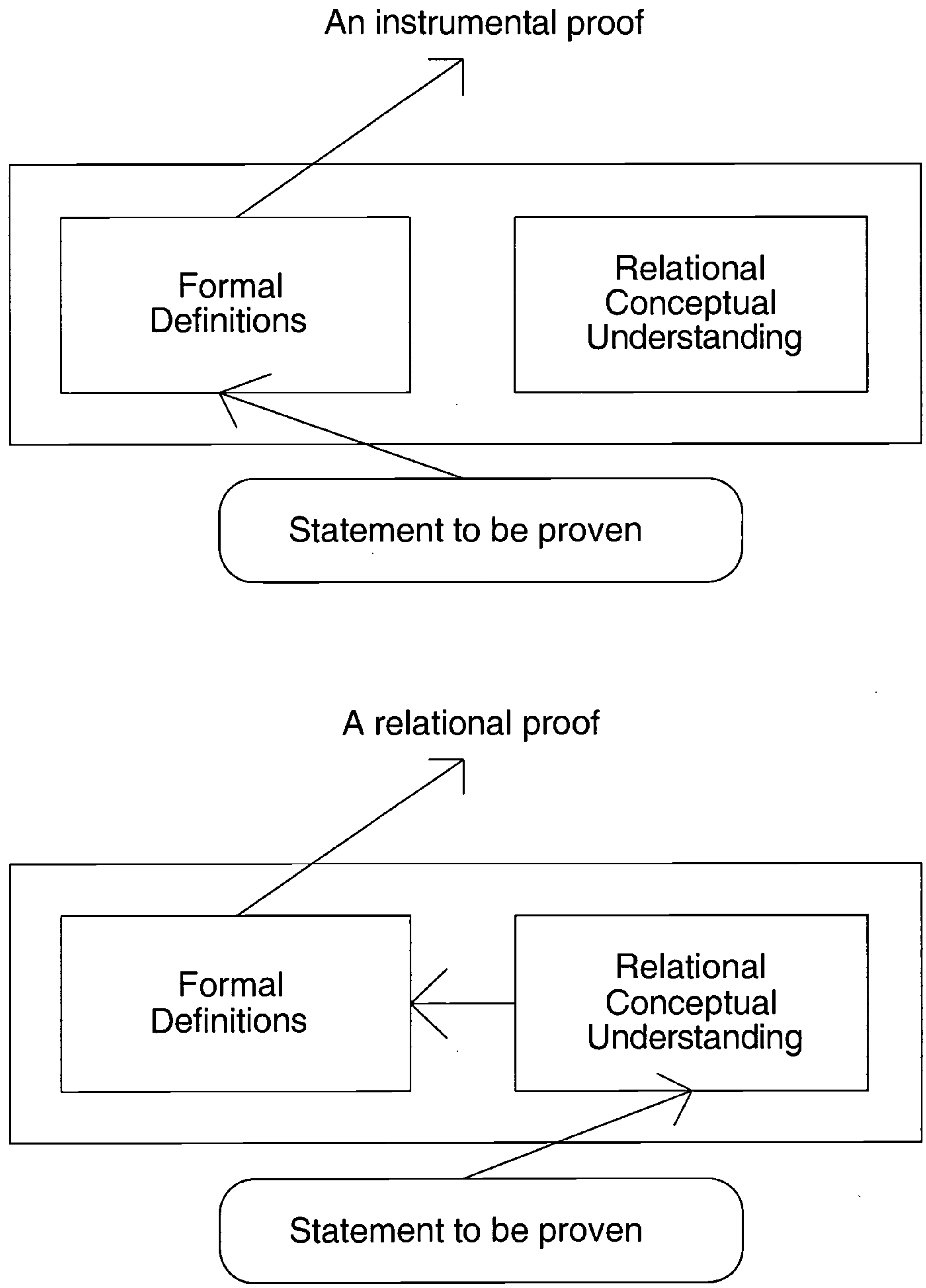
It is often said that there are two ways to understand a mathematical algorithm. An individual has an instrumental understanding of an algorithm if he or she can recall that algorithm and is capable of executing it; the individual has a relational understanding of an algorithm if he or she knows the purpose of the algorithm and why the algorithm works (Skemp, 1987).

We extend these types of understanding to include advanced mathematical concepts. We say an individual has an instrumental understanding of a concept if he or she can state the definition of the concept, is aware of the important theorems associated with that concept, and can apply those theorems in specific instances. We say an individual has a relational understanding of a concept if he or she understands the informal notion this concept was created to exhibit, why the definition is a rigorous demonstration of this intuitive notion, and why the theorems associated with this concept are true. (A relational understanding of a concept is somewhat akin to Tall and Vinner's concept image (Tall and Vinner, 1981)).

We use these types of understanding to describe two different types of proofs, as illustrated in Figure 1. An instrumental proof is a proof in which one primarily uses definitions and logical manipulations without referring to his or her intuitive understanding of a concept. A relational proof is a proof in which one uses his or her intuitive understanding of a concept as a basis for constructing a formal argument. An instrumental and a relational proof are essentially what Vinner (1991) calls a purely formal deduction and a deduction following intuitive thought.

We illustrate our definitions within the context of isomorphic groups, the concept used in our investigation. An individual with an instrumental understanding of isomorphic groups would know that the groups G and H are isomorphic if there exists a bijective homomorphism f from G to H, know basic theorems associated with isomorphic groups (e.g. an abelian group is not isomorphic to a non-abelian group), and be able to apply these theorems (e.g. S_3 is not isomorphic to Z_6). An individual with a relational understanding of isomorphic groups might recognize that isomorphic groups are "essentially the same" and that one is simply a re-labelling of the other. The definition

Figure 1. An instrumental and a relational proof



of isomorphic groups follows as the mapping f serves as the re-labelling (it is obvious then that f should be bijective and respect the groups' operations). The justification of many of the theorems about isomorphic groups, such as isomorphic groups must share all group theoretic properties, become self-evident once one views isomorphic groups as "essentially the same".

In our study, we ask participants to prove or disprove that two given groups are isomorphic. An instrumental proof of these propositions might consist of proving the two groups are isomorphic by constructing a bijective homomorphism between the groups or proving the groups are not isomorphic by demonstrating that no bijective mappings between the groups are homomorphisms. A relational proof would consist of first determining whether or not the groups in questions are essentially the same and then formalizing this intuitive reasoning.

In this paper, we observe undergraduates and doctoral students proving propositions about isomorphisms. We illustrate many examples where undergraduates failed to construct a proof despite possessing the instrumental knowledge required to do so. Further, we analyze both groups' proof attempts to shed light on the roles that instrumental and relational understanding play in proof construction.

3. Methods

Participants

Two groups of participants participated in this study. The first group of participant consisted of four undergraduate students at a university in the northeast United States. These students had recently completed their first abstract algebra course. Each student had also completed two linear algebra courses - the second of which stressed abstract vector spaces and rigorous proofs.

The second group of participants consisted of four doctoral students completing dissertations in an algebraic topic at a university in the mid-west United States. These students had approximately four more years of schooling than the undergraduate students.

Materials

Participants were first asked to prove the following Basic Propositions:

Basic Propositions

B1. Let G and H be groups and f be a homomorphism from G to G. Prove that for all x and y in G, $[f(xy)] = f(y^{-1})f(x^{-1})$.

B2. G is a group and f is a mapping from G to G such that $f(g) = g^{-1}$. Show that f is a homomorphism if and only if G is abelian.

The Basic Propositions were included to determine if the participants possessed an ability to construct rudimentary proofs. Participants were then asked to prove the more difficult Isomorphism Propositions:

I1. Prove or disprove: $\mathbf{Z}_{n!}$ is isomorphic to S_n

I2. Prove or disprove: **Q** is isomorphic to **Z**

I3. Prove or disprove: $\mathbf{Z}_p \times \mathbf{Z}_q$ is isomorphic to $\mathbf{Z}_{pq}$ (when p and q are coprime)

I4. Prove or disprove: $\mathbf{Z}_p \times \mathbf{Z}_q$ is isomorphic to $\mathbf{Z}_{pq}$ (when p and q are not coprime)

I5. Prove or disprove: S_4 is isomorphic to D_{12}

(where $\mathbf{Z}_p$ represents the integers under addition modulo p, **Z** the integers under addition, **Q** the rationals under addition, S_n the set of permutations of n elements, and D_{12} the dihedral group with 24 elements).

Procedure

This procedure is similar to the one used in an earlier study reported in Weber (in press).

- Using verbal protocol analysis (Ericcson and Simon, 1993), participants were asked to 'think aloud' as they attempted to prove the propositions listed above. At any point, the participants were allowed to refer to the textbook used in the undergraduate abstract algebra course.

- After attempting to prove the propositions, the participants completed a paper-and-pencil test about the facts needed to prove the propositions in this study. This test contained open-ended questions (e.g. "State the definition of isomorphic groups") as well as yes-or-no questions (e.g. "Can an Abelian group be isomorphic to a non-abelian group?"). After each questions, the participants were asked to indicate how confident they were of their answer with an integer between 0 and 2, where 0 represented "just guessing" and 2 represented "absolutely certain".

- If participants had been previously unable to prove a proposition, they were invited to try again by making use of their work on the paper-and-pencil test.

Each proof attempt was coded using the following scheme:

Correct- The participant produced a valid proof

Failure to apply instrumental knowledge- The participant failed to construct a proof. However, the participant indicated that he or she had the instrumental knowledge to construct the proof by answering the relevant questions on the paper-and-pencil test correctly with some degree of confidence (1 or 2). When told to use his or her work on the paper-and-pencil test, the participant produced a valid proof. Therefore, the participant could construct a proof if specifically told which facts to use, but failed to construct a proof without this prompting.

Lack of instrumental knowledge- The participant failed to construct a valid proof and either indicated that he or she was not aware of a fact required to prove the theorem (or indicated that he or she was aware of the fact, but was just guessing), or the participant could not prove the theorem when told to use the facts on the paper-and-pencil test.

Invalid proof- The participant produced an invalid proof.

4. Results

All participants in this study could prove the Basic Propositions. Although these proofs were not difficult, the participants' success indicates that they all had some basic notion of proof, familiarity with group theoretic concepts, and an ability to logically manipulate symbols.

Each doctoral student was able to prove or disprove every Isomorphism Proposition in this study. The undergraduates' performance on each of the Isomorphism Propositions is presented in Table 1. Collectively the undergraduates were only able to prove two of the Isomorphism Propositions. However, there were nine instances where the undergraduates failed to construct a proof because they did not apply their instrumental knowledge. To be specific, when the undergraduates were specifically told to use the facts needed to prove the propositions, they were able to construct a proof. When they had previously attempted to construct proofs without this prompting, they failed to construct a proof. Hence, the data indicate that even if one has an accurate conception of proof, possessing an instrumental understanding of a mathematical concept does not imply that one can effectively prove statements about that concept. There were eleven instances in which the undergraduates demonstrated an instrumental understanding of isomorphisms and the groups in question; in only two of those instances did they produce a valid proof.

Table 1. Undergraduates' performance on proving the Isomorphism Propositions

Proposition Number	Valid proof	Failure to apply instrumental knowledge	Lack of instrumental knowledge	Invalid proof
I1	1	2	1	0
I2	0	4	0	0
I3	0	1	3	0
I4	1	2	1	0
I5	0	0	4	0
Total	2	9	9	0

To investigate the role that instrumental and relational understanding plays in constructing proofs, we analyzed the behavior of the participants as they attempted to construct their proofs. Below, we present a brief description of the undergraduates' and the doctoral students' behavior for each of the propositions. We conclude by offering a summary of both groups' performance in this study.

Prove or disprove S_n is isomorphic to $\mathbf{Z}_{n!}$

Each doctoral student proved these two groups were not isomorphic (when n was greater than two) within forty seconds. Three doctoral students did so by realizing that $\mathbf{Z}_{n!}$ was abelian and S_n was not. The other student pointed out that S_n had no element of order n!.

After attempting to inappropriately apply Cayley's theorem, one undergraduate was able to disprove the proposition (by noting that $\mathbf{Z}_{n!}$ was cyclic and S_n was not). Another undergraduate made no meaningful progress on this problem. The other two undergraduates tried unsuccessfully to construct a bijection between the two groups.

*Prove or disprove **Q** is isomorphic to **Z***

The protocol of one doctoral student's proof is given below:

"**Z** is isomorphic to **Q**? That's false. Let's see… why? Well **Q** is dense and **Z** is not. No wait, denseness isn't a group property. Well then **Z** is cyclic and **Q** is not. So they can't be isomorphic".

Two other doctoral students proved that the groups could not be isomorphic because **Z** was cyclic and **Q** was not, with one adding, "I was tempted to add something about **Q** having a field structure, but that's not really the point". The final doctoral student proved the proposition by demonstrating that no homomorphism from **Z** to **Q** could be bijective.

The following excerpt of one undergraduate's protocol is given below:

"Um I think that **Q** and **Z** have different cardinalities so... no wait, **R** has a different cardinality, **Q** doesn't. Well, I guess we'll just use that as a proof. Yeah so I remember like seeing this proof on the board. I just don't remember what it is. There's something about being able to form a uh homomorphism by just counting diagonally [the student proceeds to create a complicated bijection between **Z** and **Q** by using a Cantorian diagonalization argument] Yeah I don't think we're on the right track here. Um... what you are describing is... it's um a bijection, but not a homomorphism"

This excerpt was representative of all four undergraduates' proof attempts. Upon realizing that **Z** and **Q** were equinumerous, all undergraduates constructed or attempted to construct a bijection between the groups. They seemingly showed little regard as to whether their bijections would respect the groups' operations. None successfully proved the groups were not isomorphic.

Prove or disprove $\mathbf{Z}_p$ x $\mathbf{Z}_q$ is isomorphic to $\mathbf{Z}_{pq}$ (assuming p and q are coprime)

An excerpt from one doctoral student's proof attempt is given below:

"OK, sufficient to find an element (g, h) in $\mathbf{Z}_p$ times $\mathbf{Z}_q$ that has order pq, because $\mathbf{Z}_p$ times $\mathbf{Z}_q$ has order pq and so if there's an element with the same order as the group, the group is cyclic and must be the same group as $\mathbf{Z}_{pq}$. OK um the element we're looking for is going to be (1, 1)."

The student then proceeded to show (1, 1) had order pq. The other doctoral students all proceeded to prove these groups were isomorphic by first observing that equinumerous cyclic groups were isomorphic and then showing that $\mathbf{Z}_p$ x $\mathbf{Z}_q$ was cyclic. No doctoral student constructed an explicit isomorphism between the two groups.

The two undergraduates that made progress on this problem attempted to construct a bijection between the two groups, one of which was a somewhat absurd mapping that mapped (a, b) in $\mathbf{Z}_p$ x $\mathbf{Z}_q$ to ab (mod pq) in $\mathbf{Z}_{pq}$. This mapping was neither bijective nor a homomorphism. Neither of these undergraduates used the fact that p and q were coprime. The other two undergraduates did not know how to begin their proof attempts.

Prove or disprove $\mathbf{Z}_p$ x $\mathbf{Z}_q$ is isomorphic to $\mathbf{Z}_{pq}$ (assuming p and q are coprime)

Three doctoral students proved that $\mathbf{Z}_p$ x $\mathbf{Z}_q$ was not cyclic and therefore could not be isomorphic to the cyclic group $\mathbf{Z}_{pq}$. The other doctoral student disproved this proposition by noting that $\mathbf{Z}_2$ x $\mathbf{Z}_2$ was not isomorphic to $\mathbf{Z}_4$.

One undergraduate disproved the proposition by offering the same counterexample. The other three undergraduates made no attempt to prove or disprove this proposition, explicitly reasoning that they made no useful progress on the last proposition, and there was nothing indicating their techniques would be more successful on this proposition.

Prove or disprove S_4 is isomorphic to D_{12}

The undergraduates had little familiarity with the dihedral groups so none were able to make much progress on this problem. Upon noting that the both groups were equinumerous, non-abelian groups, the doctoral students attempted to identify a distinct property of one group and demonstrate that the other group did not share this property. Some of the doctoral students' efforts were ineffective, as these groups do share some surprising properties. However, eventually all doctoral students were able to determine the groups were not isomorphic by finding a structural property possessed by one group that the other group did not share.

Summary

In most of the cases where the undergraduates seriously attempted to prove an Isomorphism Proposition, their proof attempt was of the following form: Upon realizing that the groups in question were equinumerous, they attempted to construct an arbitrary bijection between the groups. If this construction was successful, they were dismayed to find that the bijection did not respect the groups' operations and abandoned their proof attempts. If the construction was unsuccessful, they also gave up as they did not know how to proceed. Rarely did the undergraduates employ structural information about the groups in question. In our view, these types of proof attempts would be classified as instrumental, or purely deductive. Given the definition of isomorphic groups, the approach the undergraduates took was a logically viable option, perhaps the most viable option. However, we should note that this approach is unlikely to be successful. If one constructs an arbitrary bijection between two isomorphic groups, rarely will this mapping happen to be one of the few bijections that preserves the groups' operations. For this to occur, the bijection that one constructs must be based upon one's knowledge of the two groups. Likewise, it is a nearly impossible task to demonstrate that every bijective mapping between two groups is not a homomorphism without using structural information about the groups.

On the other hand, we would classify many of the doctoral students' proofs as relational proofs. The doctoral students seldom employed the definition of isomorphic groups; in fact there was only one instance where a doctoral student made any mention of an explicit mapping between the groups with which he was working. The doctoral students seemed quite consistent with their proof attempts: Prove that the two groups were the same or find a way that they were different. To show the groups in proposition three were isomorphic, the doctoral students did not attempt to construct an isomorphism between the groups, rather they tried to show the groups had the same essence- that they were both equinumerous cyclic groups. When the groups were not isomorphic, the doctoral students almost always attempted to find a property that one group possessed and the other did not. This was illustrated most sharply in their proofs of the last proposition. In the second proposition, one doctoral student recalled that **Q** was dense and another recalled **Q** formed a field. These observations were irrelevant from a group theoretic point of view, but they were indicative of the doctoral students' strategy.

5. Conclusions

There are three limitations of this study preventing broad conclusions. First, this study employed a small number of participants proving theorems within a narrow mathematical domain. More research is necessary to determine how general the effects observed in this study are. Second, it is unclear whether the undergraduates lacked a relational understanding of isomorphisms or simply declined to use it during their proof attempts (our paper-and-pencil tests are too crude to measure something as complex as relational understanding). Third, one reason that the doctoral students performed better than the undergraduates was that they had more mathematical experience. It seems unreasonable to hope that we can design short-term pedagogy lead undergraduates to achieve the doctoral students' level of performance, as the undergraduates will always lack the doctoral students' experience.

Leron, Hazzan, and Zazkis (1995) suggest students be taught a "naï ve" conception of isomorphisms long before learning their formal definition. Vinner (1991) offers similar advice in a more general setting; he advocates building an intuitive understanding of a mathematical concept before giving a precise definition. Skemp (1987) also endorses this view, recommending that students learn the essence of a concept through the judicious use of examples before learning the rule that defines the concept. We concur with these suggestions. We believe that students will best build a relational understanding of isomorphic groups if we present them with carefully selected examples of isomorphic and non-isomorphic groups. After students understand the essence of this concept, a formal definition can be given to them. Perhaps the students can generate this definition themselves. Whether this suggested pedagogy would improve students' ability to prove statements about isomorhpisms is a testable hypothesis and would be an interesting topic of future research.

Formal definitions play a crucial role in advanced mathematics. However, relying exclusively on definitions has severe weaknesses. Vinner (1991) notes that except for students well-versed in technical mathematics, students will use their intuitive understanding of a concept far more than the definition of the concept in their work. Therefore, a definition that is not consistent with a student's intuitive understanding of a concept will seldom be used. Our results indicate that students with a strong logical background can prove very little with definitions, facts, and theorems, if they do not also use relational understanding.

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COOPERATIVE LEARNING AS A TOOL FOR ENHANCING A WEB-BASED CALCULUS COURSE

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ABSTRACT

One aspect of concern when presenting a web-based course is the lack of personal contact. Group work or cooperative learning is a means of addressing this problem. We work with large groups of students of between one and two hundred, mostly residential students and mostly students who repeat the course. At present we present three such Calculus courses, on both first and second year level.

Our courses present a number of group-based activities such as assignments and projects. Students are divided into small groups of three or four and it is expected of them to get together to discuss the subject matter and work on assignments and projects as a group. The group is awarded a mark that contributes to their individual grading.

At the end of such a course a questionnaire was issued to establish the success of the cooperative part of the course. In this paper we discuss our findings on the successes and pitfalls of our model. We firstly discuss the process of forming groups. We then investigate how students experience the cooperation with fellow students, their work ethics and how trustworthy the cooperation between students is. We discuss our concerns and critically evaluate our model.

1. Background

Four semesters ago we started running our first web-based Calculus course at the University of Pretoria. The target market is the so-called anti-semester students, students who have failed first time round and need to repeat the course, although first-timers are also welcome. Our experiences and findings are reported in (Engelbrecht & Harding 2001(1) and (2)). Due to the success of the project we have expanded to presenting three successive web-based courses. We thus have students now that have completed three semesters of Calculus on the web. We work with large groups of students, between one and two hundred students per course.

One aspect of concern when presenting a web-based course is the lack of personal contact. It is often difficult for a student to stay committed and motivated when completely on his or her own, especially in a subject such as Calculus where discussion of the subject enhances understanding considerably. Group work or cooperative learning can be applied as a means of addressing this problem.

An important factor, that simplifies matters slightly, is that all our students are residential. Some of them share accommodation, others commute daily and attend some of the other courses together, but there is also group of students who have no real contact with any of their fellow students.

2. Cooperative learning

An extensive introduction on cooperative learning is presented in (Hagelgans et al 1995) and although this book was written in the pre-web era, anyone venturing this way would benefit from reading it. According to these authors, cooperative learning happens when a large group of students gets divided into small groups of say three or four students each, assigned for the duration of the course. Students then learn cooperatively as they perform activities such as homework assignments, computer assignments, etc as a group.

The value of having students learn mathematics in group regard through discussing mathematics with each other has been substantiated by many researchers (Arzt 1999), (Webb 1989) and positive teaching experiences using cooperative learning have been reported by various teachers e.g. (Qin et al 1995). In addition to this, a very important skill that we rarely include in our learning outcomes is the ability to get along with other people. Johnson & Johnson (1990) emphasize that "having a high degree of technical competence is not enough to ensure a successful career. A person also has to have a high degree of interpersonal competence."

To employ group work successfully is not an easy task. Even for students attending lectures, "cooperative learning, like most teaching techniques, is a complex strategy with no simple formulas for success" (Arzt 1999). In a distance learning situation, cooperative learning becomes even more difficult, mainly because of a lack of physical contact.

Kaufman et al (1997) identify six elements as essential to successful cooperative learning, namely

- positive interdependence
- social skills
- face-to-face verbal interaction
- individual accountability
- group processing, and
- appropriate grouping.

We have tried to accommodate as many of these elements in our web-based courses as possible, with varying success.

3. Course Description

All three our web-based Calculus courses are run along the same model. We prescribe a textbook (Stewart 1999) and guide the student through the course on a dynamical day-to-day basis. We provide for one discussion hour per week, a contact session, but this has, somewhat surprisingly, proved to be fairly poorly attended. We use WebCT as a platform, the reason being that our university subscribes to this software and they provide the necessary infrastructure and support. We break the study material down, firstly into themes and then into units, each of which provides for more or less a daily portion. For each of the units we provide study objectives, short lecture notes and problems of the day. None of these activities are monitored by us and therefore requires a fair amount of self-discipline from the student's side.

We do provide a number of activities that "assist" students in keeping up to date. One such activity is a weekly quiz, done on the web with immediate feedback. Students do these quizzes individually and we have had excellent response to this. Although there is no security check on this, we do let it contribute 10% to the semester aggregate and students soon get to use it as a fair judge of their progress. The quizzes also serve as a preparation for the two term exams and final exam, each of which consists of a written as well as a computer based section. The latter, again an individual activity, is done in a computer lab under supervision. Other activities are done cooperatively as explained subsequently.

4. Group Activities

Students are presented with two types of group activities - assignments and projects. Each assignment, at least four of which have to be handed in during the semester, comprises of problems, mainly selected from the textbook and requiring a substantial amount of work. It is expected of a group to get together to discuss the subject matter and then to work on the assignments. Each group hands in one copy and all members are awarded the same mark.

The projects, of which there is one or two per semester, normally consist of some related application that requires use of technology. The purpose of this is to familiarize students with graphical software and the use of computer algebra systems to broaden their knowledge laterally.

The non-stated implication with these cooperative activities is explained in (Hagelgans et al 1995): "The longer the students work within a cooperative group environment, the less dependent on the instructor they become. They become more willing to explore problems on their own - particularly to explore new, non-standard problems. And they become more willing to try to explain their ideas to others." For a course taught via the internet this is a crucial factor. Students necessarily need to be less dependent on the instructor and we feel that group work can be applied as a successful means to achieve this.

5. Group Selection

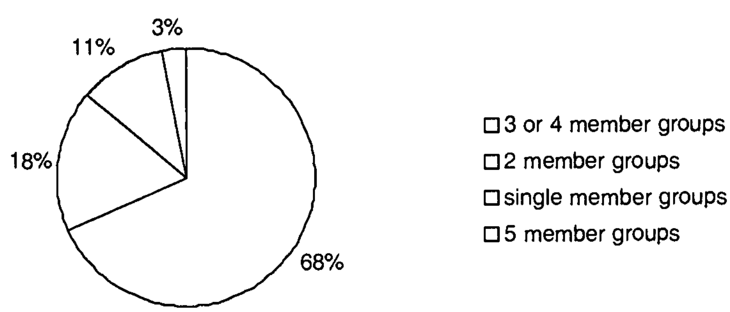
Leikin (1999) says that heterogeneity is one of the most important issues when planning a cooperative learning setting. It is recommended in (Hagelgans et al 1995) that instructors distribute the talent, expertise and various social characteristics represented in the class to form heterogeneous groups, mostly done with the aid of a questionnaire. It is also recommended to have permanent groups established by the end of the second week in class. In a distance learning model this is, unfortunately, not feasible. Just to have everyone in a group of close to two hundred accessing the web and familiarizing him or herself with the web environment in the first week or two is no mean achievement in itself, let alone have them fill in a questionnaire on the web.

Another option, that of involving student self-selection into groups, seems preferable in our case, especially because most students have no real contact. Students notify us, via the web, of their group members and they are then assigned a group number. For the students who do not know anyone at the onset there is the option of "advertising" on the website (in the Discussion Forum). Apart from a few "stragglers", the formation of groups takes about a week to ten days to be completed. It has to be added that because of experience in the early days we make a point of pressing the urgency of the matter upon them and we remind them daily. The group formation at the onset of the course is not cast in stone and, obviously, a few changes result during the semester.

In the recent semester we ran two web-based Calculus courses simultaneously, a first year single variable Calculus course and a second year course on multivariable Calculus, and issued a questionnaire towards the end. From the results of this questionnaire issued to students, a number of conclusions can be drawn on the selection process.

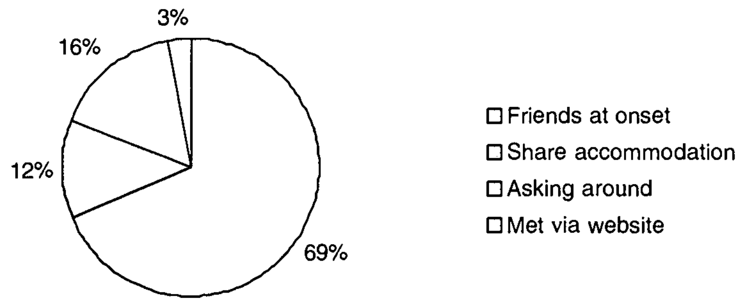
We were firstly curious as to how well this "natural" selection process works. The results are shown in Figure 1. It appears that just over two thirds of the students belonged to 3 or 4 member groups with the rest more or less evenly split between 2 member groups and single member groups and a very small percentage belonged to 5 member groups, mainly because of "orphans" (other members quitting) joining other groups.

Figure 1: Group sizes



As for how the groups got together, it seemed that convenience was the main factor here. The majority of students joined forces because they were friends at the onset, a much smaller percentage came together by simply asking around on campus for someone to work with and an even smaller percentage because they shared accommodation or lived close by. A surprisingly small group of students met via the website. See Figure 2.

Figure 2: Group selection



Although "single member groups" were discouraged at the onset and we required a motivation for every case, a bigger percentage than we had hoped for still worked on their own. Of the single member groups, 34% were left behind as "orphans" because of the higher than normal attrition rate (which was discussed in Engelbrecht & Harding (2001(2))) and 28% preferred to work alone. The latter group consists mainly of students who have no contact with fellow students because of their geographical location, a definite factor in a web-based model.

6. Students' Attitude to Group Work

In agreement with the findings of the survey done in (Hagelgans et al 1995), we generally experienced a positive feeling amongst students concerning their group activities; in fact, two thirds of all students expressed a positive feeling towards group work and the same percentage felt that their groups worked well together. Furthermore, 27% of the students were of the opinion that their positive feeling towards group work improved through the semester whereas a smaller percentage (17%) responded that this feeling deteriorated through the semester. We found this response encouraging in total, aided by the fact that only 4% reported that their groups did not work together "well at all".

Arzt (1999) reports that, "Although students are members of the same group, they may have different perceptions of how well they worked together and the solutions at which they arrived". We had the same experience and ascribe it to one of two reasons: on the one hand weaker students may have experienced the group collaboration more positively because they may feel that they have learned more, where this may not be the case with the better students; on the other hand lazy students that did not bring their side may feel that they scored in the sense that other students took over some of their responsibilities.

Listed amongst other activities they had done as a group during the semester were "revising past exam papers" and "consulting with senior students".

7. Work method

As to the procedure followed by the different groups when meeting, clearly one of three methods was followed, each with a more or less equal following.

The first method was to divide the assignment between the group members. Each had to take responsibility for his/her section of the assignment. In these cases they would have a group discussion on the inputs of the individual members before submitting the assignment.

The second work method was to split the work between the members and in these cases they simply "trusted the group member to be spot-on with his/her inputs". This, of course, was not exactly what we had in mind and these groups misused our intention, only to reduce the amount of work required by each individual.

The third procedure was that everybody tried everything before the meeting. At the meeting they would "compare notes". Normally one individual, sometimes called the "group leader" would then put everything together and finalise the assignment for submission. This was closer to our intention.

In a few cases there were complaints that "one person had to do all the work and the others had a free ride", but this was the exception rather than the rule. This is an important aspect, which leads us to the issue of assessment.

8. Assessment

The authors in (Hagelgans et al (1995)) are quite clear that group work should contribute to the evaluation of the student and recommend a contribution of 20-50% of the total grade. In our model we come in at the lower end of the scale with a contribution of 20% (assignments contributed 15% and projects 5%).

An issue not addressed in (Hagelgans et al (1995)) is whether some students do not benefit unfairly by being assigned the same mark as the rest of the group (or the reverse). Perhaps this is even more of an issue in a web-based environment where there is less control over the activities. To address this issue, we had all students attach a signed declaration with each group assignment, verifying equal (more or less) input by all group members. Afterwards we calculated the correlation between the marks that were allocated to students for group work activities and the combined mark for the two semester exams - done individually. There is strong positive correlation (Pearson correlation with 1% significance level) for both our first and second year groups of students. Not surprisingly the correlation was stronger for the second year group than for the first year students. As was feared, there were instances where students performed remarkably well in the group activities and poorly in the individual part of the assessment, in particular in the first year group. The second year students are probably a little more mature. Some consolation can be drawn from the fact that these cases were isolated and negligible percentage wise.

Linked to this is the common objection to group work that the workload is not shared equally between the group members. We asked the students whether each member of their group did his/her share in the group activities and by far the majority of the responses (73%) indicated that this happened "always" or "most of the time". This was encouraging and it confirms our belief that groups of students will spontaneously sort out issues like these themselves. Our experience is that in most cases a group tolerates a passenger perhaps once, but if on a second occasion a group member does not make his/her fair contribution, this student is either kicked out of the group or "disciplined" in some or other way.

9. Areas of Concern

The role of the instructor in a web-based course is distinctly different to that in a classroom situation, especially where group work is concerned. Quoting Hagelgans et al (1995) once again: "The instructor must play an active role in becoming aware of how the groups are operating. It can be expected that, left to their own devices, students may let their groups fall into non-productive modes of operation." In a web-based teaching it is not possible for lecturer to play such an active role. Luxuries such as " ... the instructor may move from one group to another to observe their progress and to provide assistance by giving hints, ..." are simply not possible. Yet, students do seem to find a framework for themselves within which they function fairly successfully. We also maintain that group work is probably more of a necessity than a nice-to-have when teaching web-based students compared to teaching classroom-based students. This is, in most cases, their only opportunity to verbalise mathematics and their only real contact with students doing the same course.

Other problems such as difficulties connected with modes of operation or with group dynamics, difficulties arising from organisational issues, and difficulties to do with individuals are even more distinct in a web-based than in a classroom-based course. In a web-based course it is also more difficult to deal with such problems since there is less contact between the instructor and the students. Students need to mature quicker and deal with these issues themselves. Our experience is that this happens indeed.

One disappointment was the fact that most groups met only before an assignment was due. We were hoping that the groups would progress to working together in other aspects of the course such as studying together, having more unforced discussions on the work, but unfortunately this was not the case. In a sense this was to be expected. Left entirely to his or her own initiative, the average student will follow the path of necessity, the path that leads to survival. The majority of groups (70%) would only meet when required - when an assignment was due.

10. Student Comments

In spite of a few reservations about group work expressed by students, it is significant to note that the majority (69%) testified to the importance of the group activities for the success of doing a web-based course in mathematics.

A few representative comments from students are:

"Group work is wonderful if every member does his share, otherwise it sucks."

"I was the driver and the rest were passengers."

"Those who understand the work help us a lot."

"It was the best way in which I fully tested my understanding of the material when explaining my method of solving a problem to my fellow students."

11. Conclusion

We have been experimenting with web-based courses for two years now and are convinced that this mode of teaching is here to stay and offers a fine alternative to the conventional model. However, in the absence of lectures there is a need amongst students to communicate in some way. The group work model that we introduced offers (part of) a solution to this need. Although it is clear from our research that students rate the importance of these collaborative activities as very high, not everything is working perfectly yet and we will have to build on the positive aspects of our experience in order to further develop this teaching model.

There is a need for monitoring the group activities from the instructor's side and better work ethic from the students' side. In a classroom situation the instructor has an ongoing opportunity to develop the dynamics of the collaborative learning process. In a web-based situation the instructor does not have the hands on opportunity to monitor the cooperative activities and therefore we need to provide better guidance on what the purpose of group work is and on how a group should function, perhaps in some handout in the beginning of the course.

We are convinced that collaborative learning is very important in a web-based mathematics course but more difficult to implement. Quoting Arzt (1999): "For teachers to use cooperative learning strategies effectively, they must become sensitised to the many complexities of the technique." In a web-based mathematics course it seems as if these "complexities" are even more complex.

In retrospect, it is clear that there are some important benefits that the student gains form this collaborative learning model:

- Learning a variety of approaches for solving a problem
- Opportunity to discuss and clarify ideas
- Improve communication and social skills
- More enjoyable teaching environment than conventional lectures
- Increasing confidence
- An opportunity to communicate, especially in this teaching model

These benefits correspond closely to those experienced by Kaufman et al (1997).

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DISTANCE LEARNING COURSES ON NUMERICAL METHODS WITH ACCESS TO SOFTWARE LIBRARIES

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ABSTRACT

An approach to development and technology of realization of distance learning courses for mathematics subjects is suggested. Courses "Methods of optimization" and "Elements of calculus" are described. These courses are used in the Sumy State University (SSU) for teaching the full-time and correspondent students and require usage of software. For computer practical lessons in the distance courses we have provided description of libraries of numerical methods programs, organization of access to them, adaptation of the programs to the user.

Let's have a look at the opportunities of the course "Methods of optimization". The course includes 4 stages of learning: studying of theoretical material; self-control of received knowledge; mastering practical skills; evaluation of received knowledge and skills.

Lectures are designed as a tree structure of text documents. Testing is organized both on each topic and on the course as whole. Evaluation procedure is possible through an e-mail or on-line answer.

For fulfillment of laboratory tasks we have provided an access to programs of numerical methods. Software was written in C++, Pascal and Object Pascal by students of SSU. Moreover, original texts of programs are well documented and directions on self-programming of numerical methods of optimization are given. System of access to library files is offered. The library of programs is divided into blocks and can be accessed in the distance courses as archive files. Students can copy these files to his computer, unpack them and start working with programs immediately.

The course has a chapter with examples of applied engineering, economical tasks and their solutions. Features of applications such as Maple, Excel and scientific Fortran-library are described.

The distance course is open for free usage on the following address:
http://dl.sumdu.edu.ua/mo/index.html and designed for Russian-speaking audience.

Keywords: Distance learning courses, numerical method, optimization, mathematics, the structure of the lecture, self-control, test, hyperlinks, program complex, software library.

1. Introduction

An approach to development and technology of realization of distance learning courses for mathematics subjects supposing use of application software libraries is offered. The features of construction and presentation of the material in mathematics courses are taken into account. The mechanisms of access to the software on studied themes are thought over, and methods of programming for solving applied tasks are described.

The realization of courses "Methods of optimization" and "Elements of calculus mathematics" is described. These subjects are included into the bachelor's educational plan on teaching the students of Sumy State University of "Applied Mathematics" and "Computer Science" specialties, both full-time and correspondent. The educational programs stipulate lectures, laboratory work and monitoring of knowledge. The laboratory work supposes computer realization of appropriate numerical methods by use of application packages, and also writing the programs with the help of algorithmic languages Pascal, C, Fortran. As the final task students are offered to solve one of the applied engineering or economic tasks. In distance courses the appropriate sections - lecture, laboratory work are foreseen, and most important - the problems of description of the numerical methods software libraries , organization of access to them and adaptation of the programs to users are solved, the methods of programming of applied tasks are explained.

The innovation consists in the following: in our course we have tried to offer complex solution to the question of electronic course filling. This means that theoretical description of the methods of optimization are made and access to computer programs is made possible (most of them are being developed by us). We had put a lot of examples of usage of this methods and programs to solving of application tasks.

There exist similar electronic textbooks, but they only contain description of software (http://ahp.tstu.ru/ido/Matiss/matiss.htm, http://www.srcc.msu.su/num_anal/lib_na/libnal.htm), or set of lectures and tasks without software maintenance (http://www.mpri.lsu.edu/bookindex.html, http://www.path.berkeley.edu/~varaiya/).

Our solution is innovative among Russian-language distance courses.

2. Structure of the distance course

Let's look more close on possibilities of a course "Methods of optimization".

The course is composed as a tree structure of text documents. It is divided on topics, each of them in it's turn contains lecture material, test for self-control, exercises and questions. All subsections are connected with each other. The connection is carried out with the help of hyperlinks due to use of hypertext technologies. I.e. each subsection is a text file with appropriate mark-up, so-called html-file. The course consists of hundreds of such files. The division of a lecture material was made mainly to reduce the load time of the file into the computer (as the speed of data exchange in Ukraine keeps to look for the best). The material is divided into the logically completed parts.

The material is represented in 2 languages - Russian and Ukrainian. Division into fragments is the same in both versions. Language can be selected on the title page of a course (http://dl.sumdu.edu.ua/mo/).

Work with the course begins with acquaintance with the structure and description of possibilities. I.e. student opens "About a course" section first. During work with this section student gets acquainted with the purposes and tasks of a course, its contents, peculiarities of work with a course (section "Work with a course"). Also the contact information about the developers, teacher and tutor of a course and access to auxiliary material is represented to his attention. As an auxiliary material the list of the references is offered which can be useful in work with a course. (For example, reference to the table of derivatives, to the used archiver.)

After acquaintance with this section student immediately goes to work with a thematic material. This work assumes 4 grade levels:

- Study of a theoretical material of a course (Lecture);
- Self-control of obtained knowledge (Test);
- Mastering of practical skills (Laboratory work);
- Monitoring of obtained knowledge and skills (Monitoring).

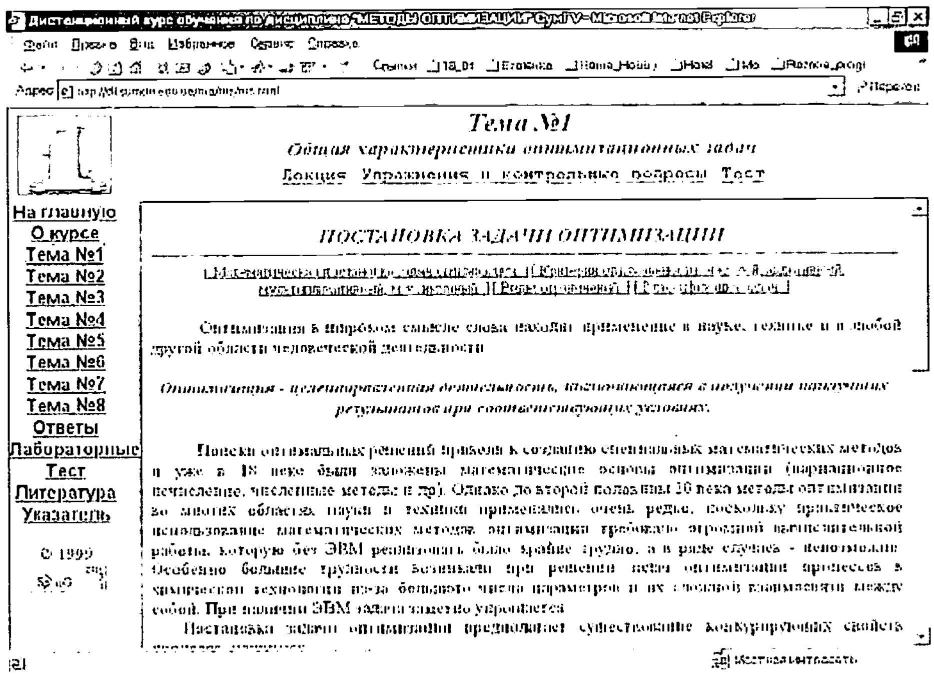

Figure 1: ***View of one of the themes of a course***

Lecture material is presented as sequence of loaded files. Student selects the order of files by himself. He can use buttons "Next", "Back" and others, located in the bottom of each page, for navigation in the course. In this case he will look through material in a strict sequence determined by the teacher. When Mouse is moved on a button "Next" there appears prompting about the following section, which will be loaded after pressing this reference. Student can use submenu of the lecture located above on the first page of a theme and below - on all remaining pages.

The structure of the lecture is presented as a scheme (Figure 2).

The course contains the following themes: "Formulation of optimization problems", "Single variable search techniques", "Nonlinear programming", "Linear programming", " Application of optimization methods", "Elements of calculus of variations multivariable optimization procedures", "Optimization of the graphs". Let's stress, that the authors did not have a purpose to develop a full course "Methods of optimization". Some topics, such as, for example, "Linear programming" or "Optimization of the graphs" are presented in abstracts. Some, for example, "Dynamic Programming", - are absent. In a distance course it is recommended for a student to address to more detailed electronic issues (such as http://www.dvo.ru/studio/linpro/ – Å.Nurminsky. Course of the lectures on linear programming), http://www.srcc.msu.su/num_anal/index.htm - server of library on numerical analysis SUCC MSU, http://www.mpri.lsu.edu/bookindex.html - electronic text-book "Optimization for Engineering Systems" Louisiana State University). Moreover, the course contains a subsection: the information review of Internet-resources on problems of a course. In particular, it has a link to such materials, as existing kinds of software for problem solving of nonlinear programming and conditions of its purchase and use (for example, paid software for problems of quadratic programming can be found on the server of Optimization Technology Center of Northwestern University and Argonne National Laboratory - http://www-unix.mcs.anl.gov/otc/Guide/faq/linear-programming-faq.html or free distributed listings of Fortran - programs and procedures for mathematical calculations on the server of Network library NetLib - http://ww.netlib.org/). The authors have paid attention to problems of numerical realization of optimization methods, description of the software, technology of programming.

Any section of a course can be accessed also from "Pointer". It contains references to all materials of a course.

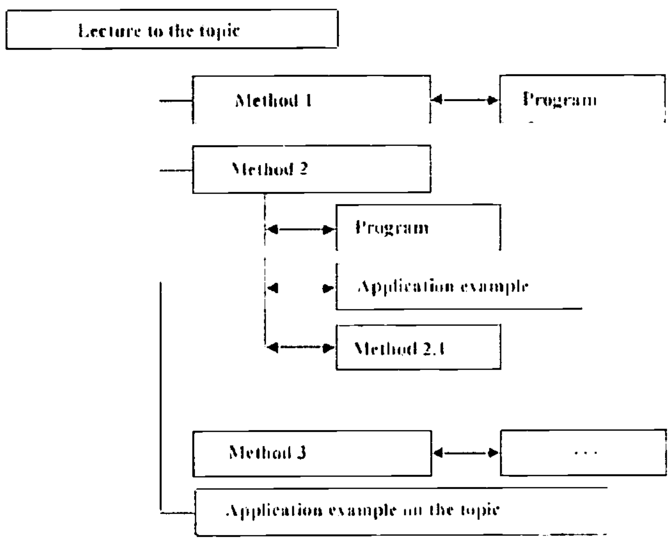

Figure 2: *Scheme of the lecture*

3. Program Complex

Software library which are presented in a course, we shall name a program complex, to prevent confusion. The program complex consists of several parts (libraries), each of them has subsections. These subsections also have ramifying. Such structure of a complex is connected with features of it's development. Firstly, the parts were written by various people and in a different time. Secondly, each part was planned as a separate software product. Therefore a distance course is presented as a whole complex, and as separate parts. The complex contains:

- Software Library, which is realized in Pascal;
- Software Library, which is realized in Ñ;
- Optimization methods software library, which is realized in Fortran;

- Software Library, which is created with the help of Delphi environment;
- Software package LP-88 for problem solving of linear programming.

A lot of the methods are programmed in different languages and are compiled with the help of various compilers, what allows to use the same method by the people who have skills of work with only one programming language, for example, Pascal. In a course student is offered the programs for methods of one-dimensional optimization, Hooke-Jeeves method, Nelder-Mead method, gradient method, Davidon-Fletcher-Powell method, Fletcher-Reeves method, complex method, penalty function method.

These software libraries in C ++, Pascal, Object Pascal are composed by the students and post-graduate students of SSU. Moreover, the original texts of the programs are described, and explanations on self-programming of the numerical optimization methods are made. In a course it is possible to work with the programs from libraries on several schemes. First: to use already compiled exe-files (for example, for realization of the comparative analysis of application of different optimization methods for solving standard set of functions). Second: to take advantage of the original texts from the site and set of procedures (modules) - i.e. to make the program using compiled modules, which contains a set of certain procedures. In this case student writes the main program by himself (probably by analogy with those offered). He specifies the function, optimal solution to which is necessary for get, and modules, which he will use in his solution (for example, module of multivariable optimization methods). After writing the program student compiles it with the help of any accessible compiler, which understands language of the program (for example, Borland C ++ 3.1 or Watcom C/C ++ 11.0, if the program is written in C or C++). Third: to write the program, which realizes this method, possibly more improved, on the basis of suggested computer realization of method. I.e. the distance course can be considered as a simulator on programming for improvement skills of writing programs realizing the numerical methods.

The system of access to software library files is offered. The software is divided into blocks and is accessible in a distance course as archive files. Student can write this file on his computer by several standard methods. For example, open the context menu of the link to such file (such menu occurs on right button click), select an item ("Save As ...") and select a path, where the file will be saved. The archive file with zip extension will be saved on the computer. Student can unzip it with the help of any archiver, which supports unpacking of the zip-format (libraries are added to archive by WinRar of version 2.5). After unpacking it is possible to begin to work with the program on the spot.

The programs do not require additional installation of any software. The software library is designed for a COMPUTER with simple performances (i.e. even for low-power computers working only with DOS operation system).

4. Applied character of a course

The possibilities of software packages Maple, Excel and scientific Fortran - library on problem solving of optimization are described in a course. Algorithm of work with an Excel procedure "Search of a solution" is described in details. There is also a section with examples of applied engineering, economic problems and their solution. In particular, the examples of problems from [3] are realized. Except verbal statement of examples of applied problems, the mathematical statement of a problem, algorithm for solution and recommendations to choose a method of a solution, outcomes received in solution by different methods are presented. One of such examples - problem of optimal designing of the disk of a turbine.

Statement of a problem is the following: it is necessary to design the disk of the turbine with other equal conditions, that the mass of the disk should be minimum and the request on durability should be satisfied, and the accounts should be made with specific restrictions on limits of modification of the disk thickness.

The minimum is achieved by selecting a function of modification of the disk thickness from a radius - h(r) so that the requests to durability of the disk under forces induced during its rotation, and also restriction on limits of modification of the disk thickness were executed. The appropriate mathematical model is composed. To present a task of optimization as a task of variation of vector of construction parameters, the approximation of dependence h(r) by some piecewise linear function is made. With this purpose a partition of a radius of the disk on m of parts is made. On each section the linear function, approximating appropriate section of a modification h(r) is created.

The criteria of optimality is represented as nonlinear function of finite number of variables, which defines geometry of the disk. A task of optimum designing of the disk of the steam turbine consist in search of a minimum of a optimality criteria with restrictions and by variation of the vector of parameters. For its solution the program is developed, in which a method of a random search, method of penalty functions and combine method are programmed. The program is written in Borland Builder C ++ Professional.

5. Testing

Testing is organized in a course. The test is an independent (autonomous) program developed by Khazhanez V.A. (programmer, graduate of Dep-t of Ap.M. SSU) for testing of various educational disciplines in SSU. The program is written in Perl. The test envelope includes program for testing, base of test questions and outcomes of testing. The envelope is placed on one of the university servers. For example, for our course the test problems allow to pass the test on each topic, and also so-called general test, in which questions concerning all course are gathered. When addressed to the reference "Test" the form for input of a surname and initials is loaded to the screen of the computer. Registration in base of tested students happens first. After verification of the entered data the test begins. The questions for a students answer are loaded on a screen sequentially one after one by pressing a button "Next". Under each variant of the answer there is a small window, in which student should mark correct to his judgment variant. There can be several variants of the correct answer. Student can interrupt testing, by pressing a button "stop the test". By pressing a button "complete" student confirms, that he has answered all questions and is ready to view his results. At end of testing the student is offered to look through the results of his answers, where the incorrect answers to the question are marked in red, and correct in green, and also the answer, selected by the student, is indicated. We offer the student to use this test for self-control of obtained knowledge after study of the lecture.

For final monitoring the scheme of dialogue with tutor is foreseen. The control tasks are composed with account of answering them by e-mail or on-line. Forming of control session is possible. Time and mode of its realization are negotiated in the beginning of tutoring. The session can be conducted in on-line mode.

6. Conclusion

The distance course is constantly improved. It has been used in educational process of the students for 3 years. During this time about 80 persons of "applied mathematics" speciality have passed tutoring. The students worked with a course on computers, connected to a local network in Intranet mode and having access to the server of university. The course is also actively used through Internet. It is confirmed by recalls, obtained by us. The users of the course mainly were the citizens of CIS (Russia (Moscow, Byjsk), Byelorussia (Bryansk), Armenia, Kazakhstan (Karaganda)).

The Distance course is open for free use at the address http://dl.sumdu.edu.ua/mo/index.html and is designed for a Russian speaking audience.

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DEMONSTRATIONS, EXPERIMENTS, AND SUPPLEMENTARY NOTES TO MOTIVATE STUDENTS IN DIFFERENTIAL EQUATIONS COURSES.

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ABSTRACT

We describe a series of demonstrations and experiments that are used in our differential equations course. These experiments are designed to be low-tech, low-cost alternatives to illustrate and motivate the modeling/prediction aspect of the course. A series of lecture notes are also being prepared to relieve students from the task of taking notes, and allowing them instead to concentrate on understanding what is being discussed. We have found that there is great enthusiasm for the experiments and demonstrations, even though most of them have been performed before in the Physics Lab. Apparently, many aspects of the experiments were not understood during that first encounter in the laboratory. Also, an effort is made to present the mathematics and the experimental confirmation simultaneously to enhance the effect. On the other hand, we have not noticed any increased performance from the students due to the introduction of these demonstrations. In any case, the instructor certainly has fun with the demonstrations.

1. Introduction

A series of experiments and demonstrations used by the author in the course *Introductory Differential Equations* offered at the University of Puerto Rico at Mayagüez will be described. These experiments do not require a sophisticated setup, computers, or measurement devices. They are designed to be low-tech, low cost alternatives for instructors who wish to illustrate the predictive and modeling aspects of differential equations.

Our institution is the Engineering campus of the University of Puerto Rico. A consequence of this is that most of our students in the course are Engineers. We also service Physics, Chemistry, and of course, Mathematics students. Because of the demographics it is particularly important for us to address the aspect of applications in the course. The demonstrations serve as a driving force, taking the course in this direction. We try to make the course very relevant to the students.

2. The experiments

The demonstrations/experiments are listed chronologically, in the order in which they are presented in the course.

First order differential equations.

For the topic of first order differential equations, no demonstrations are currently presented. However, possible demonstrations that could be introduced are: the leaking tank to illustrate Torricelli's law, or a cooling object to illustrate Newton's law of cooling. The reason we have not implemented these demos is that we have not found convenient, no- mess, low-cost ways of setting them up in the classroom, on the fly.

Second order, linear, constant-coefficient equations I: the harmonic oscillator.

For this topic we have two demonstrations: the mass/spring, and the pendulum.

Mass/spring

For the mass/spring the governing equation is derived assuming no friction, Hooke's law and no external forcing. The general solution is obtained and it is shown that regardless of the initial conditions, the solution is periodic with constant circular frequency. Immediately several experiments are performed: a mass/spring system is put in motion under different initial conditions and the frequency of oscillation is experimentally determined each time: it is found to be constant. The term natural frequency is then introduced and motivated. The differences between the assumptions made and the real setup are noted: no friction, Hooke's law. The surprise that even after making those "unrealistic assumptions" the model still yielded important qualitative information, verified by the experiment, is emphasized. Also, static and dynamic (using the natural frequency) methods for the determination of the spring constant are introduced and implemented.

<u>Experimental setup</u>: We bought a spring from our local Pep-Boys (auto shop) which cost \$5.00, and a large bolt from our local Home-Depot which cost \$2.00. A wooden classroom ruler is also convenient for measuring lengths and also serves as an anchor for the mass/spring (\$2.50) .Total cost: \$9.50.

Pendulum

Again, the governing equation is derived assuming no friction. A non-linear equation is obtained, and the idea of linearization is introduced. Assuming small oscillations, the equation is replaced by a linear, constant coefficient equation. It is noted that the equation is identical to the one obtained with the mass/spring, except for the interpretation of the coefficients. Several experiments are performed confirming the harmonicity of the oscillations, and the relationship between length of the pendulum and resulting natural frequency is confirmed by doubling or halving the length of the pendulum. Again, the assumptions are analyzed and the robustness of the analysis in emphasized by stressing that even under such assumptions, the model and analysis yielded conclusions confirmed by the experiments. Some commentary can also be made on dimensional analysis: from the parameters of the setup (length of pendulum, mass, gravity) one could conclude the dependency of the natural frequency on length and gravity.

Experimental setup: We bought a large nut from our local Home-Depot which cost $0.50, and nylon fishing line for the pendulum length in Walmart ($4.00). A wooden classroom ruler is also convenient for measuring lengths and also serves as an anchor for the pendulum ($2.50) .Total cost: $7.00.

Second order, linear, constant-coefficient equations II: forced oscillations and resonance.

Here, the emphasis will be on sinusoidal forcing of varying frequency, and the phenomenon of resonance. The mathematical analysis for undamped and damped mass/spring systems is performed predicting that a single frequency has the largest effect on the response. This resonant frequency is interpreted as dangerous for structures, in the sense that a large response can destroy the system. In the case of the RLC circuit, resonance is interpreted in terms of amplification and filtering.

Mass/spring

The same setup as before, except that we begin with trivial initial conditions and gently tap the ruler (from which hangs the mass/spring) from above and below in a periodic fashion. This experiment is imperfect in the sense that the force is not actually applied to the mass directly, but it is the best we could come up with in keeping with the low-frills philosophy! However, the results are quite spectacular. If the frequency of the stimulus is too slow or too fast, the response is negligible, but if it approximates the natural frequency (which can be observed by free oscillations), the response is considerable. One can remark that if stimulus and response are out of resonant phase, cancellations result in small response. This is visible in the experiment.

RLC circuit

We bought two kits from Marlin P. Jones & Assoc. Inc. (there are many distributors of electronic supplies that would do): a function generator and a small amplifier. Both kits and a small speaker cost less that $12.00. One has to put them together, which involves a bit of tinkering, but in less than two hours it is done. The idea is to use these kits to force an RLC circuit and to "hear" resonance as amplification and lack of it as filtering. Unfortunately, we have yet to construct the inductor of the RLC circuit to be stimulated, so this experiment has not been implemented yet.

Tuning forks, whistling tube, singing bars

These three experiments have been chosen to illustrate the fact that resonance is a fairly universal phenomenon: all objects are subject to vibrations, and all prefer to vibrate at certain frequencies. In fact, the natural frequencies give us information about the vibrating object. The demonstration kits were bought from Pasco (distributor of educational materials), and are the most expensive items (more than $300.00). However, the whistling tube is sold at toy stores for $3.00 and one can replace the catalog singing bars with an aluminum bar used as support in labs or with a crystal wine glass. The crystal wine glass demonstration is very spectacular and one does not even need resin to perform it: simply wet you finger and rub your finger slowly around the rim of the glass will holding with the other hand at the base. The glass will respond to the stimulus, and the response is usually quite audible. One can even tune the glass by adding water: More water means lower response frequency. In fact, with the frequency generator and an amplifier one could even tune the frequency generator to the natural frequency of the wine glass and perform the opera singer breaking of the glass!

The tuning forks are used to illustrate resonance in the following way: they are tuned with the adjustable mass until they resonate at the same frequency. They are placed close together, and one is struck with the mallet. After a few seconds it is muted. The second fork will still sound: it has been stimulated by the first fork at a resonant frequency and thus responds. Also, the phenomenon of beats can be illustrated by moving the mass slightly so the forks are no longer in tune, and striking both with the mallet. The periodically varying amplitude can be heard clearly. It is important to derive both phenomena from the governing equations of the mass/spring before or immediately after performing the experiments to achieve the desired effect.

The whistling tube is a corrugated hose. When spun, a pressure gradient is produced and a flow is induced. When spun at particular speeds this produces a tone. The faster the particular spinning speed, the higher the tone. The frequencies of these tones are in rational proportion. In fact, they happen to be related to the Pithagorean theory of harmony. One can hear an octave, a perfect fifth, etc. These tones are the result of resonant vibration patterns of pressure waves inside the tube (standing pressure profiles). This principle is what makes all winded musical instruments work. The moral is that resonance makes music and musical instruments possible. One can also illustrate with a rope or string to simulate string instruments, and show the standing waves or modes and corresponding natural frequencies, and their relationship with tension and mass in the rope.

Finally, the singing aluminum bars are held at some node, and stroked with fingers full of resin. The result is a very loud and powerful response at the corresponding natural frequencies (depending where one holds the bar). If one does not hold the bar close to a node of the first natural modes, then one does not hear a response. One can draw a parallel between the bar and any rigid structure (like a bridge or building) and talk about the dangers of resonance and the importance of avoiding it in certain situations.

Whirling rope

We no longer talk about boundary value problems in our introductory course, but the whirling rope is a perfect application/experiment for this topic. One can demonstrate what the mathematics predicts: the rope has a preference for whirling at certain natural frequencies. These frequencies depend on tension and mass (density). The whirling modes are also predicted by the theory. The experimental setup is as simple as a long piece of rope and the help of a couple of volunteers to spin the rope.

3. Lecture notes

We are in the process of completing a series of lecture notes. The purpose of these notes is not to replace the textbook, but to relieve the student from having to copy every little thing that is written on the blackboard. This will allow the student to concentrate on understanding what is being discussed and once in a while add a marginal note to the printed notes. Also, in our specific case, most of our students' mother tongue is Spanish, and we have found that for some of them it is difficult to understand the English textbook. The notes also give the student support in this respect, since they are written in Spanish. Finally, the notes conform to the specific blend of topics covered and limit themselves to them, presenting a concise, concrete synthesis of the course material.

4. Conclusions

There are many low-cost, practical alternatives to help bring the differential equations course to life. For some students this makes the difference between another typical mathematics course and a very relevant and interesting learning experience. Even for instructors, performing these quick experiments breaks the routine and makes lectures very pleasant. We must admit though, that we have not seen any noticeable improvement in students' performance due to use of the demonstrations. However, we hope that the impact on students' perceptions of mathematics and science has been and will continue to be positive.

One should note that some of the experiments (mass/spring, pendulum, RLC circuit) should be preceded and followed by complete mathematical derivations and analysis. However, some very spectacular demonstrations (singing bars, whistling tube, wine glass, etc...) cannot be fully justified in an introductory course in ordinary differential equations.

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MAKING THE CONNECTION: UTILISING MULTIPLE INTELLIGENCES TO MEASURE TEACHING AND LEARNING SUCCESS IN MATHEMATICS

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ABSTRACT

Why do educators connect with some students and not others? The answer lies in the fact that each student is a unique individual with his or her own learning style who will learn best from a teacher who utilises a suitable teaching style. Is it the role of students to adapt their modes of learning to capitalise on the teacher's offerings or should the teacher be trying to connect with all students by employing a variety of teaching styles?

Howard Gardner, in his theory of multiple intelligences, asserts that everyone is intelligent and capable of learning, but that an individual will favour some modes of learning over others.

The factors influencing these modes may be genetic, environmental, or experiential, but they are beyond the teacher's control as the favoured learning style is already formed by the time the student walks into the classroom. It is therefore the responsibility of the educator to adapt his or her own preferred teaching style and use a variety of modes in order to make connections with each student.

This paper analyses various strategies utilised in the teaching, assessment and examination of the Preparatory Mathematics Course at the University of Sydney, Australia, measured with reference to the multiple intelligences.

KEYWORDS: multiple intelligences, visual / spatial, verbal / linguistic, mathematical / logical, bodily/kinesthetic, musical, intrapersonal, interpersonal, naturalist, existentialist

1. Introduction

Recent research and anecdotal evidence from personal experience suggest that Mathematics and the learning of it is one of the most challenging of all subjects and often appears most inaccessible to a large proportion of students. In the plenary lecture at ICME 9 in 2000, Mogens Niss of Denmark stated "As mathematical education was provided to new and growing groups in society it became important to cater for categories of students that, in the past, were mostly neglected or dismissed. However, ...large groups of students seemed to experience severe problems at learning and benefitting from the mathematics taught to them". One suggested reason for this is that the language of Mathematics and its presenters is so esoteric that it leaves the ordinary student bamboozled, thus allowing Mathematics to remain a subject only for the elite. This was the sentiment expressed by Garbayo-Moreno, et al, at ICTM in 1998: "In our opinion, mathematics teaching should move from the kind of topics (we) mathematicians like to teach to the kind of topics society demands as general knowledge." Another explanation may be that the learning of Mathematics will be dependent on the student's strengths and whether or not the teacher can appeal to those strengths to achieve relevance or a connection.

The author was given the task of teaching a tertiary mathematics preparation course to a group of adult learners who historically have not achieved great success in their previous study of mathematics. Traditionally, this course has a large drop-out rate as students cannot relate to standard classroom teaching strategies. The challenge for the lecturer is to utilise a broad range of teaching methods to connect with these students. The purpose of this research was to analyse the learning strengths of the students and the various teaching strategies employed throughout this course to determine the suitability of the pedagogy to the capacities of the students. Howard Gardner's theory of Multiple Intelligences was used to measure and verify the success of the course teaching and assessment strategies in appealing to the individual learning strengths.

This paper is divided into three parts. The first is a general discussion of the multiple intelligences identified by Harvard researcher Howard Gardner in his book, *Frames of Mind: The Theory of Multiple Intelligences*. The second is an analysis of the various intelligences possessed by a group of adult learners, hoping to commence tertiary study in Mathematics, using the survey developed by Walter McKenzie. The third section is a discussion and analysis of the students' responses to a questionnaire (in the appendix) provided on completion of the tertiary mathematics preparation course. The thrust of the questionnaire was to examine the teaching methods and assessments utilised in the course attended by these adult learners with a view to finding their connection with the multiple intelligences. Connections made should result in enhanced learning.

It is to be noted that a good deal of research has been done into the relationship between the multiple intelligences and children's learning of mathematics, and the multiple intelligences with adult literacy. However, research on the connection between the multiple intelligences and adult learning of mathematics appears to have been, as yet, untouched.

2. The Multiple Intelligences

Measuring intelligence has always been a challenge to psychologists and educationists. Trying to determine exactly what is being measured, the definition of intelligence, avoiding cultural or socio-economic bias often inherent in the standard IQ test, have all been aspects which detract from the value of the result. People's intelligence quotients were placed on a scale with a number, which was then used in a variety of circumstances.

Howard Gardner uses the definition of intelligence as "the capacity to solve problems or to fashion products that are of consequence in a particular cultural setting or community" and asserts that it is not so much how intelligent a person is, as described by the IQ scale, but how a person is intelligent. A person uses a "variety of intelligences working in combination to carry out different tasks, solve diverse problems and progress in various domains." Gardner's theory assumes that everyone has some measure of the nine intelligences listed below:

- **Linguistic/verbal intelligence** – the ability to use verbal or language skills to express or communicate ideas
- **Logical/mathematical intelligence** – the ability to think logically, to analyse patterns and relationships in a scientific way
- **Spatial/visual intelligence** – the ability to represent ideas in a visual or graphical way, to think visually or have an understanding of space
- **musical intelligence** – the ability to use music as a mode of expression, to appreciate rhythm, melody and pitch
- **bodily/kinesthetic intelligence** – the ability to utilise one's body to express ideas, to manipulate or create physical objects
- **interpersonal intelligence** – the ability to understand and respond to other people's feelings in an appropriate manner
- **intrapersonal intelligence** – the ability to understand oneself and have an awareness of one's own feelings, strengths and goals
- **naturalist intelligence** – the ability to appreciate and understand the environment and its relationship and importance to humanity
- **existential intelligence** – the ability to see the "big picture", having an appreciation of culture, spirituality and historical perspectives.

An individual's level of strength in these various intelligences, together with how a concept is presented, will determine how well he or she will connect with a particular concept. This is where the role of the teacher becomes crucial. The teacher must appeal to the different intelligences when trying to explain a concept in order for each individual to reach a level of understanding. For instance, a student who possesses a high level of musical intelligence, according to Gardner's theory, will respond well to learning Mathematics when it is explained in terms of musical concepts. These could be through the use of songs, patterns, rhythms, instruments, pitch or melody.

Mark Wahl notes that in the United States, the National Council of Teachers of Mathematics still supports the approach to learning Mathematics via the logical-mathematical intelligence, even though it admits there is a problem. He writes "In some students this is not the strongest asset,.........We must tap the other intelligences of all the students in our quest to engender a 'felt sense' in mathematics."

The duty of the educator is therefore to analyse the various intelligences each student relies on. Then the challenge for the educator is to adapt his or her teaching style or use of examples to tap into the different intelligences so that each student can reach an understanding.

3. Student Intelligences

Gardner's philosophy, "that it is not how intelligent you are, but how you are intelligent", can be more useful in analysing how a student learns or what is relevant to him or her than the standard IQ test. Once a student's major intelligences are identified, the teacher can then alter his or her teaching style, resources and assessment tasks to make a more appropriate connection with the student.

In the research for this paper, the current teaching practices were analysed in terms of their appropriateness related to the nine multiple intelligences, rather than setting about to teach a course based around the multiple intelligences.

Most of the students in this study were preparing for tertiary entrance into mathematical or science related fields. Others were hoping to gain entrance to Law or Arts. The common thread amongst them, however, was that they, adult learners, had not had a great deal of success in Mathematics study previously and in fact, some of them had extraordinary anxiety just at the thought of coming to lectures, let alone attempting the coursework! A sub-group of this cohort had the added difficulty of trying to learn Mathematics in English as their second language. While this sub-group tended to have had historically more success in Mathematics than their anglo-background counterparts, they were most concerned about achieving success in the course. The eleven students ranged in age from 20 to 47 years, male and female, and a number of them had not studied Mathematics for at least 20 years. Most of them did not possess organisational or study skills, nor, at the commencement of the course, did they have very much confidence in their ability to succeed.

The students were happy to be analysed in terms of their multiple intelligences using Walter McKenzie's survey. Of the nine intelligences listed in the previous section, not one of the students displayed the greatest strength in the mathematical/logical intelligence section. Their greatest strengths lay in the other intelligences. These are not the characteristics displayed by students who are successful in the traditional classroom, which favours students with major strengths in the verbal and mathematical/logical intelligences. When teaching a course involving sophisticated mathematical concepts such as logarithms, exponentials, trigonometry, differential and integral calculus to adult learners such as this group, whose strengths do not lie in the mathematical/logical realm, one would think some modification of the traditional teaching style must be necessary to make a connection with each student.

The survey revealed the following results:

Student	A	B	C	D	E	F	G	H	I	J	K
Visual	40	60	20	80	**100**	**80**	10	60	50	80	40
Verbal	10	30	30	80	90	20	10	50	20	**90**	50
Logical	50	60	60	***80***	70	60	60	60	50	***80***	50
Kinesth	70	60	20	80	80	50	50	70	**60**	70	**100**
Music	**80**	60	50	80	70	50	40	60	50	40	40
Intrapers	40	**100**	**90**	**90**	60	50	30	60	60	80	90
Interpers	80	40	20	90	80	10	10	50	50	80	20
Natural	40	30	30	70	60	40	**70**	**100**	20	50	20
Exist	80	30	70	70	90	50	20	60	40	50	80

One can see from these scores (each out of 100) that there was a large range of intelligence strengths in the class suggesting that each would respond differently to different scenarios. For instance, an example on male versus female salary scales might appeal to a student with greater existential intelligence than one with kinesthetic intelligence.

The **bold** figures indicate the highest scores for each student on the various intelligences. While these maximum scores range from 60 to 100, none of them are on the logical/mathematical intelligence. The ***italicised*** figures of 80 represent the highest scores on the logical/mathematical intelligence. Overall, the scores indicate that, while these students may have some mathematical ability, their strengths lie in the other intelligences.

4. Analysis of Course Strategies

The preparatory mathematics course consisted of twenty-six lectures each of two hours duration given in the evenings. The students had to complete four assignments, a test and a group project chosen from a range of topics, such as "Managing your Mortgage", "Fractals", "The Greenhouse Effect". Finally the students sat a three-hour examination which, together with the other assessments, determined which tertiary course they were eligible to enrol in. Being adult learners, the students had to juggle their studies with family life, career and other commitments, so it was important for the teacher to provide a stimulating environment and enjoyable experience, otherwise they would fall asleep! The course moves at a fast pace, beginning with simple fractions and algebra, and finishing with integral and differential calculus of trigonometric, logarithmic and exponential functions.

At the end of the course the students were provided with the questionnaire in the appendix in order to evaluate the teaching strategies utilised in relation to their multiple intelligences.

In answer to Question 12, "Try to think of an example from lectures which appealed to each of the various intelligences", the following responses were forthcoming:

- **visual/spatial intelligence** – *drawing graphs and charts, volume generated by rotation, areas under curves, Euclidean geometry and Cartesian plane, unit circle, tangent to curve, numberline, the video, drawing*
- **verbal/linguistic intelligence** – *video shown in class on the history of calculus, word questions given on board, one-to-one explanations,* ***humour in lectures****, the teacher explaining concepts, , reading, writing, speaking and listening*
- **mathematical/logical intelligence** – *reasoning, problem-solving, algebra, understanding calculus from first principles, solving quadratic equations, liked the boss's logic*
- **bodily/kinesthetic intelligence** – *movement, doing questions ourselves, rotating curves around the x and y axes,* ***drawing gradients on curves by hand****, hands – on tasks, not much in this area available, the test and assignments*
- **musical intelligence** – ***the video****, did not do it, songs, series or sequences, trig, symmetry, sine waves*
- **intrapersonal intelligence** – *going through things step by step, values, project,* ***giving own opinions in project****, home study*
- **interpersonal intelligence** – *comparing notes when handing in assignments, project work meetings, working in group on project,* ***didn't like this as it was difficult with groups etc. plus I was here to do a job not really socialise***
- **naturalist intelligence** – *can't remember, library tour, exponential equations, project, none*
- **existential intelligence** –*.(integrals) big picture was good, none,* ***every example where the last statement in the explanation is "don't you think that's amazing?"****, calculus – didn't realise it had revolutionised the world until now, video, could not make connection.*

It is clear from these responses that while some students were able to relate the activities to a category of multiple intelligence, others were not, possibly due to language difficulties. It is pleasing to note, however, that while the lectures appealed mainly to the verbal, mathematical and visual intelligences, the students were able to identify instances where the other intelligences were utilised.

The two students who scored 80 on the logical/mathematical intelligence, when asked which activities appealed to them most, both replied that they liked doing the exercises, assignment questions and test. These are the traditional modes of learning and assessment in mathematics education. The other students, whose strengths did not lie in the mathematical realm, preferred the other non-traditional activities, for various reasons. The group research project proved popular: one student explained that it made use of his language skills, another felt that it enhanced his interest in the topic, while another said it enabled her to socialise, help and be helped by others and gave her a sense of comfort during the course.

Interestingly, the student who made the comment that he did not like to socialise scored only 20 on the interpersonal intelligence. It appears that the various activities did appeal to different intelligences.

Humour is not one of the multiple intelligences identified by Gardner, but the responses to Question 16, asking students what effect humour in lectures played on their learning/attitude, were enlightening. Students said it provided light relief, a break from heavy concepts, alleviated stress and aided memory as things are easier to remember if they are funny.

A marked change in attitude was evident in the response given by one of the students (who was suffering severe maths anxiety earlier in the course and achieved 90% in the final exam) to Question 9: Describe how your feelings about Mathematics have changed over the course. She commented *"I have felt everything from extreme gloom and worry, plus fear, to pleasure and elation when I felt I was getting somewhere. The past has been hard to overcome."* The responses to Question 15, where students were asked to identify which intelligences were utilised in the various aspects of the course, revealed that the assessment tasks alluded to the intrapersonal and interpersonal, while the textbook appealed to the existentialists. According to some students the project made use of all of the multiple intelligences.

These responses are encouraging because the writer's objective was to connect with students' individual learning strengths although the course was not designed specifically around Gardner's theory. Yet, according to the students' analyses, different activities utilised in the course did allude to the multiple intelligences. Upon reflection, the types of examples and questions to which students are exposed in this course tend to be drawn more from a broad diversity of practical and contextual situations than the purely theoretical or formal proof types of question. In fact, gender, ethnicity and other cultural considerations are taken into account when devising and presenting examples and questions. For instance, the concept of *linear functions* could be presented through the statistic:

Premature baldness is one of the greatest fears carried by men. By 30 years of age 30% of men are balding, and by 40 years of age 40% of men are balding. Draw a graph of percentage of men balding versus age.

Another statistic: *31% of women sleep in the nude.*

And a question on exponential decay:

A woman's uterus normally weighs 60g. During pregnancy it expands until, at birth, it weighs 1000g. It then shrinks exponentially and by the 14^{th} day weighs 350g. It keeps shrinking until it finally reaches its normal weight. Find an equation to describe this scenario.

That it is imperative for the teacher to attempt to relate examples to the multiple intelligences (or interests) of individual students is borne out by the response to Question 21, where they were invited to make additional comments, given by one of the students:

"*I was keen to do this course as a stepping stone to a degree course (possibly in Nutrition) but it is very hard to learn something when you're not directly interested. It is much easier to learn when it is directly relevant or something which you're very interested in.*"

A final measure of success in enabling these students with diverse backgrounds to learn mathematics was the examination. All students performed well, with a number of them achieving High Distinctions (over 85%).

In summary, students with learning strengths in areas other than mathematics can be taught mathematics. Carefully chosen examples, tasks and assessments, which allude to specific intelligences, allow the students to be stimulated in a meaningful way, empowering them to learn and enjoy mathematics.

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APPENDIX: STUDENT QUESTIONNAIRE FOR ictm2 PAPER

Janet Hunter 2001

Please answer the following questions as accurately and with as much detail as required.

1. What were your reasons for enrolling in the Preparatory Mathematics course?
2. Describe the highest level of Mathematics you had studied prior to this course, and at what age?
3. How many years is it since you studied the level in Question 2?
4. Are you male or female?
5. In the past, what sort of Mathematics student would you have described yourself as?
6. What were your feelings before the course began?
7. How confident were you of success?
8. How developed were your skills before enrolling in this course?
9. Describe how your feelings about Mathematics have changed over the course.
10. Describe the skills you think you have developed over the course.
11. What sort of Mathematics student would you describe yourself as now?
12. Try to think of an example from lectures which appealed to each of the various intelligences:

- Visual/spatial (graphs, art, eye-catching, drawing) –
- Verbal/linguistic (speaking, writing, reading, listening) –
- Mathematical/logical (numbers, reasoning, problem-solving) –
- Bodily/kinesthetic (games, movement, building, hands-on tasks) –
- Musical (songs, patterns, rhythms, instruments) –

- Intrapersonal (own intuition, values, ideas, feelings) –
- Interpersonal (work with group or partner, socialise) –
- Naturalist (field trips, subtle meanings) –
- Existential (philosophical, big picture, "why is it so?") –

13. What aspects of the lectures enhanced your learning?
14. Give an example of the one of the various intelligences which could have been used in lectures to enhance your learning, giving details.
15. From the list in Question 12, choose which intelligences were utilised in

(a) the assignments –
(b) the project –
(c) the test –
(d) the exam -
(e) the handouts -
(f) the textbook –

16. What effect did humour in lectures have on your learning/attitude?
17. What were the 3 intelligences you rated most highly on?
18. Which activities in the course appealed most to these intelligences?
    Why?
19. Can you suggest an activity to add to the course which would have aided your learning experience?
20. What was your favourite activity – most meaningful to your learning experience?
    Why?
21. Please make any other comments you wish.

TEACHING MATHEMATICS USING THE IDEA OF "RESEARCH PROBLEMS"

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ABSTRACT

Most mathematical tasks found in high school and early college textbooks begin with the words: "simplify the following algebraic expressions ...", "calculate the following..." or "solve the inequality...". Mathematicians, however, more often deal with more open problems, where the main aim may be to establish whether the object with the given properties exists at all, or whether the given assertion is valid in principle rather than to simplify or calculate something. Undergraduate mathematics courses for preparing high school teachers might benefit from including a number of such "higher – order" tasks.

By the "research problem tasks" concept we shall mean those that are based on subjectively difficult theorems or mathematical constructions that are initially not known to a particular student (or he is unfamiliar with the proof modus operandi). There are such tasks that a student, when solving them, encounters the necessity to investigate mathematical models of configuration which are new to him, non-standard connections, existing between such models, properties of figures, and at the same time he has to find and establish a logical scheme of reasoning. Solution of a research problem task results in the established and well-founded algorithm of solution for the total class of similar problems or heuristic device, the scientific idea that, after being justified and generalized, can be used and recommended for the solution of other similar nonstandard problems. The proposed method is found to considerably intensify and advance the process of students' mathematical training, to upgrade their knowledge, skills and habits. The conducted investigation has been reflected in Applebaum (2001) Ph. D. Thesis "Research Problem-based Mathematical Training Intensification and Advancement in Gifted Students ".

Rationale

Formation of mathematical thinking in children, training their minds in criticism, development of convergent and divergent faculties in an individual with simultaneous high-level support and enrichment of their knowledge, skills and habits is the key objective, task and challenge of a mathematics teacher.

Many researchers ([2], pp. 222 –248) noted that convergent intellectual faculties reveal themselves, first of all, in the efficiency of information processing, and in the capacity of quick finding the proper way out of the given situation. Divergent intellectual faculties manifest themselves in the ability to put forward a number of equally correct ideas concerning the same problem solution. Convergent and divergent intellectual faculties thus characterize the adaptive opportunities of individual behavior in the hidebound activity conditions.

The researcher A.D. de Groot has come to the conclusion that any creative act or product was in no way the result of intuitive inspiration or inherent geniality, but rather appeared as the result of specific individual development combined with long term accumulation and differentiation of experience, useful for the given sphere of activity. ([3], p. 68)

R. Gardner came to similar conclusions while describing the phenomenon of the "experience crystallization" ([4], p.26). It should be noted that Poincare ([8], p.79) has asserted similar ideas in his famous report in the Psychological Society in Paris. "The thing that surprises us first of all, I mean a visibility of a sudden inspiration, is an obvious result of the long unconscious work of intelligence in the field of the analysis of knowledge and experience that have been received in this time or another..."

Thus summarizing the foregoing, we shall emphasize:

1. The modern community increasingly more demands convergent and divergent intellectual faculties of a personality mental activity. At present, the tendency to enhance the role of these intellectual faculties is especially marked when choosing among the applicants for an office in different areas of human activity.

2. Convergent and divergent intellectual faculties of a personality can not be manifested and realized on "a blank place". The person's skills and habits of work in the chosen field of activity can effectively be manifested and developed only on the basis of solid knowledge mastered at the level of profound comprehension rather then just formally.

3. Convergent and divergent intellectual faculties of a personality are able to essentially improve his mental activity and make it constructive only then when these two branches of an individual facilities develop in parallel, supplementing and enriching each other.

Basis concepts and notations

By the "scholastic research tasks" concept we mean the subjectively difficult theorems or mathematical constructions that are not initially known to a particular student (or he is unfamiliar with the proof modus operandi).

These are such problems that a student, when solving them, encounters the necessity to investigate mathematical models of configuration which are new to him, non-standard connection, existing between such models, property of figures, and at the same time he has to find and establish the logic scheme of reasoning. As it is customary in the majority of mathematics methodical courses, all scholastic tasks can be divided into two main groups – heuristic and algorithmic tasks. Solution of a scholastic research task results in the established and well-founded solution algorithm for the total class of similar problems or heuristic device, the scientific idea that, after being justified and generalized, can be used and recommended for the solution of alternative nonstandard problems.

The technique of each such problem solution assumes that there exists an initial opportunity of splitting given problem into a chain of comparatively easy lemmas. This technique also assumes the opportunity to derive and analyze intermediate results of the received solution both by the student and by his tutor.

Thus, we shall notice that the concept of "scholastic research tasks" is always considered in a relative sense, in the context of a concrete student's personality. It allows individualizing and intensifying the process of intellectual education of a particular pupil or student on the given material. Our experience (that has shown its advantages in the educational structures of various countries) enables us to assert that it is possible to train the pupil in self-education and scientific creativity skills as well as in the elements of experimentalist or researcher work on the research type tasks.

We emphasize that, as it was noticed above, the basic motives for the choice of such tasks were dictated, on the one hand, by the considerations based on our own positive pedagogical experience. On other hand, this choice was the consequence of study and analysis of the results of well-known psychologist L.S.Vygodsky.

Vygodsky has justified the following fact: the condition of person development as a whole, and the level of his mathematical thinking in particular, is determined not only by a personality current state. Not only what the child has already learned to do is essential, but also what he is capable to learn. Here, as Vygodsky has shown, two parameters are necessary to be accounted for:

1) How a student solves the offered tasks independently, by himself.

2) How he solves the same tasks with the help of adults.

Certainly, with the help of adults the child can solve only such tasks that lay in the scope of his own intellectual abilities.

The divergence between these two parameters would also be a parameter that defines the so-called "zone of proximal development".

Tasks that a child is capable to understand or solve with the help of a tutor specify the area of his nearest development.

"What a child can perform with the help of adults today (that is what does currently lay in the area of his nearest development), will tomorrow be the thing he would manage to perform independently (that is, that will tomorrow proceed to the level of his authentic development)". ([10], p.92)

The idea of taking into account not only that was already achieved but that would be achievable in the nearest future as well, that is to work on an advancing, – has appeared rather fruitful not only in the researches of others scientists (such as psychologist V. A. Krutetsky, for instance), but in our everyday pedagogical practice as well.

Expediency and necessity of the students' training in the solution of research tasks. Problem urgency.

1. Essentially always the process of any research task solution reconstructs the atmosphere of scientific work in the most realistic way. It can be ascribed to any analysis in general and to a mathematician's work, in particular. Hence, a child can receive general notion about the research work since his school age. This is obviously rather significant for his vocational guidance. B. N. Delaunay, the brilliant representative of Moscow mathematical school, declared in this connection that "great scientific discovery differs from serious scholastic research task only in one feature: a child spends

several hours or even several days to solve his problem, whereas a real scientific discovery may sometimes take the whole scientist's life".

2. Statistical data confirm that mathematicians accomplish their most important discoveries at the age of 22 to 26. Therefore, from our point of view, it is promising enough to teach children the scientific analysis methods at their early school age.

3. In the course of training to solve research tasks students learn to master the special schemes of plausible and provable reasoning and gain high level of knowledge, skills and habits of work from numerous mathematics divisions, as well. That is very important per se, of course.

4. The process of search for the scholastic task solution will demand from a student to undertake corresponding intellectual efforts. Thus, the intellectual facilities of a pupil receive a powerful impulse for development. "You see that anything you are compelled to discover independently, by yourself, leaves a path in your mind which you can always use to take advantage when a necessity would arise".([6], p.23)

5. "Scholastic research tasks" allow individualizing and intensifying the process of intellectual education of a particular pupil or student in the given material. Use of material of the research type tasks makes it possible to train a pupil in self-education and scientific creativity skills as well as to accustom him to the elements of experimentalist or researcher work. ([1], p.5)

6. Course of the research tasks solution, as it is, opens up the majority of heuristic solution procedures that are valuable for the mathematical personality development. Later, the skills obtained can be extended to any mathematical material or to any sphere of scientific interests of a future specialist. From the aforesaid, follow the urgency and practical prospects of the declared problem study.

Selected methods of the students' training in the research task solution

We shall refer to the whole well-founded assembly of mathematical actions as to an *approach to the mathematical problem solution*. We shall refer to the well-founded logical scheme that lays in the basis of a particular mathematical problem solution as the *method of mathematical problem solution*.

From the declared task point of view, the "Method of Heuristic Training" is of particular interest.

Obviously, G. Polya may be rightfully considered as the author of this method of training in its modern interpretation and justification. (See, for instance, his book "How to solve it"). The essence of the method is that a student is offered to carry out the search for a particular problem solution in accordance with the *sui generis* invariant set of general questions. Answers to these questions should draw the student near to the guesswork or to the solution discovery.

In the due course, some students would manage to master the proposed scheme of "reasoning through substantiated questions", and quite often they would gain success in the solution of scholastic problems.

But... from our point of view, for the necessities of the general mass-scale pedagogical practice, his method in its stated interpretation can sometimes appear as unacceptable.

Let us find the cause of this.

Heuristic schemes, which in their different variants and on different stages were given to the students, have certain common features. But abstract advice of general type such as: "...apprehend an offered problem..." or "formulate a relationship between the known and the unknown..." are of little help to the student, when searching for the concrete problem solution, if any sufficient experience in problem solution is absent.

Below we offer our vision of the development of main ideas for the given method of education. The realization of the following methods and devices of students' education is its base:

- Inductive method of teaching the research task solution.
- Method of teaching the research task solution by the way of analogy.
- Deductive method of teaching the research task solution.

In practice, the choice of teaching method depends mainly on the particular task features and on the particular purposes of the tutor.

Let's briefly consider the essence of each method.

Inductive method of teaching the solution of research tasks.

(This method implies a transition from the specific to the general.)

The inductive method of teaching is based on some mathematical experiment. This method requires much more time compared to the two other approaches, as the greatest difficulty of such an approach is to promote the credible hypothesis. Nevertheless, the advantage of this method lies in its maximum vicinity to the real scientific mathematical activity and in the fact that the inductive method develops intuition and creates conditions for the insight and impressing rise. The inductive method of teaching can considerably activate the pupils' creative activity. As the psychologist V. A. Krutetsky has shown in his investigations: "Despite the primitive character of the trail and error approaches, they are underlying the large class of creative processes at the research task solution... It should be noted that the trails could be taken at any level of analytical or synthetic activity. Only at the lowest level of trails these trails are blind – that is they are just guessing, when the pupils fail to realize why namely this test is conducted and what they should receive as a result of this trail." ([5], p. 510).

In practice, we aspire to organize an inductive method of teaching as an experimental step-by-step pedagogical process.

Let's illustrate our method on several variations of the same particular problem solution. The problem was offered to the 12 – 13 year-old children who attended lessons at the mathematical club in a Beer-Sheva (Israel) school.

Example 1. For anyone natural n calculate the sum

$$S(n)=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n\cdot(n+1)}$$

The first stage. **Experiment.**

At the first stage, we offered to analyze and study the values of the sums S, as the functions of number n: $s = s(n)$. At this stage, we didn't establish the amount of tests and didn't restrict pupils in doing those tests.

Results of these tests are collected in the table 1.

n	1	2	3	4	5	6	7	8	9	10	11
S(n)	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	$\frac{8}{9}$	$\frac{9}{10}$	$\frac{10}{11}$	$\frac{11}{12}$

The second stage. **Promotion of a hypothesis.**

At this particular stage it is extremely important to enable the pupil to realize his own independence and scientific activity. After performing a set of experiments, comes the stage of a hypothesis promotion. It is clear that this is a crucial step. The speed of the task solution depends on a correct hypothesis. In the given example it is really easy to notice a rule: numerator of any fraction in the second row of Table 1 is equal to the number of addends of the sum $S(n)$, and the denominator of this fraction is by a unit more than the numerator.

Thus, we receive a working hypothesis: prospective answer is described by the correlation:

$$S(n) = \frac{n}{n+1}$$

The third stage. **Experimental check of the hypothesis validity**

The purpose of this stage is to ensure the necessary feeling of reliance in the logic completeness of the task solution in a pupil. Our pupils acquire rather quickly the firm confidence that to assert invalidity of any hypothesis that has been put forward, sufficient is to show that it is invalid in one particular case. Hypothesis validity must be proved by rigorous deduction. It will be accomplished at the next stage.

The fourth stage. **Proof of the hypothesis validity**

In general it is possible to prove the correlation:

$$S(n) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$$

by the incremental method or with the help of some other scheme. The pupils get acquainted with these ideas earlier and are trained to use these schemes carefully.

Method of teaching solution of the research tasks by the way of analogy

The method for construction of the theory of a research task solving by the way of analogy is one of the major methods of training in our pedagogical practice. The value of mastering this approach is not only the investigation of a particular educational material, but also a valuable opportunity to teach the mathematically gifted schoolchildren to the framework of scientific activity and to develop their mathematical thinking. The most difficult and important for the teacher is to pick up the maximum convenient source of analogy.

Example 2. Prove the correlation:

$$S(n) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1} \qquad (1)$$

At a training stage, the initiative in the choice of a suitable theory or analogy belongs to the teacher. In this case, it is expedient to explain an idea and scheme of application of an incremental method of the sum calculation for numerical series to the pupil.

By the way, the pupils can easily apprehend an idea of the incremental method application by the following problem solution:

Problem. Calculate the sum of numbers:

$$S(n) = 1^2 + 2^2 + 3^2 + \ldots + n^2$$

Solution. Let's consider an identity:

$$(n+1)^3 = n^3 + 3 \cdot n^2 + 3 \cdot n + 1$$

It follows from this identity that

$(n+1)^3 - n^3 = 3 \cdot n^2 + 3 \cdot n + 1$.

In the left-hand part of the last identity we have received the difference of cubes of two consecutive natural numbers. By substituting consecutive numbers 1, 2, 3,... , n instead of n, we shall receive n identities of the same kind:

$$2^3 - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1$$
$$3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2 + 1$$
$$4^3 - 3^3 = 3 \cdot 3^2 + 3 \cdot 3 + 1$$
$$5^3 - 4^3 = 3 \cdot 3^2 + 3 \cdot 4 + 1$$
....
$$(n+1)^3 - n^3 = 3 \cdot n^2 + 3 \cdot n + 1$$

The preparatory work is completed. What remains now – to total the sequentially left parts of equalities and separately their right parts. Thus in the left part of the new identity all members, with the exception of the two of them, will be mutually cancelled out. So we shall receive:

$$(n+1)^3 - 1^3 = 3 \cdot (1^2 + 2^2 + 3^2 + \ldots + n^2) + 3 \cdot (1 + 2 + 3 + \ldots + n) +$$
$$+ (1 + 1 + \ldots + 1)$$

Now, it is easy to receive the correlation for any natural n:

$$s(n) = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n \cdot (2 \cdot n + 1) \cdot (n + 1)}{6}$$

Applying the method of teaching by analogy, we aim the pupil to look for the appropriate decomposition of the common term of numerical series from *example 2*:

$$a_n = \frac{1}{n \cdot (n+1)} = c_{n+1} - c_n$$

in the difference of two consecutive terms of some new sequence with common term C_n. It is easy to show that in our case $c_n = \frac{1}{n}$.

So, to prove the correlation (1) an equality

$$a_n = \frac{1}{n \cdot (n+1)} = c_n - c_{n+1} = \frac{1}{n} - \frac{1}{n+1} \qquad (2)$$

can be used just as equality

$$a_n = c_{n+1} - c_n = (n+1)^3 - n^3 = 3 \cdot n^2 + 3 \cdot n + 1$$

was used in the previous problem solution.

By substituting consecutive numbers 1, 2, 3... n instead of n in the left-hand side of (1) and using each time the decomposition (2), we shall obtain the desired answer. The solution was reached by analogy with the previous problem solution:

$$s(n) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n \cdot (n+1)} = 1 - \frac{1}{n+1}, \text{ QED.}$$

Deductive method of teaching the research task solution
(From general — to partial!)

While training to understand the idea of approach to the research task solution by this method, we offer the pupil consecutive series of theoretical tasks, resulting in the construction of the theory elements. This technique is extremely important because it simulates and sometimes reconstructs the way of reasoning along which the pioneer scientist has gone. This method teaches to pick out the major stages of the proof. As a rule, it is possible to save training time by this pedagogical technique, because introduction of the new material takes place simultaneously with its consolidation.

Example 3. Calculate the sum:

$$s = \frac{1}{98 \cdot 99} + \frac{1}{99 \cdot 100} + \frac{1}{100 \cdot 101} + \ldots + \frac{1}{1998 \cdot 1999} + \frac{1}{1999 \cdot 2000}$$

a) Realizing a deductive method of teaching, we initially aim the pupil to search for "the Whole" by its visible, that is, by its known "Part".

In this instance the student first should try to complement his task condition up to a general form, which is to write down the general dependence of sum S on number n.

After that, the student should prove his correlation validness by using appropriate technique and calculate the difference between two values of $S(n)$ for number $n = 1999$ and $n = 97$, correspondingly.

According to the above-mentioned, we shall formulate the general task:

Calculate the sum for any natural n:

$$s(n) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n \cdot (n+1)}$$

b) As it was already shown,

$$s(n) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n \cdot (n+1)} = 1 - \frac{1}{n+1}$$

Therefore, $s(1999) = \frac{1999}{2000}$, similarly, $s(97) = \frac{97}{98}$.

The final answer is:

$$s(1999) - s(97) = \frac{1999}{2000} - \frac{97}{98} = \frac{951}{98000}$$

To accomplish this brief review of the teaching methods, it is necessary to emphasize the presence of developed internal philosophical bonds between those methods. In other words, if, for example, the method of training by analogy is selected as the main method, then all other methods are used indirectly. It was demonstrated in the analyzed examples.

Didactic support for the scientific research process of teaching and development of pupils

Here we are going to show some representative examples that illustrate our basic principles of selection of the didactic contents for the effective development of convergent and divergent intellectual faculties of the personality mathematical thinking.

Problems of the solution existence definition.

During our 20-year pedagogical practice we have carried out a series of experiments that brought us to the conclusion that if our aim is to teach a pupil the technique of the research task solution, it would be advantageous to begin with the problems that sound as "*whether it is possible or no*?"

Let's dwell on this type of tasks in detail.

Previously it is necessary to explain to the pupils the following:

Most of mathematical tasks that may be found in a school textbook begin with the words: "simplify…", "calculate this…" or "find what is…". In mathematical sciences, however, investigators very often deal with the research problems, where the main aim is rather to establish whether the object with the given properties exists at all or whether the given assertion is valid in principle than to simplify or calculate something.

This type of tasks usually begins with words: "whether it is possible to (do something)", or "whether (something) exists?" and so on.

When searching and justifying the solution of such tasks, it is necessary to stick to the following rule:

1) If we assert that something can be made it is well enough to specify the concrete way that allows fulfilling that.

2) If we assert that under no conditions something can be made, then the examples by themselves wouldn't help here. It is necessary to construct a rigorous deduction in this case.

Let's consider some representative examples of problems that we used in our work with the children.

Example 4. Several marble blocks with overall weight of 14 tons are to be transported from one site to another. Exact weights of individual pieces are unknown, but it is known that none of these blocks weighs more than 400 kg. Three questions are set:

1. May it be asserted that 12 trucks with the weight-carrying ability of up to 1500 kg will be actually enough to cope with this task?

2. What is the minimal number of trucks with the weight-carrying ability equal to 1500 kg each to be ordered to transport the cargo?

3. If 9 identical trucks are actually enough to transport this cargo, what should be the minimal weight-carrying ability of one truck?

Solution (with the elements of discussion).

Answer to the first question. We shall "throw a trial stone", that is we shall begin with an attempt to construct an example demonstrating that 12 trucks with weight-carrying ability equal to1500 kg each will cope with the given task. The problem lies in the fact that the weight of each individual block is unknown. However, it is intuitively obvious that the maximum efficiency of the trucks used means maximum possible loading of the each truck. That results in the minimum number of trucks needed. On the other hand, the less is the weight of each piece, the larger may be the weight-carrying ability of each truck used (ideal case would be if each marble block would have a grain size). The simple calculation shows that if all marble blocks have the same weight, for example, 350 kg, one truck could carry 4 blocks with $350 \cdot 4 = 1400kg$ gross weight. And this means that 10 machines would be enough to transport all marble. This calculation shows that under certain conditions we can "save" minimum two trucks of 12. We tried to think up a situation facilitating an effective utilization of transport. But it is also clear also that the calculation was carried out under "favorable conditions".

How would the situation be solved if those favorable conditions would not realized? Evidently, such favorable conditions do not take place in the general case. Let's try to construct a suitable example, reasoning from the end.

Let us assume that 12 trucks would not be enough for the given task. For example, we can imagine a situation when each truck transfers **less** than $14000:12 = 1166\frac{2}{3}kg$

of a marble. It is possible if the weight of each block does not differ much from, for example, 388 kg. In this case, the truck is able to carry three blocks as $388 \cdot 3 = 1164 < 1500 kg$ but it cannot carry four as $388 \cdot 4 = 1552 > 1500 kg$.

In that case, 12 trucks wouldn't be actually enough to cope with the task as:
$1164 \cdot 12 = 13968 < 14000 kg$.

Now we have only to define the figures more accurately. If the weight of each block is assumed to be equal $14000 : 36 = 388\frac{8}{9} kg$ and the consignment consists of 36 marble blocks, then 3 blocks can be loaded on a truck: $388\frac{8}{9} \cdot 3 = 1166\frac{2}{3} < 1500 kg$

(and $388\frac{8}{9} \cdot 4 = 1555\frac{5}{9} > 1500 kg$) and we obtain that 12 trucks would be enough. But if there would be, for example, 37 blocks of identical weight, then:

a) the weight of each piece would be equal to $378\frac{14}{37}$ kg;

b) no more than 3 blocks may be loaded on each truck because

$378\frac{14}{37} \cdot 3 = 1135\frac{5}{37} < 1500 kg$ and $378\frac{14}{37} \cdot 4 = 1513\frac{19}{37} > 1500 kg$

This means that 12 machines can carry only $12 \cdot 3 = 36$ such blocks. One block will not be transported. The answer to the first question is negative.

Answer to the second question

As by the problem conditions, the weight of each block does not exceed 400 kg, any truck is able to carry above 1100 kg of a cargo. This permits to claim that 13 trucks would be actually enough for the transportation of all marble blocks:
$13 \cdot 1100 = 14300 > 14000 kg$

Answer to the third question

If the weight-carrying ability of each truck is equal to M and the underload should not exceed 400 kg, then the maximum load that the truck should transport can exceed

$(M - 400) kg$. From an inequality $9 \cdot (M - 400) > 14000 kg$ we shall obtain $M > 1955\frac{5}{9} kg$. So, we got the lower limit of the truck weight-carrying ability.

Thus, the answers to all questions set in the problem are obtained.

Conclusion

Theoretical prospects of the given problem research.

The results of this study are expected to stimulate the development of the major methods and principles of the mathematical education organization. For instance, the investigation and development of the heuristic educational methods. We believe that the most interesting line of investigation consists in the study of motivations and reinforcement of interest to the problem solution. The central point is the stimulating atmosphere of scholastic process created by a teacher. We believe that the existing organization of students' activity does not sufficiently promote the development of deep personal interest in such an activity in the majority of students.

Students' training in the scientific methods of research problem solution stimulates the process of shaping and development of person's mathematical thinking, promotes the quality of his knowledge, brings up and drills his intellectual endurance, arouses a young person's deep personal interest in his own mental activity, and prompts a person to self-education.([1], p.21)

In conclusion, we would like to emphasize and direct reader's attention to the following: any method of teaching should be used creatively, in view of the interests of the learning person. In this connection an idea stated by Maier N.R. ([7], p.65): "the person can fail to solve a problem not because he is not capable to find the solution but rather because the habitual mode of action restrains the correct decision development", deserves merit.

From this, the need for the improvement of obsolete teaching methodology as well as the necessity of search for the new effective methods of teaching which are able to develop a creative person are naturally ensued.

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MODERN PROBLEMS OF MATHEMATICS TEACHER TRAINING IN RUSSIA

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ABSTRACT

The problem of qualitative changes in the training of a mathematics teacher of a secondary school in Russia is defined. In the course of the experimental psychological research we have found three crucial periodsof professional development of students. Teachers should be treated at the formation of an integral system of professional-pedagogical activity: professional knowledge, founding stage, technological structure of professional activity and shaping a teacher's personality. Conception of founding of school mathematical elements (knowledge, skills, abilities mathematical methods) presupposes development in the process of mathematical training of students. In the proposed didactic system of mathematical education of prospective mathematics teachers a fundamental role is played by a pedagogical technology of visual-model teaching of mathematics and activity of students.

Key words: teacher's training, mathematics, innovation technology, founding process, visual model technoligy

1. Introduction

Despite numerous attempts to alter curricula and syllabus, introduction of a state educational standard, development of tendencies to democratise higher pedagogical education, no significant qualitative changes in training of mathematics teachers in secondary school have been taking place in Russia within the last ten years.

In the course of the experimental psychological research of teacher training and independent professional activity of mathematics teachers three crucial periods of their professional development have been defined. The first period is connected with the end of the 1-year at college; the second – with the end of the third and end of the fourth year at college; the third one – the end of the first and beginning of the second year of the independent work at school.

Variance, factor and cluster analysis of the obtained results proves that every crisis is connected with overcoming of corresponding contradiction. The first crisis is connected with contradictions between the form and content of education at school and at college, the second crisis is evidence of contradiction between fundamental and professional-methodological training at college. The third one reflects contradictions between professional preparedness for work at school and professional requirements, which a teacher faces at school. All this results in inadequate professional training of a teacher: formality of knowledge, lack of soundness in professional knowledge of skills and habits in the subject, poor knowledge of methodology and technology of teaching, low creative activity and insignificant interest to innovations in planning and organising educational process at school.

It is necessary to give an adequate characteristic to professional formation, in other words, describe it as a systematic, continuous process, which is determined by a complex of internal and external factors and is being realised on the basis of various psychological mechanisms and pedagogical technologies.

At present the problems under consideration are being approached in various ways. **The first way** is to increase volume of mathematical and pedagogical courses at colleges. The authors of this approach are probably of an opinion that the more fundamental and pedagogic theories and methods the student are taught, the faster the system of their professional-pedagogical activity is being formed. But practical experience proves that it is not always the case.

The second way, which is more promising, admits that methodological training of students should be started as early as possible – even in their first and second year. This training is to be based on various professional trials and tests, still not grounded either on theory or experiment.

In our opinion, the pedagogical process of training mathematics teachers should be treated at formation of an integral system of professional-pedagogical activity. The first, professional, stage should be devoted to formation of the subject knowledge and skills, aimed at formation of the nearest specific generalisation of basic educational elements of school mathematics. The second stage, when knowledge of mathematics became fundamental, they acquire profound theoretical generalisation, which on the third, methodological stage is integrated into the structure of professional activity as a means of realisation of the pedagogue's teaching and educational functions. To ensure painless inclusion of the generalised knowledge they must be organised in a way that best suits school children. Founding and visual modelling performs exactly this function of reorganisation of knowledge of a certain subject according to aims and tasks of teaching activity.

2. Psychology of Mathematics

In the last decades, mathematics as a pedagogical problem has been under unprecedented pressure on the part of society concerning the subject matter of teaching as well as teaching methodology. The problem is that the depth of its formalization and following the inner structural regularities contradict both the ontogenesis and socialization of a single individual and needs of society in terms of providing for its visualization, modeling and revealing the social status.

What is mathematics nowadays?

Mathematics as a science presents methodology and language of other subjects, as well as connections between abstract objects, which unsynonymously reflect practical reality. That's why mathematics occupies a special place among the sciences.

Mathematics not only promotes new knowledge about nature and personality, but also finds a practical stimulus for development in applied sciences. Thus the development of the theory of locally convex spaces in functional analysis was stimulated by the physical problems of quantum electrodynamics in finding generalized solutions of equations in mathematical physics, the theory of unboundary operators in the Banach space – by problems of quantum mechanics, tensor analysis – by the problems of mechanics of elastic media, the theory of the function of many complex variables – by the problems of the quantum theory of the field etc.

Therefore consequences of the strenuous tendency of fundamental mathematical knowledge are closely connected with intensive application of mathematical methods in other sciences (including the humanities). Some of them spontaneously influence vital activities and socialization of a personality in the modem world.

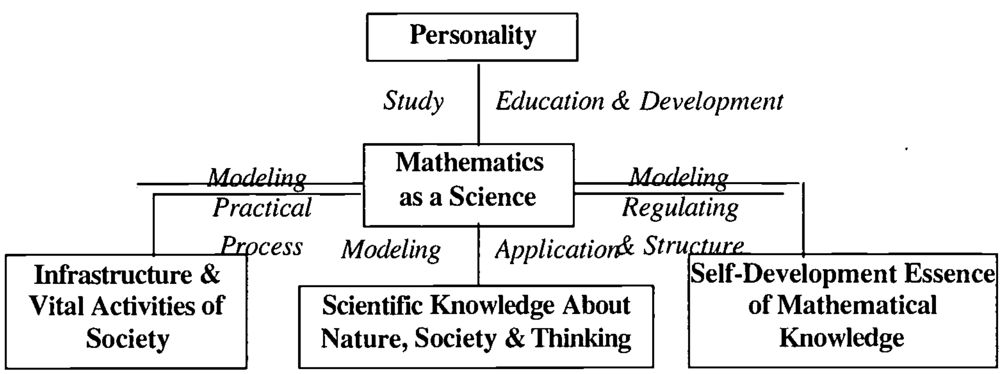
On the other hand, during the recent ten years some new areas have been developed in mathematics. These areas have independent subjects and specific methods of research. Among them are: artificial intelligence and the theory of mass service, the theory of random processes and functional analysis, the theory of games and mathematical programming, algebraic geometry and set-theoretical topology etc.

The cardinal means which promote new formations is modeling as a high form of the sign-symbol activity leading to new knowledge about nature and the technical process in production, about the laws of social development and the regularities of human thinking, perception and memory.

During the recent years we have seen reinforcement of the role of mathematics as a means of humanization and socialization of personality-oriented education in modern society as a necessary attribute of the educational paradigm of the 21st century personality.

Considering mathematics as a pedagogical problem, we face the problems of an adequate notion (idea), distinguishing, formation, stable perception and reproduction of mathematical knowledge in all the 3 hypostases of mathematics (Figure 1).

Figure 1
The Model of Three Hypostases of Mathematics



How can we show in the process of teaching mathematics its role in the substantiation of space flights and security of air transportation? How can we show that physics is a powerful supporting component of vital human activities, which come only to observation and experience without mathematical knowledge and that psychology is only fortune-telling and subjectivism without statistic methods of experimental data analysis and modeling of mental processes? What is the best way to tell a pupil that A. Wiles proved Fermat's Last Theorem in 1995, and that trisection of an angle and squaring the circle are impossible with a ruler and a pair of compasses? How can we develop the pupils' thinking operations (logic, analysis, synthesis, generalization, concrete definition, analogy etc.) in the process of teaching mathematics (which objectively must be the most powerful of the developing means which unfortunately can't be observed yet) more effectively?

The problem of an integral notion and adequate acquisition of mathematical knowledge under the conditions of immediate perception, developing and making the structures of a pupil more active is of paramount importance in organizing didactic and cognitive processes.

3. Innovative Process of Knowledge Founding

We based our assumptions on the fact that concordance or optimisation of interaction of fundamental and professional components in the general structure of pedagogical education is a key moment in training of a student at a teachers' training college. It is obvious, that fundamentalisation of mathematical knowledge without considering it as a pedagogical aim will hinder professional training of a teacher. At the same time it is beyond doubt that a teacher is not in able to fulfil his professional functions successfully without a certain volume, structure and quality of fundamental knowledge. Thus, in a nutshell, the problem is to find means, forms and ways to bring to concordance fundamental and professional lines in the process of pedagogical education.

Founding is a process of creating conditions (psychological, pedagogical and methodological-organisational) for actualisation of basic structural units, which reveal their essence, integrity, relations between the subjects in the direction of professionalism of knowledge and shaping a teacher's personality.

Conception of founding of school mathematical elements (knowledge, skills, abilities mathematical methods) presupposes development in the process of mathematical training of students in the following components:

- determination of contents of basic schooling element (knowledge, skills, abilities mathematical methods, ideas, algorithms and procedures) level of organisation;
- determination of contents of levels and stages (professional, fundamental and technological) of basic schooling element development at college;
- determination of founding technologies (diagnosable aim-finding, visual modelling of global structure levels, local capacity for modelling, control of students' creative and cognitive activity, blocks of motivation of basic schooling elements);
- determination of methodological adequacy of basic school and college (founding) schooling elements on the basis of modern methodological concepts.

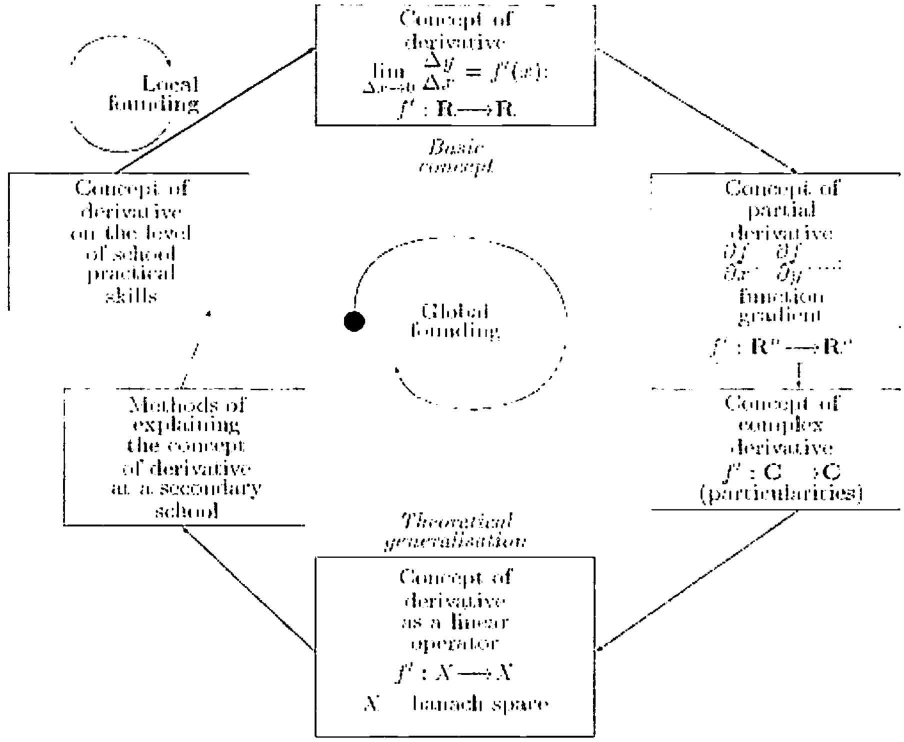
In order to realise the principle of founding it is necessary to define the basis for helical diagram of the basic knowledge, skills and experience of mathematical training of students at teachers' training colleges modelling. If founding of various school subjects is to be carried out layer by layer, then volume, content and structure of mathematical training must undergo considerable changes in respect to practical realisation of theoretical generalisation of school knowledge based on the principle of a "boomerang". If the knowledge is being founded in such a way, the teacher, who possesses knowledge of the subject, together with the student will master the methodological side of teaching. The school knowledge will act as a structure-formation factor, making it possible to select theoretical knowledge from mathematics of a higher level, via which school knowledge has been founded. The layer of founding provides perfection and extension of practical skills, projected by approximate basis of learning activity. In the activity aspect of pedagogical process realisation of foundation principle acquires a helical character, which corresponds to dialectical understanding of a system of knowledge development.

Development of spirals of basic school subject elements via construction of ancestral generalisation and technological comprehension of its specific manifestation render integrity and orientation to the projected didactic system.

For example, this chain of founding can exist.

Figure 2

Diagram of school knowledge founding (Concept of derivative)



The value of the present model of founding (conception of derivative on the level of "data" to its deep theoretical generalisation on the level of "essentiality") for training processes at college as well as prospective professional activity for a math student is beyond doubt. It seems such models will find their place in syllabuses of mathematical analysis and individual technologies of teaching mathematics at school.

At the same time construction of such a model absorb in its unique and particular manifestation all main features of theoretical knowledge about foundation process of basic educational elements of maths at school. Creation of system-genetic block of spirals of founding makes it possible to define a stable nucleus of educational information content, which projects elements of approximate basis for educational activity of students.

4. Visual Methods of Teaching in Application to Mathematics

In teaching mathematics the teacher encounters the following problems:

– a high degree of abstraction in the perception of new material;

– an information overload of sign-symbolic forms;

– a limited period of time for active perception of new material;

– weak motivation in studies.

The first problem results in getting only formal knowledge, and its isolation from the essence of the basic phenomena and theorems. The second makes it difficult to recognize the object of perception. The third increases conflict in teaching. The fourth means that pupils' interest in studies is decreasing.

Strange as it may seem, all these problems can be solved when we pay special attention to the first stage of cognitive perception, immediate perception to be exact. An in-depth study of this stage of perception requires a full understanding of neuro-physiological aspects of thinking, a psychological and pedagogical analysis, as well as personal experience and modeling.

The main purpose and result of this paper is the creating of conception on visual modeling of mathematical objects (in particular, complex and symbolic) and creating conditions for suitable perception.

It is essential to know that perception has three stages: stimulation, operation and understanding. Moreover, the process of perception may be concerned with two forms of acting: simultaneous (as an entire view of the essence of an object) and successive (as a sustained process perception is open not only on the level of stimulation but on more higher levels: analysis, choosing pattern, comparison, reaction formation).

According to our concept utilisation of visual methods in teaching of mathematics of a prospective teacher is to be treated as a special property of psychological images of mathematical objects, the essence of which is considered in a integral paradigm of perception of the basis of the following criterions:

- diagnosable aim-finding of integrity of a mathematical object;
- adequate perception (learner's comprehension of essence of the mathematical object in accordance with aims of teaching);
- stability of the perceptive image and presentation under conditions of direct perception;
- cognitive and creating activity on the basis of comfort, successful teaching.

The process of perception of the given material presupposes all the key qualities of a mathematical object. It is especially important when information is of great volume. It is necessary to keep in mind such actions when separate pieces of knowledge or an arranged set of knowledge are given. We can deal with proving theorems, teaching some parts of mathematical analysis in its various logical correlations, with a single lesson, a lecture etc.

As has already been mentioned, according to A.N. Leontyev, when visual methods of teaching are used, it is necessary to proceed from the psychological role, which they (methods of teaching) play in the perception of new material. He chose two functions of visual methods of teaching:

– the first is aimed at extending the sensible experience;
– the second is aimed at developing the essence of the processes or phenomena under study.

In connection with that, external teacher's actions are divided into bearing and structural actions depending on the orientation of the sensible or rational element of perception.

The external bearing actions can be as follows:

– writing down formulas, tables, displaying models, drawing up graphs, formulating theorems, using text-books or manuals.

The structural external actions can be as follows:

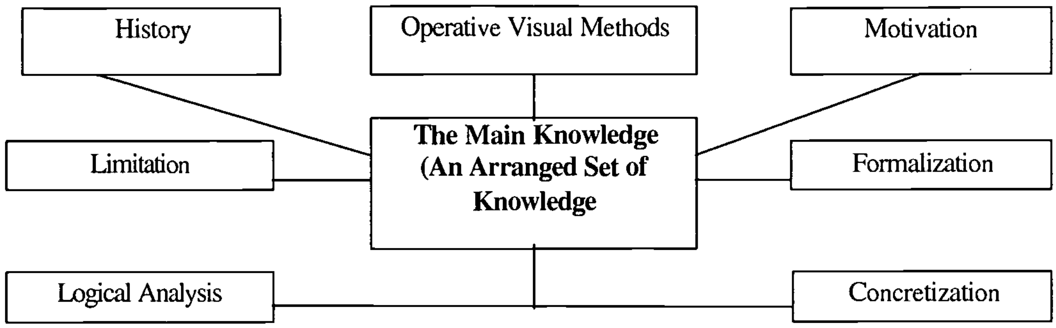
– proving theorems, choosing the main theoretical notions and methods, realizing links between different subjects.

Each external structural action is an arranged set of bearings for external actions connected with each other on equal terms.

Namely, a classification of teacher's external actions makes it possible to choose the following visual methods in teaching mathematics:

– operative visual methods of teaching (e.g. the use of computers, drawings, calculators, schemes, graphs etc.),
– structural visual methods of teaching (e.g. looking at the logical analysis of a theorem or a notion may require a whole lecture)
– formalizing visual methods of teaching,
– background visual methods of teaching (creating the necessary conditions for checking the figure from the background by elements of this background).

Figure 3
The Structural Analysis of the Key Knowledge



Developing these key qualities of the object of perception is the main point of visual teaching methods. That is why when we deal with visual methods of teaching we must not forget that the stage of immediate perception follows the stage of choosing the key qualities in the object of perception (i.e. the goal). This "a priori" point of view presupposes modeling the object of perception by means of the neuro-physiological mechanism of memory and psychology of perception. Some attributes of visual teaching methods may be as follows:

– computer displays, logical analysis of theorems, key theorems, key notions, data banks of problems and investigation methods, bearing code tables, block-schemes for proving theorems, concrete background, a list of topics or parts of mathematics.

Systematic realisation of all means of visualisation in the process of teaching mathematics is an important factor of formation of integral images of mathematical objects, which means that it considerably facilitates understanding of mathematical knowledge and development of cognitive abilities and mathematical thinking.

A mathematics teacher's activity in the process of teaching due to an abstract character and complexity of mathematical material presupposes detailed concrete definition of utilised principles of teaching and their systematic use. It leads to the necessity to work out a common interpretation of visualisation principles in mathematics, developing techniques of a teacher's activity in the process of visual teaching, detection of the visualisation specifics in teaching of mathematics. All of this is based on positive experience of progressive teachers and scientists.

We hope that results of present direction of research will be useful not only to university teaching staff , but to secondary school teachers as well. These groups will be able to rely on it as a basis of creative design of teaching processes and samples of innovation methods in teaching mathematics.

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THE USE OF ONLINE INTERACTIVE MODELS AND SIMULATION IN ASSISTING STUDENTS' DEVELOPMENT OF MATHEMATICAL CONCEPTS.

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ABSTRACT

The authors of this paper teach mathematics to engineering and business students at the Higher Colleges of Technology (HCT) in the United Arab Emirates (UAE). This system of post-secondary colleges for UAE nationals was established to provide students with vocational and technical education. To further enhance student success, the HCT, in line with current trends, has moved to a more learning-centred education, with the creation of independent learning centres, custom designed labs for integrated learning and increasing use of technology and the Internet. Laptops are now becoming quite commonplace in classrooms around the system, and laptop technology is being integrated into learning goals. The development of online modes of learning is being encouraged in order to provide students with more flexibility in their learning. Students are familiar with the Internet and, in general, quite comfortable with computers. Students' laptops are fitted with PCMCIA cards, enabling a wireless intranet connection. The majority of classrooms are equipped with smart-boards, and teaching with technology is being encouraged.

In teaching Arabic-speaking students, learning through the medium of English, particular attention is needed in providing meaningful understanding and sound concept development. With students having immediate access to the college intranet via their laptops, on-line materials can be used effectively as supportive tools in the classroom to enhance learning. The authors are involved in the development of such 'online' materials, which offer students opportunities to interact meaningfully with mathematical content. The emphasis is on concrete and visual approaches with a high level of interactivity. This paper outlines some of the methods the authors have found useful in creating interactive and simulation models for the learning of engineering and business mathematics, and presents examples of such models.

Keywords: Online learning, Interactivity, Pedagogy, Concept development.

1. Introduction

The authors teach mathematics to both business and engineering students at the Abu Dhabi Men's College (ADMC), Higher Colleges of Technology (HCT), a system of post-secondary colleges for nationals of the United Arab Emirates (UAE). Students at the HCT are currently able to enter into four types of programs: Certificate, Diploma, Higher Diploma, and the Bachelor in Applied Science. The Certificate programs are two years in length. They introduce students to general and specific occupational skills and develop basic proficiency in English, computing, and mathematics. The Diploma requires a further year of study in which English proficiency is further developed and occupation-specific skills at the technician level are emphasised. Students following a Higher Diploma (HD) program are required to successfully complete a one-year Foundations course before being permitted to enrol in HD. The HD programs are three years in length and involve a combination of theoretical knowledge and practical applications at the technologist level.

All classes are delivered in English, and entry to programs is dependent both on high school performance and ability in English. All students are native Arabic speakers and are required to successfully complete all their courses in English. Students, coming directly from schools where Arabic is the mode of instruction. At the HCT, therefore, constant attention must be paid by teachers in assisting students from making the transition from students' previous traditionally based classroom experiences to the more student-centred HCT learning approach using the medium of English. Many of the learning difficulties in mathematics are closely related to limitations in the English language. While students at the HD level have less trouble in coping with English, teachers, nonetheless, have to be alert with their diction and phraseology. The language and terminology associated with a particular mathematical topic are easily open to misinterpretation and much care has to be taken to ensure that each of the related mathematical concepts are treated with a meaningful approach to students. Sensitivity to the social and cultural background of students would bring further relevance and meaning to the students' learning.

In the span of its relatively short history, the HCT has grown and developed at a rapid pace in trying to meet the educational needs of the UAE. As needs of students and employers continue to change, there is a continuous review and adaptation of the curriculum and teaching and learning methods. This is being achieved through a series of strategic plans, which identify priority issues for each academic year. One of these priority issues for the academic year 2001 – 2002 is evolving the learning paradigm with a focus on technology. In line with this, one of the goals at the Abu Dhabi Men's College is to expand the use of technology in the curriculum and look for innovative ways to enhance learning. This has brought about a shift from providing instruction to producing learning, with the creation of independent learning centres, custom designed labs for integrated learning and increasing use of technology and the Internet. An online e-Forum has been created with its main aim of promoting independent learning among our students.

Technology is a rapidly becoming a way of life for our students. They are familiar with the Internet and, in general, are quite comfortable with computers. Many now possess laptops and these are quite commonplace in classrooms. The integration of laptop technology into learning goals and the development of online modes of learning are being encouraged amongst teachers in order to provide students with more flexibility in their learning. Classrooms are also now equipped with wireless cover, giving students instant access to the college intranet via PCMCIA wireless cards connected to their laptops. Students can obtains these cards on a refundable deposit basis through the college IT services. Students show eagerness to incorporate technology into their learning. While this may be so, students have yet to develop effective techniques for learning independently. Much teacher guidance and direction is still necessary, and an effective pedagogy needs to be developed before our students can be expected to cope with an online course in its true independent sense.

2. Online Pedagogy

The term 'online' is open to a wide range of interpretations, and there are many 'modes' of online learning. Modified and newer ones continue to evolve with improvements in access and usability. In a standard 'fully' online course, the students would be quite diverse and unpredictable, coming from a range of backgrounds and cultures. The online resources and material would be accessed and interacted with in a variety of sequences, differing times and locations. Each student's knowledge, skill level and learning style will have an impact on how he/she relates to the material and has success with the course. The geographic and time constraints could create additional difficulties.

In comparison with this, our task in the UAE is relatively easier than that of the 'fully' online course designer. We are catering to a particular class of users and with predefined assumptions on students' behaviour and styles of interaction. We know our students and can create learning activities that build on differences in students' learning styles so that students can be directed to the learning activities most suited to their preferred learning styles. Although our pedagogy involves using the available technology and working with our students to foster learning and independent thinking, there is a real need for face-to-face interaction between teacher and student as an integral part of the learning. Our UAE students come from a 'non-western' culture with their particular customs and experiences, where traditionally they have learned from family and elders. Much of their learning was and is done through doing, listening, observing, and imitating parents and others. Interaction with their peers as well as with the environment play further developmental roles. Any instructional environment which incorporates teaching and learning methods related to these existing familiar traditional approaches would have a greater likelihood of succeeding. They suggest more hands-on practical work, more concrete approaches, more collaboration and group work amongst their peers, and face-to-face student and teacher dialogue. Human contact is necessary, not only for learning content, but also more importantly for encouragement, praise, feedback and assurance that students are on the right learning path.

English is not the first language for the majority of students, and considerable care is needed in keeping communication at their level, both in the classroom as well as on any medium for independent learning. Most teacher's at present are 'western' and, quite often without realising it, make assumptions, or take for granted, that certain background knowledge or ideas pre-exist in students' minds and are expected to be automatically understood as in any western setting. The authors have been working in the UAE for several years and have developed some familiarity with the students' background and perspectives on their learning. If we want to communicate meaningful ideas to our students, it is essential to know what they are thinking and visualising, and how they are interpreting our words and actions. It is essential that our online pedagogy should include simple and meaningful language while not losing sight of the students' experiences and extant knowledge. The following design features have been adopted by the authors for an online mode:

- Use of language which is both simple and meaningful in the local context
- Student ability to investigate concepts and ideas through interactive learning models

- Provision of immediate feedback to student responses
- Adaptability to face-to-face classroom use
- Student ability to concentrate their efforts on areas of particular difficulty

These suggest that a pedagogy that is likely to be successful in the UAE is therefore one that employs a complementary online model in which the material provided online adds to face-to-face classroom delivery. The material is seen a supportive and provides interactive learning experiences to investigate concepts more deeply. They could be used both in by the teacher and students in the classroom, or by students working and exploring independently. The teacher's role of a provider of information, becomes more as one of a guide and facilitator of information. The classroom interaction with the online components would further stimulate work in small groups and discussion. Students, either individually or in groups, need to interact with learning materials that allow them greater choices that meet their particular learning needs. They need become engaged in active "doing" in the learning process, which goes beyond merely reading text. The authors believe that a richer online learning environment can be provided to students through interactive simulations that can be actively manipulated, provide engaging and challenging tasks, and that supply instant feedback on performance.

3. Interactivity and Simulation

The interactivity we are concerned with in this paper is that related to student interaction with teaching-learning resources, as opposed to the social interaction between student and teacher. Bates (1991) referring to these, respectively, as social and individual, states the need for a balance between the two in a distance-learning context. This can equally well apply to any form of learning, whether it is distance, online, or in the classroom setting.

Computer-based learning programs often follow a page-turning format where material is presented very much as an electronic textbook. Such formats are frequently encountered on web pages and are far from being student-centred and present little or no interactivity. They are static and unable to personalise the interaction with the user. Rather than trying to replicate a teaching model online, the idea is to create what has been called a 'resource' model, an environment in which students interact and wrestle with learning materials directly (or in teams), under the tutorial guidance of a mentor (Twigg, 2001). The rapid feedback that computers can provide needs to be tapped as much as possible. It is through interactive resource-based models that this can be made possible. The use of animations, simulations and virtual environments that may simulate real world settings can help to simplify a concept by way of interactive processes and bring the concept to life. For such interactive models to succeed, they must be pedagogically sound, engaging and flexible. They should enable students to focus on their areas of weakness and provide practice and investigative opportunities at different levels of complexity.

The authors see these interactive mathematical models as being central to the learning process. Students will be able to investigate mathematical situations by varying parameters to explore different possibilities and interpret outcomes. It is hoped that these models will assist students in developing appropriate learning techniques and help to improve investigative thinking. Students will be in control of their learning and can experiment with a model as long as they need. This could stimulate the students' interest in the given topic and further motivate learning. Their interaction with simple real-world type modelling can bring insight to the mathematics met in the classroom and promote independent learning.

The interactive learning models are being developed mostly with the use of Toolbook Instructor[1], a very powerful authoring package, which the authors have used extensively and which is highly suitable for interactive learning. Toolbook uses a plug-in called Neuron, which enables it to be used in a Web browser. Each Toolbook model is designed with a small number of interactive pages to minimise its download time. Brief descriptions of some examples follow.

4. Examples of Interactive Models

1. Break-Even Analysis: Figure 1 shows an example of an interactive graphical model to investigate the concept of break-even analysis. Students are able to input values for the selling price per unit, variable cost per unit, fixed costs for the period and the capacity for the period. They can then observe the total revenue and total cost graph and investigate the break-even point.
2. Volumes of Revolution: Figure 2 shows an example of an interactive model to investigate finding volumes of revolution by summing up disks. The model allows students to visualise rotations about the x- or y-axes, as well as the number of disks. By changing different parameters, students can investigate inner and outer volumes and develop a concept of the integration technique used.
3. Compartmental Analysis: Figure 3 shows an example of an interactive simulation model to investigate the mixing of fluids is a tank. A basic one-compartment system consists of a function $x(t)$ that represents the amount of a substance in the compartment at time t, an input rate at which the substance enters the compartment, and an output rate at which the substance leaves the compartment. In mixing problems we have a fluid flowing into a tank, along with the concentration of the substance in the fluid. We also know the initial concentration of the mixture in the tank. The problem is to determine the concentration of the substance in the tank at any given time if we are given the exit rate of the mixture. Students will usually solve this problem by setting up and solving the differential equation
$$x'(t) = \text{input rate} - \text{output rate}$$
Students to visualise the rate of change in concentration, by colour variation, and can simulate the mixture problem by changing input and output rates, and concentration levels.
4. Linear Regression: Figure 4 shows an example of an interactive model to investigate the linear regression line. Students are able to add up to a maximum of twenty points, each with a mouse click, anywhere on the grid. The line of best fit is shown, and varies with each additional point. At the same time, the slope, the y-intercept, and the equation of the line can be observed.

In addition to the type of interactive models described, further interactivity is provided through the use of WebEQ[2] to create interactive equations, and LiveMath Maker[3] to enable step-by-step stages in working through calculations and solving equations.

[1] Click2learn.com, Inc. (formerly Asymetrix), http://home.click2learn.com
[2] Design Science, Inc., http://www.dessci.com.
[3] Theorist Interactive, LLC., http://www.livemath.com.

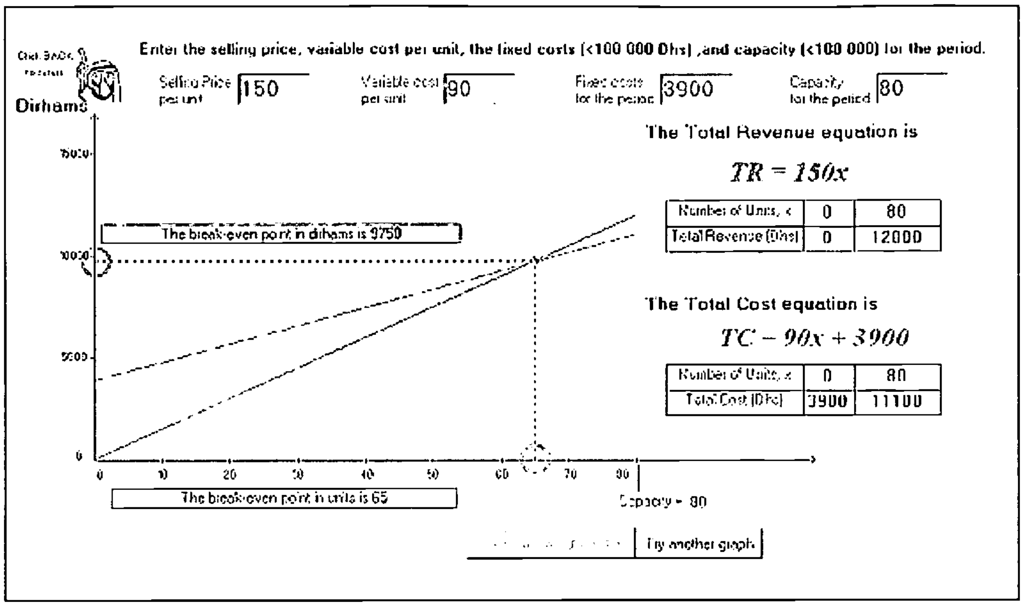

Figure 1

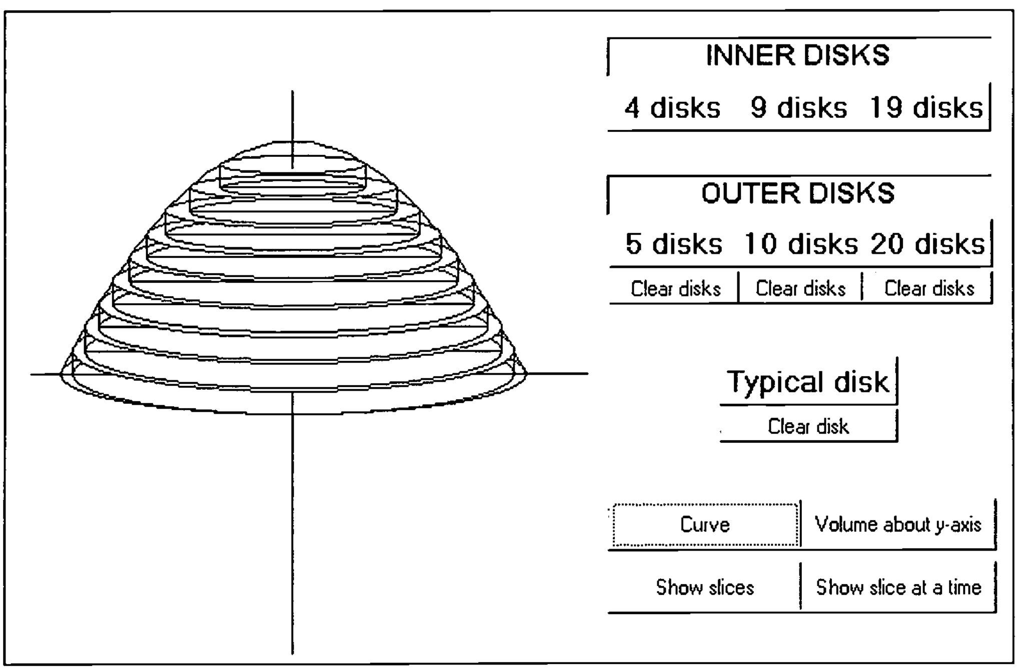

Figure 2

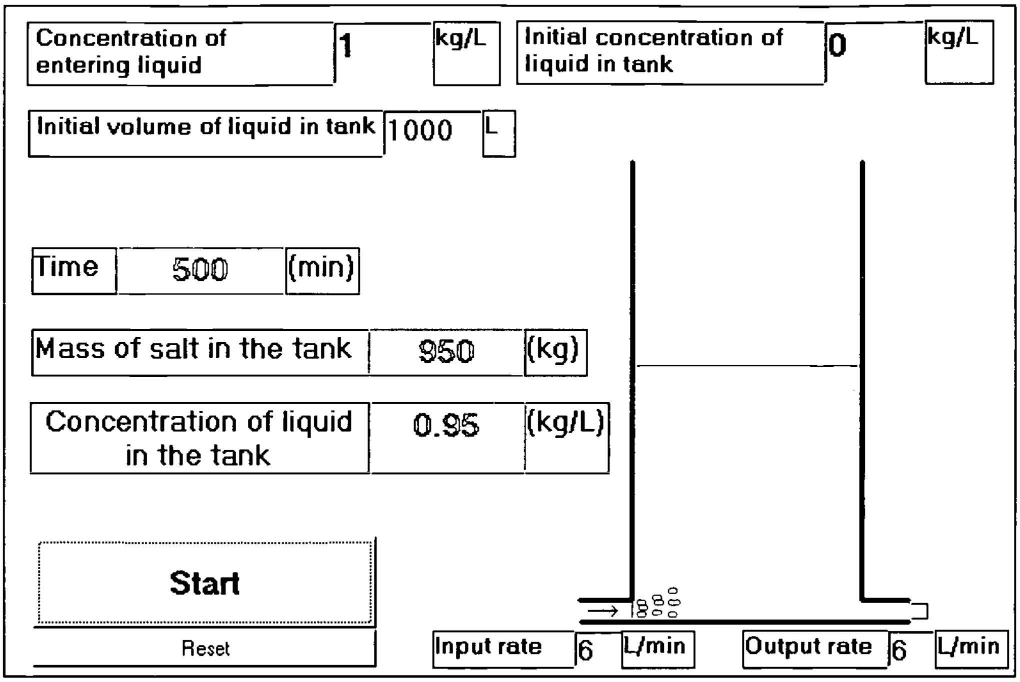

Figure 3

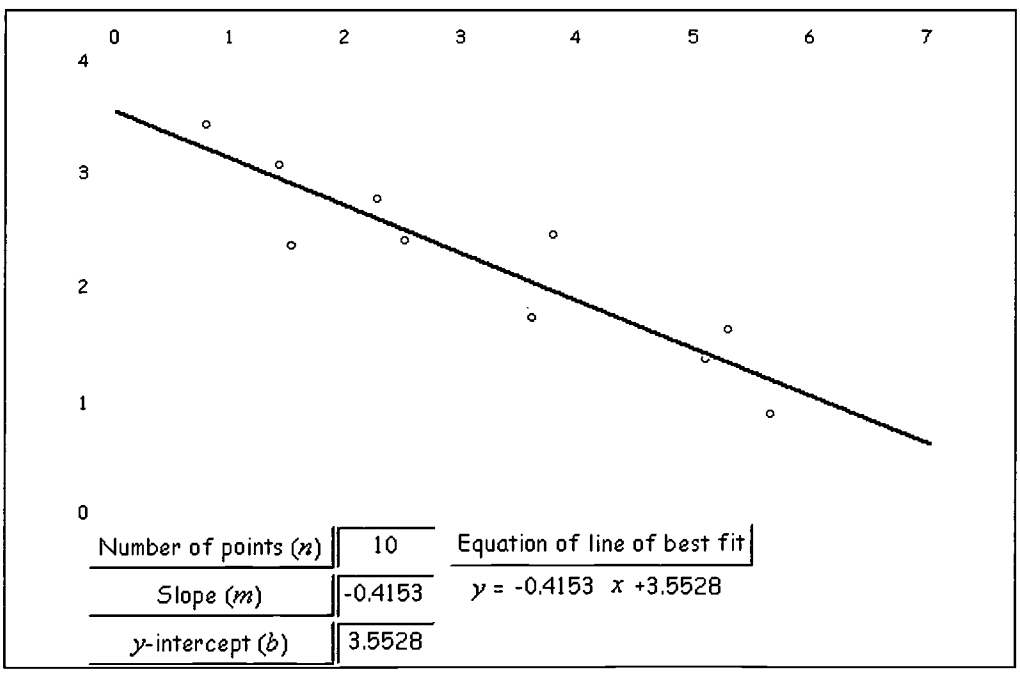

Figure 4

5. Conclusion

The authors are working on projects to produce online courses in business and engineering mathematics. The initial stage is to develop something fairly compact, but effective, manageable and applicable to the 'computer classroom'. Students should have the opportunity to use interactive learning models as central to their learning process. The intent in developing these on-line courses has been to structure them in a way that they can be used as effective learning tools primarily in the classroom, making it possible for students to participate in a synchronous communication learning environment. It remains to be seen if this modification to the learning environment brings further motivation to students and stimulates their interest in the mathematics.

As teachers foremost, who are interested in bringing mathematics to students in a meaningful and enjoyable way, we believe that an online learning environment that is developed from the students' enquiring perspective and allowing investigation of concepts through interactivity can produce successful outcomes. However, what goes on in the students' minds is not visible and far from clear, and as designers, we must be constantly aware that what really matters is 'the quality of the instructional message, rather than any inherent characteristics of the instructional medium used' (Taylor, 1996). Clark (1983) also reminds us that educational technologies are 'mere vehicles that deliver instruction but do not influence student achievement any more than the truck that delivers our groceries causes changes in our nutrition'. We can surround ourselves with technology without producing a significant increase in pedagogical efficacy. It is therefore important to ensure that we are not simply providing students with learning resources and materials online, rather we are providing them with the means and techniques to get the most learning out of those resources. In terms closer to the UAE, the analogy would be "It's not how fast you ride your camel; it's how you ride your camel fast!"

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"USING PC AND TI-92 IN TEACHING STATISTICS IN AUSTRIAN SECONDARY SCHOOLS"

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ABSTRACT

After the recent reforms of the Austrian curricula in mathematics, **statistics** and **the use of the computer** were fixed in mathematical instruction **for ten to fourteen year-old students** (1993, 2000).

In grades 5 and 6 the concepts of absolute and relative frequency, mode, arithmetic mean, median and different possibilities to plot graphs (pictogram, pie graph, bar graph, line graph, polygon) were integrated. When they work on the computer the students are allowed to use hand calculators and they can use spreadsheets. As spreadsheet the teachers generally use **EXCEL** if computer science is a new subject in grade 5 or 6.

Since the Austrian CAS II project in 1997/98, the use of the **TI-92** has been tested in many classes. With the TI-92 it is possible to get a boxplot with the different quartiles of a set of data very quickly. But it also offers the teacher a good chance to acquaint the students of **grade 8** with such difficult concepts as **linear and geometrical regression** and **correlation**.

In my lecture I will show the way I have worked with students of grade 8 and with teacher students at university. It is very important not to take sets of data out of the school books only. I allow the students of grade 8 to work with their own data (length and mass) or I let them find real data with the help of **CBL (calculator-based laboratory)** in an experimental way. Thus they get a better understanding of the concepts of regression and correlation.

1. Introduction

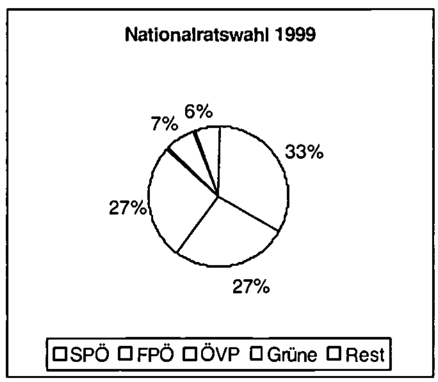
The use of computers in mathematical education in schools depends on some very important conditions. The use of the computer has to be required in the **curriculum**, sufficient **hardware** and good **software** has to be bought and the computer is to be admitted in **oral and written exams**, including the final examinations. All these conditions have been fulfilled to a great extent for grammar schools, business academies, secondary technical and trade schools in Austria in the last ten years. Therefore it depends primarily on the mathematics teacher, in which way and how intensely the computer is used in mathematical education.

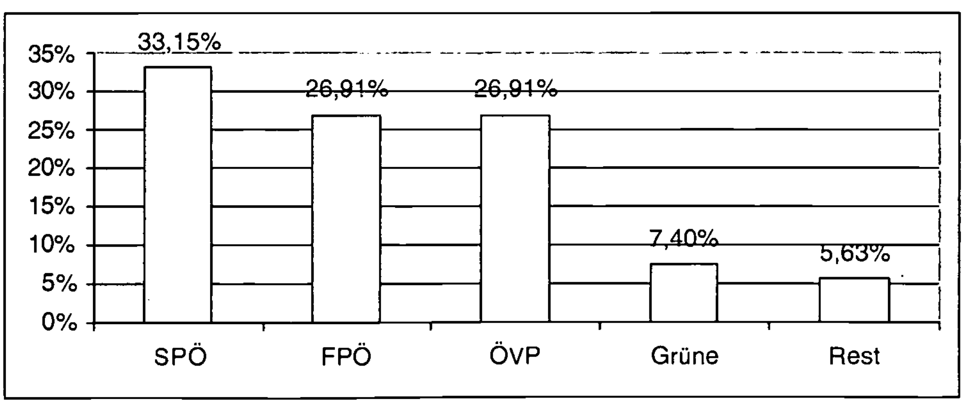
Based on the recent reform of the Austrian curricula in mathematics in 2000, **statistics** and **the use of the computer** have to be combined in the schoolbooks. This is only possible by putting parts of the book on the internet. In the schoolbook the students find a www-address and with it they can call for the additional parts called "schoolbook plus". If the twelve-year-old pupils want to call such parts of the widely used book by Reichel-Litschauer-Groß, they have to enter www.e-Lisa.at and can choose the hyperlink to the EXCEL-programs.

2. First use of a spreadsheet

I would like to illustrate this with a problem taken from Reichel-Litschauer-Groß (2001) concerning the parliamentary election. Problem 213 asks the twelve-year-old pupils to calculate the relative frequency in %, the pie graph and the histogram. With a hyperlink they can study the solution given on the internet and try to get the same result.

Problem 213 - Nationalratswahl 1999		
Right to vote:		5838373
valid:	79.17%	4622240
Partei	**Prozente**	**Stimmen**
SPÖ	33.15%	1532273
FPÖ	26.91%	1243845
ÖVP	26.91%	1243845
Grüne	7.40%	342046
Rest	5.63%	260232
Summe	**100.00%**	**4622240**





Histogram of the parliamentary election taken from the internet

The EXCEL-concept for ten to twelve-year-old pupils has been developed and tested by H. Groß and was presented at a meeting of the Austrian mathematics teachers in Vienna in 2001.

3. Linear regression line

In grade 8 the pupils make the first steps towards **Two-Variable-Statistics**. The first example in Reichel-Litschauer-Groß (1998) is the following:

The Millers want to buy a building plot and study the plots advertised in their newspaper. With the help of a map they have made the following table. In the first row they write the distance from the town center in km, in the second the size in m². ***x = distance, y = size***

Distance	2	3	5	10	20	25	30	41	49
Size	321	158	513	805	520	780	1800	1725	2540

Without a computer the pupils plot the data x and y as coordinate pairs by hand. By doing so they realize that y has the tendency to be directly proportional to x. This makes the pupils try to draw a straight line, which fits the points. It has proved useful to tell the pupils to draw this straight line through the point (mean of x, mean of y). Afterwards the pupils should try to rotate this line in such a way that it is close to the points.

If a computer is available in the classroom, you can demonstrate the solution of this problem easily with the help of a software program like MATHEASS. After the input of the coordinate pairs you get the function term of the approximation curve, the coefficient of determination, the correlation coefficient and the standard deviation together with the diagram.

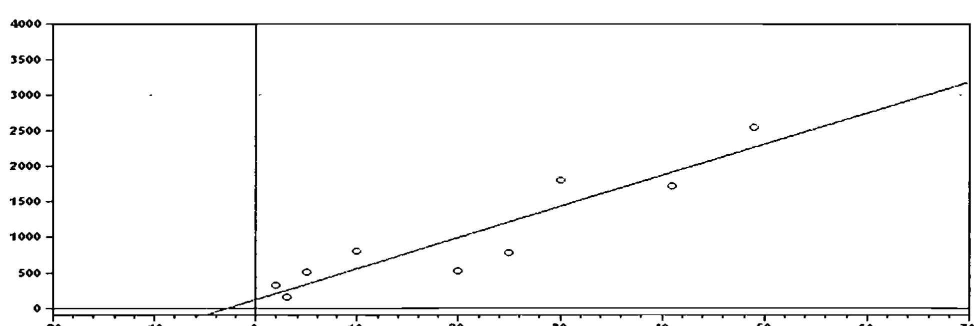
Linear Regression: y = a . x + b

y = 43.755882 x + 118.57353

9 Values
Coeff. of Determination = 0.85281527
Coeff. of Correlation = 0.92347998
Standard Deviation = 332.5105

Inverse Function : y = 0.022854x - 2.709888

x:	f(x):
2	206.08529
3	249.84118
5	337.35294
10	556.13235
20	993.69118
25	1212.4706
30	1431.2500
41	1912.5647
49	2262.6118



Scatter and regression line found with the software MATHEASS

The regression line thus found fits the data points. If all points were on the regression line, the coefficient of regression r and the coefficient of determination r^2 would be 1.

In this situation the pupils normally ask two questions:

How do you calculate: 1) the slope a and the y-intercept b, 2) the regression coefficient r?

The derivation of the formula for a and b is too difficult for 14-year-old-pupils. Therefore only the formula for the calculation of a and b is given in the schoolbooks. The formula for the calculation of r can be explained with the **concept of the covariance** without a and b. R. Diepgen has made such a suggestion in the journal "*Stochastik in der Schule*" (Heft 3, 2000).

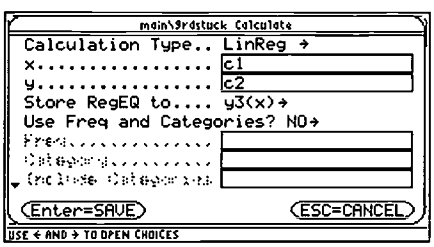
In their schoolbook "Angewandte Mathematik 4" for 18-year-old-students Kronfellner/Peschek show a concept with two regression lines. With the slopes of these two lines, r^2 can easily be calculated. With a computer algebra system (CAS) like DERIVE (TI-92) you can realize this concept even with 14-year-old pupils, if they have experiences with the handling of the data-matrix-editor and the graphic window of the TI-92.

4. Two linear regression lines – Referendum Temelin

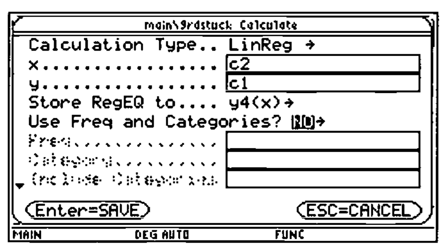
The citizens of Austria and Germany are very afraid of radioactive radiation. In Temelin (CR) an atomic reactor has been built near the borders in the last years and the test phase began. In January 2002 an Austrian party started the referendum "Veto against Temelin" to stop the operation. In the results published in the newspapers on January 22, 2002, the tendency the nearer to Temelin, the higher the percentage of people who went to sign the referendum, was remarked by the students of form 11. They suggested to check this. They took the data published in the newspaper and found the distance to Temelin with the help of the software program geothek which they often used in their geography lessons. The distance they put into the table was always the distance from Temelin to the county capital town.

Counties	B-land	Vienna	U. Austria	L. Austria	Salzburg	Tirol	Carinthia	V-berg	Styria
main towns	E-stadt	Vienna	Linz	St.Pölten	Salzburg	I-bruck	Kla-furt	Bregenz	Graz
percentage	14.8	15.4	23.5	16.9	13.4	8.7	15.5	6.7	12.0
distance	203	170	74	116	169	295	254	381	223

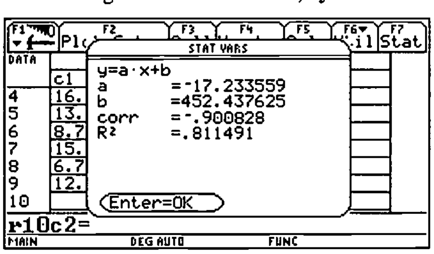
The data were entered as columns c1 (percentage), c2 (distance in km) in the data-matrix editor.



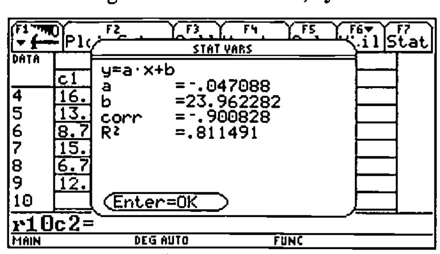
Linear regression with **x = c1, y = c2**



Linear regression with **x = c2, y = c1**



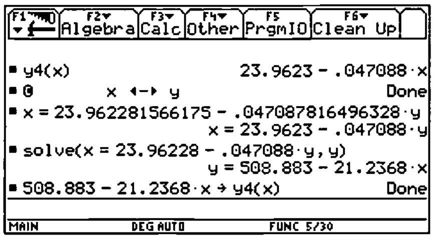
1. regression line has the slope **a = -17.233559**



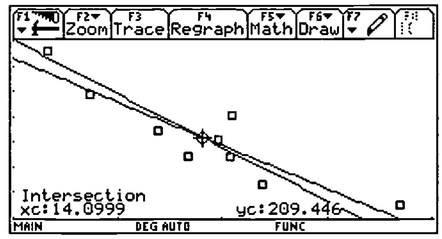
y4(x): slope **a = -0.047088**

Coefficient of determination $\mathbf{r^2 = (-17.233559)\cdot(-0.047088) = 0.811491}$, $r = \sqrt{(0.811491)} = \mathbf{0.90083}$

To find the right **2. regression line** you now have to take the **inverse linear function of y4(x)**.



2. regression line: y = -21.2368 x + 508.883



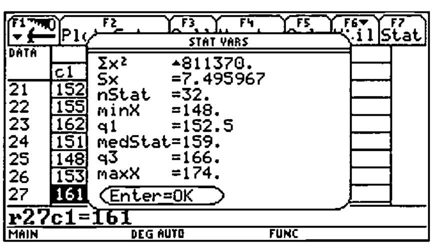
Intersection point: [mean of x, mean of y]

The coordinates of the intersection point of the two linear regression lines are the mean of x 14.1% and the mean of y 209.5 km. The angle between the two lines is small and the correlation coefficient r 0.90. From this follows that the inverse proportion is very strong.

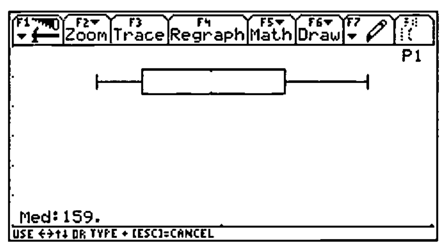
5. Relationship between length and mass of a person

It is very important not to take sets of data out of the school books only. I make my students of grade 8 work with their own data (length and mass). Looking at their own data allows the students to get a better understanding of the concepts of statistics, especially of linear regression and correlation.

The concepts mean, median and histogram have always been well known in Austria, but the concept of the quartiles has not. The quartiles q1, q2 and q3 divide the ordered set of data into four equal parts. The quartiles and the boxplot were contained in the new curriculum of form 7 in 1987 for the first time. In the commentary to the curriculum the chairman, H. Bürger, explained these concepts taking the data of population and area from the European countries 1988. In 1995/96 the TI-92 came on the market in Austria and for the first time a calculator could calculate q1, q2, q3 and could make a boxplot. In the boxplot you can see the value of minX, q1, medStat, q3 and maxX with the help of the cursor if you use the TRACE-Mode.

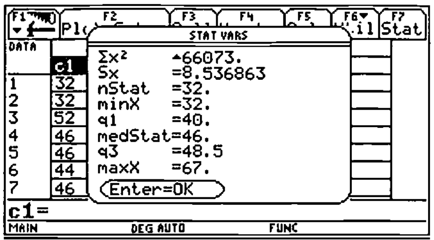


all data for the boxplot are calculated and shown

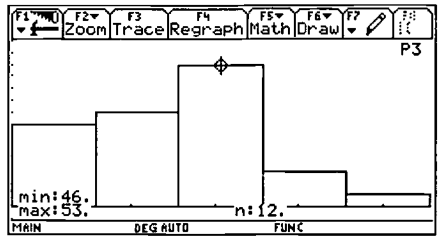


Boxplot of the length

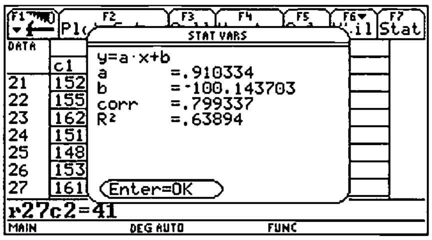
You can find all the important statistical data of the mass of the pupils with the command ShowStat on the screen of the TI-92. You can also very quickly make a histogram with the TI-92 and study it in the TRACE-Mode.



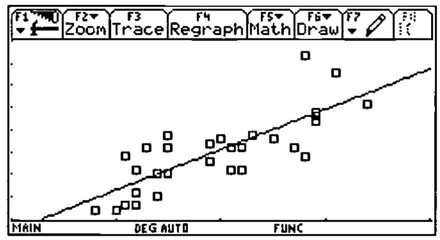
Calculation results of the mass



Histogram of the mass



correlation coefficient r = 0.799337

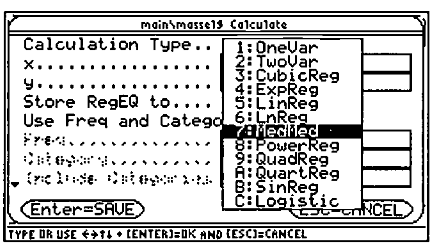


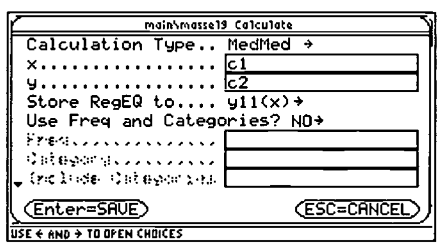
Points with the linear regression line

You can make a scatter plot with the TI-92 (x = length, y = mass) and find the equation of the linear regression line. The correlation coefficient is r 0.80. If r 0.8, you can rightly say that the mass becomes greater if the pupil is taller. But there are some exceptions, which have a great influence on the position of the regression line.

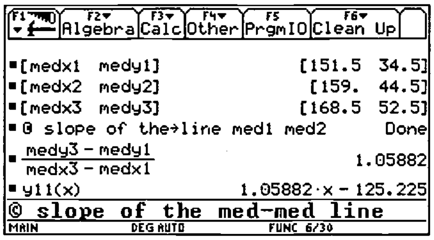
6. Med-Med Line – a new regression line

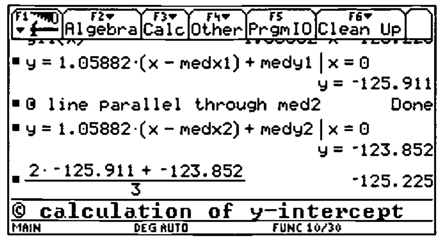
The TI-92 offers the possibility to choose another type to fit the points with the **med-med regression**. It is a linear regression, which is not so sensitive against run away data as the linear regression. The equation of the med-med line can easily be derived with the knowledge of the linear function only.



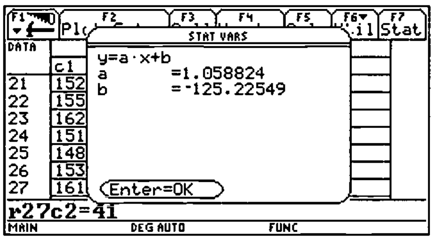


In the TI-92 guide book (1995) the calculation type MedMed is described in the following way:
"Median-Median - Fits the data to the model ***y = a . x + b*** *(where a is the slope, and b is the y-intercept) using the median-median line, which is part of the resistant line technique."*
The ascended ordered set is divided into three subsets which have an equal number of elements (if possible). In each of the three subsets the median is taken. In any case it is possible to control them. The TI-92 calculates and stores them, but they are not displayed on the screen. You have to call them: [medx1, medy1] [medx2, medy2] [medx3, medy3]
Afterwards the slope of the med-med line can be calculated with the points med1 and med3, the y-intercept with the y-intercept of med1 med2 and the y-intercept of the parallel line through med2.

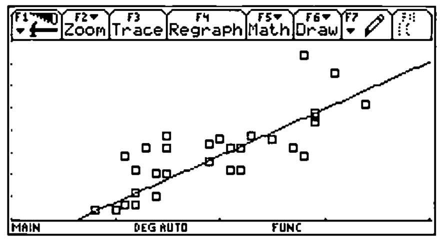




The med-med line is not only easier to calculate, it also fits the data better, which can be seen in the next figure.



slope a and y-intercept of the med-med line

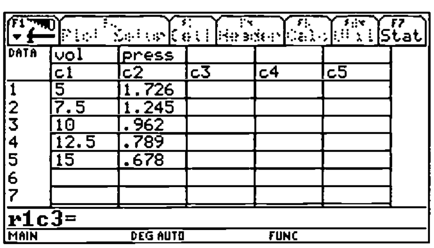


points with the med-med line

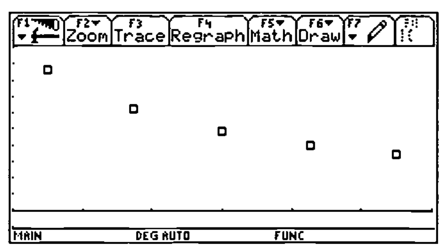
If you compare the med-med line with the linear regression line you see that the line fits the points better and is not so sensitive against the run away data.

7. Evaluation of Physical Experiments – The Law of Boyle

In many grammar schools a new subject, named **science-lab**, has been created in grade 8. In this subject the pupils make experiments in little groups. One new possibility is to work with the TI-92 and a **Calculator Based Laboratory (CBL)** which allows to collect data during physical and chemical experiments (B.&A. Aspetsberger, 2001). E.g. the students can discover Boyle's law, they can find the relationship between pressure and volume of a confirmed gas. In the subject science lab, the pupils start collecting data by varying the volume of the gas and simultaneously measuring the pressure. Both, volume and pressure, are stored to a data matrix of the TI-92: volume V in the column c1 and pressure p in c2. In mathematics the pupils define **a scatter plot** and visualize the data in a graphical window.

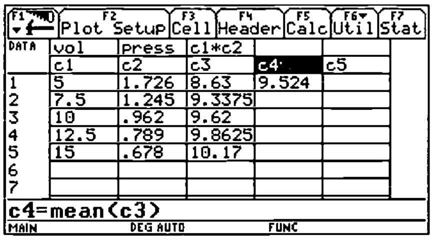


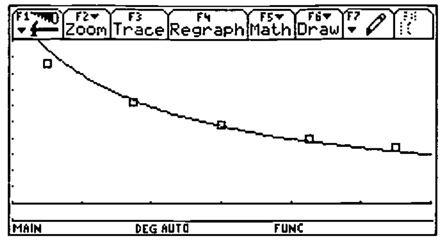
the data for volume and pressure



a scatter plot of the points [v, p]

Now they try to find a good regression type. The pupils can find out that the volume in c1 is increasing f the pressure in c2 is decreasing and they notice that the points are not lying on a straight line. The pupils of this age have the advantage that they know only one type which is not linear and in inverse proportion. It is the type **x . y = a** (=constant). They have sometimes solved such problems before, e.g. they had to find the time for different speeds if the distance was constant (e.g. 60 km). Therefore they try to multiply the volume with the pressure. The product in c3 is nearly constant. Which value is to be assumed for a? They try a plot with a = 9.524 (=mean(c3)).

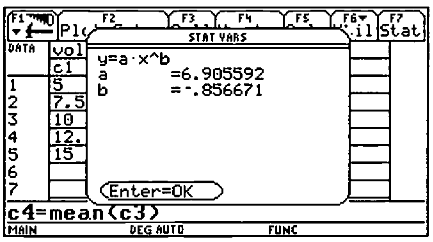


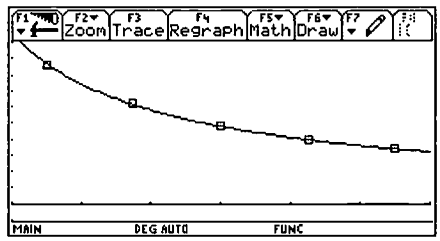


c3 = c1 ⌡c2 and mean(c3) = 9.524

the graph of x ⌡y = 9.524 fits the points

B. & K. Aspetsberger (2001) have worked with students of 17 to 18. They made experiments in chemistry and physics in their science courses. The students were not content with the results because the data points for small volumes deviated from the graph. They tried next to calculate with the type **power regression**. The graph fitted a little better, but the exponent with -0,86 deviated too much from –1.

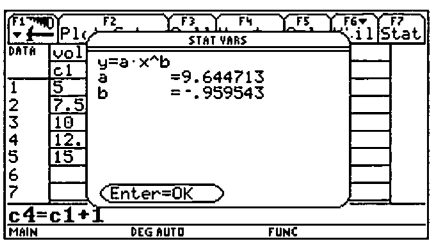


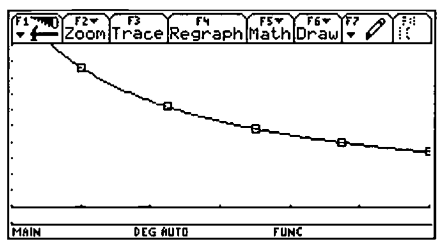


the exponent b 0.856671

the graph fits better the points

The students controlled the experiment stepwise and found reasons to add 1 to c1. Now the calculation type **power regression** fitted the data points quite well.





the exponent b 0.96 is near -1

the graph fits the points quite well

8. Statistics and Computer – a Chance for Teacher Training

Probability & statistics are a compulsory part in the final grades of all schools in Austria which prepare students for university level. Up to 2001/02, **Computer science** has been a compulsory subject for all pupils in grade 9 only. The Ministry of Education now plans to make computer science a **compulsory subject for all pupils of grade 5 in autumn 2002**. In the other grades the pupils can choose computer science as a voluntary subject. The teacher students at university have a very different knowledge in working with a computer and they rarely have experience how to teach statistics with the help of a computer at school. Therefore many teacher students find it very difficult to plan such a statistics lesson.

The use of computer packages at school results in **more independent productive pupils activity**. Individual pupil activity has nearly erased pupils calculating on the blackboard. **Computer lessons imply less class teaching and more partner and individual work as well as less note taking and more production**. On the other hand a lot of time is devoted to the pupils to enable them to work with the computer and the program (Nocker, 1996). But the use of computer packages is a great temptation for the pupils, because such functions and programs can quickly be used as a blind tool. (Wurnig, 2001)

In one of my lectures at university, I work with the teacher students in a computer lab with statistics software and try to develop concepts of the school curriculum with them. In this lecture they do not only have to learn how to use the computer in the right way, but, in addition, they also have to learn that **a mathematical concept has different levels of precision** (R. Fischer, 1985). They have to experience personally how computer algebra systems change their learning. H. Heugl (1997) states **three stages in learning mathematics** if students use symbolic computation systems in the classroom: the heuristic stage, the exact stage, the application stage. He points out that the experimental or heuristic phase often does not exist in the traditional mathematics education.

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AN ANALYSIS OF A WEB FORUM IN DISTANCE AND FACE TO FACE TEACHING OF A FIRST YEAR MATHEMATICS SUBJECT

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ABSTRACT

Almost all of the subjects taught at Charles Sturt University (CSU) are supported with electronic communications. The electronic communication facilities provide students with communication tools such as direct e-mail to the subject lecturer and a subject web forum to enhance student-student-lecturer communication and, hence, learning.

In this paper, we discuss the benefits of a subject web forum for both external and internal students. In addition we present the results from a questionnaire designed to discover the perceptions of students regarding their experience with a first year mathematics web forum. The usage statistics of the subject web forum are also presented. Web forums provide the opportunity to establish a frequently asked questions (FAQ) database for first year mathematics subjects. Questions posted by students over recent years can be examined and accumulated into a FAQ database.

We will also discuss the infrastructure and human resources needed to develop such a subject web forum in Atilim University, Turkey and to make this forum available to other universities through Turkish Higher Education Council (YOK). In the Turkish distance education system, subjects (including mathematics) have enrolment numbers in the order of thousands. The web forum discussed in this paper may be a cost effective and enhanced alternative for deliver of subjects with large number of enrolments in distance mode.

1. Introduction

In any teaching environment it is essential for us as educators to be constantly aware of the need to match what we are teaching with whom we are teaching. As academics we must recognise the need to vary our approach and style as our learners at tertiary institutions could be school leavers, mature age students, on campus or distance education students. For example, the nature of teaching a course that is offered purely by distance education requires alternate approaches to fulfil the basics of face-to-face teaching such as student/instructor interaction. Distance education technologies are expanding at rapid rate to make distance education a viable option for many tertiary institutions. The Internet has proved to be a valuable tool for enhancing the effectiveness of distance education.

Almost all of the subjects taught at Charles Sturt University (CSU), Australia are supported with electronic communications. The electronic communication facilities provide students with communication tools such as direct e-mail to the subject lecturer and a subject web forum to enhance student-student-lecturer communication and, hence, learning. A subject web forum is implemented to support external and internal students as well as students from partner institutions of CSU. Most of these partner institutions are from overseas countries such as Malaysia, Canada and England. All students enrolled in a subject delivered in distance mode receive some materials that explain how to use on-line facilities of the subject as well as other on-line facilities available at CSU. It should be pointed out here that a web forum for a subject is an additional facility to the printed material supplied to every student.

The issue of equity is one of the hurdles on-line teaching needs to overcome since not all students have on-line capabilities. Students cannot be required to access the web and use its resources unless the requirement is a university policy. Starting 2002 the communication between the CSU administration and all students enrolled in distance mode is in electronic form. Hence, every student enrolled in CSU courses is expected to have access to an electronic communication medium. For the equity principal, CSU provides dial-up modem facilities to facilitate access to the University network.

The web forums allow for open discussion, at the convenience of the students. The subject web forum is available to all students (distance and internal) enrolled in the subject and the subject lecturer to enhance their learning/teaching. The advantage of employing a subject web forum to enhance traditional distance education is that it alleviates some of the problems encountered by distance education students that internal students do not normally face (Meyenn et.al., 1996). Some of these problems are:

- absence of face to face contact;
- isolation of students;
- reluctance of students to contact instructors;
- feeling of not belonging or being part of the university;
- absence of collegiate atmosphere;
- late delivery of mail packages;
- inadequate feedback from instructors.

A carefully designed and managed web forum provides an effective tool to eliminate or alleviate some of the above problems. For example, students can access a web forum to generate a collegiate atmosphere with their fellow students. In fact, the most popular and most used feature of web sites is the communicating section for a number of subjects (Wood, 1998). In a typical subject such as a first year Mathematics subject, a lecturer at CSU may spend about two hours per week communications with his students via various on-line tools, mostly on the subject web forum. This is comparable with a tutorial component of a typical face to face teaching of a subject.

This paper is organized as follows: In section 2 we present survey results from distance as well as internal students to reflect their perceptions regarding their experience with a first year mathematics web forum. In section 3 we discuss some technical difficulties in running a mathematics web forum as well as the opportunities presented by a web forum such as establishing a frequently asked questions (FAQ) database for first year mathematics subjects. The usage statistics of the first year mathematics subject web forum will also be presented. In section 4 we discuss the infrastructure and human resources needed to develop such a subject web forum in Atilim University, Turkey and to make this forum available to other universities through Turkish Higher Education Council (YOK).

2. Web Forum: Student Perception

We have been interested in using technology/Internet in teaching our mathematics subjects at CSU since 1996. We first had the opportunity to participate in a multi-variable calculus subject in the distance education mode offered by the Harvard Extension School. This subject was completely taught with the software package Mathematica (Mathematica, 2002) based on the Calculus and Mathematica (C&M) notebooks (See the reference: Calculus and Mathematica, 2002). Jerry Uhl and Horacio Porta originally started the C&M program at University of Illinois, USA. The C & M notebooks are designed to make students think about mathematics in terms of objects that they can see, and operations upon those objects. Visualisation is also an important part of C & M notebooks. Our experience from this project was presented at the SUTMEG conference (Altas et.al., 1996).

Following this experiment we introduced the software package Maple as a symbolic calculation package in face-to-face teaching of the first year mathematics subjects in 1997 (Maple, 2002). The main reason on the choice of the software package Maple was its intuitive syntax and the familiarity of the teaching staff with Maple from their research projects. We believe that using a symbolic package in teaching has had a positive impact on student learning. A student survey revealed that 85% percent of students enjoyed the Maple component of the subject.

As educators we need to develop techniques to balance distance teaching and face-to-face teaching methods. In the past, teaching mathematics by both internal and external modes has caused problems in the presentation of dynamic mathematical concepts, to both groups of learners. The new computing technologies such as hypermedia and www, in conjunction with the use of self-instruction learning methods (for example, C&M mentioned above) and mathematical computer software in mathematics teaching alleviated any imbalance with dynamic teaching methods in both modes of teaching. A subject forum is a powerful and easy use interface to bring together such teaching tools.

In 2001 there were about 46 internal and 35 distance education students studying the first semester calculus subject. These students were mainly from the information technology and science degree courses. A student survey was undertaken at the end of the semester to understand the students' perception of the subject forum. A majority of distance education students, about 86%, used the subject forum and found it useful in their learning. However, the ratio of internal students not using the subject forum was about 54%. This was considerably higher than we expected. Further analysis of this group of internal students revealed that more than half of them, about 64%, did not need to use the subject web forum as an extra tool to enhance their learning. They found Maple laboratory sessions; tutorials and lectures were sufficient for their learning. Regarding the remaining 36% of this group of internal students we believe that two factors contributed to not using the subject forum and hence, not to make use of another learning tool to enhance their learning:

i) Some of them may not be comfortable using computers, especially environmental science degree students;

ii) Although on-campus students were provided with the distance education learning materials, in which the subject forums were explained, they may have had problems adjusting to the new environment in their first semester.

We will be closely monitoring this issue during the next teaching session in 2002.

In the survey, students were also asked to state an aspect of the subject forum they found most beneficial. The majority of the answers can be grouped under seven headings with their respective percentages as in the following table 2.1

Answers	Percentage
Group interactions and contacts	26%
Help provided to complete my assessments	19%
Ability to discuss problems with others	17%
Read questions/answers posted by others	16%
Motivation	6%
Seeing on the forum that other people are having problems as well	6%
Others	10%

Table 2.1 Some beneficial features of the subject forum identified by the students

3. Techniques, Usage Statistics and Opportunities

The e-learning environment at CSU has been developed in-house and is implemented as a standard template for all subjects offered by the University. This standardisation has significant advantages in that all students become familiar with the CSU e-learning environment and this basic environment does not change between subjects. Common features available within the e-learning environment at CSU are; subject outline & assessment information, access to on-line teaching resources, including electronic print materials, on-line submission of assignments, e-mail access to the teaching staff and other students as well as an on-line forum. While most of these features are also available in commercial e-learning packages such as Blackboard and WebCT, CSU has chosen to develop its own in-house e-learning software environment. Major reasons behind the decision to develop in-house software were the large scale of the distance education operation at CSU together with the advantages of easier internal customisation and integration with other CSU student record systems.

One of the disadvantages of this common learning environment is that the development of e-learning tools such as the subject forums has initially catered only for simple text based discussion. Clearly this environment severely limits the scope for technical discussions that require mathematical notation or diagrams. Subsequent forum development now allows for attachments to be included with the forum text message, however effective technical communication is still limited.

While attachments open the possibility for staff and students to exchange technical information in various commercial file formats such as Microsoft Word & Equation Editor, LaTeX, Maple7 and Mathematica, the cost and non-adoption of these packages by distance students pose severe problems with distributing mathematical information in these formats. After trialing many alternatives one of the most effective formats for distributing technical information to students was found to be the Adobe Portable Document Format (PDF). The PDF reader is free and readily available to students over the web and output from almost all computer packages, including those listed above, can be captured through the Microsoft Windows PDF printer driver that comes bundled with Adobe Acrobat. In addition, handwritten mathematics can be scanned and saved in either PDF format or distributed as a GIF file that can be viewed by almost any web browser.

The option of purchasing a cheap scanner, less than AU$100, and exchanging handwritten mathematics as scanned graphics files is also recommended to our students. This solution opens the opportunity for students to submit simple handwritten mathematical enquires and for the lecturer or student respondent to print the correspondence, add their own handwritten reply, scan and return. However to date most distance students have elected to simply pose their mathematical questions and replies in a simple Maple-like mark-up style such as, int(x^2 + sqrt(x) , x). The use of scanned handwritten mathematics does however considerably ease the time constraints on lecturing staff who often need to respond to numerous student forum enquires.

Over recent years there has been a significant growth in the use of subject forums by CSU students. The number of posts to forums increased from 1,000 in March 1998 to over 17,000 in March 2001, a 1,600% increase. It is interesting to note however that many more students visit and view the forums without posting, the ratio of views to posts being 20:1. This behaviour appears to parallel the experience in on-campus lectures and tutorials, where passive students are reluctant to pose active questions but are content to simply listen to the discussion going on about them. One of the major challenges facing forum administrators is to try and generate a forum environment that encourages more forum members to become active participants.

A significant opportunity associated with the growing use of subject forums is the possibility of establishing a database of frequently asked questions. As in traditional tutorial classes, there are many student questions that re-occur on a frequent basis. The electronic nature of forum enquires and associated responses opens the possibility of easily collecting this information into an FAQ database. The establishment of this database should not only ease the forum response demands on academic staff, but should also help provide more timely assistance to students with commonly occurring help requests. Analysis of the nature of specific questions entering the FAQ database should also provide valuable feed back to teaching staff on the effectiveness of the teaching program and curricula.

4. A Framework for Atilim University, Turkey

Atilim University, a private institution founded by Atilim foundation (1997), is located in Ankara, Turkey. Current enrolment is about 1100 in the faculties of Engineering, Arts and Science, and Management.

The mathematics department of the Faculty of Arts and Science offers mathematics subjects to all three. Among these calculus subjects is a 2-semester course offered to about 250 students of the Faculty of Engineering. This course has 6 contact hours/week. Four hours are allocated to the theory and the remaining two for the tutorials and computer laboratory work.

During the laboratory session students get a chance to familiarise themselves with a symbolic mathematics tool such as Mathematica, and perform predesignated calculus projects under the supervision of the course instructors. This enhances the student's ability to comprehend the theory discussed in the lectures and gives them a chance to verify their solutions to problems.

Atilim University has plans to enhance education by incorporating web technology into the curriculum starting with calculus and computer literacy courses. Obviously, in calculus subjects student-lecturer interaction and access to the lecturer outside the lecture hours are crucial.

The concept of Virtual Classrooms and Virtual Office Hours (VOH) via the Internet can be easily established by utilising an asynchronous tool such as e-mail or discussion lists. Whenever possible, voice can be integrated into VOH using for example, Microsoft's NetMeeting tool (See the reference: VOH) for synchronous communication.

An alternative option that will be taken by Atilim University is to set up web-based calculus subject forum to complement face-to-face education by taking the CSU model as a basis. This will incorporate the following features:

- on-line course material (lecture notes, problem sets, solution sets, etc.)
- on-line exams
- access to Mathematica
- e-mail

The subject web forum will offer the students the possibility of formulating and submitting their questions into the forum and will involve the classmates and the instructor.

To achieve this objective Atilim needs a new technology centre, and faculty training. Faculty training should start at an earlier stage of the project. An information technology support officer has been already employed for this purpose. Instructor involvement in the design phase is crucial to the over all success of the on-line learning program. We plan the involvement of instructors in the design phase of our web-based learning project. Some of the required prerequisites for a successful execution of the program are listed below:

1. The establishment of a Centre for Educational Technology (CET);
    - to take an initiative in improving the existing infrastructure for web based distance education;
    - to inquire, manage, and update the technology required;
    - to train supporting staff (facilitators/assistants);
    - to help in the design of web material/documentation;
    - to allocate resources .
2. To design new diploma programs in the existing vocational school to support CET to train/educate
    - web masters, web designers/developers;
    - technical staff for telecommunications/networking.
3. In addition to the existing computing facility, new computer multi-media laboratories for easy access to facilities provided by the web courses from within the campus.
4. Appointment of instructional designers to help lecturers in the preparation of distance education materials.

We believe that having a database formed with frequently asked questions based on the questions raised in a maths subject web forum will be a useful tool in delivering the subjects to classes with a large number of students. This approach may be a cost effective and enhanced alternative to the delivery of subjects with large number of enrolments in distance mode. However, a distributed farm of web servers should have been established in several cities to handle large number of student accesses.

5. Conclusions

We believe that web forums provide an opportunity to establish a frequently asked questions (FAQ) database for first year mathematics subjects. This will be a valuable resource for students to enhance their learning and for lecturers to deliver their subjects efficiently.

Our analysis indicates that majority of distance education students are benefiting from web forums in a number of ways such as; group interaction with their fellow students and reading questions/answers posted with other students. This is also supported by the usage statistics of web forums. The number of posts to forums increased from 1,000 in March 1998 to over 17,000 in March 2001, a 1,600% increase at CSU. It is also interesting to note that many more students visit and view the forums without posting, the ratio of views to posts being 20:1.

Initial analysis with Atilim University indicates that using web forums may help developing countries to deliver their teaching efficiently and effectively, especially for subjects with large number of enrolments.

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A PROJECT IN EUCLIDEAN GEOMETRY

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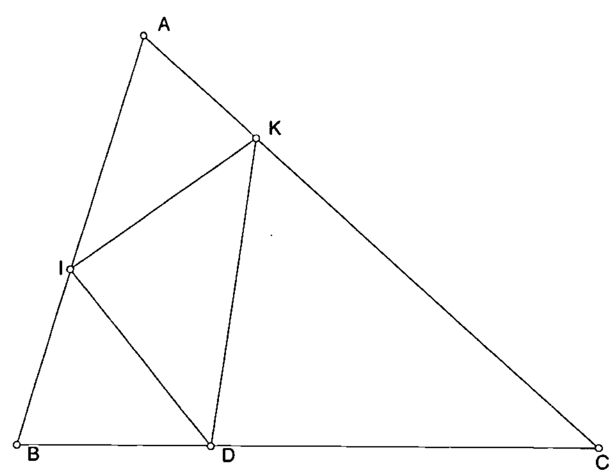
ABSTRACT

One of the most effective instructional approaches in teaching Mathematics is project work, which, in an earlier paper, I connected with active learning, see Klaoudatos (1998). And this approach is going to be more interesting for the students if the project has been developed in collaboration with them. At the same time, these kinds of projects include 'dangers' for the teacher because of unexpected demands that might be found within. In this paper, I will describe such a project, which had been created in a problem solving class during the year 2000-2001 first semester.

Through successive generalizations of a simple geometric task, the students developed the following problem: *'In an ABC triangle, D is a point on BC from which we construct segments that form equal angles at the sides AB, AC, at the points I, K respectively. Which is the position of D so that the length of IK will be minimum?'*, see figure. The problem is expressed in terms of classical Euclidean geometry so that, at first glance, there is no evidence of the hidden difficulties.

The aim of the paper is twofold: first, to attract the attention of mathematics teachers and educators to the problem above, in order to be considered for project work, especially in teacher training courses. Second, to search for conditions under which the project could be effective.

The presentation will consist of four parts. In the introduction I will give some necessary information about the problem, while the second part consists of the solution of the problem. In the third part I will describe, shortly, the general theoretical framework in which the project took place. In the fourth part I will describe the way the project was conducted and some observations that emerged from the implementation phase. The same part contains tentative conclusions, as the research is still in process, that give hints on the factors that affected the successful accomplishment of the project. The research showed that the main factors were the time duration of the project, the previous experiences of the students in research projects and the belief systems of the students about mathematics.



1. Introduction

The project aroused from the following simple problem: In a right angle triangle ABC (A=90°) the point D is moving on BC. From D we construct segments perpendicular to the sides AB, AC at the points I, K respectively. Find the position of D in which IK has the minimum length.

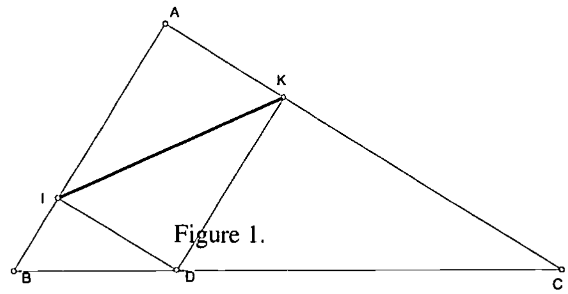


Figure 1.

The position of D we are asking for can be found when AD becomes the height of the triangle from A. The same point is the solution of the generalization of the problem in every triangle, see Honsberger (1996, p. 43-45). At that moment I asked the students to form a new generalization, while not having any specific idea in my mind. Then, after a heated discussion, the students developed the following proposition: *'In an ABC triangle, D is a point on BC from which we construct segments that form equal angles, ω, at the sides AB, AC, at the points I, K respectively. Which is the position of D so that the length of IK will be minimum?'*

The problem, which I have not yet found in the bibliography, is stated in terms of classical Euclidean geometry, it is the result of the generalization of the two previous problems and for this reason is not easy to recognize the difficulties 'hidden' in it. Indeed, it demands very much experience in this kind of problem for someone to suspect that the solution requires most of the developments that signified the revival of modern Euclidean geometry after 1873, see Honsberger (1995, p.88). I focused my attention to this area only when I noticed in the computer screen that the segments IK, as D moves on BC, seem to move on a parabola, which, also, seems to be their envelope, see figure 2.

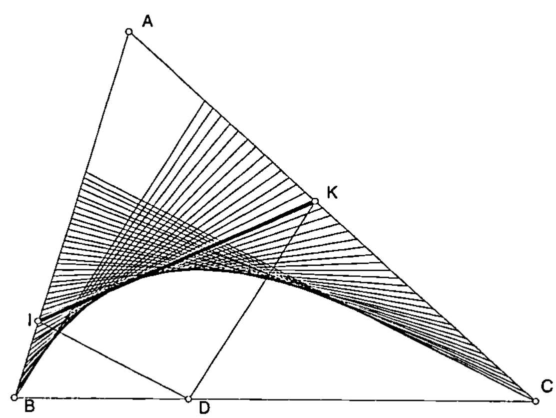


Figure 2.

One of the most powerful heuristics is 'search for relevant bibliography'. In F.G.-M (1952, vol.2, p.540, Greek edition) there is the following proposition: *'In a triangle ABC, every line IK which divides the sides AB, AC in segments inversely proportional from the vertex A, is tangent to a parabola, which is also tangent to the sides AB, AC at B and C'*. There is not any given proof, but there is reference in two papers, Brocard (1885) and Longchamps (1890), that I do not have up to now. On the other hand, Bullard (1935, 1937), gives the way in which we can construct the focus and directrix of this special parabola, see also Honsberger (1978, p.236-242).

According to this information, I developed the following plan:

1. I will prove that there is always a triangle in which IK segments divide the two sides in segments inversely proportional.
2. In this triangle, the IK segments have the parabola as an envelope.
3. I will search for the connection of these questions to that position of D in which IK has the minimum length.

The plan is divided into three cases, of which the first is the main one. From now on, ω will signify the angles which have been constructed from D to the sides AB, AC. Moreover, the solution is somehow condensed due to limited space, leaving the reader to clarify a few minor issues.

2. The solution of the problem

2.1: 1st case: ω=A.

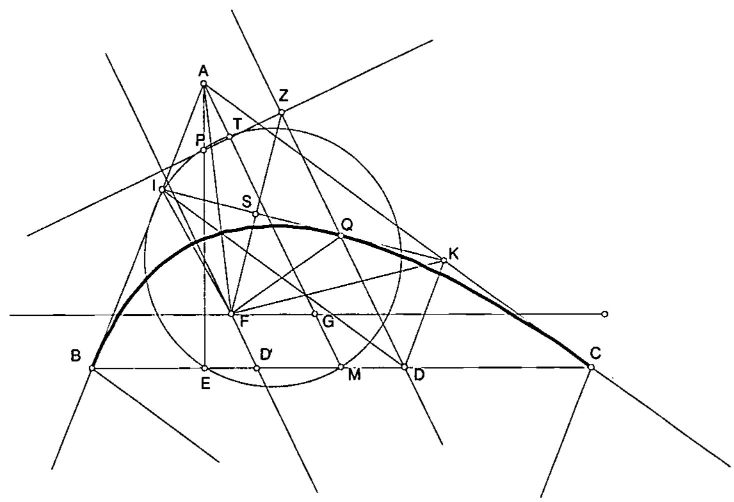


Figure 3.

In figure 3, following Bullard, PT is the directrix of the parabola, where AE is the height, AM the median and the circle is the Euler circle, of the triangle ABC. Then the focus F is the intersection of the symmedian AF and the line FG parallel to BC so that MG=AT.

I constructed IK as perpendicular bisectors of the segments FZ, Z any point on the directrix, and it is easy to realize that every Z corresponds to a point D through the ZD perpendicular to PT, which is also parallel to AM. In figure 4, Q is the point of intersection of ZD and parabola, and it belongs to IK. This point is the only common point of IK and of the parabola, so that IK is tangent to it as the point Z is moving on PT or, as the point D is moving on BC. But the point D can also be determined as the intersection of the parallel lines to the sides of the triangle from I and K.

So, the following two propositions are equivalent:

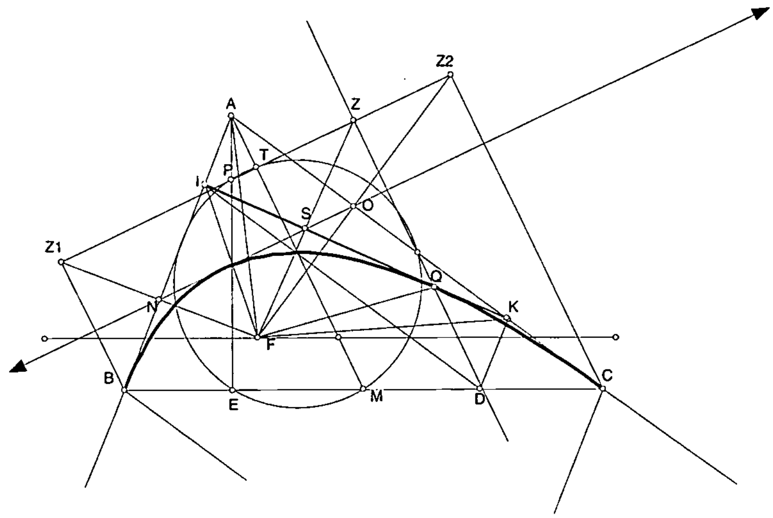


Figure 4.

1. The perpendicular bisectors IK of ZF, are tangents to parabola, which is the envelope of these segments.

2. The IK segments divide the sides AB, AC in segments inversely proportional, namely $\frac{BI}{IA}=\frac{AK}{KC}=\frac{BD}{DC}$.

Moreover, in figure 4, the points Z1, Z2, are the reflections of F through AB, AC respectively, so the points N, O are the midpoints of FZ1 and FZ2. Then, the segment NO is one of the positions of IK. On the other hand, NO is the Simson line of the triangle AIK and for this reason F is a point on the circumcircle of this triangle. As a result, the angles of IFK triangle are constant and then, as the Z moves, the triangle IFK remains similar to itself. From the similarity of the IFK and NFO triangles we have the proportion $\frac{IK}{NO}=\frac{IF}{FN}\geq 1$. So, the minimum length of IK happens when IF=FN, in other words, when D coincides to D′, which is the intersection of the axis of the parabola to BC, see figure 3. In the following I will regard D as 'the position of minimum length'.

2.2: 2nd case: $\frac{A}{2}\leq\omega\leq A$

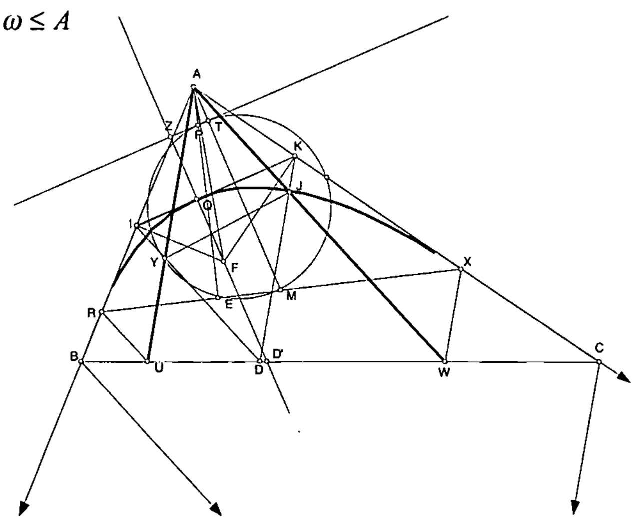


Figure 5.

In figure 5, angle BAW = angle UAC = ω.Then the proportion $\frac{UY}{YA}=\frac{AJ}{JW}$ is transferred to $\frac{RI}{IA}=\frac{AK}{KX}$. Thinking by analogy to the first case, in triangle ARX, the position of minimum length is D≠D′, which is determined as the point of intersection of the parallels to AW, AU from I, K.

2.3: 3rd case: ω> A.

Working in the same way it is easy to recognize that again D≠D′, as can be sown in figure 6, where the triangles UAW and RAX are constructed exactly in the same way as in the previous case.

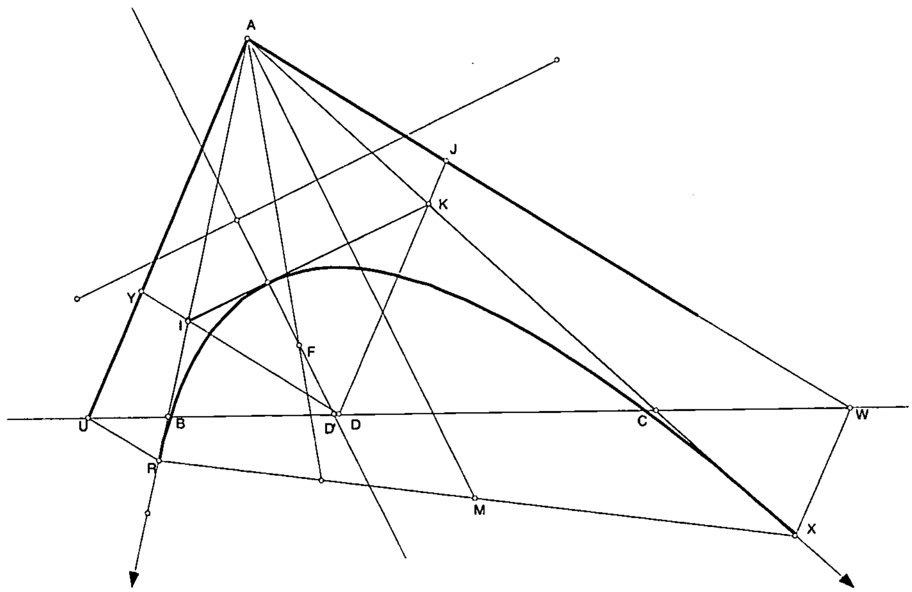


Figure 6.

3. Some theoretical notions

One of the most effective ways to teach Mathematics, in my view, are problem solving, see Klaoudatos (1994), Klaoudatos and Papastavridis (2002). Through this kind of teaching the students could develop the feeling of personal contribution to their learning and at the same time they could acquire the 'inner freedom to act', Sierpinska (1995, p.8), which are both components of active learning, see Anthony (1996).

A necessary condition for problem solving to be successful as a way of teaching is 'to get involved in problems', and for this reason I consider project work as an effective teaching strategy, Klaoudatos (1998). But how can we form an appropriate classroom environment to promote such strategies? Actually, this question refers to the 'situation context' which together with the 'task context' are the two kinds of context that Wedege (1999) has discriminated.

More specifically, I developed in a problem-solving classroom, an experimental teaching environment through which the students, working in groups, could develop a general approach for problem solving together with the understanding of specific mathematical ideas. The development of such an environment was influenced by the ideas of Polya, which I have condensed to the following steps, Polya (1973, see in particular the first three chapters):

1. Experimentation and observation
2. Pattern recognition

3. The development of a guess
4. Testing the guess

It is expected that students, working through the steps above, will be led to the acts of understanding, see Sierpinska (1990, 1994 p.56). The acts of understanding describe the crucial moments in the process of concept formation and are the following four: *identification, discrimination, generalization and synthesis*. And, although Sierpinska connects these acts with the notion of epistemological obstacle, in the case of the specific problem of the paper I use them only to give a short description of the students' thinking and reasoning. This theoretical framework is interesting because it is in close connection to problem solving and offers some tools for the evaluation of this process.

The problem of the paper was developed in the first semester of 2000, in a problem solving class of undergraduate students, where I had proposed it as a project, not knowing the solution and not having any idea of the difficulties hidden within. The students accepted it at once and were enthusiastic because of their contribution in its development. Thus, I considered it as an ideal teaching situation in which the teacher would act 'as learner', as constructivists claim. But soon I realized that, working in the broader scheme of 'teaching mathematics through project work', I could not help the students because I did not know what mathematics were involved and which ideas I had to organize and put forward. The result of the endeavor was the abandonment of the project after three weeks of students' work, partly due to students'frustration and partly because of the limited available time and the students' other obligations. The situation reminded me of Brousseau's observation that is, in words of Sierpinska (2000), *'if the teacher finds it useful to act as if she did not know how to solve the problem, this should only be good acting and not the actual state of the teacher's mind'*, Brousseau (1997, p.45-47).

4. Observations and tentative conclusions of the implementation phase

Only when I solved the problem I felt confident to propose it again as project work. The aim of the project was to search for the conditions under which the project could be effective. Indeed, in the first semester of the next year 2001, I proposed it to the post-graduate students of my problem solving class of Mathematics Education Section. Roughly speaking, the 17 students of the class could be divided into two groups: those who had the required knowledge or, at least, part of it, who were experienced secondary teachers, who I will call teachers, and those who had just finished their graduate studies and did not have that knowledge, who I will call young students.

The time duration of the project was one month, in which the students formed four groups that had to submit a weekly progress report. These reports were discussed in the classroom every week, where all the students made comments and discriminated between potential and non-potential ideas and strategies. I gave the minimum relevant bibliography only after the first week, allowing them to work on their own ideas for a week. By minimum bibliography, I mean, the F.G-M proposition and Bullard's two papers to start their research. At the same time I warned the students that they had to learn some new material, in sources that they had to find by themselves. It was agreed that this project would be included in the evaluation of the course. Finally, before I proposed the problem, the students had already solved the two previous problems, from which it was developed.

I proposed the problem, making a short presentation of it in the class, where I gave some information and used the figure 2, where there is an unusual combination of a triangle and a curve that seemed like a parabola.

The required knowledge consisted of the geometric properties of parabola, the Euler circle, the Simson line, the notions of symmedian and envelope and the triangles of Brocard. All the teachers were familiar with the parabola, the Euler circle and the Simson line, while some of them knew all the require knowledge. So, the main part of their job was to synthesize this knowledge in order to manage the project. But the young students encountered special difficulties because, not only did they have to find new sources and learn new material, but also they had to synthesize it.

In general, I can say that all the teachers, with the exception of two who did not proceeded in the fourth step of the Polya's scheme, solved the problem working in groups by using the Geometer's Sketchpad software in the first three steps. On the other hand all students had to overcome many difficulties throughout the task. In the following, I will concentrate on the young students.

Six of the seventeen students of the class were young students who participated in two groups, group A, which consisted of two teachers and two young students, and group B, which consisted of four young students and one teacher. The young students of group B showed an unexpected behavior: After the second week, they ignored the bibliography given to them and accepted a proposition made by the teacher of the group, to follow a different path towards the solution, which remained unfinished. A similar behavior showed by the other two young students of group A, who, although they solved the problem based mainly on the work of the other two members, they submitted their own 'solution', avoiding the given bibliography, which, also, remained unfinished.

The discussions in the classroom, the observations during the project work and the interviews that followed the end of the task, led to three main factors: The time duration of the project work, the previous experiences in research projects and the belief systems about mathematics. More specifically:

1.The time duration of the project affected the cognitive and meta-cognitive processes as well as the affective domain. The 'distance' that the young students had to cover in new knowledge seemed too long in a month. A student described the problem as *'A well without a bottom' because by the time we had completed a piece of new knowledge, a new area was opened.* For those who, typically, covered this distance, referring to the two young students of group A, there was another obstacle. They had to transfer the new knowledge to the actual problem. But the time for them was too short to make the *synthesis,* namely to connect appropriately the various pieces of new knowledge to the problem. So, while I could recognize the acts of understantig in the various steps of problem solving, in the final step the synthesis was absent. On the other hand, the heuristics did not work for them. A student said that *I tried to follow the heuristic: if you cannot solve the problem, try a simpler one. But soon I realized that I had already solved all the simpler problems before the project.* As a result, as time passed, the students' positive attitudes transformed into anxiety and then frustration.

2. The interviews uncover a rather unexpected fact. All the seventeen students insisted that it was the first time they faced a project with a 'sense' of genuine research work. Especially the young students had difficulties in finding the relevant bibliography and the way they could use it. Most of them had never heard of the F.G.-M book. A student said that *up to that time, I believed that all Euclidean geometry was included in the school textbook and there was nothing beyond that.*

3.The beliefs system about mathematics of the young students had an impact on their failure. One of the most striking points was the presence of the parabola in the triangle. A student said *I could not accept the parabola together with the triangle, because the parabola does not belong to Euclidean geometry, it belongs to analytic geometry.* I believe that this was one of the reasons, the young students of group B immediately accepted the proposition for another path towards the solution without using the parabola. On the other hand, their experiences in project work up to that time, created the belief that the solution of every problem should be based on familiar and specific knowledge and, in the case of given bibliography, the bibliography should contain all the necessary information for the problem. And, as a student said, *in every case we knew that, eventually, we would solve the problem within a few hours' work. But, then, after some time, we began to realize that, perhaps, we could not solve it.*

According to the above points, we can conclude that the young students were unprepared to handle this research project. This conclusion poses an important question: How can the teacher create an experimental environment in his classroom without having any experience in it? On the other hand, I stressed time as a decisive factor and not the process of the development of the required knowledge, because this knowledge is of elementary nature and I believe that it will be learned in a matter of time. It is another question, not of this paper, about the 'type of knowledge' that can be developed through this kind of work, as Sierpinska (1998, p. 58) proposes. At the same time, I recognized many of the results stated in international bibliography about the decisive role of the beliefs and affects in problem solving, which survive in tertiary education. See, for example, the review of the domain in Barkatsas and Hunting (1996), the compartmentalization perspective of mathematics and the dichotomy between 'theory and exercises', Schoenfeld (1992, p. 342), the presence of the parabola that acted as an epistemological obstacle, Sierpinska (1994, p. 125). In my opinion, the teacher has a limited influence on the factors above, except, perhaps, time. They mainly depend on the view of mathematics that an education system has adopted and the education praxis that arises from it. The research is still in progress and for this reason I have described the conclusions as tentative ones.

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TEACHING MATHEMATICS TO ENGINEERING STUDENTS WITH HAND-HELD TECHNOLOGY

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ABSTRACT

The École de technologie supérieure is an engineering school specializing in applied engineering and technology. Since 1999 fall semester, graphic calculators TI-92 Plus or TI-89 have been a compulsory purchase for every new student. This paper will give two examples of how we use these symbolic tools and will show how the calculator has changed the type of questions we ask our students. We will focus on Calculus and Differential Equations. We have been using computer algebra systems (like *Derive*) for 10 years now. But our students have to go to the computer labs if they want to use it. This is one of the major reasons for choosing hand-held technology like the TI-92 Plus. It can be used by the teacher and the student in order to teach and learn mathematics in a very original manner. One of our goals is to continue to teach "classical maths" with innovative approaches. This is possible if you can make use of technology in the classroom, when you need it, when you want it, without having to wait to go to the computer labs. And, when the teacher thinks that technology should not be used in some parts of an exam, students are not allowed to use the calculator! Finally, for some problems, hand-held technology can not compete with fast computers. There is no problem to switch from one to the other, when systems are close together.

1. Introduction

In this article, we will attempt to demonstrate, using two specific examples, that it is difficult to teach mathematics to future engineers if they don't have access to a symbolic calculator **at all time in the classroom**. We don't think it is absolutely impossible to efficiently teach mathematics without a symbolic calculator, and that such a calculator should be allowed during all exams. We only think that engineering students would better understand and appreciate different mathematical results if such a calculator is always available. Obviously, learning the use of the calculator is compulsory to the course. For this reason, it is essential for the system to be easy to manipulate by students and not be an hindrance for the teacher in order to realize the course syllabus. We must admit that TI-92 Plus and TI-89 are powerful calculators, they nevertheless remain user friendly. As for computer algebra systems, the *Derive* system is also a relatively easy tool to use and is among the current computer algebra systems most resembling TI symbolic calculators.

Another aspect we want to emphasize is the following: several teachers are still very reluctant in using this technology in their teaching, or simply don't realize the need. Indeed, they are most of the time very well prepared and competent teachers, and highly appreciated by their students. So why should they change their way of teaching and introduce this new technology? Our two examples will address this question. In fact, the use of technology not only allows answering more complex and general problems, but it also permits to do more mathematics. Let us be clear: the use of a symbolic portable system is more than scaffolding for weaker students! It's use can and must become a part of teaching and learning mathematics.

2. Taming a general result, graphically and symbolically

Our first example comes from our ODE course. Our students attend this course after their first single variable calculus course. Before the advent of computer algebra systems, ODE courses were generally focused on resolution techniques. Students didn't have to produce graphical solutions, nor did they have to use numerical methods and the emphasis wasn't concentrated on problem solving. The advent of computer algebra systems allowed teachers to introduce computing projects to their students. Introduction of symbolic calculators created a new dynamic in the classroom. Students can work on these projects **without leaving the classroom**. This is what we will demonstrate in the following example. When studying mechanical vibrations spring-mass problem in our ODE courses, we have to solve the damped forced oscillation problem.

Example1 : let m, b, k, F be fixed positive constants such that $0 < b^2 < 4mk$, let ω be a non negative real number. Students must show that the amplitude A (function of ω) of the particular solution of the ODE

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=F\cos(\omega t)$$

is given by

$$A(\omega)=\frac{F}{\sqrt{\left(k-m\omega^2\right)^2+b^2\omega^2}}.$$

Then, they have to show that, if $b^2 \geq 2mk$, then the preceding function A decreases from F/m to 0 when ω goes to infinity, whereas if $b^2 < 2mk$, then the preceding function reaches a maximum value at

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} \, .$$

This constitutes a type of project students were often asked to solve (Nagle & Saff, 1993, give all the details on pages 254 to 258). Nevertheless, it would have been much more interesting and gratifying to solve this problem in the classroom using first concrete numerical values. But this was practically impossible before. Because since September 1999, all our students have to buy a TI-92 Plus or TI-89 symbolic calculator, they now have the possibility to find a particular solution of the ODE considering fixed values for the parameter, before attempting to prove the general result. In this way, the « experimental aspect » of mathematics is emphasized. It is important to recall that, too often, in mathematics courses for engineers, we forget that the students prefer concrete examples. It will be easier to « sell » mathematics to students if the problem was presented in a concrete way in the first place.

We will fix the value of F to 2 and compare the graphs of A, as a function of ω, for each of the following situations: $m = 3/2$, $b = ½$ and $k = 1/16$ for the first case and $m = 1$, $b = ¼$ and $k = 2$ in the second case. Each situation yields the underdamped case ($b^2 < 4mk$) but only the second one satisfies $b^2 < 2mk$. But even considering such numerical values, it remains quite laborious and boring to find the particular solution using only pencil and paper techniques. In order to obtain these results, we think that students should use their calculator. This way, the teacher can encourage students to use the method of undetermined coefficients. We can divide the classroom in two groups, each one working on a case. Both groups have to do the following steps:

a) They have to define a differential operator (and, in order to do this, they have to understand what it means). They have to think about the candidate for the particular solution (the teacher can remind the students that there is no way to get mechanical resonance because we have a damped oscillation, so a linear combination of sine and cosine will do the job). Students will ask the following questions: does the differential operator need to be a function of one or two variables? If they are satisfied by the independent variable t, they can simply set

$$op(y) = m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky$$

and let the calculator perform $op(a\cos(\omega t) + b\sin(\omega t))$. Figure 1 at the end of the paper shows this for the second case.

b) When the system simplifies the expression $op(a\cos(\omega t) + b\sin(\omega t))$, we still have a linear system of equations to solve. Here again, it is a good opportunity for the teacher to remind students that sine and cosine functions are linearly independent and the teacher should ask the students to solve the following *linear* system of equations with two unknowns a and b using matrix approach and not the « solve » function of the system. For the second case, we obtain the following system:

$$\begin{bmatrix} 2-\omega^2 & \omega/4 \\ -\omega/4 & 2-\omega^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

c) Finally, they have to find the amplitude A, which is $\sqrt{a^2+b^2}$ and maximize this function, so students have to use results from the first calculus course. Even if they can plot the graph of $A(\omega)$, we think that it is important that they find, using exact arithmetic, the coordinates of the maximum. As figure 2 shows (for the second case), the function $A(\omega)$ is

$$A(\omega)=\frac{8}{\sqrt{16\omega^4-63\omega^2+64}}.$$

Here something strange happens. Many students will use their symbolic calculator in order to find the critical points of the above function and will check if these points are maximum values. Just a few will simply minimize the expression under the radical, and this does not require the use of the calculator! But, we don't have to forget that, when students have access to a symbolic calculator, you have to let them work with it, even if, sometimes, its use is not appropriate. And, finally, a plot of the function $A(\omega)$ is always possible and we encourage our students to do so (figure 3).

Some students will try to find a particular solution of our ODE using, instead of the method of undetermined coefficients, the method of variation of parameters. They will just be amazed by the complicated trigonometric expressions that the calculator will produce! Others will want to make use of the Laplace transforms methods. It is important, as a teacher, to tell them that, if they use this method, they will find the entire solution, provided they know the initial conditions; and even if they know it, why should they obtain the transient solution?

When students have experimented with different values of the parameter, we can ask them to **prove** the general result: that is to show that, if $b^2 < 2mk$, then the function

$$A(\omega)=\frac{F}{\sqrt{\left(k-m\omega^2\right)^2+b^2\omega^2}}$$

reaches a maximum value of

$$\frac{F}{b\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}}$$

at the point

$$\omega=\sqrt{\frac{k}{m}-\frac{b^2}{2m^2}}.$$

3. Connecting Multivariable Calculus with Single Variable Calculus

Our second example will try to show that, too often, students don't make all the connections they are supposed to do in order to get a good understanding of a problem. For instance, they are solving Lagrange multipliers problems with no idea of what is going on. One of the major reasons for this is that they rarely have to produce a graphical analysis of the problem. If a computer is a must for beautiful 3D graphs, we can't underestimate the graphing calculator for many 2D graphs related with the situation.

Example2 : we have to use the method of Lagrange multipliers in order to find the closest and the farthest points from the origin on the curve of intersection of two surfaces. Students are asked to verify their answers by finding extreme values of a one variable function. The surfaces will be the paraboloïd $z^2 = x^2 + y^2$ and the plane $z = 1 + x + y$; secondly, the cone $z = x^2 + y^2$ and the plane $x + y + z = 10$. What are the pedagogical interests of such of a question? Let us mention the following:

a) Our students are told that the condition $\nabla f = \lambda \nabla g$ (or, here, $\nabla f = \lambda \nabla g + \mu \nabla h$) is a necessary but not sufficient condition in order to get an extreme value of *f*, subject to the constraints *g* and *h*. For the cone and the plane, one could check that we find two critical points, no one being a maximum value (these two points are both minimum, one being a global minimum). For the paraboloïd and the plane, we find again two critical points ; one of it corresponds to to maximum value and the other one is the minimum. Regardless of the nature of the critical points, there is a challenge in solving the system of equations arising from the equality $\nabla f = \lambda \nabla g + \mu \nabla h$ where *f* is the square of the distance, $f = x^2 + y^2 + z^2$ and where *g* and *h* are the two constraints. One of the advantages of using a symbolic portable calculator consists of the possibility of solving such systems by pushing on a « solve » button. The system's solver will make use of the lexical Gröbner/Buchberger elimination method and, most of the times, will find, in exact arithmetic, the candidates for extreme values. We have to admit that solving the system of equations $\nabla f = \lambda \nabla g + \mu \nabla h$ in not easy and, before the era of symbolic calculator, the chapter about min/max problems for functions of several variables was not receiving all the attention it deserved. Students were getting lost in solving systems of equations instead of thinking about what the situation should be. As teachers, we cannot blame them for their poor ability in solving such kind of problems. Their attention was focused on algebraic skills instead of problems solving.

b) There is a very important interest for the graphs of the surfaces. The «basic surfaces» should be recognized by the students. But, with the computer and an appropriate software, we can plot, on the same window, more than one surface at the time. So we can plot both surfaces and see the curve of intersection. As teachers, we find important to ask our students to do this. They appreciate to see the ellipse of the intersection of the paraboloïd and the plane and the hyperbola of intersection of the cone and the plane! They now understand why there is a point on the ellipse closest and farthest from the origin and why there is no point on the hyperbola farthest from the origin!

c) There is an important link with parametrized 3D curves, and even 2D curves. Students should be able to find parametric equations for the curve of intersection of two surfaces. If we take the case of the paraboloïd and the plane, they need to complete a square and use their first trigonometric identity in order to obtain the parametric equations for the ellipse of intersection. Students love to see the projection of this ellipse, onto the *xy*-plane, which gives the circle they parametrized before (see figures 4 and 5). After, they are surprised to note that they can produce, using single variable calculus, the graph of the square of the distance. They don't always think that the norm of the vector describing the ellipse is a one variable function!

Unfortunately, we rarely see such approaches in our textbooks. People seem to think that basic concepts are not so important when we deal with multiple variable calculus. And we have to admit that, without appropriate technology, it will remain good intentions but practically impossible to propose many solutions for that kind of problem. Solving the system of equations generated by the vector equality $\nabla f = \lambda \nabla g + \mu \nabla h$ remains difficult and/or long to do by hand, plotting nice 3D surfaces without computer is quite difficult and finding extreme values of a single variable function without a graphic calculator is tedious and not quite interesting! These are reasons why solution to the above problems **should be done using technology**. Let us look at the problem involving the plane and the paraboloïd. So, let the three functions be $f = x^2 + y^2 + z^2$, $g = z - x^2 - y^2 = 0$ and $h = x + y + z - 10 = 0$. This leads to the following system of 5 equations in 5 unknowns

$$
\begin{aligned}
&2x = -2\lambda x + \mu \\
&2y = -2\lambda y + \mu \\
&2z = \lambda + \mu \\
&z = x^2 + y^2 \\
&x + y + z = 10
\end{aligned}
$$

and we ask the symbolic calculator to solve this system. Of course, students have to learn how to use the « solve » or « zero » function of the calculator. This requires them to correctly write the syntax. One could find the following vector function for the curve of intersection:

$$\vec{r}(t) = \left[\frac{\sqrt{42}\cos t - 1}{2}, \frac{\sqrt{42}\sin t - 1}{2}, 11 - \frac{\sqrt{42}\cos t}{2} - \frac{\sqrt{42}\sin t}{2}\right] \ (0 \le t \le 2\pi).$$

We ask our students to check first if the above vector function is correct, simply by substitution into the equations of both surfaces. And we want our students to understand that the (x, y, z) solutions of the system of equations are connected with the extreme values of the function $\|\vec{r}(t)\|$ (see figure 6). Speaking of parametric curves, finding the above vector remains a good exercise for students: they are told, in multiple variable calculus, that parametric curves are important subjects, but they rarely see parametric curves that come from an intersection of surfaces ! This gives the opportunity to recall some concepts, like vector valued function, norm of a vector, total distance, extreme value of a function of one variable (we can even make use of the second derivative test). Without technology, it would be too long to try different approaches. In order to be convinced of this, the reader should perform the calculations involved to get the zeros of derivative of the above function $\|\vec{r}(t)\|$: with the graph of $\|\vec{r}(t)\|^2$, it is so simple!

4. Conclusion

Teaching engineering mathematics with technology constitutes a good opportunity to teach « classical subjects with a new taste ». It allows teachers to adapt their teaching methods to the new technological reality. Most important, technology helps the teacher to present live examples of what mathematics are, how beautiful they are. Students will much more appreciate theorems and general results if they can visualize concrete examples. We have experimented this for the last three years in our single calculus course, our multi-variable calculus course and in our ODE course. Other colleagues are experimenting the same in a probability and statistics course, where both Excel and the TI Statistics package are used. Students don't feel that they get lost in all this technology: having the same kind of calculator surely helps, and being aware that, for some exams or some parts of an exam, they won't be allowed to use it, is a way to remind them that we still want them to learn basic concepts, do some basic manipulations by hand or, even, learn definitions by heart. But they also know that they have to learn how to use their symbolic calculator.

We are now experimenting the teaching of a graduate course of mathematics to engineers, dealing with systems of differential equations, eigenvalues problems, Fourier analysis and complex analysis. Such subjects are much more interesting if many computations involved can be done in the classroom. If not, the course remains quite theoretical. For example, studying the pointwise convergence of a Fourier series or analyzing the Gibbs' phenomenon should now be investigated first, graphically, and then, we can prove some results, like we were doing before technology. For a final remark, technology is changing the way we teach mathematics, but not so much: it simply gives some teachers the opportunity to continue to teach, year after year, the same subjects without having the impression of «déjà vu».

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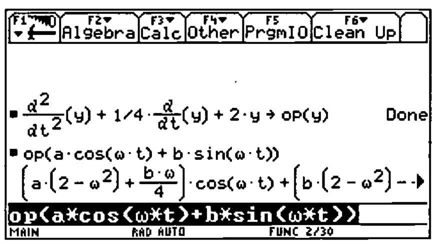

Figure 1. Differential operator with the TI-92 Plus

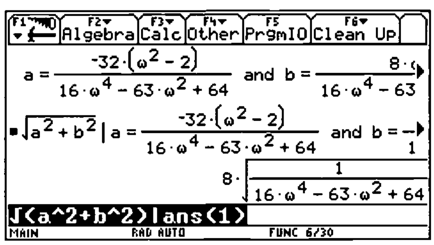

Figure 2. Amplitude $A(\omega)$

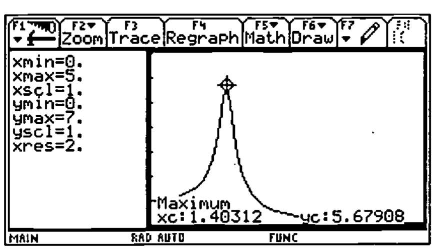

Figure 3. Graph of $A(\omega)$, showing maximum value

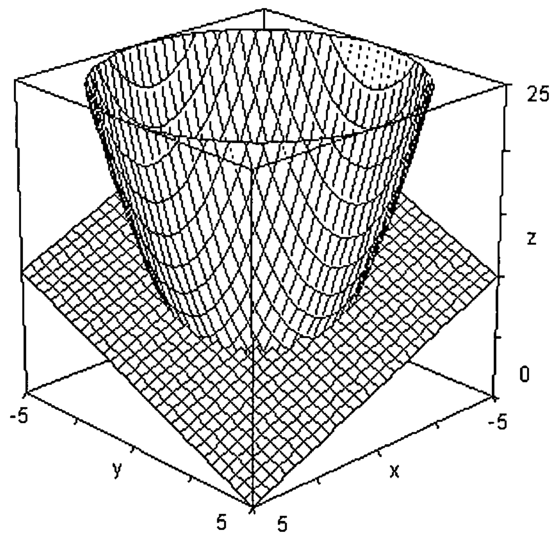

Figure 4. Intersecting surfaces with *Derive* 5

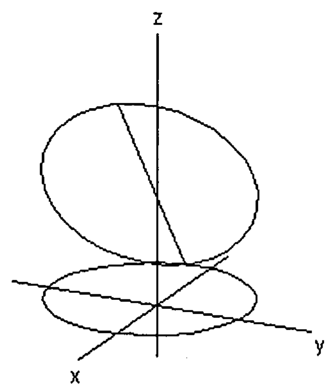

Figure 5. Curve of intersecting surfaces and its projection onto xy-plane. The line segment connects the points we are looking for.

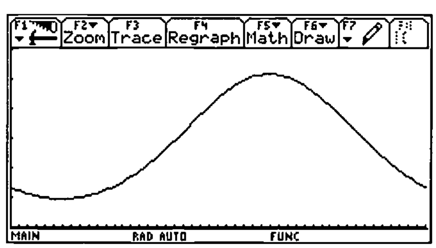

Figure 6. Graph of $\|\vec{r}(t)\|^2$.

If t_0 is any extreme value of this curve, $\vec{r}(t_0)$, when considered as a point in space, is one of the two connected points in figure 5.

TWO COMPONENTS IN LEARNING TO REASON USING DEFINITIONS

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ABSTRACT

This paper discusses the transition to the use of formal definitions in mathematics, using the example of convergent sequences in Real Analysis. The central argument is that where in everyday contexts humans categorize objects in flexible ways, the introduction of mathematical definitions imposes a much more rigid structure upon the sets so defined, and hence upon the acceptability of different types of argument. The result is that, in order to have their reasoning accepted in proof-based mathematics courses, students must do two things:

1. align their notion of what mathematical objects belong to a given set with the extension of the defined set, and
2. (more fundamentally) learn to express their reasoning about such sets exclusively in terms of the definitions or other results traceable to these.

The importance of these two components is illustrated using two examples. First, a student whose idea of what objects belong to the set of convergent sequences does not closely correspond with the definition, and whose reasoning is therefore insufficiently general. Second, a student whose set corresponds well to that given by the definition, and whose work is arguably more mathematically sophisticated, but who still does not "succeed" since he fails to reason using definitions in the required way.

Finally, pedagogical implications are discussed, with particular reference to tasks that require exploring the extension of defined sets. We consider the role of collaborative student work in promoting awareness of a broader range of examples within such sets. Further, we suggest that there is often a gap in the structure of the tasks that students are asked to complete; that many would benefit from tasks which begin with a term and require students to generate examples, in addition to the more usual task of beginning with an example and establishing its membership of a set.

Keywords: definitions, proof, advanced mathematics, imagery, Analysis

Human categorization and mathematically-defined sets

Human cultural categories are usually not "classical", in the sense that their extension is not determined by necessary and sufficient conditions for membership. Instead, many have "fuzzy" boundaries, such as the category described by the phrase *tall man*. They may also exhibit "prototype effects", as exemplified by the category *bird*, in which case there is general agreement that a robin is a "better example" than a penguin. Such effects may be attributed to considerable complexity in the internal structure of these categories (Rosch, 1978, Lakoff, 1987).

By contrast, mathematically defined "categories" or sets of objects do not have these attributes: the selection of a defining property precisely delimits a set, and does not distinguish any members as "better examples" than others[1]. This does not stop mathematicians regularly using certain examples in reasoning or explanation, and does not mean that it is necessarily easy to determine membership or otherwise in any particular case. However, in the logical structure of the subject, no special status is accorded to any particular examples, and this impacts upon accepted standards of argumentation in the subject: once a definition for a mathematical term is agreed, work that purports to establish results about the associated category must do so via arguments traceable to this definition (Tall, 1995).

This logical status of definitions should make some aspects of tasks set for students simple. Proof problems encountered at beginning university level generally either require showing that a particular object is a member of a mathematical category (e.g. "show that the sequence $(1/n)$ is convergent"), or showing that one category is a subset of another (e.g. "show that all convergent sequences are bounded"). The existence and status of definitions renders the "top level" (Leron, 1985) or "proof framework" (Selden & Selden, 1995) required in these cases very simple: one must either show that the object satisfies the definition, or show that one definition implies another. However, it is well recognized that students not only struggle with such tasks, but regularly employ alternative and less mathematically appropriate strategies such as generalization from an example or a "concept image" (Moore, 1994, Vinner, 1992, Harel & Sowder, 1998).

This paper examines the behaviour of such students, identifying two things they must accomplish in order to move from their existing reasoning habits, which are well adapted to everyday argumentation, to a mathematical approach to the use of definitions.

Research context

The students used as examples in the following took part in a research study in a top-ranking UK university. They were attending two pedagogically different first courses in Real Analysis, each of which covered work on sequences, completeness and series. The first of these courses was given in a traditional lecture format, the second was a new course in which students worked in groups in a smaller classroom, attempting to answer a structured sequence of questions which led to them proving the majority of the major results for themselves[2] (Alcock & Simpson, 2001). A number of students from each course attended biweekly

[1] The word "category" will be used rather than "set" from now on in order to highlight the fact that student behaviours would often be appropriate when handling everyday categories: it does not refer to categories in the sense of Category Theory.

[2] The course was based on Burn, 1992.

interviews in pairs. The interviews were semi-structured and comprised an introductory discussion of recent material and the students' experience of the course, a task-based section in which the pair worked largely without intervention from the interviewer, and a final section in which they reviewed their work on this task as well as responding to questions about their more views of proof and definitions in general.

One task that generated particularly rich data was the following, which was set in week 7 of the course:

Consider a sequence (a_n). Which of the following is true?
a) (a_n) is bounded $\Rightarrow (a_n)$ is convergent,
b) (a_n) is convergent $\Rightarrow (a_n)$ is bounded,
c) (a_n) is convergent $\Leftrightarrow (a_n)$ is bounded,
d) none of the above.
Justify your answer.

The interview excerpts presented in the following two sections show students who have decided upon the correct answer to this question, and are now attempting to produce justifications.

Generalization from a "prototype": Wendy

In everyday argumentation it is often acceptable to make statements about entire categories of objects based on generalization either from a specific example or from a more generalized "prototype" representing what is considered typical of the category in question. This is sensible in everyday life, where categories are not delimited by definitions, but is often inappropriate in advanced mathematics, at least in contexts such as beginning university courses where the student is required to learn about mathematical concepts as they are currently understood by the community. We can see what happens when students try to apply this strategy it in the following interview excerpts, in which Wendy's justification for her answer to the question involves a generalization from an image of a monotonic convergent sequence.

> W: Well if it converges, you get closer and closer...
> *Pause (drawing).*
> W: Is that enough to like, justify it...a little diagram, what have you?

Prompted for a proof, she does not do much more than describe her picture:

> W: [*Draws a monotonic increasing convergent sequence*] It's convergent... yes so if it's convergent it's always...or...say it could be the other way round it could be...going down this way [*draws a monotonic decreasing convergent sequence*]. It converges, so it's always above that limit.

In the context of the material she is supposed to be learning, Wendy's argument is inadequate in two ways. First, it is based on inviting the listener to agree with the generalization, without further explication of properties of convergent sequences from which one can deduce the conclusion. Second, her reasoning seems to indicate that she is only considering monotonic sequences, and hence is not properly arguing about the whole category. Such problems are well recognized in studies of students' use of visual imagery, in which it is noted that focus on a particular image can lead to a fixation with irrelevant details or even the introduction of false data (in this case, the assumption that all the sequences concerned are monotonic) (Presmeg, 1986).

In this case it is not clear whether Wendy thinks that all convergent sequences are monotonic, or whether she simply considers this subcategory more important in some way than other kinds of example (this would not be unreasonable, given that a great many of the sequences she has encountered so far will have been monotonic). It may also be argued that this is preliminary reasoning, much like any mathematician would perform, and that Wendy can be expected to refine her argument. Unfortunately this is not the case. It proves difficult to dislodge Wendy's fixation with monotonic sequences: despite repeated prompts from the interviewer to consider other types of example, she keeps returning to reasoning depending upon this property. Essentially, she *acts* as though she is unaware of the extension of the category of convergent sequences as delimited by the definition, and the result is that her reasoning is insufficiently general.

Abstraction of properties from a prototype: Cary

In everyday argumentation, if a generalization is questioned, we may provide extra justification by citing some properties of objects in the category in order to clarify why our conclusion must hold; saying, in effect, "I am correct *because...*". Depending upon the parties present, these properties are likely to be chosen spontaneously in order to draw on mutual experience.

We see this in Cary's attempt at the same problem. He begins in a way similar to Wendy, by making sketches in order to reach a first hypothesis:

> C: I've drawn...er...convergent sequences, such that...I don't know, we have er...curves... er...approaching a limit but never quite reaching it, from above and below, and oscillating either side.

However, he is not content to assume a generalization. Instead he first performs a mental check for any possible counterexamples, postponing his conclusion until he has completed this to his own satisfaction:

> C: I was trying to think if there's a sequence...which converges yet is unbounded both sides. But there isn't one. Because that would be...because then it wouldn't converge. Erm...so I'll say b) is true.

Following this he begins trying to formulate properties that will hold for all the objects he wishes to consider, and that can be used to demonstrate that his conclusion is correct:

> C: If it converges...that has to be...well I don't suppose you can say bounded. It doesn't have to be monotonic.... Erm...Yes, I'm trying to think if there's like...if you can say the first term is like the highest or

lowest bound but it's not. Because then you could just make a sequence which happens to go…to do a loop up, or something like that.

Not surprisingly, finding an appropriate property proves difficult, and Cary rejects several possibilities (that the sequences must be monotonic, that the first term would serve as a bound). Note, however, that not only are his attempts at argumentation more sophisticated than Wendy's, but that he also appears to have a better awareness of what kinds of object are classed as convergent sequences. He is aware that such need not be monotonic or even necessarily have the property that each term is nearer to the limit than its predecessor.

A teacher would recognize that the property Cary needs is the definition of convergence, but this appears not to occur to him. When eventually prompted for the definition by the interviewer, he writes down an incomplete version, and then returns to his previous attempts to abstract properties from his prototypical images. Eventually however, he is persuaded to complete his definition, at which point he realizes that this is useful, and is able to quickly construct the essence of an appropriate argument:

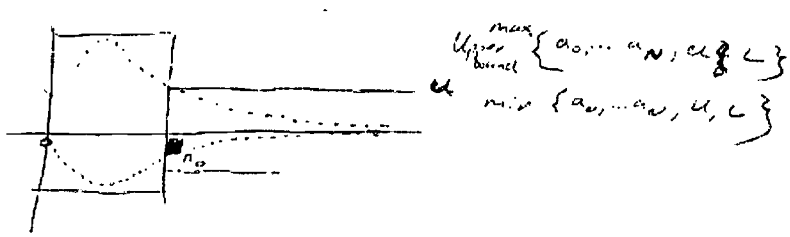

Figure 1: Cary's diagram to illustrate his definition-based argument that all convergent sequences must be bounded

C: …Yes, your n_0…that could just be called your n_0 instead, so going back to your definition up there, there exists this point here, such that after that point, i.e. when n is greater than n_0, the sequence…that statement there won't be less than any epsilon which you just happen to pick…. And so it's…and so the upper bound – so because there's finitely many terms before n_0, then er…your upper bound will either be plus or minus epsilon, or it'll be the maximum of those finite terms beforehand.

Notice that his diagram at this stage is drawn so as to illustrate some of the possible forms of non-monotonic sequences.

Pedagogical implications

Wendy's case is reminiscent of much that is seen in the literature on inappropriate uses of generalization from examples when a deductive proof is required (Chazan, 1993, Harel & Sowder, 1998). Here we would like to emphasize that the situation might improve for Wendy if she had a better awareness of what objects belong to the category of convergent sequences as this is defined in the course. Such an awareness should make her less likely to overgeneralize from a restricted range of cases, and more likely to recognize the potential pitfalls of relying on relatively fixed images.

Of course this may not be enough. Cary's case makes it clear that an idea of the category that corresponds closely to its defined counterpart, even when combined with a relatively mature approach to mathematical argumentation, is neither sufficient nor efficient in learning to produce the type of argument that is expected at this level.

In attempting to remedy these problems, we could simply attempt to enforce or at least heavily encourage the use of definitions. However, it might be argued that this is what lecturers already think they are doing, and that while some students do take this advice on board and become competent in using definitions, students like Wendy and Cary are far from atypical. A more student-centered approach would be to capitalize on the strategies already in use: after all, Cary is employing good mathematical thinking, and it would be desirable from a pedagogical perspective to capitalize on his existing strengths. This should not be impossible, as in the same interview it becomes apparent that on a philosophical level he already understands the role of definitions remarkably well:

I: Do you feel that you now see maths in a different way?
C: Not maths, but arguments.
I: Right...can you explain how?
C: We had this...I walked into the kitchen. I thought, I'll have an early night, I was going to make a cup of tea,
I: Mm,
C: And there was two people around the table, arguing about whether or not law came from morals?
I: Right.
C: And erm...so I was listening to them, and I thought, they're getting this all wrong. So I started joining in, and...and I found myself, *defining* stuff, and I was like, I cannot argue with you unless I have it defined, exactly what I'm supposed to be arguing about...

It appears that, with encouragement, it should not be too great a step for Cary to enact this understanding in his mathematical work. For others, the step to be made is greater, as indicated by this short continuation of the earlier extract from the interview with Wendy:

W: Is that enough to like, justify it...a little diagram, what have you?
I: Well, I'd like you to prove it, if you can.
W: Oh dear! (*laughs*) Oh right, well, if a to the n...

This indicates that Wendy does not consider proof to be a natural extension of her existing efforts at justification. This is not uncommon among students who regularly employ visual imagery in their work, and is epitomized in a remark by Fred:

F: Well it's not really scientifically proven. Because I think...I think I'm right, but it's not... it's not...if we've got to prove it then that's a different kettle of fish altogether.

However, even Wendy clearly has some idea of what is meant by convergent sequence, and we do not want to create a situation in which "proof" for her becomes any more removed from her intuitive ideas than it already is. It should be possible to help her move on from her present position without asking her to completely change her thinking: the fact that Cary's eventual answer is closely linked to his diagram indicates that building on this type of imagery can lead to the type of argument we would like to see. Indeed, the study as a whole indicates that students who reach the strongest understanding are often those who have access to both formal and imagery-based representations and who move flexibly between these. Adam is such a student, and he explicitly remarks upon this link:

> A: It's not usually enough to stick the definition down, you have to stick it down and then remind yourself of what it means.

So how can we promote such an awareness of the link between the definition and the objects that a student thinks of as belonging to a given category? One approach supported by the results of this study is the use of collaborative student work. It was found that those students in the new (problems-based) course showed more inclination to be critical of their initial conclusions, and to test these by attempting to check for counterexamples, than their peers on the lecture course. For example, Kate's initial thinking about the question described above is similar to Wendy's:

> K: ...it would be bounded wouldn't it, by its first term...
> J: We don't know if it's increasing or...
> K: And its last term.
> J: Depends if it's increasing or decreasing doesn't it?
> K: Well it would be bounded, either below – if it was decreasing it would be bounded above...

However, she and Jenny go on to question their conclusion, attempting to think of examples for which their argument will not work.

> K: But it's just, this one.
> J: Is there such a sequence that we don't know...
> K: Yes that's what I mean, is it true?
> *Pause.*
> K: Can you think of one?...Because I can't.

This behaviour does not necessarily reflect a mature awareness of the philosophy of advanced mathematics; it often appeared as an unexamined reaction to repeated experiences of being proved wrong. However it does appear that regular feedback and challenge from teachers and peers led to students developing the habit of subjecting their thinking to more rigorous checks, the effect of which is that their work reflected a better correspondence between their views of what objects belong to central categories and the formal versions of these.

A further suggestion is generated by noting a gap in the types of task required of beginning university students: we often ask students to show that some specific object is a member of some mathematical category (beginning with an object and concluding with a category), or to show that some category is a subset of another (beginning and concluding with categories). We also set tasks demanding that manipulations be performed on one specific object in order to obtain another, for instance (as an initial task) finding an N such that $|1/n| < \frac{1}{10}$ whenever n is greater than N (beginning with an object and concluding with another). Far less common are tasks that begin with a mathematical category and ask the student to provide examples (beginning with a category and requiring objects). Dalhberg and Housman suggest that example generation in response to a new definition is a feature of the thinking of better-performing students (Dahlberg & Housman, 1997). Hence it seems that, if well designed, tasks that start with a definition and ask for a range of examples might create a sense of the link between a definition and the objects included in the associated category. They could therefore help students like Wendy to bring their idea of what is in a given category into line with that determined by the agreed definition, and help students like Cary to think more readily of the definition as a natural basis for constructing arguments about mathematical categories.

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CHANGES OF NAMES, CONTENTS AND ATTITUDES TO MATHEMATICAL UNITS

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ABSTRACT

Will this material be on the exam? Why do I need to know this stuff? These are the sorts of questions that have been regularly asked by our mathematics students. Pre-service mathematics teachers often suggest that they do not need to learn anything that they do not have to teach. Generally, these students appear to have very little aesthetic appreciation for mathematics and its applications.

Currently, we teach five traditional mathematical content units that are provided mainly for pre-service mathematics teachers. These units have been adapted and modified over the years from units that were designed primarily for science students. They contained a heavy focus on calculus with a limited breadth of mathematical experience. After consulting widely on the best mathematical practices throughout Australia and internationally, it was decided to reform all of the mathematics units to make them more attractive to a wider audience.

The units that are currently being developed are: Profit, Loss and Gambling; Upon the Shoulders of Giants; Logic and Imagination; Modelling and Change; Algorithms, Bits and Bytes; Space, Shape, and Design; and Modelling Reality. The overall goal of this redevelopment is to improve student attitudes and motivation by exposing them to a wide range of topics in mathematics that are usable and relevant. All of these units will incorporate current technology, contain realistic problems, and include visiting speakers. Student assessment in these units will consist of portfolios, projects and examinations.

The introduction of these new units will result in students having a greater choice of the units they wish to study. In order to overcome potential logistical problems of a small mathematics department, innovative changes to the structure of the units will also be examined. This paper will provide the details of the establishment and content of these units.

1. Introduction

There have been many articles written on the advantages and disadvantages of using thematic approaches, modelling and applications to provoke the need for mathematical methods and techniques. The interested reader is referred to Kilpatrick (1997), Wu (1997) and Black and Atkin (1996) for discussions on reforms in mathematics education. While this paper will discuss the major issues briefly, the main focus will be on the rationale for restructuring the mathematics units, details of the units, and the actual implementation process conducted at the University of Ballarat.

2. Current Situation

In Australia, there is increasing concern about the declining numbers and calibre of students choosing to major in mathematics (Forgasz & Swedosh, 1997). The range of students' needs, backgrounds, learning approaches and mathematical abilities is very broad. Some of the Australian enrolment trends and student perceptions of mathematics are described in Forgasz & Leder (1998), Forgasz, Leder, & Brew (1998) and Forgasz (1998). Course restructures throughout the institution have resulted in most mathematics units being designated as electives rather than core units where a unit is defined as a full-time load for one semester. In the past, our main client group was science students. Since the restructure, numbers enrolled in mathematics units have dropped by more than 50% and the main interest is from education and computing students. In the last couple of years, there has been less than thirty students enrolled in first year mathematics units and less than fifteen in second and third year units.

An investigation into the reasons for falling mathematics numbers soon revealed some obvious problems. Students may choose their electives from a broad range of units. There is strong competition from other discipline areas and the mathematics units have not been successful in attracting these students. Current unit titles such as Mathematics, Pure Mathematics, and Introduction to Calculus and Computer Algebra Systems appear to deter both students and enrolment advisers. Further discussions revealed both students and advisers thought that the units would be difficult, boring, and irrelevant.

The units offered were designed principally for science students and were narrow in content. Unit evaluations as well as informal student comments over several semesters have confirmed the dissatisfaction of the remaining education and computing students with the domination of calculus in the current units. Another problem was that of assessment. Regular worksheets, which were really learning tasks, formed part of the assessments along with assignments and examinations. Examinations were worth about 50 % of the final grade, considerably more than in most other disciplines, and students often exhibited a high level of exam anxiety.

3. Proposed Solutions

Overall, it was clear that the mathematics department had to improve the image of mathematics and make it more relevant for the students undertaking these units. We wanted to encourage more education students to take mathematics as their special teaching area as well as entice students from around the university to take elective units in mathematics. We therefore needed to cover a wider range of mathematics, both in terms of topics and depth.

3.1 New content and teaching

In response to this situation, and following much discussion, the following goals were established for our teaching of mathematics:

- Students who do our units should be able to reason mathematically, communicate and solve problems, as well as master algorithms and remember facts.
- Students should understand and appreciate the role of mathematics and its applications in the real world.
- Education students should form a positive view of their potential careers as mathematics teachers.
- Each unit should incorporate up-to-date teaching technology and utilise methods that enhance student learning.

With these goals in mind, we set about the task of developing units that would expose students to a wide range of mathematical topics that are useable and relevant. The new focus of every unit would be in the use of themes, applications and some problem based learning to provoke the need for mathematical methods and techniques. Burrill (1993) summarises the nature of our new approach when he states that 'rather than memorising algorithms and manipulating symbols following explicit directions from a teacher, students must explore, investigate and interact with each other, and the teacher, as they develop strategies to resolve the problem. There is a strong emphasis on communication, and the ability to explain and justify a reasoning process'.

Generally, most teaching lessons examine a problem or application. The aim is to grab the attention and interest of students as well as 'the initial exploration leading to the development, discovery or invention of mathematical concepts' (de Lange, 1993). It is unreasonable to expect that every mathematics problem will have an engaging application, so there will also be times when a more traditional approach is needed. Once the problem is understood, the teacher and students attempt to identify possible strategies and any mathematical aspects of the problem. The actual process of discussing strategies can be considerable (de Lange, 1993). Students should investigate the advantages and disadvantages of various strategies and decide on the best solution. Once a strategy and solution is reached, students should evaluate their results and the mathematical methods and techniques used. These skills and insights gained can then be applied to a set of other problems that have been designed by the teacher to practice the material and methods learnt.

It should also be noted that teaching will be made more complicated by using real-world problems as the role of the teacher will be strongly geared around organiser and facilitator rather than deliverer of information. Some problems will have more than one answer or several strategies. Some teachers may also feel threatened with the apparent loss of authority, but in this case, all staff are willing to face the challenges of change.

3.2 New assessment

The proposed changes in content and teaching will result in students demonstrating new ways of learning and doing maths in the classroom. Therefore, it is important to develop new ways of assessing their understanding and progress. Students demonstrate their mathematical understanding through a variety of methods such as asking important questions; making abstract connections; and applying learnt concepts to new problems. Swan (1993) identifies the following aspects of learning that can be assessed: facts, strategies, skills, concepts, appreciation and awareness, and personal attitudes and qualities. The new units will contain assessment that is varied enough to test these mathematical abilities.

The broad range of our students' mathematical abilities as well as learning styles should also be addressed in assessment tasks. Cretchley (1999) accounts for the diversity of backgrounds and abilities by presenting an attempt at allowing students to select their own items of assessment. Students were invited to submit one 'good' question from the textbook exercises that best demonstrated the concepts covered in that section and justify their selection. We have, also, successfully trialled tasks of this type and plan to extend this form of assessment. Student reports, which summarise a task or result and student-created tests are other valid assessment tasks that reveal students' knowledge and understanding of mathematics.

As much of the in-class discussion will utilise group work, it seems obvious that there should be a greater use of group work in assessment. A large amount of literature discusses the benefits and achievements of students working in groups. The interested reader is referred to Duncan & Dick (2000) for references to recent literature. The assessment should also reflect the strong emphasis on communication during in-class discussions. Gretton and Challis (1999) discuss assignments that emphasise communicating a solution and using technology. For example, students had to perform a simple regression analysis for a business. They were then asked to write a letter to the owners (non-mathematicians) that explains the mathematics used in their solution. Alongside these, student directed and group assessment tasks, examinations will still be used to test basic skills and verification of student knowledge as displayed in open assessment tasks.

3.3 New Structure

Currently, there is no choice of subjects for our mathematics students. These students indicated that they would find their degree more enjoyable if they were offered some choice of the units they could take. With small class sizes and low staff numbers, offering more choices seemed to be unrealistic. To address this issue, a new structure has been proposed in which each unit may be taken at one of two levels. That is, some units will be offered at introductory/intermediate level and others at intermediate/advanced level. Students will then have some choice of which units they would like to take at a lower level and those they would prefer to take at a higher level. One teacher will take a unit at both levels. The first four hours in a week will involve all students. An extra hour will be spent extending the topic for those students who take the unit at the higher level. Laboratory classes and tutorials will be shared by students working at different levels. Some questions and learning tasks will be common to both levels while other tasks will be level specific. Since, in our current classes, students work on a range of problems, largely at their own pace, these changes are not expected to cause major logistical problems.

3.4 New Unit Names

Since the current titles of mathematics units seem to have a negative impact on students when making their choices for elective subjects, it was decided to market our units with names that might attract students' interest. A full list of the new unit titles along with the levels at which they will be taught is provided in table 1.

3.5 Details of Units

In this paper, space precludes a full discussion of the specific content, technology and problems investigated. Some examples of these issues will be discussed for the units being developed for the upcoming year.

Unit Title	Level
Upon the Shoulders of Giants	1/2
Modelling and Change	1/2
Bits, Bytes, and Algorithms	1/2
Profit, Loss, and Gambling	1/2
Shape, Space, and Design	1/2
Logic and Imagination	2/3
Modelling Reality	2/3

Table 1: List of Units and Corresponding Levels

Profit, Loss and Gambling

This unit lends itself nicely to introducing real life problems and demonstrating mathematical concepts to solve these problems. Enthusiasm can be generated by spending the first week both discussing the history of gambling and investigating probability through exploration with cards and dice. The next three weeks are spent introducing basic statistical concepts that will be required to understand gambling games. These concepts include probability rules, independence, mutually exclusive, odds, house margin, expected values, and probability intervals. The tutorials of these weeks involve illustrating the concepts with simple games like a player receiving $10 for an even number and $25 for an odd number on the outcome of a roll of a die. Students can then calculate expected values, standard deviations, probability intervals, house margins, and discuss fair games. Excel can be used to simulate dice throwing and empirically explore concepts such as expected values.

There is an assignment where students work in groups and invent their own simple dice or card gambling games and demonstrate their understanding of the statistical concepts by calculating house margins of their own devised games. A tutorial class is spent playing these games and investigating the empirical results. Higher-level students are introduced to Bayes' theorem, Chebyshev's inequality and gambler's ruin in the corresponding weeks.

Each of the next five weeks of the unit focus on one gambling game with discussion of the mathematical concepts necessary to determine probabilities, odds and house margins. The games covered are lotteries, Keno, Roulette, Two-Up, Craps, and gaming machines. Other applications are then used to reinforce the techniques learnt. For example, Lotto games demonstrate the use of combinations that will be reinforced by calculating probabilities of poker hands and drawing balls out of urns. Most of the lectures begin with a gambling game and a discussion of how to calculate various probabilities involved in the game. Once students realise the complexity of calculating these probabilities manually, simpler techniques can be introduced or reinforced. For example, the number of ways that 6 balls can be selected from 45 in a lotto game would be laborious if students wanted to write down the sample space. The discussion will turn into breaking the problem into simpler manageable tasks and investigating the patterns. Higher-level students are introduced to a variety of probability distributions, calculating means and variances of these distributions, and calculating moment generating functions.

This unit also addresses basic ideas in financial mathematics such as the time value of money, annuities, superannuation and investment strategies. Again, problems such as calculating compound interest are introduced with a simple real problem. Student exploration of the problem by calculating the yearly amount accrued, should lead to the need of a simple formula. Assignments are provided that contain real life applications such as comparing several loan providers or how to save for superannuation. The advanced level contains more content on stocks and bonds as well as linear programming.

Visiting speakers discussing the social/psychological issues of gambling and casino operations will strengthen students' appreciation of the context of this mathematics. The internet is used to play various games and empirically test different gambling strategies (a great assignment task). Packages such as Excel are also used to simulate events such as throwing dice and simulating data from various statistical distributions.

Upon the Shoulders of Giants

It is important to have an introductory mathematics unit that provides students with an overview of the fundamental skills in number, function, algebra, and geometry required in the other mathematics units. Rather than repeating the same material covered in the secondary school curriculum, the students will be guided through the development of these ideas from an historical perspective. By looking at the origin of fundamental concepts, it is envisaged that students will improve their understanding of these concepts. This unit will show students that mathematics was not discovered in the polished form of our textbooks, but often developed in intuitive and experimental fashion out of a need to solve problems (Katz, 1998).

Number theory, Euclidean and Cartesian geometry, astronomy, trigonometry, number systems, algebra, functions and probability are covered in this course at introductory levels. This unit has a one hour lecture and three hours of tutorials per week. The lecture takes the students through the historical and social context of each topic covered and presents appropriate anecdotes and biographies. The tutorials are used to present and apply the underlying mathematical skills inherent in the concepts discussed in the preceding lecture. The skills covered include basic trigonometry, algebraic manipulation, scientific notation and evaluation of functions. At the advanced level, students attend one extra tutorial hour in which they are challenged with problems which combine concepts in an applied situation. For example, one advanced tutorial problem asks students to estimate astronomical distances using scientific notation, trigonometry and algebra.

The assessment for this unit involves a group presentation on topics such as ð, mental calculation or the golden ratio. There is one assignment which covers elementary algebra, geometry, trigonometry and functions and includes an essay style question on mathematics history. Students are also expected to hand in a portfolio of problems as discussed in section 3.2. Current technology is used to replicate some of the early explorations, unlike Rhaeticus (1524-1576), a mathematical astronomer who spent 12 years with hired human computers to produce two trigonometric tables!!

Logic and Imagination

All mathematics courses need to find an appropriate place to introduce students to fundamental mathematical reasoning and proof. This unit aims to do this in a variety of topics which are often found in Discrete Mathematics courses. By starting with logical puzzles and informal discussion of paradoxes, we hope to gain the students' interest in more abstract reasoning. Elementary Number Theory is a good source of some of the simplest theorems, and will be taken far enough to enable a discussion of Public Key Cryptography. (One of our aims is to provide a course that will be useful and interesting for more mathematically inclined computing students to take as an elective.) The imagination of mathematicians over the centuries will be highlighted by a discussion of number systems leading up to complex numbers, and for the more advanced students a brief look at the quaternions. We will also include other topics of interest to computing students, such as hierarchies of algorithmic complexity.

In the initial implementation of this unit, it has not been necessary to run it at two levels, however the students form two relatively different groups. These groups have two classes in common and each have two other tutorial classes separately. One group consists of future teachers (many primary, but some aiming to be secondary mathematics teachers). These students find the abstractions in the course difficult, and so their tutorials are used mainly for practice on easier problems. The other group is mainly computer science students, and it has proved possible to go a bit further and introduce extra material in their tutorial classes.

The following paragraphs summarise the topics covered in the units that will be developed over the next year.

Modelling and Change

The major focus of this unit will be on learning and applying standard calculus techniques to model motion, growth, and change. Mathematical modelling by its very nature will be based on practical work and examples. The first couple of weeks will be spent on developing basic modelling skills. The rest of the unit will consider problems such as the analysis of velocity and acceleration for vehicles and athletes, growth and decline of populations under different environmental constraints, and marginal costs for business. Generally, each week will be focussed on a particular problem that utilises a mathematical modelling concept. Further exercises and examples will be used each week to reinforce skills learnt. This unit is one of the easiest to teach at two levels. Students that take the unit at the lower level will use algebra, differentiation, and integration to solve modelling problems. The higher-level students will also use these techniques as well as differential equations, optimisation techniques, and calculations of area and volume. These students will often work on similar problems each week to the others, but will be required to handle more demanding differentiations and integrations.

Bits, Bytes, and Algorithms

This unit is compulsory for computing students but is also valuable to prospective teachers. Students will explore the representation and manipulation of numbers and symbols, the mathematical structures that underlie the storage of information, the algorithms that underlie computer software programs, introductory number theory, matrix operations, and solving linear equations using matrices. Higher-level students will apply algorithms for traversal and optimisation of networks and graphs as well as developing recursive algorithms.

Space, Shape, and Design

This unit will investigate the patterns in the shapes of nature, art, architecture, and industry. It will provide students with some experience in the thinking and techniques necessary to establish evidence of general patterns and calculations related to spatial measurement and design. Topics that will be covered include two-dimensional and three-dimensional shapes, geometric properties, tessellations, symmetry, topology, graph theory, fractals, kaleidoscopes, and trigonometry. Activities will include constructing 3D shapes, working out fencing lines for land subdivisions, finding paths to fit constraints, and analysing optimum shapes for industrial design. Higher-level students will have further experience in the formal use of mathematics to solve spatial problems.

Modelling Reality

This is the second unit of modelling that students can undertake if they have completed the unit Modelling and Change. The topics in this unit include an introduction to multivariate calculus, numerical methods, interpolation, linear algebra, and consolidating topics previously encountered in Modelling and Change. Higher-level students will work on the same topics but will have more challenging problems.

4. Steps to Implement the Solutions

This section contains the different stages that the mathematics department actually undertook in getting the new units established. The process began in the middle of last year (2001) and the units will be running in first semester 2002.

Stage 1: Discussion and Research

There was lots of dialogue with current students, staff across the University, at conferences (for example: Delta Symposium on Teaching Undergraduate Mathematics and MERGA), and at other institutions. Staff read current mathematics education books and journals.

Stage 2: A 'brain storm' and response

A loosely structured, imaginative list of 'possibilities' was designed to elicit clear responses from the mathematics staff. Following a discussion at a meeting of all staff, there was unanimous 'in principle' agreement. Consensus was then reached on unit names and broad areas of content. Tasks were assigned to facilitate the preparing of a formal, detailed proposal for change.

Stage 3: Development of formal proposals for change

This stage involved more detailed discussions with colleagues at other institutions, current mathematics students, and students not currently undertaking any mathematics electives. The staff then had to gather resources and ideas and prepare a list of mathematical content required by our various client groups.

A planning day was arranged when all of the mathematics department could attend. A large matrix containing the unit names as rows and mathematical topics as columns was drawn on a whiteboard. A number (1,2 or 3) indicating the level at which the content would be taught was placed in the appropriate grids. This method quickly highlighted any areas of mathematical content would be omitted or repeated. It also helped to define whether a unit would be offered at introductory/intermediate or intermediate/advanced levels. Based on the agreed matrix, pairs of staff prepared formal unit outlines and a plan of the possible sequences of the unit was detailed.

There was agreement on an assessment policy that was general enough to be common to all units. For each unit, students will be required to submit a portfolio consisting of an annotated selection of their work to demonstrate their achievement of specific learning objectives. All students will be required to participate in projects or presentations, which in most units will involve group work. Tests or an examination will be directed at assessing basic skills and verification of knowledge and concepts demonstrated in non-supervised work.

5. Current Feedback

Even before the units have been taught, there has been feedback and interest shown from a variety of sources. Student responses to the names and synopses for units has been overwhelmingly positive with comments like: 'Oh that sounds interesting', 'I'd like to do that' or 'why didn't I have the chance to do those units?". Non-mathematics staff have been more circumspect, especially about the name 'Upon the shoulders of giants' for the basic unit. However, in contrast to the past, we have not been met with a neutral response. The titles have generated interest and discussion that have allowed us the opportunity to share our enthusiasm for our discipline!

6. Conclusion

The new units will provide a broad range of mathematical concepts that should be suitable for students from a variety of disciplines. Students will be provided with a greater choice of units and an increase in the breadth of mathematical experiences. There will be more emphasis on mathematics that is useable and relevant without reducing the content that pre-service teachers are expected to cover. The goal of these changes is to improve students' motivation, perception and attitude towards mathematics. At the time of writing, we have still to face the big test of implementation of this new program. All issues of the first semester in teaching these units will be shared at the conference. Reponses from both the teachers and students about the successes and problems found in structure, content and assessment of the units will be presented and discussed.

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DIFFERENTIAL CALCULUS OF SEVERAL VARIABLES WITH MATHEMATICA OR MAPLE

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ABSTRACT

The use of Computer Algebra Systems (CAS) in Mathematical Laboratories at Engineering Schools is increasing. Therefore, it has become necessary to undertake a new educational approach in the teaching of many mathematical topics. Obviously it is not possible to give the same lectures as in the 1950,s. The CAS allows us to experiment and teach in "a different way". The students' mathematical knowledge at the beginning of their studies in the University is completely different. Moreover we have fewer credits assigned to Mathematical topics in engineering curriculum.

The objective of he authors, professors of different Spanish Universities, is to try to demonstrate a new way to use the CAS, concretely *Maple* and *Mathematica*, as a support in the teaching of Differential Calculus of Several Variables. They take advantage of the computer's graphical capacities to sketch the graphs of surfaces defined in explicit, parametric or implicit form. They survey the general topics: limits, partial derivatives, differentiability, chain rule, implicit functions and extreme values. For each topic, they give a few basic instructions of the CAS used; afterwards they give several examples, in increasing order of difficulty, and finally the students must solve exercises and try to establish a general procedure for a quite general problem.

In this paper we have included several examples used in this survey.

KEY WORDS: Computer Algebra Systems, Mathematical Education, Differential Calculus.

1. Introduction

Over the past 20 years Spain has seen the advent of CASs (Computer Algebra Systems) as a teaching complement for the mathematical training of engineering students. In general, however, this has not been accompanied by necessary changes in teaching methods to obtain optimum benefits from this tool. What usually happens is that the teacher merely adds to traditional teaching contents the required instructions to do with the computer exactly the same as can be done with a paper and a pencil. This leads to an overload of work, using the CAS in a mechanical fashion but not for actually teaching mathematics. The aim of CASs (that is, to offer a tool for doing and learning mathematics) is replaced thus by a situation in which students become experts in the CAS without understanding the mathematical issues being dealt with.

On the other hand, it is much more common to find CASs being used in courses on Elementary Calculus than for the teaching of Calculus of Several Variables.

In this paper we propose a more integrated and harmonious use of a CAS (*Maple*, *Mathematica* or any other) within daily teaching practice. We have honed this down to show the possibilities of using CASs for giving a course in Differential Calculus of Several Variables. The proposal is the result of our experience as math teachers in engineering schools in different Spanish Universities, where either *Mathematica* or *Maple* are used.

2. Our starting point

In Spanish Engineering Schools, Differential Calculus of Several Variables is generally taught after a single semester course on Elementary Calculus. The usual topics in this area are: Basic Notions of R^2 and R^3, Limits, Directional and Partial Derivatives, Homogeneous Functions, Chain Rule, Implicit Differentiation, Inverse Function, Taylor's Formula, Maxima and Minima, Lagrange Multipliers.

Usually there are 30 hours of class (including theoretical and practical ones) for teaching these contents and it is assumed that the personal work of each student will match the same number of hours.

The aim sought is that our students should be able to gain a reasonable mastery of the concepts, to acquire basic manual calculation skills, for later use in mathematical issues or in applications in subjects specific to engineering, and moreover to know when and how to use a CAS for solving a mathematical problem.

Finally, it should be stressed that our students have a basic initial knowledge of the CAS to be used (*Mathematica* or *Maple*, depending on the University).

Therefore, in view of the manifest lack of teaching time, if the above aims are to be achieved this overall picture requires a new methodological focus.

3. Our teaching material

To carry through our methodological proposal, we have compiled teaching material for a standard course on Calculus of Several Variables. Concretely this material consists of a text book that systematically includes theoretical concepts and solved and proposed problems (see García, A., López, A., Romero, S., Rodríguez, G., de la Villa, A., 1996), together with a CD that includes problems solved with the help of a CAS.

Here we describe the material featured in the CD; in particular, that relative to Differential Calculus of Several Variables, and at the same time explain the strategy followed to compile it.

We note that the CD includes two alternative directories with parallel contents: one elaborated with *Maple* and the other with *Mathematica*. Accordingly, the examples we offer here may be in one system or the other.

This teaching material gives exhaustive coverage to all the above issues and allows us a differentiated use by the teacher as a function of the particular needs of each case. Based on the teacher's own criteria, the material can be used as class material or as a guide for the personal work of the students, attending to the students' own characteristics and the time available for developing the different topics.

We believe that our work should help to palliate the scarcity of material for the teaching of Calculus of Several Variables, at least in Spanish, as compared with the vast amount of material currently available for the teaching of Calculus of a Single Variable.

4. Our methodological proposal

The methodological proposal is orchestrated on the basis of the integration of traditional teaching, lectures and problem classes (with the introduction of different concepts and blackboard exercises), with laboratory classes in which the corresponding CAS is used to reinforce the theoretical concepts and solve problems.

We offer a methodological proposal that involves the students dedicating approximately one third of the total work time programmed for the subject (both as regards class hours and personal work) to guided use of the CAS, following the teacher's instructions and pursuing the following aims:

4.1. Improving visualisation

The possibility to sketch the graph of a surface immediately gives us additional information in the study of a function of two variables, making conjectures about the properties of the function, extremes, bounds, etc. For example figures 1 and 2, both obtained with an elementary *Maple* input, help to understand concepts such as non-differentiability and relative extremes.

The graphical capacities can also be used by the teacher to offer presentations that will help to introduce different mathematical concepts. However, it should be noted that many instructions are often required to obtain "good" presentations of graphics and the teacher must take into account the cost of performing these.

4.2. Experimentation

The possibility of performing long tedious mathematical calculations very quickly enables the introduction of new strategies for problem solving, contrasting numerical, analytical and graphical techniques.

Thus, for example, with a CAS it is easy to know the variables that can be chosen as dependent in the implicit function theorem. To do so, assuming that the rest of the hypotheses of the theorem are satisfied, it is only necessary to check whether the determinant of the Jacobian matrix is null or not. So, even if the system of implicit functions is difficult, the problem is reduced to an automatic calculation.

Also, in order to get a first impression about the existence of a limit of a function it is possible to sketch a graph or construct tables of values for the function, near the limit. For example, to study the existence of the limit of the function $f(x, y) = \dfrac{xy}{x^2 + y^2}$ at the origin of the coordinates, the Mathematica input

```
TableForm[Table[f[a + j h, b + k h], {j, 1, n}, {k, 1, n}]]
```

with a=b=0, h=0.001, k=0.001, n=5 generates Table 1. The dispersion of the data in this table leads one to suspect that the limit does not exist, which can be proved by using the technique of findings two subsets for which the limits will be different, for example, $y = x$ and $y = 2x$.

As part of the teaching strategy, when faced with any problem we should encourage students to experiment in a general sense. A crucial aspect is to avoid laziness in the use of different trials since the mechanical processes are performed by the computer.

4.3. Release from mechanical work

We should not overlook the need for students to acquire basic computational skills. Once it has been determined that the students have assimilated the concepts and that they are able to do "operations" manually in cases of calculus that are not too complicated, they can then use CASs as "advanced calculators" for the more tedious calculations. Thus, for example, in the Chain Rule the Jacobian matrix of the function $g \circ f$ is the product of the Jacobian matrices of g and f. When the component functions of f and g have complicated expressions, the Jacobian matrix of the function $g \circ f$, with $f: \Re^5 \to \Re^6$ and $g: \Re^6 \to \Re^4$ demands a huge "manual" effort, which can be done automatically with a CAS.

We believe that CASs, understood as "advanced calculators", can be used in most issues topics of differential calculus.

The usefulness of such tools in intermediate calculations, relative to concepts that have already been analysed and that are necessary for later stages of learning, is also unquestionable. For example, the hypotheses of the existence of an inverse function can be analysed using a CAS, since the calculations involved in checking such hypotheses are routine. We therefore see that the use of a CAS allows to save time dedicated to routine calculations, and can be used to better model problems involving real situations and to interpret, at each step of the process, the results obtained.

As an example, let us consider the following problem (taken from García, A., López, A., Romero, S., Rodríguez, G., de la Villa, A., 1996):

The temperature of a plate at any point (x,y) is given by the function $T(x,y) = 25 + 4x^2 - 4xy + y^2$. A heat alarm, situated at points on the circumference $x^2 + y^2 = 25$ is triggered at temperatures higher than 180°C or below 20°C. Will the alarm be triggered?

To solve this, the students need to optimise T(x,y) subject to the constraint $x^2 + y^2 = 25$. Then we can use the method of Lagrange multipliers. Calculations can be done by computer and the points $\{-2\sqrt{5}, \sqrt{5}\}$, $\{2\sqrt{5}, -\sqrt{5}\}$, $\{-\sqrt{5}, -2\sqrt{5}\}$, $\{\sqrt{5}, 2\sqrt{5}\}$ are obtained as possible solutions. Then the students must evaluate the function T in those points to find the extreme values and to interpret the result. Since the maximum of these values is lower than 180 and the minimum higher than 20 the alarm will not be triggered.

Moreover, symbolic capacities of a CAS allow us to work efficiently with formal expressions, like those related to the properties between operators. For example it is trivial, using Mathematica, to prove $div(grad\, f) = \mathbf{D} f$. After loading the package **Calculus`VectorAnalysis`** the input

```
Div[Grad[f[x,y,z],Cartesian[x,y,z]]]-
Laplacian[f[x,y,z],Cartesian[x,y,z]]
```

produces the output 0.

We also can prove general properties in real problems as it is shown in the following example (taken from Marsden, J. and Weinstein, A., 1985):

The specific volume V, pressure P, and temperature T of a Van der Waals gas are related by the equation $P = \dfrac{RT}{V-b} + \dfrac{a}{V^2}$, where a, b, R are positive parameters. R is the universal constant *of the gases, b represents the volume of the gas molecules in liquid state (b<<V) and* $\frac{a}{V^2}$ *represents the inner pressure due to the interaction between molecules.*

The students, using implicit function theorem, can prove that any two variables of V, P, or T can be considered independent. The students can also find $\frac{\partial}{\partial P}T$, $\frac{\partial}{\partial V}P$, $\frac{\partial}{\partial T}V$ and verify that $\left(\frac{\partial}{\partial P}T\right)\left(\frac{\partial}{\partial V}P\right)\left(\frac{\partial}{\partial T}V\right)=-1$.

Here the CAS has been used for verifying the hypothesis of the implicit function theorem and for symbolic computations.

We can ask them for a generalisation for this formula when F is a function with n variables.

4.4. Distinction between algorithmic and non-algorithmic processes

The use of CASs is not a panacea that will allow the solution of any mathematical problem. In this sense, it is important to stress that students should distinguish between processes that are algorithmic and those that are not.

When a process can be "algorithmized", students should be encouraged to perform a simple procedure consisting of translating the mathematical process to the language of the CAS used.

Within algorithmic processes, it is appropriate to distinguish between two types:

- Algorithms with an ensured answer, in which information is always obtained.

For example, it may be very simple for a student to perform a Maple procedure that will return the tangent plane at the surface, implicitly defined by the equation F(x,y,z)=0, at a point (a,b,c). It suffices to know the definition and write the instructions for finding the gradient vector and the equation of the tangent plane. A basic form of this procedure could be as follows:

```
> Tang_Plane:=proc(F,a,b,c)
  local gr;
  gr:=subs({x=a,y=b,z=c}, linalg[grad](F(x,y,z),[x,y,z]));
  simplify(gr[1]*(x-a)+gr[2]*(y-b)+gr[3]*(z-c)=0)
  end:
```

To use this procedure in later problems, the student must previously verify that the hypotheses of the Implicit Function Theorem are satisfied. The procedure can also be improved by including, inside it, the instructions necessary to see if such hypotheses are indeed satisfied and ensuring that an error message will appear if they are not. Also, it is possible to add a graphical instruction to draw the surface, together with the tangent plane. Students generally try to improve the procedure until it is as complete as possible.

- Algorithms that may include computational problems.

In this type of algorithms, the answer is conditioned to intermediate processes that cannot be performed owing to difficulties such as the impossibility of solving an equation, excessive systems requirements, etc.

Thus, for example, for the calculation of constrained extremes, the Lagrange multiplier algorithm requires the solution of a set of equations, which is not always guaranteed.

There are also non-algorithmic processes, for which interactive use of the CAS must be controlled. For example, finding limits is a non-algorithmic process.

It is possible to design alternatives using negative criteria such as reiterated limits or some positive criterion, such as the change to polar coordinates, which must be used with care (see next section). Thus, the function $f(x,y)$, after the changes $x = a + r\cos\phi$, $y = b + r\sin\phi$, is converted into the function $F(r, \phi)$. If this function has a uniform limit in the variable ϕ when $r \to 0$, then there exists limit of $f(x,y)$ at the point (a, b).

4.5. Fostering a critical spirit

It is clear that a CAS will enormously facilitate the task of checking results, since alternative methods can be easily used. Students should not develop a blind faith in the computer results, either intermediate or final; instead, they should at all times attempt to be in control of the situation in the sense that the results should be compatible with the context of the problem that is being solved in each case. It should be noted that sloppy use of CASs may give rise to errors with which the student is not very familiar, such as the errors due to previous assignations in the work session.

Additionally, the limitations shown by CASs can help us to foster students' critical awareness, presenting situations of calculations in which the answer of the CAS is unexpected.

For example, the expected output of the Maple instruction

```
>mtaylor(sin(x*y)/(x*y), [x=0, y=0],10);
```

should be $1-\dfrac{x^2y^2}{6}+\dfrac{x^4y^4}{120}$ but the computer output is $1-\dfrac{x^2y^2}{6}$.

Morever a CAS can obtain a mistaken result due to program failure or due to the consideration of conditions on variables that the user has not taken into account (see Alonso, F., García, F., Hoya, S., Rodríguez, G., de la Villa, A., 2001).

Also, the little care that students sometimes take in formal calculus may be aggravated by the use of a CAS.

Thus, if to calculate the limit at the point (1,1) of the function

$$f(x, y) = \frac{3x - 3x^2 + x^3 - 4y + 6y^2 - 4y^3 + y^4}{-2 + 3x - 3x^2 + x^3 + 4y - 6y^2 + 4y^3 - y^4}$$

we do the change to polar coordinates (see 4.4), where $a=b=1$, the resulting expression is $F(r,\theta) = \dfrac{\cos(\theta)^3 + r\sin(\theta)^4}{\cos(\theta)^3 - r\sin(\theta)^4}$.

Mathematica and Maple simplify to 1 the limit of this expression when $r \to 0$. Apparently, the limit does not depend on θ, but it is false, since if $\theta=\pi/2$, then the limit is -1. Therefore, the double limit does not exist.

Students must see the need to control the results.

5. Conclusions

We would like to propose a way of using mathematical software as a pedagogical tool. The graphical capacities of software packages can help to understand some concepts related to Differential Calculus of Several Variables.

We also show another possibilities of use: release of the mechanical work, possibility of experimentation, distinction between algorithmic and non-algorithmic processes and mainly to always have a critical spirit about the obtained results. Let us remark on the importance of fostering a change in the mentality of the teachers teaching theses kinds of topics.

We suggest that the software package chosen should not be as important as the way in which it is used; the same ideas and strategies with respect to software in education can be put into effect using different packages.

The possibilities of using new technologies are not limited to "witnessed" teaching (i.e., with the physical presence of the students); the teaching material compiled can be used in "virtual" teaching since there are no special problems involved in presenting the material online. In any case, this material can be completed with tutorials designed to facilitate its use.

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FIGURES AND TABLES

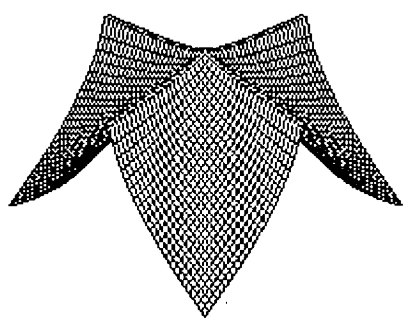

Figure 1

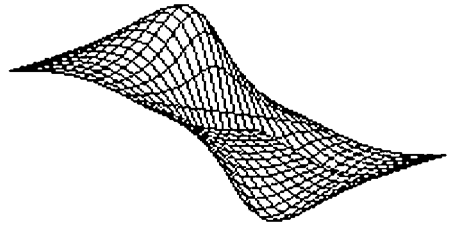

Figure 2

0.5	0.4	0.3	0.235294	0.192308
0.4	0.5	0.461538	0.4	0.344828
0.3	0.461538	0.5	0.48	0.441176
0.235294	0.4	0.48	0.5	0.487805
0.192308	0.344828	0.441176	0.487805	0.5

Table 1

"MODELLING AND SPREADSHEET CALCULATION"

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ABSTRACT

A. The process of modelling is a constant oscillation between various levels of abstraction. This can be divided into the following phases:

- get to grips with the problem (define the key questions)
- formulate a mathematical model
- generate solutions (from the mathematical model)
- validate the model (and if necessary re-formulate the model until it fits with the real-world-context)

B. Especially pupils in lower secondary schools have problems formulating their ideas or assumptions in a mathematical (algebraic) way. The transfer from the spoken language into the mathematical language becomes a difficulty as well, as it's not easy for them to generate solutions from the mathematical model. With **spreadsheets** these effects could be reduced in some fields because it is not urgently necessary to define variables and formulate equations. Furthermore the possibility of intuitive use and the splitting into modules appear as an advantage.

C. We have a classification for models.

– Models which process large quantities of data in an elementary fashion.
– Models which solve by using systematic testing.
– Models which are based on iteration and recursion.
– The qualitative and quantitative evaluation of data which requires only functional relations.
– The simulation of operations from which a mathematical solution model can be deduced.

D. Examples of the classification of models and using spreadsheets in the modelling process.
The **gas problem** (How far is it worse to drive for gas?) is an example of the class of models which are based on **systematic testing**. After formulating the model, the pupils try to solve the task by manipulating the in-data until the out-data fits the problem. The **Fermi task** (How many piano players exits in Chicago?) fits into the class of models which are based on the **evaluation** of data. Because there is no "true" answer to the question, the task is to find criterions for evaluation. To check the assumptions and the numbers in the model the use of spreadsheets provides advantages. The **bathroom problem** is an example of models which are based on **iteration and recursion**. In this model one can split of the whole process of filling up an wash-basin into modules and iterate single parts like water inflow, water outflow, total volume, water depth and outflow velocity. Other examples of the remaining classes of models can be found in the literature, listed in the bibliography. These are the **parachute jump problem** for the class of **simulation of operations** and the **financing problem** for the class of models which deals with **large quantities of data**. Of course these examples not only fit into one of the above mentioned classes.

E. Conclusions.
The earlier one begins with the concept of modelling, the better ones abilities became during the time at school and the better one recognises mathematics as a part of our world. In the modelling process spreadsheets could be used with all classes of models.

1. Introduction

This paper is based on the assumption that there is a great need for real world problems and modelling activities in schools. The paper is also based on the assumption that the use of spreadsheets in the modelling cycle offers advantages at different stages (of the cycle) and in different classes of models.

To start with modelling activities in lower secondary schools (or already in primary schools) means to start with uncomplicated but exemplary real-world problems and increase the complexity in lower and upper secondary schools, until final examinations achieve one of the most important goals of mathematical education: the arrangement of abilities to handle oncoming problems from different parts of life with mathematical methods.

More than 60% of about 100 pupils (14 to 16 year olds from a grammar school in Magdeburg) asked to solve word-problems like: *The schools solar panels collect 15kW of energy every hour. How many hours a day must the sunshine in order to collect 180kW?* "solved" the tasks without any critical reflection, only by manipulating the given numbers. More than 60% gave the "simple answer": *The sun must shine for 12 hours.* Afterwards when analysed, the pupils knew about the complexity of the tasks and were also aware of the knowledge that the "simple results" cannot be true. In summary, most of the pupils said: In *mathematics* I would find these results (because I have to calculate it with the given numbers), but in *reality* there is a great difference. That led us to our first assumption, that there is a great need of "real" real world problems.

2. The process of modelling

The process of modelling is an constant oscillation between various levels of reality. The modelling cycle can be divided in the following phases.

- get to grips with the problem (define the key questions)
- formulate a mathematical model
- generate solutions (from the mathematical model)
- validate the model (and if necessary re-formulate the model until it fits with the real-world-context)

The figure 1 shows the modelling cycle.

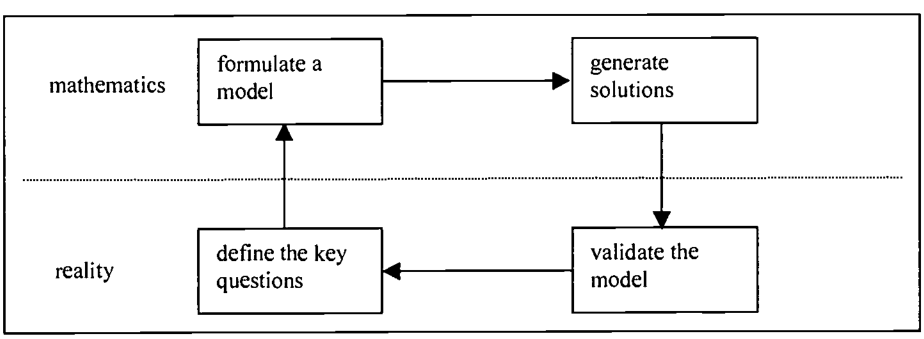

Fig. 1: modelling cycle

The process of modelling continues until the modeller accepts the validation criterion. In the cycle, one uses different kinds of languages (in "reality" one uses the spoken language and in "mathematics" one uses the mathematic language).

All four parts or steps of the modelling process can be a single component of an task for pupils or can be a topic for discussion in class-rooms as well as the whole process. For example, it's possible to concentrate on the formulation of the model without solving it or one can give a problem/task and a mathematical solution to the model and the pupils have to find and discuss evaluation criterions.

3. Modelling with spreadsheets

Especially pupils in lower secondary schools have problems formulating their ideas or assumptions in a mathematical (algebraic) way. It's not easy for them to define variables, formulate equations or inequalities.

The transfer from spoken language into mathematical language (fig. 1) becomes difficult as well, as it's not easy for the pupils to generate solutions. Often the mathematical model is unsolvable with school-skills. (For example differential equations.) Often difficulties occur and as a result displeasure or failure.

With spreadsheets these effects can be reduced in some fields because it is not immediately necessary to define variables and formulate equations. There is the possibility for a more intuitive or "real-life-use" of mathematical language.

Especially for "modelling beginners" this could be an advantage. Furthermore, the possibility of intuitive use and the splitting into modules appear as an advantage. That means one does not have to formulate a whole equation or a whole set of equations. With spreadsheets one "only" has to formulate ones ideas in very small steps. To fit the steps in the whole model becomes more and more the task of the spreadsheet. But not in a blackbox way, because every single step is visible (in the real meaning of the word). Last but not least, the algebraic equations have not disappeared, they are only hidden and for analytic discussion able to derive but with one big benefit: motivation (the pupils know, that their model is good).

Spreadsheets have further features, which are useful in the modelling process, mainly concerning modelling in schools:

- spreadsheets are simple-to-handle tools (without necessarily advanced computer knowledge)
- concentration on the modelling process without dealing with software-skills
- spreadsheets handle large quantities of data
- iterations and recursions are simple to implement
- visualisation of data, relations and functions in an easy way
- systematically testing (manipulate in-data and observe the resulting out-data)
- spreadsheets provide tools like sum, mean, max, min, etc.
- the table describes the problem, represents the model and often its solution
- spreadsheets are useful in almost every field of real-world problems

In summary of the mentioned features, spreadsheets are an educational equipment which are useful to teach efficiently the modelling process and better still, may be used in "real" real-world problems, not only in schools.

4. Classification for models

We found a classification for models if, in the modelling cycle, computers and software are used. We defined the classification as follows:

- Models, which process large quantities of data in an elementary fashion.
- Models, which solve using systematic testing.
- Models, which are based on iteration and recursion.
- The qualitative and quantitative evaluation of data, which requires only functional relations.
- The simulation of operations from which a mathematical solution model can be deduced.

Of course there are real world problems or tasks that fit into more then one of the above mentioned classes. But for pedagogical reasons the classification is useful and one can find problems or sub-problems (like the Fermi-Task) which only fit into one of these categories.

5. Examples

The first example is the so-called gas-problem. It can be solved by using a model, which calculates using systematic testing.

In Germany no week goes by without public discussion about the price of gas (and its increase). One can find in local newspapers and car magazines tables with price comparisons and a lot of people try to fill up their cars for a low price even if they don't use the nearest gas station, instead the gas station with the lowest price. Under these circumstances you have to ask the question:

Is it worth to tank up at a other gas station then the nearest?

This open problem can be modelled under different viewpoints! It depends on one understood under WORTH. At first almost everyone considers the price as a factor. Secondly one can think about factors like time or environment. Of course the assumption, one only have to go to the gas station with the lowest price doesn't work because the car use gas on the way. Now one is able to formulate the problem more precisely. How far is it worth driving to fill up the car? Or on a higher level: To what extent do the factors price difference, distance to the nearest gas station and the gas station with the cheapest petrol connect with each other? Normally one would now define variables and build up an equation and try to answer these questions. If you use a spreadsheet one can formulate the model in a more intuitive way and be able to look at the problem from different sides.

Gas-Problem						
price at station A	2,04	DM		total costs A	81,6	DM
price at station B	2,00	DM		travelling expenses	1,40	DM
gas consumption	0,07	litre/km		gas expenses B	80	DM
tank capacity	40	litre				
distance A <-> B	5	km		total costs B	81,40	DM

Table 1

Table 1 shows a simple model of the problem only considering the price, distance and tank capacity. But even at this level the pupils can check their assumptions, play with the numbers and answer questions like: How far is it worth travelling at a given price difference? How much do I have to put in the tank? What influence does the gas consumption have? Comparing this table with the 'equivalent' inequality $p_B \cdot c + 2 \cdot d \cdot g \cdot p_B < p_A \cdot c$ ($p_{A/B}$ ... price; c ... tank capacity; d ... distance; g ... gas consumption) the table is much more "real". Of course it's possible to derive the same results by manipulating the inequality, but the table is much more vivid (especially considering lower secondary schools). After looking at the model at this first level it's possible to re-formulate the model having considered of the actual position between A and B, remaining gas and on another level time costs and so on. This problem is didactically interesting because it is an example of iterative modelling (every model level fits the reality more).

The second example, mainly for lower secondary school, shows another advantage of using spreadsheets. The problem deals with the known FERMI-Task:

"How many piano tuners exist in Chicago?"

In this example we concentrate on a particular section of the modelling cycle (modelling process): the validation of the model. In this open problem one has to define a criterion to evaluate the model because there is no true answer. (Of course one could try to ask the trade corporation and hope they have the true numbers.)

Table 2, which was created by pupils of the 9th form, starts the process from their view of the world (school world).

How many piano tuners exist in Chicago?				
			number of pianos	
Population	3000000			
schools (each 12. --> pupil; 900 pupil each school)	280		280	(1 piano each)
households (4 persons each)	750000		187500	(each 4. household)
theaters/opera houses	50		100	(2 pianos each)
			187880	**total number**
piano tuner				
Pianos each day	2			
working days a year	300			
frequency per piano (once in ... years)	8			
piano tuners in Chicago	39,1			

Table 2

The interesting point is the answer to the question: Could this be true? Or better: Is it probable that this is a good estimation? Or more mathematical: Do I have a good model? Pupils have big problems with the evaluation of their solution because they are used to getting a right or wrong answer. They are trained to calculate the right numbers and not to get a good solution.

One possibility of validating a model is the method of parameter variation. (If the model reacts under small changes to the given numbers with only small or without any change to the model-solution then one has the indication that the model could be good.) In the mentioned example one can double the number of schools or theatres and find that the number of piano tuners doesn't change. On the other hand if one divide the number of working days by two the number of piano tuners double. It shows that the formulation of the given numbers and the model probably good. And it shows the potentials of the spreadsheet too. Pupils would lose the interest if they had to work out the numbers with paper and pencil (even if they used a calculator). They cannot fail and do miscalculations. They can play with the numbers and verify their ideas extremely quickly.

The last example deals with an interesting phenomenon in the bathroom and shows the potentials of spreadsheets handling large quantities of data and models, which are based on iteration. The phenomenon is:

If you turn on the tap the washbasin only fills up to a certain level then doesn't continue to rise if the outlet is open.

If the pupils consider this problem it's not as easy as it seems to begin with, because it is obvious that the water stops if there is an equivalent inflow and outflow but the outflow velocity is not constant and develops in a non-linear in comparison to the water depth. There are questions about the relation between radius of the outlet and water depth or the relation between inflow and water depth. In schools there is no possibility to set up a model with differential equations or put respect on effects like swirls and so on. Table 3 shows a model which was formulated by pupils after analysing the problem verbally, especially talking about the dependence of velocity ($v = \sqrt{2 \cdot g \cdot h}$; g ... gravitational pull of the earth; h ...height) and the outflow volume in time ($V_O = \pi \cdot r^2 \cdot v \cdot \Delta t$; r ... radius of the outlet) and the assumption that the basin is a cuboid.

wash-basin problem					
inflow	10	litre / min	166,7	cm^3 / s	
base area	900	cm^3			
outlet radius	0,7	Cm			
time interval	10	S			
time / s	V_{Inflow} / cm^3	$V_{Outflow}$ / cm^3	total volume / cm^3	depth / cm	velocity / cm s^{-1}
0,0	0,0	0,0	0,0	0,0	0,0
10,0	1666,7	0,0	1666,7	1,9	60,3
20,0	1666,7	927,9	2405,4	2,7	72,4
30,0	1666,7	1114,7	2957,4	3,3	80,3
⋮	⋮	⋮	⋮	⋮	⋮
620,0	1666,7	1666,6	5377,0	6,0	108,3
630,0	1666,7	1666,7	5377,0	6,0	108,3

Table 3

With this spreadsheet model the pupils are now able to gain play with the numbers and assumptions again without any need to do calculations by hand.

6. Conclusion

Spreadsheets provide a powerful, multipurpose tool to teach mathematical modelling. They are useful in almost all kinds of real world problems especially in connection with models, which deal with iterations, recursions, visualisation, simulation and mathematical experimentation. Pupils are almost free from technical problems (model building or model solving) and able to build a model in a very intuitive way.

The modelling process (cycle) is one didactical way of putting more emphasis on real world problems in schools even at lower levels.

If we want our pupils to develop the ability and skill to solve complex real-world problems at the end of their secondary school life, I believe we have to implement modelling activities in our classroom as early as possible. We have to start at primary school level with uncomplicated but exemplary problems and increase the complexity in the following years, in lower and upper secondary schools, until final examinations thus achieving one of the most important goals of mathematical education (besides the acquisition of elementary techniques): the ability to handle recurring problems from different parts of life with mathematical methods and to use mathematics in order to understand real-world-problems and real-world phenomena better and solve them in an (intelligent) mathematical way.

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USING COMPUTER-BASED PROJECTS WITH COOPERATIVE LEARNING IN FIRST-YEAR MATHEMATICS

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ABSTRACT

As part of a continuing 10-year, classroom research project to the effective use of cooperative learning in first-year mathematics, the use of computer-based projects has been investigated. The Maple Computer Algebra System (U of Waterloo) was adopted and the impact of various strategies has been investigated. The points discussed are: elaboration vs discovery, degree of complexity (difficulty and length), frequency of assignment, and the assessment of the teamwork.
This report discusses the impact of the various choices and suggests possible alternatives to be considered in the context of one's own institution and students. Student reactions to various choices are also discussed.

1. Introduction

This paper, which discusses the use of computer-based projects in first-year mathematics classes, consists of four parts. The next part gives the context, that is, describes the organization and operation of these classes. The succeeding parts discuss the use of projects and some conclusions about their use.

Following attendance at the now defunct "Problem Solving Across the Curriculum" at Wells College in 1992, the author undertook to revise his teaching methods and began what has evolved into a ten-year project (Rosenzweig 1994, 1995, 1998, 2001, Rosenzweig and Segovis, 1996).

About five years into this evolution, the use of computer-based projects using Maple V, a computer algebra system (CAS) that was developed at the University of Waterloo, Canada. Student reaction was not positive and the nature of these projects has been modified during successive classes.

However since the **purpose** of this course is to use the language of the course, mathematics, to have the students learn something of small group interactions and a systematic approach to problem solving, it is clear that the projects play an important role. They provide an opportunity for the student members of each team to work together to produce work that has value for their learning and their class standing. Moreover, although these are business students and not mathematics majors, nevertheless the goal of imparting a modest level of mathematical knowledge has not been ignored. In that regard, these projects make a contribution to their learning. Some students have reported that these projects, not required by every instructor, are unduly burdensome. This is due to the frustrations associated with computer use in general. Other student responses have been more positive and suggest an appreciation of the clarification of some of the ideas from the class.

2. Organization and Structure of the Class: the context

The class operates on the basis of group-work. The students are assigned to 3- or 4-person teams by the instruction. The assignments are based on a dozen-question assessment of algebra skills taken on the first day of class, gender, and living arrangement (on-campus, or not). Experience has shown that single gender groups do not do as well as mixed groups, indicating that there is a social component to group-work. In addition, each group had a student with a good score on the initial assessment, one who scored poorly, and the other one or two average scorers. It also seemed useful to associate students by living arrangement to facilitate out-of-class meetings that were expected as part of the course-work.

Each team elects a team leader whose responsibilities include: meeting with the instructor to discuss questions related to leadership and also mathematical and other questions, particularly regarding the projects. In the current iteration of this scheme, team leaders rotate among the members of the team, changing with each new project.

Generally, the class period is divided into three sections: a 15 to 20 minute introduction of new material through lecture and discussion, a 25-, or so, minute period in which the team members work together on problems, usually in the text, relating to the new material, and finally, a one or two question quiz closes every class session.

The quiz serves three purposes: it informs the instructor about the ability of the class to absorb the new material, it informs the student on his/her understanding of this material, and it is a convenient way to track attendance.

The fundamental operating principle in the class is to create a "learning organization" in the sense that "A learning organization is an organization skilled at creating, acquiring, and transferring knowledge, and at modifying its behavior to reflect new knowledge and insights (Garvin, 1993, p.80 – quoted in Rosenzweig and Segovis, 1996). In terms of this class, the goal is to look not only at what students were learning, but also *how* they were learning.

To this end, at bi-weekly "course evaluation" is distributed to determine what the instructor might do to be more effective and to ask the students to assess their level of participation and learning, These evaluations, or reviews, serve as the core of the "process evaluation" that help guide delivery of the course. They also give students a sense of control over the operation of the class that tends to reduce the anxiety that many students feel in mathematics classes.

This assessment of the process is somewhat novel in mathematics classes where traditionally only the content is assessed (Rosenzweig and Segovis, 1996), and is called "double-loop" learning in the management literature (Argyris and Schon, 1978, Bolman and Deal, 1991, Senge, 1990). Here, we examine as we go, how successful the delivery of instruction is. The advantage of this approach is to provide information in a timely way for the improvement of the class. Simple questions as, "Are the blackboards visible to everyone?", "Can everyone hear the speaker?", "Is the writing clear?", can provide surprising encouragement to students attempting to understand what is being offered. Of course, the instructor runs some risk in seeking this kind of feed-back from his/her students, however the rewards can be quite substantial.

3. The Use of Computer-based Projects

The issues relating to the use of computer-based projects that are discussed here are: investigation versus clarification, frequency and complexity, and assessment.

The first choice to be made is whether the projects are to provide opportunities for students to explore new ideas, perhaps to demonstrate or prove basic theorems, to examine material not covered in the classroom, or to explicate classroom material, perhaps at greater length or from a different perspective. This is choice is in large measure driven by the preparation of the students in the class. Students with stronger mathematics backgrounds are more likely to be capable of investigating new ideas on their own, or with limited oversight. For classes of weaker students, this approach tends to be frustrating – asking of the students more than they are able to do. In this situation, using projects to introduce new material discourages students and is likely to cause them to withdraw from the course either formally, or tacitly. This was our experience with attempting "learning by discovery".

Effective use of the projects to expand on material from the class offers interesting opportunities also. Students can demonstrate ideas graphically, taking advantage of the facility of the Maple CAS to draw graphs. It becomes a simple matter to repeat graphs any number of times. This means the limits of a graph can be changed as desired. And one can, for example, have students demonstrate for themselves local linearity, and beyond these simple situations where local linearity cannot be obtained. The library package, "Student", offers many routines useful in first-year calculus. It is a simple matter to represent Riemann sums graphically as rectangles draw between a function and the axis. It also a simple matter to change the number of terms in the sum. Therefore, students can see the convergence of upper and lower sums and make conclusions about this fact.

The frequency and complexity of projects are naturally related. For more capable students, more reliance can be placed on the computer projects to substitute for lecture time. Also, the projects should be sufficiently demanding so that the effort of the entire team is necessary. Otherwise, the stronger students tend to take over to the detriment of the others.

Projects in the context of group-work. Cooperative learning is a particular case of the kinds of group-work that may be used in mathematics classes (Davidson, 1990, Cooper, et al, 1990, Johnson and Johnson, 1991). It is highly structured, and in the classes being reported on, the students are assigned to teams that are called "base groups" in the literature. These are groups that stay together for the entire term. This is not the only alternative that is available, but has certain efficiencies, as well as, drawbacks.

Allowing the students to remain together for the entire term allows them to become comfortable with one another, and to develop a sense of teamwork. In the best cases, an esprit develops that brings the team members to class regularly in order to support their joint efforts. However, it is important to catch dysfunctional teams as early as possible in order to correct the situation. Otherwise, good students are condemned to an unpleasant experience with unproductive teammates.

In the case of group-work, questions of assessment become sensitive issues. Students can be extremely concerned on having their marks depend on the work of others, particularly more ambitious or competitive students. For projects, only one grade is awarded and each student receives it. Therefore, the incentive is for the better students to strive to take over the work, and for the others to accept "academic welfare". This requires some attention from the instructor, however can be mitigated by having grades adjusted by the amount of effort contributed by each student, or by having the project grades matter less, or by other stratagems. In this class, students are required to sign their names to the project to indicated participation, and are penalized for false statements.

In the end, it must be said that group projects encourage group-work. By holding sessions in which the team leaders are given information about the project, expect outcomes, etc, empowers them and gives them more confidence in dealing with their teammates.

As a final comment on the use of teams in first-year classes, it is sometimes the case, perhaps more often that not, that students do not possess the skills to function effectively in teams. It is particularly difficult for first-year students to perform the difficult tasks of team leadership. It may prove necessary to provide some type of support for the teams and the team leaders. In the classes here, the student leaders are required to attend a weekly session outside of class for training in the skills that may be required of them (Rosenzweig and Segovis, 1996, Rosenzweig, 2001). This training has solved many of the problems the team leaders face, and provided platforms for the exchange of ideas and experiences among the students.

4. Conclusions

The use of computer-based projects in junction with cooperative learning has proved beneficial on two counts: it has led students to an understanding of mutual enterprise, and encouraged them to participate in team activities with the knowledge that when everyone gains then it is a tautology that each individual gains. As our politicians so mischievously say, a rising tide lifts all boats, except in this case it is in fact true. Also, the projects have enriched the understanding of students by providing alternative views of the classroom material. The opportunity to visualize mathematical ideas using the graphing capability of the computer package has proven helpful, and to interested students exciting.

In our experience, the Maple V Computer Algebra System is a convenient vehicle for executing these projects. It has a "user-friendly" interface, and requires a minimum of programming skill on the part of the student. It also has sufficient power to provide the instructor with a variety of options for presenting material in the classroom and in the laboratory. There are of course alternative packages available, and there may be institutional as well as instructional reasons for choosing one or another. This is less important that the decision whether or not to require projects at all.

Student response has been variable. The students expressing approval of the use of computer projects usually enjoyed the additional exposure to mathematical ideas from a different perspective. The students that disapproved usually objected to the additional work entailed in going to the computer labs and attempting to extract results from their sometimes difficult interactions with the computers. On balance, the contention is that these projects contribute to learning and are worth the relatively modest effort required of the students.

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TEACHING FOR UNDERSTANDING FOR TEACHING: ADDRESSING THE CHALLENGE

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ABSTRACT

I present a constructivist approach that is used in some TE mathematics classes at MSU. This approach employs exclusively a model of a 'mathematical situation', a set of physical operations and a physical language to reason about **all** students' mathematical doings while a unique system of reinforcements, grading and assessment methods, support the learning experience.

Introduction

Elementary school pre-service teachers tend to ignore the need to learn mathematics, as 'we don't need it', 'we already know it', where the 'it' refers to what they conceptualize as the mathematics that is taught in k-5.

By and large, students are puzzled when they are confronted with a 'weird' question like 'what is...' which very few of them would answer and even fewer would be able to explain "why..." by means of a concrete example or by means of abstract reasoning. All the more when one gets to fractions; which usually involves intense emotions. The Math-Educator reader is probably familiar with the Mantra 'but I know HOW to do it...' which one could frequently hear in my office especially during the period of learning of fractions. Though the student stated that it is extremely important for her/him as a future teacher to be able to explain WHY she/he 'flips' the ½ and WHAT does the 6 stand for in $3 \div \frac{1}{2}$ still when 'forced' to try and construct some kind of explanation she/he would cling to the Mantra 'but I know how...'.

More often than not it feels as if the students are trying as hard as they can to 'protect' their 'fragile' mathematical assets, to keep it intact and away from me. Thus in the beginning of the course it seems that the students perceive me as the 'destroyer' of their knowledge rather than the one who is supposed to help them to construct it. This is supported by a student's comment in a first-day-of-class Attitude-Questionnaire: 'I'm a little unsure about fractions so I'm nervous that we'll spend so much time on them...' referring to the Syllabus that shows that a large fraction of the course will be dedicated to fractions.

Therefore I've employed a somewhat 'aggressive' constructivist approach in order to get the students to unpack their fragile mathematical assets and to re-construct a more flexible and a deeper understanding of the different basic mathematical concepts. The main goals of this approach are to 'force' prospective teachers to understand their math-doings and to develop a reasoning attitude towards the learning and the teaching of mathematics.

Though at this time I do not have any formal research results, I will provide findings that could indicate that this approach is effective in achieving its goals. Also, I'll bring findings that could indicate that it is also the students that find this approach helpful.

However, I would like to emphasize that this approach was developed for pre-service elementary school teachers and not for elementary school students. It was meant to help the future elementary school students to make sense of their mathematical knowledge and doings, and not to teach them the basic concepts. At the same time this approach provided them with an efficient tool to analyze and to understand their future students' doings of mathematics. Nevertheless, I believe that there are some aspects of this approach that could be adjusted to help young students in their learning of mathematics.

The Approach

Much of Math-Education is about understanding students' difficulties in grasping the Abstract-Formal Mathematical concepts and algorithms. Thus our proposed approach retreats to natural-physical doings in trying to promote a "natural" or an intuitive understanding of the basic mathematical concepts as a well-established, firm base for the understanding of the more abstract, formal concepts.

Therefore our approach is based almost entirely on natural operations that our mind can perform without any formal learning ("Join", "Take Away", instead of Addition and Subtraction). Furthermore we have used a "natural" physical language, assuming that we "don't know" any formal mathematics and so we cannot use words that mathematicians "invented" such as addition, subtraction, division etc.

Thus the main principles of our approach are:

a Constructing a Natural-Intuitive understanding of the mathematical concepts:

- a.1 Using only "Natural" operations, which require no formal learning of "how", such as 'to Join'.
- a.2 Using "Physical" language and avoiding Formal mathematical language: JOIN but not ADD, CUT INTO but not DIVIDE etc.
- a.3 Using Visualization tools such as drawings and 'role acting': 'I'm the first set and you are the second, How many of 'you' can be made out of me' or acting out the 'joining'/ 'taking-away' of the sets/elements etc.

b Building on a deep understanding of simple Whole Number situations as a basis for all further learning:

- b.1 Using a "Whole Number" Language: 'we have <u>HALF groups</u> ...' and not 'we have <u>half a group</u>'.
- b.2 . Fractions are 'just' Numbers: ½ or 1½ is as good of a number as 1, 2 or 10...
- b.3 Using Whole ("Natural") Number models to deal with "new" non-natural kinds of numbers: $\frac{2}{3} \div \frac{5}{7} \approx 6 \div 5$ so 'DO THE SAME'.
- b.4 Sequencing a continuum from whole number situations to fractions: Something like $4 \div 1, 4 \div 2, 4 \div 4, 4 \div 3, 1 \div 4, \frac{1}{2} \div 1, \frac{1}{2} \div 2, \ldots 1 \div \frac{1}{2}, 1\frac{1}{2} \div \frac{1}{2}, \ldots 1\frac{3}{4} \div \frac{15}{17} \ldots\ldots$

c Using an abstract model, which conceptualize all basic mathematical situations as the same, to channel and to mold the reasoning.

d Promoting a Reasoning attitude:

- d.1 Any statement must be accompanied by reasons: Why is it true/false, Why is it important/not important, Why is interesting/not interesting?
- d.2 Making as many statements as possible about any given situation: Declarative statements: 'we have 3 sets', descriptive statements: 'there are 3, 2 and 3 elements in the sets', or relational statements: 'these 2 sets are of equal size' or 'this is larger than that' etc.
- d.3 Exploring each situation thoroughly rather than employing a "task-oriented" exploration: All the Why's, and How's questions as well as 'What this means', 'What if we change this...", "Which units are used" etc. .
- d.4 Concentrating on a few carefully chosen examples rather than investigating many situations.
- d.5 Massive Reasoning tasks (using our physical language and our model exclusively): Why $\frac{2}{3} \div \frac{5}{7} = \frac{14}{15}$.
- d.6 Massive investigations (trying to reason) of other student's understandings and solutions.

e Developing awareness of the meta-cognitive processes:

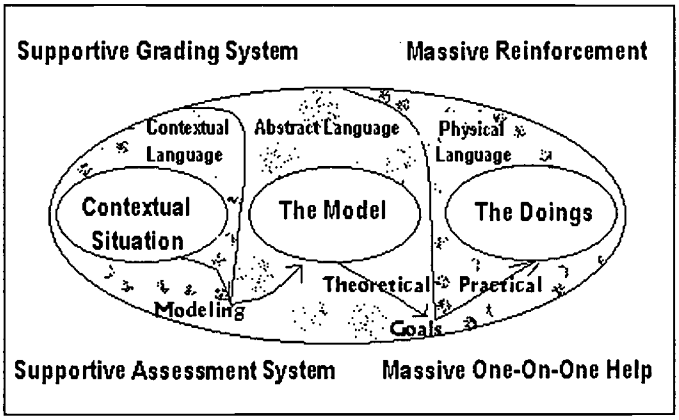
- e.1 Using the model to compare the same/different situations, reasoning and doings.
- e.2 Using the same language/doings in comparable situations.
- e.3 Reasoning about the Thinking: 'Why this procedure and not another', 'what made me think of this...' 'Does it remind me of anything...' etc.

f Employing a holstic approach working up through the contextual to the Abstract and then 'back' to the "Abstract-Physical" plan and again to the contextual stratums.

The Learning Space

We perceive the core of the Learning Space in which the learning-teaching experience is taking place as consisting of three sub-spaces: The contextual, The Abstract and The Physical-Abstract sub-spaces. These three sub-spaces differ in the kinds of reasoning that are employed in each of them; in the language that is used in each of them and in the kinds of activities that are performed in each of them. The transition among the three sub-spaces is done by goal-driven means of re-phrasing into the language of that used in the 'new' sub-space. This model also provides the frame for relating the contextual concepts (Altogether), the abstract-mathematical concepts (Addition), the Abstract-Set concepts (Union) and Physical-Abstract concepts (Join). The core of the Learning Space is surrounded by a comprehensive Support System, as it is shown in Figure 1:

Figure 1 The Learning Experience



The Support System

We used intense Verbal Reinforcements in order to boost the student's self-image and their confidence as well as to reduce their anxiety level. To illustrate, we frequently used sayings such as: '… the future of the mathematics knowledge is in your hands…' or '...If you were just an engineering student I wouldn't mind, but I expect more from you...', or '…you can't understand it NOW but you WILL in a short while…', or "…anything I can do YOU can do better…". A relaxed and an informal atmosphere encouraged students to discuss as openly, and as frankly as they could their mathematical ideas, their feelings and their attitudes towards the learning experience.

Furthermore, the students had as many one-on-one instructional sessions as they needed and many more by e-mail consultations. Though it is noteworthy that few of the students perceived the 'generous' office hours policy as a negative one, some expressed feelings that could be summed up as: 'we paid for learning in class, so YOU need to make sure that WE will not need office hours'...

In addition, we employed a supportive grading system in which the 'learning processes' (rather than the "results" or "products") were assessed and graded and in which most of the points were given on 'proven hard work'. To illustrate, the final exam was only 25% of the total grade, 10% was assigned to the weekly Home Work assignments (which were returned fully checked and with relevant comments), 12.5% was assigned to two papers, and approximately 50% was assigned to the quizzes and the exams that students could re-do.

The Contextual Sub-Space

By the contextual sub-space we refer to the sub-space of the mathematical story-problem. Thus the language here is contextual and is related to the 'story' (Apples and oranges in one instance or velocity and distance in another), while the activities here are 'real' (Picking or eating in the apples case or driving in the other) and so is the reasoning (We have less apples since few were eaten or -3 can't be the velocity since the car is moving forward).

Though there is much to be said about the nature and the constructs of this sub-space (choosing the problem, sequencing issues etc.) we will limit our current presentation to the two other sub-spaces.

The transition from the contextual sub-space to the Abstract sub-space is motivated by the "problem" which is first stated in the "contextual language" (How many fruits do I have altogether? Or How far did he drive? Etc.) and then modeled by the Abstract Model and Re-phrased into Abstract Sets-language. This is also the point in which we relate the Formal mathematical concepts to the contextual concepts.

The Abstract Sub-Space and the Abstract Model

By modeling the contextual situation we move to work in the Abstract sub-space and to use Abstract Sets-Language. Complex situations involve staging (breaking down) procedures and using the model iteratively, but the Basic Model refers only to the basic binary mathematical operations ($+, -, \times, \div$) and it consists of three components, and two types of goals.

The Model Components are:

a. First Component - The Number of Disjoint Sets that are involved in the situation.
b. Second Component - The Number of Elements in each of the disjoint sets.
c. Third Component - The Total Number of Elements in the situation.

The Model Goals are:

a. The Theoretical Goal is: Either to 'expose' the 'omitted' value of one of the model's components (i.e. # of sets or # of elements) or to 'describe' the relations between two of the model's components.
b. The Practical Goal is: To 'physically' do something in order to achieve the theoretical goal (i.e. to JOIN sets, to TAKE-AWAY elements, to PUT-EQUALLY into a few empty sets, to compare/correspond sets, to measure one set using the other, etc)

Hence we define two basic types of the model: The *Additive Model* and the *Multiplicative Model.* While a **CONSTANT NUMBER** (2) of disjoint sets that are involved in a situation characterizes the *Additive Model,* it is the **EQUAL SIZE** of the disjoint sets that characterizes the *Multiplicative Model.* In both models the third component is the number of elements of the **Union Set** of all the disjoint sets.

The Contextual Space determines the Theoretical Goal for the Abstract Space, which is therefore expressed in Sets-Language. If the theoretical Goal is to 'expose' the third component, then the model represents an addition (*Additive Model*) or multiplication (*Multiplicative Model*) situation, whereas if the theoretical Goal is to 'expose' the second or first component, then the model represents a subtraction (*Additive Model*) or a division (*Multiplicative Model*) situation.

Furthermore, in the case of finding the First Component in a multiplicative situation, the model describes what is usually referred to as The Measurement Division approach, while if it is to find the second component (# of elements in each of the disjoint sets) then the model describes what is usually referred to as The Sharing or Partitative approach.

Relational Theoretical Goals in the *Additive Model* could describe either additive relations (bigger-smaller) that are basically subtraction situations or Multiplicative relations, which are Ratio situations.

The reasoning in the Abstract Space is based on the relations between the Sets and it is done in order to support the doings in the Physical-Abstract sub-space. For example a contextual situation of 2/3 x 15 will consist of 2/3 of a set that contains 15 elements, hence since the Union set is comprised of less than one such set it must be that its number of elements is less than the number of elements in one set, which is 15. Similarly in the case of 15 x 2/3; each of the 15 sets "contribute" to the Union Set less than 1 element, so it must be that the number of elements in the Union-Set is less than the number of sets. Hence, if in the Physical-Abstract sub-space we get a solution of more than 15 elements, we'll know that we are mistaken.

The Physical Abstract Sub-Space or the Doings

The theoretical Goal determines the practical Goal in the transition to the Physical-Abstract sub-space, where the 'Doings' take place. The most significant features of this space are the 'physical' doings and the language that are employed on the abstract objects (Sets and Elements), such as "Put (elements) Equally", "Put Proportionally", "Take Away (elements)", "Make sets of size x", "Break Down (sets)", "Stage (the doing)" etc.

These physical doings also serve to reason "practically" about the situation. For example: '$3 \div \frac{1}{2} = 6$ since I've MADE 6 sets, each having ½ elements until I've exhausted my resource set of 3 elements'. Alternatively; 'I've PUT-EQUALLY all my 3 elements into all of my ½ empty sets. So now each of the (whole) sets in the situation has 6 elements, since each of its 2 halves has 3 elements'.

At this point, I will present a few (partial) examples that best illustrate our method and than we will describe a typical class discussion to provide the context in which we use this approach:

1. Abstract Space: 145+324= , Sets Language - 2 sets of 145 (Set A) and 324 (Set B) elements - An Additive Model. The Theoretical Goal is to reveal the number of elements in the Union Set. Hence in the Physical-Abstract Space the Practical Goal (expressed in "physical" Language on Abstract objects) is to JOIN both sets. The Doing of the JOIN will be STAGED:
   a. Reasoning: since I need to JOIN all sets of the situation I can do it in any way that I wish to as long as all the elements of all the sets will 'get' into the Union-Set eventually, so:
   b. First Stage – BREAKDOWN Set B into 3 Sub Sets B_1 of 300 (Ones) elements, B_2 of 20 (Ones not Tens) elements and B_3 of 4 (Ones) elements.
   c. Reasoning - I am comfortable with these "nice-round" numbers which I can manipulate mentally, so:
   d. Second Stage - JOIN A and B_1 (445 elements) and then JOIN this with B_2 (465 elements) and finally JOIN this with B_3 to have the Union-Set with 469 elements.

2. 364-79= Abstract Space - 2 sets of 79 (Set A) and of ? (Set B) elements and the Union-Set have 364 elements - An Additive Model. The Theoretical Goal is to reveal the number of elements in Set B. Hence in the Physical-Abstract Sub-Space the Practical Goal is to "Take Away" the 79 elements of Set A from the Union-Set so only the elements of set B will be left there. Figure 2 illustrates partial doing in this situation. It is noteworthy that the reasoning that leads the DOINGS is the wish to work with small numbers and so the doings involve Tens and Hundreds and not only Ones as in example # 1:

3. $3\frac{1}{4} \div 2\frac{1}{5} =$: First Component- # of Sets-?- Second Component- # of elements in each set - $2\frac{1}{5}$, Third Component - Total Number of elements $3\frac{1}{4}$ Hence the theoretical Goal is to reveal The value of the first Component. So the practical Goal is to MAKE SETS of $2\frac{1}{5}$ elements each TO EXHAUST our RESOURCE SET of $3\frac{1}{4}$ elements, as is shown in Figure.3.

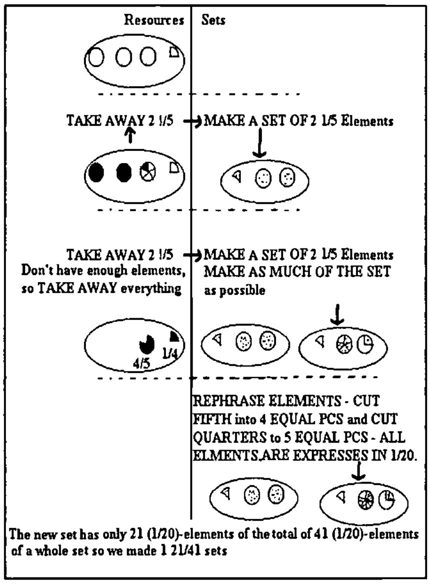


Figure 3 Example No.3

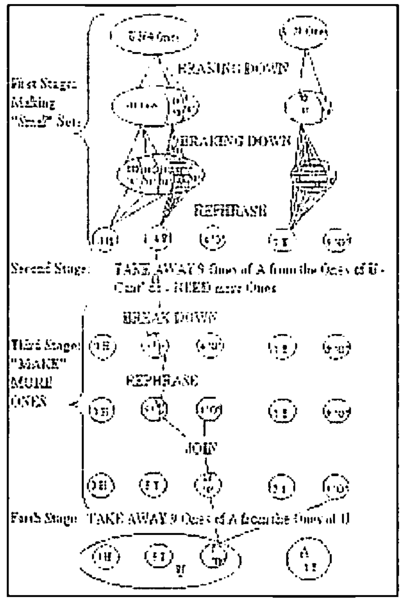


Figure 2 Example No.2

4. $12\frac{1}{6}+1\frac{3}{4}=$: First Component- # of Sets - $1\frac{3}{4}$ - Second Component- # of elements in each set -?, Third Component - Total Number of element $12\frac{1}{6}$. Hence the theoretical Goal is to reveal The value of the Second Component. So the practical Goal is to PUT EQUALLY all the $12\frac{1}{6}$ elements, to EXHAUST our RESOURCE SET, into all the ('empty') sets ($1\frac{3}{4}$). Partial doings are described in Figure 4.

5. Contextual Situation of Share $27 according to 3:2 Ratio- Model- 2 sets (additive situation) - # of elements in both sets are different and unknown, Total # of elements in the situation 27. Theoretical Goal to reveal the # of elements in each set – The Practical Goal – To PUT PROPORTIONALLY (to EXHAUST our RESOURCE SET of) all the elements in the 2 empty sets. Figure 5 Describe one way of 'Doing" it:

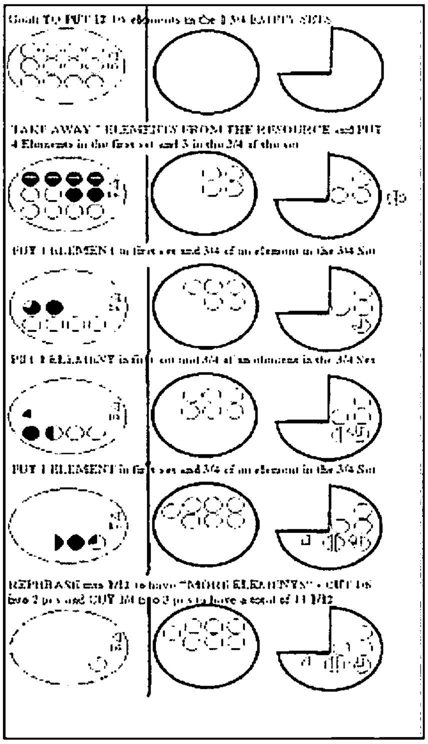


Figure 4 Example No.4

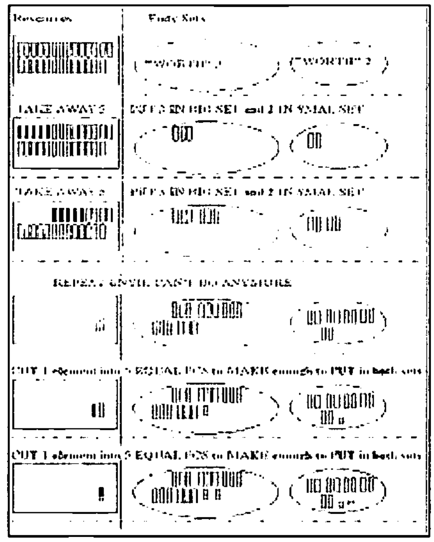


Figure 5 Example No.5

A typical Class Experience

The class discussion usually begins by presenting one or more contextual situations that lead to a specific mathematical operation, or to a few operations depending on whether my objective is to investigate an operation or if it is to compare a few operations. These situations are brought up by me or by the students as a response to my challenge.

We discuss the use of the contextual language and how it affects the way in which one perceive the mathematical situation. To Illustrate: 'I had 15 candies and I ate 7 of them. How many more…' or 'I ate 7 candies and I'm allowed to eat 15. How many more…' or 'I have 7 candies to give to my 15 guests. How many more candies…'. The language will lead us to different mathematical representations (in the Abstract-mathematical sub-space) that are all 'summed' in the 'mathematical sentence': 7+8=15.

We than move to the physical-abstract sub-space by introducing "our model" for the situation (2 sets, with 7 and ? elements each, 15 elements in total), the theoretical goal (to reveal the second component, number of elements in each of the sets), and the practical goal. The practical goal is motivated by the contextual space. Either it is to "take Away" (what I ate…), or it is to "fill in" (what I'll eat), or it is to "Pair" (candy to a guest).

We emphasize the connection between the contextual situations to the practical goals in the context of how a teacher can initiate a specific "physical-doing" by his students.

Also, when it is relevant, we discuss different ways of "Physical-Doing" to achieve the same Practical Goal such as the one in example No.2 for 364-79; in this case we compared 8 different algorithms of students, not all of which were mathematically correct. In each case we discuss All significant different options for a specific Doings, emphasizing the strengths and weakness of each one of them.

We also try to understand the connections between the motivations that lead an individual to his/her doings. For example the traditional addition/subtraction by columns could be understood as motivated by a desire to work with small numbers, not exceeding 10. Breaking down "ugly" numbers such as 364 to 300, 60 and 4 could indicate that this individual has no problem conceptualizing or manipulating 'big' numbers as long as they will be 'nice' and round. Also, using a 'counting on' (missing addend) algorithm for solving a subtraction problem (in our physical-abstract language it is referred to as 'fill in') could be understood by a strong inclination to addition algorithms and avoiding subtraction algorithms, which could be a sign of some weakness in this area. This kind of insight is something that a future teacher should be aware of while 'just' a mathematician could be satisfied with the fact that the 'mission had been completed'.

Other kinds of class discussion are constructed around a given solution to a specific mathematical sentence, which is provided by me or by the students themselves as a response to my challenge ($3 \div \frac{1}{2}$). Here the solution is purely mathematical, and we are trying to "reconstruct" the meaning, or the motivation to this solution by "justifying" each of the steps by means of our "physical language and our model". We ask: did this student think about many sets, each of them with exactly ½ elements (not half an element), and when joined together make a set with 3 elements. Perhaps he thought about a situation with as many as ½ sets (~2 sets), each of them having exactly the same number of elements (which we can't see at the moment), and 'all the ½ sets' are joined together to make a set of 3 elements. The first option would lead to a practical goal of "making" sets of ½ elements and we will look for 'evidence' of that (something like ½ + ½ + ...) or may be we will look for evidence that he is "putting equally" all his 3 elements (resource) into all of his ½ sets, and than he looks at One set to determine how many elements are in each of his sets. The students seem to enjoy this kind of discussion and they usually are very active in these discussions.

Many times the discussions are based on group activities in which groups of students try to make sense of a given solution (or to 'physically do' in order to solve a problem). Sometimes each group works on a different solution and the discussion consists of presenting the different findings and trying to gain a comprehensive picture. In other instances the different groups will work on different problems (63+45, 63-45, 63x45, 63÷45) and the discussion consists of comparing the different doings in the different situations and how our model explain could these differences and similarities.

In addition to discussions of the more practical kind ("doing" to solve, analyzing and comparing different "doings") we also have theoretical discussions. In some of them we discuss the theoretical-mathematical rules (associative, commutative and distributive) and how we can "prove" them by our "practical-doing" methods.

While in other theoretical discussions we compare the mathematical concepts of the different basic binary operations of Arithmetic by means of 'our' model and 'language', we also consider the different approaches to teaching Arithmetic that exists in the literature and we 'connect' them to our models. For example, the Measurement approach to division is tied to our Multiplicative model where the theoretical goal is to reveal the first component - the number of sets in the situation. Furthermore, these approaches contribute a significant insight to our approach. For example, the "making of sets" could be understood as "using a measuring set/cup". These discussions offer the students opportunities for consolidating their knowledge in which they can make sense of the many "different mathematics details" that they have collected through years of studying mathematics and to construct for themselves the 'big picture'.

Though some of the students had complained that this approach 'makes things harder than they really are', I believe that these types of comments reflect a misunderstanding of the main goal of a basic college-level Mathematics course for future elementary school teachers. I believe that the goal of such a course is not to 'teach' addition/subtraction/multiplication/division[1] but rather it is to offer a substantive basis for understanding of the knowledge or algorithms that the students already have (and which therefore they consider to be 'easy').

This approach offers informal "proofs" or justifications for the knowledge that the students already pos sess, but are unable to explain or to justify. The model, the "Physical-Doings" and the "Physical-Language" serve us instead of the formal theorems and logic which are used by mathematician to prove/understand their mathematical knowledge. By "physically" tracing each step of the 'statement' (solution algorithm, commutative rule, etc.) we prove it is "True" or "False". Moreover the 'physical-doing' serves as what mathematicians referred to as insightful proof, a proof that offers a 'deep' understanding of the situation on hand.

Also one of the student's tasks as future math-teachers will be to identify difficulties of their future student's doings of mathematics and to help their students to resolve these difficulties. Our approach provides them with a tool that makes tracking down and pinpointing these difficulties easy as well as offers them ideas to help their future students by means of "physical-doings".

Discussion

It is rather difficult to put into two-dimensional paper the full picture of a teaching-learning philosophy, which entails many dimensions simultaneously (mathematical, physical-doings, cognitive-reasoning, cognitive-procedural, affective, class interactions, individual aspects etc..).

The students were constantly engaged in verbally explaining each step of their contextual, abstract and physical doings and their motivation for doing it at different levels and in the different "languages". They were constantly required to relate various representations (I.e. contextual, formal-mathematical, the abstract - physical) and doings across situations and across concepts. Students were encouraged to construct their own individual 'doings' and they were challenged to try also 'awkward' procedures and not only the most 'efficient' one. For example: '...try to put ½ elements in each set first, even if it will make the 'leftovers' in the resource set an 'ugly' number...'(We were frequently using 'ugly' numbers).

Also, they had to 'finish' their colleagues' doings, or to come up with reasoning for their colleague's doings. Alongside, we had to work on the emotional dimension as well, since confusion and frustration were frequently threatening to interfere with the learning.

It seems that many of the students tend to appreciate the concise view of the situation that the model grants them, as well as the rigid frame it provides to lead the doings and the reasoning in a new situation. Also they seem to enjoy the flexibility that using the non-formal mathematical language permits. Nonetheless, they appear to not be very enthusiastic about the less structured and what they have conceptualized as less 'directed' teaching or "teaching-less" teaching.

[1] Though sadly enough too often we find ourselves in a position that we are obligated to verify that our students do know these basic mathematics concepts and algorithms.

By the end of the semester more than 70% of the students in both classes (50 students) were able to present good reasoning (>60%) while about 40% of the students presented very good to excellent reasoning capabilities (> 75%). Also, by the end of the semester it was evident that the quality of the discussions in the class was changed to the better, considerably. By then, students could discuss the whole mathematical situation from the constructing of a "good" problem-story up to the "doing" to solve it and they could reason about all its different aspects.

However the picture is not all so bright, and this approach calls for persistence in implementation in order to be effective, as 'It's too hard' for the students, and for the most part they prefer 'just tell me what to do and I'll do it' as can be seen in a few of the students' comments in the evaluation: '... She is very frustrating although she makes you think... I feel she makes things more difficult than they are, ..this class is too challenging for the type of class, ...but be aware the course is a lot of hard work...'.

As mentioned before, we found this approach to be very efficient, particularly in promoting students' understanding of fraction's situations. Students that first resisted my 'extremist' initiative eventually 'discovered' that 'It's so simple, I can't believe I did not understand it before'. So the following collage of students' comments might offer an optimistic closure for this paper: '... Teaching was excellent and she puts everything into context and helped me understand why ...have to admit that in the past I have been afraid of math - you have taught me that it can actually be "fun." ...was my favorite class this semester, which really surprised me because I didn't think I liked math....at the beginning of the semester I gave you a below average grade of your teaching ... at the end, not only me, but a lot of my other classmates... saw this pattern... After using it (the model) continuously, ...it makes problems a lot easier to solve and easier to explain..'.

TRAINING OF UNDERGRADUATE TEACHERS IN NIGERIAN UNIVERSITIES: FOCUS ON PROBLEMS OF EFFECTIVE INTEGRATION AND ATTITUDE OF STUDENTS TO COMPUTERS IN MATHEMATICS INSTRUCTION

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ABSTRACT

It has been argued that the ways teachers were taught and the behaviour patterns they developed for coping with feelings exert a long pull on their teaching. It follows, therefore, that if teachers are to comprehend and appreciate the nature of mathematics and mathematical thinking, they must experience as learners, the kinds of mathematical knowledge and thinking that they are expected to teach. Similarly, if teachers are to appreciate the use of instructional materials and tools, especially the new technology (computer), in mathematics instruction, they must experience as learners, and be exposed to the use of such technology in mathematics teaching, as a model of what they themselves might do. The study was designed to investigate the attitude of undergraduate mathematics education students to computer usage and the problems facing the effective integration of computers into mathematics instruction in Nigerian Universities. Three hundred undergraduate mathematics education students and thirty mathematics educators were selected for the study through stratified random sampling technique. Two sets of questionnaires- one for the students and the other for the mathematics educators were used, for data collection. Percentage, means and t-test statistic were used for data analysis. The results revealed non-availability of manpower and computers in the universities for the training of mathematics education students due to inadequate funding of higher education in Nigeria. Therefore, the student teachers are not exposed to computer usage in mathematics instruction. Some of the recommendations made include: more money should be made available for the universities to enable them acquire both human and material resources for effective integration of computers into mathematics instruction at undergraduate level.

Introduction

The issue of poor performance of students in mathematics has become a perennial problem, both elsewhere and in Nigeria. In Nigeria, the performances of students in external examinations in mathematics have continued to slide on a downward trend. Learners continue to manifest weak understanding of mathematics concepts, skills, generalizations, etc, not only in external examinations, but also in internal examinations and classroom exercises (Bot, 2000).

One contributing factor to this problem is the teacher factor. For instance, Agwagah (1993), Harbor-Peters and Ogomaka (1991), highlighted the issue of teaching methods adopted by teachers. This could arise from the fact that many teachers are not competent to teach mathematics, and are not able to provide and use necessary instructional materials for teaching, especially the new technology (computer), which is beginning to have a significant impact on almost every aspect of our lives, especially the education sector (Perl, 1990). As is well known, the basis for acquisition of knowledge in subject matter and ability to provide and use appropriate instructional materials is provided in the methodology courses teachers go through in their training. Thus, one objective of teacher education, as stated in the National Policy on Education, is to provide teachers with the intellectual and professional background adequate for their assignment and make them adaptable to changing situations (Federal Republic of Nigeria, 1998).

However, it has been recognised, in Nigeria and elsewhere that the undergraduate program is not adequately preparing teachers to function as professionals (Reid, 1997). This supports previous reports (e.g. Bigum, 1990), which have indicated that many of the inappropriate uses of IT in schools are the result of lack of preparation and training for teachers.

In Nigeria, the computer is slowly finding its way into the public school (Fafunwa, 1991), and it has been found very useful. However, not much seems to have changed since 1991, especially with respect to the training of teachers on the use of computers. If computers are known to have positive influence on education and especially the pedagogical aspect of education, teachers ought to be familiar with how to effectively utilize them for instructional purposes, and this they ought to acquire during training. It has been argued that the ways teachers were taught and the behaviour patterns they developed for coping with feelings exert a long pull on their teaching (Hyde, 1989). It follows therefore that teachers must experience good mathematics teaching as a model of what they themselves might do. They must experience as learners, and be exposed to the use of instructional materials, such as the new technology (computer) in mathematics teaching, as a model of what they themselves might do.

Given this stand, how are undergraduate mathematics education students in Nigerian Universities prepared to cope with the teaching of mathematics after their training, especially as it relates to the use of computer in mathematics instruction? Are computers available in education departments of Nigerian Universities for mathematics education program? To what extent are students exposed to the use of computer in mathematics instruction in the mathematics method course?

Also, it has been pointed out that teachers' beliefs, attitudes and feelings about mathematics are equally important in developing in them the confidence and competence they need to be able to teach mathematics. Thus, a survey of teachers' attitudes toward the use of computer in mathematics education is necessary. Harbor-Peters (1997), found that majority of mathematics teachers in Nigerian secondary schools are not ready and not in support of the use of computers for fear of being displaced from job. Would undergraduate mathematics education students have the same feeling? Would the feelings of students be influenced by gender of student or the type of University (Federal or State), in which the student is studying? What are the problems militating against the effective integration of computers in mathematics instruction? These questions constitute the problem of this study.

Research Questions

1. What proportion of Universities has computers for mathematics education program in Nigeria?
2. To what extent are mathematics education students exposed to the use of computer in teaching mathematics in the mathematics methods course?
3. What proportion of mathematics educators in Nigerian Universities is computer literate?
4. What proportion of mathematics educators own computer machines?
5. What are the attitudes of undergraduate mathematics education students to computer usage in mathematics instruction?
6. What are the problems hindering the effective integration of computers into mathematics instruction in Nigerian Universities?

Hypotheses

The following hypotheses were tested at 0 .05 level of significance.

Ho_1 : The mean attitude rating of male undergraduate mathematics education students on the use of computer in mathematics instruction , does not differ significantly from that of female students.

Ho_2 : There is no significant difference in the mean attitude ratings of students from Federal Universities and those from State Universities, on the use of computer in mathematics instruction.

Method

Sample: The subjects were selected from thirty Nigerian Universities that run the mathematics education program. A total of three hundred and thirty subjects (300 undergraduate mathematics education students and 30 mathematics educators were selected through stratified random sampling. The unit of stratification was ownership of university (Federal-owned and State-owned universities).

Instrument: Two sets of questionnaires – one for the students and the other for the mathematics educators were used for data collection.

The questionnaire for students had 3 sections. Section A sought information on personal data – students' gender and ownership of university. Section B sought information on availability of computers in education departments, and extent of exposure of students to computer usage in mathematics instruction. Section C sought information on general attitudes of students toward computer in mathematics instruction.

The questionnaire for mathematics educators also had three sections. Section A was on ownership of university. Section B was meant to collect data on availability of computers for mathematics method course, and extent of exposing students to computer usage in mathematics instruction. The items were also meant for collecting data on computer literacy level of the educators, mode of training in computer literacy and access to computer in homes. Section C sought information on problems of integrating computer in mathematics education program.

The questionnaires were found to have an alpha reliability of 0.89 and 0.83 respectively. Two types of validity were assessed: face validity and content validity by a panel of 3 judges.

Results

Table1: *Percentage of the Universities where computers are used for mathematics education*

University ownership	Percentage
Federal (n = 23)	17.39
State (n = 7)	00.00
Total (n = 30)	13.33

Table 1 shows that 17.39 per cent Federal Universities have computers, no state university has computer and 13.33 per cent of all the universities used for the study, have computers.

Table 2: *Response of both students and educators on the extent to which the students are exposed to computer in mathematics instruction*

Extent	Frequency (n = 330)	Percentage
Very great extent	0	00.00
Great extent	0	00.00
Little extent	24	7.27
Very little extent	306	92.73

Table 2 shows the percentage of both students and mathematics educators who indicated the extent to which the students are exposed to the use of computer in mathematics instruction.

Table 3: *Percentage of mathematics educators who are computer literate (n = 30)*

Item	Yes		No	
	Frequency	Percentage	Frequency	Percentage
Are you computer literate?	14	46.67	16	53.33

In table 3, 14 (46.67 per cent) of the mathematics educators indicated that they are computer literate, while 16 (53.33 per cent) stated that they are not computer literate.

Table 4: *Mode of training of mathematics educators on computer literacy (n = 14)*

Mode	Frequency	Percentage
a. In-service training in regular institutions	1	7.14
b. Self-development through reading of computer manual	4	28.57
c. Self-development through training by computer vendors/technicians.	8	57.14
d. Workshops	1	7.14

Table 4 indicated that 1 (7.14 per cent) of the mathematics educators who are computer literate, had in-service in a regular university, 4 (28.57 per cent) were trained by self-development through reading of computer manuals, 8 (57.14 per cent) were self developed through training by computer vendors and technicians, while 1 (7.14 per cent) became computer literate by attending workshops.

Table 5: *Percentage of mathematics educators who own computer machines (n = 30)*

Item	Yes		No	
	Frequency	Percentage	Frequency	Percentage
Do you own a computer machine?	9	30.00	21	70.00

In table 5, only 9 (30.00 per cent) of the mathematics educators have access to computers in their homes, while 21 (70.00 per cent) indicated that they do not have access to computers in their homes.

Table 6: *Mean ratings of students (n = 300) by gender on their attitudes to the use of computer in mathematics instruction*

Item	Total $\overline{X}$	Male $\overline{X}$	Female $\overline{X}$
i. Integration of computers in mathematics will threaten the job of teachers	4.24	4.26	4.18
ii. Computers can greatly improve learning in mathematics	4.78	4.86	4.57
iii. Some mathematics topics can not be taught with computer	2.75	2.67	2.98
iv. Computers are only useful as computational tools and therefore cannot be useful for effective teaching in mathematics	2.33	2.46	1.98
v. The use of computer in mathematics instruction can have a significant motivating effect on students	4.94	4.97	4.85

vi. Computers offer a cost-effective way of individualizing mathematics instruction	4.95	4.98	4.86
vii. With the use of computers the teacher can cover a lot of work to be done within a short time.	4.42	4.57	4.01
viii. Students might perceive mathematics more abstract in computer aided instructions	3.06	2.97	3.31
ix. The use of computers would waste more time, and it may not be possible to cover the scheme of work.	2.75	2.69	2.93
x. The learning of mathematics would become easier with the use of computers	4.02	4.15	3.69
xi. The use of computers to teach mathematics might make students to loose their sense of reasoning and thinking ability.	3.02	2.98	3.13
xii. Computers are very important and necessary in mathematics instructions	4.13	4.11	4.17
xiii. Computers will help to increase socialization among students in the mathematics classroom	3.16	3.19	3.10
Grand Mean		3.76	3.67

Table 6 shows the mean ratings of the respondents on the attitude items, and their mean ratings by gender. While items (i), (ii), (v), (vi), (vii), (viii), (x), (xi), (xii), and (xiii) had mean ratings above 3.00, others had mean ratings less than 3.00. The grand mean of the male students was 3.76 while that of females was 3.67.

Table 7: t table for difference in mean attitude ratings of male and female students

Gender	**n**	$\overline{X}$	t_{cal}	t_{crit}	**Decision**
Male	218	3.76	0.825	2.064	NS
Female	82	3.67			

Table 7 shows that the calculated t value for the difference in the mean attitude ratings of the male and femalc students is 0.825. This value is less than the critical value of 2.064, at the 0.05 level of significance. Hence we fail to reject the null hypothesis.

Table 8: *Mean ratings of students (n =300), by ownership of university, on their attitudes toward the use of computer in mathematics instructions*

Item	Federal $\overline{X}$	State $\overline{X}$
i. Integration of computers in mathematics will threaten the job of teachers	4.46	3.29
ii. Computers can greatly improve learning in mathematics	4.91	2.12
iii. Some mathematics topics can not be taught with computer	2.63	3.53
iv. Computers are only useful as computational tools and therefore cannot be useful for effective teaching in mathematics	2.81	2.26
v. The use of computer in mathematics instruction can have a significant motivating effect on students	4.98	1.56
vi. Computers offer a cost-effective way of individualizing mathematics instruction	4.93	3.07
vii. With the use of computers the teacher can cover a lot of work to be done within a short time.	4.99	2.87
viii. Students might perceive mathematics more abstract in computer aided instructions	2.95	4.67
ix. The use of computers would waste more time, and it may not be possible to cover the scheme of work.	2.94	3.06
x. The learning of mathematics would become easier with the use of computers	4.98	2.09
xi. The use of computers to teach mathematics might make students to loose their sense of reasoning and thinking ability.	3.87	3.07
xii. Computers are very important and necessary in mathematics instructions	4.98	3.21
xiii. Computers will help to increase socialization among students in the mathematics classroom	4.99	1.76
Grand Mean	4.19	2.81

Table 8 shows the mean attitude ratings of subjects, by ownership of university. Whereas there was an agreement on the mean attitude ratings of students from both the Federal- and State-owned Universities on items i, iv, vi, xi, and xii, they differed on items ii, iii, v, vii, viii, ix, x, and xiii.

Table 9: *t table for difference in mean attitude ratings of students from Federal- and State-owned Universities*

University Ownership	n	$\overline{X}$	t_{cal}	t_{crit}	Decision
Federal	230	4.19	3.82	2.064	S
State	70	2.81			

Table 9 shows that the calculated t value for the difference in the mean attitude ratings of students from the Federal- and State- owned Universities is 3.82. This exceeds the critical value of 2.064. at the 0.05 level of significance. Hence, we reject the null hypothesis.

Table 10: *Mean rating of mathematics educators (n = 30), on the problems facing the effective integration of computers into mathematics education program in Nigeria.*

FACTORS	MEAN
1. Inadequate funding of higher education in Nigeria	4.83
2. None-availability of computer laboratory in the Universities	4.83
3. None-availability of computer facilities in the Universities	4.83
4. Many mathematics educators are not familiar with the use of computers in teaching	4.70
5. Inability of mathematics educators to attend international conferences on the teaching and learning of mathematics	4.70
6. None-availability of computer technologist to handle problems emanating for the use of computers	4.83
7. None-availability of relevant text books on the use of computers in teaching mathematics	4.83
8. High cost of telephone and internet services	4.93
9. Irregular supply of electricity in Nigeria	4.93
10. Inadequate security of University properties	4.67

Table 10 shows the means of the response of mathematics educators to the problems facing the effective integration of computers into mathematics education program in Nigeria. The means are between 4.67 and 4.93.

Discussion

Results of this study show that only 13.33 per cent of the universities studied have computers for the implementation of mathematics education program. A further analysis of data indicated that 17.39 per cent of the Federal- owned Universities have computers, while no State University has computers for the mathematics education program (table 1). This could be the basis for exposing the undergraduate mathematics education students, to a very little extent, on the use of computers in mathematics instruction (table 2).

The results of the study also indicated that fewer proportion (46.67 per cent), of the mathematics educators are computer literate (table 3), the few who are computer literate acquired this mainly by self-development through training by computer vendors and technicians (table 4). This finding is consistent with previous reports, such as Forcheri and Molfino (1997, p. 1), who observed, "... External stimuli led an increasing number of teachers to develop and use activities involving IT.

Moreover, very few (30 per cent) of the mathematics educators studied have access to computers in their homes (table 5). Definitely, the inability of teachers to have access to computers and the lack of opportunity to be computer literate would hamper their effectiveness in the mathematics education program. Renzulli (1998) observed that more rigorous curriculum standards, without improved curricular materials and teachers able to use them would not yield significant improved academic performance.

The results of this study indicated that generally, the mathematics education students have positive attitudes towards the use of computers in mathematics instruction. They believe that computer can greatly improve learning in mathematics; the use of computers in mathematics instruction can have a significant motivating effect on students; computers offer a cost-effective way of individualizing mathematics instruction; with the use of computers, the teacher can cover a lot of work to be done within a short time; the learning of mathematics would become easier; computers are very important and necessary in mathematics instruction; and computers would help to increase socialization among students in the mathematics classroom. However, on the negative aspect, the students in addition to believing that some mathematics topics cannot be taught with computers and computers cannot be useful for teaching for understanding in mathematics, believe that integration of computers in mathematics instruction will threaten the job of teachers. This result supports Harbor-Peters (1997) finding that Nigerian Secondary school teachers are not in support of the use of computers for fear of being displaced from job.

Analysis of the differences in the attitudes of the subjects, by gender, and ownership of university showed that no significant differences existed in the attitudes of male and female students, but a significant difference was found in their attitudes on the basis of ownership of University (table 8). Students from Federal-owned universities were found to have better and more positive attitude to the integration of computers into mathematics education program, than students from State-owned Universities. This result can be attributed to the fact that Federal-owned university students have more access to computers and already familiar with the role of computers in education, than the State-owned university students.

On the issue of effective integration of computers into mathematics education program in Nigeria, the problems identified include- inadequate funding of higher education in Nigeria;

none-availability of computer laboratory in the Universities; inability of mathematics educators to attend international conferences on the teaching and learning of mathematics; None-availability of computer experts/technologist to handle problems emanating for the use of computers; none-availability of relevant text books on the use of computers in teaching mathematics; high cost of telephone and internet services; irregular supply of electricity in Nigeria; and inadequate security of University properties.

The major cause of these problems may be traced to the inadequate funding of higher education. If enough funds are made available to the universities, most of these problems may be solved. For instance, most of the mathematics educators interviewed stated that the inability to attend international conferences was due to lack of financial support from the universities. With adequate funds, the university should be able to provide powerful generator to take care of irregular supply of electricity. Besides, the provision of adequate security for university properties will cost some money.

Conclusion and Recommendations

It has been found that computers are not widely available in the Nigerian Universities for the training of undergraduate mathematics teachers, and the student teachers are not exposed to the computer usage in mathematics instructions. Also, very few of the mathematics educators are computer literate, and have access to computers in their homes. If the Nigerian government should achieve its goal of integrating the computers into education especially mathematics education in Nigerian schools, then the teachers must be empowered through training in the use and application of the new technology. The authors therefore make the following recommendations.

1. The government should adequately fund the universities in Nigeria. Besides, industries and some "well-to-do Nigerians" should be involved in the funding of higher education in Nigeria.

2. Mathematics educators should be supported to attend at least one international conference on the teaching and learning of mathematics every year.

3. Universities should form linkages/exchange programs with Universities in the developed nations so as to help train the mathematics educators in the areas of using computers in teaching mathematics.

4. Universities should restore oversea training for their lecturers, so as to be exposed to current methods and materials for teaching and learning of mathematics.

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CALCULUS MACHINA: AN INTELLIGENT TUTOR PROVIDING COMPUTER BASED SUPPORT FOR TEACHING UNDERGRADUATE CALCULUS.

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ABSTRACT

Students arriving at University are far from homogeneous and there is a growing need to assess their active mathematical ability on entry to any course **and** provide suitable support materials when necessary. This paper explores how emerging technologies can provide an environment for diagnostic testing and follow up support material for such students. In particular, it discusses a new Computer Algebra System, called *Calculus Machina*. Although many Computer Algebra Systems are excellent at "Doing" mathematics they leave something to be desired when it comes to teaching and supporting learning in first year undergraduate mathematics, as many of the intermediate steps involved with basic calculus are not revealed. *Calculus Machina* is capable of solving many of the problems that arise in the standard Calc I and II sequence, but also disclosing the steps and processes by which these results are obtained. *Calculus Machina* can also function in tutorial mode where students are required to take an active role in learning, and where the program can "look over the shoulder" of a student as the steps in a calculation are performed, checking each step, and offer help when required. Finally, there is always a certain element of inertia when considering the adoption of any new teaching material so we conclude this paper with an evaluation of *Calculus Machina* in a teaching environment.

Keywords: Innovative Teaching, Technology, Computer Algebra Systems (CAS), Teaching Calculus, Diagnostic Testing of mathematics skills.

1. Introduction

When students enter Higher Education courses in Science and Engineering, instructors frequently have to make assumptions relating to their ability in a range of topic areas and mathematical skills. (See Kitchen (1996), Hirst (1997), and Lawson (1997).) Such courses also tend to recruit large numbers of students with a rich diversity of intake qualifications and prior experiences. In addition, over the last decade the nature and background of the students who arrive at our universities each September has changed markedly. The structure of a modular A-level curriculum, the main entry vehicle for students in the U.K., and in particular Curriculum 2000, has meant that students have a considerable range of mathematical experience and limited exposure to mathematical ideas that were once taken for granted. (See Porkess (2001). Furthermore, there is substantial evidence to suggest that schools are being selective in which A-level modules they opt for in order to maximise the overall performance of the student cohort. As a consequence of all these factors, students arriving at University are far from homogeneous. The need to assess individual students on entry and assess their current active ability of students to any course is crucial.

In a previous paper, one possible approach that uses technology for diagnostic testing and follow up support was described. (See Quinney (2001)) This paper explores how emerging technologies can provide support material for students at a time when they most need it and in a form that may encourage them to become independent learners.

2. Diagnostic Testing

The need to provide suitable diagnostic testing of mathematical skills is taken for granted in a wide variety of different institutions for two distinct but inter-related reasons.

(i) To provide students with useful individual feedback **before** problems escalate.

(ii) To provide teaching and tutorial staff with a global assessment of the current active ability of each student on a chosen range of topics.

The Heads of Departments of Mathematical Sciences in the UK (HoDoMS) funded a WWW site giving information, contacts and case studies of existing diagnostic tests. http://www.keele.ac.uk/depts/ma/diagnostic/ in 1996. This site contains links to the diagnostic tests used at a number of universities and a selection of case studies which give details of how diagnostic testing is carried out and, just as importantly, how students are supported thereafter.

Diagnostic testing is now being introduced in many universities, some use paper-based tests that are frequently optically marked to minimise the staff overheads, others have opted for computer-based testing often in the form of Multiple Choice Questions (MCQs). MCQs are attractive to those looking for a way of assessing students arising from their ease of marking by providing a computer-based form of assessment. (See Brydges & Hibberd (1994) and Beevers, Bishop & Quinney (1998).) At Keele University we have used a MCQ diagnostic test for a number of years in order to identify any students who may need particular attention. The test consists of 20 MCQs selected randomly from a bank of about 50 questions each of which is randomised. A typical question is shown in figure 2.1.

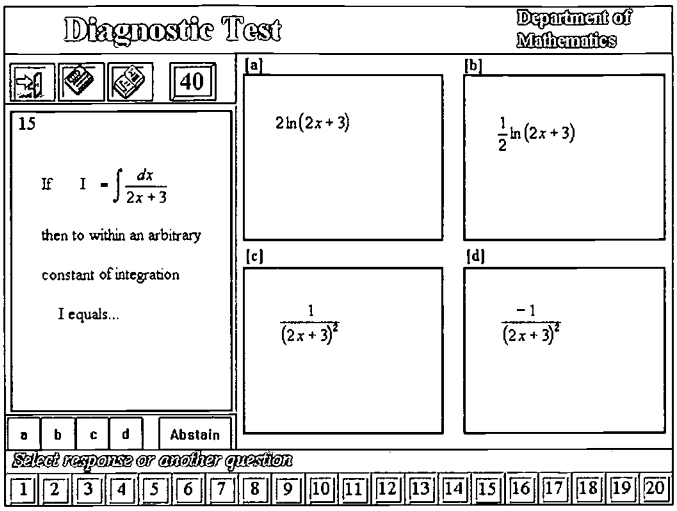
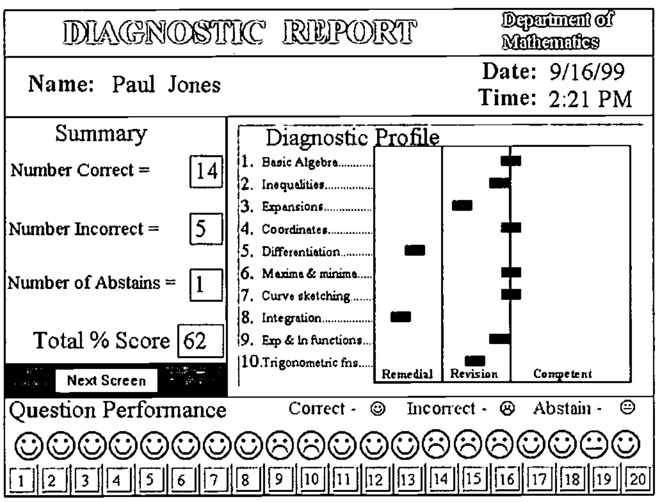


Figure 2.1: Sample Question

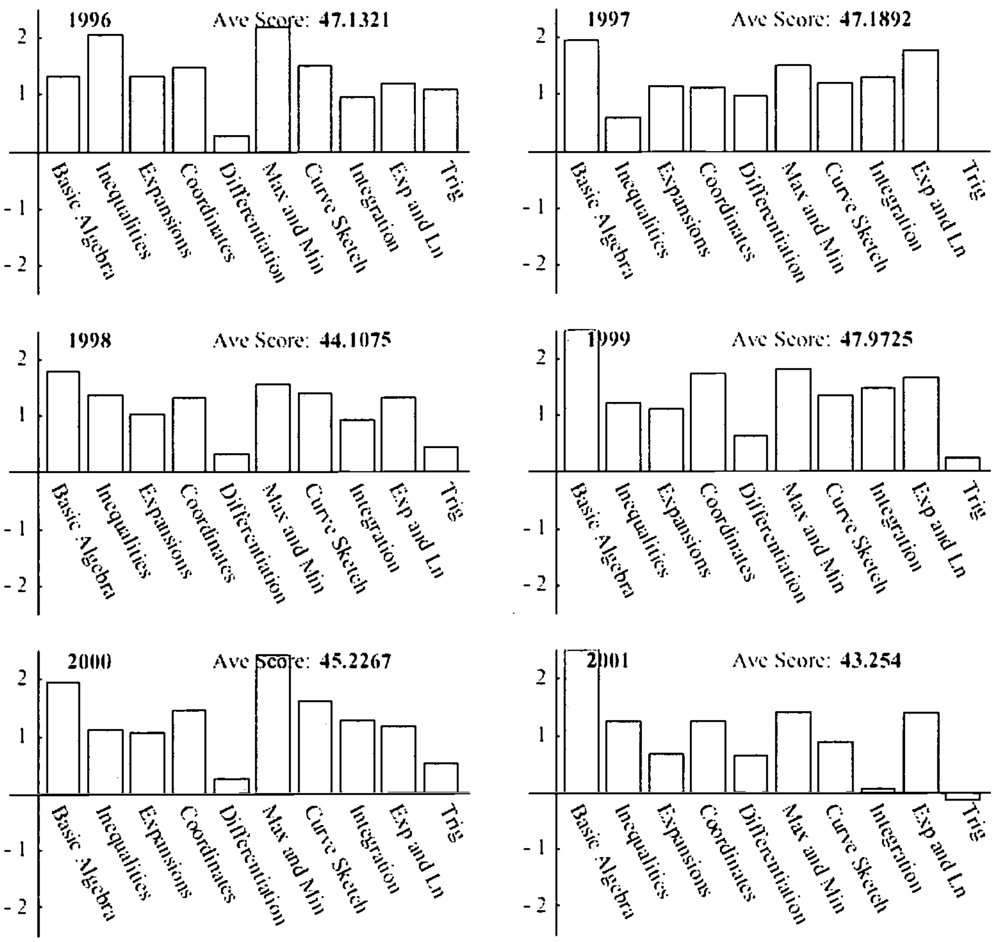
The aim of the test is not simply to return a numerical mark; its primary aim is to identify skills that might be lacking. The test is designed to give partial credit by grading the skills that might lead a student to select one of the incorrect answers and rewarding them accordingly. The student can decide to abstain from a question; in which case they are not penalised for selecting a wrong answer. However, such a decision indicates a deficiency of a particular skill and this is reflected in the final diagnostic report. Each student's responses are analysed to determine the student's capabilities in 10 distinct skills and the results are presented with a diagnostic screen as shown in figure 2.2.



Figures 2.2: Student Diagnostic Report

During the academic year 2000-2001, in an attempt to discover whether the diagnostic test described above provides a realistic indicator of individual students' capabilities, students were asked to take both the diagnostic test and a written paper and the results compared. All 87 students entered Principal Mathematics took both the diagnostic test and completed a written test that involved a large number of problems involving differentiation at various levels of difficulty. A statistical comparison of the written and diagnostic test showed that the scores are highly correlated (r=0.75, p<0.001) and that a simple linear regression model accounts for 55% of the variation of the marks. We conclude that the diagnostic test is a good predictor of individual an individual student's skills in differentiation. (See Quinney (2001).) This is significant, as the reduction in workload required in using the automation provided by the CBL diagnostic test can be significant, but more importantly because the CBL gave immediate feedback to each student.

A diagnostic test described above has been operating in the Mathematics Department at Keele University during 1996-2001; figure 2.3 illustrates results of profile skills for the student cohort in five successive years. The wide discrepancy, year by year, indicates that simply providing common remedial courses will not be suitable. It seems appropriate, therefore, to look at the microscopic scale and try to focus on individual students and attempt to assign each student suitable support material. Providing individualised programmes of study using computer based self-study programmes based on the results of the diagnostic test may provide a solution to this problem.



Figures 2.3: Cohort Profiles 1997-2000

The results of the diagnostic test between 1996 & 2000 were sufficiently encouraging that it was decided to integrate the process of diagnosis and support into the first year programme. The response from students has been exceptionally positive, in that the students have requested similar material to extend the diagnostic process to consider integration in more detail.

3. Online Web Support

At the end of the diagnostic test students were asked to reflect on the result to see if they considered it fair. Many did and excused their poor performance on the grounds that it was several months, over the summer vacation, since they had actually done any mathematics. In order to remedy this in future years, students that have been accepted onto the course at Keele will be given access to WWW-based mathematical quizzes that will enable them to hone up their skills before they arrive at university.

There are a large number of WWW based tutorial systems currently available but we shall be encouraging students to use *eGrade*. (Published by John Wiley (2002).) This system provides a large number of prepared tests but in addition it gives the facilities for instructors to enter their own questions and manage the delivery of both quantitative and technical problems. The questions can be either multiple choice or free text and the software provides facilities for students to preview answers in "pretty print", i.e. mathematical layout. *eGrade* system has been class-tested for several years at the University of Michigan where in excess of 8000 students have used the system. (See LaRose (2001).) Students can access banks of problem sets and view example problems, which are integrated with some of the better-known texts. The software provides immediate scoring of student work and individualized feedback.

The advantages of such WWW based systems are manifold.

(a) Students can practice their skills and enhance their confidence prior to any formal testing.

(b) The questions are available anywhere and anytime and are therefore more attractive to a generation of students who delight in the availability of the WWW.

(c) The performance of individual students can be tracked and analyzed, though in some cases the latter can be a deterrent if students believe their every mistake is being recorded.

The first of these reasons is by far the most attractive and the availability of a large bank of reliable test problems can be extremely beneficial when coupled with immediate marking and feedback.

4. Computer based support material

Gains made from the implementation of diagnostic testing or the provision of on-line preparatory quizzes is limited without providing suitable learning support material. Such support materials needs to be tailored to each student's individual needs and yet cover the broad range of core mathematical knowledge at this level. This can be accomplished through human tutors, drop-in clinics, supplementary lectures, and mathematics resource centres, etc. (Lawson, Halpin and Croft, (2001).) However, experience has shown that even though the weaknesses of individual students can be detected using diagnostic testing the restrictions of individual and teaching timetables make it difficult to allot specific times when students can be supervised to ensure that any remedial work is carried out.

During 1996-1999 the mathematics department at Keele University pioneered the use of the TLTP material, *Mathwise*, to provide individual study profiles which were automatically allocated following the diagnostic test. (Hibberd, Looms & Quinney (2001)). However, many students are becoming familiar with computer algebra systems (CAS) such as *Mathematica, Maple, Derive*, etc. Although these systems are excellent at "Doing" mathematics they leave much to be desired for teaching and learning mathematics. To this end we have been investigating the use of a CAS system that concentrates on teaching and learning, and how such a system can be integrated to provide the student support needed to follow up a diagnostic test.

A new software package called *Calculus Machina* has been developed, which has been designed to have a full range of computer algebraic skills in basic calculus but is also capable of revealing the steps that are required to evaluate derivatives and integrals. Furthermore, the interface between the student and software has been designed to be as simple as possible and yet remain very versatile. Students are able to type in their own expressions and see them displayed immediately in a "pretty print" form, or select and edit the current expression using "point and click". Alternatively, mathematical expressions can be entered using simple templates. (See figure 4.1.)

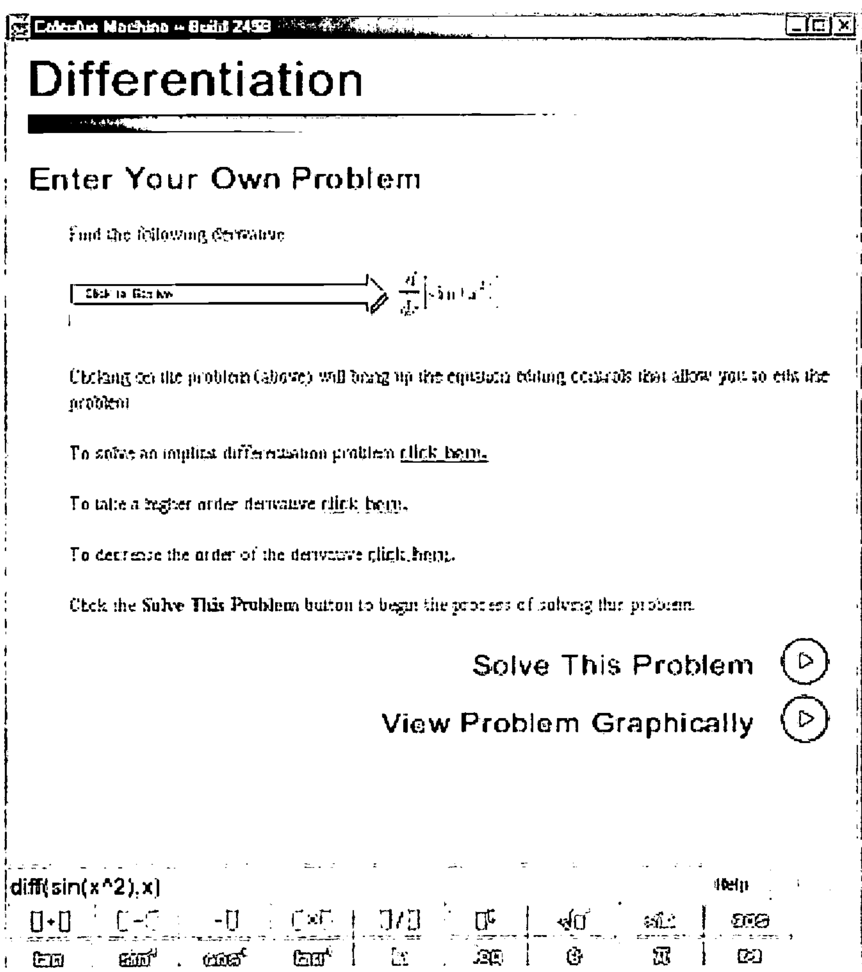


Figure 4.1: ***Calculus Machina's*** **input tool**

Once a function has been defined the software will either display the steps required to determine the derivative, as shown in figure 4.2. In figure 4.3, the *Calculus Machina* has been asked to differentiate $\sin(x^2)$. Notice that it recognises that it is necessary to use the Chain Rule (flagged by the text Derivative of Composite Function) and then reveals the steps needed to continue. These flags also provide a hypertext link to context sensitive help that allow the student to "drill down" and gain additional help as shown in figure 4.4. These pages are derived from "*Calculus*", Hughes Hallett, et al (2002) or the "*Calculus*", Anton (2002). Future versions of the software will enable an instructor to add links to alternative texts and additional material. The advantage with *Calculus Machina* is the ability for the students to type in their own problems or for it to generate practise problems for the student to attempt to re-enforce their skills in this topic.

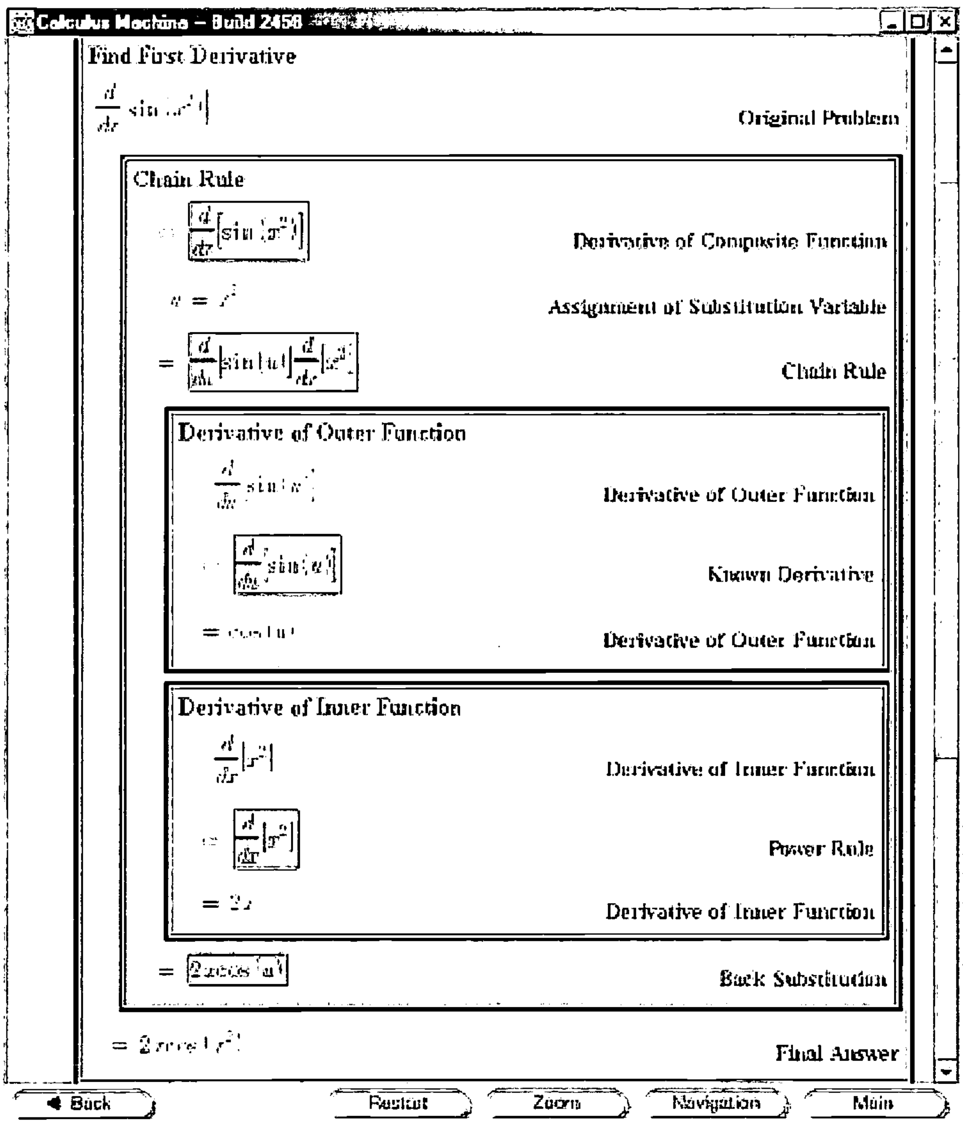


Figure 4.2: *Calculus Machina* output revealing the steps in finding a derivative

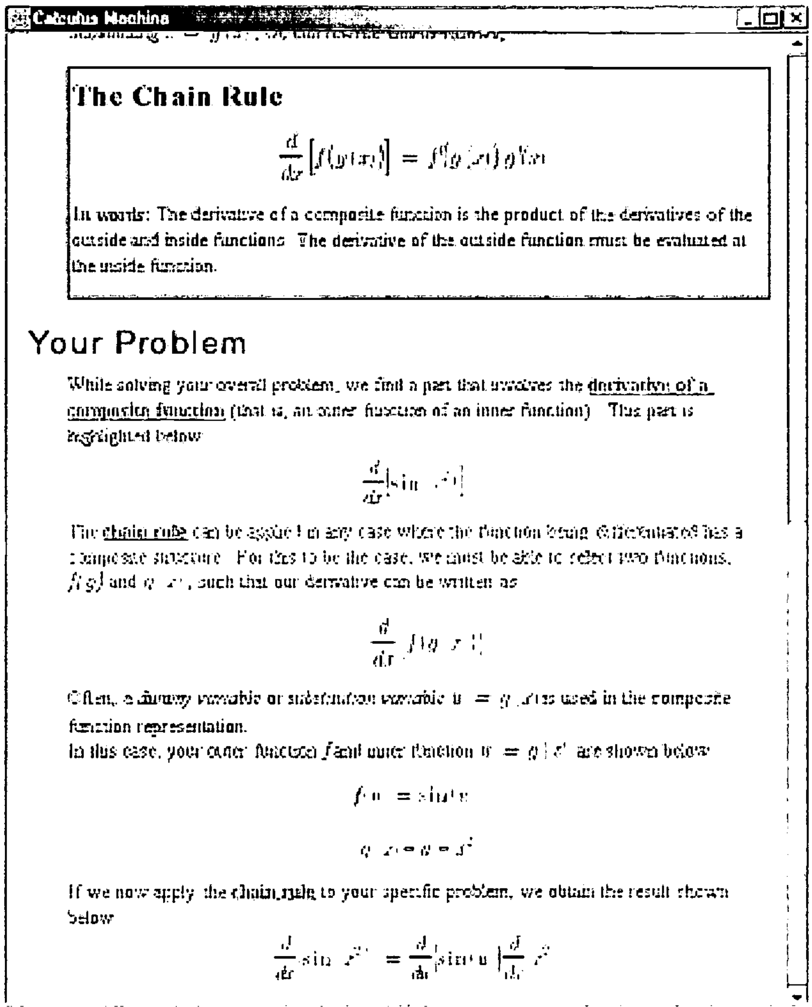


Figure 4.3: Context sensitive help file – note that the example reflects the current problem being solved.

Since *Calculus Machina* is able to differentiate almost all functions met in first and second year mathematics *and documents all the steps involved,* it might be thought that this will encourage students to take a very passive role and allow the computer to do the work. However, *Calculus Machina* has a second, more educational, mode in which the student has to take a much more active part in the process. This mode, called Udo, is illustrated in figure 4.4. Once again *Calculus Machina* has been asked to differentiate $\sin(x^2)$ but now the student has to supply the requisite substitution which is then checked before they are permitted to proceed. In this mode *Calculus Machina* can play the part of an individual tutor checking on each step and allowing students as much practise, as they need.

Finally, the software includes the ability to generate further problems that are closely related to the current problem to give further practice.

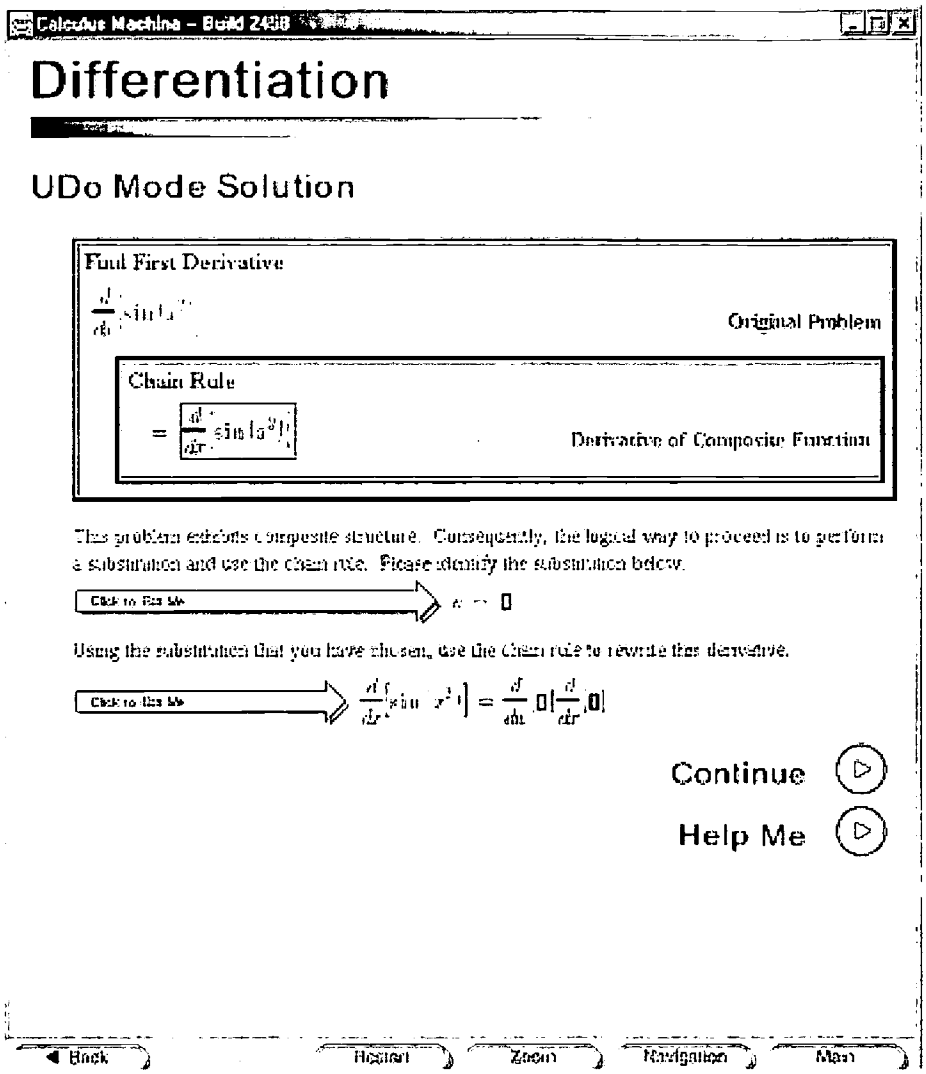


Figure 4.4: ***Calculus Machina*** **in tutorial (Udo) mode**

5. A Case Study 2000-2001

To investigate the effectiveness of the *Calculus Machina*, the students studying Principal Mathematics at Keele University during the academic year 2000-2001, were divided into two groups. Those scoring in excess of 65%, on the diagnostic test, were asked to look at a *Mathwise* Module called *Applications of Mathematics*. (See Beevers et al, 1998). The remaining students were further randomly sub-divided into two further groups (B1 and B2). Group B1 was asked to study a *Mathwise* Module: *Rules of Differentiation* and Group B2 was asked to use *Calculus Machina.* The aim of the project was to compare the performance of groups B1 and B2 to see if there was any statistical difference in performance of the two groups. To do this Groups B1 and B2 were asked to retake the diagnostic test at the end of their study and also complete a paper-based questionnaire.

5.1 Results

28 students completed the pre and post-diagnostic test though somewhat fewer also completed questionnaire. The students in Group B1 had a mean baseline score of 49.53 whilst those in Group B1 scored slightly less, 43.3 though this difference was not statistically significant, ($p=0.23$ using a t-test). 2 students in Group B2 were not included in the analysis, as they would have skewed the result even further in favour of the *Calculus Machina*. To investigate the effectiveness of the packages allocated to the two groups the mean paired absolute differences of the two groups were analysed.

The results of this trial are given in Table 5.1, and suggest that Group B2 have improved significantly better that Group B1 ($p=0.005$) even though their pre-test score was slightly poorer. Analysing the relative improvement in diagnostic score after using the software gives a similar result. Even though there is substantial variation in the results observed and the sample sizes are relatively small we can conclude that, based on these results, the *Calculus Machina* appears to be the more effective software when used in this context.

Group	Number	Software	Pre-test score	SD	Mean Difference	SD
B1	13	***Mathwise***	49.53	14.61	5.38	10.39
B2	13	***Machina***	43.30	10.94	22.4	17.02

Table 5.1: Results of comparative trials using the *Calculus Machina* and *Mathwise*: Rules of Differentiation.

It must be noted that a direct comparison between the *Calculus Machina* and *Mathwise: Rules Of Differentiation* is a little unfair as they are several generations of software apart and the *Calculus Machina* is designed specifically for the Calculus whereas *Mathwise* covers a wider remit. Nevertheless, the mathematics department at Keele University has invested substantially in its use of *Mathwise* and there is substantial inertia in changing to a new system, however, the evidence of this study provides some credence for changing to *Calculus Machina*. A similar experiment was conducted during the academic year 2001-2002 and the results were very similar. The major advantage of the *Calculus Machina* is its ability to accept problems entered by the student and disclose and document how the derivative or integral is found.

5.2 Questionnaire Results

18 completed questionnaires were returned; 9 from Group B1 and 9 from Group B2. Respondents reported a wide range of reasons for studying Mathematics or Statistics and a wide variety of topics in which they had perceived strengths and weaknesses. Most of the students regarded the diagnostic test as accurate. Students varied widely in their attitudes to the use of computers in teaching and learning. Some appreciated the fact that the computer allows them to work at their own pace, provides instant feedback, and was able to lead them step-by-step through methods; others found the experience somewhat stressful. A similar questionnaire in 2002 found fewer students in the latter category; further investigation has shown that, as might be expected, students are becoming more acclimatised to using courseware.

6. Conclusion

Courseware is now available to help detect areas of mathematical weakness at individual student level, provide individual testing at the convenience of the student and provide individualised support. In particular we have shown:

(1) That the simple diagnostic test that we have used is a good predictor of student performance and may thus be used to support differentiated teaching. Although discussions with course tutorial support staff are vital, the computer-based profiles provide a pro-active mechanism for the early identification of student weaknesses. Of course, the basis of this paradigm is dependent on the development of study skills by individual students and the inclusion of both summative and formative assessment can help re-enforce this. The same software can also be used to gather information on the cohort as a whole and also to track the performance of students on a year-by-year basis.

(2) Although the department has made use of several modules from *Mathwise* over the last 5 years and invested quite heavily in such materials there is sufficient evidence to show that the capabilities of more recent software, *Calculus Machina,* are more beneficial. Accordingly we aim to build it into the week that the Department has set aside for developing the students' skills in Introductory Calculus from the academic year 2002-2003.

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SUCCESSFUL INTERDISCIPLINARY TEACHING:
Making One Plus One Equal One

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ABSTRACT

Interdisciplinary courses are widely commended to help students acquire the mental agility and critical thinking skills needed for success in the modern world, but mathematics is seldom one of the interdisciplinary players. This paper uses evaluation data from ten mathematics and humanities courses developed as part of the Mathematics Across the Curriculum project at Dartmouth College to show that interdisciplinary mathematics and humanities courses did more than help students achieve an interdisciplinary perspective. By involving students actively in learning interesting mathematics, they were more successful than more conventional courses in promoting positive attitudes about mathematics. Connecting student outcomes with faculty strategies in developing and teaching these courses yields guidelines for developing successful interdisciplinary mathematics courses.

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Keywords: Curriculum development, pedagogy, collaboration, humanities, liberal arts, interdisciplinary, quantitative literacy, math phobia, math avoider, evaluation.

Introduction

In the last decade, the call for an interdisciplinary perspective has risen from a suggestion to an exhortation. From all quarters, colleges are urged to breach barriers between departments by developing more interdisciplinary courses and programs. Reviewing the 1997 *Handbook of the Undergraduate Curriculum,* Klein (1998, p. 4) writes, "For the most of this century, the dominant trend in higher education was the growth of specialization and the proliferation of programs and courses. At present, we are in the midst of a historic reversal of this trend, and interdisciplinarity is at the heart of it." The need for interdisciplinary teaching and learning is a leit-motif in Rhodes' (2001) prescription for the college of the future. If the sciences led the way in specializing, they now especially feel the need to reintegrate knowledge. In *Shaping the Future* (1996), the Advisory Committee to the National Science Foundation repeatedly commends interdisciplinary learning as a strategy for keeping the United States' workforce competitive.

The driving rationale is that success in the contemporary world demands an acrobatic intellect capable of constant readjustment. Interdisciplinary approaches, it is reasoned, exercise the mental muscles needed for this kind of thinking. Recent literature catalogues the benefits believed to accrue from interdisciplinary courses. These courses will show students how to address complex issues and help them think more critically (Newell, 1994; Davis, 1995; Klein, 1998; Rhodes 2001). They will encourage faculty to be pedagogically adventurous, promote the synthesis of knowledge, and help to draw the campus community closer together (Austin and Baldwin, 1991; Davis, 1995, Rhodes 2001). In mathematics and the sciences, they will increase student interest by relating those fields to other accessible and engaging questions, and they will increase student numbers by attracting students from outside the traditional mathematics and science majors (National Science Foundation, 1996; Ganter and Kinder, 2000).

This is a tall order for any pedagogical strategy, especially one that goes against the structural grain of most universities. Apart from the organizational challenges of apportioning faculty time and rewards among departments (itself no small consideration), the pedagogical value of interdisciplinary courses remains moot. In interviews about interdisciplinary teaching, Dartmouth College faculty from all disciplinary corners described their own scholarly work as highly interdisciplinary, but in the next breath many voiced reservations about the value of interdisciplinary courses for their students, especially at the introductory level. A physicist who felt graduate school was the appropriate location said, "We have to get through this essential material before [students] even have anything to think *with*." A humanist agreed: "The student has to have some grounding already in a discipline."

In this skeptical environment, mathematics has historically been the discipline least likely to succeed. Elementary and high schools that integrate all other subjects still teach math as a standalone offering. Interdisciplinary courses at the college level often connect disciplines where communication is already close, a matter more of overcoming dialectical differences than of learning a new language. Courses linking English, history, philosophy, and drama are common. Math and physics are also a frequent (and usually successful) pairing, but as one student insisted, "Physics **is** math." But interdisciplinary mathematics beyond "math applications for science" courses are viewed suspiciously by mathematicians, who cannot believe such courses could be rigorous enough to teach real math, and by humanists, many of whom have made math-avoidance a lifelong endeavor. Dartmouth's decision to link these two ends of the curricular spectrum in interdisciplinary mathematics and humanities courses was largely unprecedented.

The Dartmouth Project

The determination to make mathematics and humanities courses a cornerstone of the large, multi-year National Science Foundation-funded Mathematics Across the Curriculum project was serendipitous. Responding to the NSF's call to promote broader mathematical competence, the project's goals were to make mathematics accessible, interesting and relevant to students in all disciplines. Coincidentally, the project's Principal Investigator, Dorothy Wallace, along with an artist, had just created "Pattern," a course that used pattern in art to generate interest in and to illustrate elementary group theory. Wallace was convinced that other humanities could provide topics that would similarly motivate students by showing the relevance of mathematics to their other interests and by allowing them use more familiar non-mathematical material as a springboard into math. Her belief was supported by the regnant constructivist educational theory which asserts, put simply, that students are stimulated to learn when they are actively engaged, with others, in addressing material with personal relevance and that they learn most easily by building on what they already know (Bransford, Brown, & Cocking, 2000; Phillips 2000; LaRochelle, Bednarz , & Garrison 1998). The goals of the mathematics and humanities courses thus incorporated all the interdisciplinary goals noted earlier, with a constructivist twist. While improving analytic abilities and learning real math (and other real stuff) were clear goals, faculty also believed that making students receptive to studying more math in the future—a job that often involved undoing old fears and broadening constrained perspectives—was also a valid goal.

Over five project years, fourteen faculty members (half mathematicians, half humanists) created nine new courses connecting mathematics with literature, cultural history, music, art, architecture, drama, and philosophy.[1] Course developers expected the usual challenges of creating interdisciplinary courses to be magnified for them: greater substantive differences between the two kinds of content were accompanied by equally sizable pedagogical and linguistic differences. They also knew that they ran the risk of being seen as (and, in truth, of becoming) examples of "marshmallow math"—soft, sweet and toothless. But there was one wrinkle they didn't anticipate. They imagined that these courses would attract mostly students who were anxious about mathematics. In fact, perhaps because they were labeled "mathematics **and** humanities" courses (not "math for humanists" or "humanistic math"), when opened to an unrestricted population, they drew as many competent mathematics students as fearful ones—and few in between. (Three of ten course iterations were presented as first-year writing seminars, drawing only strong math students.) A population bimodally distributed between strong mathematics students hungry for new perspectives on a favorite subject and apprehensive ones hoping for a soft landing on their quantitative requirement posed yet another challenge for instructors. What math could engage both? In her paper in this volume, Wallace discusses how instructors selected interesting mathematical topics and made them accessible to a varied audience.

[1] Descriptions of these courses, and syllabi and materials for most, can be found at the MATC website http://www.math.dartmouth.edu/~matc/

Each faculty pair had complete independence in course development, and the resulting variations on the theme provided an excellent laboratory for evaluating the effectiveness of different approaches. Not all were unqualified successes, especially early in the project. However, since it's often easier to identify strategies that don't work than to tease out the components of success, less successful efforts were particularly instructive. Student data from 75 in-depth interviews with randomly selected students in nine course iterations and from 134 matched pre-post mathematics attitude surveys from the last four (and most "mature") courses offered[2] were linked with pedagogical strategies documented through faculty interviews, observation of planning sessions and classroom observation. Here is what we learned.

Student Results

The critical questions in evaluation are always, "compared to what?" and "for whom?" Nearly half the population in the surveyed math and humanities courses was math-phobes (necessarily non-science majors), who saw these courses as alternatives to introductory calculus for meeting the College's quantitative requirement.[3] The remainder was about equally divided between math or science majors eager to discover any new angle on a subject they enjoyed and strong mathematics students whose interests and majors directed them away from science and the calculus. For this latter group, mathematics and humanities courses offered interesting and challenging math without a calculus prerequisite.

Survey data show that in sustaining desirable attitudes about mathematics, the mathematics and humanities courses compare favorably to the introductory calculus course (the most prominent option for non-science majors, whether weak or strong in mathematics) and to two highly successful mathematics applications for science courses (which draw mostly science majors). Table 1 below compares the three types of courses along five indices constructed from the 35-item, 5-point Likert-scaled survey. The "Overall Index," constructed by dividing an individual's total post-survey score by the total pre-survey score, provides a gross measure of change in his/her attitudes about mathematics over the interval of a course. Indices greater than 1.00 show an overall gain in desirable attitudes; those less than 1.00 show an overall loss. The "Ability," "Interest," "Personal Growth" and "Utility" indices are similarly constructed from the four scales derived through factor analysis from the survey data and reference, respectively, students' perception of their mathematics ability, their interest in math, their belief in its importance for their personal growth, and in its usefulness in their professional lives.

[2] Six of the ten mathematics and humanities course were offered in the first two years of the project, Winter 1996 - Spring 1997, before the mathematics survey was in final form.
[3] About three-quarters of the entering class take calculus at some level.

Table 1. Mean index scores by type of course for science, social science, humanities, and undecided majors.

	INDEX	MATH AND HUMANITIES	INTRO. TO CALCULUS	MATH'L APPLICAT'N FOR SCIENCE
SCIENCE MAJORS[4]	*Number*	*(N = 34)*	*(N = 99)*	*(N = 49)*
	Overall ††	1.04	.91	1.01
	Ability ††	1.07	.93	1.02
	Interest ††	1.01	.88	1.00
	Personal growth††	1.07	.91	1.03
	Utility††	1.06	.90	.99
SOCIAL SCIENCE MAJORS	*Number*	*(N = 34)*	*(N = 38)*	*(N < 10)*
	Overall **	1.01	.92	
	Ability *	1.01	.92	
	Interest	1.00	.91	
	Personal growth *	1.03	.92	
	Utility	1.00	.95	
HUMANITIES MAJORS	*Number*	*(N = 24)*	*(N = 20)*	*(N < 10)*
	Overall	.97	.91	
	Ability*	.99	.86	
	Interest	.93	.85	
	Personal growth	.99	1.01	
	Utility	.98	.89	
UNDECIDED ABOUT MAJOR	*Number*	*(N = 29)*	*(N = 127)*	*(N<10)*
	Overall **	1.01	.92	
	Ability *	1.02	.96	
	Interest **	1.00	.87	
	Personal growth **	1.02	.90	
	Utility **	1.05	.92	

†† $p < .01$ using one-way ANOVA

* $p < .05$ using Student's t-test for independent samples

** $p < .01$ using Student's t-test for independent samples

For students in **all** majors the mathematics and humanities courses were more effective in sustaining and increasing desirable attitudes about mathematics than was the standard first-year

[4] For science majors, both the mathematics and humanities courses and the mathematical applications courses were significantly different from the introductory calculus courses, but they were not different not from each other.

calculus course or the lively advanced applications courses. While there is no substitute for calculus for students who need it, mathematics and humanities courses offer students who do not require calculus to pursue their intellectual interests—or those who simply want to try a new kind of mathematics—an alternative that nurtures their mathematical interests. This is particularly significant viewed against the inexorable decline in math participation in United States colleges.

Well and good, the skeptic might respond, but did they learn any math? Positive attitudes framed in the absence of rigorous mathematical effort are fragile at best. The answer to this, of course, is complicated. Few would equate math learning with exam performance. Faculty and students consistently report that unless math knowledge is reinforced and conceptually deepened by subsequent use, most evaporates shortly after the test. Math grudgingly learned or believed to be irrelevant disappears even faster, although distaste for the subject may linger. On the other hand, one could argue—indeed, interviewed students do so argue—for the value of the problem-solving skills developed in learning math, even if little math content is retained. While students in interdisciplinary courses spend only half their time on math (and expectably would learn "less"), their reduced exposure is offset by the fact that learning math which is intellectually engaging and relevant to their other interests encourages diligence and enhances retention. The engine of mathematics learning requires both hard work and intrinsic motivation; neither is adequate alone. Interview data suggest that the motivation-infused mix on which interdisciplinary courses run is as productive as the work-enriched fuel of many mathematics courses.

Survey results offer a broad-based but superficial understanding of **how** students responded to the courses. In interviews students detail their experiences, giving substance and depth to survey data. Even the most successful courses did not persuade all students that an unconventional approach to mathematics is worthwhile. As one dissatisfied student explained, "In terms of my perception of what math is—numerical equations and actual problems with concrete right and wrong answers—that was definitely not part of the class. There weren't concrete right and wrong answers. It was theorization and ideas." Some students never achieved the desired connection between the disciplines. Despite their shortcomings, the interdisciplinary courses resulted in stronger gains in student attitudes than the other courses surveyed. In the exemplary quotes below, students explain **why** these courses were effective.

Revealing how mathematics is embedded in other fields helps students understand the mathematics better. Whether a math concept is embodied in a painting or used as an element in plot development, seeing it instantiated provides a new avenue to comprehension. As one student explained, "Compared to [other] math courses, it's more interesting because it's not just like they give you a formula and then you give them an answer. It has some kind of applications; something you can hang onto. Some of the math in there, I hadn't seen much of at all. For instance, when we looked at infinity, infinite cardinals and things like that, I had no exposure to that whatsoever. So I felt that I would be able to understand those rather obtuse ideas better in the context of the science fiction stories, so I could see, not exactly practical applications, but just some sort of a demonstration of what they meant."

Interesting applications and different, non-calculus math stimulate student interest in mathematics. For many college students, calculus **is** higher-level math. Mathematics and humanities students were excited to discover whole new worlds of mathematics. Repeatedly, they prefaced their revelations with, "I used to think of math as cut and dried, but now...." One reported enthusiastically, "We just leapt ahead, and talked about things like the transfinite numbers, and the set theory things that I had never really heard about before." Students who came to these courses weary of a subject valued more for its challenge than for its content found their interest resuscitated. As another remarked, "This course renewed my passion for math." For another, the interdisciplinary course changed her perception of math "from black-and-white to color." Still another student concluded, "I think the reason a lot of people shy away from math or science is because it's not a tangible subject that you can relate to different aspects of your life. Which [these courses show] is very false."

Different pedagogical approaches increase student confidence. Hands-on exercises were common: students kept star journals, composed music, wrote stories, created art. These alternate entries to mathematics offered students who had not succeeded in conventional courses a second chance. "The great thing about [this] course was that it did give me confidence about math again. I learned that it's always connected and that I can do it, that I can succeed in math." Student comfort rose when faculty members functioned as a clever but inexpert "model students" because, as one student explained, "you don't feel like you're just working with an expert." Perhaps most important, the interdisciplinary dialogue between professors included students in genuine scholarly discourse. This student related, "The two of them challenged each other, which was really nice, and they weren't afraid to contradict each other, or to add things to each other's lectures, or to cut one another off. They weren't inhibited by formality. The collaborative environment they tried to foster with the students was really nice. It felt more like a partnership than 'we'll tell you stuff and you learn it.'"

The interdisciplinary approach brings an exciting new perspective. In discovering the intersection between two subjects, students developed stronger analytic abilities and achieved a broader perspective. Consider these responses to the question, "Please tell me something you learned in the course."

"...to see the world through a more mathematical eye, take a second look at the world."

"...to look at things from two different angles, and see how different aspects of a subject can fit into another subject that you would never relate before."

"...how to think more broadly, and look at things in a less than mainstream way, kind of off the beaten path, and just take a different approach to ordinary things."

"...the interdisciplinary approach—just knowing how to integrate material that doesn't necessarily at the beginning seem like it would fit together. And learning that when someone says, 'Can you do these two things?' and you say, 'No' you probably can. You just need to figure out how. "

Designing and Teaching Interdisciplinary Courses

A genuinely interdisciplinary perspective, essential to the success of these courses, is realized only as an emergent attribute of conflating their separate parts. The ordinary metaphor for interdisciplinarity, that of bridging different domains, fails to convey adequately their property of intersection. Perhaps Piaget's (or Whitehead's) concept of a hierarchical structure in which higher levels subsume and "explain" lower levels evokes a more appropriate image. Thus, if to link in an interdisciplinary way is to achieve a level of abstraction unifying both disciplinary perspectives, the challenge to co-instructors grounded in distinctively-framed worldviews can be considerable. So fundamental a shift holds a number of direct implications for teaching in such courses. Correlating student results with faculty teaching methods documented in pre- and post-course faculty interviews, observations of planning sessions and in classroom presentation provides guidelines for designing and teaching successful interdisciplinary mathematics courses.

Think differently. Making the interdisciplinary connection the armature of the course (instead of an epiphenomenon) requires approaching one's own discipline differently. Course planning needs to begin by establishing a productive point of intersection (like the concept of time, or pattern) and then choosing material—typically not standard introductory topics—to elucidate it. Interdisciplinary teaching is not an exercise in parallel play: teams who proceeded by coordinating existing topics or lectures had notably less positive student outcomes than those who began afresh. Needless to say, this is a lot of work. As one math collaborator remarked, "I think that doing this was much more work than doing two [regular courses]." (He went on to add that he would happily do it again!) For math professors, the time required to grade written work was also a revelation. The first time around, most instructors felt that they had underestimated the time required to do a job they deemed satisfactory.

Think deeply. Successful course developers exposed disciplinary linguistic and epistemological differences during the planning process, defining more sharply for themselves the contours of the relationship between the two disciplines. For a mathematician, it happened this way: "What we realized in talking to one another is that each of us has our own language that we think is English—and it's not English. It's jargon. And so we find ourselves having to explain to one another things that we each take for granted, and don't even realize we take for granted." Faculty pairs who did not tackle epistemological issues head-on were less able to negotiate the intermediate territory. As one student remarked, "The two [disciplines] just never came together. They were coherent, but they were coherent as separate entities."

Talk about pedagogy. This should emerge naturally from deep thinking (above), since different ways of knowing imply different ways of learning. Bringing pedagogical issues out into the open not only clarifies epistemological differences, it anticipates potential moments of classroom awkwardness, helping to smooth the transition from the intimacy of teaching alone to the exposure of teaching collaboratively. Trust between collaborators is critical—one described it as "like a marriage"—and it's easier to achieve if potentially divisive issues are aired and resolved in the planning stage. Whether collaborators knew one another beforehand was less important to smooth functioning than the openness of pre-course discussions.

Be committed. Faculty need to acquire the same level of knowledge in the other discipline that they expect of their students. Not only does this generate more productive and informed interdisciplinary discussions, it's a matter of voting with your feet. What message do we send to students about the value of interdisciplinary learning if we're not willing to do any ourselves? (When acting as the model student, you don't want to be the one who didn't do the homework!) As students note in interviews, "If we can learn it in ten weeks, why can't they?"

Be transparent. Like teachers, students have a clear idea of how a course should proceed. Interdisciplinary courses break many of the rules, and they can leave students bewildered about their direction and purpose. Sharing the goals of the course and the strategies you'll use to achieve them is not cheating; it reassures students and makes them partners in the enterprise. Modeling interdisciplinary thinking in the classroom—and identifying it as such—is important for introducing students to the analytic practice of finding patterns and connections where they are not obvious. Then be sure students are required in their homework to make connections on their own.

Be sparing. Don't overload the syllabus. This is a temptation in any new course, doubly so when two disciplines are involved. Much of the work of interdisciplinary courses takes place in the conceptual space between the two, so it's especially important to resist the impulse toward all-inclusiveness.

Teach math. Despite the range of student abilities and backgrounds represented, almost every student wanted to learn mathematics. Few who feared math were there to avoid it; most were there to surmount their fear. Courses that failed to challenge students, presenting math that was too easy or stressing the humanities portion at the expense of the math, left even math-phobes unsatisfied—and confirmed their conviction that real math must, after all, be too much for them.

For Faculty: "A great experience"

Despite the hard work, faculty uniformly and enthusiastically endorsed these courses.[5] The most common first response was that they were "fun." It was "fun" to work with students they would not usually encounter and "gratifying" to see them truly engaged. "They were very excited; we could see the light bulbs going on." Talking about pedagogy and learning from one another's teaching was "very exhilarating" for novice and experienced faculty. Most important, faculty found deep personal satisfaction in the opportunity to be scholars together, exploring new fields and acquiring a fresh perspective on their own. Working with colleagues in other fields was "stimulating intellectually," "very exciting," "humbling in a very good sense." As one summarized, "In present academia everyone rushes around and there's little time to talk about what one is doing. Here a very interesting dialogue is going on, and I liked that enormously."

At this point it is well to recall that these results were achieved under the most favorable conditions. Nobody was drafted for this job; faculty members who developed interdisciplinary courses had a desire to do so. Because these courses were created as part of a well-funded project, faculty was given the equivalent of one course in free time to develop them. Their enthusiasm undoubtedly reflects pleasure at simply being given adequate time to accomplish what they set out to do, as well as satisfaction with the experience itself. Like any enterprise, courses like these are likely to be more successful when faculty has the resources they need to develop and teach them.

The Dartmouth experiment suggests that interdisciplinary mathematics courses are worth the investment. They can fill a gap in the curriculum, offering a fresh start for the mathematically timid and a lagniappe, a bonus of unexpected applications and insights, for mathematically

5 Only one member of one team did not value the experience highly.

adventurous students. But students are not the only beneficiaries. As they explored new material with new colleagues and many new students, faculty found the intellectual and pedagogical challenges of these courses immensely rewarding.

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USING COUNTER EXAMPLES TO ENHANCE STUDENTS' CONCEPTUAL UNDERSTANDING IN ENGINEERING UNDERGRADUATE MATHEMATICS: A PARALLEL STUDY

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ABSTRACT

This paper addresses a practical issue encountered by many lecturers teaching first-year university engineering mathematics. A big proportion of students seems to be able to find correct solutions to test and exam questions using familiar steps and procedures. Yet they lack deep conceptual understanding of the underlying theorems and sometimes have misconceptions. In order to eliminate misconceptions, and for deeper understanding of the concepts involved, the students were given the incorrect mathematical statements and were asked to construct counter examples to prove that the statements were wrong. They had enough knowledge to do that. However, for most of the students that kind of activity was very challenging and created conflict. 127 students from two universities, in Germany and New Zealand, were questioned regarding their attitudes towards the method of using counter examples for eliminating misconceptions and deeper conceptual understanding. The vast majority of the students (96% in the German group and 84% in the New Zealand group) reported that the method was very effective. Many of the students made positive comments that using counter examples helped them to eliminate misconceptions, prevent mistakes in future, understand concepts better, and develop logical and critical thinking.

Framework

One of the main objectives of the study was to check our assumptions on how effective the usage of counter examples is for eliminating students' misconceptions in engineering mathematics. In this study, practice was selected as the basis for the research framework and, it was decided 'to follow conventional wisdom as understood by the people who are stakeholders in the practice' (Zevenbergen R, Begg A, 1999). Over recent years in some countries, partly due to extensive usage of modern technology, the proof component of the traditional approach in teaching mathematics to engineering students (definition-theorem-proof-example-application) almost disappeared. Students are used to relying on technology and sometimes lack logical thinking and conceptual understanding. 'The rapid increase of information over very short periods of time is a major problem in engineering education that seems worldwide. Misconceptions or unsuitable preconceptions cause many difficulties'. (Kolari S, Savander-Ranner C, 2000). 'The basic knowledge, performance and conceptual understanding of the students in mathematics worsen'. (Gruenwald N, Schott D, 2000). We have more than 50 years experience between us teaching first-year undergraduate mathematics using different pedagogical strategies. The research question arose from our teaching practice.

The theoretical framework was based on Piaget's notion of cognitive conflict (Piaget, 1985). Some studies in mathematics education at school mathematics level (Swan, 1993; Irwin, 1997) found conflict to be more effective than direct instruction. 'Provoking cognitive conflict to help students understand areas of mathematics is often recommended' (Irwin, 1997). Swedosh and Clark (1997) used conflict in their intervention method to help undergraduate students to eliminate their misconceptions. 'The method essentially involved *showing* examples for which the misconception could be seen to lead to a ridiculous conclusion, and, having established a conflict in the minds of the students, the correct concept was taught'. (Swedosh P, Clark J, 1997). Mason and Watson (2001) used a method of so-called boundary examples, which suggested creating by students examples to *correct* statements, theorems, techniques, and questions that satisfied their conditions. 'When students come to apply a theorem or technique, they often fail to check that the conditions for applying it are satisfied. We conjecture that this is usually because they simply do not think of it, and this is because they are not fluent in using appropriate terms, notations, properties, or do not recognise the role of such conditions.'(Mason J, Watson A, 2001). In our study, not the lecturers but *the students* were asked to create and show counter examples to *the incorrect* statements based on their common misconceptions, i.e. the students themselves established a conflict in their minds.

The Study

To enhance students' critical thinking skills, help them understand concepts and theorems' conditions better, eliminate common misconceptions and encourage active participation in class, we were giving our students incorrect statements and asking them to create counter examples to prove that the statements were wrong. The students had to refer to definitions of the basic concepts and to their geometrical illustrations because in most cases the easiest way to prove that the statement was wrong was just to sketch a graph. Often the statements were based on common students' misconceptions. Below are several examples of such statements.

Statement 1. The derivative exists at a point if the graph is smooth and continuous at the point being considered.

Statement 2. If the derivative is zero at a point then the function is neither increasing nor decreasing at this point.

Statement 3. At a maximum point the second derivative is negative and at a minimum positive.

Statement 4. The tangent to a curve at a point is the line which touches the curve at that point but does not cross it there.

After several weeks of using counter examples in teaching Calculus to first-year engineering students, 47 students from a German university and 80 students from a New Zealand university were given the following questionnaire to investigate their attitudes towards the usage of counter examples in learning/teaching.

The Questionnaire

Question 1. Do you feel confident using counter examples?

a) Yes Please give the reasons:

b) No Please give the reasons:

Question 2. Do you find this method effective?

a) Yes Please give the reasons:

b) No Please give the reasons:

Question 3. Would you like this kind of activity to be a part of assessment?

a) Yes Please give the reasons:

b) No Please give the reasons:

Findings from the Questionnaire

The statistics from the questionnaire are presented in the following table:

Number of students	Question 1 Confident?		Question 2 Effective?		Question 3 Part of assessment?	
German group	Yes	No	Yes	No	Yes	No
47	12	35	45	2	19	26
100%	26%	74%	96%	4%	43%	57%
New Zealand group	Yes	No	Yes	No	Yes	No
80	18	62	67	13	15	65
100%	22%	78%	84%	16%	19%	81%

Table 1. Summary of findings from the questionnaire

The majority of the students (74% in the German group and 78% in the New Zealand group) were not familiar with the usage of counter examples as a method of proof. The common comments from the students who answered 'No' to question 1 on whether they are confident with using of counter examples or not were as follows:

- I have never done this before;
- I am not familiar with this at all;
- I am not used to this method of proof;
- This method is unknown to me.

The vast majority of the students (96% in the German group and 84% in the New Zealand group) found the method of using counter examples to be very effective. The common comments from the students who answered 'Yes' to question 2 on whether the usage of counter examples is effective or not were as follows:

- helps me to think question deeply;

- gives more sound knowledge of the subject;
- we can understand more;
- it makes me think more effectively;
- can prevent mistakes;
- you gain a better understanding;
- it makes you think more in-depth;
- it teaches you to question everything;
- it makes you think carefully about the concepts and how they are applied;
- it makes you think critically;
- it supports self-control;
- it requires logical thinking, not only calculations;
- makes problems more understandable.

The majority of the students (57% in the German group and 81% in the New Zealand group) did not want the questions on creating counter examples to incorrect statements to be part of assessment in contrast to the trends pointing to the effectiveness of the method (96% in the German group and 84% in the New Zealand group). The common comments from the students who answered 'No' to question 3 on whether the questions on creating counter examples be part of assessment or not were as follows:

- it is hard;
- never done this stuff before;
- confusing;
- not trained enough;
- complicated;
- can affect marks.

Most of these students were more concerned about their test results rather than acquiring useful skills.

The students who answered 'Yes' (43% in the German group and 19% in the New Zealand group) provided excellent comments similar to those made on effectiveness of the method. The common comments from the students who answered 'Yes' to question 3 on whether the questions on creating counter examples be part of assessment or not were as follows:

- it provokes generalised thinking about the *nature* of the processes involved, as compared to the detail of the processes;
- better performance test;
- it shows full understanding of topic;
- a good way to test students' insight;
- it is an extremely valuable skill.

Conclusion and Recommendations

The overwhelming statistics of the study and numerous students' comments showed that the students were very positive about the usage of counter examples in first-year undergraduate mathematics. Many of them reported that the method of using counter examples helped them to understand concepts better, prevent mistakes in future, and develop logical and critical thinking. From our experience it also made students' participation in lectures more active. All these give us confidence to recommend this pedagogical strategy to our colleagues to try with their students. There could be different ways of using this strategy: giving the students a mixture of correct and incorrect statements; making a deliberate mistake in the lecture; asking the students to spot an error on a certain page of their textbook or manual; giving the students bonus marks towards their final grade for providing excellent counter examples to hard questions during the lecture and so on.

We are very aware of the limitations of the study. It was not an international comparison. It was intended more as a pilot study to check our assumptions and share the findings with university lecturers and the mathematics education community.

Further Study

We would like to extend the study to measure the effectiveness of this pedagogical strategy on the students' exam performance. We plan to compare the performance of 2 groups of students with similar backgrounds. In one group we will extensively use counter examples, with the other group being the control group. Then we will use statistical methods to establish whether the difference is significant or not. We also would like to extend the study to other countries in order to reduce the effect of differences in cultures, curricula, and education systems and also analyse the data from different perspectives and backgrounds. This co-operation can lead to organising a Research Forum or Discussion Group at an international conference on mathematics education to discuss the issues arising from this collaborative research. Those colleagues who are interested in joining the study group are cordially invited to contact the authors.

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CONNECTING UNDERGRADUATE NUMBER THEORY TO HIGH SCHOOL ALGEBRA:
A Study of a Course for Prospective Teachers

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ABSTRACT

Most universities in the US require prospective high school mathematics teachers to major in mathematics. In most cases, these students will encounter a course in abstract algebra and number theory, usually in the third year. Though the topics studied in these types of courses are closely related to those of high school mathematics, research on teacher education indicates that students generally do not see these connections and regard these courses as completely unrelated to the mathematics they will be teaching in the future. For example, the students in my study did not appear to view linear congruences as being analogous to equations. When solving a congruence such as $5x \equiv 3 \pmod 7$, they did not tend to think of "dividing" both sides of the congruence by 5, or of using a "guess and check" strategy. A method for solving linear Diophantine equations was viewed by the students almost exclusively as an algorithm to be memorized, and they generally did not recognize the connection between this method and the solving of equations in elementary algebra. This study has several implications for teacher education. In line with current recommendations for teacher preparation, I believe that we should make explicit for future teachers the connections between the abstract algebra and number theory that they study as undergraduates and the high school algebra that they will teach. Placing emphasis on the connections between the mathematics they are learning at the undergraduate level, the mathematics they already know, and the mathematics they will be teaching will emphasize the importance of understanding why algorithms and processes work. We expect them to emphasize this understanding with their own students; thus expecting it of them is important .

1 Introduction

The teaching of algebra is arguably the largest component of the job of a secondary school mathematics teacher. Most secondary schools in the US offer at least three levels of courses in algebra, and most universities in the US require that students have completed at least two years of algebra study. In addition, algebra is the foundation for much of the mathematics that secondary school students will study. According to the National Council of Teachers of Mathematics, algebra is an "essential component of contemporary mathematics and its applications in many fields" (NCTM, 2001).

Many researchers have emphasized that in addition to studying a good deal of mathematics at the undergraduate level, prospective teachers need to develop *knowledge of mathematics for teaching* – an understanding of the underlying processes and structure of concepts, the relationships between different areas of mathematics, and knowledge of students' ways of thinking and mathematical backgrounds (Fennema and Franke, 1992, Ma, 1999, MET, 2001). However, it has become clear in recent years that this knowledge of mathematics for teaching is not easily developed. For most prospective teachers there is what Cuoco calls a *vertical disconnect* between the undergraduate mathematics that they study and the mathematics that they will teach, and that "this is especially true in algebra, where abstract algebra is seen as a completely different subject from school algebra" (Cuoco, 2001). Undergraduates do not automatically recognize that the topics studied in abstract algebra provide explanations for why certain equations can be solved and others not, and provide rationale for many of the processes of high school algebra (Usiskin, 1988). *The Mathematical Education of Teachers* recommends that prospective teachers take courses in abstract algebra and number theory in order to examine the mathematical structures foundational to algebra and number systems, noting that these connections may need to be made in other courses (MET, 2001). If these connections are not made, then teachers must rely upon their own precollege algebra education, "an experience that is likely to have been focused on an algorithmic approach to mathematics and unlikely to have contributed to conceptual understanding" (p. 441, Ball and McDiarmid, 1990).

My dissertation study focused on students' understanding of congruence of integers developed during a unit on modular arithmetic in an introductory number theory course. The topics studied in this course were chosen by the instructor because they are closely related to those of high school mathematics. For example, the students were introduced to various methods for solving linear Diophantine equations, including the method of reduction of moduli. In order to understand how to use this procedure, the students were first introduced to solving linear congruences of the form $ax \equiv b \pmod{n}$. In general, the students did not appear to view congruences as being analogous to equations. When solving a congruence such as $5x \equiv 3 \pmod{7}$, they did not tend to think of "dividing" both sides of the congruence by 5, or of using a "guess and check" strategy. Reduction of moduli was viewed by the students almost exclusively as an algorithm to be memorized, and they generally did not recognize the connection between this method and the solving of equations in elementary algebra.

2 Review of Relevant Research

There is a small body of research on the learning of abstract algebra, most of which focuses on elementary group theory. Dubinsky (1994) writes that "constructing an understanding of even the very beginning of abstract algebra is a major event in the cognitive development of a mathematics student" (p. 295). Dubinsky also argues that since a significant proportion of mathematics majors will become high school teachers, this course plays a critical role in developing teachers' knowledge of and attitudes toward mathematical abstraction. Clark *et al* (1997) write that "many who are to be ambassadors and salespersons for mathematics at the secondary level develop a negative attitude towards mathematics in general and a fear of abstraction" (p. 182). There seems to be general agreement that this type of course is a turning point in the mathematical careers of many students, and that serious investigation into the teaching and learning of abstract algebra is of critical importance. Recently, research on the development of concepts in elementary number theory has begun to appear, though this has for the most part focused on concepts related to divisibility and proof. To my knowledge, no research has focused on the topic of congruence of integers.

Research on children's interpretations of algebraic equations and the process of solving these equations reveals that there are many conceptual difficulties. Booth (1988) says that "in algebra, the focus is on the derivation of procedures and relationships and the expression of these in generalized, simplified form" (p. 21). Students have difficulty accepting algebraic expressions as "answers," preferring to pick values for the variables in order to give a numerical answer. Kieran (1981) and Wagner (1977) showed that secondary school students typically regard the equals sign operationally – as "a unidirectional symbol preceding a numerical answer" (p. 24, Booth, 1988), instead of relationally – indicating that two quantities are the same. Kieran (1988) reported that when solving equations, beginning algebra students tended to rely on a memorized procedure that appeared to disregard the role of the equals sign in the equation. Wagner and Parker (1999) describe the difficulty that students with an operational view of equality often face when solving equations in algebra, noting that most solution methods assume a relational view of the equals sign, so that students must work with the entire relation as they transform it into equivalent relations. They state that "few students fully appreciate the fact that solving an equation is finding the value(s) of the variable for which the left- and right-hand sides are equal" (p. 333).

Bernard and Cohen (1988) write that understanding how to solve equations by the equivalent-equations procedure is a conceptually sophisticated task that requires a good deal of cognitive preparation. They claim that methods typically used in pre-algebra such as guess-and-check, the "cover-up" method (viewing equations as arithmetic identities with one value covered up), and the "undoing" method (viewing equations as a sequence of reversible steps that have been applied to a number), though important activities, are not adequate preparation for learning to solve equations using equivalent-equations. Note that a student with an operational view of equality can be successful learning to solve simple equations by such methods, since the relational aspect of equality is not necessary to guess the value of the missing number, and then perform arithmetic operations to check if the result is correct. Herscovics and Kieran (1999) also report that students have a great deal of difficulty solving equations by the equivalent-equations procedure. Kieran (1999) states that though research shows that many students become quite adept at solving equations in an automatic, procedural fashion, "these studies demonstrate that the same students are generally not aware of the structure underlying the manipulations they perform" (p. 351).

3 Study Background and Methods

Since the topic of congruence is virtually unstudied, I decided to use an exploratory case study design in my dissertation study. In the spring of 2001, I was a teaching assistant in a third-year introductory number theory course at a large state university in the southwestern US. The course was taught by Dr. Thomas, the professor who had originally designed the course. The students enrolled in the course were primarily prospective secondary mathematics teachers.

Modular arithmetic was introduced in the course as a tool for solving linear Diophantine equations, and students were first taught to solve them graphically, by guessing, and by using the Euclidean algorithm. Congruence was defined in two ways: a is congruent to b modulo n if 1) a and b have the same remainder upon division by n, and 2) n divides $a - b$. Reduction of moduli was then introduced as a means to find all solutions to linear Diophantine equations.

Dr. Thomas and I chose six students that we viewed as above-average based on exam scores and our perceptions of their attitudes towards the course. I interviewed the students three times over the course of the semester about their conceptions of statements of congruence. The first interview took place approximately three weeks into the modular arithmetic unit, the second took place after the exam, approximately three weeks later, and the third at the end of the semester. The interviews were transcribed, and then these data were triangulated with written questionnaires and exams, and field notes from observations in class. Analysis of the data was primarily done via open and axial coding, followed by a modified discourse analysis.

4 Results and Discussion

In general, the students demonstrated an operational view of congruence. They tended to view a congruence statement as a transformation from $\mathbb{Z}$ to what they called the "mod n world." For example, Chris interpreted the statement $5x \equiv 1 \pmod{11}$ as:

> "I think of this [left] side of the congruence as being any possible number and this [right side] is the class of number it is, it's a 1 mod 11. That [right] side of the congruence thing means something specific to me and that [left] side of the congruence thing means something that's in that same class. But it's not as specific."

In fact, there was a shift *towards* this operational view as the semester progressed. At the time of the first interview, Chris and Barbara had demonstrated a relational view of congruence, considering congruences as statements that showed when two integers could be considered "the same." However, by the second interview, both seemed to be viewing congruences operationally. The other students had held operational views at the time of the first interview, and this did not change. This finding is interesting in light of children's tendencies to view the equals sign operationally.

Before reduction of moduli was introduced, many of the students had been struggling to understand how to operate in the "mod n world." When they realized that one could rewrite a congruence of the form $a \equiv b \pmod{n}$ as $a = b + nk$, the students seemed to grasp onto this interpretation as an alternative to the earlier definitions they had been given. Fran said, "I think that now I have a better understanding of how to put it into an equation which makes a lot more sense to me than being in mod world." Dan suggested that most of the students in the course felt this way. "People are uncomfortable with congruence arithmetic and they look at this and they say, I don't really understand the rules of congruence arithmetic. But if you put it straight out in normal equation setting, it's not a problem."

The students began to display a tendency to automatically rewrite congruences as equations, and then work with these equations as much as possible. Once this practice emerged, the students appeared to have stopped trying to understand what was going on in "mod world" and to deal primarily with equations in $\mathbb{Z}$. In some cases, the students appeared to view the "mod n" term as merely different notation for "$+nk$". When solving $75x + 27y = 12$, Fran said, "I guess that would be just 12 minus $27y$. So $75x$ is equal to 12 mod 27. Can I do that with a negative?" At this point, Fran was not sure if she could rewrite $12 - 27y$ as 12 mod 27. When Barbara was asked if she viewed $5x \equiv 1 \pmod{11}$ as similar to an equation, she responded, "I actually look at this in terms of an equation. Like when I look at that, Im thinking to myself, five x equals one plus 11 y."

Overall, there were many parallels between the students' views of the reduction of moduli procedure and children's difficulties solving equations in algebra. The fact that the procedure of reduction of moduli is analogous to the equivalent-equations method of solving algebraic equations was not seen by the students, and they had little understanding of how this process worked or why it produced a set of solutions. Barbara's comment was typical:

> "I guess I understand why we're reducing it down, but when we start introducing other variables and you know, keep trying to reduce it, reduce it, and then probably where I get lost is when we go back to unraveling it. I'm trying to figure out like why that's important to solve it. You know to me, it seems like once we get down here, that would seem like a solution but it's not because you have to go back and do that so, that's what's the mystery to me."

In general, the students did not understand that this procedure was a process of repeatedly transforming the original equation several times by viewing it modulo one of the coefficients n (and thus mapping the equation to $\mathbb{Z}/n\mathbb{Z}$), and then deriving a related equation (mapping the equation back to $\mathbb{Z}$), the solutions of which were related to the solutions of the original. Instead, they viewed reduction of moduli as a complex procedure to be memorized and applied with great care, since mistakes were easy to make. They frequently expressed frustration with this procedure, not understanding why they were getting incorrect answers or even when they were making a mistake. Dan said, "I just don't understand ... there's always a small answer. I mean half the stupid homework problems we did there was a smaller answer than the way if you did it with the reduction. So like I don't ... am I doing it wrong?"

In class, Dr. Thomas had attempted to guide the students towards viewing congruences as analogous to equations in the sense that one could operate on both sides of a congruence, but his attempts were generally met with silence and confusion. The students instead chose to rewrite congruences as more familiar equations of two variables in $\mathbb{Z}$.

Dr. T: [writes $8x \equiv 4 \pmod{12}$ on board] "What do we do here?"

Student 1: "If you divide everything by 4, you get $2x$ congruent to 1 mod 3."

Student 2: "Just add 12 to four so that you have it congruent to 16. Then you can see you would have x is 2."

Dr. T: [following the second suggestion] "Well, if we did this we'd get that our answers are of the form $2 + 12k$. But we're missing ... 5. And if we do it the first way, we get five as an answer. [pointing to the congruence divided through by 4] Why does this work? How could we prove it?"

Student: "You could rewrite it as an equation, and then everything is divisible by 4."

Dr. T: [writes $8x = 4+12k$ and divides through by 4 to get $2x = 1+3k$, then rewrites as $2x \equiv 1 \pmod{12}$.

Students: [nod in agreement]

Barbara: "Can you divide the 8 and the 16, but not the 12?"

Dr. T: "Let's try that." [writes $8x \equiv 16 \pmod{12}$, and divides both sides by 8 to get $x \equiv 2 \pmod{12}$] "So can we do this?"

Chris: "There's something in the book that says the gcd of two of the numbers has to divide ... so the gcd of 8 and 12 must divide 16. If it doesn't, you can't."

Dr. T: [writes another example on the board: $8x \equiv 6 \pmod{7}$] "So here we can do what Barbara is suggesting ... the gcd of 8 and 7 is 1, and that divides 6, so we can divide by 2 on both sides. Barbara, did that answer your question?"

Barbara: "I think so ..."

The above transcript demonstrates that at this point in the course, the students generally dealt with linear congruences by rewriting them as equations in two variables. When asked how to prove that one can divide through a congruence by a common factor, a suggestion is immediately made to rewrite the congruence as an equation, and the students readily accepted this interpretation with no need for further justification. However, the students were reluctant to treat congruences as analogous to equations. When Barbara asked if one could divide both sides of a congruence by 8, the students did not know how to respond. Chris recalled an unrelated theorem about dividing through congruences by a common factor, but when Dr. Thomas redirected this comment towards an example in which one *could* divide both sides of the congruence (but not the modulus), the result was confusion.

When solving $5x \equiv 1 \pmod{11}$ in an interview, Chris said, "Then I just subtracted the 1 from both sides and I had to think about that one. I always, I still have to do that. I think about that. The subtracting, dividing or multiplying whether or not I can or can't." When solving the same congruence, Barbara was asked if one could approach solving it as one would solve an equation in algebra. She responded, "I think if I tried to solve for x, the thing that scares me is that I would get fractions and so since these are deal thingies or equations or whatever they're called, we want integers." Similarly, when asked if she viewed working with congruences as similar to working with algebraic equations, Fran replied, "To me it's not an equation like that, but I know I can convert it into an equation. But I don't look at that as an equation."

5 Conclusions and Implications for Teaching

It is striking that there were many similarities between the students' lack of understanding of the reduction of moduli procedure and children's difficulties solving equations in algebra. This may indicate that there are common underlying reasons for these difficulties. In addition, it is clear that these students did not make connections between the mathematics they were studying and the mathematics they will teach, as suggested by the research on abstract algebra. At the very least, addressing these difficulties with undergraduates in such a course may provide an opportunity to make connections with secondary mathematics.

Zazkis (1999) advocates having pre-service teachers re-examine familiar mathematical processes and objects in unconventional number systems as a means to get students to "reconsider their basic mathematical assumptions and analyze their automated responses. [These types of activities] constitute an essential tool for the development of critical thinking in mathematics teacher education" (p. 650). She uses a language analogy, saying that studying another language helps one to better understand the structure of one's own language. "Working with non-conventional structures helps students in constructing richer and more abstract schemas, in which new knowledge will be assimilated."

I strongly agree with this perspective and suggest that the study of congruence provides an ideal opportunity to examine teachers' fundamental understandings of algebra. For example, studying the properties of functions and equations in the rings $\mathbb{Z}/n\mathbb{Z}$ could enable students to explicitly make connections with and deepen their understanding of the ways in which algebraic structures underlie the processes of secondary school algebra, such as modeling situations with functions and equations, finding roots of polynomials, and using various procedures for solving equations.

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INTEGRATION OF MATHEMATICAL MODELLING AND TECHNOLOGY IN MATHEMATICS TEACHER PREPARATION COURSES

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ABSTRACT

Common themes in requirements for prospective mathematics teachers include mathematical modelling, problem solving, technology, and communicating mathematics. In this presentation we will discuss student presentation of projects to model and communicate mathematics. Technology is used as an integral tool in developing and solving the model as well as a medium for effective presentation.

Key Words and Phrases: teacher preparation, mathematical modelling, technology, computers, calculators, communicating mathematics, problem solving, student presentation of projects.

1. Introduction

It is difficult to correctly and concisely summarize the current state of mathematics education in the United States. However, it is possible to provide a few themes common to many of the contemporary reform movements in mathematics education. For example, mathematical modelling, problem solving, and communicating mathematics are three such themes. Moreover, the role of modern technology in the teaching and learning of mathematics is significant.

In our courses we assign student projects to provide the opportunity for students to communicate mathematics. We stress that mathematical modelling is an ongoing and dynamical process that is useful in daily life.

We require that students work in teams of two or three and report to the class on their projects. Our objectives are twofold: to enable the students to learn about the wide variety of mathematical models and to provide experience in communicating mathematics. Moreover, we require each team to make a formal presentation of their project to the entire class. We encourage the students to use the technology of their choice in the preparation of the paper and for their presentation.

For example, the students can use the technology of the TI-89 computer algebra system to explore differential equations and interpret solutions from three different points of view: graphical, numerical, and analytical. Slope fields and graphs of solutions or direction fields and solution curves in the phase plane contribute to better understanding of long-term behavior of the model. Tables of approximate solutions using Euler or Runge-Kutta methods also provide information. Graphs and tables of exact or approximate solutions can be compared on a split screen. The **deSolve** command of the TI-89 can be used to compute exact symbolic solutions to many 1st- and 2nd-order ordinary differential equations. Matrices, eigenvalues and eigenvectors are also easily handled on the TI-89 to determine the exact solutions to systems of ordinary differential equations.

A variety of computer software packages produce similar results.

Many of our students use a computer with presentation software and/or access to the Internet to present their projects. Others use calculators, posters, and transparencies on overhead projectors.

Some of the topics we have covered are: population models (including several different models of one population as well as models of competing populations from ecology), models of social choice (how groups make decisions), economic models, models of the epidemiology and the immunology of AIDS, and simulation models in planning and development.

2. Illustration

The struggle for existence among species has been studied for centuries. According to the theory of Charles Darwin, the average number of a species of prey depends on how many of the species are consumed by their predators. In the 1920's and 1930's Vito Volterra and Alfred Lotka, independently reduced Darwin's predator-prey interactions to mathematical models. The Lotka-Volterra predator-prey model is the system of first order differential equations:

$x' = (-a + by)\,x = -ax + bxy$ and

$y' = (c - dx)\,y = cy - dxy$, where

$a, b, c,$ *and* d are positive constants and

$x = x(t)$, $y = y(t)$ are populations at time t of a predator and a prey, respectively.

This is the simplest predator-prey model. It includes only exponential growth or decay and the predator-prey interaction. All other factors are assumed to be insignificant.

The Italian mathematician Vito Volterra developed the model in response to a problem posed to him by his son-in-law, the Italian biologist Umberto D'Ancona. D'Ancona was researching populations of species of fish that interact with each other in the Adriatic Sea. He had data on percentages of the catch of various species of fish brought into the Mediterranean ports of Trieste, Fiume, and Venice during the years of World War I, a period of reduced fishing from these ports. D'Ancona expected that a period of reduced fishing of food fish would be beneficial to the population of food fish. Yet the data seemed to indicate, in a relative sense at least, that reduced fishing was not beneficial. Instead there was a large increase in the percentage of predator species, selachians (sharks, skates, rays, etc.), which depend on their prey, the food fish.

Let's consider a specific case of Volterra's model, which we will analyze with the TI-89. We choose $a = 1$, $b = 0.1$, $c = 1$, and $d = 0.2$. This predator-prey model is represented by a system of two first order differential equations with constant coefficients:

$x' = -x + 0.1\,xy$ and

$y' = y - 0.2\,xy$, where

$x = x(t)$ represents the amount of selachians (predators) and

$y = y(t)$ represents the amount of food fish (prey) at time t.

Let the initial populations be represented by $x(0) = 8$ and $y(0) = 16$. Note: In this setting, it is more realistic to use units of pounds or tons rather than the number of fish i.e. biomass. So "8" might be 8 tons etc.

To enter the differential equations in the equation editor of the TI-89 Calculator, the built in variables $y1$ and $y2$ are used for x and y, respectively.

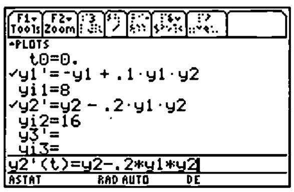


Figure 1

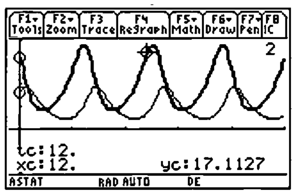
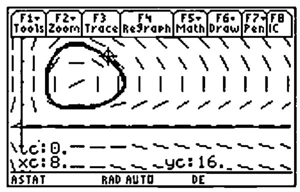


Figure 2



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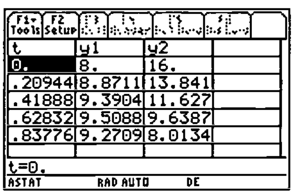

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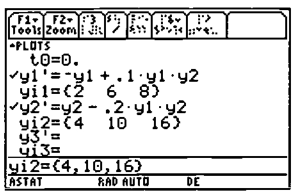

Figure 5

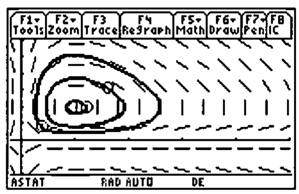
Figure 6

Figures 1-6 illustrate how the students can analyze the model numerically with a table (Figure 4) and graphically with a time graph (Figure 2) and phase portraits (Figures 3 and 6). Figure 2 portrays how the graphs representing the populations of food fish and selachians can be plotted simultaneously with thick and thin lines respectively. The "trace" feature of the TI-89 enables the students to see the values of the coordinates of points on the graph (Figures 2 and 3). Figures 5 and 6 depict how phase portraits with different initial conditions can easily be graphed simultaneously.

The amounts of the food fish (prey) and the selachians (predator) appear to be periodic. Moreover, the trajectories of solutions to the system seem to be closed loops—even closed loops that are "quasi- elliptical". In fact, these specific cases are representative of the general case.

Now return to the Volterra model provided above. The students are now prepared for a serious discussion of such a system of first order differential equations. (In our curriculum, the first course in mathematical modelling does not have systems of differential equations as a prerequisite so many of the students have not had such systems in previous courses). Part of the discussion is the proof that the average value of x (t) is c/d and the average value of y (t) is a/b. (For a proof, see Borelli and Coleman (1998), Chapter 5). What happens to these average values when "fishing is introduced"?

The Volterra model with fishing is

$x' = -ax + bxy - ex = -(a+e)x + bxy$ and

$y' = cy - dxy - fy = (c-f)y - dxy$, *where* $a, b, c, d, e,$ and f are positive constants.

Here e is a constant that represents the effect of fishing on the predator and f represents the effect of fishing on the prey. Note that if f is less than c we have the same setting as before since the coefficient of y is positive! The average values are x (t)=$(c-f)/d$ and y (t)= $(a+e)/b$. Thus, a "moderate" amount of fishing will increase the average amount of the prey (food fish)— "moderate" means that the fishing rate on the prey (food fish), f, is less than c (this forces the constant $c - f$ to be positive, so we can use the previous result on the average values). Note that the constant c is directly rated to the growth rate of the prey so "moderate" fishing is a rate less than the growth rate. But, if the fishing rate f is reduced (e.g. no fishing, $f = 0$), the average amount of prey (food fish) will decrease. This was Volterra's resolution of D'Ancona's problem.

There is another nice application of this model, which explains one of the deleterious effects of the pesticide DDT. The scenario is set in the mid 19th century when an insect was accidentally introduced to America from Australia. The insect had no natural predator in America. Its population grew at a rate sufficient to threaten the existence of the citrus industry. A natural predator was imported from Australia. The pest's population was reduced to a level where the citrus industry flourished again. However, the pest was not eliminated. With the introduction of DDT it was assumed that the pest population could be totally eliminated. The DDT is the "fishing agent". Since a "moderate amount of fishing" was "good" for the prey (the pest), as discussed above, the pest population increased rather than decreased — in particular, it was not eliminated!. See Braun's (1993) excellent book for a complete discussion.

The Lotka-Volterra model can be employed in a wide variety of different scenarios. One of the points we stress in our teaching of mathematical modelling is this very principle; namely that the same model can be employed in very different settings. A good project is to have the students investigate other applications of the Lotka- Volterra model.

3. Conclusion

There is evidence that appropriate use of technology does help students to learn mathematics better. Some studies which provide examples of the use and effectiveness of technological pedagogical tools are included in Connors, 1995; Connors & Snook, 2001; Dunham, 1998; and Hurley, Koehn, & Ganter, 1999. It makes sense, therefore, to provide prospective teachers with the opportunity to utilize technology in their mathematics and teacher preparation courses.

Projects provide an extra dimension in the learning process. Students work together on a problem that is not completely laid out for them. In some cases, they have a broad choice in topic selection and, therefore, they acquire a sense of ownership. They are required to analyze the problem, do research, if necessary, make decisions, and find results. Sometimes they are asked to make recommendations based on their findings. This provides an opportunity for them to interpret their results as it relates to real life. They are encouraged to criticize their work and, in some cases, are asked what else they would have done if they had more time or more resources.

Students often comment that projects helped them to understand better and also to recognize the importance and relevance of the mathematics studied in the course. Some report a sense of accomplishment and personal pride. Most importantly, they are doing mathematics and that is the best way to learn mathematics!

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AVOIDING MATHEMATICS TRAUMA: ALTERNATIVE TEACHING METHODS

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ABSTRACT

Children in primary education often encounter mathematics having picked up a general fear of mathematics from the society around them; this results in lack of confidence, avoidance of non-standard thought processes, weakness in problem solving strategies and other negative consequences. We offer an alternative approach; presenting mathematics as dynamic, interactive entertainment. The Mathematics Society, a student club at Izmir Institute of Technology (IZTECH), has developed a Mathematics Drama program addressing elementary mathematics. How successful has this group been in addressing the needs of the pupils attending their shows? This presentation will first consider possible causes of the fear of mathematics, then look at the work of the Mathematics Society and discuss its validity as a possible educational model. Finally we will present and analyze data from a survey of 500 pupils.

1. Sources of Mathematics Anxiety

Over the doors of Plato's Academy was inscribed the motto "Let no one ignorant of geometry enter here". Was this, in accordance with a rule of the almighty gods high above on Mount Olympus, beyond the scope of ordinary humans? Was it a mental inference that only the best philosophers of the time could work on? Were there other factors? Such a forbidding perception of the subject has permeated society since those days. Over 5500 years, from Ancient Greece to the beginning of the third millennium, ordinary people have passed on this fear to their children: Mathematics is the unknown, the unfathomable. While the philosophers of the Academy confidently indulged themselves in their elitist abstract competitions, an ordinary villager, Zeno, who showed with his paradoxes that the masses could understand and participate successfully in these intellectual activities, challenged them. The public could have been informed clearly about the field of mathematics, the methods of mathematical thoughts, mathematical objects and their properties, and how these relate to nature and society. Instead, Mathematics has been conveyed as difficult, abstract, and requiring intellectual curiosity. Hence it has become generally accepted that mathematics was not for the average mind, a perception unchallenged through the generations. As a result, instead of strategies of investigation, something unattractive and awkward appeared as rules and methods developed. In general the widespread assumption is that people are either good with numbers or with words; they could not be good at both. Besides, math is "dreary, never fun".

One cause of math trauma for students is the teaching style in the mathematics classroom. Pupils complain that mathematics offers little opportunity for debate or discussion. Teachers say pupils prefer literature and social studies to mathematics since they can participate more in class and are under no pressure to find the one right answer. Teachers may create anxiety by placing too much emphasis on memorizing formulae, learning mathematics through drill and practice, applying rote-memory rules and setting out work in the traditional way rather than understanding and reasoning (Greenwood, 1984).

People fail to do their best work when scared. Math anxiety or trauma develops from uncertainty and from a lack of confidence. With this anxiety or tension, understanding and recall pathways become cluttered by emotions, resulting in an inability to think. As the teacher persists in asking questions, the learner's brain stops functioning altogether. Although mathematics aims at right answers, these can be reached through open-ended problems, mathematics being experienced as a series of discoveries to be made by the learner. Rather than mathematical methods and rules, learners need to acquire abilities to analyze, question, test and find solutions: knowledge and skills relating to the processes, which can later be applied in any situation. But who will bring about this change, and how? Which methods of instruction or approaches to learning can bring mathematics to large numbers of people, in particular within the reach and interest of a significant section of young minds? Many authors have looked at the causes of mathematics anxiety and alternative teaching techniques to aid in student understanding. (Greenwood, J., 1984; Newstead, K, 1998; Hembree, R, 1990; Hopko. D.R and Ashcraft, M.H., 1998; Tobias, S, 1978)

2. The Math Show

What pupils learn is always less than what we teach. How much they learn is determined by native ability, background and learning style – which may or may not match our teaching style. There are many different types of learners: sensing, intuitive, sequential, global, active, reflexive, inductive, deductive, visual and verbal. To maximize student learning, the factor most readily within teacher control is his or her own teaching style. The IZTECH Math Society was founded in 1998 by the first author with a group of undergraduate mathematics, science, or engineering majors. Many of these students had suffered various forms of traditional teaching and were keen to search for better alternatives by researching mathematics, having fun with mathematics, and increasing its popularity in everyday life.

The critical age for the development of mathematics trauma is between 9 and 11 (McLead, 1993). Although trauma may deepen or change throughout schooling, generally once formed, negative attitudes and anxiety are difficult to change and may persist into adult life, with far-reaching consequences in the form of avoidance of mathematics, distress, and interference with conceptual thinking and memory processes. Possible sources of trauma, namely teacher anxiety, societal, educational or environmental factors, failure and the influence of early-school experiences of mathematics (Newstead, K. 1998), were taken into consideration as we designed a math show to relieve mathematics anxiety and mathematics trauma.

Cooperative learning is a key concept in the entire process: not only in the performances themselves but also in the preparation. Encouraging people to work with peers in small co-operative groups may have important affective consequences, including a reduction in anxiety for both Math Society members and pupils. In the preparation, questions dealing with everyday events are collected from libraries and the internet, and are set to music or prepared as stories. Within the week prior to our visit to a school, the conditions at the school, the situation of the children and parental attitudes are investigated in order to choose suitable questions. In the final rehearsal, show leaders attempt to anticipate all possible questions and reactions that may arise as well as deciding the mathematical games (can be used to reinforce mathematics skills of pupils) to be included.

Shows were generally performed for groups of 20 to 100, although in some schools between 300 and 900 pupils, teachers and parents have watched the show. The math show usually involves opportunities for social interaction, independent investigation and study, and the expression of creativity, as well as provision for different learning styles; in the first ten minutes, we brainstorm "What is mathematics?" with the young people, before "the History of Mathematics" unfolds. It has become clear to us that the pupils know nothing about the history of mathematics and have little knowledge of the background of the subject. Despite a minimum of three hours of mathematics per week during the 8 years of statutory education, pupils have insufficient knowledge of what mathematics is. History is a good vehicle for reflecting on cognitive and educational problems, for working on students' conceptions of mathematics and its teaching, and for promoting flexibility and open-mindedness in mathematics (F.Furinghetti, 2000). Thus it was decided the presentations should be given through the eyes of famous mathematicians to establish important events, the roles of significant mathematicians and key concepts, all at a level and in language suited to pupils.

Music is a key feature of the show. One member is always a musician, using either guitar or flute to draw the audience into new territory, melting away fear. The musical narrator leads the plot, introducing the different mathematicians. Each has several possible questions in their repertoire to be able to respond to the interests of the crowd. Any member of the audience who wishes to try solving a question comes up on stage to explain their reasoning. Some reach the answer, others ask for clues and others give up. However, the fundamental principle is that Math Show asks open ended questions "What if, Why, How...?", "What is the meaning of...?", "How would I use ... to ...?", "What is the difference between ... and ...?", "Why is this problem difficult for you? How can we make it easier?" Pupils have a chance to watch – on the over-head projector, a new experience for most of the spectators. As their friends progress in their reasoning, those watching think about the logical thoughts being expressed in words, actions and pictures. It becomes apparent that more than one approach may be used to reach the answer. As pupils reflect on their own learning styles, they become more adept at discovering flaws in their thinking. The aim of the show is to move away from the true/false focus, towards exploration of the subject, individually or as a group, developing problem solving strategies and collective thinking; the focus is on process rather than on the outcome. During the 1 to 2 hour-long show, transitions between activities are lightened with 'mathemusic' (musical riddles), topological games (knots to untangle), coloring puzzles (4-color problem), etc.

The first show, performed in the autumn term of 1998, rapidly drew the attention of the usually mathematically uninterested press. The following headlines appeared in the daily newspapers: "Nightmare masters", "Mathematics warriors" "Thought provoking entertainers"; a television channel announced that thanks to our shows "Children will no longer have nightmare about mathematics". Local newspapers began to publish mathematics puzzles. Faced with excessive demands for Math Show, the Mathematics Society of IZTECH was soon forced to limit performances to one per school. The society has supported the formation of mathematics groups in a number of schools.

3. Sample and Data Collection

Between October 1998 and June 2001 over 10,000 pupils, teachers and parents at 15 schools and institutions attended Math Shows. The questionnaire was given to a sample of 500 pupils (250 from state elementary schools and 250 from private schools). They were asked 10 questions after the show.

TABLE-1

1) Which year are you in?
5th grade: 20% 6th grade: 40% 7th grade: 30% 8th grade: 5% Others : 5%.
2) What was your grade last semester?
BA-AA: 20% CC-BB : 50% DD-DC : 20% FF –FD : 10%.
3) How much interested are you in Mathematics?
Very much: 26% Fairly : 50% Not Much : 14% Not at all : 10%.
4) How much interested are your parents in Mathematics?
Very much: 44% Fairly : 30% Not much : 14%

Not at all : 12%.
5) What do you think of Math Show?
Excellent: 76% Good : 20% Not bad : 2% Poor : 2%.
6) How similar are Math Show and your class activities?
Totally different: 70% A little similar : 25% Very similar : 5%
7) Would you like your math lessons to be like Math Show?
Yes : 80% No : 5% No idea :15%.
8) What did you like in Math Show? (Each pupil was asked to choose, in order of priority, the 3 factors that most impressed them)
Math Games : 90% Music : 80% Interesting problems : 70% Math Society Members : 40% Friendly Atmosphere : 40% Group Activities : 34% Technological equipment: 20%.
9) Now, do you think that Mathematics lessons can be fun?
Yes : 90% No : 4% No idea: 6%.
10) What do you feel about Mathematics?
I like it : 20% It can be interesting and fun: 50% It is hard and frightening : 30%

Discussion of Sample and Data

Questions 5 and 7 indicate very clearly that the Math Show is popular with the students. There is a strong indication that students would prefer to have non-traditional methods employed by their teachers in their math lessons. The children were so engrossed in the show that they did not realize that time had passed. The show always generates great enthusiasm, with many requests for more mathematics. Many children who had never considered that mathematics lessons could possibly be fun had changed their minds by the end of the show. Question 9 reports that 90% of students see how mathematics lessons can be fun when offered in a non-traditional format.

Question 4 indicates that the students' perceptions of parental interest are high. These pupils are aware that their parents value success in mathematics. The students' themselves were fairly interested in mathematics, as question 3 reveals.

We had anticipated 'novel equipment' might have attracted higher ratings; however, this option had the lowest level of interest. As question 8 shows, the highest levels of interest were in mathematics games, music and interesting problems, reflecting a focus on the essence of the show.

We also asked the teachers their opinion of the show. Teachers who attended the show were impressed, though many feel they cannot teach in such a dynamic way in their regular classes. This reflects what the literature says. Most mathematics teaching is done in a traditional manner. In addition, question 6 indicates that most in-class activities are not similar to the math show.

4. Conclusion

Overall, the math show benefited the teachers, the elementary school students, and the undergraduate students participating in the show. The show enabled teachers to realize that their teaching styles did not always match the learning styles of their pupils, and that a broader more varied approach can increase pupils' attention and interest during lessons. The data supports this conclusion. Abstract mathematical concepts can be better grasped if presented using drama, music and concrete applications of the concept, thus facilitating internalization and generalization.

The pupils not only developed greater awareness of their learning styles, but also learned not to be afraid of making mistakes, and to persevere in problem solving. Furthermore they began to be able to understand what was blocking their thought processes and avoid the obstruction. They also realized that self-confidence and the ability to generate ideas towards solving a problem are more important than getting the answer. Math anxiety was reduced or eliminated with this method of teaching mathematical concepts.

The undergraduate students participating in the Math Show also gained positively from the experience. They became more effective learners and teachers, both individually and in a group. They learned to develop and utilize different teaching techniques. Key concepts of each problem were discussed and became the focus of learning. In addition, they developed independent thinking skills and better self-confidence.

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TEACHING MATHEMATICS TO PRIMARY TEACHERS IN AUSTRALIA

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ABSTRACT

Primary teacher education is a relatively low status option for school leavers in Australia, when judged by competitive rankings of university entry scores. Although undergraduate primary education students have completed 12 years of study in mathematics, their knowledge is not always secure and their understandings are largely instrumental. Although most students are continuing from school, there are also mature-aged students (mostly women) among this cohort who have been out of education for many years. Not surprisingly, mathematics anxiety is manifest among many students, young and old. This paper will give details of an innovative approach to strengthening the foundation of mathematical knowledge as well as broadening the students' perspectives on the nature of mathematics itself, with a view to influencing the pedagogical approaches that the students will eventually adopt. The course content is based upon Bishop's (1988) 'six universals': counting, locating, measuring, designing, explaining, and playing. As with all educational endeavours, the paper represents a work-in-progress. It will outline the theoretical foundations of the course structure, describe student responses, and evaluate the progress of this course which has run since 2000.

Keywords: Teacher education, Curriculum, Culture, History, Adult learners

1. Introduction

Internationally, the preparation of primary (elementary) school teachers appears to be faced with the problem of teaching students who are insecure in their mathematical knowledge and frequently lack confidence in the subject; Australia is no exception. Primary teacher education is a relatively low status option for school leavers in Australia, when judged by competitive rankings of university entry scores. At Monash University the course *Exploring Mathematics* is attempting to address the issue of broadening and deepening students' knowledge of the discipline. Typically, students who are continuing from school have studied the less demanding options in the final years; others (mainly women) are returning to study after decades away from a mathematics classroom. Indications of mathematics anxiety are common — exacerbated by the requirement that 50% of the assessment is a written examination. Not surprisingly, students exhibit preferences for an instrumental approach to learning (Skemp, 1978): "Just give me the rules and I will memorise them" is a common plea from those less confident.

This paper will detail one approach to strengthening the foundation of mathematical knowledge as well as broadening the students' perspectives on the nature of mathematics itself, with a view to influencing the pedagogical approaches that the students will eventually adopt. It will outline the theoretical foundations of the course structure, describe student responses, and evaluate the progress of this course, which has run since 2000.

2. The Course Structure

Theoretical foundations.

Grugnetti and Rogers (2000) assert that school mathematics should reflect aspects of mathematics as a cultural activity:

— from the philosophical point of view: mathematics must be seen as a human activity, with its cultural and creative aspects.

— from the interdisciplinary point of view: when mathematics is linked with other subjects, the connections must be seen not only in one direction. Students will find their understanding both of mathematics and their other subjects enriched through historical liaison, sympathies and mutual aid between subjects.

— from the cultural point of view: mathematical evolution comes from a sum of several contributions. Mathematics can be seen as having a double aspect: an activity both done within individual cultures and also standing outside any particular culture. (p. 61)

In Australia, although there are national and state-based curriculum statements supporting these aims (e.g., Australian Education Council, 1990; Board of Studies, 2000), the reality is that they are peripheral in terms of the implemented and assessed curriculum; this is reflected in the range of commonly used textbooks. The three aspects listed above provide a summary of the theoretical foundations of *Exploring Mathematics*. However, it should be noted that in Australia, with less than 300 years of European settlement and where policies of economic rationalism prevail, the history of mathematics is not a major area of study in universities where departments of mathematics (and history) themselves are struggling to survive (Thomas, 2000).

Course content and assessment.

Rather than attempting to match directly the mandated curriculum strands — algebra, chance & data, number, measurement, and space — the course content was based upon Bishop's (1988) 'six universals': counting, locating, measuring, designing, explaining, and playing — in order to better gain access to the metacognitive perspectives listed above. Nevertheless, we remain cognisant of the content our students will be expected to teach, as well as the kind of mathematical/statistical written and graphical texts, emanating from ministerial and other research sources, that they will be expected to interpret and act upon as professional teachers. These six broad topics provide a sociocultural-historical basis to underline mathematics as a human construction (including its explicit and implicit values), with particular emphasis on non-European cultures such as those of Asia, the Pacific region, and Australian Aborigines. Lectures were predominantly focused on transmission of these aspects, with intermittent whole-group activities to keep students engaged. Tutorials were focussed on worksheet activities to be completed during the week. One aim of the course was to develop in students a sense of exploration through a problem-solving approach and the encouragement of appropriate web searches.

The course was presented over 10 weeks, with a one hour combined lecture and a one hour tutorial for two groups of approximately 30 students. The assessment consisted of a major project (20%), a folio of completed mathematical activities together with weekly reflective journals (30%), and a final examination (without calculators) (50%). In the reflective journal, students were required to address four items:

1. A list of mathematics content I learned for the first time, or had forgotten about. [Note anything that is still unclear, or that you are worried about, or you would like further work on.]
2. How I felt this week as a learner of mathematics. [Give reasons.]
3. How the topic relates to the primary school curriculum [mathematics **and** other subjects].
4. One teaching idea that I have developed from this week's work. [Give details of activity and approximate age level.]

The examination dealt with the mathematical processes that the students would be expected to be competent in (no higher than the upper secondary curriculum but attempting a greater depth of understanding through explanation), as well as questions concerning their knowledge of historical and cultural aspects of mathematics. For the latter, the questions were more open-ended. For example: "A primary student says to you: 'Where did our numbers come from.' How would you respond?" Problem-solving was assessed through tutorial work only — to the great relief of many.

The first two weeks were focused on revision of arithmetic and statistical knowledge supposedly covered in school, but which can never be taken as assumed knowledge. A .pdf file was loaded on to the Monash University intranet, detailing arithmetic algorithms with annotations and calculator keystrokes; students could use it as a self-paced module to update their skills. Both weeks included problem-solving or investigative work. The final week was revision, and all other weeks were devoted to one of the six themes listed above, with Counting given two weeks. Each weekly worksheet had about six activities, each developed through a range of increasingly open-ended or abstract questions.

3. Responses from Student Journals

As mentioned above, mathematics anxiety is a significant feature of students enrolled in courses such as these, and may be portrayed in emotionally-charged behaviours in tutorials, or through considered written reflections. However, in both years of presenting this course every student who made a sincere effort ended up gaining a pass mark, or higher credential (around 97% of students). It also happened that mature-aged students, initially among the most anxious, actually achieved excellent results on account of their serious attitude towards the study of mathematics (see FitzSimons & Godden, 2000). In the (non-random) selections from student journal entries below, two students are mature-aged [M1 & M2] and two are young, around their early twenties [Y1 & Y2].

Journal responses have also been used as supporting evidence for the three categories listed above as offering a theoretical framework for the course. In addition, there are entries, which highlight shortcomings in the course to date, and signs that it might be achieving some of its goals.

Anxiety

How have I felt this week as a learner of maths? Confused, dumb, like [I] have a mountain to climb, insecurities about teaching maths when I [have] feelings of being incompetent. ... Because I have forgotten much of the terminology and formulae, this adds to my insecurity. However, I do not want to pass on any self doubts to the students I will teach in the future, and intend to work hard at this subject to improve my maths on a personal scale, as well as my confidence in teaching maths to others. [M1, week 1, Arithmetic revision]

I still find myself becoming anxious whenever the word 'problem solving' is mentioned. I lack the confidence to 'have a go', perhaps [a] legacy of the days when getting the wrong answer meant punishment. [M2, week 3, Counting, part 1]

I am beginning to think that Mathematics can be an enjoyable experience, especially when shared with others. I have found that by talking about my investigations with my colleagues in the staff room, I realise that I am not the only person who experiences difficulties with some concepts. [M2, week 9, Playing]

Again this week I felt confused as a learner of mathematics, simply because I was learning about things that I had never considered to be maths or maths related before. As I began to see the relationship, though, I felt comfortable with what I was learning. [Y2, week 6, Designing]

Mathematics as a human activity.

I often like to make and construct things for fun for my home. It has occurred to me that when I make something I usually consciously conside r the logistics and aesthetics of whatever project I am undertaking, but after this week's activities I feel more appreciative of the significance of mathematical properties that come into play when designing and creating something. Similarly, as a learner of mathematics, I feel more appreciative of the significant role mathematics plays in daily activities. [Y1, week 6, Designing]

I also learnt that explaining is universal (all cultures use explanations), however we all explain in different ways. [Y2, week 8, Explaining]

Before this week I was unclear about the connection between maths and playing, therefore both the lecture and tutorial were able to help me feel more comfortable as a learner of maths because they helped me see the relationship — "playing is often valued by mathematicians because rule-governed behaviour is like maths itself" (lecture). [Y2, week 9, Playing]

Interdisciplinary aspects of mathematics.

I did enjoy doing Fibonacci numbers, especially once I researched Fibonacci and how this principle can be applied to many patterns in nature. [M1, week 3, Counting, part 1]

During teaching rounds I like to have a look at mathematics software available for children to use in the classroom. Much of the software I find features games that require the use of problem solving and logic. Children seem to enjoy them without realising that the games have foundations in mathematics. [Y1, week 9, Playing]

The topic of locating obviously related very strongly to the school curriculum through the maths strand of space. However, the unit also has relevance to probably every other KLA [Key Learning Area] because the ability to locate and use corresponding terminology are valuable in everyday life and language. In particular I think it relates to SOSE [Studies of Society and the Environment] (geography), Art (drawing and painting), English (understanding the terminology) and Technology (construction and info tech). [Y2, week 4, Locating]

I could also see how the study of design could be integrated with other subjects. For example, studying Ancient Egypt in SOSE could see the students investigating the properties of the pyramids. Students studying the cultures and practices of different countries could investigate and practice the art of origami, or make a range of Chinese influenced tangrams to create pictures. [M2, week 6, Designing]

Mathematics and culture.

I found the information about cultural differences in classifying and representing information pertaining to Maths fascinating. In particular, I have tended to simplify the actions of Aboriginal people, only seeing the physical connections they have with their land and people, yet the overhead of the Family Tree of the Yolnu people shows a complexity of mathematical information. [M2, week 8, Explaining]

In a way, Mathematics can be considered an art, demonstrating an ordered way of presenting and viewing information and using its own distinctive language including signs, symbols and terminology.

Mathematics is part of our everyday activity, so can be considered as a tool for daily life. Problem solving, investigation, inquiry can all be assisted through mathematical knowledge. [M2, end of semester introduction to journals]

Complaints from students.

As noted in previous weeks, I wish we spent more time during the lectures and tutorials going over some of the basic mathematical principles for the topics ... as this would give us a basis to build upon. [M1, week 5, Measuring]

I felt less comfortable as a learner of maths this week for a number of reasons. The first of which is that I didn't know if I was in a maths lecture or a history lecture. I don't mean to sound disrespectful, I just felt that today's lecture wasn't overly relevant to what I need to know to be a primary teacher. [Y2, week 5, Measuring]

I have researched the internet for some helpful information, but there seems to be an abundance of information on how to make a box plot or stem and leaf plot, but nothing on how I can describe or interpret the information. [M2, week 2, Statistics]

Making progress.

Whilst on teaching rounds over the last three weeks, I taught students maths and enjoyed it. I researched before each lesson making sure that I used the correct terminology etc. [M1, week 6, Designing]

I needed help from my classmates with the 'fractions to decimals' as I was unsure how to do this and was not familiar with the terms 'terminating, repeating, recurring'. However, once I realised what they meant, I tested out various fractions on th e calculator. I felt I had accomplished something when I saw fractions that were repeating, recurring or terminated and my confidence with these fractions to decimals increased. [M1, week 7, Counting, part 2]

It's strange to think this is possible, but I feel that this week I learnt a lot about my own views and understandings of mathematics. When we were asked to write down what maths is in the lecture, I found it challenging to determine all the things that this subject entails, even though I studied it throughout my entire schooling from Prep to12. Often the word maths is solely related to computation, and I think that in order to make the subject interesting and fun for children we need to start seeing maths as much more than that, so that we bring some variety into our classrooms. [Y2, week 3, Counting, part 1]

I actually enjoyed my role as a learner of maths this week. I like problem solving that requires a bit of thought and time, and many tutorial and lecture questions relied on working out processes and working towards a solution. I was challenged to think for quite some time about a number of the investigation questions, and therefore when I discover the answer it gives me a sense of achievement and satisfaction. I also felt comfortable with the new knowledge I was learning about combinations and with the revision on probability because it made sense to me. It's so easy to change how one feels about themselves as a learner of maths from week to week, because it's one of those subjects that if you don't get it, it will just drive you insane. Thankfully, this week I am understanding. [Y2, week 7, Counting, part 2]

4. The Projects

For their major projects students were asked, in 2000, to design and model an adventure playground suitable for primary-aged children. In 2001, they were asked to design a 'mathematics trail' for primary children, utilising a real or hypothetical site, including activities relating each of the six 'universals,' with questions of varying sophistication according to criteria such as Bloom's (1956) taxonomy. In both years there were many outstanding projects, as well as a few of doubtful quality reflecting minimal effort. Many had actually been trialled by students on their teaching rounds, which took place for three weeks around the middle of the course. Many projects indicated that the students had taken serious account of the intentions of the course as a whole, and will provide them with an excellent teaching resource in years to come; perhaps even a folio item for future job applications.

5. Conclusion

For a variety of reasons, no formal evaluation of the course took place in either year. Clearly it is easy for the author, who co-designed the course with Alan Bishop, to highlight the positive aspects and present selective journal entries. The major serious complaint appears that the lectures were too heavily weighted on the side of illustrated narratives of historical and cultural aspects to the detriment of mathematics theory and worked examples. This point is valid. However, there are also problems associated with lecturing on technique when the range of abilities of students (not previously known to the lecturer) is very wide, both in terms of courses studied and results achieved, and in terms of length of time away from formal study of mathematics. Obviously there is still some fine-tuning to be done. Other criticisms, not expressed here, are that this series of lectures and tutorials does not model good teaching practice, according to the theories espoused in students' teaching method lectures. The second time around it was easier to head off these complaints by addressing them in the beginning. Combined lectures and tutorials of around 30 students each are not ideal, but are one of the constraints set by the university. The examination, a source of great anxiety as mentioned above, had already been mandated by the accreditation process of the university.

Could this course be described, pejoratively, as a course in mathematical tourism? What are the borders/boundaries between improving disciplinary knowledge in terms of content and process, and enhancing pedagogical content knowledge in terms of offering a 'bigger picture' (Ernest, 1998) of mathematics — in relation to the three aspects of philosophy, interdisciplinarity, and cultural awareness as outlined by Grugnetti and Rogers (2000)? What evidence is there that these students will be more confident and competent as classroom teachers in years to come? The evidence so far, at least, is that new ways of seeing and knowing mathematics have been opened up to at least some of the students. A longer-term research project is needed to answer some of the other questions. As noted in the abstract, this paper has sought to describe a work-in-progress.

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MODERN STATE AND CHANGES IN MATHEMATICS TEACHER TRAINING IN RUSSIAN FEDERATION

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ABSTRACT

Serious changes in social and economical life in Russian Federation during the last decade had remarkable impact also on mathematics teacher education. Instead of strict and uniform curricula for mathematics teacher preparing, new standards have been elaborated by the Ministry of Education and since 1996 are being adopted by pedagogical universities. On the basis of these standards, universities construct their curricula for themselves. The example of a new course "Psychological and pedagogical foundations of mathematics teaching» within the new teacher education program at the Moscow State Pedagogical University is described.

Keywords: mathematics education, teacher education, curricula development, Psychological foundations of mathematics teaching

1. Overview

Serious innovations in educational system and, generally, the steps to the educational reforms were caused by crucial changes in political and social life during the last 10-15 years.

The Ministry of Education declared many good intentions.

First, two problems were stated:

1) To formulate a purpose of school education which is not reducible to preparing pupils for entering universities and colleges.

2) To give a priority to the developing function of education, to teaching pupils for the life in conditions of democracy.

New slogans were raised: openness, democratization, decentralization and humanization.

In curricula elaboration, the decentralization meant:

1) the rejection of the centralization of the process of elaboration of curricula.

2) the permission to use alternative curricula, textbooks, teaching methods etc.

3) the reflection of the regional and national specific in curricula.

The humanization demanded the turn towards needs of children and, in particular, abolition of the compulsory character of homework.

Also, the humanization asked for the differentiation of education, especially at the upper secondary level.

The differentiation is the central part of discussions on the reforms which continue since 1990 till today. Two kinds of the differentiation are being discussed:

1) w.r. to the amount of mathematics to be studied;

2) w.r. to the inclination of classes: mathematical, for engineering and natural sciences, for the humanities.

Now the process of the differentiation of schools and higher education takes place. Many of schools are converted into gymnasiums, "lycee"s, vocational schools etc. Various supplements to programs, special and optional courses are included in curricula of schools.

Serious changes in social end economical life in Russian Federation during the last decade had remarkable impact also on mathematics teacher education. Generally, approaches to the higher (university) education have changed. Instead of strict and uniform (all over Soviet Union) curricula for mathematics teachers training, new (preliminary) standards have been elaborated by the Ministry of Education and since 1996 are being adopted by pedagogical universities. On the basis of these standards, universities construct their curricula for themselves. Many pedagogical and other institutes are converted into pedagogical universities, technical universities, agricultural academies etc. In some of pedagogical universities, two-stage curricula have been elaborated: after first 4 years, students become Bachelors and may teach at lower secondary schools. After 2 years of additional studies, they become Masters and have the right to teach at upper secondary schools.

Obligatory assignments to institutes for annual production of young teachers, percent of satisfactory marks etc. are abolished. Institutes can almost independently work out their curricula.

As earlier in Communist times, mathematics teacher education programs cover all contents, issues, methods and resources one can only imagine.

Although, generally, higher education is now more popular among young people, teacher's profession is not popular because of bad employment possibilities, low salaries (15-40$ depending on region, experience and loading) which are, moreover, often delayed for many months. On the other hand, it is difficult for young graduates of pedagogical universities to find a job, because most of old teachers do not want to retire. Most popular in our country are now professions of finance managers, bankers, bookkeepers and lawyers. Many new private universities aimed at preparing for these professions appeared. Most of them compromise the idea of private educational institutions by their low level of organization and teaching. However, teachers are being prepared only at state universities, and thus rather high standards of teacher education are saved. The serious problem is the bad financing of educational institutions and low salaries of university teachers.

2. The structure of curricula

Generally, in standards recommended by the Ministry of Education in 2000, the amount of classroom hours for the whole course on pre-service teacher education program is about 4000, half of which (about 2000) are devoted to mathematical disciplines, one quarter (about 1000) to general cultural (e. g. social, philosophical and medical) sciences and one quarter (also about 1000) to psychological and pedagogical (including mathematics education) disciplines. However, the total amount of mathematics education is usually only about 170 hours (i.e. less than 5% of the whole program). The standards are used both in traditional 5-year course and in new two-level 6 year courses completing with magister's degree. The most remarkable features of these standards are greater freedom in distributing the classroom hours (which are assigned by standards only to whole blocks such as mathematical block or general cultural block) between different subjects, larger place for alternative special courses that can be freely chosen by students, larger amount of independent work of students.

However, democratical traditions are not very strong in Russian universities yet, and often such distribution (of classroom hours) and, generally, elaboration of particular curricula for concrete faculties is being authoritatively accomplished by deans. In conditions of freedom in composing curricula specialists in calculus who constitute a majority in mathematical departments are now reducing amounts of algebra and geometry lectures. Generally, both in school and pedagogical institutes' curricula the whole amount of hours devoted to mathematics is gradually lowering. The amount of lectures and seminars at universities has decreased, and the amount of hours for the independent studying by students has increased. Thus, the Government obtained the possibility to decrease the number of university teachers. On the other hand, university teachers find new possibilities to teach in many new private universities and institutes.

We will consider the new approaches to pre-service teacher education in a leading pedagogical university in Russia – in the Moscow State Pedagogical University.

3. The pre-service teacher education program in the Moscow State Pedagogical University

a. New approach to pre-service mathematics teacher education.

The well-realized by the Russian community need of inclusion of the educational system of Russia into world educational space compels us to think over the perspectives of the taking into account the world standards, federal, regional and national components of teacher education, that can be expressed in the international approach to the concept of professional competence of the prospective teacher.

In Russia, traditionally always rather serious attention to the teacher preparation has been given. There is a well-established system of the continuous pedagogical education: professional orientation at school (pedagogical circles at pedagogical institutes, classes with a pedagogical bias at schools), preparation of the elementary school teachers at pedagogical colleges; a system of preparation of the secondary school teachers at classical universities, at pedagogical universities and institutes.

During the last years some modifications in this preparation are taking place connected with multilevel structure of higher education: two-years' incomplete education, four-years' study for the Bachelor's degree; additional two-years' preparation to the Master's degree; alternative (traditional) way is five years' professional training for obtaining teacher's diploma. Besides, the rather important role belongs to three-years' graduate study, three-years' post-graduate study, and also to the ramified system of in-service professional training of each school or university teacher (ideally - once every five years of work).

The new system of continuous education for teachers is only arising, but both in traditional and modern systems there is a lot of unsolved problems, which are caused by the lack of the complex approach to the development of the professional competence of teachers. For the solution of these problems, at the Moscow pedagogical state university the international scientific conference "Pedagogical education for the 21st century" was held. There were many treatments and approaches to the model of the 21st century's teacher. Here are some parameters of such model having explicitly expressed communicative character:

– Our society and higher school will choose as a priorities democratic development in social life and market relations in the economic sphere, therefore giving up many traditional stereotypes;

– Generally, the importance of education is being realized now, that raises new demands on the system of teacher education;

– The transition to the preparation of the teacher-humanitarian will be gradually carried out; the thinking will promote the establishment of humanistic education.

For the last few years at the Moscow pedagogical state university the problems connected to the realization of the complex approach to mathematical, psychological, pedagogical and methodological preparation of the mathematics teacher have been studied.

In particular, the program of a course "Psychological and pedagogical foundations of mathematics teaching" is elaborated. This course has already been taught for several years to the students of the third year of study (i. e. to students having certain amount of knowledge on special mathematical disciplines, on pedagogy and psychology). The purpose of this course is the complex approach to the education of the teacher of mathematics.

Let's describe some features of the structure and contents of the course.

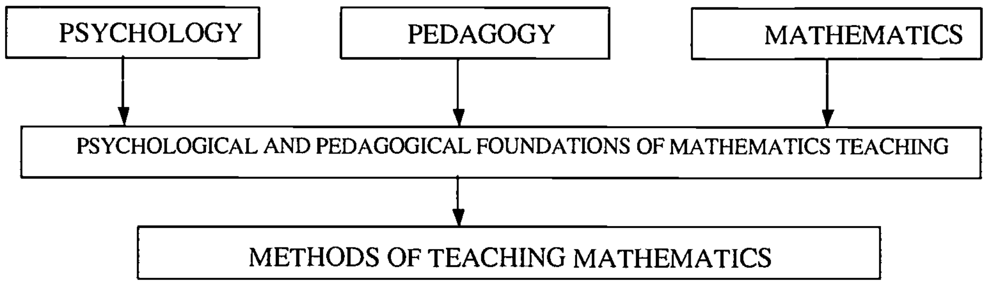
b. The general conception of the course.

1) It is possible to speak about three theories of teaching (or about three levels of the theory of teaching): psychological (the pedagogical psychology), general pedagogical (didactics) and methodological (methods of teaching a subject).

Traditionally, the course on methods of teaching mathematics is divided in two parts: general and particular (special) methods, but the experience has shown, that such division is not very useful. Essentially, the traditional general methods duplicate didactics, not concerning at all psychology of teaching. On the other hand, the particular methods consist of exact prescriptions for teaching certain themes of school mathematics, sometimes simply describing the school course. This is the really existing situation not providing pre-service teachers with necessary professional training.

Necessity of the intermediate course, serving as a bridge between psychology, pedagogy and mathematics, on the one hand, and methods of teaching of mathematics, on the other hand, is obvious. This necessity is caused by the impossibility of effective reference to general psychology and pedagogy.

2) It is assumed that the course "Psychological and pedagogical foundations of mathematics teaching" will be studied by the students, that have already learnt psychology, pedagogy and some part of mathematics, before the course on methods of teaching mathematics (see the scheme below):



In accordance with its purposes, the course "Psychological and pedagogical foundations of mathematics teaching" only briefly reminds the essence of already learnt concepts and statements of psychology and pedagogy, paying attention to their concretization in view of specific properties of mathematics. Therefore, the course "Psychological and pedagogical foundations of mathematics teaching" essentially differs from general courses on psychology and didactics. It is also essentially different from the course on methods of teaching mathematics, the base for which it constitutes.

3) The main contents of the course "Psychological and pedagogical foundations of mathematics teaching" consist of the series of the extremely important and interesting concepts: the purposes of teaching mathematics directed on all-round development of the personality of the pupil; theoretical foundations of the individualization and differentiation of teaching mathematics; the theory of abilities and, in particular, mathematical abilities; thinking, means of thinking, mathematical thinking; the activity approach to teaching mathematics; mathematical learning activity; the essentials of the developing instruction, mathematical development of the pupils etc.

The traditional preparation of the mathematics teacher actually does not give a possibility to study the above-mentioned concepts: in courses on pedagogy and psychology they are not considered, because teachers of these disciplines are not familiar with mathematics and its methods; on the other hand, mathematicians can not deeply study these problems, because they do not have a good knowledge of psychological and pedagogical theories.

4) The course "Psychological and pedagogical foundations of mathematics teaching" inevitably should include also logical foundations of teaching mathematics, because a separate course on logic is not present in curricula (the course on mathematical logic has other purposes). At the same time, the logical foundations are inseparable from psychological. For example, the process of learning concepts is connected partly to psychology and partly to logic. Searching for a proof is a complicated psychological process, but a proof itself is a logical construction.

5) The course "Psychological and pedagogical foundations of mathematics teaching" consists of lectures and exercises. Taking into account the fact that this course lies on the crossroad of scientific disciplines, it seems useful to stimulate also conducting research by the students, writing by them term and final research papers on psychological and pedagogical foundations of teaching mathematics.

c. The contents of the course.

1) The course "Psychological and pedagogical foundations of mathematics teaching", its interrelations with other subjects (psychology, pedagogy,mathematics).

2) Mathematics as a science and as an educational subject. Methodological foundations. Teaching mathematics and development of the personality of the pupil. The purposes of teaching mathematics at school.

3) Thinking. Psychology and logic of thinking. Kinds of thinking. Mathematical thinking. Levels of mathematical thinking on various stages of learning mathematics at school. Means of thinking. The role of mathematics teaching in developing of basic means of thinking.

4) Process of teaching/learning. Psychological, informational, logical and didactical aspects. A model of process of teaching mathematics. Activity: the teaching activity of the teacher, the learning activity of the pupil. The activity approach to learning mathematics. The cognitive activity in the field of mathematics. A model of learning mathematical activity. The individualization of learning mathematical activity. Didactical principles in teaching mathematics.

5) Methods of teaching. The problem of methods of teaching. Reproductive, empirical, logical methods in teaching mathematics. Psychological and pedagogical foundations of differentiation of teaching mathematics. The problem method of teaching. Problem situations. Basic types of problem situations in teaching mathematics.

6) Mathematical knowledge and skills. Scientific and educational knowledge. Transformation of scientific knowledge into educational (didactical transposition). Basic results of teaching mathematics: mathematical knowledge and mathematical development. Relations between mathematical knowledge and skills. The principles of the selection of the contents of school mathematical course. Psychology and logic of the process of learning the concepts, of proving propositions and solving problems in teaching mathematics. Psychology and logic of questions and answers. Algorithms and heuristics in teaching mathematics. The role of problems in teaching. Means of searching for a solution of a problem. The systemic and structural analysis of school mathematical problems. Complexity and difficulty of mathematical problems.

7) Mathematical development. Various treatments of the concept "mathematical abilities of the pupils". Detection and development of mathematical abilities during the process of teaching. Means of reaching certain levels of the mathematical development. Investigational activities in teaching mathematics. Motivation of learning. The principle of the best stimulus (G. Polya). Independent work of pupils with the elements of creativity. The role of non-standard tasks in mathematical development of the pupils.

4. Further problems

The complex approach to the solution of the above-mentioned complicated problems, and also multidimensional character of the preparation of teachers compels us to combine the work of the lecturer and the student, all possible forms and variations of training in colleges and higher pedagogical educational institutions, to determine the work and responsibility of various subdivisions in the system of in-service teacher education. In this connection a global problem arises – a problem of the detection of the levels of professional readiness, of the evaluation of these levels, of the detection of mechanisms and technologies of the transition from one level to another etc. It is possible to formulate rather primitive, but useful initial statement: it is well known, that the professional skill comes with experience; at the same time it is clear, that the teacher who comes to work in school, should have the skill in a certain initial level (so that below this level one does not have the right to teach).

The system of preparation of the mathematics teacher consists of a series of blocks, and now in conditions of the multilevel system of education these blocks are even more distinct, their structure and contents are subjects of thorough attention during the process of evaluation of curricula and of work of a faculty or an institute in general.

As already noted above, it is necessary to begin with the detection of interrelations (in actions and in effect) of psychological and pedagogical, on the one hand, and methodological, on the other hand, preparation of the teachers of mathematics. The transition to the multilevel system of preparation of the teachers demands elaborating levels of readiness to professional activity corresponding both to stages of education and to general fundamental conceptions of professional skills of teachers.

DESIGN OF THE SYSTEM OF GENETIC TEACHING OF ALGEBRA AT UNIVERSITIES

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ABSTRACT

For teaching on the basis of the genetic approach,, one should accomplish the analysis consisting of two stages: 1) genetic elaborating of a subject matter and 2) analysis of arrangement of a material and possibilities of using various ways of representation and effect on students. The genetic elaborating of a subject matter, in turn, consists of the analysis of the subject from four points of view: a) historical; b) logical; c) psychological; d) socio-cultural. In designing of the system of genetic teaching very important is to develop problem situations on the basis of historical and epistemological analysis of a theme.

Keywords: mathematics education, teacher education.

For teaching on the basis of the genetic approach, we offer to construct didactical system of study of a mathematical discipline (a part of a mathematical course, important concept or system of concepts) consisting of two parts 1) preliminary analysis of the arrangement of the contents, of didactical means and 2) concrete design of the process of teaching.

The preliminary analysis consists of two stages: 1) genetic elaboration of the subject matter and 2) analysis of the arrangement of a material and possibilities of using various ways of representation and effect on students. The genetic elaboration of a subject matter consists of the analysis of the subject from four points of view:

historical;

logical;

psychological;

socio-cultural.

The historical analysis frequently encounters with large complexities because of insufficient knowledge of the history of the origin and development of many branches of modern mathematics included in university curricula, inaccessibility of the literature on the given subjects. Therefore, it is necessary to conduct research of the history both of appropriate areas of modern mathematics, of their inclusion in the university curricula, to study educational literature, text- and problem-books, the history of the teaching of modern mathematics. As more or less accessible sources for the teachers and students the monographs and other scientific works – books and articles, books on the history of mathematics and mathematics education, manuals and encyclopaedias can serve. Very important is also to study original works of great mathematicians, classical textbooks, popular scientific literature, journal and magazine articles. The purpose of the historical analysis is to reveal paths of the origination of scientific knowledge underlying the educational material; to find out, what problems have generated need for this knowledge, what were the real obstacles in the process of the construction of this knowledge.

In designing the system of genetic teaching very important is to develop problem situations on the basis of historical and epistemological analysis of a theme.

The major aspect of rational (in the sense of Toulmin, 1972) organisation of an educational material consists in organising a material so that to reveal the necessity of the construction and of development of concepts and ideas. It is necessary to arrange problem situations or tasks, for which the important concepts or ideas, which should be studied, would serve as the best solutions. It is necessary to analyse those problems of knowledge, for which the considered concepts and ideas serve as the necessary solutions. For this purpose, both historical analysis and epistemological considerations, and special search for appropriate problem situations and tasks can help.

In our view, for the logical organisation of a system of concepts and propositions of a theme, of the teaching unit of a mathematical discipline, one should carefully analyse the deductive structure of such system, required, for example, for the construction of a concept or for the statement of a proposition. We will name the results of such analysis a logical genealogy of a concept or a proposition. In the university mathematics, especially in higher algebra, such genealogies may be rather complicated.

Clearly, such complicated structure of concepts and statements, needed for understanding the theorems of large difficulty, requires well-designed activities for successful learning.

Therefore, very important is also the psychological aspect of the genetic approach to the teaching of mathematical disciplines.

The psychological analysis includes determination of the experience and the level of thinking abilities of the students (whether they can learn concepts, ideas and constructions of the appropriate abstraction level?), possible difficulties caused by the beliefs of the students on mathematical activities (for example, the students can bear from school views on mathematics as mere calculations aimed at the search of (usually unique) correct answers with the help of ready instructions etc.). The psychological analysis has also the purpose to plan a structure of the activities of the students on mastering concepts, ideas, algorithms, to plan their actions and operations, and also to find out necessary transformations of objects of study.

When studying university algebra courses, the students usually are encountered with sequentially growing steps of abstraction - with a «ladder of abstractions».

À. À. Stolyar (1986, p. 58-60) has revealed 5 levels of thinking in the field of algebra and has noted, that "the traditional school teaching of algebra does not rise above the third level, and in the logical ordering of properties of operations even this level is not reached completely". The following is the description of the third, fourth and fifth levels according to À. À. Stolyar (ibid., p. 59):

"On the 3-d level the passage from concrete numbers expressed in digits, to abstract symbolic expressions designating concrete numbers only in determined interpretations of the symbols is carried out. At this level the logical ordering of properties is carried out "locally".

On the 4th level the possibility of a deductive construction of the entire algebra in the given concrete interpretation is become clear. Here the letters designating mathematical objects are used as variable names for numbers from some given set (natural, integer, rational or real numbers), and the operations have a usual sense.

At last, on the 5-th level distraction from the concrete nature of mathematical objects, from the concrete meaning of operations takes place. Algebra is being built as an abstract deductive system independent of any interpretations. At this level, the passage from known concrete models to the abstract theory and further to other models is carried out, the possibility of existence of various algebras derived formally by properties of operations is accomplished".

Thus, to the 5-th level the deductive study of groups, rings, linearly ordered sets etc. corresponds. The highest degree of abstraction here is the study of general algebraic systems with various many-placed operations.

To the 4th level corresponds, for example, a systematic and deductive study of the sets of natural numbers or integers. Therefore, taking into account, that in school teaching even the 3-rd level is not completely reached, it would be certainly a big mistake to omit in pedagogical institutes the 4th level (systematic study of an elementary number theory) and immediately pass to the deductive study of groups, rings and even of general universal algebras (as is done in a text-book by L. Ya. Kulikov, 1979). Therefore, systematic study of the elementary number theory can serve as a good sample of the construction of a deductive theory for preparation for the further construction of the axiomatic theories.

À. À. Stolyar built his classification of levels from the point of view of teaching school algebra. In our view, development of algebra as a science in the last decades (after the World War II, under the influence of works of S. Eilenberg and S. MacLane, 1945, and A. I. Maltsev, 1973) allows to distinguish one more higher, the 6th level of algebraic thinking - we will namc it the *level of algebraic categories*. At this level the entire classes of algebraic systems together with homomorphisms of these systems - varieties of universal algebras, categories of algebraic and other structures (for example, topological spaces, sets and other objects) are considered. Thus, the abstraction from concrete operations in these structures and from the nature of homomorphisms and generally of maps takes place; morphisms between objects of categories are considered simply as arrows subject to axioms of categories – for example, the associativitiy law for the composition. Moreover, the functors between categories – certain maps compatible with the laws of the composition of morphisms, and natural transformations of functors are considered.

Note that J. Piaget in the last years of his life was interested in the theory of categories as the highest level of abstraction in the development of algebra (Piaget and Garcia, 1989).

The teaching of algebra at this level (theory of categories and varieties of universal algebras) is not included into the obligatory curricula even of leading universities and happens only on special courses. But, nevertheless, the presence of this level demands that the students should master algebraic concepts in obligatory courses in a sufficient degree for understanding the algebraic ideas on the highest level of abstraction.

Essential in teaching algebra and number theory in pedagogical institutes are the 4-th and 5-th levels in the classification of À. À. Stolyar. First of all, the 4-th level (which is already beyond the school curricula) should be reached. Therefore, during the first introduction of the definition of group in the beginning of the algebra course, one should not immediately begin the full deductive treatment of the axiomatic theory of groups. Only after the experience of the study at the 4-th level of thinking in the field of algebra, namely of the study of the elements of number theory, it is possible to consider a deductive system of the most simple constructions and statements of the group theory, and the systematic account of complicated sections of the theory should be postponed to a later time, after studying at the 4-th level of such themes as complex numbers and arithmetical vector spaces.

J. Piaget who developed the classification of levels for thinking in the fields of geometry and algebra ("intra", "inter" and "trans"), noted that it is possible to distinguish sublevels inside each level (Piaget and Garcia, 1989).

According to the theory of A. N. Leontyev (1981), actions on learning concepts, as well as any actions, consist of operations, which are almost unconscious or completely unconscious. These operations are essentially «contracted» actions with the concepts of the previous level of abstraction. As M. A. Kholodnaya (1997) noted, «a contraction is immediate reorganisation of the complete set of all available ... Knowledge about the given concept and transformation of that set into a generalised cognitive structure».

The theories of E. Dubinsky (1991) and A. Sfard (1991) are close to the Soviet conceptions of actions and operations as contracted actions in mathematics teaching.

In our view, for reaching a contraction of an action with algebraic objects into (automatic) intellectual operation it is necessary, after sufficient training with of this action, to include it in another action, connected with the construction of objects of the next step of abstraction.

One more purpose of the psychological analysis of the subject matter is finding out the ways of the development of motivation of learning.

The socio-cultural analysis has a purpose to establish connections of the subject with natural sciences, engineering and economical problems, with elements of culture, history, public life, to reveal, whenever possible, non-mathematical roots of mathematical knowledge and paths of its application outside of mathematics.

During the second part of analysis, considering the succession of study, it is necessary, in accordance with the principle of concentrism (Safuanov, 1999), to find out, on the one hand, which earlier studied concepts and ideas should be repeated, deepened and included in new connections during the given stage, and, on the other hand, which elements studied at the given stage, anticipate important concepts and ideas that will be studied more completely, become clearer later, to check, whether there are possibilities of such repetitions and anticipations.

The principle of multiple effect requires also the finding out possibilities of multiple representation of concepts studied, of use of active, iconic and verbal-symbolical modes of transmission of information, of other means of effect on students (the style of the discourse, emotional issues, elements of unexpectedness and humour).

After two stages of analysis, it is necessary to implement the project of the process of study of an educational material. We divide the process of study into four stages. The first two stages (construction of a problem situation and statement of new naturally arising questions) constitute the process of the rational organisation of the educational material confronted to the 3-d stage of the logical organisation of the educational material.

1) Construction of a problem situation.

In the genetic teaching, we search for the most natural paths of the genesis of processes of thinking and cognition.

According to the activity approach to the process of teaching, usually "the initial moment of the mental process is the problem situation ... This problem situation involves the person in the thinking process; the thinking process is always directed to the solution of a problem" (Rubinshtein, 1989, p. 369). Therefore, the main purpose of the teacher is to construct a problem situation. The necessity of the construction of a problem situation was underlined by many prominent educators – by constructivists (creation of "disequilibrium") and representatives of the "French didactique" ("didactic engineering", directed on the creation of the didactical situations, on determination of the "epistemological obstacles") as well.

2) Statement of new naturally arising questions.

According to the theory of the activity approach to teaching, "the arising of a questions is the first sign of the beginning work of the thinking and the first step to understanding ... Every solved problem generates a lot of new problems; the more a man knows, the better he realises what more he should know" (Rubinshtein, 1989, p. 374-375). Therefore, it is important, after the solution of the initial problem situation, to constantly consider new, naturally arising questions. It was well understood by N. A. Izvolsky (1924) in his version of the genetic approach. Thus, in the design of the process of study of a subject the statement of new, naturally arising, questions is necessary.

Actually, both stages – construction of a problem situation and the statement of new, naturally arising questions – are aimed at the same purpose - to help students in the independent mastering of a concept. Therefore it is necessary to organise a construction of problem situations and also statement of new, naturally arising questions in such way that in a certain moment of time (we will name such moment "the hour of truth") the students could, independently or with the minimal help of the teacher, discover the new concept for themselves. It is similar to the moment of the selection in a subject of "the initial universal relation", leading to the theoretical generalisation in the theory of learning activity of V. V. Davydov (1986, p. 148), and also to the act of reflective abstraction (as the of interior co-ordination of operations of the subject in a scheme) in the theory of J. Piaget (Dubinsky, 1991), and also to a moment of a *reification* (Sfard, 1991). Such organisation of teaching frequently may be quite difficult and not always completely possible. For this reason we admit appropriate help from by the teacher.

3) Logical organisation of an educational material. Here, after the problem situation has been dealt with, the paths of its solution, various aspects and natural arisen questions have been discussed, the appropriate motivation has been reached, the construction of the elements of the theory - precise definitions, statements (axioms and theorems), conclusions takes place. At this stage deductive reasoning plays the great role.

4) Development of applications and algorithms. After the logical organisation of mathematical objects of a studied theory, it is possible to consider various interesting and useful applications of the theory in practice and in mathematics itself. According to the principle of multiple effect (Safuanov, 1999), it is necessary to solve the sufficient number of exercises on the variations of signs of concepts, on the inclusion of concepts in new connections and contexts, on various transformations of mathematical objects under study.

During all stages of study of the teaching unit or theme it is important to help the students to develop their own language for expression of their reasoning and ideas. For this purpose each proposition (definition or statement) should be stated (at lectures and in textbooks), whenever possible, in various languages: logical-symbolical and verbal (this suggestion complies also with the principle of multiple effect).

It is necessary also to give the students the exercises on development of mental operations (analysis, synthesis, generalisation, comparison, analogy, abstraction and concretisation). For example, the exercises on extraction of conclusions from theoretical positions will be useful. Such exercises promote development of abilities of the synthetic reasoning.

Finally, it is very important to encourage reflection in minds of students, i. e. the ability to realise the foundations of their own activities, reasoning and conclusions, to be aware of the structure of their thinking process.

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ON-LINE ASSESSMENT IN UNDERGRADUATE MATHEMATICS

An experiment using the system CAL for generating multiple choice questions

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ABSTRACT

In this paper we report on two experiments of assessment of Linear Algebra courses, each involving more than 300 students, led at *Instituto Superior Técnico, Lisbon*, in 2000/2001 and 2001/2002. We used a system for authomatic grading and generation of multiple choice questions. In this way we were able to assess the students weekly.

Multiple choice questions were chosen because they are easy to grade automatically. However, this does not necessarily mean that the questions themselves are boring and uninteresting. The students have, of course, to provide the right answer, but they have enough time to find it. Each individual exercise list has a week time to be solved and students usually discuss the questions with their colleagues and teachers.

The most important goal of this on-line assessment model is not to assess the students but to provide them a weekly stimulus for learning the subjects taught at lectures at that time. The final grade takes these grades into account but also the grade of a final written exam and written tests.

The results were very convincing [3]. Students were highly motivated and kept asking specific questions about the subject matter. The rates of success in the course grading were higher than those of previous years.

We have used a system (CAL-Computer Aided Learning) that randomly generates different instances of the same template question. Six to eight template questions are organized in one exercise list. Each student is assigned a different instance of this exercise list.

Although CAL has proved to be very useful it is not the best solution since it requires a considerable effort and knowledge of programming from the teacher who is implementing new questions. A simpler approach is to re-use and to slightly adapt template questions already available.

1 Introduction

Exercising in Mathematics, especially in basic courses such as Linear Algebra and Calculus, is important. The more a student is exposed to solving problems, the deeper is his/her understanding of a given subject. Moreover, the exercises a student tries to solve shape the way he/she understands the subject.

The web provides a way of displaying material that is more appealing to new generations of students, plus giving them the freedom to choose the time, place, and style of study. As instructors of Mathematics in an engineering school we are sensitive to all these possibilities offered by the web [1,2,4].

For that purpose it is necessary to have a system that is able to produce automatically many questions, to make these questions available in the form of web-based exercise lists and to automatically grade the answers.

In the next section 2 we present the system CAL that we used for the purposes just described, including a simple example that illustrates the main concepts. In section 3 we report on our experience using the system and in section 4 we conclude and refer to future directions of work.

2 Computer Aided Learning - CAL

The CAL system allows one to write template multiple choice questions, and generates random instances of these questions, thus producing individual web-based exercise lists for students to solve and that can be used for assessment, and as a training basis for first-year undergraduates.

During the last two years, we created a database of multiple choice questions on Linear Algebra. The text of each template question depends on parameters and is written in *Mathematica*, directly using the algebraic operations made available by *Mathematica*. The program randomly determines the parameters, and also determines the right answer. In this way we are able to get different instances of the same template questions. In the following we illustrate how to write template questions. We chose a very simple example that refers the basic concepts involved in a template question. We hope to convice the reader that this is not a too difficult task.

Another example is presented in order to show that it is possible to construct interesting questions (even in a multiple choice setting). This and other examples (in portuguese) are available in http://www.math.ist.utl.pt/~cal2000.

2.1 Determinant of a Matrix

This first example is a very simple one. The student is asked to compute the determinant of a given matrix. The question has the following form:

Consider the matrix ■. Its determinant is:

1. *wrong answer 1*
2. *right answer*
3. *wrong answer 2*
4. *wrong answer 3*

The ■ represents a concrete matrix that is randomly chosen each time that a new instance of the question is needed. The possibility of choosing randomly a particular matrix (or other mathematical object) with certain properties is the main facility of the system that we use.

2.2 Types

In general one defines previously the *type* of the parameters to be randomly determined. In this example we firstly define the type of the entries (they can be any integer between -2 and 2) and the type of the matrix (a 3×3 matrix with those entries):

TypeOfEntries = INTERVAL[Integer, {-2, 2}];
TypeOfMatrix = MATRICES[TypeOfEntries][{i, 3}, {j, 3}];

In order to make life easier for the students (and to illustrate another construction) we refine the type of matrices and consider the subset of those matrices that satisfy the further condition that their determinant (in absolute value) is less or equal than 10:

NewTypeOfMatrix =
SUBSET[TypeOfMatrix, Function[m, Abs[Det[m]] <= 10], 1000];

(The number 1000 is the number of times the system will try to find a matrix satisfying these conditions).

These are all types we need for this example. A concrete instance of such a type is obtained by the command **RandomChoice** as shown in the next section.

2.3 Random instances and errors

Recall that we want the student to find the determinant of a certain matrix and then select the right answer from among four different ones. That means that we have to a) generate a matrix and determin its determinant and b) generate 3 other numbers to be used as "wrong determinants" of that matrix. The sequence

Matrix = RandomChoice[NewTypeOfMatrix];
determinant=Det[Matrix]

assigns a new matrix to the corresponding variable and then determins its determinant.

There are several ways to define the "wrong determinants". For simplicity these are the determinants of 3 new matrices. Since these numbers have to be all different and different from the right determinant we repeat the choice of new matrices until this condition is met.

DetError1 = DetError2 = DetError3 = determinant;
While[Not[determinant ≠ DetError1 ≠ DetError2 ≠ DetError3],
DetError1 = Det[RandomChoice[NewTypeOfMatrix]];
DetError2 =Det[RandomChoice[NewTypeOfMatrix]];
DetError3 = Det[RandomChoice[NewTypeOfMatrix]]]

At this point we have all data necessary for the question.

2.4 Text of the question

The text of the question (written in a text *Mathematica* cell) is simply:

Consider the matrix Matrix. Its determinant is:

Above *Matrix* is a *Mathematica Inline expression* meaning that the value of *Matrix* (the randomly determined matrix) will be placed at that position in the final text.

The right and wrong answers are treated in a similar way.

2.5 Generation of a question

By running a particular command (**SaveAsHTML**) on a *Mathematica* notebook containing the code previously described a new instance of the question is generated. We show two examples in the following:

Consider the matrix $\begin{pmatrix} 0 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix}$.

Its determinant is:

−3 0 2 1

Consider the matrix $\begin{pmatrix} 2 & 1 & 0 \\ -2 & 1 & 1 \\ -2 & -1 & 1 \end{pmatrix}$.

Its determinant is:

4 6 0 −1

Each example is shown with the list of choices (authomatically generated). The first is the right one. The list of choices is processed (see below) so that their order is randomly changed.

Besides the implementation of several "types of mathematical objects" specific packages for Calculus and Linear Algebra have been also developed by the authors.

2.6 Another example

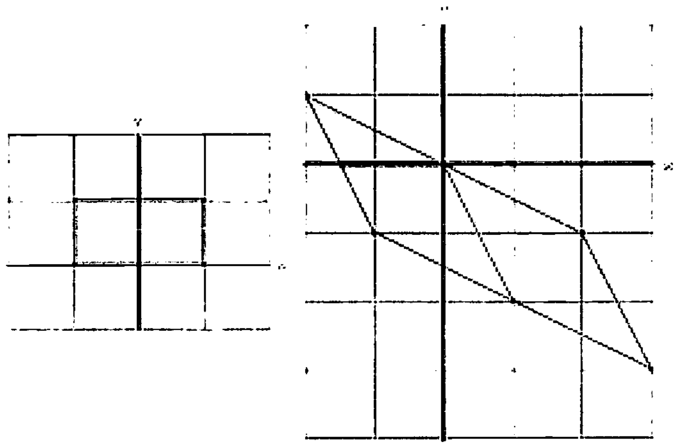
The following graphically appealing question is by no means easy to solve and shows that it is possible to have interesting questions also in a multiple choice setting.

The text reads:

Let $T : \mathbb{R}^2 \Rightarrow \mathbb{R}^2$ be a linear transformation that maps the paralelogram on the left to the one on the right. Find the cosine of the least angle between the eigenvectors of T.

1.

Considere a aplicação linear T de $\mathbb{R}^2$ em $\mathbb{R}^2$ que transforma o paralelogramo da figura à esquerda no da figura à direita. Qual o coseno do menor ângulo entre as direcções próprias da aplicação T ?



(2 valores)

- 0
- 1
- $\frac{1}{\sqrt{10}}$
- $\frac{2}{\sqrt{5}}$

2.7 Making the questions available

After the generation of enough instances of the exercises these are again randomly assigned to the students (identified by a username). The Perl script that does the random assignement is a modification of a preliminary version of *Web Assign* from Larry Martin and is also responsible to change the order of the choices of the multiple choice.

2.8 User Interface

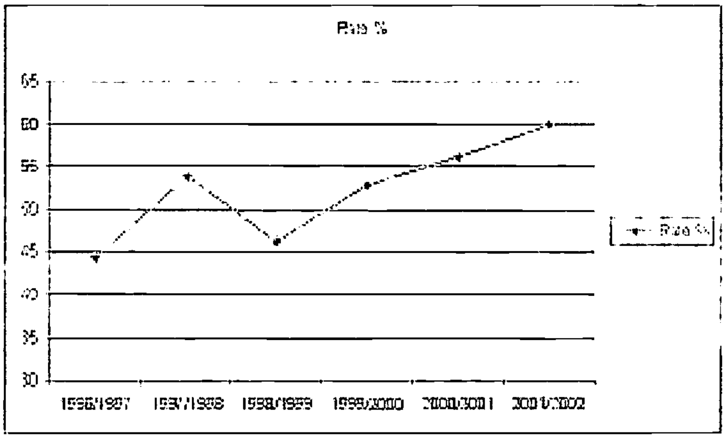
The time given to the student for solving each individual list of exercises is decided by the teachers. We chose one week for each list. The student enters the system with his/her username and password and has access to his/hers individual exercise list. The selection of a choice is by clicking on the corresponding box (we chose always to have four possibilities). During the week the student can obtain a hard copy of the list of exercises, exit and return to the electronic list even without answering some or all questions. However, by clicking Submit, his/her present selected choices are sent to the system and considered final. An immediate feedback is given about how many right/wrong answers the student has submitted. Afterwords, the student can still return to the list, and select choices for other questions but not for those already considered final. At the end of the week the list of exercises is closed. The student can still access his/hers individual list of exercises and be informed on the right choices for the questions of that individual list of exercises. This quick feed-back helps identifying subjects that need further study and that he/she probably would not be aware otherwise.

3 Assessing First-year Undergraduates

We used CAL in a large scale (400 students) for the first time in the Linear Algebra course of the first semester of 2000/2001.

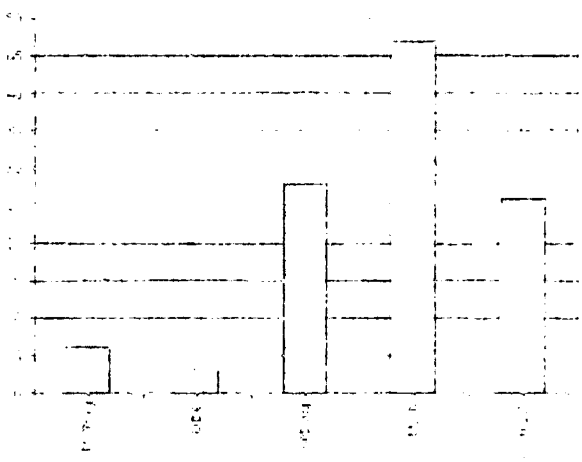
The students answered eight lists of exercises (each open for a week) covering many different subjects on Linear Algebra. These exercises contributed 30% to the final grade in Linear Algebra. The other 70% of the assessment was traditional, that is, through grading of written tests and exams. We took great care that the questions in these exams and tests were at least of the same difficulty level of those of previous years.

As we stated before our goal was to provide students a weekly stimulus for learning. The results were very convincing. Students were highly motivated and kept asking specific questions about the subject matter. The rates of success in the course grading were higher than those of previous years. The next figure shows those rates. Recall that the system CAL was used in 2000/2001 and 2001/2002.



Rates of success from 1996/1997 to 2001/2002

During an evaluation survey (in 2000/2001), students were highly positive about this experiment and its contribution to their own learning process. From about 100 volunteers, who answered the survey, 70% considered that the on-line exercises were very important to their understanding of the course content (see next chart).



Answers to the question: How much did the on-line exercises help you understanding the subjects?

4 Conclusions and Future Work

We have described an on-line assessment model whose main purpose is not to assess the students but to provide them a weekly stimulus for learning the subjects teached at lectures at that time. It improves learning by facing students with several exercises that they have to solve throughout the semester.

This helps mostly the less motivated students that, in other situations, loose track of the sujects after the first two weeks and only try to learn some days before the final exam (and fail). In principle this type of assessment might be done traditionally, without computers. For the teacher the advantage of this system is the possibility of generating many different questions (a different exercise list for each student) and, furthermore, without the otherwise extreme effort to grade them. Moreover, the fact that the exercises are available electronically seems to be more apealling to students than traditional lists of exercises.

Our conviction, supported by the positive reaction and the rate of success of students, is that this or similar systems do help students to learn.

There are several directions of future work: a) to improve the system with more questions possibly about different subjects; b) to use the system not only for assessment but also for self-learning and c) to improve technically the system.

We are cooperating with the University of Madeira in order to reproduce and improve the assessment experience of Linear Algebra there, adapting and creating more template questions. This will improve the variety of the database of questions in this subject. In cooperation with the same university we plan to make available exams of mathematics of interest to secondary school students and teachers. Moreover, we are already constructing a database of questions (not only multiple choice ones) on Calculus II for first-year undergraduates. Cooperation with teams from other universities is welcome (our questions are in Portuguese but should be easy to translate).

On another direction we are planning to create an electronic textbook for secondary school mathematics that uses some template questions for ilustration and self-assessment purposes. In this case the questions are not only graded but a short explanation of how to find the solution should be given. The textbook is intended to help to fill the gap between secondary school mathematics and undergraduate courses.

We are currently improving the interface with the teacher in many ways: selecting the questions for a test in an easier way, entering student data and entering student scores in the grade book.

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TEACHING PERIODIC DECIMALS IN TERTIARY EDUCATION

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ABSTRACT

Periodic decimals (pd) constitute a fundamental representation of rational numbers. Additionally, their study leads to the understanding of basic characteristics of the decimal system and to a more profound comprehension of the operation of division. Furthermore, the a-priori analysis of the properties of the pd indicates that this area is adequate for learning activities, in which students may work by combining the use of inductive and deductive methods. This claim is supported by the historical development of Number Theory.

However, only a very limited part of the Mathematics curriculum in elementary and secondary education is devoted to this area. Often, this is also true for the pre-service undergraduate studies of (elementary and secondary) schoolteachers of Mathematics in Mathematics and Education Departments.

In the first part of this work, we present results of an empirical study of the knowledge, which (elementary and secondary) schoolteachers of Mathematics and students of Departments of Education in Greece have on pd. Our results point out that the properties of pd are largely unknown both to the teachers and to the students who have been asked. For example, 96% of 207 students of Departments of Education don't know that in all non-terminating divisions, the quotient is a pd.

In the second part of this work, we give an outline of a teaching approach concerning pd, which has been applied in the Department of Education of the University of Crete. In the context of this teaching approach, students become able to discover the basic properties of pd and to prove many of them, by combining the use of inductive and deductive methods. Apart from a significant improvement of the students' knowledge concerning the pd and the decimal system, we have also observed an important improvement of their understanding of inductive methods and of the fruitfulness to combine such methods with deductive methods, in order to study problems in Arithmetic and elementary Number Theory.

1. Introduction

Apart from the classical representation of fractions as ratios of integers (a/b), students encounter them as decimals with a finite or infinite number of digits (finite or periodic decimals respectively). Students come in first contact with periodic decimals (**p.ds**) at the age of 10-11. For all the rest of their mathematical education, p.d. is one of the most frequently appearing mathematical objects in students' work. (e.g. It is possible to encounter them every time they work on a non-terminating division of entire or decimal numbers)

In compulsory education (10-15), the exploration of the properties of the p.ds can contribute to a better understanding of the decimal system, of the decimal development of fractions and of the division algorithm (see Kourkoulos M. 1999 ii). Nevertheless, as we will see in [2], not only students but also teachers know very little about p.d.

The a-priori analysis of the basic properties of p.d. (see Kourkoulos M. 1999 i, ch 4), as well as the experimental data presented in [3] indicate that p.d. is an adequate domain for the development of students' Mathematical culture (see also Tzanakis, Kourkoulos 1998)

This is because, the exploration of p.d.:

-Leads to fertile questions, which instigate students' interest.

- Gives the possibility to form conjectures and hypotheses

- Is suitable for the organization of experimental (inductive) exploration of the formulated conjectures. The degree of difficulty of the experimental exploration of the properties of p.ds varies. This offers important possibilities to the teacher to design activities in which an experimental research on mathematical conjectures is asked (This is an important but neglected issue in the current conditions of Mathematics education, see Polya 1954, Lakatos 1976).

It is worthwhile to note that the experimental investigation of the properties of the p.ds often can be facilitated by "the intelligent use of the calculator and the computer in teaching activities" (see also Bruillard, Vivet 1994).

-It creates the desire to look for the explanations and the justification of the properties, which are found empirically (Balacheff, 1982, Duval, 1993)

- With the use of appropriate teaching activities, already from the level of the compulsory education, students can combine experimental and deductive methods of work fruitfully. The students of this level can discover the elementary properties of the p.ds[1] and understand their explanation. (The results of an experimental teaching, which we realized with two classes of 13-14 years old pupils, on p.d. confirm the aforementioned concerning the abilities of Junior High School (J.H.S.) students, Kourkoulos 1999,ii).

A complete justification of other properties[2], can be taught at the High School (H.S.) level or higher (depending on how the curriculum is related to Number Theory). In parallel, this teaching can lead students to discover and/or understand better important properties of Number Theory (e.g. property 5 and Fermat's little theorem).

[1] Such as: The quotient of every non-terminating division is a p.d (property1). Which are the terminating divisions (property2). The way that we transform a p.d. to the form of an ordinary fraction (property3). The length of the period (**l.p.**) of 1/a is equal to the l.p. of a/b, when a, b are primes between them (property4).

[2] Such as: The l.p. of 1/p divide p-1, when p prime (property5). The l.p. of 1/(a*b) is equal to the LCM of the l.p. of 1/a and of the l.p. of 1/b, when a, b primes between them (property6). The l.p. of $1/p^n$ is equal to $m*p^{n-k}$, when p prime, m the l.p. of 1/p, $(1/p)^k$ the bigger power of 1/p for which the l.p. is equal m and $n \geq k$ (property7).

The assertion that the domain of p.ds is appropriate for the combination of experimental (inductive) and theoretical (deductive) treatment is corroborated by the historical development of Number Theory (see Dicson 1971).

From the end of the 18^{th} century until the end of the 19^{th} century there is a vigorous interest and activity of the mathematical community on the subject of p.ds (However, we can find research works on this subject dated from the end of the 17^{th} century (e.g. G.W. Leibnitz 1677, J. Wallis 1685) until the beginning of the 20^{th} century (e.g. Weixer 1916, Hoppe 1917), see Dicson L.E., 1971).

When we consider the path that research has followed on this subject, we observe that there often appears the following scheme: formulation of conjectures, experimental research, possible modification of the conjectures, proof.

A characteristic example is the path which led to the discovery of the properties 6 and 7: Wallis in 1685 claimed, without proof, that the l.p. of 1/(m.n.) is equal to the LCM of the l.p. of 1/m and of 1/n, if m and n have prime factors different from 2 and 5 and gives as example the 1/(3*7) (This example constitutes also a counter-example to an anterior assertion of Leibnitz).

In 1771 J. Bernoulli published a table with the periods of 1/p, where p is an odd prime smaller than 200 and a table with the periods of $1/(d_1*d_2)$, where p_1, p_2 are odd primes, smaller than 25. From this table he confirmed Wallis' claim when $p_1 \neq p_2$ but he rejected it for the case that $p_1 = p_2$. He also remarked that if p>3 the l.p. of $1/p^2$ is m*p where m is the l.p. of 1/p (This proposition is not valid in all cases, e.g. it is not valid for p=487, but it is correct when the l.p. of 1/p and the l.p. of $1/p^2$ are not equal. A more general answer to the problem of the l.p. of $1/d^n$ is given by property 7).

Finally, in 1843, Thibauld formulated the properties 6 and 7 correctly in the same work. In this work only property 6 is proven. Property 7 is proven some time later (1846) by E. Prouhet.

2. The Teachers

Although, the p.ds constitute an important aspect of fundamental notions such as the rational numbers and the decimal system and despite of the fact that the investigation of their properties can have very positive effects on pupils' mathematical culture, the curricula in use in primary and secondary education, in Greece, reserve a very limited place to the study of p.ds.

In primary education the official instructions for the curriculum indicate that it must simply mentioned the existence of p.d. along with the existence of non-terminating divisions that have as quotient a p.d. There is no mention that further explanations on the subject, or any other properties of p.d. should be taught in primary education. According to the official instructions, in the J.H.S. level , only properties 1, 2, 3 should be taught (their teaching is placed to the 2^{nd} year of J.H.S.). At the H.S. level the official instructions of the curriculum don't mention any property of p.d. that should by taught (or any other activities on p.d. that should by realized).

Furthermore, the data presented in Appendix 1 indicate that, concerning the fruitful educational use of the subject, certain additional difficulties can come the teachers: In the investigated samples teachers appears to know very few things about p.d.

After answering to the questionnaire, all secondary education's teachers were individually interviewed. In these interviews the teachers of J.H.S. (18 out of the 32) stated, that p.d. is one of those subjects that are taught briefly or not at all ("it is taught briefly", "I don't teach it when time presses", "it is usually exempt of the exams",...). Only 2 out of the 14 H.S. teachers explain the transformation of p.d. to fraction by using Geometrical Progression (2nd year of H.S.). Also, no-one don't teaches topics concerning p.d. in the course of Number Theory (course taught only to students with a Scientific or Technological orientation).

The aforementioned point out that, moreover the fact that the teachers asked know a few elements[3] concerning the p.d., they have not consecrate some systematic work on the teaching of the subject.

3. Experimental Teaching

3.1. In order to investigate the possibilities and the difficulties that students encounter concerning the p.d. we have realize an experimental course with 24 students of 3rd and 4th years of the Department of Education of the University of Crete. Below we present the outline of this course and some significant elements of its realization.

The course was optional and lasted for 11 weeks (one meeting of 3hours per week).

The course was not in the classic form of a series of lectures. Instead, students worked in groups and the emphasis was given on their research work. In such a course, the learning of properties and procedures and the production (or reproduction) of well-done proofs are not the only elements considered as valuables. Experimental research, the formulation of conjectures and questions are also considered as important elements of doing mathematics. In traditional teaching these elements are disregarded, because the teaching focuses mostly in the learning of properties and algorithms or methods and little attention is given to the procedures leading to these properties and algorithms or methods. One, in order to appreciate the importance of the aforementioned elements and to begin to understand their role in doing mathematics, it is necessary to consecrate a relatively long period to the research of the properties of a mathematical area. This is important especially for prospective teachers' mathematics education. However, our students had never done that, since they had followed a traditional mathematical education. This fact as well as the elements presented in [1] led to use the aforementioned non-conventional form of course.

3.2. Before the beginning of the course, a questionnaire was given to the 24 students in order to evaluate their knowledge related to our subject. Some significant results of this questionnaire are presented in Appendix 2.

Taking into consideration the knowledge of the students, as it appears in the questionnaire, and the amount of inductive and deductive work that they had to do, we concluded that their work would have been more efficient if they had worked in groups of four. So, in the first meeting six groups of four students were formed. The students determined the formation of groups. However, the teacher interfered in the formation of 2 groups, in order to avoid the formation of groups in which basic knowledge and/or skills would be completely absent.

The rest 2 hours of this meeting were dedicated to the reviewing of necessary knowledge (algorithm of primes factors analysis of integers, algorithms of LCM and GCD, property "if a,b,c integers, GCD(a,b)=1 and a divides b*c then a divides c").

In the 2nd meeting, students initially made certain divisions (11/4, 5/7, 6/11, 3/17) and found some digits of their quotient. After this, the teacher told them that, as it is apparent, for some divisions the quotient is a finite decimal **[f.d.]**, for others the quotient is a p.d. and maybe (maybe not) there are divisions whose quotient is a decimal non-finite and non-periodic **[n-f.n-p.d.]**, so

[3] It is interesting to note that the only properties known by the majority of secondary education's teachers who have been asked, are the properties 1,2 and 3, which, are , also, the only properties contained in the schoolbooks of mathematics (Mathematics' Schoolbook of 2nd year of J.H.S., O.E.D.B. 1995, pp 58-62)

the purpose of this course is to explore this subject and to research related questions. The teacher also gave some indicative questions:

i) Which divisions have as quotient a t.d. and which a p.d. ?

ii) Are there divisions for which the quotient is a n-f.n-p.d.?

iii) What can we know for the period of a p.d. before we made the division (such as the length of the period)? Can we find methods permitting the quick calculation of the period (or a part of it), in all or in some cases?

The first question that students chose to investigate, concerned the divisions which have as quotient a f.d., as they knew some examples of this kind. The experimental investigation on this question permitted them to find that all tested divisions with divisors2, 4 and 8 have as quotient a f.d. and the same holds for 5. They also saw that all divisions by powers of 10 are terminated, the explication was easy for the students because of the particular algorithm of the division in this case. These results leaded some of them to conjecture that the same holds for the other powers of 2.

Some students, considering treated examples (such as 6/11, 2/3, 5/14, 7/12), introduced a new distinction: p.d. whose period starts immediately after the decimal point (**i.s.p.d.**) and p.d. whose period starts later (**l.s.p.d.**)

As every group of students had to continue alone their research until the next meeting, many of the students were anxious about the way of choosing a sample of examples which would permit them to obtain interesting results, especially concerning the questions for which specific conjectures had not yet emerged (such as i,ii above)[4]. So the problem was discussed in class. Two students proposed to take all the divisions with Dividend and divisor between 1 and 20. Some students proposed to link Dividend and divisor with a simple relation (they proposed D=d+1, D=d+2 and D=d-1), apparently because they found it difficult to vary in a systematic way two independent variables. Others objected that, in this way, they will have only one example for every Dividend and divisor. After some discussion, they concluded that it was better to keep first constant either the divisor or the Dividend and to vary the other, and then, after having finished with one divisor (or Dividend), to proceed in the same way to the next one. Concerning the kind of numbers to be tested, they proposed even and odd numbers, numbers bigger or smaller than 20 and only four proposed to test samples of prime numbers and samples of composite numbers.

<u>In the 3nd meeting</u> Students had found that in all tested divisions with divisor of the form 2^n ($1 \leq n \leq 9$) the result was a p.d. , and the same holds for the powers of 5 (tested until 5^5).

One group (the 2nd) claimed that they had found the explanation of this property and they presented it with the following example:

$7:64=\frac{7}{16}=7\times\frac{1}{16}=7\text{x}\frac{1}{2}\text{x}\frac{1}{2}\text{x}\frac{1}{2}\text{x}\frac{1}{2}=7\text{x}0{,}5\text{x}0{,}5\text{x}0{,}5\text{x}0{,}5$, they remarked that the initial division gives the same result as a series of multiplications of f.ds, so the result is necessarily a f.d. (in the example 0,4375). They remarked, also, that this procedure will function in the same way with any other power of 2 as divisor and any other Dividend.

Other students said that the procedure could also be applied for the powers of 5 and some time later they remarked that it could be applied for the divisors of the form $2^n\text{x}5^k$.

The teacher remarked that other interesting results could be found from this procedure; for example results concerning the decimal part or other characteristics of the quotient. After these

[4] How to choose the sample of cases which will be examined, is an important problem of experimental research and the students who have received a traditional education have important difficulties on this subject even in simpler cases, such as the test of a precisely stated conjecture

instigation three students (not of the 2^{nd} group) found that the procedure points out that, if a fraction $a/2^n$ (or $a/5^n$) is irreducible, the decimal part of the quotient of the corresponding division have n digits, and its digits, regardless of the decimal point, form the number $ax5^n$ (or $ax2^n$). [5]

The procedure conceived by the 2^{nd} group uses the consideration of a division as a fraction, the transformation of a fraction to a multiplication (and to a series of multiplications) and the transformation of a fraction to a decimal. These transformations and changes of point of view and their inversions are basic tools in the research of the explanations for the properties of p.d. The majority of students realized this fact and began to use systematically these transformations in their research after the 5^{th}-6^{th} meeting. [6]

Another element creating difficulties in students' work was that they focused their attention in what they considered as "principal elements" of a division (Dividend, divisor and quotient) and they paid less attention to the remainder and even smaller to the sequence of partial remainders obtained during a division. This behavior changed slowly. This fact was the reason for which they significantly delayed to find the proof of the property that all non-terminating divisions give a p.d. (at the 9^{th} meeting)[7].

From the 3rd until the 6^{th} meeting they posed the question "how one can directly perform the four operations with p.ds.?" and they arrived to produce well organized algorithms, for the addition the subtraction of p.ds and the multiplication of a p.d. with a f.d. The patterns of these algorithms pointed out that if the quotient of 1/a is an **i.s.p.d.** and l.p.(1/a)=n then there is periodic repetition of the n first decimal digits of the quotient of b/a. After this, the remaining question: "In which cases the l.p.(a/b)< l.p.(1/a) ?" led them to conceive couples of examples of the type $1/(p_1xp_2)$, $p_1/(p_1xp_2)$ and of the type $1/p^2$, p/p^2 (where p, p_1,p_2 are primes). On the one hand these examples led them to discover property 4 and on the other hand, these examples and their extensions, conducted them to the discovery of the properties 6 and 7 (later they achieved to prove property 4 and 6 but not property 7)

During the construction of the aforementioned algorithms, they had often used p.ds, which they constructed directly by repetition of an arbitrary period. So they posed the question if these p.ds come from a division and how one can find it. The 5th group found the first part of the answer, which was presented in the 6^{th} meeting with the following example:

0,353535... = 0,35 +0,0035+0,000035[8]....=

=357/100+357/10000+357/1000000...=357x(1/100+1/10000+1/1000000....)=

---

[5] The students of the group which conceived the procedure, had not found these elements not because their discovery is difficult but because they regarded the procedure only as an answer to a specific conjecture ("When the divisor is 2^n, is the quotient a f.d.?"). They didn't examine if elements of the answer can enlighten other relative questions. This restricted way to look at answers and questions is characteristic of students who have received a traditional mathematical education. In the beginning of the course the students presented this attitude very frequently but progressively their behavior changed and after the 8^{th}-9^{th} meeting the majority of them, having an answer, looked for elements of this answer, which could help answering other questions.

[6] The issue about terminating divisions was not settled until the 7^{th} meeting, when students arrived to pove that the only irreducible fractions transformable to f.d. are those of the form $a/(2^n x5^k)$. At this moment, the experimental investigations performed so far had convinced students that very probably these divisions are the only ones, which terminate. This conviction led them to reasoning in which was used the reductio at absurdum. This and the aforementioned transformations led them to the proof of the property.

[7] From the beginning of the course until the 9th meeting, from time to time, students proposed fractions that they considered as probable **n-f.n-p.d**. The experimental investigation, which some times was long, had always showed that they were p.ds

[8] this analysis of a p.d. was often used during the construction of the algorithms

357x(0,01+0,0001+0,000001)= 357x0,010101... here they claimed that the remaining problem was to find a division (fraction) having as quotient the decimal 0,010101... They also explained that this procedure can be applied when the period is longer or different. Other students had the idea to apply the same procedure on 1/11, which they also knew as 0,0909..., so they obtained 1/11 =9x0,0101...and from this 1/99=0,0101...After this, they made directly the division 1/99 and some others of the form 1/9..9 and they understood that they give the prescribed quotients. Following this they found easily the rule to transform an **i.s.p.d.** to a fraction (the case of **l.s.p.d.** was treated in the 7^{th} meeting)

During the last meetings, students discovered the property 5 and the property of complementarity between the first and the second half of the period of a/p (where p is prime and l.p.(a/p)=2xn). They also searched extensions of this property and other ways to find the period and its length faster, because, at this stage of the course, these problems were considered as major ones by the students.

Furthermore, some groups considered as an intriguing problem the question "Are there other primes than 3 for which the l.p.(1/p)=l.p.($1/p^2$)?" and they realized, using Excel, extended empirical researches on this question. They arrived to found p=487 but having examined the primes until 1000000 they didn't arrive to find another prime of this kind. Students had understood that properties 4, 6 and 7 permit to reduce the problem of finding the l.p. of a/b to the problem of finding the l.p of the prime factors of b. This conception had reinforced their interest in the aforementioned problem because it was considered as the last element in order to complete this reduction.

3.4. At the end of the course students' knowledge concerning the p.d. and the operation of division has been considerably improved (see Appendix 2).

They also have a more profound understanding of the decimal system (e.g. they understand its limitations concerning the representation of rational numbers).

Furthermore, their ability to select a sample of cases in order to perform an experimental investigation in arithmetic as well as the way that they consider the treated examples have significantly evolved.

At the domain of p.d. most of them have formulated questions and conjectures.

Moreover, the majority of students have began to express evaluations on problems and properties, characterizing them as more or less important or interesting. These evaluations depend on the links that they conceive between the characterized problem (or property) and the other properties and problems known to them.

Concerning algorithms, they appreciate other elements besides correctness; especially they take under consideration the "cost" and the efficiency of an algorithm and this consideration can push them to further research even in cases that an algorithmic answer already exists.

Finally, some of them have spent considerable time working on problems just because they found them intriguing, which is an indication that they begin to find some fun in mathematics.

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Appendix 1

A questionnaire was given to 110 Primary Education Teachers[9] (PET), to 32 Secondary Education Mathematics Teachers[10] (SET) and to 207 students[11] (S) of 1^{st} and 2^{nd} year of the Department of Education of the Univ. of Crete. The following three questions were included in the questionnaire:

Q1) When we have an irreducible fraction a/b<1(a,b, are integers) and we want to transform it into a decimal number by dividing a with b, in which cases the division terminate [12]?

Q2) The a/b and c/b are irreducible fraction (a,b,c,d are integers) . The length of the period of a/b is n. What can we say for the length of the period of c/b?

Q3) Which others properties of periodic decimals do you know?

The results are the following:

Q1	Correct Answer	Partial answers[13]	Wrong Answer	Answer that they don't know[14]	No answer
PET	20% (22)	18% (20)	24%(27)	34% (37)	4% (4)
SET	56% (18)	9% (3)	0	35% (11)	0
S	2% (4)	23% (47)	30% (62)	42% (86)	3% (6)

Q2	Correct Answer	Answer with a conjecture[15]	Wrong Answer	Answer that they don't know	No answer
PET	14%(15)	3% (3)	6% (7)	72% (79)	5% (6)
SET	25% (8)	0	0	75% (24)	0
S	1,5% (3)	2% (4)	16%(33)	67,5% (140)	13%(27)

Q3	Give property1	Give prop.3	Give no properties	Report property 4, 5,6 or other
PET	23% (25)	14% (15)	68% (75)	No-one
SET	88% (28)	78% (25)	13% (4)	No-one
S	3,5% (7)	1,5% (3)	95% (197)	No-one

The questionnaire to the students was given after we have looked at teachers' answers. Therefore, a more elementary question were added:

Q4i) The division 157:47 A) terminates B) it doesn't terminate but the decimal digits of the quotient are repeated periodically C) it don't terminate and the decimal digits of the quotient are not repeated periodically D) I don't know. Q4ii) The same question for the division 453:67

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[9] The in service carrier of these teachers was 2-11 years (average 6,3 years).

[10] The in service carrier of these teachers was 10-29 years (average 18,5 years).

[11] Pre-service school teachers of primary education

[12] The term "the division terminate" was explain orally, for the case that some one don't clearly understand it. The same hold for the term "length of the period" of the 2nd question

[13] They present particular cases in which the division is terminated such as: "when we divide by 2 or 5", "when the divisor is 2, 4, 5 and 10", "when the divisor is 10,100, 1000 etc "

[14] We asked them to check a corresponding "don't know" box in case that they didn't know.

[15] "Probably they have the same number of digits...", "Maybe, they have the same number of digits"

Q4	Answer B in both divisions	Ans. A in both divisions	Ans. C in both divisions [16]	Ans. C in one division and B in the other[17]	Ans. C in one division and A in the other[18]	Ans. C in one division and no answer for the other	Ans. E in both divisions	No answer
S	4% (8)	1,5%(3)	63%(131)	10% (20)	2% (4)	7% (14)	5%(10)	8%(17)

Appendix 2

Some results of the initial and the final questionnaire of the experimental teaching

Q1[19]	Correct Answer	Partial answers	Wrong Ans.	Answer that they don't know	No ans.
Initial	0	17% (4)	38% (9)	45% (11)	0
Final	79% (18)	8% (3)	8% (2)	13% (2)	0

Q2	Correct Answer	Answer with a conjecture	Wrong Ans.	Answer that they don't know	No ans.
Initial	0	0	13% (3)	79% (19)	8% (2)
Final	84%(20)	0	8% (2)	8% (2)	0

Q4	B in both divisions	A in both divisions	C in both divisions	C in one division and B in the other	C in one division and A in the other	C in one division and no answer for the other	E in both divisions
Initial	0	0	71%(17)	8% (2)	4% (1)	0	17% (4)
Final	84%(20)	0	4% (1)	4% (1)	0	0	8% (2)

Comment In the initial questionnaire, the percentage of correct answers of these students is, in all common questions, a little smaller than the percentage of correct answers of their younger colleagues presented in the previous page [20]

Q5) Analyze an integer in primes factors. Q6) Find the LCM of two integers Q7) Find the GCD of two integers. For each one of Q5, Q6, Q7 two examples were asked (e.g. Analyze in primes factors 5940 and 13260). Success (S) is considered the correct answer in both. [21]

Q8) Determine the remainder of a division when the quotient found have a decimal part

[16] All students of this category found some digits of the quotient and as they didn't find periodicity or termination of the divisions they concluded that C is the correct answer.

[17] The students of this category acted as the students of the precedent category for one division. For the other division, errors in the execution of the division or conceptual errors (such as the misinterpretation of the repetition of one digit) led them to select B

[18] Errors in the execution of one division led them to select A for this division.

[19] The questions Q1,Q2,Q4 are the same as Q1,Q2,Q4 in Appendix 1

[20] This slight, but systematic, difference is probably duc to the reform of the curriculum of mathematics in H.S., realized between 1997 and 2000. Because of the reform the younger students coming from the "Theoretical orientation" of the H.S., have received a significantly longer and reinforced mathematical education, during their H.S. studies, than the older ones. (More than 80% of the students in the Departments of education in Greece come from the "Theoretical orientation" of the H.S.)

[21] These algorithms were taught to the students on the elementary and JHS level and they were re-taught in the University, as part of compulsory courses

Q9)Make correctly the verification of a division when the quotient found have a decimal part.

	Q5 S	Q5 F	Q6 S	Q6 F	Q7 S	Q7 F	Q8 S	Q8 F	Q9 S	Q9 F
Initial	11	13	9	15	7	17	1	23[22]	3	21
Final	19	5	18	6	17	7	19	5	20	4

Q10, Q11) Find the l.p. of $13/(3x13^2x11^2)$ and of $28/(37x3x11^3x7^3)$.[23] (The l.p of 1/3, 1/13, 1/11, 1/7, 1/13 were given.)

	Q10 S	Q10 F	Q11 S	Q11 F
Final	17	7	16	8

[22] At the initial questionnaire 21 out of the 24 students believe that the remainder of a division is in all cases an integer. The failure also in Q9 is related to this misconception.

[23] To answer this question is necessary to combine properties 4,6 and 7.

"RATIO": RAISING TEACHERS' AWARENESS OF CHILDREN'S THINKING

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ABSTRACT

We believe that the problem of teacher preparation is one of development of "pedagogical content knowledge" rather than "subject knowledge" per se. It has been found that even experienced teachers may not be aware of the misconceptions that learners tend to exhibit, or at what stage of development and in what areas of the curriculum these are likely to be manifested. This pedagogical knowledge is, we believe, important to teachers' mental models of their learners, and hence their teaching effectiveness.

In this study, we aim to contribute to teachers' awareness of their pupils' strategies and misconceptions in the field of "ratio": a topic that is difficult to teach and learn in the middle school years.

Towards this aim, we constructed a diagnostic instrument which reveals children's proportional thinking. Our instrument contains two versions, one with "models" thought to be of service to children's proportional reasoning and one without. It is also designed to function as a questionnaire for assessing teachers' pedagogical content knowledge. We use the same items that form the children's diagnostic instrument, but we ask the teachers to predict the children's errors and likely explanations and to comment on the difficulty of the item.

We present data on Year 6,7,8 and 9 (aged 10 to 14) children's performance at three items of our tool and we compare them with data on trainee teachers' pedagogical content knowledge with respect to children's thinking in these particular items. We also present the trainees' perception of difficulty hierarchy of our instrument as a whole and contrast it with the learners' difficulty hierarchy.

Our data indicate a gap between pupils' strategies and errors and their future teachers' perception of those. Further research is needed to investigate the use of such an instrument in teaching and in teacher education.

Key words: Mathematics Education, Ratio and Proportion, Misconceptions, Teachers' Awareness, Teachers' Preparation.

1. Introduction

Extended research from as early as 1966 until now (Lunzer & Pumfrey 1966, Hart 1981, Hart 1984, Tourniaire & Pulos 1985, Singh 1998) in the field of proportional reasoning reveals that solving ratio and proportion problems is a very difficult task for most pupils in the middle school years throughout the world. The above research studies identified common errors and misconceptions in pupils' proportional reasoning which affect their learning.

We believe that a starting point for the effective teaching of the topic of ratio is the teachers' awareness of these misconceptions. In previous work we have found that even experienced teachers may not be aware of the misconceptions that learners tend to exhibit, or at what stage of development and in what areas of the curriculum these are likely to be manifested. This knowledge is, we believe, important to teachers' mental models of their learners, and hence their teaching effectiveness (Williams & Ryan 2000, Hadjidemetriou & Williams, 2001).

Thus, a significant aspect of teacher preparation is one of development of what Shulman (1986, 1987) calls "pedagogical content knowledge" rather than subject knowledge per se. "Subject matter content knowledge" refers to "the amount and organization of knowledge per se in the mind of the teacher" (Shulman 1986, p.9) whereas pedagogical content knowledge refers to "subject matter knowledge for teaching" (p.9) and includes "an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. If those preconceptions are misconceptions...teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners" (Shulman 1986, p.9-10)

In this study we aim to contribute to teachers' awareness of their pupils' strategies and misconceptions by developing an assessment instrument for proportional reasoning. This instrument was designed to assess pupils' performance at simple ratio and proportion tasks: to reveal their strategies and to locate significant misconceptions that need to be addressed in teaching. We also aim to explore whether this instrument would be suitable for assessing this aspect of teacher's pedagogical content knowledge. Particularly we were interested in the function of our instrument as a tool for teachers' training in mathematics.

Twenty-four, "missing-value" type, items were used to construct the instrument. All the problems were selected having as criterion their "diagnostic value", their potential to provoke a variety of responses from the pupils, including errors stemming from misconceptions already identified in the literature. As a result of this selection, errors indicative of common and frequent misconceptions such as the "additive strategy" (which will be described later) were expected to occur.

On the other hand, since it is recognised that children's methods differ in varying circumstances, we tried to use a variety of problems as far as "numerical structure", "semantic type" and "local context" is concerned. Thus, we hoped that less frequent misconceptions or even ones that are not mentioned in the research literature would also occur.

Some of the items have been adopted with slight modifications of those used in previous research and others have been created based on findings of that research. (CSMS 1985, Lamon 1989, Lamon 1993, Tourniaire 1986, Cramer, Bezouk & Behr 1989, Resnick & Singer 1993, Kaput & West 1994, Ryan & Willliams 2000, Singh 1998)

Finally, two versions of the instrument were constructed (both of these versions can be seen in full on the web at http://www.education.man.ac.uk/lta/cm/index.htm). The first version ("W Test") contains all the 24 items presented as mere written statements. The second version ("P Test") contains the same items supplemented by "models" thought to be of service to children's proportional reasoning. These models involve pictures, tables or double number lines, which can be used in modelling ratio problems. Lamon (1993) advocates the use of pictures, Middleton and Heuvel-Panhuizen van den (1995) support the use of ratio tables and Streefland (1984) suggests the use of double number lines. Our purpose was to compare the difficulty of the parallel items for the children and test the awareness of future teachers' of mathematics of such models.

2. Method

In order to be able to administer more items to the same sample of pupils, each version of the test consisted of two separate test forms with common linking items. Thus, Test W was divided in Test W1 and Test W2. Test W1, designed to be easier, consisted of sixteen items and Test W2 has the same number of items, but was designed to be more difficult. Eight of the items were common for both tests. Exactly the same pattern applies for tests P1 and P2 into which Test P was divided. Finally we equated Test W1 and P1 through common items and we did the same for Test W2 and Test P2 in order to be able to compare the difficulty of the parallel items for the children.

The pupils' data presented here come from a sample (N=232) of Year 6,7, 8 and 9 pupils (aged 10 to 14) from 4 schools in the North West of England.

Before administering the tests to the pupils, their teachers were asked to comment on the suitability of the test items for their classes. They found that although they differed in difficulty the items were generally acceptable for the pupils' age. They viewed them as valid assessment of the curriculum they are teaching.

Nine trainee teachers of mathematics participated in this study. These are people that have already obtained a university degree in mathematics and are trained in order to work as mathematics teachers at schools. In order to assess their pedagogical content knowledge the form W of the test (all the 24 items) was administered to them. They were asked to complete it and to provide additional information: to predict possible correct and erroneous strategies at each item and to suggest on tools, methods or activities that could help the pupils overcome their difficulties.

Firstly a qualitative analysis of the tests' results was conducted. For each item, all the pupils' answers, correct and erroneous, were recorded. Each answer in the list was accompanied were possible, by the strategies that pupils followed to obtain it. Then these answers and strategies were cross-examined with the ones that were suggested by the trainee teachers for the corresponding items.

The qualitative data provided interesting indications concerning the trainees' pedagogical content knowledge. In illustrating the essence of these data, we decided to present in detail one item, the one we named "Paint 1" and then present in summary the results from two more items, which we named "Mr Short and Mr Tall" and "Printing Press". Finally, we present a comparison between teachers' estimates and actual pupils' difficulty for all the items.

3. Results

Item: "Paint 1"

Presentation of the item

The "Paint 1" item was presented in the Test W1 as follows:

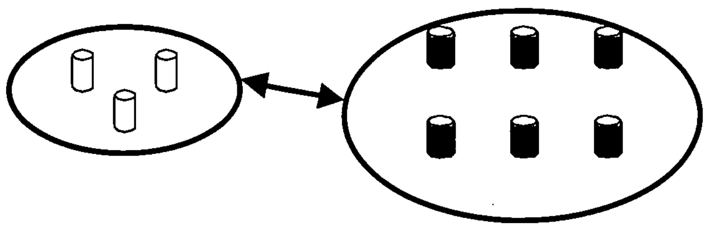
Sue and Jenny want to paint together.
They want to use each exactly the same colour.

Sue uses 3 cans of yellow paint and 6 cans of red paint. Jenny uses 7 cans of yellow paint. How much red paint does Jenny need?
Answer:

How did you find this answer? Please show your working out below.

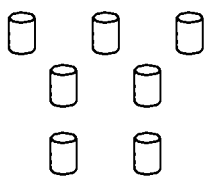
The presentation of the same item in the Test P1 is given below:
Sue and Jenny want to paint together.
They want to use each exactly the same colour.
Sue uses 3 cans of yellow paint and 6 cans of red paint.

3 cans of yellow paint *6 cans of red paint*



Jenny uses 7 cans of yellow paint.

7 cans of yellow paint *How many cans of red paint?*



?

How much red paint does Jenny need?
Answer:

How did you find this answer? Please show your working out below.

Pupils results

The qualitative analysis of the pupils' data yielded the following list of pupils' answers and strategies (all the percentages for the correct and incorrect strategies refer to the W form of the item):

Correct strategies (Correct answer: 14)

1. "Doubling" and "For every" strategy (Tourniaire, 1984).

These, multiplicative in essence, strategies were used by 17.2% of the pupils. The doubling method can be applied simply as: 3x2=6, therefore 7x2=14. Employing the "for every" strategy means finding the simplest ratio that expresses the relationship of the problem. In the case of an integer ratio this method is equivalent to the "unit value" method. In the "Paint 1" item the simplest ratio that expresses the relationship of the problem is the ratio 1:2 and by multiplying both of its terms by 7 the answer can be found

Incorrect strategies

1. "Constant Sum" strategy (Mellar, 1987) (Answer: 2)

This was the most common pupil strategy since it was used by 34.5% of the pupils. In this item, the pupil who applies the constant sum strategy thinks that the sum of Sue's cans should be equal to the sum of Jenny's cans: 3+6=9 therefore 7+2=9 and so the answer should be 2.

2. "Constant difference" or "Additive" Strategy (Tourniaire & Pulos, 1985)(Answer: 10)

This is a frequently used error strategy that has been mentioned by Inhelder and Piaget (1958) and has been widely observed ever since (Hart 1981, Hart 1984) "In this strategy, the relationship within the ratios is computed by subtracting one term from another, and then the difference is applied to the second ratio." (Tourniaire & Pulos 1985, p.186)

Here, this was the second most common strategy employed (20.7%).

In this particular problem, the answer 10 can be obtained either by thinking that 3+4=7 so 6+4=10 or by thinking that 3+3=6 and so 7+3=10.

3. "Incomplete Strategy" (Karplus, Pulos & Stage, 1983) (Answer: 6)

This strategy was used by 3.4% of the pupils. For them, the number asked should be the same as the one given from the same measure space: that is 6, since 6 are the cans of red paint given.

4. Incorrect application of build up method

3.4% of the pupils could not apply a build up method correctly.

For example, the answer "13" was obtained as follows:

" 3 yellow	6 red
6 yellow	12 red
6+1=7	12+1=13"

The rest of the pupils either gave answers that derived by strategies that we recorded as "random operations" because they were not justified properly or did not answer at all.

A tool that could facilitate pupils' thinking.

The pupils' performance on the W form of the item was compared with the pupils' performance on the P form using the data from the overall Rasch analysis of the items. The percentage of correct answers on the W form was 17.2% whereas this percentage for the P form was 55.2% which seems definitely higher. We believe these data are enough to hint that a pictorial representation of a ratio problem might influence positively pupils' proportional reasoning.

Trainee teachers' results

All the teachers provided the correct answer "14" to the "Paint 1" item, except one who wrote down as an answer the phrase "Depends on the size of Jenny's room".

They offered as the **correct strategies** that pupils would use the following:

1. Doubling strategy

Three of the student teachers predicted that a possible correct strategy used by pupils would be "doubling"

2. For every strategy

Only one trainee suggested that this problem could be solved by "noticing that the ratio of red paint to yellow paint is 2:1"

3. Multiplicative (within measure space approach)

One trainee teacher offered as a second possible strategy apart from doubling a multiplicative, within measure space, approach. In his own words: "Jenny used 7/3 x as much paint as Sue therefore red=7/3 x 6 = 14"

4. Cross multiplication method

One predicted as a probable strategy "setting up a proportion 3/6=7/x and then 3x=42 so x=14."

The **incorrect strategies** that the trainees predicted where the following:

1. Additive Strategy (Answer: 10)

Only two teachers suggested that an incorrect strategy that would be used for this item would be the additive strategy.

2. Incomplete Strategy (Answer: 6)

One wrote that an erroneous approach would be "being unable to recognise the ratio of red to yellow can be used to find the answer". We presume that she had in mind the incomplete strategy.

No one could predict the constant sum strategy and finally one wrote "perhaps they would reverse one part of the proportion".

A tool that could facilitate pupils' thinking.

Just one teacher suggested the provision of pictorial help as a tool that would facilitate pupils. She suggested that "drawing the problem out" could help the pupils find the correct answer.

Comments on the results for the item "Paint 1"

Only two of the trainee teachers could predict the well documented and many times replicated in the research literature additive strategy. No one could predict the most common incorrect strategy for this item, which was the constant sum strategy and all but one, had no suggestions about tools or activities that could aid pupils' thinking.

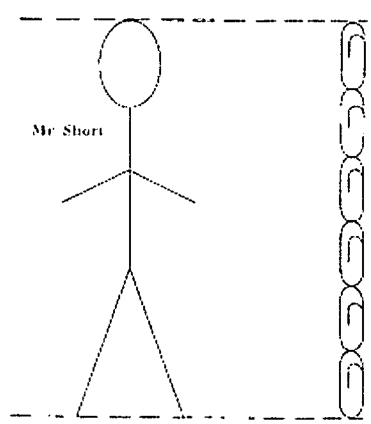
Item: "Mr Short and Mr Tall"

Presentation of the item

The "Mr Short and Mr Tall" item was one of the items that linked the P and W forms of the test and was presented in both versions as follows:

You can see the height of Mr

Short measured with paper clips.



Mr Short has a friend Mr Tall.

When we measure their heights with matchsticks:

Mr Short's height is four matchsticks

Mr Tall's height is six matchsticks

How many paper clips are needed for Mr Tall's height?

A summary of trainee teachers and pupils' results

All the trainees gave the correct answer "9" and the rest of the data are summarised in the table below:

	Correct Strategies	Incorrect Strategies	Appropriate Tool
Pupils	1. For every and multiplicative strategy (11.2%) 2. Build up method (4.3%) 3. Unit value method (1.7%)	1.Additive strategy (38.8%) 2. "Magical doubling" (6%) 3. Incomplete strategy (4.3%)	The use of actual models(paperclips and matchsticks) +appropriate teacher intervention
Trainee Teacher 1	Cross multiplication method	Did not mention any	Did not mention any
Trainee Teacher 2	For every method	Did not mention any	Did not mention any
Trainee Teacher 3	Did not mention any	Did not mention any	Did not mention any
Trainee Teacher 4	Multiplicative (within measure space) approach	Additive strategy	"Drill them with lots of unitary proportion sums so that they always find what the ratio for 1 unit is"
Trainee Teacher 5	Multiplicative (within measure space) approach	Additive strategy	Did not mention any
Trainee Teacher 6	Did not mention any	Did not mention any	Did not mention any
Trainee Teacher 7	Did not mention any	Did not mention any	Did not mention any
Trainee Teacher 8	Multiplicative (within measure space) approach	"Not recognizing that it is necessary to calculate the ratio between Mr Short and Mr Tall's height in matchsticks and then applying that ratio to the paperclips."	Did not mention any
Trainee Teacher 9	**Cross multiplication method**	1."Miscounting paperclips" 2. "Setting up the proportion wrong"	Did not mention any

The "magical doubling" method (Mellar, 1987) mentioned in the table means that the pupil doubles (when doubling is inappropriate) one of the data of the problem in order to find an answer. In this case, the answer obtained was "12".

Comments on the results for the item "Mr Short and Mr Tall"

A characteristic of this item is that it provoked the highest occurrence of the incorrect additive strategy compared with all the other items of the test. This strategy was mentioned by only two of the trainees. It is also notable that none of the pupils used the cross multiplication algorithm whereas two of the trainees suggested it as a possible correct strategy.

Item: "Printing press"

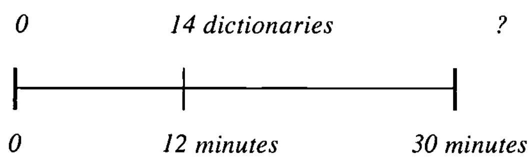
Presentation of the item

The "Printing Press" item was presented in the Test W2 as follows:

A printing press takes exactly 12 minutes to print 14 dictionaries.
How many dictionaries can it print in 30 minutes?
Answer:

How did you find this answer? Please show your working out below.

The presentation of the same item in the Test P2 is given below:
A printing press takes exactly 12 minutes to print 14 dictionaries.
How many dictionaries can it print in 30 minutes?
(You may use the figure below to help you find the answer)



Answer:

How did you find this answer? Please show your working out below.

A summary of the trainee teachers' and pupils' results

All the trainees gave the correct answer "35" and the rest of the results are presented in summary, below:

	Correct Strategies	Incorrect Strategies	Appropriate Tool
Pupils	1.For every strategy (6.9%) 2. Build up method (5.2%)	1. Additive strategy (15.5%) 2. Magical doubling (13.8%) 3. Using as a unit value the value of the quantity the problem starts with (3.4%) 4. Incorrect application of build up method (3.4%)	Maybe the use of the double number line (correct answers at Test W2=15.5% whereas correct answers at the Test P2=20.7%)
Trainee Teacher 1	Cross multiplication method	Did not mention any	Did not mention any
Trainee Teacher 2	Multiplicative, within measure space approach	Did not mention any	Did not mention any
Trainee Teacher 3	Multiplicative, within	Did not mention any	Did not mention any

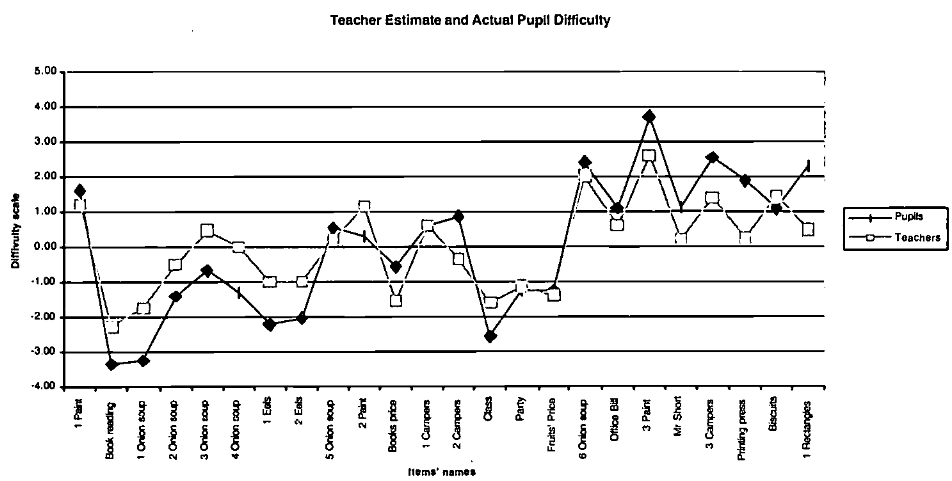
	measure space approach		
Trainee Teacher 4	Multiplicative, within measure space approach	Additive strategy	Did not mention any
Trainee Teacher 5	Unit value method	Additive strategy	Did not mention any
Trainee Teacher 6	Multiplicative, within measure space, approach	Did not mention any	Did not mention any
Trainee Teacher 7	Multiplicative, within measure space approach	Did not mention any	Did not mention any
Trainee Teacher 8	Multiplicative, within measure space approach	Did not mention any	Did not mention any
Trainee Teacher 9	1. Cross multiplication method 2. For every strategy	Did not mention any	Did not mention any

Comments on the results for the item "Printing Press"

Again, only two of the teachers mentioned the occurrence of the additive strategy, none of them predicted the incorrect strategy "magical doubling" and none of them mentioned any tools that could help pupils perform better.

Comparison between teachers' estimates and pupils' difficulty for all the items.

The trainee teachers recorded their perception of the difficulties of the items on a five point Likert scale. Their data were subjected to a rating scale analysis and the results were correlated with the children's difficulty estimated by the test analysis. The results can be seen at the figure below:



Although there are some discrepancies the high correlation (rho=0.88) is encouraging since it shows that the trainees were able to predict in general the difficulty hierarchy of the items.

3. Conclusion

Due to the small sample of pupils and trainees examined so far, the aim of this paper is not to generalise about teachers' pedagogical content knowledge. Instead, it aims to suggest a tool for evaluating and even developing this knowledge.

The data that were presented here showed that these nine "teachers to be" do not possess integrated mental models of the pupils' learning about ratio and proportion. There seems to be a gap between pupils' strategies and errors in proportional reasoning tasks and their future teachers' knowledge of these. The existence of this gap gives us reason to believe that a well-designed diagnostic instrument may be a tool that will help the training of future teachers of mathematics in two ways. First, they can be informed on their pedagogical content knowledge about ratio and proportion by trying the teachers' version of such an instrument themselves. Then, they might be able to enhance that knowledge, by delivering the same instrument to pupils and by comparing the actual data with their previous predictions.

Consequently, the next stage of the research should be to try and provide robust research findings about the use of the instrument in teacher education and in teaching in general.

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STUDENTS' UNDERSTANDING OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

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ABSTRACT

Exponential and bgarithmic functions are pivotal mathematical concepts that play central roles in advanced mathematics. Unfortunately, these are also concepts that give students serious difficulty. In this report, we describe a theory of how students acquire an understanding of these functions by prescribing a set of mental constructions that a student can make to develop his or her understanding of these concepts.

We analyze students' understanding of these concepts within the context of our theory. Our main result is that while all of the students in our study could compute exponents in simple cases, few students could reason about the process of exponentiation. Thus, according to our theory, these students' knowledge of exponential and logarithmic functions will be limited.

We conclude by describing instructional activities based on our theoretical analysis designed to foster students' understanding of these concepts.

1. Introduction

Exponential and logarithms functions are important concepts that play crucial roles in college mathematics courses, including calculus, differential equations, and complex analysis. Unfortunately, these are also concepts that give students considerable difficulty.

Researchers and educators alike have recognized the need to improve the way we teach exponential and logarithmic functions; both have proposed alternative instructional techniques to supplement or replace traditional instruction. (For examples, see Confrey and Smith, 1995; Rahn and Berndes, 1994; Forster, 1998). Other than these instructional techniques, our literature search has found little research on exponential and logarithmic functions in the mathematics education literature (Confrey and Smith, 1995, is a notable exception). In particular, little is known on what mental constructions students can make to develop a meaningful understanding of exponents or logarithms. The purpose of this study is to describe a theory of how students might develop their understanding of these topics and to analyze students' understanding of these concepts within the context of this theory.

This paper is organized as follows: In section 2, we propose a set a theoretical constructions that a student could make to understand the concepts of exponents and logarithms. In our view, it is critical that students be capable of understanding exponentiation as a mathematical process and exponential expressions as mathematical objects that are the result of this process. In section 3, we report an empirical study in which we investigate students' understanding of these topics within the context of our theory. Our investigations reveal that students' understanding of exponents and logarithms is rather limited and that most students are incapable of understanding exponents and logarithms as processes. In section 4, we briefly describe instruction based on our theoretical analysis designed to teach students these concepts.

2. Theoretical analysis of exponents and logarithms

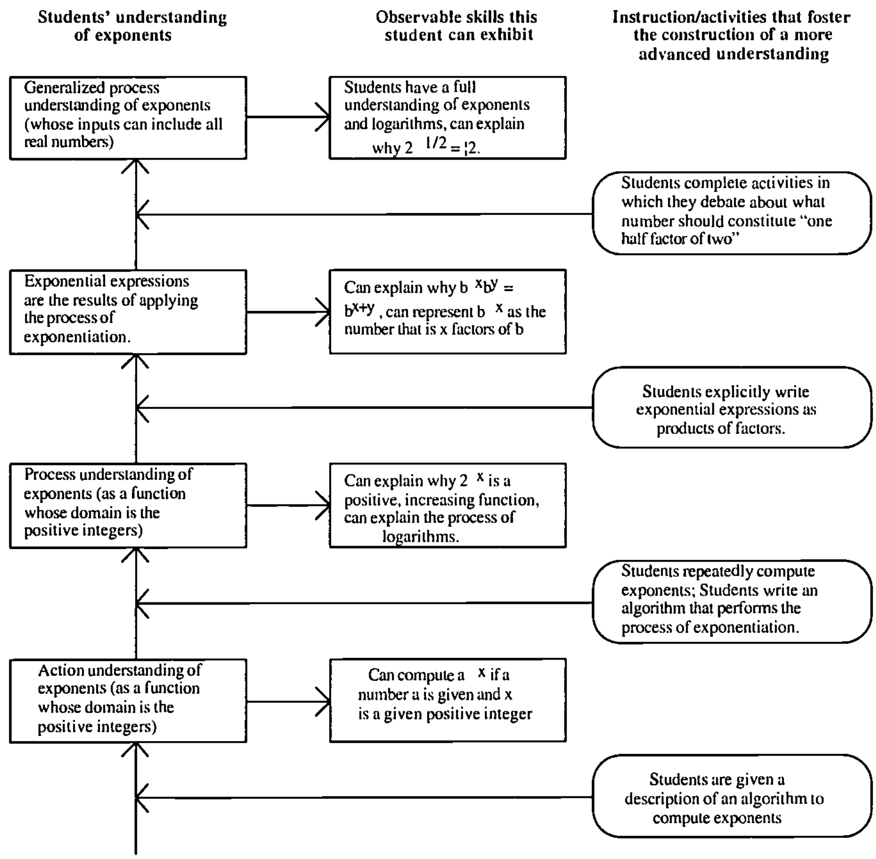
In this section, we propose a set of specific mental constructions a student might make to develop an understanding of exponents. In our view, the most plausible way that a student can learn to understand real-valued functions is to first understand exponential functions with their domain restricted to the natural numbers. The student must then generalize his or her understanding of this process to make sense of what it means to be "the product of x factors of a" when x is not a positive integer.

We present our theoretical analysis in Figure 1. In the leftmost column, we describe stages that we believe students progress through as they develop an understanding of exponential functions. In the middle column, we describe observable skills that students with each level of understanding can exhibit. We describe both of these columns in more detail below. In the right column, we propose instructional techniques to lead students to progress through these stages. We briefly describe these instructional techniques in section 4.

We use Dubinsky's APOS theory (Dubinsky, 1991) to understand how students develop their understanding of exponentiation and logarithms as functions. Our analysis of understanding exponentiation as actions and processes is very similar to Breidenbach, Dubinsky, Hawks, and Nichols (1992) analysis of how students view functions in general.

Exponentiation as an action- An action is a repeatable physical or mental transformation of objects that obtains other objects. In the case of exponents with powers that are specific positive integer coefficients, computing b^x involves repeatedly multiplying by b x times. A student limited to an action understanding of exponents will be able to evaluate exponential functions only in the cases when the power is a given positive integer. These students will not be able to do much with exponents besides compute these values and manipulate their formulas.

Figure 1. Stages of students' understanding as they develop an understanding of exponents



Exponentiation as a process- After an individual repeats an action and reflects upon it, the individual may interiorize the action as a process. Individuals with a process understanding of a concept can imagine the result of a transformation without actually performing the corresponding action, and can reverse the steps of the original transformation to obtain a new process. Students with a process understanding of exponentiation can view exponentiation as a function and reason about properties of this function (e.g. 2^x will be a positive function since you start with the integer one and repeatedly multiply this by a positive number; it will be an increasing function since every time x increases by one, 2^x doubles). They can also imagine the process obtained by reversing the steps of exponentiation to form the process of taking logarithms.

Exponential expressions as the result of a process- Terms such as 2^3 can be viewed in two distinct ways. On one hand, this can be interpreted as an external prompt for the student to compute two times two times two. However, this can also represent the output of applying exponentiation- that is, 2^3 represents the mathematical object that is the product of three factors of two. Research indicates that students are not capable of viewing 2^3 in this way (e.g. Sfard, 1991).

Representing b^x as the number that is the product of x factors of b is necessary for understanding laws of exponentiation such as $b^x b^y = b^{x+y}$. In a similar vein, students can think of computing $\log_b x$ as answering the question, "x is the product of how many factors of b?"

Generalization- Until this point, students' understanding of exponential functions only makes sense when their domain is restricted to the natural numbers. Of course, a full understanding of exponential functions involves interpreting situations where the number to be evaluated is a fraction, a negative number, or even an irrational number. To understand these situations, the student must generalize his or her understanding of b^x representing the number that is "the product of x factors of b". For instance, consider the function $f(x) = 2^x$. To interpret f(1/2), the student must make sense of what "one half factor of 2" would be. What is critical here is that students do not reason that $2^{1/2} =$ 2 because of an arbitrary rule given by a teacher or a textbook. Rather, they should reason that 2 is the only logically consistent number that would qualify as "one half factor of 2".

3. Students' understanding of exponentiation and logarithms

In this section, we report the results of a study in which we analyze students' understanding of exponents and logarithms within the context of our theory. 15 students enrolled in a traditional pre-calculus course at a university in the southern United States volunteered to participate in this study. Three weeks after learning about exponential and logarithmic functions, the students agreed to be interviewed about these topics. In the interviews, students were asked a wide range of questions: Students were asked to recall properties of exponents and logarithms, explain why these properties were true, and to perform standard and non-standard computation. The students were also asked open-ended questions designed to probe their conceptual understanding of these topics.

Although this was not the point of this study, it should be noted that students' performance on the traditional questions was poor. For instance, when asked to simplify $b^x b^y$, only six students correctly recalled that this simplified to b^{x+y} and only six students recalled that $\log_b x + \log_b y = \log_b xy$. No student saw any connection between the previous two rules. Just eight students recalled that $x^{1/2} =$ x and no student could compute log $_x$x. Not a single student could explain why any of the rules of exponents and logarithms were true.

Every participant in this study could compute 2^3 and was able to correctly specify how they would compute 7^6. Hence all students were capable of understanding exponentiation as an action. The main finding of this study was that most students could only understand exponentiation as an action and did not understand this concept as a process. We argue this point by presenting students' responses to some of our questions below.

What does the function $f(x) = a^x$ mean to you? What do you think of when you see this function?

This was an open-ended question designed to probe students' general understanding of exponential functions. One student noted that, "This is a multiplied by itself x times". Another student gave a similar response.

The rest of the responses were varied, and somewhat idiosyncratic. Examples of some of these responses are given below:

Student: This is a to the x^{th} power, where a is a constant.

Interviewer: Can you elaborate on that?

Student: No.

Student: It's a certain number raised to a certain power.
Interviewer: Can you elaborate on that?
Student: It's like suppose a was two. If x was two, it would be two times two.

Student: It's a variable raised to another variable.
Interviewer: Can you elaborate on that?
Student: Um, it's one variable taken to the power of another variable.
Interviewer: OK, can you expand on that?
Student: I don't think so. I don't know what you mean.

Besides the first two students, no students gave a response demonstrating any understanding of exponentiation as a process, or a^x as a function. In particular, unlike the first two students, no student explicitly stated how the term x was used in computing a^x without first assigning x a concrete value. If nothing else, this indicates that these students are not very articulate when speaking of exponential functions.

Is 5^{17} an even or an odd number?

Answering this question correctly requires a process understanding of exponentiation. Clearly this number cannot be explicitly computed, but one could reason that you are repeatedly multiplying by an odd number, and an odd number times an odd number is always an odd number.

Only three students answered this correctly, and they all did so by examining a small number of cases. A representative response is given below.

Student: 5 is odd. 25 is odd. 5 cubed would be... 125 which is odd. And it would keep being odd so it's odd.

Interviewer: Are you sure that it would keep being odd?

Student: Um, I think so, yeah.

Interviewer: Can you explain to me why it would keep being odd?

Student: [laughs] I don't know.

10 other students guessed that the answer was even, often conjecturing that an odd number raised to an odd number was odd and an odd number raised to an even number was even. Two students did not know how to approach this problem and refused to hazard a guess at all.

Is $f(x) = (1/2)^x$ an increasing function or a decreasing function?

All 15 students correctly answered that this was a decreasing function. Explaining why this was a decreasing function requires a process understanding of exponentiation- as x increases, you are multiplying by more factors of 1/2; hence, f(x) decreases. Only two students were able to give a mathematical explanation for why $(1/2)^x$ was a decreasing function. One student said, "Every time you multiply by 1/2, it keeps getting smaller and smaller". The other student correctly reasoned that the denominator of $(1/2)^x$ would grow as x increased, while the numerator remained constant.

10 students could not move beyond looking at specific cases (usually only x = 1 and x = 2) to determine the general behavior of $(1/2)^x$. A representative response is given below:

Student: It's a decreasing function.

Interviewer: OK, can you explain why it's decreasing?

Student: If it was like, 1/2 squared, it would be smaller than 1/2.

Interviewer: Will it always get smaller as x gets bigger?

Student: I think so.

Interviewer: Can you tell me why?

Student: I don't know.

The remaining three students appeared to know that a^x would be a decreasing function if a was a positive number less than one, but could not offer an explanation of why this was true. As one of these students said, "I'm not sure why this is decreasing. I think it has something to do with 1/2 being less than one, but don't quote me on that". (My apologies to this student for quoting him).

Suppose you didn't have a calculator. How would you go about computing $\log_5 78125$?

Answering this question requires a process understanding of exponentiation as it requires reversing its process. Correct responses might include continuously multiplying by five until you reached (or exceeded) 78,125. A more sophisticated response might involve dividing 78,125 by five repeatedly until 1 was reached (this is more akin to reversing the process of exponentiation). Unfortunately, no student gave responses of these types.

Four students knew that they must find an x such that 5^x equals 78,125, but were unable to find a way to determine what this x was. One student's response is given below:

Student: This involves solving $5^x = 78125$.

Interviewer: Do you have any ideas how you would solve such an equation?

Student: Um, a lot of trial and error?

Interviewer: OK, can you think of any other way to solve this equation?

Student: Um... no. Just trial and error.

Three students mistakenly believed that the answer would be the fifth root of 78125. The other eight students were unable to propose a way for computing its value. Clearly, the students' understanding of logarithms was quite limited.

4. Teaching suggestions and conclusions

In this section, we describe instructional designed to foster students' understanding of exponential and logarithmic functions. These activities are based primarily on our theoretical analysis reported in Section 2. As these activities have yet to be evaluated, we will mention them only briefly.

Understanding exponentiation as a process- An effective tool for leading students to interiorize an action as a process is to have them write a computer program that performs that action (Tall and Dubinsky, 1991). Our first activity involves having students program a graphing calculator to perform exponentiation (when the power is a positive integer). We do not anticipate this to be difficult, as the program is a basic "for loop". In the previous section, we report that students have difficulty explicating the role x plays in the function $f(x) = a^x$. Writing a program that performs this computation will require the students to reason about the role of the variable x. Our second activity involves having students answer basic questions which require students to view exponentiation as a process. (e.g. Why is $(-1)^x$ negative when x is odd? Why is 2^{x+1} twice as much as 2^x?) When students in our study were confronted with unfamiliar problems, they could only resort to crude symbolic techniques, such as looking at specific cases and trial-and-error. We hope that by completing these exercises, students will be introduced to a more powerful technique for thinking about exponents.

Exponential expressions as the result of a process- Students will be asked to write terms such as 2^3 as $2\cdot2\cdot2$ and "the product of three factors of two". The students will then have to use these representations to solve problems, such as to demonstrate that $2^3 2^4 = 2^7$.

Generalization- The class will discuss what it means to be "a half factor of 2". Students will propose possible values for "a half factor of 2" and analyze the validity of their choices. This will continue until students become convinced that "a half factor of 2" must be 2. Students will also discover why other properties of exponents and logarithms are true, such as why 2^{001} should be a number very close to one.

These instructional activities are currently being implemented in an experimental pre-calculus class. The effectiveness of these activities will be the subject of a future report.

In this paper, we proposed mental constructions that a student might make to develop his or her understanding of exponential and logarithmic functions. We also analyzed students' understanding of exponents and logarithms in the context of our theory only to find most students have not progressed beyond an action-level understanding of these topics. Understanding exponentiation as a function is required if one is to fully understand calculus and advanced mathematics. But understanding exponentiation as a function first requires understanding this concept as a process (Breidenbach et. al., 1992). As most students in our study were unable to view exponentiation as a process, their future in calculus is in jeopardy. Hopefully, employing our suggested instruction will better prepare our students to succeed in college mathematics.

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NEW APPROACH TO THE USE OF SOLUTION MANUALS IN THE TEACHING OF HIGHER MATHEMATICS

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ABSTRACT

This paper presents a new approach to the teaching of undergraduate mathematics in which the students are allowed a free access to the complete solutions manual. Our method consists in assigning a very substantial amount of homework problems and allowing the students to consult with the solutions manual while doing it. Our philosophy is that the purpose of the homework is not to test the student's knowledge but to give her/him the opportunity to acquire and experience knowledge. The student is thus being exposed to a very large number of examples in which she/he actively participates, with the comfort that if unsuccessful to do the problem alone, there is a resource which can help. Remarkably, we have so far not experienced blind copying from the solutions manual, which was our fear to start this program. We attribute this fact to making clear early in the semester that the tests are very challenging both in content and level of difficulty. We observe a big jump in the students motivation, interest in the subject and performance on exams. All students in the classes in which we used this method improved their tests scores very significantly. Additional benefit which we observed is a remarkable increase in the students' self-confidence and study-independence. .

1 Introduction

This paper presents a new approach to the teaching of higher mathematics. We describe it in detail and compare it with other approaches. Several statistics which show the advantages of this approach to other methods of teaching are presented and a comparative analysis is done.

The new method, described in this paper, consists in allowing the students completely free access to the solution manual containing the complete solutions to all problems in the textbook and simultaneously assigning an exceptionally large number of homework problems, ranging from easy to the most challenging. Our philosophy is that we use the homework for a learning tool instead of testing acquired abilities. The intellectual challenge level of the exams is raised as high as the most abstract and the most sophisticated problems which the textbook involves. We use these exams to assess the acquired skills and knowledge of the students. The homework is mandatory and the students are being given credit for doing it, but it is used solely as a tool in the learning process and not to test acquired skills. We use the university tutoring center for the place where the solution manual is being made available to the students. The solutions manuals which we use are published by the same company which publishes the textbook in use of the particular course.

The statistics which we have done on the effectiveness of this new approach to the teaching of higher mathematics demonstrate its advantage to the "classical" approach of "rediscovery", in which the solutions of the assigned home work is strictly unavailable to the student before the homework is collected. It was not without an opposition that this new method of teaching established itself in our institution. Our work benefited greatly from it, since it was this opposition that forced us to perform comparative statistics on the effectiveness of the new approach and to analyze the results. These results show consistent and very significant improvement of the performance of the students on exams. To give an example of the comparative statistical analysis which we made - we subject a class (like for example Calculus 3) to this new method of teaching and compare their performance on exams with the performance on exams of *the same group* of students in the previous mathematics course (Calculus 2) taught under the "classical" method, as described above. The statistics show a "jump" in the performance of the students on exams. Under our new method, in which we allow free access to the solution manual, the majority of the students improve their exam grades with 1 - 1.3 letter grades. In addition we observe a jump in the students' motivation, self-confidence and study independence. Surprisingly, we did not observe *any* blind copying from the solution manual, which was our main fear to start this program. Further, we were delighted to be able to raise the challenge level of the tests much higher than we could when using the "classical" method of teaching. Thus, our statistics were done not only on the *same group of students* but also using significantly more difficult exams. Even though this paper concentrates on the remarkable improvement of exam performance resulting from the new method of teaching which we use in our mathematics courses, the jump in the students' motivation, self-confidence and interest in the subject should also be emphasized.

2 Origins and Motivation

The idea for the method of teaching higher mathematics, which this paper presents, has its origins in the Russian educational system.

In the pursue of effective ways of building knowledge in the abstract and challenging subject of higher mathematics we experimented with different approaches to teaching. Here in the United States, historical reasons have firmly established what we call the "classical" method of teaching mathematics, in which revealing any part of the solution to a homework problem, before collecting the homework, is a tabu. This "classical" approach to the teaching of mathematics has its many positive sides, and this paper has no intention of ruling out or raising doubts in its benefits. The feeling of discovery is an important one, and forcing the student to rediscover the solution of a problem by herself/himself by making the solution unavailable, has shown to work with many students. The drawbacks of this method are well known. One significant drawback of this "classical" method is that it lowers the standards of the students toward an assignment- many students submit incomplete homework, when they are unable to find the right solution of a challenging problem. Only the strongest students manage to complete the whole assignment. This is actually not the biggest drawback of the "classical" method of teaching mathematics. The students who submit incomplete assignments have built a criterion on whether a solution is correct or incorrect and it is actually an asset that they do not submit an incorrect solution. These students are a small percentage though. The greatest majority of the students, forced by the fact that the homework carries some weight in calculating final grades, submit wrong solutions. This lowers their standards in the quality of their work, and thus has a very negative effect on their mathematics education and more generally on their grout as professionals and humans. A third drawback of the "classical" method of teaching is that "discovering" takes a lot of time to the student. Even the strongest, most motivated student, needs a lot of time to discover completely by herself/himself a correct solution to a challenging problem. This limits the number of mathematical problems which the student can be exposed to and the types of "situations" which she/he can "experience".

In our efforts to maximize the benefit of a mathematics course we looked into other approaches to teaching and more specifically an approach which will resolve the difficulties which the "classical" way of doing home work in higher mathematics presents. The experience which the Russian educational system provides in the teaching of mathematics was an exceptionally valuable recourse. Even though mathematics is the same everywhere, the philosophies behind teaching it differ drastically from one to another educational system. The task of assessing different approaches to the teaching of mathematics is an enormous one and has been a focal point of our educational research for many years. Much has been written on different approaches and by now we all are convinced that each proposed model has its advantages and its drawbacks. In our exploration of the different methods of teaching higher mathematics we found that the Russian method avoids many of the problems which the "classical" approach described above is subject to. The idea of **"discover by yourself"** is missing from the Russian mathematics teaching philosophy. In its place is the **"strive for performance excellence"** teaching strategy. The students are given everything available and are asked to master the most challenging. All kinds of study resources are made available, like for example books consisting of a collection of problems. These books contain the complete solutions to all problems with detailed explanations. In the Russian system of teaching mathematics there is absolutely no fear that the students are not sufficiently challenged, because the level of exams can be raised as high as the individual instructor desires. The students are completely aware of the fact that copying the solution of a problem is not improving their abilities to perform on exams. The "discover your self" strategy is replaced by the "who is going to do better on the test" race, which is easily translated to "who is going to learn more" race. As is well known, the Russian approach has proved itself to produce excellent results at all levels of mathematics education. It was an exceptionally valuable resource for ideas in our search for more effective teaching strategies.

3 Description and Statistical Analysis

The method of teaching which we are describing in this paper is based on this Russian approach to teaching mathematics. Our philosophy is that we use the homework as a **tool for learning** and not for assessing skills or knowledge. We assign an **exceptionally large number of homework problems**, much larger than one can assign when the solutions manual is not available. If the "classical" method of teaching was to be used with this amount of required homework, no student is able to do even half of the assignment. The problems range from easy to the most challenging. The large number of problems allows us to expose the student to a much greater variety of problems. The way in which we make it possible for the student to go through this large series of exercises is that we provide the complete solutions to all the assigned problems. For this purpose we use the complete solutions manuals which are published by the same publisher as the one which publishes the textbooks. We use the university tutoring center to provide the access to the complete solutions manual. The students can use the manual freely at their convenience. We provide the solutions manuals in the tutoring center and do not post the solutions on the web and, thus, avoid any possible interference with copyright laws.

The fact that we have used this new method of teaching during the last 5 years we had the opportunity to apply it in classes of predominantly engineering students as well as classes of liberal art students. At Oregon State University the Differential Calculus, Integral Calculus and Vector calculus are about 85% engineering students with the rest being science majors. we have used our method extensively with these classes. Pacific University is a liberal art college and so, the students in the classes in which we have used this new approach are considered liberal art students. For the reader interested in specific data: these classes had about 15% mathematics majors, 20% physics majors, 30% premedical/science majors and 35% humanities/arts majors.

All of the classes in which we have used this approach were required classes. We have not made any specific suggestions on whether or not the students should work together. Our observations were that the students at Oregon State University worked almost exclusively individually, while about 30% of the students at Pacific University worked in groups. These groups seemed to form based on existing friendships. We have noticed no influence of the "working together/working individually" variable on the effect of the solution manuals. The homework is required, collected weekly and graded. The weight of the homework in the final grade is between 10% and 15% with the rest being performance on exams.

At Oregon State University the students had access to the solutions manuals every week day 9 am to 5:30 PM. At Pacific University they have access to it Thursday through Sunday 5:30 PM - 10 PM, which are the working hours of the tutoring center. We work on extending these hours.

At the beginning of each course we make very clear to the students that the exams are very challenging, and that the grades are calculated based on 85 - 90 % exams and only 15 - 10 % homework. A course always includes at least two midterm exams, the first of which is scheduled very early in the semester, so that the students can get a first-hand feed back on the high level of expectations. We often reinforce the testing of acquired skills and knowledge by weekly quizzes. Thus, the homework is a tool in the study process. It is a part of the learning experience. Some of my colleagues view it as forcing the student to being exposed to a very large number of examples in which the student participates actively. Our main fear in starting this program was the anticipation of blind copying from the solution manual. We were afraid that only the most motivated and already advanced students will have the true understanding of how to make use of the provided solutions and how to benefit from them. Encouraged from the success of the Russian system we decided to try our idea and monitor the results. We were surprised and very pleased that we did not observe *any* blind copying from the solutions manual. The performance of the students on exams jumped with comparison with their performance on exams under the " classical" method of teaching. Their performance improved very significantly from the first midterm to the second midterm and from the second midterm to the final exam. We require the students to submit all the assigned homework problems. The excuse "I couldn't solve this problem" is completely eliminated, because of the availability of the solutions manual. The students must consult the manual if they are unable to solve the problem themselves. We observe, that most students do not need much of stimulation in this respect. Knowing that nothing is being hidden from them , they strive to take it. We observed in several classes subjected to our new method of teaching, that a competition is being created between the students in the class. They compete about who is going to "get more" out of the solutions available, who is going to be able to do better on the test. This competition has been a healthy one in every respect in the classes which we have observe it. It makes the class more exciting, more of a race. This method of teaching seems to appeal to the students because of its correlation with their naturally youthful impatience, curiosity and need to compete. Even though the number and level of intellectual challenge may be overwhelming to some of the students, doing the home work is a positive experience, since they do not feel left alone to struggle with the difficulties. They have the help provided by the solutions manual and they approach the long and challenging process of doing all the assignment with the feeling of security. We think that one reason for the success of this method of teaching is the fact that it emphasizes positive, encouraging attitude to the learning process. The fear of "punishment" because of inability to do the problems is completely eliminated. The number of problems to which the student is being exposed is several times larger than in the "classical" method of teaching. This is due to the fact that with the help of the solution manual the students can finish a much larger number of homework problems, and thus we, the teachers, can assign a much larger number of homework problems, than if no solution manual was provided.

We would like to stress that this new method of teaching higher mathematics eliminates the drawbacks of the "classical" method described above. The students do not quit doing the homework. They consult the manual when they get stuck on a problem. Thus, they submit complete assignments. Especially pleasing is the fact that this method eliminates the difficulty which is inevitably present in the "classical" method of teaching, which allows submitting incorrect solutions and expecting credit for them. With the availability of the correct solutions, the excuse "This is the best I could do", is eliminated. This asset of the method which we propose in this paper and which has established itself in our institution is one of the most valuable. It benefits not only the mathematical education of the student but also his/her growth as a professional and as a person.

In this paper we present the comparative statistical analysis which illuminates the new teaching method described above as opposed to the "classical" method. In the statistical survey shown below we present our observations performed on the same groups of people. Our analysis is based on collecting data on the performance on exams of a class in a given course, say Calculus 3, taught with the new method of teaching, and comparing this data with the performance on exams of the *same* group of people in the previous course, Calculus 2, taught with the "classical" method of teaching.

We have used the method which this paper describes in classes of 25 to 40 students. We hope that this paper will inspire others to continue this line of work and test this approach in classes larger than 40 students as well as on small classes of less than 20. The observations which we have collected are very consistent. Below we show some of these statistics.

In the Calculus 3 course taught in the Fall semester of 2001 at Pacific university there were 28 students enrolled . We subjected the class to our new method of teaching in which we allowed a free access to the solution manual and collected data on the performance of the students on the exams. We then compared this data with the performance of the *same* students on exams in the preceding course, Calculus 2. To keep the statistic accurate we eliminated form our calculations the performance of students in the Calculus 3 course, who did not take Calculus 2. These we only a couple of the students in the Calculus 3 course. The majority of the students improved their letter grades with 1 - 1.3 letter grade from the first midterm to the second midterm in the Calculus 3 course. 61% of the students improved their performance on exams from a test score in the 70-80 range on the Calculus 2 final exam to a score in the 90-99 range on the Calculus 3 final exam. 7% of the students improved their performance from a test score in the 60-70 range on the Calculus 2 final exam to a test score in the 90-99 range on the Calculus 3 final exam. Another 7% of the students improved their performance on these exams from the 60-70 range to the 85-89 range. Another 28% of the students improved their performance from the 89-94 range to the 95-99 range. There was no student who lowered his/her performance on exams under the new method of teaching.

The statistics collected at Oregon State University were made on classes between 35 and 40 students. We compared the performance on exams of the *same group of students* taking Vector Calculus in which free access to the solution manuals was allowed to their performance on exams in the preceding Integral Calculus class. We did the same with Integral Calculus versus the preceding Differential Calculus. The statistics were very close to the ones above, demonstrating consistent and significant improvement of the performance on exams.

In conclusion we would like to share our delight with the benefits of this new approach to teaching higher mathematics and hope that this paper will serve to encourage other institutions to apply it in their courses.

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CALCULUS COURSES AT THE COMPUTER SCIENCE FACULTY, THE UNIVERSITY OF INDONESIA

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ABSTRACT

Computer Science students at The University of Indonesia are among the top high school graduates. However, two years ago a report showed that for several semesters around twenty percents of them failed in calculus courses.

The responses to questionnaires given by students and lecturers said that the students have lack of enthusiasm and they have low motivation in learning calculus, they questioned about the importance of calculus for their subsequent work in computer science, and they found that calculus is difficult and less challenging.

This paper presents a new approach in teaching calculus given in the last three semesters, its effects, and obstacles. The approach is devoted to give students strong background in calculus and greater capacity to use the methods and hence better prepared to complete their degree in computer science. The approach is focused on helping students to better understand calculus conceptually, having higher problem solving and computational skill, and appreciating the relevance and the importance of calculus.

The effort to achieve the goals includes encouraging independent learning, presenting relationship between calculus and computer science, providing computer related examples, using Maple for calculus projects, using computer science terms and style in explaining some calculus concepts.

This approach has improved the grades and the students' perception about calculus. However, there are still some obstacles faced by both students and the lecturers.

Introduction

Due to the wide spread use of computers in modern public and business institutions, computer science graduates become more and more needed in the job market. As a result, schools that offer computer science programs become very popular among high school graduates and getting into such schools become more competitive. One of the schools that attract more and more top high school graduates is the Faculty of Computer Science, University of Indonesia. It offers the best program, has the best lecturers and facilities, and the tuition fee is affordable (much lower than tuition at private universities).

The calculus courses given at the CS faculty are two four-credit courses; calculus I and II are offered in the first and second semester consecutively. Two two-hour per week was dedicated for conventional way of lecturing followed by problem solving and discussion. One hour per week for tutoring, which is focused on problem solving. One hour per week for computer laboratory work, even though, in practice, the students require a lot more than one hour. Calculus for remedial classes is offered during the semester break. It is an intensive course given six hours each week in the form of conventional lecturing.

The average entrance test scores of the computer science students at the University of Indonesia is always within the top three among the thirteen faculties. The entrance test to enter state universities is held nationally once a year, taken by more than 500,000 high school graduates and the competitiveness to be admitted into the CS program is one out of 40.

Two years ago, a report revealed that around twenty percent of the CS students failed in the calculus courses. Even though no thorough investigation had been done yet, the problems seemed real. The high percentage of failure in the course contradicted the fact that the students are among the best. To identify the main factors to this failure; I gathered information by listening to complains and suggestions from students, lecturers, tutors, and members of the faculty. Other good source of information is the course feedback from the students. The feedback is given in two forms. The first one is the standard questionnaire and the second one is a narrative comment on a piece of paper. They are free to anonymously express their opinion about the course or the lecturers in a piece of paper at the end of the course.

Most of the students wrote that they were not well motivated. They found that calculus was only a list of formulas and rules need to be memorized. They were not aware of the importance and relevance of calculus for their future work, especially for other computer science courses. There were two boxes in students' heads, one contained calculus the other contained other computer science and nothing connected those two.

The students were hard working. They spent hours in front of computers doing programming and other computer related homework. However, they gave much less time doing calculus work or any other subject for that matter; they were just computer freaks. They found that calculus was less challenging. They did not know that they were expected to have problem solving and computational skill. They were not aware that understanding the concept, given in class alone, was not enough. They needed to build problem-solving skill by doing exercises.

Other problems were big classes, the heavy teaching load of the lecturers, and the differences between high school and university learning environment. It was difficult to maintain a big class consists of about one hundred and thirty students. Not every student got enough attention from the lecturer. As a result, students felt they were not part of the learning process. Moreover, they were used to more spoon-fed teaching style as opposed to student-centered learning.

In order to address the above problems, several approaches have been devised. The following are some of them.

1. The Beginning

There are profound differences between high school and university learning process. The main difference is that in high school the teaching material were spoon-fed to them and they had scarcely had any opportunity to express themselves. All teaching and learning process is done in the run of the mill mode; the school and teachers act as the conductor. In the university, the process is very much independent; the students are expected to be proactive and independent. Therefore, when the University admitted about one hundred and ten new students, it somehow is responsible to change the incoming students' paradigm and mindset. Normally, the responsibility rests on the shoulders of the lecturers of the freshmen classes, including calculus. Students will encounter many kinds of problems when they have not gone through an adequate preparation process.

It is a standard practice that lecturers have to inform clearly the rules of the game on the first day of the lecture. Students should be well informed about the purpose of the course, the expectation of the lecturers, outline of the courses, the marking scheme, and the importance and relevance of calculus in their future work. Hopefully, the students will understand the direction of the course and prepare themselves to succeed in the course.

The purposes of calculus courses are to make the students have strong foundation in mathematics for their subsequent work in computer science. Thus, students are expected to understand conceptually, be able to solve various problems, and have computational skills. The problem solving skill can only be accomplished through exercises. They should realize the importance of doing exercises.

Students are encouraged to be proactive in class. Reward are given to those who asked good questions, give suggestions or do in-class exercises. Class participation contributes to final scores. Therefore students need to be well prepared not only for final examination but also for attending the class. The preparation is the responsibility of the students, and this is one of the changes that they must be accustomed to.

2. Instructors and Students Interactions

Healthy and productive relationship among students and the lecturer makes it easier to motivate them and generate a discussion. The lecturer should make the initiative, and then this relationship should be maintained inside and outside the classroom. Students having good relationships with the lecturers outside the classroom tend to be "nicer" in the class compared to those having no contact to the lecturers outside the classroom.

However, remembering the names of more than one hundred students is difficult. A list of names with photographs attached is very helpful. Every student was asked to give her or his personal data with a photograph attached. By the end of the third month, I know most of them individually. To give more attention to students and to keep tract of their progress, more tutors are assigned. There are four tutors, each responsible in helping a group of consisting around thirty students. The tutors should have good communication skills.

The questions, exercises, and examples given in tclass are easy ones, so that students can solve them. This will increase students' self-confidence and make them more comfortable with the subject. The level of difficulties is increased gradually. Moreover, there will be no punishment for a bad questions or false solutions. Never embarrass students in front of their friends. It is important to convey a message that we learn more from other people's mistakes. Sometimes, mistakes arises from miscommunication, in such a case the mistake is the responsibility of both the students as much as the lecturer. The lecturer should make an effort to rectify such miscommunication, since the she or he is the one who presenting the teaching material. Often, students' mistakes invite more profound discussion. They must realize that making mistakes is part of learning; so, they must not afraid to try new methods and to express their opinion.

3. Course Preparation

Copies of transparencies notes are given in advance. It gives student a chance to review the last lecture and prepare for the next lecture. The transparencies contain important concepts and problem solving procedures without examples of solutions. Solutions are given step by step in the whiteboard. Students are encouraged to adjust themselves to taking note skills, since the explanation in the lecture is fast. Therefore, students need to attend the lectures to get the explanations in details and the examples how to apply the procedures. Students sometimes feel that having the notes (from the transparencies) is good enough, no need to attend the class. It is a great mistake! Notes are not substitute to classroom lectures. The following is an analogy between the calculus course and a jigsaw puzzle. Course material transparencies, references, lectures, quizzes, homework, and class projects are pieces that resemble the puzzle. Students should take all the pieces to make it a complete.

4. Concept to Natural Phenomenon Relationship

To make it challenging and interesting, the calculus courses should be brought closer real life, especially to other computer science courses. Most importantly, it should be presented in such a way that will challenge students' mind, make it easier to understand and more enjoyable. Students love stories. Historical stories related to mathematics and mathematicians are both entertaining and bringing calculus close to everyday life. For instance, story about Tantalus and the removable discontinuity of a function, the story behind the witch of Agnesi, the derivative songs, proofs without word, and proof by poem. Students are given chances to share their calculus related stories during the last five minutes of the class.

5. Calculus-Computer Science Relationship

Bringing calculus closer to computer science courses can be done in three correlated ways: first we use computer science terms, such as algorithm and program, to explain the concepts of calculus. Second, students are asked to investigate the application of calculus, third, we use computer to do calculus projects. For example, the procedure for divergence and convergence of infinite series is given in algorithmic manner using flowchart. Students understand this topic more easily than if we give them a long list of theorems. The words "theorems" often intimidate them. Moreover, students with enough programming experience will be interested in implementing such algorithm into a program. First year students are not expected to do numerical programming, but, at least, they must be exposed to it. They should know that many mathematical solutions are done by using computers. Numerical treatment of Ordinary Differential Equations and Partial Differential Equations are both major mathematicians and computer scientists' great works.

Preliminary Results

The approach had been applied for three consecutive semesters since September 2000. Students get better grades. The percentage of students failed in the class is around ten percents. Even though improvement in students' grade is debatable, the students' response to questionnaires is getting better. There is no complain about the lecturing style, but there are complain about the size of the class and the final examination. As for the final examination, some of them wrote that the questions are too many and too difficult. More in depth study is needed to measure the improvement and standardize the grading methods.

Obstacles

There is no thorough research in the calculus teaching at the University of Indonesia. It is hard to get up to date and comprehensive data about calculus teaching. Therefore the approach designed to meet computer science students' needs were heavily dependent on the comments and suggestions given by students, lecturers, and faculty members. To design the best approach in improving the quality and standard of calculus, an adequate and specific data are required.

The lecturing style should always be adapted with the new demand and technology. The lecturer must have enough background in computer science. It needs a lot of preparation and hard work.

The response to the approach varies between freshmen and the repeaters. It seems that the repeaters are not responding well in classical lecturing. They get bored, and hence more difficult to motivate. They got lower marks in the examination. However, they did very well in programming-related homework. Students who have taken more computer science courses, often asked about implementing computational procedures given in class in a program.

It is difficult to get qualified tutors. The financial reward for tutors is not competitive. Qualified final year students prefer to find part time job outside the campus. To invite more students for becoming a tutor I asked the faculty to increase the reward, to give them certificates, as a reference to find a job in the future, and to make them realize that they benefit much more than just the money they received.

Ideally, one tutor takes care of fifteen to twenty students. With small number of students in a group a tutor have enough time for preparing the tutorial and correcting homework and quizzes. Some actions should be taken by the faculty to increase the number of good tutors.

Future Plan

So far, no intensive talk between the calculus lectures and the CS lecturers about the applications of calculus in their field. Such information is needed to strengthen the bridge between calculus and various fields in computer science. Hopefully, this kind of talk will take place soon.

Some plans have not been optimally applied. Careful investigation is needed to design a more innovative and suitable lecturing methods and using technology to support the lecturing. The improvement will be done step by step. First, we will concentrate in helping students to build problem-solving skills. Qualified lecturers instead of student tutors will give tutorial class. We will provide a more organized and comprehensive worksheets. The problem sets will be renewed annually since students tend to learn from previous year solutions, done by their seniors, without actually doing the exercises by themselves. Finally, we want to provide a lecture note and a project guide, a guide to do the projects by computers.

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IMPACT OF FORMATIVE FIELD RESEARCH WITH CHILDREN ON APPLICATIONS OF MODULO STRUCTURES UPON PREPARATION OF TEACHERS

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ABSTRACT

Researching directly with children in school settings on the accessibility of mathematical ideas is analogous to the laboratory of a scientist where theory is discovered and validated. Discovering accessible ideas for children leads to researching potential applications for teachers and preparation of teachers. This has been central to the evolvement of research on applications of modulo structures to arithmetic over the past several years.

The authors have extended the idea of applications of modulo structures to checking arithmetic of rational numbers expressed in various numeral bases. The following problem, in base seven, has been chosen as an example because it represents a difficult problem to solve and check in rational number arithmetic: -6.0432 divided by 0.34 is –14.65 WR 0.0034, and checks with the cast out of 6 being 3. The authors have determined that the idea of cast outs is accessible to children as soon as they are conserving a one-to-one correspondence and can engage in developmental numeral structures. Also, the authors have determined several enhancing techniques for implementation of the idea as children become progressively more mathematically sophisticated, numerically. Not only can children access and apply the ideas, but also, the ideas/techniques impact conceptual understanding/applications of numeral structures. NCTM and others consider checking of arithmetic by application of calculators an abuse of technology as an educational tool.

Formative field research with teachers and pre-service teachers, based upon these applications of modulo structures, has led to significant changes in teacher preparation courses. Further, the implications suggest a renewed interest in modulo structures for pre-calculus.

The authors propose to share the fundamental accessibility of applications of modulo structures to arithmetic and how such has impacted the preparation of teachers, with implications for pre-calculus courses.

Formative Research Impacts Curriculum and Preparation of Teachers

Formative field researchers are often the avant-garde in curriculum innovations which impact teacher preparation programs. Sharing mathematical ideas directly with students in their classroom environment is the laboratory for formative research mathematics educators. Discovering accessible ideas and strategies for enhancing such accessibility requires knowing mathematics, having insights, intuition, imagination, and having lots of experience with sharing ideas with students. Some accessible ideas may be appropriate for curriculum. As professionals become aware of such ideas, then such ideas may have potential for impacting curriculum and teacher preparation programs. One such pioneer field researcher whose research impacted curriculum, the preparation of teachers and indirectly relates to the research of this paper is that of E. Glenadine Gibb (Gibb 1954, Van Engen and Gibb 1956). She perceived how children partition collections in relation to subtraction and repeated subtractions which not only impacted the teaching of subtraction but also the teaching of long division. This long division idea is related to the concept of cast outs. Field research by the authors on cast outs and applications of cast outs which impacts teacher preparation has occurred over several academic years and at all of the grade levels first through seventh. Research sessions were usually problem oriented involving guided discoveries by students with hands on manipulability, i.e. a constructionist mode. This mode is endorsed by the National Council of Teachers of Mathematics, "This constructive, active view of the learning process must be reflected in the way mathematics is taught." (NCTM 1989 p. 9). Aspects of the authors field research on applications of cast out structure have been reported over the years and recently (Edgell 2000, Edgell 2001, Edgell and Magnuson 2002).

Cast Out Structure and Applications

The cast out structure concept is imbedded in the idea of algebraic modulo structure. The cast out idea is independent of a specific base related numeral structure and can be applied at the verbal/word and tally numeral stages as well. Ordinarily though, the cast out idea is stated in terms of division by one less than a base when applied to historically and developmentally important base related numeral structures such as: simple grouping, multiplicative grouping, or base n (positional notation where co-factors are digits and powers of n which are determined by the position relative to a point which separates the non-negative powers from the negative powers) numerals. The cast out of a number, expressed in terms of a counting number base greater than two, is the remainder after dividing the number by one less than the base. This statement may symbolically be expressed:

$(B_n\overrightarrow{n-1}R) \Leftrightarrow (B_n = (n-1)Q + R, 0 \le R < (n-1)$ and $n > 2)$. It is clear that the number expressed in terms of base n, B_n, when divided by one less than the base, $n-1$, has a non-negative remainder, R, which is the cast out of B_n. "The cast out of multiples of $(n-1)$ of B_n is R.", is represented by $B_n\overrightarrow{n-1}\,R$. When one is doing several problems in terms of one base the statement is often abbreviated to, "The cast out of B_n is R.", or $B_n \rightarrow R$. The arrow represents the equivalence relation "the cast out of multiples of $(n-1)$", which is directly related to the algebraic modulo equivalence relation of congruence. The cast out of any number expressed in base n is one of the fundamental digits of the base which is less than $n-1$.

There are lots of algorithmic strategies, other than the definition, for determining the cast out. As with algebraic modulo structure, there are important statements about cast out structure which apply to arithmetic operations such as: the cast out of the sum of the cast outs of addends is equal to the cast out of the sum, and, the cast out of the product of the cast outs of factors is equal to the cast out of the product. Since subtraction is defined directly in terms of addition and division is defined directly in terms of multiplication, the process of cast outs easily extends to these operations in terms of corresponding definitions. That is: the cast out of the sum of the cast outs of the subtrahend and difference is equal to the cast out of the minuend, and, the cast out of the product of the cast outs of the divisor and quotient is equal to the cast out of the dividend. Further, the cast out process can easily be extended to the operation of division with remainder. In the preceding statements one should be aware that the cast out of a number and another number may be the same, i.e. have the same remainder when divided by $n-1$. This implies that an incorrect result could have the same cast out as a correct result.

A Review of Alternatives for Checking Arithmetic

Traditional checking techniques include those that are based upon: re-doing the process with more focus, perhaps using more details, applying the commutative property of an operation, applying the definition of an operation in terms of another operation, applying an alternate algorithmic form of an operation, applying the cast out technique usually in terms of casting out nines (the cast out of nines process for checking addition or multiplication of counting numbers is documented to have been used prior to the ancient times of the Hindu arithmetic, possibly as early as the time of Euclid, (Boyer 1968, Cajori 1914, Eves 1953, Smith 1951, Smith 1953, and others)), and/or applying the calculator. In checking arithmetic with techniques such as demonstrating more details, increasing focus, applying the commutative property, applying a different operation, or using an alternate algorithm, the situation of having two different results can always occur. What are students then advised to do? There might have been a time when applying calculator/technological instruments had been considered as a checking technique. Influential mathematics education organizations such as the National Council of Teachers of Mathematics, the Mathematical Association of America, and others are not currently recommending the use of technology for the purposes of checking routine arithmetic. The underlying principle seems to be related to overuse/abuse of technology in general. The rationale is to reserve technology for situations where technology may be one of the best tools or even a unique tool for assisting a student in learning significant ideas of mathematics. Conversely, although calculators could be used for checking arithmetic of numbers expressed in base ten numeration, most are not programmed to perform base n arithmetic when n is not ten.

There are several issues which have emerged over the years which tend to influence mathematics educators to endorse more developmental strategies of introducing students to the idea of representing numbers, rather than starting with base ten numeration. Concerns about U.S. students not comparing well, internationally, on questions involving decimals and place value on the Third International Mathematics and Science Study, TIMSS, (TIMSS 1995) has researchers and educators considering viable alternatives. Such alternatives include those which tend to gradually incorporate the co-factor role of digits and exponential powers of a base. As a result, students may be introduced to the idea of representing number through a historical-developmental approach which could start with verbalization (number names directly associated with numbers of objects), enlarge to tally structure (a direct one-to-one correspondence between symbols and objects) experiences, incorporate simple grouping (trading power base groupings with addition) numeral structures, gradually introduce the need for digits and overt exponential co-factors as with multiplicative grouping numeral configurations, and finally incorporate base n numerals. Calculators would not ordinarily be used for checking arithmetic with such numeral structures.

As mentioned before, applying the cast out structure to checking arithmetic has a long and rich history which predicated the development of the idea of a congruence relation between numbers as described by Gauss in *Disquisitiones Arithmeticae*, (Reid 1992, pg 132). Further this eventually led to the law of quadratic reciprocity as proven by Gauss who described the law as the gem of arithmetic, (Reid 1992, pg 139). If one gets a different cast out in the final step of checking arithmetic by casting out nines (or in casting out $(n-1)$ as with base n numerals), then one is confident that an error has occurred in the initial operation. But, there are concerns about being sure that the initial result is correct when the final step of checking yields the same cast out, since two different numbers can have the same cast out. This can occur when digits get reversed, (for instance, the cast out of nines of fifty-one is the same as the cast out of nines of fifteen, or the cast out of nines of one hundred twenty-three is the same as the cast out of nines of three hundred twenty-one). Having errors such as these occurring very often can sometimes lead to diagnosis and modification. Alternately, the cast out of nines of sixty-seven is the same as the cast out of nines of thirty-one, which is not so likely to occur. The bottom line is that it seems to be an inherent property of all checking techniques to not be infallible or without confounding issues. It seems, with respect to casting out nines, that a major concern would be with the restrictions to checking just addition or multiplication of numbers in base ten. These restrictions are completely unfounded. The principal author has determined that cast out techniques can be applied to checking addition, subtraction, multiplication, and division of rational numbers, which include the counting numbers, integers, and rational numbers expressed in the base n related format, at least, and starting with verbalization.

Research Impacting Teacher Preparation

The principal author has been involved with the preparation of teachers of mathematics at all levels for the past forty or so years and has been involved in multiple field research programs focused upon several issues with students at the public school level for about twenty of those years. In general, impacting the preparation of prospective mathematics teachers with innovative ideas seems to be about as slow and difficult as incorporating innovative changes in public school mathematics education curriculum. Since the principal author directly teaches pre-service teachers there are opportunities to also use the classroom as a research laboratory from time to time, thus opportunities to incorporate the latest research findings can occur.

In sharing mathematical ideas with pre-professionals there are usually several options. In many instances, particularly when ideas relate directly to ideas to be shared at the public school level and when one has personal experience in formative field research with sharing the ideas with public school level students, one may decide to use techniques similar to those that have enhanced the accessibility of such ideas with public school students when sharing such with pre-service teachers. This has been the situation with respect to helping pre-professionals to discover the concept of cast outs and applications of cast out to checking arithmetic. The underlying principles for sharing have been the same for these pre-professionals as for students involved in the field research. These principles are essentially the constructionist perspective, that is, guided discoveries with active access to hands-on manipulative objects revolving around problems where students are expected to conceptualize and apply the concept of cast outs in terms of personal algorithms and as problem situations vary students are expected to modify their personal strategies accordingly.

Stages and strategies for acquiring and applying the idea of cast outs for public school students, which translate to strategies for the preparation of teachers as, determined by the authors are as follows.

1. One can introduce the ideas as soon as students are verbalizing numerals, refer to Example 1 for applications, in context with objects by physically removing groups of objects while describing the action in terms of casting out the number of objects removed from the group. Although a specific cast out group size is not required, one usually selects a group size that is consistent with an anticipated base associated with stage three. For instance, the cast out of fours of the number seven is three, because there are three objects left after physically removing four objects (the focus is upon the number of objects that are left after physically removing all possible groups of one less than the base). Some students start with a personal cast out algorithm which is essentially a form of repeated subtractions of the same number.

2. One continues the same kind of activity in terms of removing tally symbols, refer to Example 2 for applications which includes an example of division not appropriate for students usually at this stage of development but appropriate for potential teachers, again in context with also physically removing objects and verbalizing. Usually, since tally numeral expressions tend to represent relatively small numbers so as to not confuse students in understanding the numeral, students do not have to significantly modify their personal cast out strategy.

3. Simple grouping, the sum of multiple powers of a base should be introduced in conjunction with physical base power blocks. One should guide students to discover that the cast out of any power of the base is one. This might be aided by physically demonstrating trading a larger power base block for base of the next smaller power base blocks and removing base minus one of these smaller power blocks as related to casting out one less than the base. Also, one can introduce the graphing of the cast outs of consecutive counting numbers, see Graph 1, and help students to discover a geometrical as well as numerical pattern. Usually, since the numbers encountered tend to be larger numbers than those expressed in Stage 2, students start modifying their personal tactic for determining cast outs. Refer to Example 3 for applications of the cast out structure to checking arithmetic.

4. One builds upon the power of a base idea associated with simple grouping to incorporate co-factors with digits when introducing multiplicative grouping numerals. Students are led to discover relationships between the digit co-factors and the cast outs (there tends to be less focus upon the power co-factors since the cast out of such is always one). One continues to focus upon geometric and numeric patterns. Students, having gained a range of cast out experiences, tend to become rather sophisticated with personal algorithmic casting out strategies. Refer to Example 4 for applications of the cast out structure to checking arithmetic.

5. When one makes the jump to base n numeration, leaving out the overt expressions for powers of the base as co-factors, students are usually already focused upon the digits as primarily impacting the cast outs. Usually students have acquired sophisticated personal cast out strategies which need virtually no modification. And, they seem readily able to apply cast outs to the ordinary operations with counting numbers.

6. Students making the transition to integers and the cast out of integers seem to experience somewhat of a mental quantum jump. As before, one can usually help guide students to discover modifications of their personal cast out strategy by leading students to believe that the geometric and numeric pattern previously established for cast outs of consecutive counting numbers is consistent and discretely continues for integers. Having students involved in graphing some consecutive integers around zero while maintaining the discrete consistent geometrical pattern seems to be helpful to students in the transition, see Graph 2. Also, trial and error in conjunction with the geometric pattern evident in the graph and the emerging numeric pattern generally tends to assist students in modifying personal cast out strategies. Some students discover a principle such as: the cast out of the additive inverse of a counting number is equal to the difference of one less than the base and the cast out of the counting number, which is interesting.

7. Making the transition from determining the cast outs of integers to rational numbers expressed in digit-point numeration, refer to Example 6 for ordinary applications and refer to Example 7 for an application to division with remainder, is usually easy for students. Students recall that the co-factor powers involved with counting numbers and integers did not impact the cast outs. When they comprehend the role of the point in merely separating non-negative co-factor powers from the negative co-factor powers, they realize that whatever stratagem they were using to determine a cast out could be continued.

In the process of learning about cast outs students should also engage in applying cast outs to checking arithmetic at every stage, beginning with addition. Early in the process of checking addition students should make a verbal association with the discovery: the cast out of the sum is equal to the cast out of the sum of the cast outs of the addends. Similarly, with respect to the process of checking multiplication, students should make a verbal association with the discovery: the cast out of the product is equal to the cast out of the product of the cast outs of the factors. A group of thirteen first grade students progressed through the first three stages applying the cast out of fours in related base five simple grouping numerals to addition and multiplication over sixteen one hour research sessions, one per week (not consecutive weeks), over a period of an academic year. These same students as second graders seemed to have retained the information over a summer break and when combined with ten other second graders, new to the program, were able to share their previous experiences. This group of twenty-three students was able to continue through the next two stages in base five and enlarging the scope of operations to include subtraction. Other second grade groups were similar in scope of accomplishments. A group of fifth and sixth graders were able to start at Stage 1 and continue through the seven stages with more than one base of numeration. Fifth graders tend to have considerable difficulty with Stage 7. This was not with reference to determining cast outs or applying such, though. The issue for the fifth graders rested primarily with not having experience with rational numbers, which included the negative rational numbers, and also with inexperience with rational digit-point numeral expressions. A class of focused seventh graders taught by an innovative mathematics teacher had no problem with jumping directly into Stage 5 and using various bases for numeral expressions flexibly within a couple of hour sessions. Workshops of three hours with teachers having extensive mathematical backgrounds are usually required to share the cast out concept with limited applications at Stages 5 and 6 with bases other than just base ten. Ordinary teachers at the elementary school level involved in in-service training workshops usually require about a week at the first five or six stages to become functional and reasonably confident primarily with base ten. Since pre-professionals in this university system are required to have had prerequisite mathematics of at least college algebra recently, they are usually functional with the cast out concept and applications to checking the four fundamental operations (addition, subtraction, multiplication, division) starting with Stage 2 and continuing through Stage 6 in terms of base five, base ten and sometimes a couple of other bases after five or six class sessions. These sessions are of one hour fifteen minutes duration meeting twice a week. Elementary students and teachers tend to really feel mathematically empowered by these experiences and seemed to be enlightened or more enlightened as to the role of digits as co-factors of powers of a base in numeral expressions, i.e. the role of digits with respect to place value.

Clearly the direct impact of such field research upon the preparation of teachers is appropriate. But, there are usually mathematical prerequisites to such teacher preparation courses. Traditionally, congruence-modulo structures are included in algebra and elementary number theory based courses, which may be part of prerequisite mathematical courses. When it is known that teacher preparation students are participants of such courses, professors might consider somewhat the possibility of some emphasis upon applications of such structures and perhaps include some ideas associated with applications to the cast out idea.

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Examples of Applying Cast Outs to Checking and Graphs

Example 1. The cast out technique for checking arithmetic of counting numbers expressed in verbal numeral form is independent of any specific base. In the following examples the following will represent what number is being cast out, $\overrightarrow{number}$.

$$\begin{array}{lll} thirteen & \overrightarrow{five} & three \\ \underline{+nine} & \overrightarrow{five} & \underline{+four} \\ twenty-two & \overrightarrow{five} & two \quad seven\ \overrightarrow{five}\ two\ CK \end{array} \qquad \begin{array}{lll} seven & \overrightarrow{six} & one \\ \underline{\times five} & \overrightarrow{six} & \underline{\times five} \\ thirty-five & \overrightarrow{six} & five \quad five\ \overrightarrow{six}\ five\ CK \end{array}$$

Example 2. The cast out technique for checking arithmetic of counting numbers expressed in tally numeral form is independent of any specific base. In the following examples a slash, /, will represent one (a tally mark). To check subtraction or division one merely applies the definition of each in terms of addition or multiplication and checks such.

$$\begin{array}{lll} /////////////////\ \overrightarrow{seven}\ /// \\ \underline{-/////////}\ \overrightarrow{seven}\ // \\ ////////\ \overrightarrow{seven}\ \underline{+/} \\ \qquad ///\ \overrightarrow{seven}\ ///\ CK \end{array} \qquad \begin{array}{l} \quad // \\ ///////\overline{\big)//////////////} \quad // \quad \overrightarrow{three} \quad // \\ \qquad \underline{\times///////} \quad \overrightarrow{three} \quad \underline{\times/} \\ //////////////\ \overrightarrow{three}\ // \quad //\ \overrightarrow{three}\ //\ CK \end{array}$$

The following examples will all be in base five and consequently the arrow will simply indicate the cast out of fours.

Example 3. This example is in terms of a simple grouping numeral structure where / represents one, f represents five, t represents twenty-five, etc. (the powers of base five), and 0 represents zero.

$$\begin{array}{lcl} ff\ /// & \rightarrow & / \\ \underline{+f\ //} & \rightarrow & \underline{+///} \\ ffff & \rightarrow & 0\quad //// \rightarrow 0CK \end{array}
\qquad
\begin{array}{l} fff\ /// \rightarrow // \\ \underline{-f\ //} \rightarrow /// \\ ff\ / \rightarrow \underline{+///} \\ \qquad f\ / \rightarrow //CK \end{array}
\qquad
\begin{array}{lcl} f\ // & \rightarrow & /// \\ \underline{\times f\ /} & \rightarrow & \underline{\times //} \\ tfff\ // & \rightarrow & //\quad f\ / \rightarrow //CK \end{array}
\qquad
\begin{array}{l} \qquad\quad //// \\ f\ /\overline{)ffff\ ////} \\ \qquad //// \rightarrow 0 \\ \qquad \underline{\times f\ /} \rightarrow \underline{\times //} \\ ffff\ //// \rightarrow 0\quad 0 \rightarrow 0CK \end{array}$$

Example 4. This example is in terms of a multiplicative grouping numeral structure where 0, 1, 2, 3, 4 are the digital co-factors of a term and / represents one, f represents five, t represents twenty-five, h represents one hundred twenty-five, etc, which are the exponential co-factors of a term.

$$\begin{array}{lcl} 3f2/ & \rightarrow & 1/ \\ \underline{+4f1/} & \rightarrow & \underline{+1/} \\ 1t2f3/ & \rightarrow & 2/\quad 2/ \rightarrow 2/CK \end{array}
\qquad
\begin{array}{l} 1t2f2/ \rightarrow 1/ \\ \underline{-3f3/} \rightarrow 2/ \\ 3f4/ \rightarrow \underline{+3/} \\ \qquad 1f0/ \rightarrow 1/CK \end{array}
\qquad
\begin{array}{lcl} 3f2/ & \rightarrow & 1/ \\ \underline{\times 2f3/} & \rightarrow & \underline{\times 1/} \\ 1h3t4f1/ & \rightarrow & 1/\quad 1/ \rightarrow 1/CK \end{array}$$

$$\begin{array}{l} \qquad\quad 3/ \\ 3f2/\overline{)2t1/}\quad 3/ \rightarrow 3/ \\ \qquad\quad \underline{\times 3f2/} \rightarrow \underline{\times 1/} \\ \qquad\quad 2t1/ \rightarrow 3/\quad 3/ \rightarrow 3/CK \end{array}$$

Example 5. This example is in terms of integers expressed in base five positional numeration.

$$\begin{array}{lcl} {}^{-}342 & \rightarrow & 3 \\ \underline{+{}^{-}413} & \rightarrow & \underline{+0} \\ {}^{-}1310 & \rightarrow & 3\quad 3 \rightarrow 3CK \end{array}
\qquad
\begin{array}{l} 142 \rightarrow 3 \\ \underline{-{}^{-}334} \rightarrow 2 \\ 1031 \rightarrow \underline{+1} \\ \qquad 3 \rightarrow 3CK \end{array}
\qquad
\begin{array}{lcl} {}^{-}41 & \rightarrow & 3 \\ \underline{\times 34} & \rightarrow & \underline{\times 3} \\ {}^{-}3044 & \rightarrow & 1\quad 14 \rightarrow 1CK \end{array}
\qquad
\begin{array}{l} \qquad\qquad 211 \\ {}^{-}102\overline{)^{-}22022}\quad 211 \rightarrow 0 \\ \qquad\qquad \underline{\times {}^{-}102} \rightarrow \underline{\times 1} \\ \qquad\quad {}^{-}22022 \rightarrow 0\quad 0 \rightarrow 0CK \end{array}$$

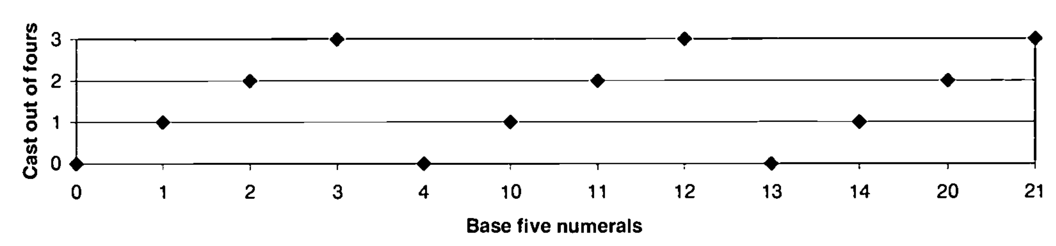
Example 6. This example is in terms of rational numbers expressed in base five.

$$\begin{array}{lcl} {}^{-}3.301 & \rightarrow & 1 \\ \underline{+4.232} & \rightarrow & \underline{+3} \\ 0.431 & \rightarrow & 0\quad 4 \rightarrow 0CK \end{array}
\qquad
\begin{array}{l} {}^{-}12.03 \rightarrow 2 \\ \underline{-{}^{-}31.41} \rightarrow 3 \\ 14.33 \rightarrow \underline{+3} \\ \qquad 11 \rightarrow 2CK \end{array}
\qquad
\begin{array}{lcl} 1.23 & \rightarrow & 2 \\ \underline{\times {}^{-}23.4} & \rightarrow & \underline{\times 3} \\ {}^{-}40.442 & \rightarrow & 2\quad 11 \rightarrow 2CK \end{array}
\qquad
\begin{array}{l} \qquad\qquad {}^{-}2.03 \\ {}^{-}31.4\overline{)120.302}\ {}^{-}2.03 \rightarrow 3 \\ \qquad\qquad \underline{\times {}^{-}31.4} \rightarrow \underline{\times 0} \\ \qquad\quad 120.302 \rightarrow 0\quad 0 \rightarrow 0CK \end{array}$$

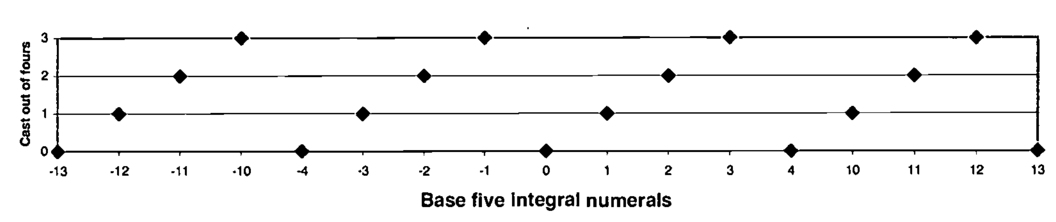
Example7. This example is in base seven where the arrow represents the cast out of six.

$$
\begin{array}{rrr}
 & \overline{{}^-14.65} & \\
0.34\overline{)\,{}^-6.0432} & {}^-14.65 \rightarrow 2 & \\
\underline{-{}^-6.0466} & \underline{\times 0.34} \rightarrow \underline{\times 1} & \\
0.0034 & {}^-6.0466 & 2 \rightarrow 2 \\
 & \underline{+0.0034} \rightarrow & \underline{+1} \\
 & {}^-6.0432 \rightarrow 3 & 3 \rightarrow 3CK
\end{array}
$$

Graph 1. Graph of the cast out of fours of base five counting numbers.



Graph 2. Graph of the cast out of fours of integers in base five.



THE INFLUENCE OF THE FAMILY IN THE LEARNING OF MATHEMATICS

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ABSTRACT

I analysed the issue of mathematical learning focusing on the relations, which involve the school and the family ideological apparatus. The political and economic consequences that we recognise in these relations are due to the work of signification that transforms an unskilled into qualified labour-force, through the incorporation of sign-value to labour force commodity. Pedagogical mathematical practice is based on scientific practice and represents a standard for the entire school system, therefore acquiring a higher standing. The coherence that permeates the family judgement practices may be absent from pedagogical mathematical practice; therefore, judging the students' performance means to extract plus-value from those who do not get the sign-value (get failing grades) but have contributed with their working hours.

Introduction

With the theme "the influence of the family in the learning of mathematics" as a starting point, I raised two questions to guide the research. One of them deals with the superior status of formal mathematical speech[1]. Exercising formal mathematical speech signifies "being specialised labour", and therefore being part of a group of people that has a higher economic status (with different levels within the profession) than that of groups of people that do not have the qualification. In this way, I understand that the hierarchical positions of work are also determined as a function of this status, which produced the differentiation between social positions and income. The other question is whether or not the family participates in the qualification of the workforce.

As a foundation for the work, I base the theoretical standpoint of Ubiratan D'Ambrosio, Louis Althusser, Jean Baudrillard, Roberto R. Baldino, and Tania C. Cabral.

With respect to the methodological procedures, I used Michel J.M. Thiollent (1987) as a reference.

To analyse the influence of the family in the process of qualification of the workforce, I considered it sufficient[2] to analyse one of the school years in primary school. I collected data related to three students in the fifth grade[3] in a public school in the city of Rio Claro[4]. Since I was interested in the influence of the family in the constitution of codes of prestige and discredit in the process of students' academic performance, I selected students who were classified by the school as *excellent, good,* and *poor*, respectively.

I adopted the following methodological procedures to obtain the data: indirect observations and semi-structured interviews (Thiollent, 1987). Keeping in mind that these students were inserted in the ideological apparatuses of the family and the school and that their practices would be consonant with the ideology that permeates these apparatuses (Althusser, 1980), I considered that the students would express these practices, in which speaking is the form of expression that underpins the "subjectification" of the social order.

I understand speech in the realms of the enunciated and the enunciation (Vallejo and Magalhaes, 1991). The enunciated is limited by what is expressed in the manifest discourse. In the enunciation, the subject positions himself beyond what he intentionally means. According to Vallejo and Magalhaes (1991), "There is a subject who enunciates the message, and there is an enunciating subject that diverges from the first" (p.42). When the subject[5] submits her/himself to formal evaluation[6], to the extent to which s/he speaks or writes, s/he pronounces her/himself and commits her/himself in relation to the other (a commitment in relation to the structure of the message and an implacable, inexorable code). The subject, when s/he makes use of the written or the spoken word, makes recognition possible. Thus, "...every discourse that is carried out has as a

[1] Formal mathematical speech is that which is exercised in educational or bureaucraticized scientific instances with the objective of increasing the exchange value and/or use-value of the labour force by way of the sign-value.
[2] During the period the study was carried out (1993-94), each school year was a necessary (bureaucratic) condition for the qualification of the labour force. Elementary school completed, currently Basic Education (*Ensino Fundamental* - Federal Law n. 9.394, Dec.20, 1996, Title V, Cap.II, Section III, Article 32 - Lei de Diretrizes e Bases da Educaçao Nacional) signifies one of the degrees for the qualification, since it addresses one of the attributes required in the field of labour, and is a necessary condition for secondary school, currently *Ensino Medio* (Federal Law n. 9.394, Dec.20, 1996, Title V, Cap.II, Section IV, Article 35 - Lei de Diretrizes e Bases da Educação Nacional).
[3] Children in this grade range from 10 to 12 years of age.
[4] Rio Claro is a city of 150 thousand inhabitants located in the central region of the state of São Paulo, Brazil.
[5] The category of subject for Althusser (1980): "...is constitutive of all ideology, but at the same time, is immediate - we add that the category of subject is constitutive of all ideology, to the extent to which all ideology has as its function (is what defines it) to 'constitute' concrete individuals into subjects. It is in this game of double constitution that the functioning of all ideology is located, the ideology not being greater than its functioning in the material forms of existence of this same functioning" (p.87).
[6] The evaluations I refer to are means of judgement. It does not matter to me if they can be considered arbitrary, since both evaluation x and evaluation y have the same proposal: to differentiate.

result a determined position of the subject that is relative to the discourse and which cannot be disassociated from the structure of the message"[7] (Vallejo and Magalhaes, 1991, p.42).

Thus, the subject is always at the mercy of what s/he is able to articulate when facing the other, at the moment s/he speaks. If the speech denounces a lack of knowledge of the basic rules of passing through the academic/school hierarchy, then the subject is a strong candidate for not belonging to this order. Therefore, the speech pronounced stipulates the privilege that the labour force commodity assumes.

I understand that the family is responsible for the introduction of the codes of ideological recognition, which may or may not coincide with the codes adopted for bureaucratised instances, instances of the production of knowledge, such as the School. The recognition of these signs (culture) is going to say what can be learned as codes that can be deciphered. However, the production of meanings that occurs in the family makes the differentiation in the labour force commodities[8], once their qualification depends on access to texts that can or cannot be read deciphered, dialogues that can or cannot be experienced. Thus, such commodities assume use-value[9] and exchange-value[10] beginning with the attribution of sign-value[11], which is exactly what guides differentiation in academic evaluation, and which depends on the insertion of the "subjectification" proposed by the family. Thus, the subjects who share in the informal speech, which coincides with the formal in the diverse fields of valid knowledge, will have greater advantage over those who do not share it.

It cannot be denied, however, that mathematical speech forms part of the practice of any subject, but it is perfectly possible to affirm that there is a difference between that which has to do with academic content and that which does not. If it is absolutely natural to discuss mathematical problems proposed in the classroom with one´s brother or sister, with one´s mother or father and learning to utilise the necessary resources to deal with them efficiently, in this case it would naturally be easier to achieve the documentation needed for qualification of the labour force. I am not saying that constituting use-value is a necessary condition for qualification, but it almost always signifies advantage in the approval process. What I want to emphasise is that the family provides the subject with social survival, leaving it to the learner to "communicate his/herself", exercising the speech as s/he learned it. If the grammar used was that which is considered formal, s/he will possess privilege, effortlessly. If the environment in which he was raised had a library and people considered to be "cultured", the subject will function in this way; he had no choice; it constitutes, without the slightest effort, possessing privilege. The same can be said with respect to mathematics.

Thus, the usurpation regarding formally-instituted mathematics lies in the superior status that this field assumes over others, once academic programmes have been based[12] on it. However, the subject whose speech is in agreement with the formal has an advantage in the process of qualification of the labour force, and in this way, the family participates in the attribution of the sign-value to the qualified labour force.

The study I propose could certainly be developed in any area of formal knowledge[13]. I chose mathematics because my field of study is mathematics education, and therefore I am also able to analyse data is related to mathematics.

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[7] Translation by myself.

[8] "The labour force value is determined, like that of all other commodities, by the labour time needed for production, and consequently, also for reproduction of this specific article" *(Das Capital*, I, Cap.VI, 1982). "The individual is an ideological structure, a correlative form to the commodity/form (exchange value) and to the object/form (use value). The individual is not more than the subject thought of in terms of economy, re-thought, simplified, abstracted by the economy" (Baudrillard, 1972, p.165).

[9] Ibid.

[10] Ibid.

[11] Ibid.

[12] For more on didactic transposition, consult CHEVALLARD, Y. *Aspects dún travail de théorisation de la didactiques des mathematiques*. Fac. Des Sci. de Luminy, Univ. d'Aix-Marseille II, 1989.

[13] The extension to any area of knowledge can be made as it is a legitimate area in the school or university spheres.

Data collection procedure

1. I worked during 36 classroom hours with fourth grade students[14] in the "Marcelo Schmidt" public school, located in downtown Rio Claro in the state of Sao Paulo, Brazil. The theme being addressed was fractions. I evaluated each student individually. Based on this work, and with analysis of the official academic evaluations, I selected the students who would take part in the study from among those who presented the conditions as suggested by the school´s own classification[15]: *Excellent* (the student plainly achieved all the objectives); *Satisfactory* (the student achieved the essential objectives); and *Poor* (the student achieved only a small number of the objectives). We will call the students who were selected Patricia, considered excellent, Juca, considered satisfactory, and Marcos, considered poor. (Period: December, 1993)

2. I observed the children who were now enrolled in the fifth grade[16] in the same school. (Period: 1994)

3. I carried out analysis of the school documents related to them: the school notebook[17] and the folder (use was not mandatory), which we will call the student's dossier. (Period: 1994)

4. I interviewed the students, their mothers, their teachers, and the "Inspector"[18]. The fourth grade teacher, who I will call D.Marta, had worked for ten years in the school and had taught the students for three years (second, third, and fourth grade); the fifth grade maths teacher, D.Isabel (pseudonym), and the Inspector had been working in the school for five years. (Period: 1994)

The students mentioned above were considered middle and low middle class according to the 1992 School Plan[19].

The observation of the class was focussed on one student at a time, taking into consideration mainly their involvement in the activities related to the program, as well as those not related to the program, which I called parallel activities[20]. The attitude of the teacher and classmates in relation to the subject observed were considered relevant.

The objective of the student's folder (dossier) was to record the participation of the parents in meetings, the problems presented by the students, and the warnings they received. The school notebook was a document that was kept by the student whose objective was to record examination grade and test, attendance, and has a bi-monthly signature from the parents.

The interview with the children made it possible to evaluate the interview with the mothers, and also gave the students' opinions of the school, the evaluation processes, the learning of mathematics, and their judgement of themselves and their classmates with respect to their position in the classroom.

With the interviews with the fourth and fifth grade teachers, it was possible to establish their views of the ability and achievement of each student.

The interview with the Inspector was important in that she provided information about the relation of the parents with the school, of the teachers with the students, and among the students themselves. She had worked in that institution since the students initial enrolment up until the time of the study.

[14] Children in this grade range from 10 to 12 years of age.
[15] *Caderneta Escolar* of the Marcelo Schimidt public elementary school, 1994.
[16] Children in this grade range from 11 to 13 years of age.
[17] SAO PAULO, *Caderneta Escolar* of the Marcelo Schimidt public elementary and secondary school, 1994.
[18] According to the Common Guide Rule of the State Elementary and Secondary Schools (1998), Section III - Administrative Support, Article 41, the function of the Inspectors is to monitor and attend to students.
[19] Document produced by the administration of the teaching staff of the school. The most recent school plan was from 1992.
[20] Parallel activities was a term used to designate the conversations and joking that occurred during the class, which had nothing to do with the subject being studied.

Results of the fieldwork:

Ideology operates in the sudden of individuals as subjects, producing students, mothers, teachers, and inspectors. The family apparatus plays a fundamental role in the constitution of the subject as student, since it attributes to her/him habits, behaviours, appearances, and knowledge, enabling the school apparatus to judge and classify him.

It is through the establishment of sign-value (Baudrillard, 1972) for the student that approval becomes possible, and, conversely, disapproval through the establishment of codes of discredit and failure. We were able to verify the recognition by all those who were interviewed of the superior status of academic maths, thus creating passivity on the part of the parents and the students regarding the possible classifications (approved or failed). This is not because parents believe mathematical knowledge is above all others, but because the guarantee of prestige resides in the degree to which they accept its superiority, which guarantees, in the final instance, greater exchange value, or in other words, a superior position.

Juca, Marcos and Patricia were recognised in the opinion of their teachers, the inspector, their classmates, their mothers, and their own voices, based on the code of prestige that establishes sign-value in order to differentiate the qualified from the non-qualified. The truth of the equivalence between approval in maths and learning the essential concepts is found in the testimony of approval of Juca and Patricia and the failure of Marcos, which can be seen in the written assessment of the bureaucratic authorities that assist in the control of academic documents.

In the case of Marcos, the codes of failure were established by the Labour Force Agent; that is, by the teacher (who also imposed her domination by way of the bureaucracy), by the inspector, by his classmates and by his own family.

With respect to Patricia, the Labour Force Agent established the sign-value. The practices carried out by Patricia's family provided her with the possibility of being judged as a student able to articulate pedagogical mathematical speech, resulting in the establishment of sign-value on her labour force, with is in the process of qualifying.

In the case of Juca, the Labour Force Agent also established sign-value. He was not considered as good a student as Patricia, but he fulfilled all the conditions for approval.

Thus, according to the testimonies obtained, there is no way to deny that the objective of the school is to increase the purchase value by establishing the sign-value for the labour force. As Baldino and Cabral (1991) said, the students seek to decrease study time, and in this sense, act as workers, and when they are approved, value their labour force, going on then to act as capitalists, which increases their capital; and in this sense that some extract a plus-value from the approval, resulting from the failure of the others. At what point does the family enter into this movement? It is exactly in the possibility of the student being the one who is exploited or the one who exploits. It is in the family that the subject learns to survive socially, and it is in this survival that the signs exist that delimit the way, the possible choices, and hence, the possible social and economic positions. In this sense, the students experience participation in the educational practice in completely different ways, and this depends very much on the families they are part of. In this respect, some are historically exploited, and others exploiters[21].

Conclusion

The family, as an instance of the determinations of rules of social survival, guarantees the difference in students' effort in school. Through educational practice, the subject learns the signs of the culture of which

[21] This statement does not eliminate the possibility of alternation between exploiters and exploited. We are not dealing with individuals but with positions.

the daily rituals form a part, in which she plays a role, which is demanded of her daily. The educational practice can produce meanings that are similar or are very different from those spoken in the school, and the disparity is made precisely in this aspect. I believe that family practices perform the trimming, the first order that situates the subject, which makes her/him feel the reason (discursive order). In this way, the pedagogical mathematical practice can be significant in different ways. Based on the empirical part, it is possible to say that some tactics for performing the pedagogical mathematical speech can be reading the text proposed by the teacher, several times, until one is able to repeat it for the test, or it could be to understand it enough to get an adequate grade (approval) even on a surprise test, or the tactic could even be unsuccessful, that is, the grade could be insufficient for approval.

For the reasons explained here, in the theoretical realm as well as in the analysis based on the empirical research, I attribute to the family the meaning of these possible articulations in such a way that the qualification (or lack thereof) of the labour force of the student of mathematics can remain subject to the conditions that sustain the familial order. This being so, the new members of the family integrate themselves to the other social practices, having the family practices as a first option. After the forced choice[22], they will go on, certainly, to be exposed to new ideological postures, but there will be no way for their judgement to lose its first reference. Human transformation is peculiar to the apprehension of this choice, of which ideology is intrinsic from the first moment, and which is only possible in the presence of language.

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[22] It is always forced for everyone, since the search is in the other; thus, the choice is sought in the other, in the other from whom they learned the language that submits them as a subject.

USING ENVIRONMENTAL SCIENCE TO BRIDGE MATHEMATICS AND THE SCIENCES

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ABSTRACT

Environmental issues can provide an excellent way to connect mathematics with the sciences. At Florida Atlantic University's Honors College, faculty are working together to build an interdisciplinary curriculum for lower-division mathematics, biology, chemistry and environmental science. Traditional calculus and statistics courses introduce environmental materials, some adapted from outside sources and some developed through collaboration between mathematicians and scientists in the college. Many of these materials are small projects, designed for students to explore collaboratively, with the assistance of a graphing calculator, computer algebra system, or statistical software. Complementing the mathematics program, lower-division science courses bring science and mathematics out of the classroom and into the community, using local ponds, lakes, forests and greenways as science laboratories. Student and faculty teams collect data on the water quality in dozens of area ponds, the diversity of wildlife in more than 250 acres of nearby preserves, and the impact of a growing population on the environment. They then bring their studies back to the classroom and use mathematics and statistics to analyze and model their data. A series of three new "links" – one-credit courses that are team-taught by scientists and mathematicians – focus on the analysis of student-collected data using increasingly sophisticated tools.

The project is supported by a National Science Foundation grant. The project goals are for students to understand the interdependence of mathematics and the natural sciences, and to be able to apply what they learn in the classroom to hands-on scientific studies. For both faculty and students, the project aims to integrate teaching, learning and research in a holistic form of scholarship. Preliminary data were collected in the fall of 2001, and a first assessment of the project's goals will be completed in the late spring of 2002.

Key words and Phrases: Interdisciplinary mathematics, discovery-based learning, environmental science

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1 Introduction

In the last century, the undergraduate curriculum in the United States has experienced tremendous growth in specialization of programs and courses. While this specialization has allowed great advances in many fields, it has also, unfortunately, led to a great deal of fragmentation in the learning experience for students [10]. Like faculty at many other institutions across the country, mathematicians and scientists at Florida Atlantic University's Honors College are making a concerted effort to generalize and integrate the undergraduate curriculum, and, in the process, build a sense of community among both students and faculty on campus. The project described here focuses on using environmental issues to bridge mathematics and the natural sciences.

The National Research Council Committee on Undergraduate Science Education calls for the primary goal of institutional efforts to reform science, mathematics, engineering and technology undergraduate education to be the following [19]:

> Institutions of higher education should provide diverse opportunities for *all* undergraduates to study science, mathematics, engineering, and technology as practiced by scientists and engineers, and as early in their academic careers as possible. [Emphasis in the original.]

In order to study science and mathematics as practiced by professionals, and to study the process as well as the content, students must learn to integrate, rather than compartmentalize knowledge, and they must be engaged in real scientific studies. To these ends, the Honors College program has students

- engage in science and mathematics as active investigators rather than as mere spectators or passive consumers of information;
- experience the sciences and mathematics as interconnected and mutually informing areas of human knowledge rather than as isolated "fields" (or "stovepipes" as Rita Colwell [6] aptly called them) separated by impervious disciplinary boundaries; and, finally,
- build an understanding of the connection between science and the world beyond the classroom, especially by exploring local and regional environments.

We want to give students hands-on field and laboratory experience in collecting and interpreting data, and, at the same time, to present them with a valuable opportunity to contribute to an ongoing scientific investigation of their own environment.

The desirability of experiential, connection-building learning activities is well-documented, and many successful reforms have been implemented at schools across the United States. Relevant activities in mathematics range from the adoption of reform calculus texts and courses ([21], [15], [2], for instance), which make natural and social science applications central to the courses and/or involve students directly in exploring and explaining physical phenomena; to the development of integrated courses in mathematics and science, generally offered as an alternative to the traditional sequence of courses ([3], [13], [8], for example); to the offering of programs, such as that at Evergreen State College [9], where the traditional role of a course has been replaced by wholly integrated semesters of study.

Lacking both the student body and the faculty resources to offer an alternative track to traditional calculus or science sequences, and restricted by the state university system from overhauling the overall course structure, we strive toward smaller successes within a fairly traditional program: an integrated lower division mathematics and science curriculum, valuable for *all* students, but particularly well-suited for majors in the natural sciences and related disciplines. In this endeavor, the Honors College is taking a two-fold approach to integration within a fairly traditional curriculum. First, faculty are working to weave an environmental thread through the introductory mathematics courses: precalculus, calculus and statistics. Second, the college is offering a series of new one-credit courses that are team-taught and that focus on the analysis of data from student and faculty science projects using increasingly sophisticated tools. These are described in more detail below.

2 The College and its Setting

The Honors College at Florida Atlantic University opened in the fall of 1999 as an autonomous, residential, liberal arts college within the larger Florida Atlantic University system. As of January 2002, the college enrolled approximately 240 students at the freshman, sophomore and junior levels. At full capacity (Fall 2005), the Honors College will enroll 500-600 students and employ some 50-60 faculty members, numbers that strongly promote close faculty-student interaction and discovery-based approaches to learning. Present enrollment trends suggest that a significant fraction of our students (roughly 40%) plan to concentrate in areas of the natural sciences that include pre-medicine, biology, and marine and environmental studies.

The Honors College is located in Abacoa, a 2055 acre, master-planned, mixed-use community that is currently under development. The planning of Abacoa has been guided by the philosophy of the "new urbanism," which seeks through architectural strategies to facilitate a sense of community among residents and to provide a connection to the natural environment via provision of greenways and lakes. To this end, some 259 acres of preserves have been retained, along with observation points, walking trails, and protected native habitats. A significant area of the development consists of varied aquatic systems, including lakes, ponds and connecting streams, providing the opportunity for faculty to bring science and mathematics out of the classroom and into the surrounding community.

3 The Courses

3.1 Introductory Mathematics and Statistics

To create stronger connections among the introductory science and mathematics courses, we are integrating in the introductory mathematics courses materials which emphasize environmental science. Some of these materials are small examples or exercises used in classroom discussions, but many are larger projects in which small groups of students explore, analyze and model environmental data.

For statistics, educators are nearly unanimous in their encouragement for students to work with real data, and preferably student-gathered data [14], [4], [17], [5], [24]. As long-term research projects are incorporated in the science program, we will naturally develop a large bank of environmental data sets that can be used for examples and projects in the introductory statistics course. So far, students in statistics have worked on projects analyzing pollution levels in Lake Erie (materials adapted from [22]), aggressive behavior in the Giant Damselfish and characteristics of Abacoa's gopher tortoise population, the latter two projects based on data collected by biology professor Jon Moore.

In the precalculus course, students work on a series of projects in which they model the physical characteristics, individual growth, and population growth of the gopher tortoise population in Abacoa, employing linear, quadratic, exponential and logistic models. We have also employed materials from Mooney and Swift's text [16], in which students investigate migration patterns of squirrels.

At the calculus level, our efforts are supported by the Harvard Consortium text [15], which is particularly strong in its inclusion of examples from the biological sciences. For projects, we have used modules on logistic growth, air pollution, and the SIR model from Duke University's *Connected Curriculum Project* [18], which contains a collection of web-based environmental science modules as well as some longer projects. We have also used materials from *Project Intermath*'s collection of modeling problems [20], including, *Rising Mercury in Water*, in which students use difference equations to investigate the bioaccumulation of mercury in humans, and *Lake Pollution*, in which students investigate levels of pollution in a river and lake system.

3.2 Current Linked Courses

To help students make connections between their science experiences and mathematics, we are developing new "linked" courses. In its planning document, the Honors College emphasized the importance of establishing connections between disciplines and created linked courses to provide a structure for making these connections. Typically, "links" are one-credit offerings, co-taught by instructors in different disciplines, in which students discuss common themes, examine disciplinary assumptions, and explore areas of conflict in topics which cross disciplinary boundaries.

Data Analysis. The Data Analysis link provides students with the opportunity to statistically analyze data they have gathered for their science projects. The course is available to students who have had one semester of statistics and are currently enrolled in a first or second year science course that includes a project component. To analyze their data, students are expected to make appropriate choices in the application of statistical methods. Through the discussion and critique of a variety of projects in different disciplines, the students evaluate choices made in the design of studies and in the collection of data.

For example, the current Data Analysis course is working with data concerning 1100 sea turtle nests on an 11-mile stretch of beach directly east of the college. Biology professor Jim Wetterer and some of his students collected data on the species of sea turtles, the locations of the nests, types and number of ants found on the nests, number of eggs in the nest, and number of live hatchlings, among other variables. Students are analyzing the data, using techniques learned in the introductory course, to answer questions about, for instance, the relationship between the number of ants found on a nest and the nest's location (distance from vegetation or high tide mark), the relationship between the number of eggs laid and the species of the turtle, and the relationship between the presence of ants and number of live hatchlings. The students will soon discuss multi-linear regression (a technique that is new to them), then use what they have learned to build a model for predicting the percentage of live hatchlings from a nest, based on the variables they determine are important to the model. Later in the semester, students will analyze data sets individually (data from their own projects, or from a project of a faculty member in their chosen discipline), present their analyses to the class for discussion and refinement, then write a final paper discussing their findings.

Environmental Science Seminar. The Environmental Science Seminar is designed to introduce students to multidisciplinary collaboration and peer-review. The seminar also helps prepare students for the writing of their senior theses by involving students in the design and critique of their own projects. Junior year participants do directed reading, and develop and present ideas for projects (working towards identifying a senior thesis project), while senior year participants present results from their ongoing research projects. This year, because we do not yet have seniors, a portion of the seminar is dedicated to faculty discussing their current research, emphasizing potential student projects. The seminar is attended by faculty members and students in chemistry, biology, mathematics, physics, economics and psychology.

3.3 Future Linked Courses

Mathematical Modeling. The modeling link, first to be offered in the Fall of 2002, will require significant planning and close cooperation among participating science and math faculty. For a student to enroll in this course, it will be necessary to have on hand a data set which lends itself to a modeling approach. For example, a student who wishes to explore the fluxes of phosphorus (or, indeed, any number of other elements, especially redox-sensitive elements like iron and manganese) from lake sediments into the water column or do more sophisticated diffusion modeling using Fick's first and second laws, will need to bring to the course a coherent set of measurements of dissolved phosphorus in sediment pore waters, determined at appropriate cm-scale intervals [1]. In any one-semester modeling link, we expect that students will be introduced to several types of modeling—steady state box models, non-steady state box models, chemical equilibrium models, population models—depending upon the projects carried out in chemistry and biology. Examples of environmental modeling materials abound, including the rich introductory texts of Harte [12], Mooney and Swift [16], and Hadlock [11], all of which are largely accessible to students with a background in calculus, which will be a prerequisite for the Modeling link.

Geographical Information Systems (GIS). Beginning in Fall 2002, the Geographical Information Systems (GIS) link will be offered primarily for students in their junior year, and will bridge the natural and social sciences. GIS technology is an increasingly useful and popular way of recognizing and studying relationships in our environment by analyzing spatial patterns. This is becoming a standard research tool among environmental professionals and in graduate institutions. The increased demand for its use has led to the incorporation of GIS-based curricula into undergraduate education.

For example, students may use GIS to examine the spatial arrangement of gopher tortoise burrows and grazing areas and determine how this arrangement relates to topography and vegetation. As in the case of other wildlife, increased human population and traffic may directly or indirectly impact not only the population of tortoises, but also where they burrow and graze [23]. As another example, data collected for chemistry projects on fertilizer application could be used to study the leaching of nitrogen and phosphates and eutrophication of local water bodies. GIS will allow the students to incorporate distance from fertilizer application and topographical variables to the chemical study of the water bodies.

4 Assessment

At the time of this writing, the project is just beginning its second semester, and the data available are very preliminary. In the late spring of 2002, after our first round of evaluation, substantially more data will be available, including comments from outside evaluators who will be reviewing student projects, testing instruments, faculty and student reactions and criticisms, and the overall contribution of the project to the mission of the college. Thus, a more substantial interim assessment of the project will be presented at the conference.

The data currently available are student survey responses from the beginning and end of project-related courses in Fall 2001. The surveys asked students to respond to statements about their academic interests, their beliefs about the degree of connectedness between math and the sciences, their understanding of inquiry-based learning, their ability to cite examples of the use of mathematics and statistics in science, their facility with writing and library research on scientific issues, and so on. Analysis of pre- and post-responses to the mathematics and statistics questions gives some evidence that the courses are making progress toward meeting the project's curricular goals. For instance, in response to the statement "I have a clear idea of the role that mathematics plays in scientific research," students responded on a scale of 1 (strongly agree) to 5 (strongly disagree). The mean difference in responses from the beginning and the end of the semester was significant at the 10% level, and two of the three other statements specifically geared towards mathematics and statistics generated similar differences in responses.

Our sample was relatively small, and many students had completed only one course of our 4-course requirement (2 mathematics, 2 natural sciences, including one with an environmental emphasis), so while the data are by no means conclusive, we are encouraged that our courses seem to be contributing to students' increased belief in the interdependency of mathematics and the sciences. By the end of the sophomore year, when most students will have completed the 4-course mathematics and science core requirement, we hope to see more conclusive data.

5 Challenges

Specific to mathematics and statistics courses, a major challenge is getting the students involved in hands-on data collection. Unlike most science courses, which include a three hour lab in addition to three hours of "lecture," mathematics and statistics courses generally meet three times each week for a total of four hours. The short class periods make field work nearly impossible, and a lack of laboratory space makes it difficult to gather data in the lab. Thus, we have been unable to employ as many hand-on activities as we would like. The students still have opportunities for discovery in many of the projects, but the discovery is structured, with data provided instead of gathered.

Overall, the biggest challenge to the project may be the disciplinary training of the faculty members involved. Universally, we are willing to take the intellectual risks necessary to teach and learn outside of our disciplines, but many are concerned about the consequences of these choices at tenure and promotion time. Out of eleven mathematics and science faculty members involved in the project, only one has tenure, so the risk to individuals is considerable.

6 Conclusion

As a residential liberal arts college, the Honors College of Florida Atlantic University strives to provide students with a broad education, to demand critical thinking, to promote inquiry across disciplinary boundaries and to engender the desire for life-long learning. Our project has potential to make outstanding contributions to the mission of the college, by engaging students in discovery-based, interdisciplinary projects, and by providing faculty role models who, on a daily basis, exhibit the process of inquiry-based discovery, of continuing education, and of building and maintaining cross-disciplinary collaborations. Moreover, we believe the use of data gathered in student and faculty projects creates a sense of student ownership of the curriculum, and helps build a sense of community among students and faculty in mathematics and the sciences.

A collection of links to materials we have adapted or adopted for use in statistics and calculus, as well as some of the materials the college has developed, is available at `http:www.fau.edu/~sfitchet/ccli/ccli.html`.

7 Acknowledgements

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ELECTRONIC ACCESS TO LITERATURE IN MATHEMATICS EDUCATION

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ABSTRACT

Fast and comfortable access to literature in teaching mathematics at all levels is discussed. For this purpose the most important bibliographic database for theory and practice in mathematics education, the database MATHDI, is presented in more detail. The literature search in MATHDI is exemplified by identifying literature relating to the topic of this conference.

Mathematics Education Subject Classification of ZDM (MESC): A50

Keywords: mathematics education, information retrieval, Databases, online searching, online systems, educational research, abstracts

1. Introduction

There has been a substantial increase in publications dealing with research in mathematical education in general and in particular on experiments in various countries, new pedagogical concepts and insights, topics, and teaching concepts. One of the features of the growth is the increasing number of conference proceedings (this conference is an example of new conferences established during the last decade), collections of papers, reports, etc. being published. The penetration of calculators and computers in education led to the creation of whole new areas of research. Another aspect is the expansion of journals in this field in both number and page count. Journals are of great importance for everyone interested in national developments as well as in an international exchange of ideas. About 400 journals on mathematics education and/or computer science education serve worldwide as channels for scientific communication (see an overview in http://www.fiz-karlsruhe.de/fiz/publications/zdm/zdmzs.html).

This ever increasing flood of information is a problem encountered in most fields of science: for example, some 120,000 books and papers on physics and engineering are published every year and some 60,000 on mathematics and its applications. It is well known that the production of what we may call scientific literature will continue to increase exponentially unless there are drastic changes in the practice of scientific research. Educational professionals like other scientists are thus faced with the problem of how to extract those items which they need for their own work from a vast pool of potential information.

The purpose of this paper is to provide an insight into how to cope with this flood of information. The reader is given some information on the international services which may help him or her keep up to date with the current progress in elementary mathematics and mathematical education: abstracting journals and on-line databases.

2. On-line Literature Databases in Mathematics Education

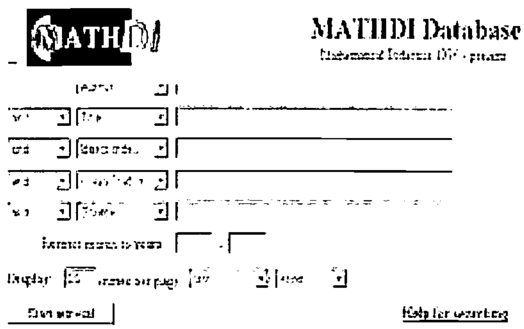
If one wants to read research studies in mathematics education or experience reports on proposed curriculum changes, how does one find them?

It is possible to look for published research in mathematics education by browsing through recent issues of internationally well-known journals such as *Journal of Research in Mathematics Education,* or *Educational Studies, or ZDM* (see paragraph 4), or by scanning national core journals. Browsing is haphazard at best and a time consuming method for searching for a particular subject. Searching so-called bibliographic databases – automated indices of published literature – is the most efficient and effective way to identify literature of relevance to a particular question or aspect.

2.1 Database MATHDI

The most important bibliographic database for research in mathematics education is MATHDI (MATHematical DIdactics) produced, designed and offered by FIZ Karlsruhe. On the Internet MATHDI is offered through the World Wide Web via the EMIS service of the European Mathematical Society (EMS). The rich array of useful literature information is available through http://www.emis.de. Another possibility of availability is through the host STN International.

MATHDI provides the quickest and most convenient access to literature in mathematics education and computer science education. It contains literature reviewed since 1976, totaling 95,000 references (31.12.2001). Some 6,000 items are added each year.



Subject coverage of MATHDI:

- Research in mathematics education,
- methodology of didactics of mathematics,
- mathematical instruction from elementary school to university teaching and teacher training,
- elementary mathematics and its applications,
- computer science education,
- basic pedagogical and psychological issues for mathematics and science education.

MATHDI is intended for:

- didacticians of mathematics in research and education,
- trainers and lecturers,
- educational technologists, instructional designers, and curriculum experts,
- policy-makers and educational administrators,
- teachers in general, special and vocational schools,
- librarians and information specialists.

2.2 Other Bibliographic Databases of Interest

Another important bibliographic database for research in mathematics education is ERIC. Sponsored by the U.S. Department of Education, the Education Resources Information Center database contains more than a million citations to education related documents and journal articles. It covers educational research at all levels in all subjects published as journal article or report or dissertation. The bias is on US-American publications.

Other databases of interest are PsycINFO and Zentralblatt MATH. PsycINFO is produced by the American Psychological Association and covers international literature in psychology and related behavioral and social sciences, including education. Zentralblatt MATH is also multilingual and international in scope. It contains references to the worldwide literature drawn from more than 2,300 journals and serials, from conference proceedings, reports, and books. Zentralblatt MATH input is about 65,000 items per year, produced by more than 7,000 scientists. Although the emphasis is on pure and applied mathematics literature on undergraduate mathematics is indexed as well.

Summarizing, if a mathematics educator needs an overview on relevant scholarly publications for writing an article, delivering a conference paper, or approaching a new working field, the search in MATHDI, complemented sometimes by the other databases mentioned, will help to be up to date. Especially with the computer on-line search the searcher has almost unlimited flexibility to tailor the results to precise specifications, to be as broad or as narrow as desired, to include or exclude certain factors, or to combine search terms.

3. CD-ROM MATHDI

MATHDI is also available on a CD-ROM. This alternative electronic medium of scholarly information offers the following attractive features:

- reviews and bibliographic data from MATHDI, from 1976 to 2000 (about 90,000 citations in mathematical education),
- time-independent searching,
- no additional costs e.g. telecommunication costs.

CD-ROM MATHDI allows to search with a command language (retrieval language used on the STN International host) or with an independent easy-to-use menu system.

MATHDI

Now, for the first time, one can have instant access, every hour of the day, to literature about mathematics education throughout the world. The CD-ROM MATHDI is the most appropriate medium of output when you need information on your desk, your working place or directly on your computer. It is also comfortable in libraries for students use.

4. Printed Abstracting Service in Maths Education: ZDM

MATHDI is the online computer file of bibliographic information compiled by ZDM. ZDM is the acronym for *Zentralblatt für Didaktik der Mathematik* / International Reviews on Mathematical Education. This well established information and abstract journal started in 1968 within the field of mathematical education and expanded its scope ten years ago to computer science education. The journal is published every two months, each issue containing an *articles section* with articles of particular interest to educational professionals and a *documentation section*.

4.1 Documentation Section of ZDM

The main part of ZDM is dedicated to documentation. The documentation section is an abstract service and reference tool providing ready access to worldwide publications on topics such as mathematics teaching, basic pedagogical and psychological problems, elementary mathematics and its applications as well as computer science education and recreational computing. The information presented is extracted from all relevant documents. The publications are announced in the documentation section by bibliographic data and abstracts mostly in English.

The bibliographic part of ZDM is followed by an *index section*, facilitating pinpointed retrieval of documents according to different criteria: author, subject, corporate and source/affiliation, journal title.

In 1976 its online database MATHDI was introduced. As described in paragraph 2 it provides bibliographic information on entries in ZDM from 1976 to the present.

4.1.1 Mathematics Education Subject Classification (MESC)

To arrange entries in the printed service, a classification scheme for mathematics education was developed in the late sixties. The number and terminology of subject headings have changed over the years. The last revision has been in 1999. All subject categories are represented by a three digit notation, consisting of headings (determined by a capital letter) each with 10 subheadings. In the third position the special field of education is indicated such as primary education, secondary education, vocational education, or teacher education.

Nowadays, the MESC-classification still serves for the ordering of the documentation items in ZDM, another aim, however, is to facilitate searching for a particular item in MATHDI.

A. General
B. Educational policy, system and research
C. Psychology of Mathematics Education
D. Instruction, Goals, Teaching Methods, Assessment, Curriculum Development
E. Logic, Language, Proofs
F. Arithmetic, Numbers, Measures, Ratio
G.-K. Mathematical fields
M. Mathematical Modelling ,Applications
U.Educational Media, CAI, Technology

4.1.2 Languages

English dominates the publishing output in mathematics, but this is not as much the case in mathematics education. Journals published in English-speaking countries restrict themselves with few exceptions to articles written in English. Some scientific journals in mathematics didactics published in other countries prefer to publish mainly articles in English. But mostly, the language of choice for journals in mathematics education is the native language. So, articles of some 30 languages are indexed and reviewed in ZDM/MATHDI.

The bibliographic details for each entry, especially the title, are in the original language. For each title other than English there is an English translation. English is the primary language in the information services ZDM/MATHDI, but abstracts in French, German and Spanish are also included.

4.2 Articles Section of ZDM

The articles section of ZDM is an international journal with contributions in English, French or German. It provides survey articles and state-of-the-art reports on educational problems, discussions of current issues and problems in mathematics and computer science education, literature reports as well as reports on international conferences. Furthermore in the book reviews, selected publications are discussed in detail by experts in this field

One emphasis in the articles section of ZDM is on surveys. Expository writing brings together research results, often treating both theory and applications. It also pays special attention to the consolidation of related results, simplification, and the development of relationships in a general body of theory. It also involves presenting mathematics education research to non-specialists ("Non-specialist" meaning a mathematics educator who is not a worker in the specific area of research being treated). Therefore the articles section of ZDM is useful for many mathematics educators to get overviews on many research areas in maths education.

The articles section of ZDM is published electronically on the Internet via WWW. The full text of the ZDM articles section is online available (as PDF files) free of charge to individuals or institutions which subscribe to the print version of the current ZDM volume.

All other mathematics educators and interested persons can now also retrieve previous articles, free of charge, after the first year of publication. The articles of issues of volume 22 (1990) to volume 31 (1999) can be accessed by everyone via the Internet through the EMS server under www.fiz-karlsruhe.de/fiz/publications/zdm/zdmp1.html. Our chief aim is to extend scholarly communication, and we think the electronic medium offers new possibilities for this purpose.

Enjoy surfing through the articles of ZDM articles section on EMIS and thus get an impression of the developments in mathematics education during the past decade.

5. Literature on Some Conference Themes

The abstracting service ZDM/MATHDI, possibly complemented by other ones, enables specialists in mathematics education to keep up with the literature in their subject by providing them with a manageable source of information on current developments, controversies and advances, selected from virtually the whole of the international literature. In addition ZDM/MATHDI assist in maximizing the use of the time scholars have available for reading. They spend their available reading time scanning core journals and can then use abstracting services covering their field to identify other papers.

In addition online databases can be scanned to highlight trends in research. Mathematics and mathematics education, like other subjects, suffer fashions and a given topic may be an active research area for a time and may then be neglected temporarily. Such a topic is now "The Impact of Computer or Calculator Technology on Mathematics Education". By identifying the annual total of articles published in the past five years one can see an increasing interest in this general subject.

Emerging technologies are changing different aspects of society and science in different ways and their impact in mathematics is ever growing. On the one hand, progress in hardware speed permits the visualisation, simulation and animation of complex systems, and on the other, mathematical software is now able to reason algebraically and symbolically by means of computer algebra systems. In addition educational software in form of geometry software has been developed. This causes a plethora of publications stored in MATHDI dealing with discussions about technology in the twenty-first century classroom, or investigations of the effectiveness of technology-based instruction.

Other trend-topics in research, discovered through searches in MATHDI, are mathematical applications at all levels, teacher education, and innovative curricula.

Here are a few examples of sample searches and the number of documents retrieved in MATHDI:

Title	Number of documents
Use of calculators in grades 10-13	423
Use of CAS in mathematics education	1,320
Teaching with Technology	3,835
Cooperative learning/teaching	444
Computer-assisted instruction in geometry	332
The International TIMS-Study (TIMSS)	186
Writing in mathematics	833
Trends in teacher education	107
Mathematics and other disciplines	2,852
Distance education	259
Curricula Innovations	275

In the following there are some samples of MATHDI records.

ANSWER 1 OF MATHDI COPYRIGHT 2002 FIZ KARLSRUHE
TI Teacher education and investigations into teacher education: a conference as a learning environment.
AU Krainer, Konrad (University of Klagenfurt (Austria))
SO European research in mathematics education I.III. Vol. 3. On research in Editor(s): Krainer, Konrad; Goffree, Fred; Berger, Peter
Forschungsinstitut fuer Mathematikdidaktik e.V., Osnabrueck (Germany) 1999. p. 13-39 of 250 p. Available from Forschungsinst. fuer Mathematikdidaktik, Osnabrueck.
Conference: 1. Conference of the European Society for Research in Mathematics Education (CERME-1), Osnabrueck (Germany), 27-31 Aug 1998
ISBN: 3-925386-55-6
DT Miscellaneous; Conference
CY Germany, Federal Republic of
LA English
IP FIZKA
DN ZD3331967

TI Writing about life: Creating original math projects with adults.
AU McCormick, Karen Hicks; Wadlington, Elizabeth (Southern Louisiana State University, LA (United States))
SO Adult numeracy development. Theory, research, practice.
Editor(s): Gal, Iddo
Creskill, NJ: Hampton Press. 2000. p. 197-221 of 377 p.

Ser. Title: Series on Literacy: Research, Policy, and Practice.
ISBN: 1-57273-233-4
DT Book Article
CY United States
LA English
AB This chapter presents a model for integrating learning of writing, reading, speaking, and listening with learning of mathematical concepts in ways that are meaningful for adult students. Three questions guide this chapter: How can adult educators integrate mathematics and language arts skills so that students perceive learning as a whole rather than in distinct, isolated parts? How can adult educators make mathematics relevant to students' daily lives so they become confident, competent problem solvers? How can adult educators provide activities that teach language processes and mechanics in such a way that learning is transferred to other areas, including mathematics and real life?
CC *M18 MATHEMATICAL MODELLING. INTERDISCIPLINARITY (FURTHER EDUCATION)
ST INTERDISCIPLINARY APPROACH; ADULT EDUCATION; FURTHER EDUCATION; LEARNING; AFFECTIVE VARIABLES; WORD PROBLEMS

ANSWER 3 OF 135 MATHDI COPYRIGHT 2002 FIZ KARLSRUHE
TI Reflections on the changing pedagogical use of computer algebra systems: assistance for doing or learning mathematics?.
AU Pierce, Robyn (University of Ballarat, VIC (Australia)); Stacey, Kaye (University of Melbourne, Parkville, VIC (Australia))
SO Journal of Computers in Mathematics and Science Teaching. (2001) v. 20(2) p. 143-161.
CODEN: JCMTDV ISSN: 0731-9258
DT Journal
CY United States
LA English
AB This article documents a change in the use of a Computer Algebra System, (CAS), with a group of first year, undergraduate, mathematics students. CAS was initially used as an assistant for doing mathematics, enabling students to solve difficult problems. During the period of the study it came to be used as an assistant for learning mathematics, as a partner in the teaching and learning process. This article notes the changes required in organisation, teaching materials, and assessment, then reflects on changes in students' attitudes and learning outcomes. Surveys, interviews and teacher observations suggested that students' attitudes toward the use of CAS for learning mathematics were positive and that they believed that it aided their understanding. Students appreciated the availability of CAS for examinations. There was no demonstrable change in student achievement resulting from the changed pedagogical use of CAS. However changes in learning goals and assessment procedures mean that no simple comparison is possible.
CC D35 OBJECTIVES OF MATHEMATICS TEACHING (UNIVERSITIES, COLLEGES, POLYTECHNICS)
ST COMPUTER ALGEBRA; TEACHING METHODS; TEACHING-LEARNING PROCESSES; MATHEMATICS AND COMPUTERS

And here an example from the CD-ROM MATHDI

ZD3264099; MI1460482

Mathematics and new technologies. Matematica e novas tecnologias.

Ponte, J.P.; Canavarro, P.

Lisboa: Universidade Aberta. Aug 1997. 344 p.

ISBN: 972-674-207-2

Book

Portugal

Portuguese

This book is a resource for teacher education in new information technologies. It discusses the role of information and communication technologies (ICT) in society, in the activity of professional occupations and in education. It also analyses the relationships between ICT and mathematics, specially concerning scientific research and technological applications. It pays attention to the use of ICT in mathematics teaching, with reference to a number of curriculum topics and providing classroom examples. It also presents software and equipment useful for mathematics education.

6. Concluding Remarks

Scientific work depends mainly on information and exchange of ideas. In this time of abundant information there is a need to get a quick overview over relevant published articles or books in mathematics education, either in order to locate studies or to get inspired by a classroom experiment, or to be better informed about the accomplishments of scientists working in the same field. To access information in the field of mathematics education you should simply use ZDM/MATHDI with its 95.000 citations.

THE ROLE OF PHYSICS IN STUDENTS' CONCEPTUALIZATIONS OF CALCULUS CONCEPTS: IMPLICATIONS OF RESEARCH ON TEACHING PRACTICE

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ABSTRACT

This paper discusses the implications of research on undergraduate calculus learning for calculus teaching practice. In particular, this paper addresses the challenges of using research results and adapting instructional materials in diverse classroom settings, and aligning research-based conceptions of teaching with practice. In a previous research study, the author investigated students' use of physics experiences and concepts as they construct calculus concepts in an interdisciplinary calculus and physics course. The results of this study suggest that students frequently draw upon physics experiences and concepts as they develop understandings of average rate of change, but that students less frequently make use of physics experiences as they develop understandings of derivative and antiderivative. In addition, other researchers have alluded to the importance of prior physics experiences on students' conceptualization of the average rate of change concept (e.g. Nemirovsky & Noble, 1997). However, implications of these research results for calculus teaching practice have received little attention. The combination of the author's research findings and those of other researchers suggest four major implications for calculus teaching practice. Research results: 1. Modify pre-existing instructional design theory. 2. Influence the design of classroom activities and development of learning sequences. In particular, research provides information about the potential mismatch between the experiences students bring with them to the classroom and teachers' assumptions about students' past experiences. 3. Influence how the teacher conceptualizes the role of the students in the classroom community. 4. Alter the role of technology in the classroom. The results of the author's previous research as well as other research results are discussed relative to these four outcomes for undergraduate calculus teaching practice.

1. Introduction

This paper discusses how the results of recent research on calculus learning inform new ways of teaching undergraduate calculus. The broad question that this paper discusses is: What do we know about the learning of calculus and how does that knowledge inform calculus teaching? Specifically, this paper presents a discussion of recent research on the role of physics understanding in the learning of calculus concepts and the implications of research for the calculus curriculum. This paper is divided into two sections and reflects the continuum of research informing practice and practice informing research. The first part of this paper describes research that grew out of the author's evaluation of an interdisciplinary calculus/physics program (practice informing research). The second part of this paper discusses the implications for practice of the results of the research described in the first part of the paper, along with other recent research in this area (research informing practice).

2. The role of Physics in Students' Conceptualizations of Calculus Concepts

Background and Purpose

Students' understanding of calculus concepts lays a foundation for their future study of advanced mathematics, science, and engineering courses. The idea of change – both how things change and the rate at which things change – plays a particularly important role in students' conceptualizations of calculus concepts. Students must understand the concept of rate of change in order to understand the derivative and differential equations. Furthermore, students must understand the idea of total change to understand the integral. Finally, students must understand the relationship between rate of change and total change in order to understand the relationship between derivatives and integrals outlined by the Fundamental Theorem of Calculus.

In order to grasp abstract ideas of rate of change, students might rely on physical interpretations of change (Nemirovsky, Tierney, & Ogonowski, 1992). Students may have encountered some of the underlying calculus concepts informally in everyday life; thus students often enter the calculus classroom with some intuition about concepts such as rate of change and derivative (Nemirovsky & Rubin, 1992; Nemirovsky & Noble, 1997). Furthermore, many students experience the mathematical concepts of average rate of change, derivative, and integral in physics classes as they study concepts such as motion, force, and electricity.

Physics, a typical introductory course for most engineering, science, and mathematics students, provides a context for which students can study change in a concrete setting. Research has shown that mathematics understanding enhances the learning of physics concepts (Hudson & McIntire, 1977; Champagne, Klopfer et al., 1980), and more recent research has begun to examine how physics understanding affects the learning of calculus concepts (Thompson, 1994; Marrongelle, 2001).

During the late 1990's the National Science Foundation of the United States of America funded several initiatives aimed at exploiting connections between mathematics and other disciplines at the undergraduate level. One such project, which took place at a large, public research university in the Northeast, integrated the curriculums of differential and integral calculus and introductory calculus-based physics (NSF-DUE 9752650). The integrated Calculus/Physics program was offered to first-year students as an alternative to enrolling in separate calculus and physics classes. The program was developed during the spring and summer of 1998 and first offered to students during the fall of 1998. The Calculus/Physics program development was informed by recent research in the areas of calculus and physics learning (c.f. McDermott, 1984; Ferrini-Mundy & Graham, 1994), ideas from the work in Cognitively Guided Instruction (Greeno, 1997), and research in the area of problem-solving (Larkin, 1980; Schoenfeld, 1985; Arcavi, Kessel, Meira, & Smith, 1998).

The ordering of the calculus and physics topics contributes greatly to the integrated nature of the curriculum. The curriculum is designed for the students to see the applicability of the calculus as they learn it and conversely that they have all the mathematics they need to solve physics problems. In order to coordinate the calculus and physics topics in the class, the presentation of calculus topics was reordered. The four basic threads of calculus (function, continuity, derivative, and integral) are discussed first for polynomial functions only and then again for other classes of functions (logarithmic, exponential, trigonometric) as they arise in the physics curriculum. This reordering of the calculus curriculum allows for the presentation of the physics and calculus content in a more unified way and gives the mathematics a rich context.

The author began examining students' learning in this context in her role as an evaluator of the calculus/physics program. As part of the evaluation of the Calculus/Physics program, the author conducted clinical interviews with students enrolled in the Calculus/Physics class and student enrolled in a traditional[1] calculus course. An analysis of the clinical interview data uncovered differences between the manner in which Calculus/Physics students and traditional calculus students approached average rate of change and derivative tasks. The Calculus/Physics students tended to use physics terminology and concepts as they solved average rate of change and derivative tasks. The traditional calculus students, who were either concurrently enrolled in a physics class or had completed a physics class, tended to rely on their memorization of mathematical formulas and processes as they solved average rate of change and derivative tasks. The Calculus/Physics students seemed to make more connections to their knowledge of physics as they solved the average rate of change and derivative tasks than the traditional students.

Research Question and Theoretical Perspective

As a result of her work evaluating the Calculus/Physics program, the author designed a research study to investigate the question: How do students draw upon physics concepts to inform their understanding of rate of change, derivative, and integral? The study was guided by the basic constructivist view that knowledge is constructed through a process of experience and reflective abstraction (Noddings, 1990). The consequence of holding a constructivist perspective is the assumption that mathematics is built from human activity; thus students informally experience mathematical ideas in their day-to-day, culturally situated experiences.

The theory of *transitional tools* was employed as a means to analyze and discuss role of physics in the students' constructive activity Nemirovsky and Noble (1997) put forth the notion of

[1] 'Traditional' is used here to describe those students who were not enrolled in the integrated Calculus/Physics program at the university.

transitional tools as part of their emerging psychological perspective that allows for the analysis of an individual's constructive activity by challenging the convention that any given object or picture must reside either inside or outside a person's mind. By rejecting the notion that a visualization must be either internal or external, Nemirovsky and Noble (1997) overcome the common difficulty that arises from the need to describe whether a visual image is internal or external to the student. Transitional tools are experiences or objects in the environment that both separate the learner from another physical object and strengthen his/her understanding of the object using mathematical contexts such as symbols and graphs. For example, a student who talks about the motion of a cart on a track to help him/her conceptualize properties of the derivative is using the cart and track as transitional tools. Note that the cart and track are tools that reside both internally (in the student's memory of the cart's motion on the track) and externally (the physical existence of the cart and track).

Recent Research in Calculus Learning

Much of the research on calculus learning has shown that students are able to successfully carry out methods of differentiation and integration but sometimes lack the conceptual underpinnings necessary to explain procedures, work though problems using multiple strategies, and make connections between concepts (Orton, 1983; Vinner, 1989; Ferrini-Mundy & Graham, 1994). Throughout the literature, researchers have alluded to the importance of prior experiences on students' conceptualizations of calculus concepts (Thompson, 1994; Nemirovsky & Noble, 1997; Noble, Nemirovsky, Tierney, & Wright, 1998). These experiences refer to both mathematical and non-mathematical episodes and situations encountered both in and out of the classroom. Additionally, some researchers have stressed the need for investigations into the effects of introducing substantial physical examples and applications in the calculus course (Ferrini-Mundy & Graham, 1991).

More recently, Ricardo Nemirovsky has undertaken a number of projects aimed at investigating the effects of physical graphing devices on students' calculus learning. Nemirovsky, Tierney, & Wright (1998) found that students will broaden their use of motion graphing devices as they become more familiar with the graphing devices. Steve Monk, Ricardo Nemirovsky, and Paul Wagoner have developed a set of computer controlled interactive physical devices. The devices enable students to explore calculus concepts in a physical context. Underscoring these projects is the assumption that students' physics experiences and knowledge will shape how and what they learn.

Methodology

Over a two-semester period, eight first-year students enrolled in the Calculus/Physics program participated in this study. Students were invited to participate in this study based on their reported backgrounds in mathematics and physics. The goal of the participant selection was to generate a sample of students whose range in abilities spanned the abilities represented in the Calculus/Physics class. Seven males and one female participated in the study. The gender balance in the study reflected the gender balance in the Calculus/Physics class.

This study utilized a multiple case study design with analysis by and across cases. Data sources included: four audio taped, task-based interviews with each student; classroom participant-observation; and photocopies of students' class notes, in-class activities, homework assignments, and examinations. The interview tasks were designed to help reveal students' ways of thinking about average rate of change, derivative, and integral. All of the interviews were transcribed and all data was coded into categories using micro-analytic and constant comparison methods (Strauss & Corbin, 1998).

Transcripts of the students' initial interviews were selected as the primary data source for the microanalysis since these were the earliest pieces of data collected. It was necessary to conduct a microanalysis on early pieces of data in order to generate a scheme by which to classify the ways in which the students used physics as they solved calculus problems. The classification scheme was further tested and refined with data collected later in the year. The classification scheme will be discussed in more detail in the following section.

A within-case analysis of each student was conducted in order to test the stability of the classification scheme that emerged during the microanalysis. The classification scheme (See Table 1) was used to analyze transcript episodes, students' homework assignments, classroom activities, and examinations. Selected pieces of the data were re-coded by three independent raters to check for inter-rater reliability.

Results

The results of this study indicate that when students used physics as transitional tools, they used physics in one of four (not necessarily disjoint) ways: Contextualizers, Example-Users, Mis-Users, and Language-Mixer (see Table 1).

Physics Use	**Description**
Contextualizer	Student works and talks through calculus problems as if it were a physics problem. Majority of technical vocabulary used to solve problem is physics terminology. There is evidence that student is thinking about the problem in terms of physics.
Example-User	Student uses physics examples to justify solutions to problems or to help make sense of part of the problem. Actual problem at hand is solved using mathematical concepts. Student does not submerge the problem in a physics context. Majority of technical vocabulary is mathematical terminology.
Mis-User	Student's use of physics misconceptions interferes with student's solution to the problem. Student uses physics misconception to incorrectly solve the problem at hand.
Language-Mixer	Student intersperses physics and calculus terminology as he/she solves problem. Student does not immerse problem in physics context or use physics examples to justify solutions or help make sense of problem. Rather, student intermingles physics and mathematical language as he/she solves the problem.
Non-User	Student does not use physics concepts to language to solve calculus problems.

Table 1: Physics Use Classification Scheme

Contextualizers show evidence of immersing calculus problems in a physical context in order to solve them. Example-Users use physics examples to justify solutions to calculus problems or help make sense of part of a problem, but do not show evidence of immersing the problem in a physical context. Mis-Users allow physics misconceptions to interfere with the solution to calculus problems. Language-Mixers intersperse mathematics and physics terminology as they solve calculus problems but show no evidence of using physics other than a communication tool. Another interesting finding is that students frequently used physics concepts as transitional tools to construct meaningful conceptualizations of average rate of change but less frequently drew upon physics concepts as transitional tools to aid in their understanding of derivative and integral.

A discussion of the implications of these research findings for teaching and curriculum development will be presented in the next section. The results of the research study described above, as well as the results of previously reported research have been synthesized in order to identify areas of possible modification in undergraduate teaching practice and curriculum development.

3. Implications for Practice

A number of observations arise here that suggest implications for calculus teaching practice and curriculum development.

1. Instructional design theory should reflect the notion that students build their mathematical understanding from human activity. Students often encounter calculus concepts through participation in physical situations in their environments. The 'human' aspect of calculus concepts is often ignored in the design of calculus course. Instructional design theory needs to incorporate students' prior experiences with calculus concepts.

For example, the instructional design theory of Realistic Mathematics Education is rooted in Freudenthal's interpretation of mathematics as a "human activity" (Freudenthal, 1991). From this perspective, students should learn mathematics by mathematizing subject matter from realistic situations (i.e. from context problems or from mathematically real objects for students) and by mathematizing their own mathematical activity. As a global, guiding theory, Realistic Mathematics Education provides a framework for considering the use of context problems and the role of mathematization in the learning process.

2. Curriculum changes should be made to address the close connection between calculus and physics as well as students' reliance on physics to help them make sense of calculus concepts. While integrating calculus and physics in a single course is not always possible or appropriate, the physical context of calculus cannot be ignored. The calculus curriculum should reflect that students' experiences with physics are valued and an important part of the learning process. Currently, many calculus textbooks continue to use physical examples only as 'applications' of calculus or as a follow-up to discussions about calculus concepts. Physics concepts and examples should be used to *initiate* discussions about calculus concepts, especially in calculus classes designed for science and engineering students. Drawing on students' previous experiences with physics will help students create more meaningful conceptualizations of calculus concepts. This suggestion should not lead to the exclusion of other calculus applications from the curriculum, such as biology, business, and economics. Because the results of the current research project were based on data collected from students with a predisposition to physics, further research is necessary to determine if the results of the present study are generalizable to a larger student population. Specifically, future research should investigate the effects of a physics-based approach to calculus on students majoring in such areas as business, economics, and biology.

3. Students' previous experiences and knowledge should actively shape the classroom learning environment. The research on calculus learning underscores the idea that "calculus students will actively formulate their own theories, build their own connections, and readily construct meaning for problem situations" (Ferrini-Mundy & Graham 1994, p. 43). Thus, the calculus curriculum should be informed by the experiences and knowledge that students bring with them to the classroom. In particular, students must be afforded opportunities to link their past experiences with physical phenomena to calculus concepts. Students should be given opportunities to share their experiences with and knowledge about calculus concepts, validating the student's role as a learner. Additionally, as students discuss their experiences and knowledge, they will begin to consider calculus concepts from multiple perspectives.

4. The role of technology in the classroom should be under continual modification. Students in the present study used motion detectors, graphical interfaces, and graphical software to create, analyze, and explore properties of derivative and anti-derivative functions. The use of such technological learning tools, while standard in many physics laboratories, has only recently been explored in mathematics classrooms (Nemirovsky, Tierney, & Wright, 1998; Huetinck, 1992). Motion detectors allow students to visually, audibly, and kinesthetically engage with calculus concepts. However, using motion detectors and related hardware and software requires additional funds and space for students to move about the classroom. Not every school or classroom can accommodate technology such as motion detectors. New technological advances as well as web-based programs are allowing for easier interactions with powerful teaching tools similar to the motion detector. As new technological tools are made available, educators need to consider their appropriate use in the classroom.

4. Summary and Conclusions

The combination of the author's research findings and those of other researchers suggest four major implications for calculus teaching practice. Research results suggest: 1. The modification of pre-existing instructional design theory. 2. Influence the design of classroom activities and development of learning sequences. In particular, research provides information about the potential mismatch between the experiences students bring with them to the classroom and teachers' assumptions about students' past experiences. 3. Influence how the teacher conceptualizes the role of the students in the classroom community. 4. Alter the role of technology in the classroom.

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USING WEB-BASED INTERACTIVE GRAPHICS TO ENHANCE UNDERSTANDING OF PARAMETRIC EQUATIONS:
Lessons Learned

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ABSTRACT

Like many instructors, we have increasingly employed web-based graphics both for in-class demonstrations and for use by students on out-of-class assignments. Web-based interactive graphics have the potential to enrich learning in ways that print resources lack. Furthermore, web-based interactive graphics offer several advantages over other technological devices: they can be more adaptable than graphing calculators, can use familiar and readily accessible interfaces (web browsers), and (in principle) can be run on any computer using any web browser (either from remote websites or from local secondary storage). In the process of employing such web-based interactive graphics, we have learned some lessons about issues such as: "How do students interact with Web-based interactive graphics?" and "What kinds of activities facilitate learning with such graphics?" In this paper, we will show examples of web-based interactive graphics that we have used and we will offer experience-driven recommendations for future implementation and development.

1. Introduction

The use of technology in teaching college mathematics is a pressing issue, currently with more questions than answers. Although some instructors continue to resist the use of technology, for many of us the issue is not whether to use technology but how to use it in efficient ways for effective student learning. The authors of this paper hope to fuel conversations about the development of technology, the evaluation and use of technology, and circumstances under which various media are appropriate for teaching and effective for student learning.

The authors have a history of integrating graphing calculators and computer algebra systems (e.g., *Mathematica*) into calculus courses. These experiences, combined with conversations with other instructors and researchers, have led us to conclude that there are activities for which graphing calculators are insufficient but computer algebra systems are excessive. For example, many standard college algebra and calculus courses introduce curves given by parametric equations x(t) and y(t). Students are often puzzled by the relationship between the parametric equations and the resulting curve. Graphing calculator features such as TRACE provide students some ability to explore relationships dynamically. In addition, graphing calculators can be used to examine the graph of x as a function t and the graph of y as a function of t. However, it is unwieldy on graphing calculators to simultaneously display all three graphs (x, y, and the curve given parametrically by x and y) to examine relationships graphically. On the other hand, students need not go so far as to become proficient with a computer algebra system in order to gain the insight they seek.

Web-based interactive graphics have the potential to fill needs in these areas because they

- can be more adaptable than graphing calculators,
- can use familiar and readily accessible interfaces (web browsers), and
- (in principle) can be run on any computer using any web browser (either from remote websites or from local secondary storage).

A Java applet, for example, is a computer program ("applet" means "small application") that can be run as part of a web page. QuickTime movies are another, perhaps more familiar, example of such programs. The web browser plug-ins needed to run Java applets (and QuickTime movies) are free and, with well-designed applets (see the mathlet review criteria at the JOMA website), users can get started more quickly than they can with computer algebra systems, which tend to have considerably steep learning curves.

The *Journal of Online Mathematics and its Applications* (JOMA) defines a "mathlet" to be a "small, interactive, platform-independent tool for teaching math" (see the JOMA website). Many mathlets are Java applets. In this paper, we restrict our attention to mathlets that are not only interactive but also graphical and that can be run via web browsers. In section 2, we offer an example of how web-based interactive graphics might contribute to teaching and learning in combination with other media. In section 3, we share thoughts on some of the lessons we have learned, including some advantages and disadvantages of web-based interactive graphics. Finally, in section 4, we make recommendations for future work in this area. At the end of the text, we provide a list of related websites.

2. A Role for Web-based Interactive Graphics

In June 2000 and 2001, Murphy attended the Mathematical Java Workshops and Conference at Emporia State University in Kansas, U.S.A. Faculty members Joe Yanik and Chuck Pheatt of

Emporia State University (with funding from the National Science Foundation) had written a collection of Java classes, The MathToolKit, with the intention of teaching mathematicians to use the toolkit to produce Java applets for use in teaching mathematics (see the Mathematical Java website). Using the MathToolKit, Murphy wrote the Parametric Curves Applet (currently, version 3.0) shown in Figures 1 and 2, for use in her calculus classes at the University of Oklahoma (OU).

This applet behaves much like a graphing calculator, allowing the user to input parametric equations x(t) and y(t) as well as to adjust the viewing window. Unlike a graphing calculator, the applet can simultaneously display, on separate sets of axes, the graphs of t versus x(t), t versus y(t), and x(t) versus y(t). The user has additional options to (1) show the progression of points being plotted on all three graphs as t increases (Figure 1); (2) trace simultaneously along all three curves; and (3) trace simultaneously along all three curves with tangent lines (Figure 2). More details about revisions inspired by use are offered in the "lessons learned" section below.

Fall 2001 was the first semester during which Murphy used (an earlier prototype of) this applet with her calculus classes. To give some indication of possibilities for using multiple kinds of media (e.g., chalkboard, graphing calculators, applets), each lending itself well to some activities and less well to others, we describe one of the class sessions:

Projected on an overhead screen was the context for the problem (designed by White): "Milo the Mouse is out for a walk. The coordinates of his position at a time t (in minutes after noon) are given by the equations $x(t) = t^2 - 1$ and $y(t) = t^3 - 5t$."

> Reason for choosing overhead projector: keep the context available for the duration of the exercise; save time by not having to write it out during class.

With Murphy as scribe at the chalkboard, the class calculated Milo's position at times t=0, t=1, and t=2, plotting the corresponding points on an xy-coordinate system on the board.

> Reason for choosing chalkboard: build computational understanding of the relationship between the parametric equations and the resulting curve; writing calculations and plotting points all in one convenient area.

Murphy pointed out that x is a function of t and, with the Parametric Curves Applet, graphed that curve (upper left graphing space in Figure 1). She asked the students to describe orally the shape of x(t), emphasizing the use of "increasing" and "decreasing" rather than "up" and "down". After doing the same for y(t), she chose the "plot points" option in the applet. This option shows, as the user types the $\boxed{\rightarrow}$ key, the progression of points being plotted on all three graphs as t increases (Figure 1). As she typed the key, Murphy asked the students to predict, based on the behavior of x(t) and y(t), what the corresponding behavior of Milo's path would be: e.g., as x(t) decreases and y(t) increases, Milo should be heading left/west and up/north. She asked them to warn her when they expected a change in direction to occur: e.g., when x(t) has a minimum, Milo changes from heading in a westerly direction to heading in an easterly direction.

> Reason for choosing Parametric Curves Applet: build graphical understanding of the relationships between the parametric equations and the resulting curve; ability to display all three graphs dynamically and simultaneously.

Murphy then showed the students how to graph parametric curves on their calculators and the students spent the rest of the class session working in groups on examples.

Reason for choosing graphing calculator: opportunities to analyze graphs dynamically; students will have regular access to graphing calculators more readily than applets or computer algebra systems.

After the in-class demonstration, Murphy felt that using the applet had been effective, but as she had not tracked the participation of individual students, she really had only her perceptions as a basis for her judgment.

Furthermore, the authors had nagging suspicions that students should interact with the applet for maximal learning. An in-class demonstration (rather than a lab) had been chosen as the initial use for the applet, in part because access to suitable computer classrooms at OU is limited, in part because it was not clear that the applet was "ready for prime time," and in part because we had not yet determined what, if any, activities would be effective to enhance student learning. Thus, we wanted to design an activity that would encourage the students to interact with the Parametric Curves Applet without imposing undue frustration. We finally agreed that, under the circumstances, the activity initially should be an out-of-class extra credit assignment to be completed in groups. Figure 3 gives one version of this activity (written by White). Details about the activity and lessons learned are offered in the next section.

3. Lessons Learned

In addition to using Java applets (along with graphing calculators) with classes, the authors formally gathered data related to the use of technology (by "formally" we mean with approval from the OU Institutional Review Board to use human subjects in research). Data collection consisted of (1) an instructor journal, (2) written student work, and (3) observations of several students working on an activity related to the applet. Specifically, we wanted to address two questions: (I) How do students interact with web-based interactive graphics (i.e., what expectations do students have for the technology)? and (II) What kinds of activities facilitate learning with such graphics? These data informed the redesign of both the applet and the related activity.

(I) How do students interact with web-based interactive graphics?

(A) Java applets written using the Emporia State MathToolKit require a particular (free) browser plug-in (see Murphy's Calc 3 website for details). Prior to the Fall 2001 semester, Murphy made sure that the student computer labs at OU had the plug-in properly installed so that students could use the applets in these labs. As Murphy was relatively new to Java programming herself, and not inclined to become an expert, she hoped that this effort would be sufficient. The extra credit assignment related to the parametric curves applets specified, "The applets require a special browser plug-in. Your best bet is to use the applets in the computer lab in PHSC 230 (Murphy has tested them there and knows that they work there – she makes no guarantees about getting them to work anywhere else)." Nevertheless, several students indicated that they had tried to use the applets on their own personal computers. Few realized that they needed the specific plug-in. Surprisingly, almost none of the students attempted to use the applets in the specified student computer lab.

LESSON LEARNED: Students want to be able to work on their own personal computers. Instructors need to know enough about the applets (and other browser-based interactive graphics) to ensure that students can access the needed software.

SOLUTION: One of the students who successfully installed the needed plug-in wrote an instruction sheet, now posted on Murphy's Calc 3 website.

(B) Lacking sufficient experience with Java programming and user expectations, Murphy initially enabled all of the trace features to trace only for increasing values of t, thinking that students would also have their graphing calculators accessible if they wanted other options. However, in observation sessions, students did not simultaneously use their graphing calculators and the applets. Also, thinking that simple was best, Murphy originally separated the tracing options into two applets. In watching students work on the activity, however, we were reminded that users prefer not to switch windows more often than necessary.

LESSON LEARNED: Users prefer to have resources available in one convenient, multi-feature module. If designers believe that a feature is important, then it should not be in a separate, isolated spot. Students also expect web-based interactive graphics to include features and behaviors familiar from graphing calculators.

SOLUTION: Murphy combined the options into one applet and enabled more trace features.

(C) As with graphing calculators and computer algebra systems, Java applets expect input to use specific syntax (e.g., in the MathToolKit, multiplication is represented by an asterisk: 3*t). In addition, Java applets can be somewhat intuitive to use but, as with other technology, they need documentation explaining operation procedures. The web page that houses the Parametric Curves Applet includes instructions detailing the expected syntax as well as explaining the features available. During observation sessions, the students consistently bypassed the instructions, instead going straight to the applet, then asking questions when they got stuck on how to use it. On the other hand, when this item came up for discussion, the students indicated that they did want the instructions provided and that they did not want the instructions on a separate web page (see lesson (I)(B) above).

LESSON LEARNED: Students expect to go straight to using the applet without reading instructions first.

SOLUTION: Murphy moved critical instructions to the applet itself and added syntax error dialog boxes, with references to the instructions provided above the applet. In keeping with student requests not to have multiple windows to navigate among (see lesson (I)(B) above), the detailed instructions were left on the same web page that houses the applet, rather than linked from that page. One option that was not discussed by this project team but that has been implemented by other applet designers is to have detailed instructions linked to small pop-up windows that can be viewed at the same time as the primary window.

(II) What kinds of activities facilitate learning with web-based interactive graphics?

(A) As in Figure 3, the original Milo the Mouse problem asked "Where is Milo at 11:59 a.m.?" We wanted this item to help the students think about the meaning and appropriateness of negative values for t (see lesson (II)(B) below). In practice, however, the students that we observed either calculated the point using just the formulas for x(t) and y(t) or did not answer the question at all. This student behavior raises at least two issues: the first relates to helping students think about appropriate domains and is discussed in lesson (II)(B); the second involves the motivation students have to complete a problem. When this item was asked as a part of the extra credit assignment, all of the groups that did the assignment answered this question. Yet when the activity was used just during an observation session, with no stakes attached, students were inclined to skip it, thinking (for the most part, correctly) that they already knew how to answer it and did not need to practice that skill. If the students do not complete the item, then our "hidden agenda" for the item is lost. This raises concerns about using such an activity as an ungraded in-class lab.

LESSON LEARNED: Students may not complete items that they believe they know how to do unless there is a reward for doing so.

SOLUTION: One possible solution is to grade the activity. Another solution is to write problems that students believe will contribute to their learning.

(B) Originally the first Milo the Mouse question did not include parts (b), (c), or (d) as in Figure 3. The Parametric Curves Applet has as a default domain for t the interval from 0 to 2π, as most graphing calculators do (see lesson (I)(B) above). Using this default domain, students believed that we had mis-worded the question because the graph of x(t) did not appear to go from increasing to decreasing since that part of the x(t) curve shows up when the domain includes negative values for t. We had hoped that the question, "Where is Milo at 11:59 a.m.?", which preceded the increasing/decreasing item, would prompt the students to alter the domain from the default to a domain that included negative values. As noted in lesson (II)(A) above, the students calculated x(-1) with the x(t) formula. Apparently, the students did not make the connection from this exercise to the idea of altering the domain for t. A related phenomenon appears to occur when students graph parametric curves on their calculators. For example, if an exercise asks students to find the area enclosed by a loop, but the loop does not show up with the default domain, the students get confused. For instructors, one question triggered by these observations is: What activities will help students to consider altering a domain in order to see a different part of a curve?

LESSON LEARNED: As when they use graphing calculators and other technology, calculus students do not automatically analyze whether they have an appropriate domain.

POTENTIAL SOLUTION: We changed the problem to include parts (c) and (d). As the part of x(t) that goes from decreasing to increasing shows up using the default domain, we hope that students will at least gain confidence that they can answer such questions, before they get confused by the item asking when x(t) goes from increasing to decreasing. We also added part

(b) to increase the number of instances of negative values of t, hoping that this addition will improve the chances that students will consider negative values of t. [Note: we will test this solution early in March.]

4. Discussion

We are certainly not saying that graphing calculators and computer algebra systems should be usurped by web-based interactive graphics. Rather we want to emphasize that each tool has advantages. Graphing calculators provide substantial functionality all in one small easily portable machine. Computer algebra systems have powerful graphing and computation abilities. Yet, for narrowly focused activities, web-based interactive graphics may a better choice.

We want to emphasize all three aspects: web-based, interactive, and graphical. We have long believed in the power of visualization for student understanding, especially approaches that interweave multiple representations (symbolic, graphical, numerical). On various occasions we have used static graphs as well as dynamic animations (e.g., animated GIFs) to enhance visualization skills. Yet we are convinced that students learn best by doing rather than by just watching. Thus, we prefer tools that allow students to interact with the graphics. Finally, web-based resources can be readily accessible to students and instructors anywhere through any networked computer (and/or can be available on secondary storage such as CDs). To these ends, web-based interactive graphics have advantages over graphing calculators, computer algebra systems, animated GIFs, chalkboards, and print.

Websites

Journal of Online Mathematics and its Applications (JOMA): http://www.joma.org

Mathematical Java website: http://mathcsjava.emporia.edu/

Murphy's Calc 3 website: http://www.math.ou.edu/~tjmurphy/calc3.html

Figure 1. Parametric Curves Applet with "plot points" option selected.

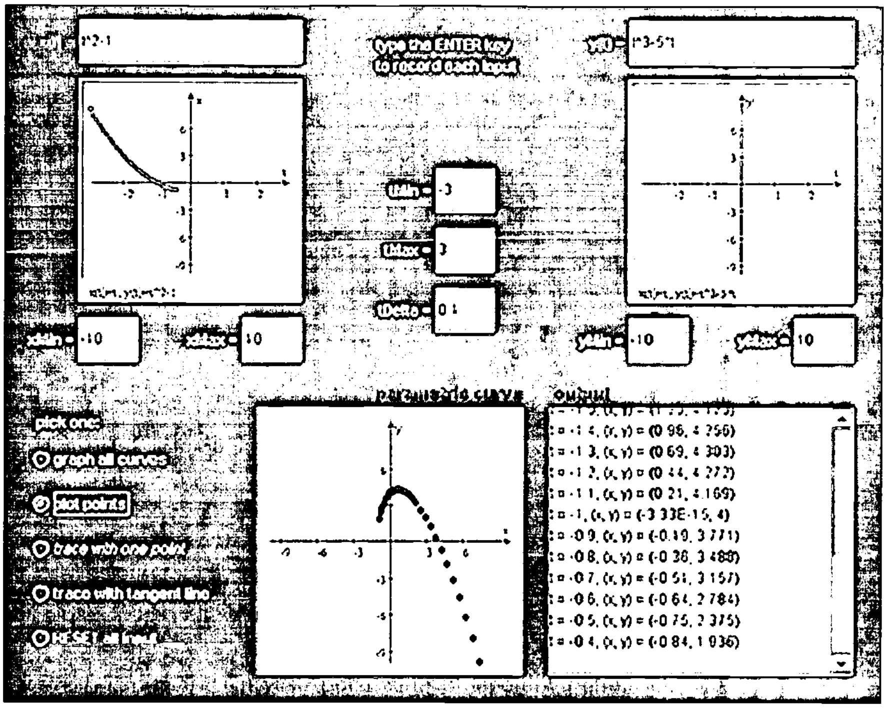

Figure 2. Parametric Curves Applet with "trace with tangent line" option selected.

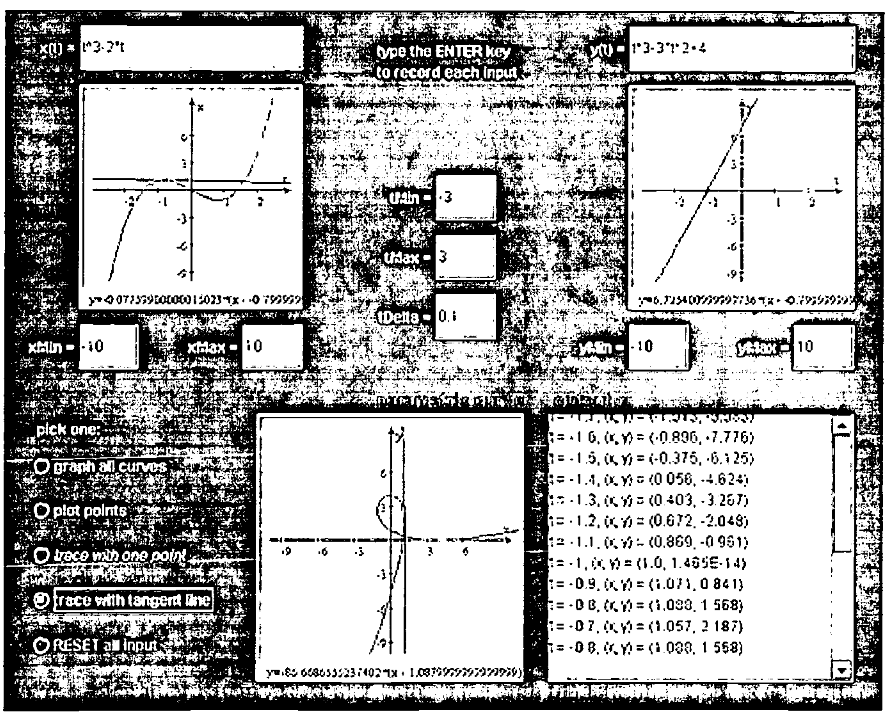

Figure 3. An activity for use with the Parametric Curves Applet.

Parametric Equations Activity

For use with the Parametric Curves Applet at http://www.math.ou.edu/~tjmurphy (follow the 2433 link, then follow the link to the applet). If you are using your own computer, be sure that you have installed the required plug-ins.

For each item below, show and briefly explain your work. Reproduce any graphs you used to think about the questions. Your explanations must include reference to any resources (e.g., people, books, technology) you used and how you used them .

Milo the Mouse is out for a walk. His path is given by the parametric equations

$$x(t) = t^3 - 2t$$
$$y(t) = t^3 - 3t^2 + 4$$

where t is in minutes after noon (or before noon for negative values of t) and where the positive x direction is East and the positive y direction is North.

(a) Where is Milo at 12:01 p.m.? At 11:59 a.m.? (Note: "where" means "at what point (x,y)".)

(b) When is Milo at the coordinates (-4, 16)? (Note: "when" means "at what time t".)

(c) Look at a graph of t versus x(t). At which value of t does x(t) go from decreasing to increasing?

(d) Look at a graph of x(t) versus y(t) (i.e., look at Milo's path). When does Milo stop heading West-ish and start heading East-ish?

(e) Look at a graph of t versus x(t). At which value of t does x(t) go from increasing to decreasing?

(f) Look at a graph of x(t) versus y(t) (i.e., look at Milo's path). When does Milo stop heading East-ish and start heading West-ish?

(g) What can you say in general about what happens to a parametric graph x(t) versus y(t) at a t value where the graph of t versus x(t) goes from decreasing to increasing? What can you say about t values where the graph of t versus x(t) goes from increasing to decreasing?

(h) What can you say in general about what happens to a parametric graph at a t value where the graph of t versus y(t) goes from decreasing to increasing? Increasing to decreasing?

DEVELOPMENTAL MATHEMATICS

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ABSTRACT

The appearance of developmental mathematics in the United States has brought low-key but still intensive debate about its place in higher education, its content, its methodology, and its costs. This paper presents the reasons for such reaction and examines developmental mathematics programs in post-secondary and higher education institutions. This examination addresses the nature of developmental mathematics, the reasons for its emergence, the philosophical principles on which it is based, its historical background, and the way it is practiced in American institutions. The material presented comes from review of some of the relevant literature, anecdotal information from the writer's colleagues around the country, and the thirty-five-plus years of experience of the writer as a mathematics educator and administrator in public secondary schools, two-year colleges and four-year colleges and universities, as well as from the last five years he has spent as instructor in and supervisor of the developmental mathematics program of Northern State University.

Terms Applying to American Education as Used in this Paper

Campus. The total physical parts of an institution, but it is also used to imply the collection of every entity of the institution (including students, faculty and staff).

College vs. university. Interchangeable terms; college, however, is also used to designate a university school (i.e. college of medicine), and the term college, by itself, may refer to any postsecondary institution, from community college to university.

Community colleges. Two-year comprehensive (academic, technical/vocational, and community service programs) institutions granting associate degrees.

Freshman. First year college student.

General education. A body of courses, in the basic disciplines, required of every college student.

Higher education. Education provided by four-year colleges and universities granting at least bachelor degrees

Open admissions. Matriculating any high school graduate in a college.

Postsecondary education. Education/training beyond high school, but normally used to designate associate degree-granting institutions (community colleges, technical/vocational schools).

Private education. Private and denominational schools and colleges are independent of state control.

Public education. Primary and secondary schools are controlled and funded on the local level, under the supervision of each state. Colleges and universities are funded and controlled on the state level.

Secondary education (high schools). Normally, the last four years of pre-college education.

Introduction

The American education scene is a paradox.

On one hand, the need for well-educated citizens has been constantly increasing since the end of World War II. This is especially true in mathematics and science, because of the demands that galloping technological advances have placed on such education. Yet, on the other hand, the preparation of American high school pupils for entering college is progressively deteriorating—especially in mathematics.

The underpinnings of the American educational philosophy are essentially based on the credo of egalitarianism—all citizens should have an equal opportunity for access to education. In post-secondary education this has been translated by many higher education institutions as open admission. Philosophically, most colleges and universities believe that the poorly prepared high school students should be given a second opportunity. And, in the five-year experience of the writer, such a second chance works for some students.

Besides philosophy, however, there are also practical reasons for this situation. Student body size has direct relationship to financial resources colleges may have. This is especially critical for public institutions, as many states base their financial support on their number of students. Additional students bring more tuition income. Often there is economy in size (larger classes) and possibly some gain in higher prestige and political influence, as the number of students' increases`.

American collegiate institutions require their students to successfully complete general education, which includes at least one mathematics course. Institutions found out that large sections of entering freshmen are not able to complete successfully the study of even college level algebra.

This situation has been disturbing to mathematics and science educators as well as to officials of both state and federal governments. Efforts have been made by the National Science Foundation, the Mathematical Association of America, the National Council of Teachers of Mathematics and others to improve the teaching of mathematics in public schools but, by and large, their results have been spotty at best.

Developmental Education

During the social and civic upheaval in the 1960s and 1970s, concerns were raised about the manner in which higher education institutions were treating their students. Universities reacted by enacting student-friendly policies. Some such policies centered on students with special needs or, as they came to be known, "students at-risk academically." Through the years the list of students at-risk expanded and today it includes students with physical disabilities and emotional, psychological, mental, and learning difficulties, as well as students from minorities and students who lack preparation for college academic work, lack study skills, and have difficulty fitting into the campus environment.

In the past, remedial academic work was given rather haphazardly and with inconsistent efforts. What colleges and universities have done is to reform such work by expanding its scope, sharpening its focus, codifying its application, and consolidating its efforts to the overall goal that the assistance to the student at-risk is effective. Thus, they elevated efforts for the college educational development of the student at-risk to official professional status, and remedial education was metamorphosed into Developmental Education. Since lack of preparation in mathematics has become one of the most critical educational needs in the nation, it provided impetus to facilitate remedy to it; thus, the emergence of Developmental Mathematics.

In the United States, if there is a constituency there must also be a national organization representing that constituency. It's hardly surprising, then, that the National Association for Developmental Education (NADE) emerged. NADE provides information to developmental educators through electronic mail, newsletters, and a journal. It also sponsors workshops and seminars, holds an annual meeting for its membership at large, and lobbies Congress and the Executive Branch in the interests of developmental education. It has also created regional subdivisions to facilitate communications of professionals across the states, as well as networks for each discipline called Special Professional Interest Networks (SPINs). Thus, the mathematics educators have the MathSpin network through which they communicate, mostly by electronic mail.

Causes and Extent of the Problem

About 29% of students entering four-year institutions need to enroll in developmental mathematics. In some institutions that figure comes higher than 40%, while in two-year institutions the average is 43% (Carriuolo, 2001). The causes of this situation are complex. And as it normally happens, when difficulties occur that cut through and reflect on the society at large, one entity or constituency blames the other for contributing to this conundrum.

Part of the problem for high school graduates is not necessarily that they did not receive instruction in mathematics, but that instruction was not adequate. A few years back, the writer read a study done by the National Council of Teachers of Mathematics (if his recollection is correct) that many high school mathematics teachers openly criticize pupils for not doing well and especially female pupils. Criticism when constructive is helpful, but the criticism referred to in the study was malicious.

It is no surprise then that many of these students have developed "mathematics anxiety." The term was coined by Professor Sheila Tobias and is accepted as a psychologically induced phobia. Such a phobia renders the student incapable of tackling mathematics. The student believes in earnest that he/she does no possess the intelligence peculiar to understanding mathematics. The writer had students that just thinking that they had to take mathematics made them physically ill.

Another reason for the mathematics problem is the increasing graying population of college students, commonly known as "non-traditional students." These aging adults come to higher education in such numbers that they are fast replacing the traditional college students (Carriuolo, 2001). They have not attended school or college for ten, twenty or more years, and most of them are deathly afraid to tackle mathematics.

This writer also considers that the affluence of American society and the uneven distribution of that affluence contribute to the fact that both high school pupils and college students avoid difficult academic subjects as mathematics and science.

Theory

Developmental mathematics differs from remedial mathematics, according to developmental mathematicians and behaviorists, in the fact that remedial education addresses "student weaknesses or deficiencies" and carrying the connotation that the student needs "fixing" in a specific area. Developmental mathematics, on the other hand, addresses the problem by a "comprehensive process focussing on the intellectual, social, and emotional growth" of the student (Kinney, 2001).

Part of the general problem with graduating high school pupils is that they have been conditioned to be followers rather than leaders, both in their studies and social behavior. Thus, part of the responsibility of mathematics education on the collegiate level is to develop the students to be original thinkers. Stahl, Simpson, and Hayes have found that developmental mathematics instructors should "strive to help students to become independent learners: autonomous, self-regulated, and good strategists." (Kinney, 2001)

Generally, high school pupils, especially in inner city schools as well as in small rural schools, are not trained to respond positively to demands of their studies and their environment. The position of mathematics educators is that a developmental mathematics program should include the following elements of demandingness and responsiveness, which in turn should help the student to reach a responsive and responsible self-regulation (Wambach, 2000)

A. Demandingness
   1. Standards for excellence and expectations for appropriate behavior are clearly stated and enforced.
   2. Skills courses are challenging and clearly connected to the curriculum.
   3. Content competence is demonstrated by required reading, writing, and computation.

B. Responsiveness
   1. Responsiveness is exhibited by delivering timely and useful feedback.
   2. Responsiveness is exhibited when the development of self-regulation is intentionally fostered.
   3. Responsiveness is exhibited when a wide variety of learners are accommodated.
   4. Responsiveness is exhibited when the program staff gets to know the learners as individuals..

Content

The type and extent of the content of developmental mathematics programs is normally organized by mathematics faculty of each institution. Thus, developmental mathematics programs differ from institution to institution in form, extent of the material covered, and the level of difficulty the material is treated.

Normally, the program includes two (or more) courses in sequence. Typically, a first course includes arithmetic, concentrating on the set of rational numbers and their properties and operations. Then, it moves to a rather light treatment of basic algebra. This includes linear equations and inequalities, systems of linear equations, introduction to Cartesian Geometry, some treatment of Euclidean Geometry, polynomials and functions and possibly algebraic fractions or even further. The second course raises the difficulty of the algebraic materials covered in the first course, and moves on to cover as much of algebra as time and the students' caliber allow.

Student Placement

In order for the student to succeed in his efforts in mastering mathematics, the institution's intervention must be proactive, and measures taken must be done in a positive manner. The institution, therefore, must intervene in the student's studies as early as possible. As a rule, students entering college must take a mathematics placement test. According to results of that test, students are placed in one of the courses in developmental mathematics, or in one of the more advanced courses. In some universities, such placement is done according to scores in the national college admission tests.

In the past, the student's high school performance in mathematics, as reflected by the student's grades, was the criterion for placement in the appropriate college mathematics course. It was found, however, that high school grades were unreliable for predicting the student's achievement in college mathematics. This is also indicated in two research studies done in Utah. It was found that students who had taken high school Algebra I and Algebra II (and some geometry) still had to take a developmental course in college (Hoyt 2001).

Delivery

Some universities have established their own schools, usually named "General College" or "University College." They are charged with the first two-year college education for students who have not decided their field of study or do not meet the standard academic criteria for admission to regular programs, and with providing developmental education. The majority of the universities, however, have left that to their mathematics departments or to specially developed administrative units. Finally, the community colleges provide developmental education to their students who transfer to four-year institutions..

Delivery Techniques

According to research, the lecture method is not the most effective way for the delivery of information in the classroom. It is, however, the most efficient way and, therefore, the most economical—which budget-makers love. It is also the technique in which most of us were raised with, from grade one to doctorate level. Additionally, it offers the most expedient use of our time and caresses our ego the most. So most of us prefer it. And our institutions unabashedly endorse it.

Team teaching offers students a better approach than the lecture method. In such a case, besides the mathematician in charge of the course other instructors or even behaviorists are actively present and partaking in the instructional process.

Increasingly, universities are moving into computer-mediated instruction. The effectiveness of such instruction seems to be supported in a study that found that those students in developmental mathematics courses using the lecture technique were more likely to withdraw from the course than the ones in computer-mediated courses (Kinney, 2001).

In sparsely populated areas, the advent of distance learning, mainly through electronic media, is becoming popular and promising (Kinney, 2001). Such an area is South Dakota's northern tier, which covers almost half of the state, with dispersed ranches, farms, small towns and villages, and Native Indian Reservations, all large distances from each other. Here, the schools are small, with limited resources and often inadequate staff. Northern State University (NSU) is presently preparing distance education modules in mathematics to assist these schools in becoming effective in the remediation of their pupils. This is in conjunction with a federally funded program (Upward Bound) that provides the resources for NSU to assist high school pupils in understanding the academic demands of a collegiate institution and familiarizing themselves with the campus environment.

The best results occurred, however, when the students were allowed to move in the course at their own speed. In such individualized, self-paced instruction, students move at their own choice of speed from module to module and decide when they are ready to be tested in the modules given to them for study (Kinney, 2001).

Part of the delivery process is the Supplemental Instruction. Here, graduate assistants or other qualified staff members hold special sessions with small groups of developmental mathematics students and go over the same material already covered by the instructor, but modified according to the needs of each group.

Tutoring is an integral part of developmental mathematics. It is done both by peers of the students and by tutors paid by the institution. It is a one to one process and each student receives the full attention of the tutor. Some institutions contract tutoring to outside concerns whose expertise is in tutoring.

Electronic media tutoring in developmental mathematics is gaining ground. Some use of videotapes takes place, but interactive computer programs are becoming popular. Additionally, more and more publishing houses have developed their own tutoring programs that correspond to their developmental mathematics texts and are offered free to students through the Web

Complementing tutoring is mentoring. This is a two-pronged process. First students are trained to be mentors of other students. Second, mentors assist in the training of students wishing to be tutors and supervise them at the beginning of their tutoring duties. Mentors may also be assigned to specific students in order to assist them in their studies and campus deportment. .

Learning Communities

Learning communities use the resource of faculty from various disciplines, so that various types of expertise come to bear on a specific unifying theme Carriuolo (2001). Some students performing poorly in mathematics because of a combination of poor study habits, social problems, family difficulties, low self-esteem, lack of motivation, or even poor health. Such students who have similar academic, professional or social interests are grouped together to constitute a learning community. This is especially facilitated if students are housed in close proximity to each other, as in a dormitory. A staff member is in charge of the group and qualified staff members work with them to resolve their problems. The staff member that is in charge of the group monitors the progress of each student in the courses he/she is taking and meets individually with students whose performance does not measure up to expectations.

Reaction of the Campuses

Many campuses, especially in small to medium institutions, did not react kindly to the introduction of developmental education. The various campus constituencies are not happy to share their resources with other constituencies. So, it is not unusual that mathematics departments were not overjoyed when they were asked to fit the expenses for developmental mathematics into their budgets. And the reaction of mathematics faculties was typical of the general reaction of faculty at large not participating in the developmental program.

Aside from finances, some mathematics faculties seem to feel that students who come to the campus without a solid high school grounding in mathematics do not belong there. And the general student population seems to reflect the professors' feeling—so much so that students in developmental mathematics in small institutions are often reluctant to attend the tutoring laboratories or supplemental instruction group sessions.

Some of this animosity comes from feelings of snobbery, possibly feelings of embarrassment to have such students on campus, or from deep philosophical conviction that does not include second chances. The writer once listened to the remarks of another mathematics professor who said "We don't want these students in our university. If this [having developmental mathematics students on campus] continues it will eventually lead to lower academic standards." The feeling that the institutions' academic integrity is being violated by the presence of students who have to be specially "cuddled and pampered" runs strong among both faculty and regular students. The resources could be spent more profitably; the common logic goes, by helping the good students.

To minimize this behavior, the title of developmental education in some institutions is Transitional Academic Studies (thus, Transitional Studies in Mathematics), perhaps giving a clearer picture of what developmental education does.

Meeting the Cost of Developmental Mathematics

Besides the campus community, taxpayer groups object to developmental education programs. They have brought forth the argument that developmental education programs supported by state or federal funds constitute double taxation for American citizens. They state that they pay taxes for the student's mathematics education in high school, and then they are taxed again to have the same student take essentially the same mathematics courses in college (Saxon 9001). Because of that, in some states public colleges and universities expect the students to pay special tuition for the developmental programs.

Summary and Conclusions

In the United States, philosophical position for education and practical reasons have led universities and colleges to practice open admission. As a result, a large number of students cannot successfully study college level courses, especially in mathematics and science, because they lack adequate background for them. Additionally, some students are not ready socially and psychologically to adjust to the demanding academic environment and the social deportment prevalent on collegiate campuses. These two situations contribute to a large number of students dropping out of college or failing in their studies. And to correct or minimize this, developmental education became a part of the curriculum of the American collegiate campuses.

Since mathematics is one of the most critical disciplines for the country, and also one apparently largely contributing to students' academic difficulties, mathematics became part of the developmental education programs of colleges and universities. Developmental mathematics involves going back to mathematics (albeit with higher level of difficulty) that should have been mastered during high school.

The purpose of developmental mathematics, therefore, is accustoming students to mathematics work ethic and allowing them success in the study of at least elementary mathematics required by general education. Because of that, developmental mathematics programs are multifaceted and, besides mathematicians, faculty from other disciplines as well as behaviorists are involved.

That fact that developmental mathematics has firmly established its place among college mathematics curricula was accomplished because of critical national need for mathematics education rather than the support it received from the campuses at large.

There is plethora of research for developmental education. Specific research for developmental mathematics is wanting, however. There is need for research that is well designed, for both longitudinal and short-term studies. The value of developmental mathematics as an instrument of introducing laggard students into mathematical thought must be established, and the most effective approach to its delivery should be identified.

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HOW TO PREPARE PROSPECTIVE TEACHERS TO TEACH MATHEMATICS – SOME REMARKS

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ABSTRACT

The reform of the Polish education system initiated in 1998 is a new challenge for teachers of mathematics, especially at the early teaching level and the primary school level. Since that time mathematical content has been bound up with other items of education. It was necessary to prepare a different way of teacher training – so as to prepare teachers to go through the new content of the subject "math". There was a chance to:

- Create a new approach to teaching early geometry.
- Create new connections between arithmetic and geometry, keeping the essence of arithmetical and geometrical cognition.

Keywords: early geometry, teaching, proportions

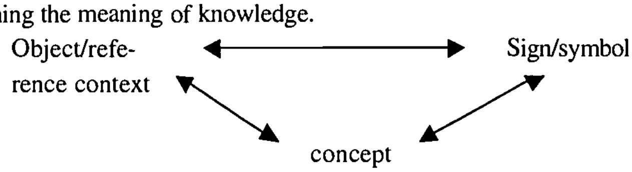
1. Introduction

The reform of the Polish education system iniciated in 1998 created a new challenge for teachers of mathematics, especially at the early teaching level and primary school level. According to the reform concept, mathematics at the early teaching level is a part of the integrated educational block. It causes a real danger of losing mathematical content in the amount of information and topics. The teachers of primary level are not good enough at mathematics – they are not sure whether they have sufficient knowledge of mathematics, they are afraid to look for their own didactical proposals.

It was necessary to prepare a different way of teacher training – so as to prepare teachers to go through the new content of the subject "math". This has created a chance to extend an offer for students – prospective teachers - in the framework of their professional preparation. Among others, there is a chance to:

- Create a new approach to teaching early geometry.
- Use a new tendency (based on recent didactical research) in teaching arithmetic.
- Create new connections between arithmetic and geometry, keeping the essence of arithmetical and geometrical cognition.

In the primary education system we try to adhere to Dienes' "deep end" idea, adjusting to the contemporary trends. It means that we think not only of issues emended in the current teaching programme. We also try to organise some of mathematical activities to create a wide intuitional basis for concept, which formally will appear in the further levels of education (Fischbein, 1987). Intuitions are built gradually. We give every student a chance to make their own individual investigations through participation in some real situations. Wide and various context, real materials and tools help to create a rich reference context. This is accompanied by a special mathematical or informal language. This model of work – in our opinion – corresponds not only to Freudenthal's idea (1973) that "mathematics is a human activity", but also to the epistemological triangle by Steinbring's (1997) model establishing the meaning of knowledge.



2. Theoretical background for didactical proposals

One of the main concepts, which we worked-out theoretically and tried to realise practically are *proportions.*

For a long period of time proportions have been the centre of interest among didacticians of mathematics (Researchers from Freudenthal Institute in Utrecht: van den Heuvel – Panhuizen M. 1990, 1991, Treffers 1991, Streefland, 1985). At international conferences the new aspects of understanding of this concept are still being referred (ICME 9 – Nunes. T, 2000, PME 25 – de Boeck et el, 2001, van den Valk T. at el, 2001).

Proportion can have a geometrical as well as arithmetical aspect. In our work we tried to realise the idea that the early beginning of geometrical and arithmetical learning should not be connected with each other – the way of learning and teaching of each type of concept is different and very specific, so making links between those two domains too early can destroy the notion itself (Tall, 1995). On the other hand – there is a huge need to enrich the geometric substance for pupils at early educational level. Pupils at this level shouldn't limit their geometrical knowledge only to geometrical figures such as square, triangle, or circle. Geometrical world, which has emerged from children's surroundings, is a lot richer.

In the book "International Handbook of Mathematics Education"(1996) three perspectives of teaching geometry are discussed:

1. Interacting with real shapes and space,
2. Shape and space as the fundamental ingredients for constructing a theory,
3. Shapes or visual representations as a means for better understanding of concepts, process and phenomena in different areas of mathematics and science (pp.161)

In this perspective shapes appear in a dynamic aspect. Students from the very beginning should not only distinguish shapes but they should also be able to perceive the position of one shape in relation to the other, and be sensitive to relations between shapes. One possible relation between geometrical objects is the relation of similarity, and similar figures can be used as the visual representation for better understanding of proportions.

The decision about the creation of the proportion supported by similar figures gave rise to the necessity of detailed phenomenological analyse of this concept.

Similarity as a transformation can be defined in two ways:

1. as a composition of isometry and homothety – this is geometrical description;
2. as a transformation changing all distances in the same way – this is an arithmetical description, based on proportions.

But the way of creating the concept of similarity cannot be directly guided by a final mathematical product.

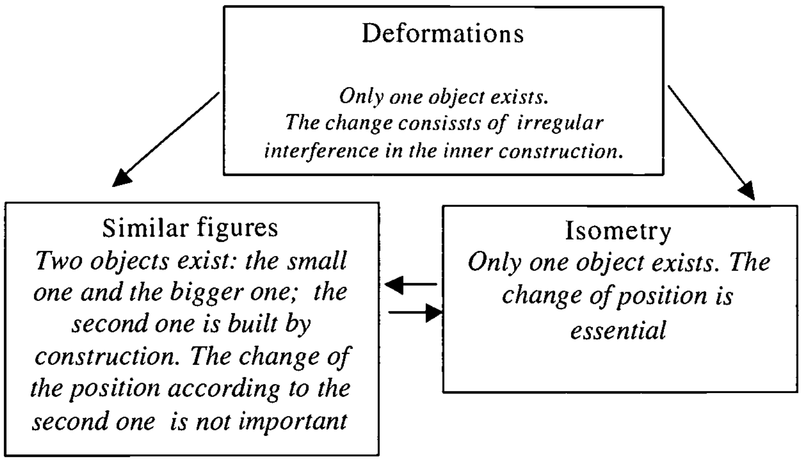
From a didactical point of view it is important that similarity is the equivalence relation, and its' abstraction class is "shape". Shape is seen visually and in such a way is closer to a geometrical way of creating the similarity. But because of actual and future student's matehematical knowledge it is important to use "shape intuition" to create a proportional description.

For several years the understanding of proportion in similar figures is the subject of our own empirical research. Results from this research brought a lot of important information about the process of forming geometrical concepts in childrens minds, and about differences between sources of geometrical and arithmetical acquaintance.

- One of the results from the research is an empirically confirmed fact, that children recognize similar figures visually. This statement concerns not only children from early educational level –in the same way react older students, who already know the formal definition of similar figures.
- The next result is that students' activities related to isometries are different than activities related to similarity. In both situations an utterance "the same shape" appears in a different meaning related to the performed action:
    - The basis of the activities related **to isometries** is a physical movement of the whole figure. Action on the object consist of moving, turning, reflecting. The object does not change itself, it changes only its position. So – the object all the time has "the same shape", because it is not changed as a whole at all.
    - The base for activities related **to similarity** is the existence of two separate objects. These objects have "the same shape" – only one of the figures is smaller and the other – bigger.
- Also onother type of transformation exists: deformations. Deformations are not a topic in the curriculum at all, but they exist in children's minds as a spontaneous concept (in Wygotski' sense, 1987). By deformations the object is changed by interference in its inner construction.

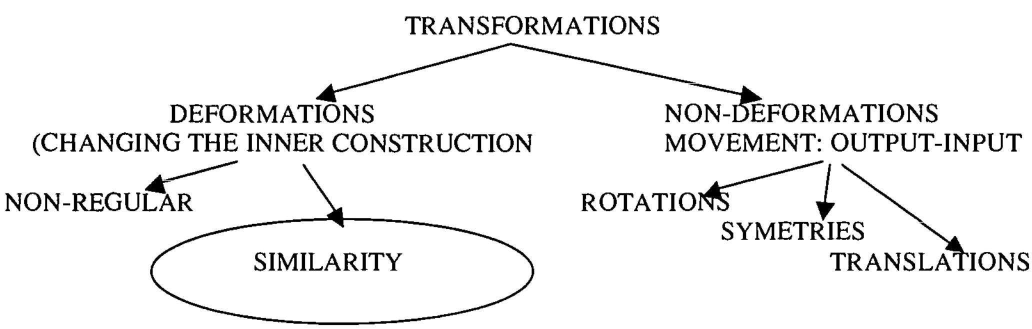
The figure is changed; it looks different than at the beginning. The change is irregular. Otherwise – the inner proportions are changed.

The diagram below shows the differences between each of these transformations:

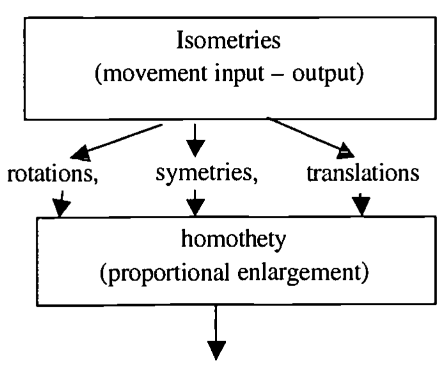


Although each of these transformations are totally different, given the specific understanding of the utterance "the same shape" it seems that similarity has more in common with deformations than with isometries. Comparing similarity with deformations gives a chance to differentiate "the same shape – the changed shape". In the area of isometries we can only differentiate "the same shape – the other shape". Comparing similarity and isometries can lead to misunderstanding of the statement "figure persists the shape".

Due to such conclusions the hypothetical students' way to similarity can be shown as follows:



It is more convinient for a student to notice the preservations of the shape of the figures by the regular deformations (that means by using proportions) than doing this by composition of isometries and homothety



Similarity → Irregular deformation

Teaching similarity through the composition of transformations is directed at the structure of mathematics as the science, but it is not guided by the psychological aspect of learning mathematics. Integration of the final results achieved in these two ways (due to regular deformations or due to composition of transformation) is the teachers' assignment in older classes. This task will be easier when both these ways become clearly mathematicaly established on appropriative level. In cognitive psychology, it is said, that we all have an assortment of mental models connected with the mathematical concept. Conceptual structures are the major factor of progress in understanding mathematics, mainly by testing the reality. Though mathematically and psychologically different, both ways of building up the meaning of the similarity enable the complex understanding of this concept. It is so, because in both these ways the concept is created in the following aspects:

- connected with real situations;
- dynamic, as a construction process of a final result (the figure is similar to the second one ⇔ the figure with the same shape);
- giving a chance to descript the basic, structured relation (inner proportion of the length of segments ⇔ external scale of similarity).

We try to implement the conclusions of these analyses in didactical proposals directed to teachers and students. Here are the basic assumptions of a didactical line concering teaching proportions from a geometrical aspect:

1. The starting point is the intuition about preserving the shape of a figure.
2. First assesment of changing the shape or preserving the shape are done visually, without any metrical aspect.
3. The activities concerned with similar figures are based on the construction of the figure, which is similar to the second one. Different tools are used and a different reference context is given.
4. Description of the mathematical properties is focused on preserving the inner proportions, mainly as the relation between the segments, which have the same length.
5. Numeral relations between lengths of the related segments are based on the intuitional understanding "the rule of three".

As can be seen, the basic assumptions of these proposals are the visual essesment of a shape. This is the core of student's individual work. The context is the construction of similar figures. The basic mathematical relations, which are discovered and described by a student in his own language, are the inner proportions of the figure (see Duval 1998, p.38). This proposal is new for teachers and students in our country.

Acceptance of these assumptions has clear didactical consequences. It seems that paying attention to the inner proportions in similar figures can help to understand the following concepts:

- **Fractions**. One of the aspects of understanding fractions is seeing it as a ratio between two quantities. In this aspect not only a magnitude of the numerator and the denominator determines the value of the fraction, but mainly their ratio.
- **Irrational numbers**. Number 2 or ð are defined as the ratio of the length of eligible elements of some geometrical figures. 2 is as the ratio the length of the diagonal of a square to the length of its side, ð is the ratio of the circle to the diameter. All squares are similar, and all circles are similar. A student who is familiar with the properties of similar figures is ready to

accept the fact that mentioned ratios are independent from the size of figures, and that such ratio have only one value.

- **Trigonometric functions.** At school the sine, cosine, tangent and cotangent is defined as some ratios in right-angle triangles. Also in this case, the knowledge about the inner proportions in similar triangles is very helpful.

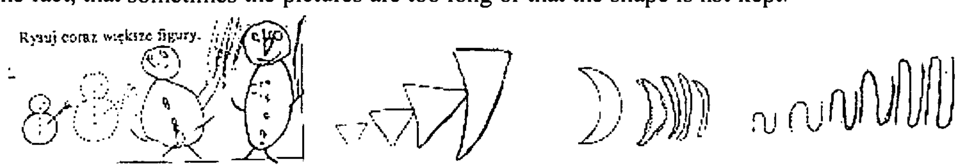
3. Didactical propositions

The following examples affirm the realisation of our proposition. Tasks were prepared for students from the lower educational level (6 – 9 years). In our opinion, learning about proportions at this level can cause more controversies. We do not show all the possible forms and methods of work – there are only examples reinforcing of our proposition.

3.1. Intuition of preserving the shape of the figure

3.1.1 Series of tasks with commission: draw figures bigger and bigger, draw figures smaller and smaller.

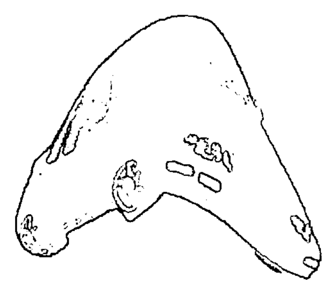
These series usually are placed at the beginning of the booklet for children from first grade, in the chapter "size relations" (Semadeni, 1992). But in analysing student's work it is good to pay attention to the fact, that sometimes the pictures are too long or that the shape is not kept.



3.2. Preparation to the understanding of an utterance „the same shape".

3.2.1 Deformations

The meaning of this preparation is the situation of the contrasting the change of the shape. During various activities with physical objects, children observe the change of the shape. For example:



- reflection of own face in water, in disturbed water;
- reflection of own face in the glass Christmas-tree bauble;
- looking at different things through water in a jar, through water in bottles of different shapes;
- stretching rubber with a picture;
- blowing a balloon with the picture.

During these exercises children describe in words the observed changes. The teacher encourages individual statements, which express the point of changes in the best way, eliciting things like: is very tall and thin, is too fat, the nose is too big in relation to the whole face.

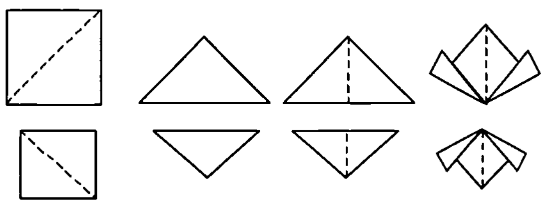
Continuation of these exercises may be with plastic works: 'my caritature", "bad witch", the world reflected in a puddle"... Drawing is here a form of transformating the information achieved during the observation of the deformed shapes. The next step in the coding, as in the mathematisation, should be the children's conversation about their own work.

3.2.2 Proportional enlargement

A: We draw advertisements. There is a huge number of situations, in which we can prepare a big poster giving information about a (for example) celebrity. It is a perfect occasion to make a spontaneous enlargement with an intuitional idea of preserving the shape

(For example fluorisation action – we draw a big toothbrush ....)

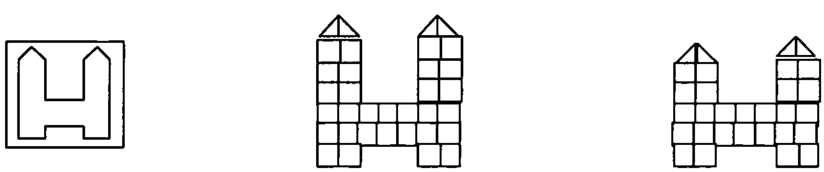
B: Origami – activities concerned with creating the same shape in two different sizes (a big flower – crocus, a small one - snowdrop. During this exercise the teacher guides the children's work to underline the fact, that the construction (the following foldings in origami) is the same for the big and small figure (the same angles are created, the lines are shared in the same proportions). After finishing work, the children code the information about the figure construction, making "a letter" for friends from other classes. In this letter they code the algorithm of making the origami (Wollring 2000, Karwowska at al. 2001).



3.3 Description of the metrical properties of similar figures

It is not easy to cross the path between a visual perception and a mathematical description of the numerous relations between lengths of the segments. Visual perception is spontaneous, natural. The mathematisation process needs a concious act of abstraction, and the ability of paying attention to the isolated parts of the figures (instead of a "Gestalt" perception). We can ease children work by preparing the tools of work, which force the preservation the proportions. That's why our next proposals lead our students to preserve the inner figure proportions.

A: *Jigsaws*. Children get squares 3cm x 3cm and a small picture. They work according to the tasks: make the same shape as in the picture.

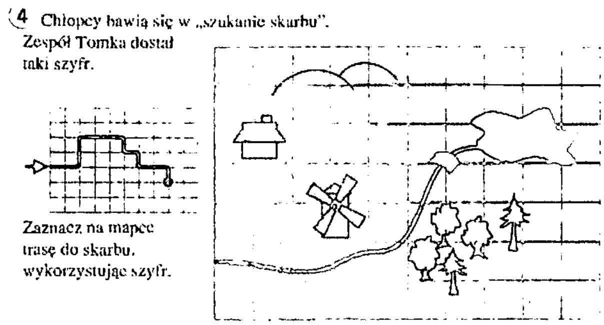


(correct example) (incorrect example)

Preserving the shape depends on preserving the inner proportion - changing the shape is done by changing the inner proportions. The teacher encourages students to talk about those facts (in informal language), during the work as well after. Child may notice that the second example is incorrect because the towers are not high enough. The towers should be as high as the width between these two towers. The tool – square – is used as the unit and forces the enlargement of the figure. Children work spontaneously, changing one small (non-existing) unit from a picture on to a bigger one.

B: *Tasks on the grid*

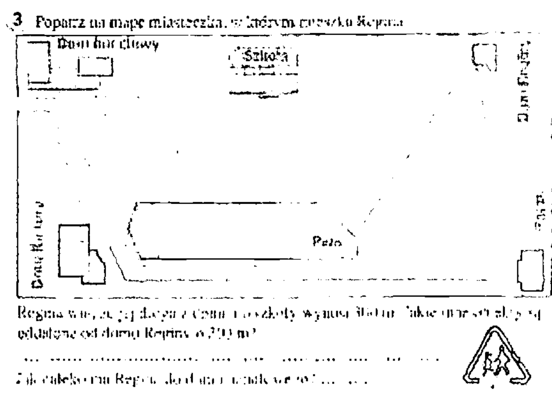
(Task: Boys play the game "searching for treasure". Tomek's team has the key-code. Find the way to the treasure on the map.)



In this task (Wilk – Siwek, Swoboda, 1996) units are given. A child can see that there are small units and big units. To solve the task, children have to count the units very carefully, because they enlarge a special type of figure: the open broken line.

3.4. Tasks paying attention to the relations between lengths of the related segments based on the intuitional understanding “the rule of three”

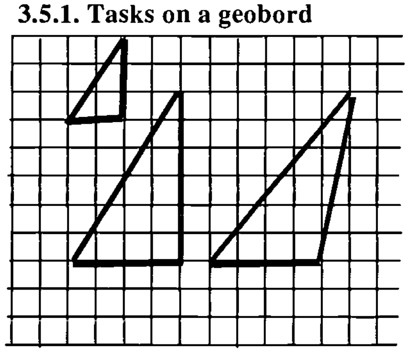
(*Task: Look at the map ofRegina’s village. Regina knows, that the distance between her house and the school is 300 m. Are there any more buildings laying in the same distance from Regina’s house? How far is the shopping centre?).*



Building the similar figure segments having the same length changes into segments having the same length; if any segment is two times longer as the other one then the dependence is kept on the enlarged figure. The distance on the picture between Regina’s house and the school is 6 cm, in reality it is 300m. Each segment having a 6 cm length determinates 300 m. in reality. This is why the school, the park, and the post office are 300 m from Regina’s house. The shopping centre is 600 m. further, because it is two times further than to the school.

3.5. First mathematisation in direction of external proportions (scale)

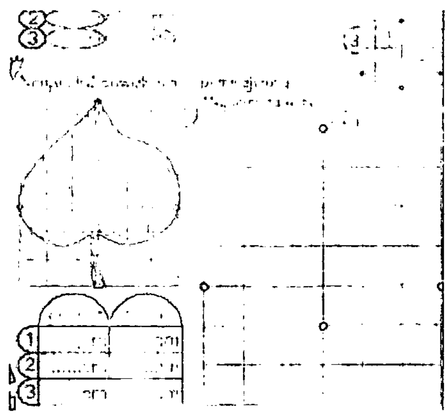
3.5.1. Tasks on a geobord



(Task: Build the triangle two times bigger)

If a child works on the one geoboard, he must count units very carefully and multiply lengths by the enlarger’s factor. In addition to that, this task gives the opportunity to discuss the connections between the inner and external proportions: not every triangle which has the base and the height which is two times longer than the original is similar to the given triangle, because not all of the inner proportions are preserved and not all of the angles are the same in both triangles.

3.5.2. Tasks for "re-counting"



(Task: Draw the enlarged and the reduced leaf. Fill the table)

This is a "scale" drawing . The realization of the work is a little different than traditionally is. The child does not need to count the "new" length of the segments, because the grid is already changed. He/she can concentrate on the shape of the enlarged figure. After drawing (and after visual estimation of the received shape), the pupil measures some segments and fills the table. It is worth discuss the results. Reduction is two times in the size – what does it mean? Which segments are two times shorter? Are there any other shorter segments? Enlargement is "strange". Shape is correct, but the numbers do not fit, the figure is not two times, not even three times enlarged ...Children can measure squares from grids and compare the results.

4. Final remarks

Proposition, presented in this paper, is a part of the project of preparation of the prospective teachers for primary educational level. We tried to connect two streams: mathematical preparation (as the answer for the question: what kind of mathematics should university students learn), and didactical preparation.

Working with students we try to show philosophy of mathematics, differnt from that they usually know. Matematics is not a set of facts for learning. It is a knowledge build individually by pupils, and the teacher's task is to create activities, which are the base for mathematical ideas. Maths on this level at school is one of the elements of integrated teaching to young learners. The process of teaching is based on "thematic areas" which allow children to study the reality in a complex way. Children store their mathematical knowledge in various cognitive situations (observation of real world, creating imaginative worlds). In this way the children create a network of associations, which let them creatively use their mathematical experiences for solving mathematical problems. The teacher's task is to organize the process in such a way that the children who study the world and regularities which exist in the world, can describe them in mathematical language Teacher should help a child to make links between the problem, the procedure of solving it and the solution.

For this reason the teacher have to be sensitive for mathematics emerged from the real world. His/her mathematical preparation has to go over of the narrow frames of traditional topics prepared for primary education. He/she has to see mathematics in a very large perspective.

The teacher's practical preparation depends on drafted theoretical establishments of the whole proposition. Tasks, commonly found in existing books help in its realisation, by projecting activities for children at school.

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HOW TO PREPARE STUDENTS FOR A SUCCESSFUL FIRST YEAR AT UNIVERSITY: AN EXPERIENCE

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ABSTRACT

During the last years, the basic mathematical knowledge with which Belgian students have enrolled has decreased a lot. This phenomenon results from various causes that we will try to outline. As a direct consequence, most of the students do not possess enough mathematical skills to follow the usual curriculum and face enormous difficulties from the start. In fact, only the better of them will go through these obstacles. In an attempt to give a chance to everyone, we have set up since 1999 a special system of support. Here are its main characteristics: a course has been added to the first year curriculum with the goal to deepen the understanding of high school mathematics. During this course, the students benefit from individual help from the teacher but also from a set of more advanced students who intend to become high school teachers. We will explain the organization of this course (unfolding, material covered,...) and will assess the students progresses. Our students also have access to another kind of support, more targeted to specific courses of the curriculum. A special session takes place once a week which focuses on the problems encountered by the students. Although this session is optional, the participation rate is high. We will show what makes this system work well and will analyze the positive effects on students successes.

Introduction

Year after year, Belgian French speaking students come to university less and less prepared. Not only is their mathematical knowledge poor but also they do not possess a good working method and are unaware of the efforts they will need to make. We do not mean they are fundamentally less capable but that without additional help, most of them will face huge difficulties to follow the usual curriculum.

These findings have been confirmed by the OECD Programme for International Student Assessment (PISA 2000) whose goal was to assess the knowledge of the fifteen years old students in the Organisation for Economic Co-Operation and Development (OECD) countries in the fields of readings, mathematics and sciences. Among the 32 countries, the performances of Belgian French speaking students are below average. The results of our best students are worse than many of the OECD countries. Moreover our High School teaching system appears unfair in the sense that socially unprivileged students have a greater chance to be among the weaker students. Would we be powerless against social inequalities? In fact, the problem essentially lies in the different demands of schools. It is true that being given the violence and the lack of interest for the studies, teaching often is an impossible mission. Will the current reforms (the formation of teachers, the material of scientific courses,...) solve a part of these problems? It is another debate on which we are not going to comment here.

This sad description does not obviously dissuade our students to enroll in university. That is why we are looking more than ever to give a chance to everyone without lowering the level of our training. To succeed, we think that is necessary to give students specific support and it is evident that this task requires a big pedagogical investment.

In 1999, on an initiative of the French Community Government, a support system was set up in universities. We will describe the one developed at the "Université de Mons-Hainaut". It is called *Système Transition Secondaire-Université*. We will give its main characteristics and explain why it works and which positive effects we can observe on our students.

1 Description of the Support System

Belgian High School aim to give a general training during six years. However the students can choose to focus on a particular field like sciences, economy, foreign languages,... each of which still allowing a panel of options. So the students who engage scientific studies, and in particular mathematics, come from very different backgrounds and therefore have uneven mathematical knowledge.

Here is the first characteristic of our support system: we added an *Elementary Mathematics Course* to the curriculum of the first year. Because of our concern in mathematical teaching, we create this course in 1994, that is long before it was made mandatory by the Government. At the beginning, the course was optional with a charge of thirty hours. Today it is compulsory and covers sixty hours. Its main goal is to bring the students at the same mathematical level. The material we teach is considered to be the basis necessary to follow the first year of the undergraduate level. The more important material covered in the course is:

- complex numbers,

- an introduction to linear algebra,
- an introduction to logic,
- study of proof mechanisms,
- elementary functions,
- analytic geometry in 2 and 3 dimensions,
- manipulation of the sum symbol.

Normally the students should not learn any new material during the course as it aims to recall High School notions. That is why the rhythm is rather fast. The course unfolding is a little bit particular. No theorem is proven. Our work is focused on the comprehension of mathematical concepts, on their fluent manipulation and the development of some intuition. The main points are exposed on the blackboard and are immediately apply to some exercises in order to confront the students to their own difficulties. During this course, the students benefit from individual help from the teacher but also from a set of more advanced students who intend to become high school teachers. We estimate that one person manages about fifteen students. This is important. Indeed the students are always supervised when they try to solve exercises and are encouraged to provide a personal effort. In return, they obtain personal help when they encounter obstacles. *Elementary Mathematics* is given during the first six weeks of the academic year, that is between mid September and the end of October. This is the first course that the students are confronted to. Time tables have been adapted, so the courses which need a certain mathematical background, like for example Analysis, start after the completion of this elementary course.

In our support system, Monday morning is free of courses. This half day is reserved to *Guidance Sessions* and *Evaluation Tests*. During the first six weeks, a two hours weekly test in relation with *Elementary Mathematics Course* is organized. These tests are followed by a correction, so the students can immediately correct their lack of understanding and that allows us to continue the course on good grounds. In November, an exam is organized. A student who obtain a note greater or equal to twelve on twenty passes. If he fails, he has another chance in January. For some people, it may be inconceivable that a student who did not meet elementary expectations can succeed in January and accesses the second year without problems. This however happens. We think an explanation of this as follows: we talked before about the difficulties for the students to dedicate enough time to their studies. Their failure to the November exam may wake them up as they realize they only have one chance left. They may thus decide to really involve themselves in their studies. To help them to prepare the January exam, facultative sessions are organized every week between November and December.

We also propose a twelve hours seminar in which we approach mathematical questions with an emphasis on algorithmic. In this way, we offer to the best students more elaborated subjects.

From November onwards, i.e. at the end of *Elementary Mathematics Course*, the Monday morning is dedicated to sessions called *Remediations*. Their goal is to help the students to achieve a good understanding of the material covered in the different courses. During these sessions, the students devote themselves to projects like drawing up cursus plans, solving additional exercises,... in collaboration with a teacher. They also have the opportunity to ask questions and to again assimilate what they have not yet grasped. Moreover the teacher gives pedagogical advice and helps the students with their working method.

Three persons are currently involved in the *Système Transition Secondaire-Université.* They share the teaching of the *Elementary Mathematics Course.* Two of them are scientific members of the Mathematical Department. The third one has been engaged at the creation of the system to manage the full time sessions.

2 An Analysis of the System

To sum up, the *Système Transition Secondaire-Université* is composed of the following activities:

- a compulsory *Elementary Mathematics Course*,
- *Evaluation Tests*,
- facultative *Elementary Mathematics Sessions* to prepare the January exam,
- *Guidance Sessions* every Monday morning in relationship with the courses,
- *Sessions* to prepare exams.

Our work is made up of two phases. In the first one, we try to make the students conscious of the important efforts they will need to provide during their studies. This takes place during the first six weeks with the *Elementary Mathematics Course* and the *Evaluation Tests.* We then focus on the mathematical evolution of our students. At this stage, the Monday morning and the facultative sessions start. Please note the graduation of our system: we begin with a daily support and pass after a few weeks to a weekly support.

The good results given by our system show that our efforts are not vain. First, our students look more active from the beginning of the year. Moreover, each activity has its own beneficial impact. The *Elementary Mathematics Course* provides the students with a good basic mathematical knowledge. For the better of them, it may also be the opportunity to discover the High School material with a different approach. The *weekly tests* force the students to have some regularity and autonomy in their work. They are also able to follow their evolution because they obtain every weeks their tests results. After the six weeks, the *Facultative Sessions* help the students to mature their comprehension of *Elementary Mathematics.* During these sessions, the small numbers of students guarantees that they receive a personal help and benefit from presentation tailored to they need. It is not rare, during the months of November and December, to see a real evolution of several students. During the same period, the Monday morning *Guidance Sessions* offer the possibility to the students to complete their notes, to have a more global view of the courses and to obtain individual explanation. Two weeks before the exams, some *Preparation Sessions* start. At this time, the students are very interested to put it to practice all the advice they have received. The exercises are especially designed to show the students the level of understanding they have reach and thus which kind of effort they have to make if they want to succeed.

The fact that a person is full time in charge of the guidance activities is a real asset. Indeed, she needs to dedicate four hundred hours of work a year to these activities. Because her contact with the students is good, they naturally come to her office to discuss their problems in various courses. They also often come to bring her additional exercises to correct.

We made a survey of the students opinion to try to analyze objectively the above described system. All the results point in the same direction. The students feel that the various kind of support contribute to increase their mathematical skill and help them to improve their work method. This investigation also shows a point we will like to emphasize: the severity of the *Elementary Mathematics Course* (constantly supervised work, weekly test,...) is not felt like a punishment. On the contrary, the students appreciate we have taken care of offering them a good basis to start university. Let us mention yet another positive point. With the exception of *Elementary Mathematics Course*, all the other proposed activities are facultative even though they are incorporated in the schedule. In spite of that, the participation rate is almost ninety per cent. Although the most activities are targeted to students who have mathematical difficulties, the better students nevertheless recognize to benefit from the system. Since its creation in 1999, our success rate is close to sixty per cent.

Some people may wonder about the necessity to create such a system. After all is not learning by oneself one concept of the university studies? Are not we neglecting this aspect? This question is fundamental especially because the management of the system requires a considerable amount of work and a great pedagogical investment. As an example, the *Weekly Tests* and the November exam represent the correction of six thousand pages in six weeks. However we strongly believe that a good success rate can not be obtained without the incorporation of a such system in the cursus. The current students are indeed much different from those we had only a few years ago. This is an inescapable fact with which we have to deal.

Provide enough resources are dedicated to this support system, we hope to have convince you that is an interesting way to explore.

In October 2001, a Commission of international experts evaluated the teaching quality of mathematic sections in the Belgian French speaking Universities. The commission was chaired by Professor Ivar Ekeland, President of Université de Paris-Dauphine from 1989 to 1994, General Director of Institut des Finances de Paris-Dauphine since 1995. The report described our system as innovated and efficient.

3 Conclusion

Guidance sessions have been created following a significant failure rate. *Elementary Mathematics*, *evaluation tests* and the several kinds of *facultative sessions* were grafted to form what we call the *Système Transition Secondaire-Université*. As we already mentioned above, the level of knowledge of the students who start university decreased. The system first aims at learning what means " to make mathematics ". Accordingly we have voluntary choosed not to incorporate technological means. For example, neither computer support or slides are used during *Elementary Mathematics*. The students have like only tools a paper sheet and a pen, so the course is more centered on the analysis and the drafting of exercises, but also on the dynamic of an oral presentation using blackboard. In our opinion, this method makes the learning more effective.

Currently we reach a point where the system is fairly stable. The actors are the same ones and the activities are well ground. That is why we now try to assess the effectiveness of our support program. We have some reasons to think that our work has some positive effect. At first, the system is completely integrated in the schedule and that contributes to have a high participation rate. With *Elementary Mathematics* and *weekly tests*, we individually follow the mathematical evolution of every student. The good success rate at the exam means that a lot of them reach the necessary level to follow the curriculum. We also have a positive feedback of the other professors. They think that *guidance sessions* somehow mature the students. In particular they become more autonomous. Finally our success rate in the end of the first year increased since the creation of the system.

The described experience is specific to our university. We think that the students probably have the same problems in other universities but we have few informations about the pedagogy developed outside Belgium. The next stage will be to discover if similar experiences are tried in other countries.

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VISUM: VIRTUAL SEMINAR FOR EDUCATION IN MATHEMATICS

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ABSTRACT

VISUM - the (VI)rtual (S)ystem for Ed(U)cation in (M)athematics is a fast growing knowledge base containing material about teaching mathematics (URL: http://visum2.uni-muenster.de). The project is funded by German ministry of science and education with 1.6 Million Euro and will cover mathematical as well a didactical content for the education of student teachers from primary to higher secondary level. Mainly written in German, the content will be translated into English in the coming years. When preparing knowledge for presentation in an Internet based multimedia system, special methods are needed to avoid - for instance - the "lost in hyperspace"-problem. The designer's answer to this problem is the so called Object Oriented Theme Analysis (OOTA) which is based on ideas coming from computer science, but adapted to the analysis of didactical knowledge about certain topics (e.g., arithmetic in the primary level, working aids, ...)

The lecture will present this method, give a survey of the constructivist background of the system and the role of media (video, audio, ...) in the system, and show examples. The VISUM software creates a navigation platform, which can be used as an authoring tool by universities world-wide - at the time of the conference there will be an online authoring tool for this purpose available. So, the lecture will contain an invitation to take part in the creation of a world wide network for teacher education in the field of mathematics.

1. Introduction

In the area of teacher education in mathematic s videos have been used in lectures and seminars for a long time. Showing sequences of lessons or interviews with children after activities of problem solving are examples for the concept of *situated learning*. Working with such authentic material has the intention:

- to illustrate didactic theories or strategies, and
- to present a starting point for the examination of learning processes.

Multimedia systems offer additional possibilities to approach didactical content by using animations, audio and video sources. Researchers started examining the new facilities in teachers education in the middle of the nineties.

The CD-ROM *Learning about Teaching* (*=LAT*) (Mousley & Sullivan 1996), introduced by P. Sullivan at the PME conference in Lahti (Sullivan 1997) is an outstanding example how to embed questions of the lessons design and analysis in an learning environment with video recordings and transcripts. By these means student teachers can approach theoretical concepts by examining authentic material with a focus on didactical questions. A video of a lesson together with a lesson plan is one example of authentic material. Authentic documents of children demonstrating a problem solving process is an other source for the student teachers.

At the end of 1998 M. Stein took this CD as a starting point for the VISUM project to use the new multimedia facilities for teachers education in mathematics in Germany. Beside the production of multimedia content for teachers education there are three basic aspects showing the differences between the *LAT-CD* and the VISUM project. .

- no limitation to a fixed commercial authoring tools (the *LAT-CD* was developed with *Authorware*),
- strong focus on developing a concept of Internet based learning, with a platform which facilitates at the same time the distribution of content on CD-ROM without a Webserver,
- presentation of the methods to generate hypermedia content from linear text information,
- the production of didactical and mathematical content is part of the educational concept. This means that VISUM is not only a learning environment with multimedia material *(LAT-CD),* but the project encourages *student teachers to produce multimedia documents* in private areas of VISUM. The learning process for the student teachers during the production of didactical content is one main aspect of the educational concept of VISUM.

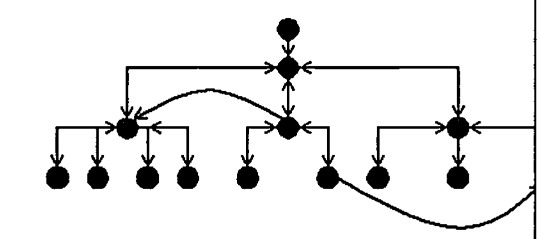
Objective: *The objective of VISUM is the Internet based production and presentation of didactical and mathematical knowledge. The issues are a knowledge base and a method of content generation to embed the production of multimedia content in the educational concept for student teachers. .*

Since 1.1.2001 four working groups (Th. Weth, Erlangen, H.-G.Weigand Würzburg,U. Tietze, Braunschweig, M. Stein Münster) are joining the VISUM project focussing on different mathematical and didactical aspects in primary, lower and upper secondary teachers education. This project is funded by the German ministry of science with 1.6 Million Euro.

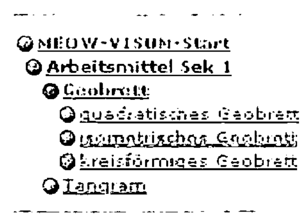
2. Knowledge Representation

Knowledge representation is one part of the objective of VISUM. The following section describes the structure of the knowledge base, in which the mathematical and didactical content in VISUM is organised.

The basic supposition of the didactical and mathematical knowledge in VISUM is that the known scopes of teacher education can be represented in a structured way. The arising structure is organised in the shape of a directional tree.



Picture 1: directional tree



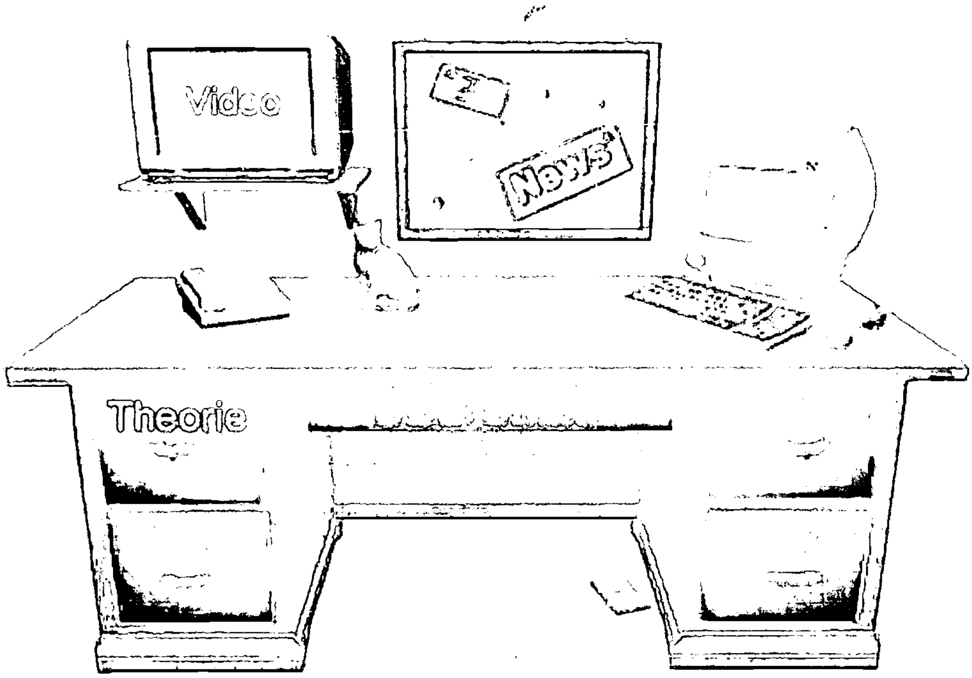
Picture 2: menu of the tree

In the general case, the arrows can be represented as hypertext links, connecting one information with an other information. In VISUM these links are not simple hypertext links, because the nodes in the tree represent a *collection* of different types *of information* (HTML-pages, video, audio, animations, ...) for a special subject. If we take as subject *"Ruler in Geometry"* this collection contains for example

- a *video* of a geometric problem solving activity with a *ruler* (recorded in a classroom),
- a collection of tasks for children in primary schools using a *ruler* as a tool,
- *tasks* for student teachers to analyse the geometric problem solving activity of children with a *ruler*,
- *Internet links* to the subject *"Ruler in Geometry"*, e.g. to geometric problem solving activities with a *ruler* in lower secondary schools,
- *theoretical* background information to the applications of the *ruler* in primary schools.
- additional *literature* references to the subject *"Ruler in Geometry"*,
- *news*, e.g. "conference July, 25., 2002 in XY -Title: Geometry in Primary Schools"

So a link between tree nodes in VISUM is a link from one collection of information to an other collection. Now we need an interface for this collection of different types of information. In VISUM we chose a Desktop as metaphor.

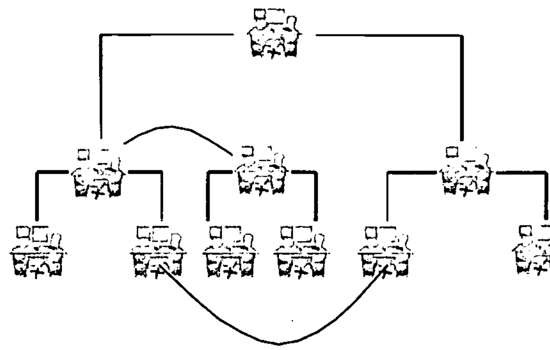
Ruler in Geometry



Picture 3: Desktop for the content "Ruler in Geometry"

In the illustration we see the "desktop"-interface representing the collection of information to the subject *Ruler in Geometry.* Some drawers are highlighted for example with a title *theory* (German "Theorie") and *video* (German "Video"). This means that the desktop provides *news, help*, video and theoretical information to the subject *Ruler in Geometry*. Some information categories are not highlighted because they are not availabe. Normally every *tree node* respectively *desktop* should contain these different types of information, so that the user can choose an individual approach to the subject *Ruler in Geometry*.

This provides a tree structure with desktops as nodes.



Picture 4: Content in the desktop depend on the tree node

So when a user is navigating from one collection of information to an other collection the desktop with the drawers is filled with *hypertext, videos, links* and *references* according to the subject the user is looking at. This *context dependency* of the desktop has the objective that the user will only get the information of a chosen subject (e.g. only literature references and videos to subject *Ruler in Geometry*). This means that the user can access the video screen only, when there is a problem solving video to the subject *ruler* offered from the author.

The basic idea of the *VISUM context dependent knowledge representation* is, that the desktop *classifies the information*. The following attributes show the chosen types of information in the VISUM system:

Survey: presents a short description about the respective subject, which could be used as an introduction containing hints for user to start with -- drawer in the middle,

Theory: the didactical or mathematical theory about the respective subject of the desktop -- drawer on the top left,

Examples: the examples are used as an illustration for the theory -- drawer on the top right,

Literature: References -- drawer on the bottom left,

Activities: Exercises for the student teacher according to the subject of the desktop -- Drawer on the bottom right

Video: Videos and animations -- monitor on the left

News: scheduled events according to subject of the desktop, -- News sign at the board in the middle,

Links: Internet links -- monitor on the right,

Help: gives advice for solving problems in the *Activities* drawer -- Help sign at the board in the middle.

Contact: presents the e-mail address of the author or the tutors of the lecture/seminar -- telephone on the desktop.

3. Internet Based Didactical Concept

At first sight it seems to be sensible that only didactical and mathematical experts should work on the construction of a knowledge base. But one step further the didactical concept of VISUM offers the embedding *of content generation* in the process of teacher education. This means that student teaches produce web pages and multimedia material for an information system, which serves e.g. as a basis of discussion in seminars. With this approach the student teacher swaps the *receptive role* with the *constructive role* of a content generator. This process is embedded in lectures, seminars and the homework of the student teachers for their examinations. The following three items illustrate the process from the receptive to the constructive role of a teacher student.

receptive - In the *lecture* material of VISUM, like animations, videos and HTML-pages, are used for the presentation. After the lecture the student teachers can access the information online. The sources of the lecture are embedded in a knowledge base, so VISUM offers additional information according to the subject of the lecture. By this receptive work the student teacher gets an idea of the knowledge organisation in the information system. Teacher students should learn to navigate and retrieve information for the lecture from VISUM.

receptive & constructive - in a *seminar* student teachers get a first contact with the content generation in VISUM. The VISUM system is designed in a way which demands (nearly) no technical knowledge for the construction of web based information, so that the didactical concept can focus on the *organisation* of web based content and the *structuring of mathematical and didactical knowledge*. Beside the fact that the work of the students is presented in a closed area of VISUM the student teachers had to *integrate their content* into the existing information system of VISUM. The individualised information system of the student teacher combines personal material with material of the official VISUM information system. For this combination of personal area and the official expert content the student teacher has to explore the VISUM information system for helpful connections (links) to the subject of the seminar. This includes major *receptive work* with VISUM.

constructive - student teachers get the opportunity to write a *homework* for their *examination* within the VISUM project. This homework consists of the construction of a *product* and a *theoretical text.* The *product* is a VISUM knowledge base about a didactical subject (for instance: practise in arithmetical lessons), the *theoretical text* describes the principles of collecting knowledge and preparing it for use in the knowledge base. This type of homework is very attractive to our students since they know that – good quality the product assumed – the product will be made accessible to other students via the Internet, as part of the VISUM project.

The following section tries to give some rough ideas how the method of analysing knowledge for presentation within the VISUM system works.

Object Oriented Analysis (OOA):

Object Oriented Analysis is a problem solving strategy developed in Computer Science. Despite of the fact that this strategy has its origin in Computer Science, it is a modelling concept *strictly independent* of a programming language. The *VISUM method of structuring knowledge* applies basic ideas of OOA to *structure knowledge* (for instance, about didactical subjects like *ruler in geometry*). It analyses a system and decomposes it into objects that are found in the system. *Decomposition* is one principle of the OOA, so that all objects can be decomposed in subobjects again. The process of decomposition provides the tree structure which was shown in section 2.

Beside the decomposition, the *objects are classified by properties they have in common*. So *classification* is another principle of the OOA. The relationship between Theory, Example, and activities (see section 2) is one derivation of the classification principle within the VISUM system – the desktop is the metaphor for this classification.

The upper mentioned decomposition defines a *relationship between the objects*. Beside this relationship the OOA model contains *associations between objects*. For example, the OOA model could contain an *association* between the *problem solving strategies in geometry* and *psychological aspects of problem solving*. These associations are represented in the VISUM system by a special type of link, so that every user can see the existing associations. So the arising structure is a web of objects organised in a tree.

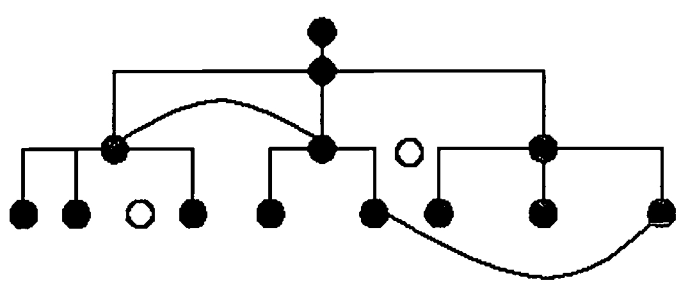
It is important to mention that a model, generated with the OOA, is dependent on the person, who did the modelling. Different views on a system generate different object oriented models.

In the process of us ing VISUM in lectures and seminars to developing material for VISUM as part of the final examination ("states exam") the *constructive aspects* of the student work increases, so more detailed knowledge for content generation is necessary. Students who wish to present their knowledge in the VISUM system as part of their final examination, have to visit special seminars in which they are trained to apply OOA to their special theme, and how to use the possibilities of *multimedia* for presentation in a web based system. The full method used in VISUM is the called *OOTA* (*object oriented theme analysis*) and described in Niehaus 2002 and Ernst, Stein 2002

Of course not all students write their homework in mathematics and didactics or get in contact with VISUM in lectures and seminars. The main focus in the developing of the didactical concept is the consistent embedding of VISUM in the education of student teachers at the university. This leads to a ***constructive competence of structuring didactical knowledge and didactical problems***. On the highest level the issues of the student work could be presented within the official VISUM information system or it could be used as a basis of discussion and further development of the students.

4. Individualisation of Knowledge Representation

In the preceding sections we focused of on the VISUM knowledge representation. Keeping the constructive aspects of the didactic concept in mind, it is necessary that students can participate in the *construction process* of VISUM knowledge without modifying the public accessible knowledge base. In the following picture we can see black tree nodes and white tree nodes symbolising one single desktop. The *black nodes* with the black lines symbolise the public accessible knowledge base with the connections between the desktops. The *grey connections* and the *white nodes* symbolise a *private modification* of the public knowledge base. The private modifications characterise an *individualisation* of the knowledge representation.



Picture 5: White nodes and grey connections symbolize the individualisation in the tree

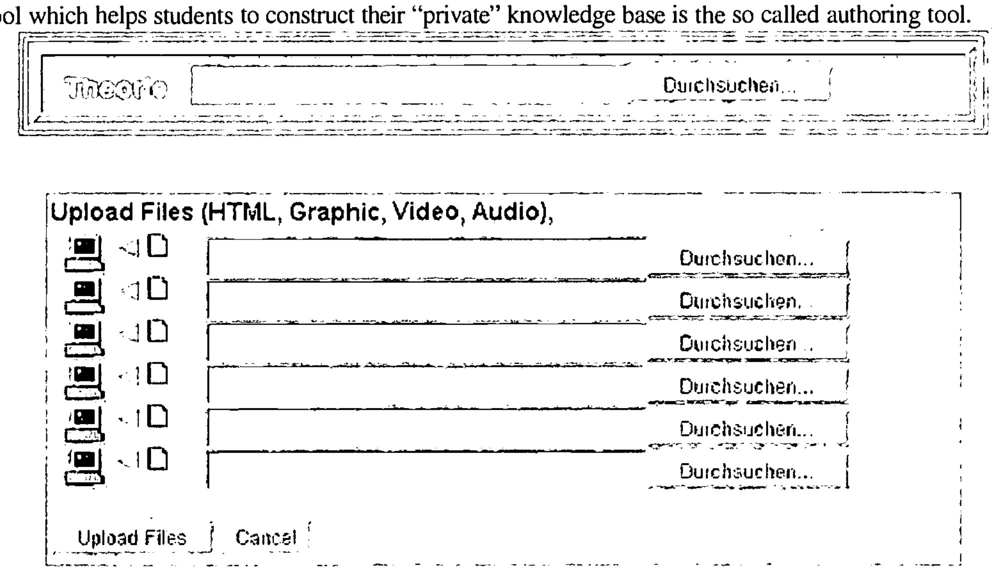
This aspect of *individualisation* gives the student teachers an option to integrate their own knowledge in a given official knowledge base. By the interaction between the *receptive* and *constructive* elements of the didactical concept the student teachers discover parts of the given knowledge base.

The private white nodes in tree represent for example an evaluation of geometric problem solving strategies of a teacher student S in the primary school XY about the application of the *ruler in the geometry lesson*.

If a *guest* is navigating through the mathematical and didactical knowledge base of VISUM, then the white nodes and the grey links between the nodes are *not visible* respectively *not accessible* for the guest.

Furthermore the visibility, the accessibility and the rewriteability of content can be extended from the author to special user groups (online workgroup of teacher students). This means that the private evaluations of a teacher students S in the primary school XY about the application of the *ruler in the geometry lesson* can be visible and/or rewriteable by the members of seminar Z about geometry in primary schools to serve as a basis of discussion.

The individualisation is also helpful for *teachers educators using VISUM in a lecture.* They can adapt the knowledge base to meet the personal requirements (focal points of the lecture). We should keep in mind that the *public official knowledge base* is in general not affected by this modification. The following picture shows a screen shot of the user interface of the authoring tool. As has been said before, the VISUM system is designed in a way which guarantees that student teachers can bring content into the system even with (nearly) no knowledge about the technical aspects of hypermedia and web-design. The tool which helps students to construct their "private" knowledge base is the so called authoring tool.



Picture 6: layout and design of the beta-version

With this interface a user can upload a HTML-file to the *THEORY-drawer* in one desktop of the VISUM knowledge base (top rectangle -- Theorie=theory). After the upload procedure the content is accessible in the *THEORY-drawer.* The Interface consists of two parts (top and bottom rectangle) because the HTML-file in the *THEORY-drawer* contains e.g. pictures and video files, which had to be uploaded as well. A click on *"Durchsuchen..."* (German for *"Browse..."*) provides the user with a file menu to choose the file the user wants to upload (transferred files to the server). After this upload procedure the information is available in the *THEORY-drawer* of ***one desktop*** in the VISUM knowledge base (password protection).

This option of individualisation in the VISUM system takes two contructivistic aspects into account.

- The ***receptive aspect*** offers the user a knowledge base, in which the user can navigate to VISUM content she/he is interested in. This knowledge should be embedded in the individual structure of the users knowledge.
- The ***constructive aspect*** offers the user the option to modify the knowledge base according to the individual structure of the users knowledge without modifying the original content.

5. Summary

The preceding sections show that VISUM is not only a knowledge base but it stands also for a didactical concept, which supports the education of teacher students at the university. To realise the *receptive* and *constructive* aspects of the constructivistic approach it was necessary to develop guidelines for the knowledge representation in VISUM to support teacher students in the constructive parts of the content generation for VISUM. This concept started with the OOA was developed further to the OOTA (Research Project: A. Ernst, M. Stein). So one focus of VISUM is integrating student teachers in the process of content generation. Therefore a simple user interface of the authoring tool was a technical precondition for this integration.

The *classification* of information in the VISUM desktop structures the generation of sources in the VISUM knowledge base. For the teacher students the desktop provides a scaffold for the constructive aspect. The VISUM system guides the user from the *receptive work* (lectures) to a constructive work in the knowledge base (seminars, homework for the examination). The underlying idea of Constructivism together with the object oriented concept presents a didactical approach which breaks the borders determined by a knowledge representation of books. Leaving out the different access and visibility rights to the VISUM system, *expert user*, *author* and a *didactical novice* share and embed content in a knowledge base, which serves at the same time as a basis of discussion for the mentioned user profiles:

- for expert users examining work of teacher students and colleagues,
- for student teachers examining and discussing the content of expert authors and/or student teachers.

Keeping in touch with the didactic state of the art and contributing ideas is a basic objective for understanding of the dynamics of Internet based content.

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PROSPECTIVE PRIMARY TEACHERS' EXPERIENCES AS LEARNERS, DESIGNERS AND USERS OF OPEN MATHEMATICAL TASKS

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ABSTRACT

This paper explores prospective primary teachers' views of open tasks. The study has been realized in the framework of developing student teachers' awareness of mathematics teaching and learning through a number of activities that aimed to relate theoretical perspectives to the mathematics teaching practice. Data was collected from students' portfolios, and those parts that refer to open tasks have been analyzed. In particular, students' ways of analyzing two different kinds of open problems, their approaches in designing and their experience from using open tasks in the classroom have been explored and aspects of their views have been identified. Overall, the study contributes to our understanding of the development of students' awareness concerning open tasks as developed through their involvement in different kinds of experiences.

Introduction

Teachers' education today is not a process of developing skills that the teachers can implement in their actual teaching, whereas most pre-service programs do encourage student teachers to get involved in the process of inquiry as learners but also as teachers during their teaching practice. The way that these two different contexts coexist with teachers' development has been discussed in a number of studies (Ebby, 2000; Georgiadou and Potari, 1999). This perspective is also shared in this study and more specifically the focus is on the development of prospective teachers' awareness of mathematics teaching (Mason, 1998). In this way, it is possible not only to see how teaching is itself a path of personal development, but also to discern the different cultural and cognitive phenomena constituting the teaching act. Mathematics teaching aims to develop pupils' knowledge, strategies and thinking tools. This cannot be done if teaching is as "telling", a well-established belief that student teachers carry with them from their school experience as pupils (McDiarmid, Ball and Anderson, 1989). Meaningful mathematics teaching means the use of a variety of teaching activities and thoughtful teaching interventions in contexts familiar to the children. One way of encouraging student teachers to develop such a view of teaching is through their involvement in open tasks as learners, as designers and as users. These three roles were embedded in all the activities of the initial training course for primary teachers discussed in this paper. This approach was in all the three parts of the course, the set of lectures, the lab-work and the teaching practice in classroom.

Within this context, student teachers' views of open tasks are studied. A debate on the meaning of open tasks is still going on (Ellerton & Clarkson, 1996; Silver, 1995). Moreover, pupils' and teachers' conceptions have recently attracted research interest (Pehkonen, 1995; Pehkonen, 1999). Nevertheless, little research has taken place in that area. In this study, we aimed to develop student teachers' meanings of open tasks through their involvement in solving, planning, using and reflecting on open situations. Through a variety of questions and activities, students experienced different aspects of such situations and we expected to encourage the development of their awareness. Their responses, their plans and their actions as described by them, were included in student teachers' portfolios and analysed.

Methodology

Student teachers' portfolios include extended data collected over a semester. Each week they had a lecture on issues of mathematics education, while every second week there was a lab meeting where the students themselves explored more concrete expressions of these issues and planned the activities for the school practice. Each lab was followed by a week of practice in school where the students implemented their plans. One hundred and twenty five student teachers participated in the course activities.

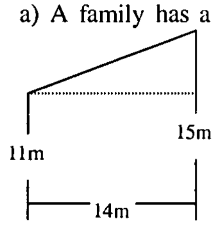
Description of the activities

Student teachers' portfolios consisted of worksheets that supported the students' involvement in different types of activities, their reflections on experiences from planning learning activities and from their classroom implementation. The worksheets, through a number of questions, asked students to focus on specific aspects of learning and teaching mathematics. A number of these worksheets referred to the meaning and use of open tasks. A sequence of questions had been developed concerning: the solution to an open problem; the identification of its characteristics; the comparison of it with another open task; the planning of their own open task (without the mathematical content being specified) to be used in the classroom; the use of this open task in the classroom; and the planning of an open task on a given mathematical content (this is not included in the analysis). This series of activities can be separated into three phases. In the first, student teachers are involved in the process of facing open problems and can be considered as *learners*. In the second, they use their previous experience in planning an open task for teaching and can be considered as *designers*. In the third, they implement the designed open task in the classroom and they evaluate this experience, so they can be seen as *users*.

The first phase

The questions on the worksheet concerning this phase are the following:

a) A family has a garden the shape of which resembles the one that is shown in the figure below. The family would like to plant tomatoes in it. Could you help this family to find ways of planting if, as the grandfather said, each plant should have around it a space of 35 cm. Try to solve this problem and write down one solution in the given space.



b) What are the characteristics of this problem? Which are the mathematical concepts involved? What could working on such a problem offer to someone?

c) In what ways does the following problem differ from the one you have already considered?

"A class of pupils would like to organize an excursion to Zakynthos. They are trying to find financial support by selling a magazine, which is edited by the class. Could you think of actions that need to be undertaken for the organization of this excursion? What kind of reasoning should they develop? Can you suggest anything that may help them?"

The second phase

In this phase of planning, the students-teachers were faced with the following tasks in order to encourage them to consider certain issues of their teaching more specifically:

a) Next time you visit a school classroom you want to organize a teaching approach, which can be considered 'open'. It would be helpful if you think of a situation from everyday life or the cultural environment or from a subject other than mathematics. Think about and describe this situation. Give two arguments for your choice.

b) Plan and describe the classroom organization for your planned teaching. What will your own teaching actions be and what will your pupils' involvement be?

c) Think of ways to evaluate the whole teaching approach. Write them down.

The third phase

This worksheet focuses on students' reflections of using open tasks in the classroom. More specifically the questions are described below:

a) You have implemented an open situation in a school classroom. What were your feelings during that experience and how do you feel now?

b) Think of the nature of your experiences at two levels: the cognitive level and the level of classroom management, and describe them.

c) Do you think that the situation was successful? In what respect?

d) If you taught this again, what would you do differently?

Data analysis

Here, we analyze part of the data referring to open problems. More specifically, the focus is on students' conceptions about the character of an open problem. The data analysed in this paper is students' responses to questions (b) and (c) of the task given in the first phase, to question (a) from the second phase and to questions (a) and (b) from the third phase. Initially, we examined each question separately. We analysed the answer given in terms of units that expressed a certain view. We studied these units and looked for different dimensions that underlined these views. A kind of categorization emerged, based on the construction of systemic networks (Bliss, Monk & Ogborn, 1983) that are presented and discussed below.

Student teachers' views of open problems as learners

Student teachers' characterizations of the problems emerged explicitly in their descriptions of the characteristics of the problem that they faced in the first task as solvers, and implicitly through their writing about its importance or by comparing the two types of problems. The categorization of their conceptions about the character of an open problem is presented in the systemic network of figure 1.

The student teachers characterized the problem both in terms of the problem itself and of the solver. In some descriptions these two dimensions seemed to coexist: "It is an open problem, it has a lot of solutions and so it is difficult for the pupils to solve." The solver was considered in two ways, one referring to the actions, either mental or physical, that he/she had to undertake and the other referring to the implications of the process of solving an open problem for the solver as individual or as a member of a group.

The problem itself was evaluated as open in terms of the kind of data given: "it is open (the second problem) as it does not give numbers"; its complexity: "the basic characteristic is the complexity of the demands which need a particular way of thinking to be understood"; its phrasing: "The phrasing is interpreted by the reader in different ways"; its applicability: "It has applications in everyday life"; the number and the kind of solutions: "It has more than one solution", "There is not a fixed way for solving it"; the degree of openness: "The second problem is more open than the first because it can be further extended according to the data that we give each time".

Concerning the solver's actions the student teachers often referred to the need to find extra information: "The children have to organize (in the 2nd problem) the procedure needed. For example, how many children are in the classroom? How much does the ticket cost? How much does the magazine cost?". They also appreciated the role of open problems in the development of thinking: "it needs critical thinking". Finally, the use of everyday or mathematical knowledge and more often the combination of the two were also included in their arguments: "It leads someone to thinking both on mathematical and practical levels"

The implications for the solver were identified mainly from student teachers' responses about the importance of the first given problem. These referred to the development of thinking, to the understanding of mathematical concepts, to the development of children's abilities to find relationships between mathematical concepts, to solve problems, to apply mathematics and finally to develop an awareness of what a problem is. Some characteristic responses are the following:

"This problem offers the deepening of mathematical thinking, the recalling and the use of multiple mathematical concepts and the relation between them."

"He will understand better the mathematical concepts, clarify their operations and this will result in his handling them in the best way."

"Someone realizes that these kinds of problems have a lot of solutions according to each one's thinking."

"He will practice his thinking, he will learn to think in a different way, to find multiple solutions."

Moreover, they acknowledged the importance of dealing with open problems through teamwork as if encourages cooperation and communication.

Student teachers' views of open problems as designers

In the second phase we analysed the problems that the student teachers designed and the classification that emerged appears in the systemic network of figure 2. Two main dimensions have been identified: the kind of problem and the arguments for its choice. The first is further analysed in terms of the openness of the problem, the mathematical content used and the context in which the problem was situated. Regarding the openness, some student teachers designed a closed problem, which they considered "open" because it involved more than one arithmetic operation or had a reference to everyday life. An example of such a problem is the following:

"A poultry-farmer has 15 hens. Each hen lays 4 eggs per week. Each egg costs 80 drachmas. How much does he earn every month, every semester, every year? If he spends 2000 drachmas per month to feed a hen, what will his net income be?"

Another group of students conceived an open problem as a teaching situation where the children had to use physical materials to understand certain mathematical concepts. An example of this was a situation where the children were asked to share a chocolate bar among a number of other children in order to develop the meaning of fractional units. In a similar situation, the children were asked to make an open exploration through drawings in order to discover properties of triangles.

A large number of problems that the student teachers designed could be characterized as open. These have been classified either in terms of the structure of the problem or of the process of solving it. So, three groups of problems have been constructed: problems without numerical data, logical problems where the development of strategies was emphasized and problems including both closed and open questions. We give below some examples:

"We know that in a weekday a pupil goes to school, attends English classes, studies, eats, sleeps etc. We want to calculate whether the hours that he plays daily (during the weekdays) are more or less than the hours that he plays during the weekend when he does not go to school." (non-numerical data given)

"At the toll stations of Patras, the buses and the lorries pay 1000 drachmas, the cars 600 drachmas and the motorbikes 400 drachmas. From 8.00 am to 12 am yesterday, the cash

taken was 200.000 drachmas. How many buses, lorries, motorbikes and cars could have passed?" (logical-combinatory)

In terms of the solution process, the problems could have a number of solutions, as in the problem of the toll station described above, a specific method of solution, as in the problem with the hours of play. There was also a group of problems, which had a specific solution but could be solved in different ways.

Most problems were arithmetical involving the use of the four operations while very few were geometrical. In terms of the context only a few problems were purely mathematical. Some of the problems like the one regarding the toll station referred to a realistic situation. The school type context included situations, which were 'dressed' in context:

"We have a basket with one orange that has on it the number 2, two apples that each have the number 4, three pears that each have the number 6, and four strawberries that each have the number 8. If we have a basket and we want to fill it with fruits that give us the following sums 10,12,6,8, which and how many fruits we will choose"?

Student teachers' arguments for choosing the problems

Arguments can be categorized in terms of their reference to the problem itself, in terms of the implications for the children and in terms of the broadening of teacher's knowledge. So, student teachers argued for their choice of open problem referring to its context or to its other characteristics. The familiarity of the context to the children, its relation to everyday life and to current social issues were arguments concerning context given by the student teachers. Some of the other characteristics mentioned were problem's "exploratory nature" and the possible extensions that the problem offered.

In terms of the implications for the children regarding their involvement in solving open problems, reasons mentioned were children's positive feelings, the development of their thinking and the appropriateness of the problem to children's cognitive needs. An example of the first category is: " the children feel happy as each solution they give can be acceptable", while for the appropriateness of the problem to children's cognitive needs arguments were like: "it is relevant to children's prior knowledge", "it is appropriate to the class age". Most reasons referred to the development of mathematical thinking skills: "the children are practising their thinking skills", "they relate different parameters", "they recognize the appropriate mathematical concept", "they are practising existing knowledge". Arguments referring to the development of other types of thinking, like "it can develop awareness of the nature of a problem" and "it can broaden children's conceptions about mathematics" were also given.

Finally, the broadening of a teacher's knowledge was also given as an argument. Some examples were, "it gives us evidence about children's thinking", "it evaluates children's prior mathematical knowledge".

Student teachers' reflections after applying open problems in the classroom

Student teachers' reflections as expressed in the third phase have been analysed and the identified aspects are presented in the systemic network of Figure 3.

The experience of using their own open problem in a real classroom created to most teachers positive feelings like, in their own words, satisfaction, certainty, and new challenge. Some of them, however, had negative feelings after the completion of this task and expressed anxiety, fear, dissatisfaction and uncertainty. There were also some students who showed mixed feelings. A few of them stated neutral feelings. A characteristic example follows: "My feelings were neither pleasant, nor unpleasant, neither during that experience, nor now. It was a normal situation. It was something different, but nothing so sensational."

Reasons offered for their feelings were related to the appropriateness of the problem, to the kind of previous experience, to the success of the whole process and to students' attitudes. The following extract is an example of the last case:

"During my experience in the classroom I felt both nice and uneasy. This happened because of the children, who at the beginning were quiet, cooperative and eager to solve the problem, while at the end they shouted, wandered all over the classroom, which means that the situation couldn't be controlled. I also have mixed feelings now, good feelings because the children solved the problem fast and in many ways, and bad because at the end we had a problem in managing the classroom."

In the descriptions of their experiences, two dimensions have been identified: implications for the teachers and implications for the students. The first dimension implies a development in teacher students' awareness concerning their children and also concerning themselves. So, an open problem can offer to the student teachers opportunities to obtain knowledge of their pupils' ways of thinking, cognitive problems and deficiencies. It also gives them opportunities to refresh previous knowledge in the classroom. The use of open problems can also reveal capabilities that low and high achievers have or do not have in problem solving. Such experiences are described in the following example:

"There were some students who answered all the questions with ease, while others met difficulties. However, even the second group, with our help and guidance managed to give the correct answer in the end".

Student teachers, in terms of themselves, think that they had a new experience, broadened the meaning of the open problem, developed awareness concerning the fulfillment of their teaching goals, or of the goals of the course. They also felt more capable in differentiating the initial problem in order to become appropriate for classroom use and more mature in managing the classroom organization more effectively. The last ideas are expressed in the following extract: " We should take care to create a more open problem... and also I think we should be less directive in handling the classroom"(The two problems they used were closed ones, referring to children's everyday life).

The second dimension concerns the benefits of the open problem for the students. Student teachers think that the use of the open problem helped children to cooperate, encouraged classroom dialogues and supported the spirit of teamwork. They also noticed that such situations encouraged the children to express their ideas and develop their thinking, offered all children opportunities to act mathematically, aroused their interest and finally supported the development of children's problem solving skills. A characteristic example is the following: "I believe that the situation we have chosen aroused children's interest. All the children have tried to give an answer, while some have managed to find more than one solution".

Concluding Remarks

From the analysis above, a number of different aspects in student teachers' thinking of open tasks have emerged. Most of the categories of the networks seem to exemplify the main categories that Pehkonen (1999) formed by analyzing teachers' responses to the question "What are open tasks in mathematics?". In our study, student teachers in expressing their ideas about what an open problem is, they seemed to appreciate not only features of the structure of the problem itself but also how these features are related to the solver. However, the majority of the student teachers concentrated on the phenomenological characteristics of the problem and neglected deeper aspects of the problem such as the thinking processes required for solving the task. In the construction phase almost all the student teachers argued for their proposed open tasks but they expressed a surface understanding of the role of these tasks in learning and teaching mathematics. Another issue that emerged was the student teachers' difficulties in designing an open problem. A number of student teachers proposed closed problems as open. Moreover, some of the open tasks that produced had a similar content or structure to those that they had experienced in the first phase. It seems that their experience with the open tasks in the course was not adequate to help them design a form of open problems that would meet their expectations. The phase of the implementation of the designed open tasks in the classroom and the student teachers' reflections revealed the difficulties that were met in a real situation. These were due to student teachers' lack of teaching experience and in particular of the use of open tasks. However, the student teachers seemed to have developed a degree of awareness of the meaning of open tasks and of their function in the classroom. In particular, they realized that with such teaching situations they had more evidence about pupil's learning, they recognized the limitations of the open tasks designed and used and they became aware of the differentiation of classroom management needed while using open tasks.

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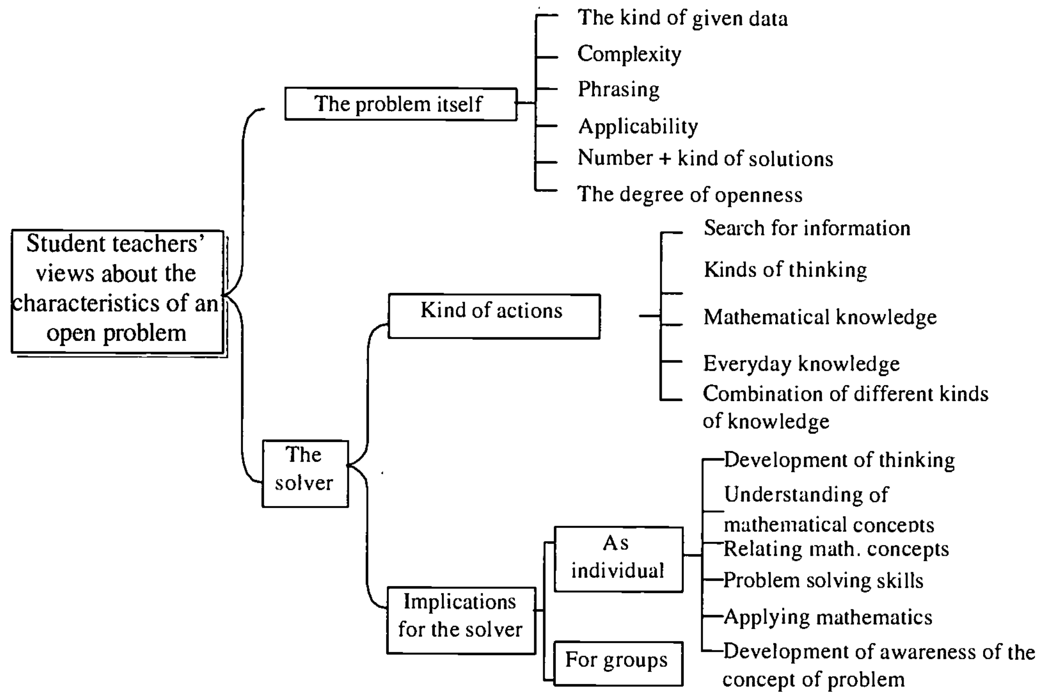

Figure 1. *Systemic network representing student teachers' views of open problems as learners*

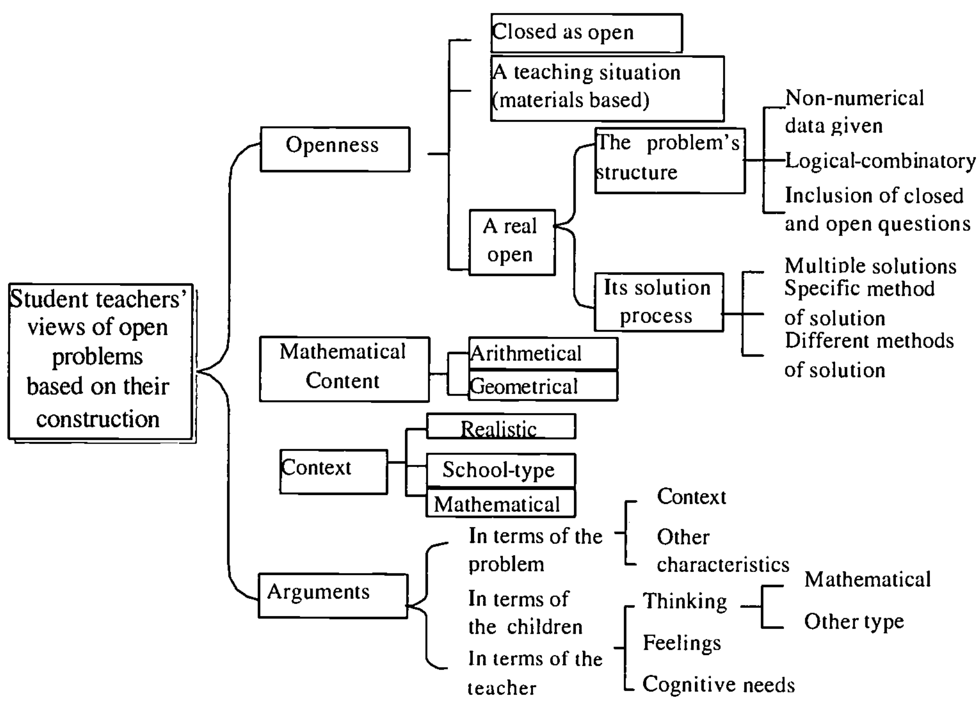

Figure 2. *Systemic network representing student teachers' views of open problems as designers*

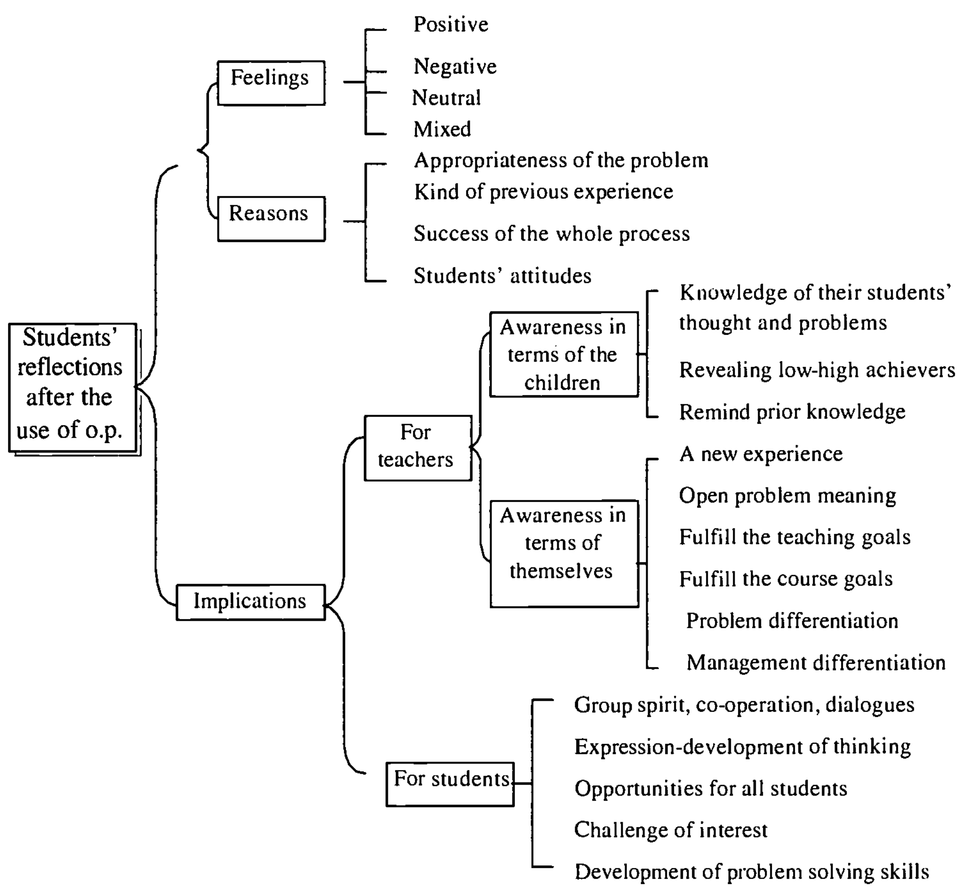

Figure 3 *Systemic network with categories emerging from student teachers' reflections after applying open problems in the classroom*

SOCIOCULTURAL FACTORS IN UNDERGRADUATE MATHEMATICS: THE ROLE OF EXPLANATION AND JUSTIFICATION

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ABSTRACT

This paper extends the study of social interaction patterns as a means to characterize mathematics learning to the learning and teaching of mathematics at the undergraduate level. We present here the analysis of teaching episodes from a discrete mathematics course to document the change in social and sociomathematical norms over the course of one semester. First, the instructor established the social norm that students justify, explain and share with their peers their thinking and solution processes. We show how the instructor of the course established an expectation for explanation and justification, and how students' interactions developed in accordance to this normative understanding through the semester. That is, we trace students' development from the passive acceptance of the instructor's authority to the expectation that students become contributors to the class and that they all share common understandings. We then shift our focus to the sociomathematical norms – normative interactions specific to mathematics. We discuss the development of students′ explanations from the procedural level to ones that are grounded in deeper conceptual understandings. We finally link the shift in the aforementioned social and sociomathematical norms in students' interactions to the development of students' ability to reason deductively.

Introduction

Schoenfeld argued that "mathematics is an act that is socially constructed and socially transmitted" (1992, p. 335). As such, it is governed by a set of norms; an etiquette for what is deemed appropriate behavior by members of the mathematics community. These can be *social* rules, that is, the ways in which members of the community interact and exchange ideas – rules that are not specific to mathematics but may characterize the behavior of members of other fields (e.g., historians). There is also a set of mathematical, or *sociomathematical* rules, that is, rules that are specific to the field of mathematics, such as what constitutes a proof (Cobb, Wood, Yackel, McNeal, 1992; Yackel & Cobb, 1996). While the second set of rules are explicit in the field, the first set can be very implicit. And yet, one may argue that social norms constitute the broad basis upon which the mathematical norms are constituted.

In recent years, we have witnessed a renewed interest in this social facet of mathematics and a growing tendency in studying social interaction patterns as a means to characterize mathematics learning (e.g., Yackel, 2001). Yet, little work has been done at advanced levels; the bulk of the research in this area has been conducted in elementary and secondary school classrooms (Cobb & Bauersfeld, 1995; Cobb, Yackel & Wood, 1992). In this paper we join the efforts of Yackel, Rasmussen and King (2000) to extend these analyses to the learning and teaching of mathematics at the undergraduate level using data from a classroom teaching experiment in discrete mathematics. We document the development of social and sociomathematical norms regarding explanation and justification over the course of one semester, and we discuss how these norms were constituted in this specific case.

First, we focus on the social norm that students publicly explain their thinking and solutions and try to make sense of other students' thinking. We show explicitly how the instructor of the course established an expectation for explanation and justification, and how students' interactions developed in accordance to this normative understanding through the semester. That is, we trace students' development from the passive acceptance of the instructor's authority to the expectation that students become active contributors to the class and that they all share common understandings. We then shift our focus to the sociomathematical norms – interactions specific to mathematics. We discuss the development of students' explanations from the procedural and empirical level to ones that are grounded in deeper conceptual understandings within the context of the course. Finally, we discuss the social interactions with respect to reformed instruction in advanced mathematics classrooms.

Methodology

Participants for the study were a group of 50 undergraduate mathematics students enrolled in a two-semester, first-year course on discrete mathematics emphasizing mathematical argumentation and proof. The course was taught by one of the two investigators, while the other investigator collected data. For homework assignments and reference purposes, the course used a broad text on discrete mathematics (Grimaldi, 1999). A typical class section begun with a problem introduced by the instructor followed by student group work. Students were encouraged to ask each other questions and help each other clarify concepts and problem requirements. The small group work was usually alternated with whole class discussion of students' approaches, thinking and questions. Throughout the course there was a concerted focus on both written and verbal expression of student thinking. Implicitly the instructor worked towards establishing the social norms that students are expected to explain their reasoning, to try to make sense of each other's explanations, and to challenge each other's reasoning and justifications.

Each class was videotaped and attention was paid to both the instructor's actions and the students' reactions, including the students' interactions when working in groups. In analyzing the data, our first goal was to demonstrate the use and change of social and sociomathematical norms in the classroom over time. In order to document these, we analyzed transcripts of classroom discourse data according to its function and pattern (Potter & Wetherall, 1987), using each speaker's turn as the basic unit of analysis. We focused our coding on the forms of explanation and justification used by students. This required detailed coding of verbatim transcripts, with the meaning of each speaker's turn interpreted within the context of the larger conversation. Additionally, students were presented with written assessments, at the beginning and end of each semester. These assessments were analyzed to identify shifts in students' proof schemes (Harel & Sowder, 1998) and each student's own competency in justifying and proving.

Social Norms

Students were initially surprised by and even resistant toward the social norm that they explain their thinking and try to make sense of other students' thinking. It became apparent that the instructor's expectations that students explain publicly their thinking ran counter to the students' earlier experiences of mathematics work. Students felt uncomfortable engaging in explanations of their thinking and even lacked the language to do so. They were initially hesitant to challenge their classmates' thinking and acknowledged that they did not know how to explain why their solutions worked. However, as the semester progressed, students got accustomed to engaging in explanations and justifications. Here, we present excerpts from two different episodes in the course from two different points in time, that sharply contrast social norms regarding student explanation and making sense of each other's thinking. In the first case that took place during the second week of the semester, students did not feel the obligation to explain their thinking nor did they expect to make sense of other students' explanations, despite the instructor's urge to do so. In the second case the students felt obliged to do so, without prompting from the instructor.

First episode (second week). The class was introduced to combinations and permutations – students were asked to find the number of different combinations of pastries one can purchase from a bakery. The instructor prompted students to "think of their thinking" and to question each other's approaches and arguments implicitly letting students know that there is an expectation that they will engage in this question and share their reasoning. Further, students were asked to work in groups. The following was the interaction among Isabelle and Josh:

Isabelle: What did you do?
Josh: You multiply them all out and you get 10×9×8×.....
Isabelle: Oh, OK.

The level of discussion described in this short episode among Isabelle and Josh is illustrative of the discussions that took place among almost all groups; when students were asked to work in groups and to collaborate in solving the problem while making sure they question each other's thinking, they, instead, tended to ask each other (or the instructor) for a method to solve the problem or for an answer–a procedural approach to problem solving–and accepted each other's solutions without further questioning. In the few cases where a student asked another student for further clarification or explanation for his answer or approach, the response often was "it worked for me!"

Second episode (seventh week). The class was discussing rational and irrational numbers and students were asked to consider the square root of 2, and to show it is irrational. As usual, the instructor prompted students to question each other's approaches and arguments. The following is the interaction among Jared, Daniel, and Mike while thinking about the problem in a whole class discussion:

Jared:	I set $\sqrt{2} = p/q$. Then I...
Daniel:	What are p and q?
Jared:	Two integers
Daniel:	Any integers?
Jared:	Two integers
Daniel:	If it's not *any* integers, then it's not true for all cases, and then someone can come up with a case where it fails and your argument is gone.
Mike:	To me, the important thing to remember is that $\sqrt{2}$ is written as a specific ratio, not any p/q. We are trying to show it can't be rational....

In contrast to the first episode where students hesitated to challenge each other, during the second episode, students expected their classmates to explain their reasoning. After Jared started sharing his thoughts, Daniel, without prompting from the instructor, asked for further explanation – what numbers was Jared considering in his proof. Jared clarified, but Daniel prompted for more – a sincere attempt to understand Jared's reasoning. Notice, however, that the interaction was not a dialogue among two naturally inquisitive students; Mike joined the discussion in an attempt to clarify the argument further. Mike's language further suggested that the argument was a collective one; he pointed that "*we* are trying to show it can't be rational" (emphasis added), it was no longer Daniel's attempt to show that the square root of 2 is irrational, it was an argument embraced by the class. It was, from that point on, the class' responsibility to clarify for each member of the community and to ensure that each member shares the ownership and understanding of the argument.

Overall, this episode illustrates how the students had advanced during the course of semester in their ability to debate with their peers. Furthermore, they had overcome their initial resistance towards public argumentation and had developed the expectation that others explain their reasoning to the class. The two social norms, that students were expected to explain their reasoning and that they were expected to try and make sense of other students' thinking were gradually constituted throughout the semester. Such discussions are essential in students' mathematical development and in the development of the classroom as a community of learners.

Sociomathematical Norms

We showed that as the semester progressed, students got accustomed to engaging in explanations and justifications. Furthermore, the *quality* in students' explanations and their capacity to express their mathematical thinking in increasingly formalized ways changed substantially over time. Students' arguments gradually shifted from empirical and procedural to deductive and conceptual. As students advanced in their ability to argue, they also raised their expectations as to what counts as a strong mathematical argument; while during the first weeks of class the instructor's request for explanation often resulted in a description of the procedure that a student used or the listing of several examples, a few weeks later students attempted to explain the generality of the argument. We argue that students acted in accordance with the normative understanding that they were expected to explain, but they also established sociomathematical norms that are very specific in the mathematics community as to what constitutes an acceptable explanation in mathematics

Once again, we discuss the two episodes in the course that were presented in the previous section to contrast social norms regarding student explanation. We now discuss these same episodes from a different standpoint; students' growth in their use of *mathematical* arguments.

First episode (second week). In the first episode presented in the previous section, Josh shares his solution with Isabelle. Isabelle's question "what did you do" is a prompt for a procedure that will produce an answer and that is precisely what Josh has to offer – a guide that will lead her to the correct solution to the problem. Josh did not see the need to give a conceptual explanation (why one should multiply out all the numbers) and Isabelle, in turn, was satisfied with the procedure and did not see the need to prompt for an explanation. The discussion among Isabelle and Josh once again illustrates the quality of the arguments that were exchanged among students during the first few weeks of the semester – students exchanged procedural explanations and recipes for solutions that appeared to produce correct answers.

Second episode (seventh week). In this episode, Jared started to share his approach to showing that the square root of 2 is irrational, but was interrupted by Daniel who questioned the generality of Jared's use of integers. Finally, Mike attempted to help Daniel in understanding the proposed solution. Their mathematical argument seems to be in determining the meaning of 'p' and 'q', specifically, whether they represent a fixed but unknown pair of integers, or whether they represent any two arbitrary integers. The excerpt shown in the previous section illustrates that the students acted in accordance with their own understanding in explaining their thinking and making sense of each other's thinking and it attests to the existence of classroom and sociomathematical norms by which such conversations can occur. Such discussions are essential in students' mathematical development and in the development of the classroom as a community of learners. In particular, we claim that classroom discussions such as this helps to build a habit of mind whereby students internalize public argumentation in ways that facilitate private proof construction.

We take shifts in student responses to one proof problem given on two occasions as part of the evidence for this claim. The problem was administered on the first day of class, before any instruction occurred, and 9 weeks later on a mid-term assessment. For purposes of interpreting the quantitative results, we provide one possible "correct" solution.

> **PROBLEM:** *Prove that the sum of an even number and an odd number is always odd.*
>
> **POSSIBLE SOLUTION:** *Let x be even and y be odd. They* $x = 2m$ *and* $y = 2n+1$*, for integers m and n. Then* $x + y = 2m + 2n + 1 = 2(m+n) + 1 = 2k + 1$*, where* $k=m+n$ *is an integer. But* $2k+1$ *is odd, by definition, so* $x + y$ *is odd. Thus the sum of an even number and an odd number is always odd.*

The problem described above was given to students as part of an individual pre-assessment at the beginning of the semester, and as part of a mid-term assessment. We do note, however, that students worked *in pairs* during the mid-term assessment. While this arrangement certainly contributed to the success students had with this problem, it also illustrates the type of socio-mathematical norms that had evolved in the classroom, norms that we take to be critical for the development of students' capacity to build proofs. Even so, the results clearly indicate significant gains in students' responses. Table 1 shows a summary of student solutions to this problem.

	Pre-Test (individual) (50 responses)	Mid-Test (paired) (51 responses)
Correct proofs		
completely correct	1 out of 50 (2%)	34 out of 51 (67%)
almost correct (minor error)	---	13 out of 51 (25%)
Total correct proofs	1 out of 50 (2%)	47 out of 51 (92%)
Incorrect proofs:		
Used examples as a "proof"	26 out of 50 (52%)	2 out of 51 (4%)
Used illogical reasoning	10 out of 50 (20%)	2 out of 51 (4%)
Looked at a narrow case	10 out of 50 (20%)	2 out of 51 (4%)
No attempt made	7 out of 50 (14%)	0 out of 51 (0%)
Total incorrect proofs	49 out of 50 (98%)	4 out of 51 (8%)

Table 1. Summary of student responses.

Only one person (2%) gave either a correct or essentially correct proof on the first attempt, while 92% of the class gave correct (67%) or essentially correct (25%) proofs on the second attempt. In addition, 52% of students on the first attempt used examples to 'prove' the conjecture, while only 4% of students used this as a strategy on the second attempt. Moreover, there was a significant increase in students' level of formalization, particularly, their capacity to express their thinking in increasingly formal ways via symbolic language. Only 16% of respondents on the pre-test used some form of symbolization, whether correctly or incorrectly (otherwise, if students attempted a proof, they used everyday language). On the mid-test, 94% of students expressed their proof or proof attempts symbolically in a manner similar to the possible solution given here.

Concluding Remarks

This study adds to the literature on the nature of cognitive and social dimensions of mathematics instruction and learning at the university level. We presented here some examples on the nature of social and sociomathematical norms that support the learning of mathematics in a discrete mathematics classroom. Our results suggest that college mathematics classrooms can potentially function as communities of learners, in which students engage in sense-making and meaning-making. In this respect, this study supports the work of Yackel, Rasmussen and King (2000), in that, over time, students' attitudes develop from the passive acceptance of the instructor's authority to the expectation that students become active contributors to the class and that they all share common understandings.

The significance of this work for mathematics reform at the university level is that it provides a different perspective to view and analyze mathematics learning that complements the work of mathematicians and mathematics educators who have focused primarily on the individual cognitive aspects of advanced mathematics learning (e.g., Dubinsky, 1992; Harel & Sowder, 1998). We are suggesting a shift towards the study of social processes as a way to understand students' development. As we continue to examine the cognitive and social dimensions of mathematics learning in college classes, it is important that we look deeper into the interconnections of social and cognitive development.

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MATHEMATICS PROBLEM-BASED LEARNING THROUGH SPREADSHEET-LIKE DOCUMENTS

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ABSTRACT

Problem-based learning is particularly suitable in fields like Mathematics. It is quite usual to find abstract descriptions of solving methods to express mathematical knowledge. However, these descriptions happen to be somewhat hard for students. For instance, when a learner faces the integration-by-parts method, one possibility is to try to understand the general formula (by using the abstract description), while on the other hand another possible scenario is to learn by watching particular examples. Both approaches are complementary and, in fact, teachers usually move back and forth in order to manage their students to understand the underlying concepts.

In this paper we introduce *ConsMath* (CONStraint-based MATH teaching), a computer system that includes an authoring tool for the creation by the teacher of interactive Mathematics documents. The teacher can establish spreadsheet-like relations between the different mathematical formulae that appear in the document, and require certain conditions to be held when used by the student. The documents are dynamic and their contents changes depending on the formulae filled in by the student. ConsMath can be used in a particular style of problem-based learning in a context where each document is a problem pattern, and students can work on those problems by practicing with them repeatedly, and deciding in each case about the different steps that are necessary for their resolution. *ConsMath* runs both as a standalone application and in an applet within a web page.

An important feature of *ConsMath* is the possibility for the teacher to create interactive documents starting from static ones. Besides, the student can also ask *ConsMath* to generate specific problem statements from a given problem pattern. Consequently, students can choose between working on a problem posed directly by themselves, or asking the system to generate different problems corresponding to some given problem patterns.

KEYWORDS: Problem-based learning, Mathematics teaching, Spreadsheet, Interactive documents, Distance learning, Authoring tool.

1. Introduction

Learners of Mathematics many times have to solve problems that involve symbolic calculations by performing appropriate sequences of steps. These steps involve the manipulation of mathematical formulae obtained from the ones that appear in the statement of the problem. There are courses on specific subjects, like Integral Calculus and Ordinary Differential Equations, in which a high percentage of the work of the students is devoted to learning different methods of problem resolution of the above kind, and learning when these methods are successful. For instance, a typical course on Ordinary Differential Equations can include methods for the solution of linear equations, solution of equations by separation of variables, homogeneous equations, etc. Similarly, a course where basic integration methods are explained will include integration-by-parts and specific methods for the integration of rational functions, among others; all these methods are processes that consist of steps to be applied to the integrand. In general, Calculus and Algebra are particularly suitable to this approach, as calculations and predefined steps are necessary for the resolution of particular types of problems, although other considerations are also needed such as applicability conditions for certain steps. Textbooks like (Simmons 1981) help students to get insight on the different methods by showing particular examples that can be generalized. The work of the student when solving problems proposed by the textbook or by a teacher consists very often in reasoning about which method can be applied to the problem at hand, and applying it, or trying to adapt a known method to a slightly different situation.

On the other hand, problem-based learning in general, (Dutch & Gron & Allen 2001), is particularly suitable in Mathematics; this is especially relevant in subjects like the ones we have described in the previous paragraph, and textbooks of the type mentioned above help teachers and students to organize the learning process along the lines proposed by this theory. One of the main difficulties students have to overcome is the understanding of abstract descriptions of solving methods that express mathematical knowledge. For instance when a learner faces the integration-by-parts-method, one possibility is to try to understand the general formula (by using the abstract description), while on the other hand another possible scenario is to learn by watching a particular example. Both approaches are complementary and, in fact, teachers move back and forth in order to manage their students to understand the underlying concepts. Problem-based learning can take place in different ways, either through standard presentations by a teacher, or by means of a book or a computer. Even a collaborative approach is possible. Among the advantages of learning in the context of representative problems, probably the most outstanding one is the fact that this method of work allows the student to build a deeper abstract idea out of particular cases. Besides this, students can also recognize different forms of similar problems more easily, and they generate active self learning attitudes that are fundamental as a global goal of the educational process. Moreover, the learning process itself can be more attractive to the student due to the possibility to select the problems to be solved.

In this paper we show how the spreadsheet paradigm can be used in computer assisted tutoring of Mathematics courses of the kind introduced above. Moreover, we also show how these ideas can be used by means of a computerized tool that allows Mathematics teachers, without the need of a specialized technological knowledge, to define sets of problems that cover subjects similar to the ones we have described. The students can work on those problems by practicing with them repeatedly, and deciding in each case about the different steps that are necessary for their resolution.

Before going into more details, we shall see what is needed in order for a computer system to accomplish these goals. Firstly, we must be able to represent math formulae, so we need a powerful language to represent them. Besides, a specific software for rendering the formulae in a convenient manner is necessary, and a WYSIWYG (what you see is what you get) formula editing tool is also necessary. Secondly, the computer system has to support symbolic computations, since much of the work in the areas of Mathematics we are interested about entails transformations of formulae by purely applying arithmetic and symbolic rules, as it happens for example when solving a second-degree equation. Finally, we must remark that both the teacher and the student have to benefit from these possibilities, the teacher using the system as an authoring tool, and the student as a learning tool. In addition, distance learning scenarios would help, since teachers and students would not have to be "at the same time in the same place", and even in such a case a collaborative environment would be thereby possible.

More precisely, in this paper we describe the *ConsMath* (CONStraint-based MATH teaching) computer based authoring tool, which is based on the previous proposals and allows students to learn suitable mathematical concepts and problem solving procedures by means of a specific guided problem-based approach, based on the repetitive practice of particular techniques or processes. Teachers can create documents and sets of problems with *ConsMath* by describing the steps involved in a certain problem-solving method (e.g. solution of differential equations by separation of variables, or differential homogeneous equations) by using constraints between parts of a mathematical text according to a spreadsheet-like fashion. The use of the system in a distance learning environment is also possible, and consequently (this is an ongoing work) in a collaborative framework, (Mora & Moriyon 2001). Moreover, an important feature of the system is the possibility for the teacher to create interactive documents starting from static ones, which shall be explained below.

ConsMath is being developed as a part of the *Encitec* project, (Encitec), that is aimed at the development of tools for the development of distance learning materials and courses in scientific fields, involving symbolic, graphic and simulation components. A first prototype of *ConsMath* has been developed that includes the functionality explained in this paper. It has been tested by Mathematics teachers in order to build practising materials on Calculus and Ordinary Differential Equations, and the use of the corresponding sets of problems by students has started recently. From this point of view the system is suitable for its use according to the initial goals, but a detailed evaluation of its possibilities is still needed. From the technological point of view, the system is completed since the first prototype has been released earlier this year. However, assessment trials are still too limited. The assessment of the technology that has been carried out from the point of view of allowing teachers to incorporate their ideas has been successful, but the result of ongoing tests with students will be essential for the development of a system that can be used by teachers without any help in order to develop interactive materials related to different subjects.

In the following sections, firstly, we describe the technologies available to address these issues (section 2), and after this we describe our approach in the *ConsMath* computer system. Specifically, a first assessment of the system in terms of its use by teachers is discussed (section 3.3), along with considerations related to the technology chosen to implement the system (section 3.2), and a description of the system in terms of its possible use both by document authors (teachers) and by students (section 3.1).

2. Technology

There are different computer languages for the representation of Mathematics formulae, like *TeX* and *MathML*. The *TeX* language is broadly used in writing Mathematics books and papers, whereas *MathML*, (MathML), was conceived for the representation of Mathematical texts in web pages and for their future interactive treatment. The web is progressively becoming an appropriate instrument for the spreading of mathematical texts. However, W3C's *MathML* has not yet become extensively used in web browsers, and in fact only a few browsers, e.g. (Amaya), support *MathML*, though there are also cxamples of plug-in software (TechExplorer) to display both *TeX* and *MathML* documents in general purpose browsers. Nowadays formulae are usually represented in the web by using image files, nevertheless, *MathML* is being used as a common language to represent formulae in several computer systems, since interoperability and reusability are guaranteed from its use.

On the other hand, there are systems such as *Mathematica*, (Wolfram 1999), which support symbolic calculations (so are *Mapple*, *Matlab*, etc). In this system, a specific language is used to represent formulae, though the user writes formulae in the usual way (e.g. by entering "x+y"). Moreover, it contains a module for the transformation of *Mathematica*-based formulae to other languages such as *TeX* or *MathML*.

Graphical WYSIWYG equation editors are also widely available. For instance, *Mathematica* incorporates a built-in one that allows the user to select from palettes the corresponding operators. Other common tool is the Microsoft equation editor, used to generate formulae included in Microsoft Word documents. Another example is *WebEQ*, (WebEQ), a Java equation editor that can be used as a program included in web pages (applet). However, all these editors are ad-hoc elements that cannot be used in a project for Mathematics learning such as the one we propose in this paper. Specially, their extensibility is the main problem since they are not open source tools.

As far as networking, web browsers are the common means for distance learning. In particular, web pages can contain formulae represented as image files, or can have Java applets inside. Networking capabilities of the Java language allow communications between web browser users (students), and either a server or other students. The *Mathematica* software, apart from being able to be used as a standalone application, can communicate with a Java program by means of *MathLink*. Therefore, it is possible to build a distributed computer system containing modules such as a *Mathematica* evaluator, and one or several Java programs (or applets). Recently, the *WebMathematica* module has been released, (*WebMathematica*), which basically enables the access to the *Mathematica* standalone application by using the web as its front-end; *WebMathematica* actually relies on *JLink* for its networking processes. Nevertheless, by using *WebMathematica* it is difficult to achieve a high level of interactivity due both to the limitations of form-based web pages obtained and the usability of the language required for creating these pages (*Mathematica Server Pages*).

Finally, it is worth mentioning the existence of a very limited amount of computer systems that can be considered authoring tools for the creation of interactive sets of Mathematics problems, comparable to *ConsMath*. *PAT*, (Koedinger 1998) and MathEdu (Díaz 2001) are the most remarkable ones. Both systems achieve a higher degree of interactivity than *ConsMath* does, since in particular *MathEdu* includes dialogs between the student and the system, and also the possibility to define subproblems that must be solved during the solution of a problem, but *PAT* is very limited about the fields where it can be applied, that is restricted to Linear Algebra, and *MathEdu* has many more limitations than *ConsMath* from the point of view of the knowledge it requires from the teachers for its use, since they have to know a non trivial amount of programming in *Mathematica*.

Extending a system such as the previous ones to fulfil the goals proposed in this paper is a hard issue. The *Leibniz* system, (Leibniz), a tool built on top of *Mathematica* for the creation of mathematical documents by carrying out evaluations or by applying operators to formulae appearing previously in the document, is an example of this kind. However, the level of interactivity the system allows is somewhat simple. Moreover, a higher level of extensibility and reusability is needed, since these conditions are important in the generation of digital interactive learning resources, (Roschelle & Pea & Digieano & Kaput 1999).

3. The *ConsMath* Approach

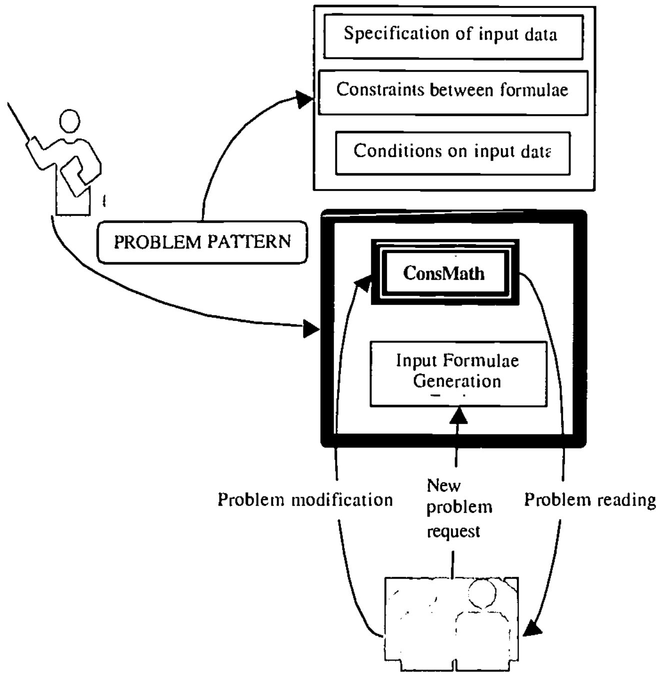
In this section, the approach used in the *ConsMath* computer tool shall be described. We shall present the system from three different perspectives: 1) a user-centred description in terms of the possible scenarios of use; 2) the decisions undertaken in relation to the integration of technologies needed, and the architecture of the system being developed; 3) a first assessment about the use of the system by teachers, and considerations about the benefits observed of using a system like *ConsMath*.

3.1. A User-centred Description

The *ConsMath* system has two types of users: the teacher and the student. As for the Mathematics teacher, he or she must have a certain familiarity with a graphical Mathematics editor, such as the Microsoft Equation Editor available in Microsoft Word. An important point is the possibility for the teacher to start with a static document, such as the description of the solution of differential equations by separation of variables. By using the graphical equation editor, he or she writes the equations (steps) involved in the method for a specific example. In a spreadsheet-like manner, when a formula depends on others, the teacher can define a constraint that links two formulae. For example, if a formula (or a portion of it) is calculated as a certain function of a previous one (e.g. on integration), this relation is defined by a constraint. The teacher can also designate the formulae that the student can modify (in our case the f function, for the differential equation y'= f y), namely input formulae, as well as demand certain conditions on those formulae. For instance, in our case the condition would be something like "dependsOnlyOn(x,f)"; as another example, in a problem where the teacher is explaining the method of rational function integration the condition to be held is that the integrand (input formula) must be the quotient of two polynomic expressions.

Students can use *ConsMath* in three different scenarios. Firstly, the student uses the material prepared by the teacher as a reading material, whether in a web browser, or in the *ConsMath* main window. Secondly, the student can interact with the teaching material containing solved problems. Specifically, the student can modify input formulae to generate alternative solved problems. In such a case, the system takes care of allowing those modifications (after checking that the conditions on input formulae defined by the teacher are satisfied), and it responds accordingly, by updating the formulae that are related to them by means of constraints. And thirdly, the student can ask the system to generate a similar problem about a solving method. The system responds by presenting a new problem where the input formulae have changed, while holding the conditions required for that solving method.

Each document created by a teacher that defines a solving method constitutes a problem pattern. The possible uses of *ConsMath* are depicted in the following figure:



3.2. Integration of Technologies, Architecture

The *ConsMath* system is implemented in Java. The reasons for this choice are the networking capabilities available in Java, along with the possibility of use in web browsers. As for the language for representing formulae, we chose *MathML*, particularly using content mark-ups, which give us a semantic representation of formulae, instead of presentation mark-ups suitable only for graphical renderings. Another reason for the choice of *MathML* was the number of efforts being undertaken to consider it as a standard, as well as that many systems, like *Mathematica*, provide support for conversion to *MathML*.

As far as the graphical equation editor concerns, we have our own Java program to graphically build equations, which are represented internally in *MathML*. We could have tried to use other software such as the *Mathematica* front end, *WebEQ*, or even the Microsoft Word equation editor. However, it was impossible to us to use them as specific libraries to include in our Java programs, so we decided to build our own editor. The first prototype supports a limited amount of operators, namely the ones that have been necessary for the tests that have been performed (though we are augmenting the available ones in our operator palette), but it has allowed us to do the tests described in this paper.

On the other hand, we rely on the *Mathematica* system for symbolic processing or evaluation of formulae. *Mathematica* has also a powerful pattern matching capability for Mathematics formulae that allows teachers to establish conditions on them. Moreover, when a student requests a new problem, the system generates random formulae that satisfy the conditions specified by the teacher. This part has been implemented in *Mathematica,* and it benefits from powerful capabilities of this system. Another interesting point is the modularity achieved in *ConsMath* in relation to the evaluation system used, as it can be substituted by a different one without any major changes in the tool.

3.3. Assessment

A complete assessment of *ConsMath* includes two different levels: on one hand, the tool must be tested by teachers in order to build their collections of problems on some specific subjects. On the other hand, the use by students of the materials developed by the teachers is also essential. *ConsMath* has been tested by Mathematics teachers in order to build practising materials on Calculus and Ordinary Differential Equations. The solution of a set of integration problems from (Spivak, 1989), and another set of problems on Ordinary Differential Equations from (Simmons 1981) have been implemented using *ConsMath*. The main difficulties found arose from the need to extend the set of operators included in our equation editor, something due to the fact that it was an unfinished prototype. The teachers who have used *ConsMath* consider that it is a highly suitable tool for teaching purposes. The corresponding sets of problems have started to be used by students recently, and no results are still available from this experience. However, we can already claim that the system is suitable for its use according to the initial goals, although a more detailed evaluation of its possibilities is still needed.

The tests that have been accomplished with teachers point out three important features of the *ConsMath* approach:

1) The spreadsheet-like fashion of problem patterns allows teachers with a certain familiarity to spreadsheets to emulate the constraint-based approach by generating documents that include mathematical formulae and constraints between them. These documents can also be used for simulation processes where the user observes the consequences of the modification of certain input data.
2) *ConsMath* allows teachers to generate documents in an environment that is similar to the one used by students. Moreover, teachers can seamlessly switch to the student role to verify the suitability of the document being generated.
3) With *ConsMath*, interactive documents can be generated from static ones, by specifying which are the input formulae and the constraints to be held between different formulae. Hence, available static documents containing *MathML* formulae are potentially suitable for its use with the *ConsMath* tool. As more an more static documents of this kind are available, the possibilities of use of *ConsMath* and the simplicity of using it will be bigger.

4. Conclusions and Future Work

In this paper, we have described our experience in the design of the *ConsMath* system. This system is an authoring tool for the creation by the teacher of interactive Mathematics documents that describe a given solving method. Teachers can establish spreadsheet-like relations between parts of documents and require certain conditions to be held when used by the student, who can benefit from the relations established by the teacher in a distance learning environment. In relation to the *ConsMath* approach, some advantages have been pointed out concerning to the ease of use by the teacher and the positive aspects of its usability, namely the spreadsheet-like manner of creating documents, the use of the same environments by the teacher and the student, and the possibility of creating interactive documents from static ones.

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TEACHING STATISTICS AND ACADEMIC LANGUAGE IN CULTURALLY DIVERSE CLASSROOMS

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ABSTRACT

The last decade has seen a substantial increase in the cultural and academic diversity of commencing tertiary education cohorts. The challenge for mathematics and statistics educators is the development of curriculum measures which address the language related difficulties of language minority students (Cocking & Mestre, 1988) and improve learning outcomes for all students. Our focus in this paper is on enhancing language and communication skills in culturally diverse undergraduate statistics cohorts. Most students have difficulty adjusting to the formal language requirements of academia. Non-English speaking background (NESB) students can have particular difficulty with the reading and assessment demands of Western universities if they are not adequately supported (e.g. Ballard & Clanchy, 1997). This is especially problematic when discrepancies between verbal and written expression and true intellectual ability result in assessment penalties. What is required are curriculum models which focus on **what students do** as opposed to deficit models which focus on **who students are** (Biggs, 1999).

We describe curriculum development in two subjects designed to teach language skills in statistics. Both subjects require students to engage with academic language and to develop statistical discourse skills relevant to modern professionals in the quantitative sciences. Methods used to encourage this include explicitly teaching academic reading techniques, and group research projects that are peer assessed. The projects are designed to develop statistical concepts within the context of professional practice and to address key competency requirements of relevant professional associations.

We will present data that suggest that NESB students have more difficulty than ESB students on "traditional" statistics assessment tasks and describe curricula interventions that assist those students to achieve their academic potential. The reaction of students to these developments has been very positive. The quality of the work is impressive and students improve both their statistical knowledge and their reading and writing skills.

Keywords: language, statistics, professional preparation

1. Introduction

Several studies have shown that students have difficulty with all forms of academic language: reading, writing, speaking and listening; and that students from a non-English speaking background (NESB) have more trouble that those from an English speaking background (ESB) (e.g. Martens, 1994). NESB students tend to have a more conserving attitude to knowledge (Ballard & Clanchy, 1997) resulting in a strong reluctance to "skim read" or "mine" articles for useful information. NESB students can also be reluctant to participate actively in class due to cultural influences which suggest that this is inappropriate behaviour (Ballard & Clanchy, 1997; Biggs, 1999). These factors are problematic when discrepancies between verbal and written expression and true intellectual ability result in assessment penalties. The drive to internationalise university curricula in Australia has created opportunities for local and international students but requires imaginative teaching solutions and sensitivity to cultural issues. In this paper we will describe work done with students studying in the English medium, though this applies to students studying in any medium that is not their home tongue.

All students, whether ESB or NESB, have to adjust to the more formal language requirements of academia. It is our thesis that these language requirements be taught explicitly rather than implicitly. We will present data that confirm that NESB students have more difficulty than ESB on "traditional" assessment tasks and describe interventions that assist those students to achieve their academic potential. It that curriculum measures for improving learning outcomes for students currently disadvantaged by language difficulties will improve outcomes for *all* students.

Two similar subjects that are designed to teach language skills in statistics will be discussed. Both subjects require students to engage with academic language and to develop statistical discourse skills (reading, writing, listening, communicating) required of globally competent professionals in the quantitative sciences. Methods used to encourage this include explicitly teaching academic reading techniques using statistical articles and independent group research projects which require students to synthesise a journal article relevant to their professional field and to present findings for peer assessment. The projects are designed to develop statistical concepts within the context of professional practice and have been popular with students. Teaching and assessment methods are also designed to address key competency requirements of relevant professional associations.

The reaction of the students has been positive. They appreciate that effort is being made, not only to improve the standard of teaching and learning, but to make what is learnt relevant to their careers ["I was surprised to see so much information about stats in the journals. Very few lecturers attempt to make the unit seem relevant to our careers but ... this is very worthwhile" and "The group project was good as it showed how statistics is applied to optometry."]. (All student quotes are reproduced here verbatim from original electronic sources.)

2. Rationale

The curriculum design for both subjects uses four theoretical frameworks. The first is a study of students' conceptions of statistics (Petocz & Reid, 2001). Petocz and Reid found that final year students who had studied statistics had three levels of conception about the subject, an *extrinsic technical* conception, *extrinsic meaning* and *intrinsic meaning*. This influenced the students' interaction with the discourse of statistics. Careful design of questions assists students to progress to higher levels of both language and statistics.

The second framework uses Systemic Functional Linguistics (Halliday, 1995). We can use Fairclough's three-dimensional conception of discourse (Fairclough, 1992:73) to design learning activities for undergraduates to help them understand:

- how the texts that they read are constructed,
- how they themselves can construct and interact with text,
- and how these constructions and interactions become the substance of that discipline.

To look at the specific needs of mathematics and statistics students, a research team gathered teaching and learning materials from a range of classes and video- and audio-taped several lecture and tutorial sessions. Some results were published in Wood, Smith & Baynham (1995). The team also collected examples of published work that exemplified the range of reading material that professional mathematicians and statisticians use, and the manner in which analysis is conducted and presented in professional journals.

Another framework to inform the design of curriculum is the increasing amount of work investigating graduate profiles and professional competencies. Both universities and professional societies are developing lists of graduate competencies that can be developed through degree programs. The development of academic and professional discourse skills connects well with the generic competencies needed by graduates of quantitative disciplines. These include: information retrieval; problem solving; application of knowledge; effective oral and technical communication; functioning as an individual and a member of a team; critical thinking; and lifelong learning skills. Fortunately, mathematics and statistics are two of only a small number of disciplines able to embed all the generic attributes in an integrated way (Challis & Gretton, 1997).

The final framework connects the ideas of equity, equal opportunity and non-discrimination which have become part of teaching and learning at university. For many lecturers these are principles they have adhered to throughout their lives. However, the recent changes have been in the legislative and social acceptance of these principles. Universities are no longer for the elite and importance is given to teaching and learning initiatives that are inclusive and assist students to reach their academic potential whatever their background.

Analysis of results data for a first year quantitative methods unit offered at Queensland University of Technology, Australia (see next section) suggested that NESB students performed, on average, at a level below their ESB counterparts on all five performance measures considered (see Appendix 1). There were, however, no significant differences in the level of high school achievement between the two groups. The teaching and learning strategies described here were motivated by our desire to improve the performance of NESB students who may be disadvantaged by language difficulties, while improving the discourse skills of all students by providing explicit instruction in reading, writing and presenting academic language.

3. Context

Quantitative Methods for Optometry and Health Science (QMOHS). This unit is taken exclusively by commencing students in Queensland University of Technology's Bachelor of Applied Science (Optometry) course. Teaching, learning and assessment practices place a heavy emphasis on developing statistical literacy (reading, writing and communicating statistics). Learning experiences are designed to develop students as "users of statistics" (practitioners) rather than "creators of statistical knowledge" (researchers). The competency standards of the Optometrists Association of Australia (OAA) (see Kiely *et al.*, 2000) relevant to the unit are drawn to the attention of students early in semester. These include an ability to: "read recent publications";

"undertake continuing professional education"; evaluate developments in clinical practice using "journals, videos, tapes, library, seminars, conferences"; demonstrate "independence in optometric decision-making and conduct"; and to "clearly communicate information to patients, carers, staff, colleagues and other professionals". This activity helps to set the tone for the unit and motivates the teaching and learning activities to follow.

A typical cohort has approximately 35 students, one-third of whom are from non-English speaking backgrounds. Of these, two (sometimes three) are full fee paying overseas students, usually from Singapore or Malaysia. On average, there are two mature age entry students per cohort. The majority of students (approximately two-thirds) are Anglo-Australian students entering QUT straight from high school. Students in the Optometry program are generally selected from the top 5% of high school achievers, or equivalent if overseas or mature age entry students.

Mathematical Practice. The second subject is *Mathematical Practice* taught at the University of Technology, Sydney Australia. Approximately 60 first year students study the subject each year, about 2/3 are from a NESB, half are male and all are studying statistics or operations research as a major. The subject aims to develop language and statistical skills, specifically:

- to distinguish between levels of formality used in various contexts, such as journal papers, lecture notes, seminars and discussion;
- to communicate knowledge clearly, logically and critically, in a variety of forms and levels, appropriate to the context;
- to analyse and criticise the way mathematics/statistics is used by the media;
- to judge the appropriate type and level of representation and language for a particular context.

The subject was developed because of the large number of NESB students studying mathematics majors and perceived problems with academic language related to statistics. We did not believe that NESB students should receive supplementary tuition in academic discourse but that *all* students would benefit from structured statistical language activities. This fitted well with the desire to introduce more communication skills into all degree programs.

4. Teaching and learning approaches

Quantitative Methods for Optometry and Health Science. The aim of the group reading and presentation project is to provide students with early exposure to the use of statistical techniques in *real* optometric scenarios. Students are required to select their own articles from amongst the professional optometry literature. This freedom allows students to follow up on topics of personal interest and encourages wide reading. Most importantly, the process requires students to make a substantial effort to transfer concepts from the "safety" of the classroom to the complex scenarios encountered in applied experimental design and data analysis. Students are supported by group consultations with the lecturer. During these meetings, statistical misconceptions are corrected and advice for the most effective way of presenting the material is proffered.

After gaining a basic understanding of the aims, methods and results of the study in question, students are asked to think critically about their articles. Are the techniques used appropriate? What assumptions have been made, and are they valid? What other techniques (if any) could have been used here? In some cases, students reconstruct the data sets from available information and redo the analysis as a check. At the end of their research phase student groups are asked to "distill" their articles in a form suitable for a 10 minute presentation to an audience of their peers and interested academic staff. These presentations are peer assessed (see Section 6). The critical thinking and presentation phase of the group project take place towards the end of semester, a typical QMOHS student's first at university.

Mathematical Practice. The teaching and learning approach is to use *real* materials. Starting at the lowest level of conception, we examine how texts are constructed. This is the extrinsic technical level of students' ideas of statistics. Students are encouraged to read widely in a range of genres, from informal newspaper articles and web sites, to research papers. The tasks lead them through structured reading activities. At this level of skill, students are technically proficient at statistics and reading.

	Fairclough (1992)	Petocz and Reid (2001)
Level 1	How texts are constructed	Extrinsic technical concept of statistics
Level 2	Construct and interact with text themselves	Extrinsic meaning
Level 3	How this all fits into their discipline	Intrinsic meaning

Table 1. Teaching approach

The next level is for students to start to construct and interact critically with texts. They are still looking at the learning process as outsiders, with the meaning external to themselves. At level 3 students have internalised the meaning. They become part of the discipline and see how the statistics and discourse fit together to communicate ideas. An example of a Level 1 and 2 learning task is listed in Appendix 2. In this task students are asked to find a recent article that uses statistics. The publications suggested are general science publications and this reflects the student body who are first year undergraduates. For a post-graduate group, I would ask them to find a professional journal in their discipline area. Section 1 of the assignment takes students through the development of reading and summarising skills. First they write down the aim of the article in one sentence, then summarise it in 3 points, then in 100 words. The 100-word summary is done for two different audiences to sensitise students to the idea that writing is for your audience. Section 2 of the assignment looks at oral presentation and so asks students to take essentially the same material and change to form of the summary for oral presentation. Again this is done for 2 different groups to encourage students to think about audience. The presentations for senior high school students have been particularly imaginative and reflect the fact that this group is close to the current age of the students so they are able to identify with the audience.

5. Outcomes

Quantitative Methods for Optometry and Health Science. The group reading and presentation projects have been popular with students since their inception in 1999. Their popularity stems from the exposure students gain to *real* applications of statistics ["the group reading project was useful both as an insight into the practical use of stats that otherwise seem fairly obscure in their application (in the field of optometry anyway). Also gives us a view into the "world of optometry" from a quite different point of view"]. They also give NESB students an opportunity, and the necessary encouragement, to contribute verbally in class. Previous research has suggested that NESB students studying in western universities tend to take on the role of passive learner, looking to the teacher for "direction, authority, and control" (Ballard & Clanchy, 1997). NESB students in QMOHS had similar reservations initially ["Most of the non-English back ground student will not ask question in the class. I think the reason is they afraid to speak out. We rarely speak out loud in the class during our high school"] but all participated enthusiastically in the end-of-semester presentations.

One of the most important outcomes of the reading and presentation projects has been a perception amongst students that group discussion of concepts aids understanding ["It was good to have a different way to help consolidate concepts learnt in class - through discussion with other group members."]. It is well recognised that the process of explaining concepts verbally is an important aid to developing higher order cognitive strategies (e.g. Rosenshine & Meister, 1992) especially when preparing students to tackle unfamiliar problems and when teaching students with a diversity of cultural attitudes towards reading and knowledge (Ballard & Clanchy, 1997). This was reflected in student surveys.

Giving students responsibility for preparing and communicating syllabus sections to other students can also engender a sense of "ownership" of the material and a perception of improved understanding and confidence (Coutis *et al.*, 2001) ["The group project was useful as [it] ensures you have a full understanding of the statistical methods used"]. These benefits were enjoyed by NESB students also ["Group reading is useful that we can discuss our problems and I think I understand more of the statistic after the group project!"].

Mathematical Practice. The *Mathematical Practice* subject has been running for 8 years. Surveys of students show that they perceive that their reading and writing skills have improved. The historical content of the subject is not popular with about a third of the students. For the oral component, the quality of the presentations is excellent and students feel more confident with public speaking. The written examinations demonstrate that students reach good levels of reading, comprehension and writing. The main presentation task is structured like a mini-conference. Students choose a theme and write papers to that theme (this year the theme is based on *The Code Book* by Simon Singh). The papers are refereed by fellow students and a book of proceedings is produced. Students appoint an editorial team who decide on the editorial guidelines and format for the proceedings. During the mini-conference, students are very supportive of their peers and particularly encourage students with poor spoken English.

Lecturers in subsequent subjects make use of the skills developed in the first year subject and require students to write and make presentations about the technical content of their areas. There is evidence that many students are reaching a Level 2 conception of statistics but few students reach Level 3 (Petocz & Reid, 2001). This is not surprising as the Level 3 conception is a professional level and we may expect to see this in final year students or graduates. We are investigating these conceptions with graduates.

6. Problems

The main problem encountered concerned the peer-assessed component of the subjects. A comparison of peer assessment marks for the QMOHS group projects in 2000 revealed that NESB students faired significantly worse in the eyes of their peers (Appendix 1). This is interesting when considered in tandem with student responses to the Likert scale item "The group reading project and presentation was a worthwhile learning experience" which suggested that the views of ESB and NESB students were comparable (p = 0.58). In hindsight the marking criteria used (see Appendix 3) may have disadvantaged NESB students who were self conscious about their spoken English ["We (non English background student) have problem while speaking to friends using

English, it's even worst when we need to talk out loud in the class. We'll get nervous and sometimes I'm not sure what I'm talking out there"].

In response to these findings, a 5-point Likert scale will be trialed this semester in place of the current criteria-based system, which requires students to give a numerical rating based on their own interpretation of the various grading criteria. MacAlpine (1999) found that this significantly improved consistency and discrimination amongst ratings given by final year Electrical Engineering students at the Hong Kong Polytechnic University because it gives more explicit guidance on relationships between level of performance and grade. Class time will also be set aside for groups to practice their presentations prior to their graded performance, and for misconceptions concerning the grading criteria to be addressed.

Our observations also suggest that students tend to avoid working in culturally mixed groups, limiting opportunities for cross-cultural perspectives (Volet & Ang, 1998). However, recent student discussions have encouraged us to experiment with mixed groups in the future ["Grouping with Australian student of course is an advantage because its help to improve our English by more frequent conversation using English. ... if we were grouped and not chose the group member ourselves, it will forced us to use English, that's better."]

7. Conclusion

The types of learning experiences described here develop discourse skills in statistics. More than that, they develop skills required for professional practice in disciplines that need the communication of technical information. They encourage deep learning by requiring students to internalise concepts, and to *explain*. It is a teaching and learning strategy that is appropriate for students studying mathematics majors as well as students studying statistics as part of another degree program. Students believe that the approach is relevant to their future work and study needs and there is evidence that the skills are transferable to other subject areas.

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APPENDIX 1

Comparison of ESB and NESB student performance in the unit Quantitative Methods for Optometry and Health Science (2000).

Item	NESB mean	ESB mean	p-value
Final Result (/100)	71.0	79.7	0.055
Final Exam (/100)	65.2	75.7	0.057
Group Presentation (/10)	9.06	9.46	0.0001
Class Tests (/30)	22.5	24.5	0.10
Group Assignment (/100)	95.1	95.6	.30

APPENDIX 2

Assessment task for Mathematical Practice

Aim: To develop skills in summarising information from mathematical/statistical writing.

Section 1: For this task you may choose an article from a recent edition of *New Scientist* or similar scientific, computing or financial journal. Only read the parts of the article that each question requires you to.

(a) Clearly reference the article

(b) Read the title, abstract (if any) and the first and last paragraph. Write down the aim of the article in one sentence.

(c) Now skim-read the article. Write down the three main points of the article in three dot points.

(d) Now read the article in detail. Write a summary of the article for your peers in about 100 words.

(e) Imagine that you are writing for students who are about to start university. Write a summary of the article for these students in about 100 words.

Section 2: Using the same article, this task requires you to prepare for an oral presentation of information.

(a) You are to deliver a 5 minute talk that summarises the article for your peers. List the points you would make and design three overhead transparencies to accompany the talk.

(b) Again, you are to deliver a 5 minute talk but this time it is to senior high school students. You will also prepare an A4 page handout to accompany your talk. Here the emphasis is on how to present the material in a way that the students will grasp the ideas. Write down how you would deliver the talk, what you would say and design the 1 page handout.

Marking:

0	1					1(a)	The aim is correct
0	1	2				1(b)	The three dot points summarise the content accurately
0	1	2	3	4		1(c)	The content and language of the summary are accurate, clear and appropriate for the required audience
0	1	2	3	4		1(d)	The content and language of the summary are accurate, clear and appropriate for the required audience
0	1	2	3	4		2(a)	Your content and overhead transparencies show accurate, clear and appropriate preparation for the required audience
0	1	2	3	4	5	2(b)	Your method of presentation, content and handout show accurate, clear and appropriate preparation for the required audience

APPENDIX 3

Previous peer assessment criteria for QMOHS group presentations.

Group Presentation Peer Assessment Form

Notes:

1) Content marks (/5) should be awarded on the basis of:

   i) quality of background information / introduction
   ii) demonstrated understanding of techniques discussed
   iii) quality of overview: the bigger picture?

2) Presentation mark (/5) should be awarded on the basis of:

   i) clarity of explanations
   ii) conciseness of summary/conclusion: take home message?
   iii) overall presentation quality
   iv) accurate citation details

MATHEMATICS FOR ELEMENTARY TEACHERS:
Making Sense by "Explaining Why"

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ABSTRACT

In order for prospective teachers to develop the reasoning and sense-making abilities of their future students, the teachers themselves must make sense of and reason about the mathematics they will teach. However, many prospective teachers have only experienced mathematics as the rote following of procedures, and are not aware that reasoning can be used to solve problems in non-standard ways, or that reasoning underlies the standard procedures in mathematics. A way to help prospective elementary teachers make sense of and reason about mathematics is to engage them in explaining mathematics. This paper discusses obstacles that arise in doing so, and recommends ways to overcome these obstacles. The paper also describes desirable features of problems asking for explanations, and gives examples. Finally, the paper gives guidelines to help students write good explanations.

1 Introduction

Recent reform efforts in mathematics education emphasize that students should make sense of mathematics and engage in mathematical reasoning (NCTM, 2000). In order for prospective teachers to develop the reasoning and sense-making abilities of their future students, the teachers themselves must make sense of and reason about the mathematics they will teach. However, many prospective teachers have only experienced mathematics as the rote following of procedures, and are not aware that reasoning can be used to solve problems in non-standard ways, or that reasoning underlies the standard procedures in mathematics. How then can prospective teachers learn to make sense of and reason about mathematics in a way that will help them to enable their own future students to make sense of and reason about mathematics? This article addresses this issue for prospective elementary teachers.

Certainly, making sense of mathematics and engaging in mathematical reasoning are intimately connected to *explaining* mathematics. Every mathematics teacher knows that when we explain mathematics, we enhance and solidify own understanding of mathematics. And every mathematics teacher knows that when we explain (or prepare to explain) mathematics, we sometimes uncover our own lack of understanding. It is only when we can explain a piece of mathematics in a way that makes sense both logically and intuitively that we feel we understand the mathematics. Thus, prospective teachers should learn to explain mathematics not only because they will explain mathematics to their future students, but also because explaining mathematics enhances their own understanding of mathematics and their own mathematical reasoning abilities.

To be an effective tool in teacher education, we should choose the explanations that we ask prospective teachers to give deliberately. What features should we seek in the problems we ask prospective teachers to explain, and why? What should we expect or ask teachers to draw on in producing their explanations? What are ways to help teachers improve their ability to explain?

2 What Kind of Explaining?

What kind of explaining of mathematics should prospective teachers engage in? Starkly different choices can be made, even when the subject matter is centered on the mathematics the teachers will teach.

One choice is to give prospective elementary teachers axiomatic developments of numbers and of geometry, and to expect the teachers to establish various facts in arithmetic and geometry by giving rigorous proofs that refer to axioms and to previously established theorems. These are not bad objectives, and they may be reasonable in small amounts, but will the teachers be able to use this learning to help their own young students reason about and make sense of mathematics? Realistically, the connection may be too long for most teachers to bridge in practice.

The Mathematical Education of Teachers (2001) recommends the following:

> All courses designed for prospective teachers should develop careful reasoning and mathematical "common sense" in analyzing conceptual relationships and in solving problems. (Chapter 2)

This suggests an *intertwining* of logical reasoning with ordinary sense-making. Thus the explaining that I advocate here is more than just logical and convincing to a skeptic; it should be truly explanatory, and it should help to make sense of the related mathematics.

For example, we can use induction to prove that

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

The proof is logical and it will convince someone who understands induction, but it doesn't show where the simple formula $\frac{1}{2}n(n+1)$ comes from. In this sense, it doesn't really explain why the equation above is true. Instead, if we imagine a "step triangle" made of n rows of squares, with 1 square in the first row, 2 squares in the second row, 3 squares in the third row, and so on, then we can see visually why the $\frac{1}{2}n(n+1)$ formula makes sense: put two step triangles together to make an n by $n+1$ rectangle.

"Explaining why" is different from proving in several ways. When "explaining why", a careful examination of several important cases is often more illuminating than an argument that covers all possibilities. For example, how should we explain why the standard longhand multiplication procedure is valid? Instead of a proof, we can examine some special cases carefully, such as some 2-digit by 2-digit products. Also, although one proof establishes truth, when "explaining why" we should seek several explanations, and we should try to coordinate these explanations. To explain why longhand multiplication is valid, we can use the distributive property; we can also draw a rectangle and subdivide it into pieces corresponding to steps in the procedure. Best of all, we can link these two explanations.

Thus I propose the following as desirable features of explanations that prospective elementary teachers should engage in:

- The explanation is logical.

- The explanation explains in a common-sense way. It is convincing, both to the person who is explaining and to the intended audience (e.g., peers, the instructor, children).

- If possible, there are several explanations, such as one using equations and one using a picture, and the explanations are coordinated.

The literature includes examples of teachers and prospective teachers engaged in sense-making by explaining mathematics. For example, Schifter (1998) describes a teachers' seminar in which teachers worked with problems such as the following:

> Wanda really likes cake. She has decided that a serving should be 3/5 of a cake. If she order four cakes, how many servings can she make? (p. 67)

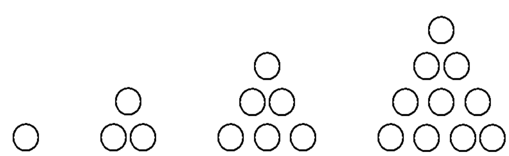
The teachers reasoned with the aid of pictures to explain why the solution made sense. Simon and Blume (1996) describe a course in which prospective elementary teachers worked on various explanations, such as explaining why the area of a shape can't be determined from its perimeter.

Prospective teachers should learn to "explain why", but what are some of the issues and obstacles we encounter in attempting to carry this out? The next two sections will address this.

3 Obstacles in Learning to "Explain Why"

Many prospective elementary teachers enter their mathematics training expecting to learn to give children clear directions for carrying out mathematical procedures. This creates an obstacle in a course that is about "explaining why", and not about "showing how". Therefore, when I teach our first mathematics course for prospective elementary teachers, I discuss carefully why we focus on "explaining why". I take students' questions of "why do we need to know this?" seriously, and address them in detail. Soon, most students see the wisdom in our approach. But this is not the only obstacle.

Initially, many prospective elementary teachers have a shallow conception of what it even *means* to explain why something is true. We often begin our first mathematics course for elementary teachers by considering triangular arrays of dots:



Every time, at least one student offers something roughly like the following to explain why the formula $\frac{n(n+1)}{2}$ gives the correct number of dots in the nth triangle:

> There is an $n+1$ in the formula $\frac{n(n+1)}{2}$ because you are adding 1 to each row in the triangle.

This "explanation" is really a mnemonic device that connects the formula to the problem in a superficial way. A student who offers it may not understand what explaining means.

A while ago, I assigned the following problem early in the semester:

> Mary says that $100 \times 3.7 = 3.700$. Why might Mary think this? Explain to Mary why her answer is not correct and why the correct answer is right. If you tell Mary a procedure, be sure to tell her why it makes sense!

Despite the instructions, and despite having discussed place value in class, most students simply told Mary that $3.7 = 3.700$ and that she should move the decimal point 2 places to the right. When I have asked students to explain why the standard multiplication procedure makes sense, some have responded with a clear explanation for how to carry out the procedure. Thus I now give my students more guidance in "explaining why" early in the semester.

Similarly, as reported in Ma's study (1999), when elementary teachers were presented with a hypothetical situation in which students mistakenly did not shift over the partial products when calculating

$$\begin{array}{r} 123 \\ \times 645 \\ \hline \end{array}$$

many American teachers suggested remedies that focused on clarifying the multiplication procedure, such as using lined paper sideways, or using whimsical placeholders to catch the students' attention (pp. 28–35). The American teachers tended to "show how" rather than to "explain why".

Thus instructors of courses for prospective elementary teachers should not assume that the prospective teachers know it is possible to give meaningful explanations for mathematical procedures and facts. Most students need time and practice to develop the notion that mathematics can be explained, and what it means to do so.

Another initial obstacle is students' beliefs about what constitutes mathematical activity. For some students, common-sense reasoning and pictures may not seem "mathematical" enough. I posed the following problem early one semester:

> Susan was supposed to use $\frac{5}{4}$ of a cup of butter in her recipe but she only used $\frac{3}{4}$ of a cup of butter. What fraction of the butter that she should have used did Susan actually use? Draw pictures to help you solve this problem. Explain your answer clearly. For each fraction in this problem, and in your solution, describe the *whole* that this fraction is associated with.

One student responded by drawing pictures to show $\frac{5}{4}$ and $\frac{3}{4}$ cups of butter, and then calculated:

$$\frac{3}{4} \div \frac{5}{4} \text{ or } \frac{3}{4} \cdot \frac{4}{5} = \frac{12}{20} = \frac{6}{10} = \frac{3}{5} \text{ out of } \frac{5}{4}.$$

She went on to explain as follows:

> ... To find the fraction of the butter that Susan used out of what she should have used you need to divide $\frac{3}{4}$ and $\frac{5}{4}$. When dividing fractions you can take the reciprocal of the second fraction and multiply it by the first fraction: $\frac{3}{4} \cdot \frac{5}{4}$. When you do that you find that Susan used $\frac{3}{5}$ of the $\frac{5}{4}$ of butter. ...

Despite the directions to use a picture to help solve the problem, the student showed (correct) calculations and discussed those calculations. Perhaps the student didn't know how to use a picture to solve the problem (although we had used both pictures and calculations in class, and the student was a consistently diligent worker), or perhaps the student didn't find a picture together with common-sense reasoning to be sophisticated enough mathematically, and therefore didn't believe she should work with a picture to explain the solution. If it was the latter, then this is similar to what Raman (2001) found in her study of students and teachers in collegiate calculus: students viewed thinking mathematically as involving algebraic tricks and formal language. Raman found that students were less willing than instructors to accept a pictorial proof that the derivative of an even function is odd.

Thus students need time and experience to develop the idea that *reasoning* is a cornerstone of mathematics, and that this reasoning can not only involve equations and formulas, but can also refer to pictures and experiences.

4 Will it Transfer to the School Classroom?

A major challenge in mathematics teacher education is to help teachers carry explaining and sense-making into their own classrooms.

The National Council of Teachers of Mathematics (NCTM) has promoted a vision of reasoning and sense-making in school mathematics (NCTM, 1989, 2000). Yet studies in which teachers are trained in accordance with this vision show mixed results when the teachers enter the classroom (Wilcox, Lanier, Schram, and Lappan, 1992; Frykholm, 1999). Frykholm (1999), in his study of secondary mathematics student teachers, found that most student teachers were not able to put the NCTM Standard's vision of reform into practice:

...the student teachers reported that, although the Standards were valuable inasmuch as they articulated a compelling vision for what mathematics instruction could be, there had been little offered in the way of practical advice and examples of innovative pedagogy that could be used as a model for implementing such instructional strategies. (p. 94)

Mathematics classes that focus on reasoning and sense-making often seem diffuse and inefficient as described in the literature. One wonders whether students will be able to pull the ideas together in the end; one wonders if class time has been used effectively. For example, Simon and Blume (1996, pp. 10–17) describe a class for prospective elementary teachers in which the students and the instructor discuss why the number of cardboard rectangles covering a table can be determined by multiplying. There is a lot of fumbling and searching; there is a lot of confusion. In the end, some of the students were able to explain clearly why it is valid to multiply, but excerpts from journals of other students show that several students left the class still uncertain and confused. Learning mathematics is necessarily messy and imperfect; it inevitably involves some fumbling and false starts. But I can't help wondering if the important class discussion described in the article couldn't have helped the students learn more effectively and efficiently if it had taken place in a narrower context. What if the instructor had given the class a definition of multiplication, and had asked the class to use the definition to explain why it is valid to multiply? In my own experiments with teaching in different ways, I have found that being too much of a "guide on the side" leaves too many students confused and unable to pull the ideas together in a coherent way.

Could it be that in our desire to help students make sense of mathematics for themselves, and in our desire not to lecture, that we sometimes give students *too little structure* in which to learn *efficiently*? And if prospective teachers view sense-making as too inefficient and unstructured, will they feel that they do not have the luxury of engaging their own students in sense-making? After all, as teachers, they will be responsible that their students achieve specific learning objectives on specific topics, which may be tested on high-stakes state or national tests.

5 Recommended Features of Explanations

In light of the discussion above, I offer the following recommendations for choosing explanations for prospective elementary teachers to engage in.

1. **Choose many explanations that are fairly closely linked to the actual practice of teaching mathematics in elementary school.**

For example:

Jim thinks that because $30 \times 40 = 1200$, and $1 \times 1 = 1$, therefore

$$31 \times 41 = 1200 + 1 = 1201.$$

Draw a picture and use your picture to help you explain to Jim how 30×40 and 31×41 are actually related. (Beckmann, 2003)

2. **Choose explanations that will help teachers organize their thinking around key principles and concepts.** In some cases, state the principle or definition to be used in order to provide structure and context.

In her study of American and Chinese elementary teachers, Ma (1999) found that some of the Chinese teachers developed what she called *Profound Understanding of Fundamental Mathematics* (PUFM). One key component of PUFM is a focus on basic ideas. As Ma explains:

> Teachers with PUFM display mathematical attitudes and are particularly aware of the "simple but powerful basic concepts and principles of mathematics" (e.g., the idea of an equation). They tend to revisit and reinforce these basic ideas. By focusing on these basic ideas, students are not merely *encouraged* to approach problems, but are *guided* to conduct real mathematical activity. (p. 122, emphasis in original.)

These key concepts include fundamental definitions, such as the definition of multiplication and the definition of fraction. In some situations, the principle or definition can be referred to explicitly in asking for an explanation. For example:

> John, Trey, and Miles want to know how many two-letter acronyms there are that don't have a repeated letter. For example, they want to count acronyms such as BA and AT, but they don't want to count acronyms such as ZZ or XX.
>
> John says there are $26 + 25$ because you don't want to use the same letter twice, that's why the second number is 25.
>
> Trey says he thinks it should be *times*, not *plus*: 26×25.
>
> Miles says the number is $26 \times 26 - 26$ because you need to take away the double letters.
>
> Discuss the boys' ideas. Which answers are correct, which are not, and *why*? Explain your answers clearly and thoroughly, drawing on the meaning of multiplication. (Beckmann, 2003)

Or:

> The grid lines below are 1 cm apart. Use the *moving* and *combining* principles about area to help you determine the *exact* area of each of the triangles below. Explain why your answers are correct. *Do not* use a formula for areas of triangles. [The problem includes a picture of triangles on a grid.]

Or:

> Use the *meaning of fractions* to explain why
>
> $$\frac{2}{3} = \frac{2 \cdot 57}{3 \cdot 57}.$$
>
> (In other words, explain why $\frac{2}{3} = \frac{114}{171}$.) *Do not* use multiplying by 1 to explain this.

We can ask not only for explanations of why things are the way they are, but also for explanations of why things aren't the way they aren't. In these cases, the underlying principles are not given, but must be uncovered in order to give a full explanation. For example:

> Frank thinks that it would be easier to add fractions by "adding the tops and adding the bottoms." So for example, Frank wants to add $\frac{1}{2}$ and $\frac{3}{4}$ this way:
>
> $$\frac{1}{2} + \frac{3}{4} = \frac{1+3}{2+4} = \frac{4}{6}.$$
>
> Frank uses the picture below to explain why his method makes sense. Why is Frank's method not a valid way to add fractions, and why does Frank's picture not prove that fractions can be added in his way? Do not just state the proper way to add fractions, explain what is wrong with Frank's reasoning.
>
> $$\boxed{X\,|\,O} + \boxed{X\,|\,X\,|\,X\,|\,O} = \boxed{X\,|\,X\,|\,X\,|\,X\,|\,O\,|\,O}$$

In order to explain what is wrong with Frank's method, prospective teachers must focus on the crucial role of the whole associated to each fraction, as in the following explanation given by a student.

> Although Frank's reasoning looks good at first, he is not using the same wholes to get the fractions $\frac{1}{2}$ and $\frac{3}{4}$. When adding fractions, it is important to consider the wholes. He starts with 2 blocks, 1 shaded [X] and 1 white [O] which is equal to $\frac{1}{2}$, but then he adds two more blocks to show $\frac{3}{4}$. The wholes (2 blocks) and (4 blocks) are not equal and therefore we cannot add these fractions [yet].

3. Give students specific guidelines for writing mathematical explanations.

I give my students the following guidelines characterizing good explanations in mathematics:

A. The explanation is factually correct, or nearly so, with only minor flaws (for example, a minor mistake in a calculation).

B. The explanation addresses the specific question or problem that was posed. It is focused, detailed, and precise. There are no irrelevant or distracting points.

C. The explanation is clear, convincing, and logical. A clear and convincing explanation is characterized by the following:

   (a) The explanation could be used to teach another (college) student, possibly even one who is not in the class.

   (b) The explanation could be used to convince a skeptic.

   (c) The explanation does not require the reader to make a leap of faith.

   (d) Key points are emphasized.

(e) If applicable, supporting pictures, diagrams, and/or equations are used appropriately and as needed.

(f) The explanation is coherent.

(g) Clear, complete sentences are used.

For example, we could respond to the problem "use the meaning of fractions to explain why $\frac{2}{3} = \frac{2 \cdot 57}{3 \cdot 57}$" as follows.

> According to the meaning of fractions, $\frac{2}{3}$ of a pie is the amount formed by 2 parts when the pie is divided into 3 equal parts. This amount is shown shaded in the picture below. [Show the relevant picture of a pie.] If I divide each of those 3 equal parts into 57 small equal parts, the pie will now be divided into $3 \cdot 57 = 171$ small parts. Because the 2 original shaded parts representing $\frac{2}{3}$ of the pie have each been subdivided into 57 small parts, these 2 original shaded parts become $2 \cdot 57 = 114$ small parts, as indicated in the picture. [Show another picture of the same pie, indicating that each piece is now subdivided into many smaller pieces of equal size.] It's still the same amount of pie that is shaded either way you look at it. So 2 of the original 3 parts of pie is the same amount of pie as $2 \cdot 57$ small parts of the total $3 \cdot 57$ small parts. This is why $\frac{2}{3}$ of a pie is the same amount of pie as $\frac{2 \cdot 57}{3 \cdot 57} = \frac{114}{171}$ of the pie.

Notice that even though we can also use multiplication by 1, in the form $\frac{57}{57}$, to explain why $\frac{2}{3} = \frac{2 \cdot 57}{3 \cdot 57}$, the explanation above addresses the specific problem that was posed, namely to use the meaning of fractions. The explanation is written to explain in a natural and convincing way, and not just to establish truth.

With attention to the matters that I have described in this paper, it is possible to teach an efficient course in which prospective teachers learn to "explain why" and make sense of mathematics.

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MATHEMATICS THAT CHANGES LIVES

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ABSTRACT

We are developing a general education mathematics course that will introduce students to mathematical reasoning and applications. The course will cover the history of, the motivation for, and an introduction to cryptography, fuzzy set theory, graph theory, and non-Euclidean geometry. Three weeks (nine discussion-lecture hours) will be devoted to each topic, and the remaining three weeks will be used for student group work and project presentations. Through discussion and exploration, students will experience some of the insights, frustrations, and excitement experienced during the development of new concepts.

The cryptography unit emphasizes the classic cryptosystems and focuses on the mathematics behind the *how* and *why* the schemes actually work. The fuzzy set theory unit will emphasize applications. Topics in graph theory will highlight applications in management science, including Euler and Hamiltonian circuits, the traveling salesman problem, minimum spanning trees, and ideas in scheduling and planning. The non-Euclidean geometry unit will teach the basics of spherical geometry and focus on developing an understanding of an axiomatic system.

Pre-service elementary and secondary school teachers will be encouraged to take this course in order to expand their understanding about the nature of mathematics. The hope is they will begin to experience mathematics as a process of finding patterns and making and verifying conjectures. We will encourage them to work on projects that can be useful in their own classrooms.

Mathematics Subject Classification (2000): 00A35, 97C90, 97D50

1. Introduction and Rationale

According to Schoenfeld (1992), mathematics instruction goals depend on beliefs about what mathematics is:

> At one end of the spectrum, mathematical knowledge is seen as a body of facts and procedures dealing with quantities, magnitudes, and forms, and relationships among them; knowing mathematics is seen as having "mastered" these facts and procedures. At the other end of the spectrum, mathematics is conceptualized as the "science of patterns," an (almost) empirical discipline closely akin to the sciences in its emphasis on pattern-seeking as the basis of empirical evidence. (p. 335)

Our conception is the latter, and we hope to induce our students to engage in a "hands-on" version of *doing mathematics*. Classroom activities will focus on *sense-making* with the hope of developing a predilection to think mathematically, i.e., to explore patterns, formulate conjectures, and test these conjectures. We hope our students will begin to develop the following attitudes toward mathematics and mathematics teaching: (Lampert in Shoenfeld, 1992)

- Mathematics is more about ideas and mental processes than about facts
- Mathematics can best be understood by rediscovering its ideas
- The main goal of the study of mathematics is to develop reasoning skills that are necessary to solve problems
- The teacher needs to create and maintain an informal classroom atmosphere to insure students' freedom to ask questions and explore their ideas
- The teacher should encourage students to guess and conjecture and should allow them to reason things on their own rather than show them how to reach a solution or answer
- The teacher should appeal to students' intuition and experiences when presenting the material in order to make it meaningful
- Coming to grips with uncertainty is a major part of the learning process
- Students should be induced to reflect on their own thought processes.

We plan to assess the change in students' beliefs about mathematics as well as their level of enjoyment of mathematics by having them complete a questionnaire on the first day of class and at the end of the semester. (Please see Tables 3 and 4 following the references.) Additionally, we will assess the students' abilities to do mathematical modeling. In our pre-course evaluation, we will ask the students to choose a situation and model it mathematically. We expect that most students will claim they cannot model anything mathematically. In the post-course evaluation, we will ask them to choose one or more situations and to model them mathematically in as many ways as they can.

2. The Cryptography Module

A unit on cryptography is ideal for enticing students to explore patterns, formulate conjectures, and test these conjectures. Initially, we will give students enciphered messages to decode using frequency tables of letters of the alphabet and common digraphs. Working in small groups, they will experience uncertainty and formulate conjectures about possible decoding schemes. Testing their schemes will come naturally as they try to make sense of the message before them.

Monoalphabetic Substitution Ciphers

The first two weeks will emphasize monoalphabetic substitution ciphers. In this type of cipher, each character of a written message (hereon refereed to as the *plaintext* message) is matched with a unique alternate character to obtain an encryption (a *ciphertext* messsage).

Several types of monoalphabetic substitution ciphers exist: *additive ciphers, multiplicative ciphers, affine ciphers,* and *keyword ciphers*. In an *additive cipher,* a plaintext character is replaced by another plaintext character whose position in the alphabet is a certain number of units away (mod 26) from the plaintext character. This number of units is called the *key*. The mathematics involved includes modular addition, additive inverses, and elementary statistical analysis. In a *multiplicative* cipher, rather than adding a number (the *key*) to the position of a plaintext character, one *multiplies* its position by a number mod 26. Students working on decoding such a cipher would be using modular multiplication, multiplicative inverses, prime numbers, the division algorithm, and relatively prime numbers. *Affine* ciphers combine both of the above ciphers by having two *keys*. One first applies an additive cipher with a key r to obtain an intermediate cipher and then applies a multiplicative cipher with key s to produce the cipher text. For *keyword* ciphers, one chooses a word and a letter as keys to create the monoalphabetic substitution cipher.

Once the groups successfully decode short encrypted messages with each of the above ciphers, they will come together as a whole class to discuss the problem solving processes they used to decipher each message as well as the mathematics behind their work. We will discuss the mathematics informally, hoping that the students find them natural to understand given the applications.

Polyalphabetic Substitution Ciphers

A polyalphabetic substitution cipher is a cipher in which the correspondence between the characters in the plaintext and those in the ciphertext is not one-to-one. Since these ciphers are more difficult to decipher, classroom activities will require more direction from the instructor. We will emphasize one example, the Vigenère Square. Students will initially *encode* a message before beginning work on a short *deciphering* assignment. Using the Friedman Test, which entails elementary probability, they will determine if a ciphertext has been encrypted using a monoalphabetic or polyalphabetic substitution cipher. If the ciphertext is polyalphabetic and a string of characters appears repeatedly in the message, the distances between occurrences *may be* a multiple of the length of the keyword. This observation is known as the Kasiki Test, and with it students will attempt to determine the length of a keyword.

R.S.A. Algorithm

A brief discussion of the RSA algorithm will conclude the unit, time permitting. The algorithm derives its name from its creators R. Rivest, A. Shamir, and L. Adelman and requires that each participant have two keys, a private key and a public key. The private key is a positive integer and the public key consists of *two* positive integers.

Group Projects

Student projects will entail researching the history of cryptography, encryption methods not discussed in class, or a more detailed exposition of the RSA method.

3. The Fuzzy Set Theory Module and Fuzzy Modeling

Introducing Fuzzy Sets

After reviewing the classical concepts of set and set operations, we will pose a question like: "Suppose you are working with the set of average daily temperatures for our area. What is the set of *warm* temperatures?"' After discussions about such a subset, we will, if necessary, ask the students if it would make sense for some temperatures to be *partially* in the subset of *WARM* temperatures. We will use the sets {0,1}, {0,0.5,1}, and [0,1] as ranges for our membership functions.

Operations on Fuzzy Sets

Our review of classical set operations will use a computer program and Venn diagrams. The program will work so that subsets of the universal set will be labeled, outlined, and shaded in a dark color, for example, in dark blue. The area within a Venn diagram but outside the sets in question will be in white. We will perform set operations, and the results will appear in dark blue with labels and with a border outlining the original sets used in the operation(s).

We will then consider fuzzy (sub)sets with the membership set {0,0.5,1}. Elements with a 0.5 membership value will have a medium shade of blue. We will ask the students to represent the union, intersection, and difference of sets using the program's color scheme. We hope it will not be difficult for the students to understand these operations in terms of the color scheme, and then we'll discuss the operations using standard mathematical notation. Next we will generalize our results to the membership set [0,1]. The program will represent membership values from 0 to 1 by using color shades from white to dark blue, respectively.

Fuzzy Conditionals

As many fuzzy control applications are based on fuzzy conditionals or fuzzy If-Then statements, we will introduce fuzzy conditionals where both the antecedent and the consequent involve fuzzy sets, i.e., fuzzy linguistic variables. Once the concept of a fuzzy set with a membership function is understood, understanding fuzzy conditionals is relatively straightforward.

Fuzzy Modeling

Our fuzzy modeling will be based on a fuzzy partitioning of the domain space, on defining fuzzy conditionals relative to the partition, on unioning the results of the conditionals, and on defuzzifying via a modified center-of-area method. The fuzzy partitioning will be explained and justified informally.

We will work though examples including a fuzzy model for designing a personal savings plan. The fuzzy conditionals will be defined with respect to age, number of dependents, and annual income, and the consequents will be defined in terms of a percent of income to be saved and/or invested. Using this fuzzy model, we will be able, for example, to suggest what percent of John's income should be invested if he is 38 years old, has 4 dependents, and earns $42,000 per year.

4. The Graph Theory Module

Topics on graph theory are chosen with a commitment to helping students acquire knowledge about the basics of management science. It is our view that such knowledge can be built with virtually no previous mathematical training and with relatively little pain. Granted that a working knowledge of basic statistics would be quite useful in dealing with complex problems, our choice of topics will include very little statistics, if any. Management science is a many-faceted subject. Its aim is to provide analysis, advice, and support to decision-makers. While our introduction to this vast subject will not be in depth, we expect that our students will find it easier to understand lots of commonsense examples, and be able to appreciate the beautiful mathematics that describe the solutions. The three main themes of the graph theory module are Euler Circuits, Hamiltonian Circuits, and Planning and Scheduling.

Euler Circuits

The first week of this module will delve into Euler circuits motivated by practical problems of street networking. Equipped with only the basic definition of an Euler circuit, namely circuits that cover each edge only once, the students will be encouraged to try out different solutions to given networking problems. The problems will be carefully chosen to incorporate a variety of possible cases. Our emphasis will be on discovery and innovative methods of solution. Collaboration will be allowed and encouraged through small group discussions. Based on their solutions, they will be asked to make conjectures about the existence of an Euler circuit for a given network. Through such a discovery-based approach, the groups (at least some) would come up with a statement similar to Euler's Theorem. After refining the students' conjectures, we will formally present Euler's Theorem, which describes a necessary and sufficient condition for the existence of an Euler circuit in a given graph. A simple elementary proof will be given. The section on Euler circuits will conclude with a few exercises on eulerizing graphs (by reusing edges). It is at this stage that the students would realize that some solutions are better than others, and this realization would lead into a discussion about "optimal solutions."

Hamiltonian Circuits

In Hamiltonian circuits, one starts at a given vertex and visits each vertex exactly once and returns to the starting vertex. Both Euler and Hamiltonian circuits are similar in that they both prohibit the reuse of some entity of the graph (edges in the case of Euler and vertices in Hamiltonian). Several beginning activities under this topic would come in the form of in-class worksheets. The reasons are two fold. Firstly, we want *discovery* to be at the helm of learning and cooperative activities can help accomplish this goal. Secondly, we want the prospective teachers among the students to gain some experience in designing in-class worksheets that are created as part of an outcome-based lesson plan.

The topics covered in the beginning of this section will include construction of non-Hamiltonian circuits, weighted and minimum cost Hamiltonian circuits, and the fundamental counting principle. The latter half of the section will include topics ranging from the traveling salesman's problem (TSP) and nearest neighbor algorithm to NP complete problems. We will use Kruskal's algorithm and critical path analysis to launch our discussions into ideas in Scheduling and Planning. Throughout this entire section we will make available to the students computer programs written on graphing calculators. With the aid of these programs the students will be able to check solutions, experiment, and confirm conjectures - practices that often form the backbones of mathematical research.

Planning and Scheduling

The final section of the graph theory module will focus on applications. We will guide students through simple optimization problems arising from applications in planning and scheduling. The emphasis will be on solving simple problems with a deep understanding of methods and principles involved. We will keenly observe the students throughout this stage to glean information about their individual skill levels. Our observations will then be incorporated into the designing of the final group projects for the course. It is our hope that each project from this module will have some aspect that would appeal to each student. The problems solved in class as well as the project problems will be similar to the ones found in Brucker (1999), COMAP (2000), Dolan and Aldus (1993), Heizer and Render (1999), and Roberts (1978). Samples of such topics include scheduling exams, planning meeting times, allocation of hospital resources, and efficient banking practices.

5. The Non-Euclidean Geometry Module

The purpose of this unit is to expand students' thinking and teach the basics of spherical geometry. They will review the concepts and definitions of plane geometry by comparing them to their corresponding ideas in spherical geometry. We assume that students will have had a high school course in geometry.All work will be done with manipulatives, presumably a class set of Lénárt spheres which have smooth write-on surfaces and tools for drawing and measurement.

Initial activities would concern basic geometric concepts: straight lines and distances, equators and pole points, angles, and parallel and perpendicular lines. Activities will be those suggested in Lénárt (1996).

Straight Lines on a Sphere

After finding the shortest distance between two points on a plane, students will sketch two points on a sphere and stretch a piece of string to find the shortest path between them. Using a spherical ruler, they will continue drawing the line, thus sketching a great circle. Class discussion would guide students to recognize great circles on the earth, with a possible connection to ancient astronomy. Students will be asked to compare characteristics of lines on the plane with those on the sphere.

Distance on the Sphere

Students will sketch pairs of points on a sphere and use a spherical ruler to measure the length of each arc. They will compare distance on plane with distance on sphere, particularly noting the units of measure. They then will find length of entire great circle and perhaps find a place on the globe that is 90° from Kent, 180° from Kent, etc.

Parallel lines

Through a guided activity, students will review lines in the plane. For example, given a line l, they would be asked to draw another straight line that has no point in common, exactly one point in common, exactly two points in common, more than two points in common with l. Then they will try to do the same on sphere. Which constructions are possible? A discussion of parallel lines will ensue, with the instructor giving some history about Euclid's fifth postulate and its role in the development of non-Euclidean geometries. Ultimately, we will learn that the first four postulates of Euclidean geometry also hold true on the sphere.

Triangles on the Sphere

- Students will investigate how many different triangles they can create by connecting three non-collinear points on a sphere and compare their results to those obtained on the plane.
- Students will investigate the sum of angles of triangles first on the plane and then on the sphere. They will try to construct a triangle with more than one right angle, then one with three right angles. They will then investigate the sum of the measure of the angles of quadrilateral.
- Students will investigate whether two triangles must be similar if their corresponding angles are congruent.

The culminating activity concerning triangles on the sphere will be finding the *qibla* (the angle one must turn from a given location in order to face Mecca) using spherical trigonometry (the law of sines and rule of four quantities).

Hyperbolic Geometry

A brief mention of hyperbolic geometry will conclude the unit. Time permitting, we would define line and distance using the Lobachevskian model. Students would be asked to compare and contrast properties in the three geometries. They will be encouraged to pursue an independent study project on the history of non-Euclidean geometry, its applications, or perhaps an elementary inquiry into hyperbolic geometry.

6. Assessment

The introduction of innovative pedagogy often prompts reevaluation of traditional classroom assessment practices. As described earlier in the Introduction and Rationale Section, our course is designed to promote mathematical reasoning and expand understanding about the nature of mathematics. Our philosophy is to minimize the anxiety that students typically associate with mathematics. We will continuously monitor students' learning, constantly provide important feedback about their progress, and encourage them through the power of systematic inquiry. With these standards in mind, we will evaluate our students in the following manner:

- During the first twelve weeks of class, i.e., during the presentations-discussions of the four main topics, assessment will be based on in-class worksheets and homework assignments. A student who completes this portion of the course successfully will receive a "C" grade.
- The last three weeks will be devoted to group projects based on material selected from each module. A student, upon earning the "C" grade, may successfully complete a single project to move to the next grade level "B."
- In order to earn an "A" for the course, a student must complete a second project (chosen from a different module) successfully.

7. Changes

Due to scheduling constraints this course has not yet been taught; it is planned that it will be taught during fall 2002. Thus, we cannot report on how students have changed as a result of taking this course. We can, however, do, at least, two things. We can report on opinions and beliefs of students like those who will take the course, and we can elaborate on the types of changes which we hope to see in our students.

To better know and understand the opinions and beliefs of the students who will take this course, we have given the "Beliefs about Mathematics" (given at the end of this article) questionnaire to 16 students like those who will take this course. The responses are summarized below.

Question	1	2	3	4	5	6	7	8	9	10
Mean	2.75	1.56	2.44	2.38	2.56	2.25	2.81	3.56	4.38	2.19
SD	1.18	0.73	1.41	1.15	0.96	1.06	1.33	1.03	0.72	1.42

Table 1. Means and Standard Deviations of Responses to "Beliefs About Mathematics" Questionnaire

The relatively low mean and standard deviation for responses to question #2 (which concerns the uniqueness of a solution to a problem) pleasantly surprised us, though we are concerned about the responses to question #3. Eight students (50%) responded to this question with an answer of 3, 4, or 5. At least two of these are pre-service elementary school teachers. Responses to question #4 confirm what students indicated to us in the qualitative portion of the survey. When asked whether or not they saw advantages of working on mathematics in groups, 13 of 18 (72%) students responded positively. The comments in the table on the following page are typical.

Question: Do you see any advantages to working in groups? If so, what are they? If not, why not?
• I think group work is great. Sometimes it is just one little concept that was missed that makes a particular problem confusing. Group work helps to clear the confusion.
• Yes, working in a group in math can help in many ways. Not only are there other ideas, but solutions can be explained through many different viewpoints. Different ways of knowing/solving exercises in math often help those that may be having difficulty.
• Yes...Working in groups helps students who are high achievers understand the mathematics by explaining it to others. Sometimes people go through the motions without truly understanding the reasoning behind mathematics. For example, smart students might memorize.

Table 2. Sample Student Responses about Group Work in Mathematics

The types of changes in which we are especially interested are beliefs about the nature of mathematics, confidence in understanding public press articles involving mathematics, and (for those who aspire to be teachers of mathematics) a willingness to freely and openly think about appropriately difficult mathematical concepts.

For the most part, our target audience believes that mathematics is the solving of equations often involving one or two variables, that mathematics deals with well known and clearly understood (though not by most students) concepts, and that mathematics in the form of equations supports science and technology. Student responses to the question "In your opinion, what is mathematics?" are listed in Table 5 at the conclusion of this paper. Fourteen of the 18 respondents referred solely to numbers, numerical expressions, or numerical calculations. Interestingly, 7 students think of mathematics only as calculating or manipulating numbers, variables, or formulas. These comments support our contention that student thinking needs to be developed and changed. Some bright spots did emerge in the data, however. Three students mentioned the study of relationships and six at least alluded to modeling. One student, albeit a masters level economics student, indicated that "Mathematics is a way of studying how things behave."

By the end of the course, we want the students to understand that mathematics embodies a rich and ever growing field of knowledge and concepts that in many cases *mold* the sciences and technology and that in most cases the basic nature and purpose of these concepts can be understood by educated individuals.

Further, we want to create a classroom environment that encourages questions and inquiry so that those who plan to teach mathematics, either as mathematics teachers or K-6 teachers who will teach mathematics as one of several subjects, will feel comfortable allowing their students to freely think and openly inquire about mathematical ideas and concepts even if it means that the teachers themselves will not be able to answer or solve all the questions.

8. Summary

In this course we want to expose our students to mathematics that has changed people's lives, and we want to present this mathematics in a way that will change the way our students think and, thus, will also change their lives.

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STUDENT QUESTIONNAIRE[1]
BELIEFS ABOUT MATHEMATICS

Students will be asked to describe their reaction to each of the following statements by using the following scale.

1	2	3	4	5
strongly disagree				*strongly agree*

1. Mathematics problems have one and only one right answer.
2. There is only one correct way to solve any mathematics problem – usually using the rule the teacher demonstrated in class.
3. When learning mathematics, I really don't expect to understand it; I prefer to memorize it.
4. Mathematics is best done by oneself.
5. Students who have understood the mathematics they have studied will be able to solve any mathematical problem in five minutes or less.
6. The mathematics I have learned in school has little or nothing to do with the real world.
7. Mathematics is less important to people than art or literature.
8. An understanding of mathematics is needed by artists and writers as well as scientists
9. Mathematics is needed in designing practically everything.
10. There is nothing creative about mathematics; its' just memorizing formulas and things.

Table 3. Student Questionnaire about Beliefs in Mathcmatics

[1] Adapted from Lampert, in Schoenfeld, 1992.

STUDENT QUESTIONNAIRE[2]
ENJOYMENT OF MATHEMATICS

Students will be asked to describe their reaction to each of the following statements by using the following scale.

1	2	3	4	5
strongly disagree				*strongly agree*

1. I enjoy going beyond the assigned work and trying to solve new problems in mathematics.
2. Mathematics is enjoyable and stimulating to me.
3. Mathematics makes me feel uneasy and confused.
4. I am interested and willing to use mathematics outside school.
5. I have never liked mathematics and it is my most dreaded subject.
6. I have always enjoyed studying mathematics in school.
7. I would like to develop my mathematical skills and study the subject more.
8. Mathematics makes me feel uncomfortable and nervous.
9. I am interested and willing to acquire further knowledge of mathematics.
10. Mathematics is dull and boring because it leaves no room for personal opinion.
11. Mathematics is very interesting and I have usually enjoyed courses in the subject.

Table 4. Student Questionnaire about their Enjoyment of Mathematics

[2] Adapted from Aiken, 1974.

- study of numbers and objects that deal with numbers
- the study of laws concerning the physical world and the ways to understand it using relatively visible methods
- study of numerical expressions
- hard and complex problems that make a person think logically
- applying numbers and formulas to answer questions and solve problems
- using numbers to get a solution to a problem
- the study of numbers, their form, arrangements, and associated relationships, using defined literal, numerical, and operational symbols *(sounds like a dictionary definition)*
- manipulating numbers to understand everyday life and make it easier
- language of using numbers and formulas to describe why things work as they do.
- It is using numerical calculations to solve problems. It is very useful to a certain extent in everyday life, but is even more important in instances where exact values are needed.
- study of calculations, numbers, volumes, dimensions, and all kinds of other everyday measurements *(pre service elementary teacher)*
- numbers *(pre service elementary teacher)*
- numbers, signs, shapes, lots of stuff I don't understand *(pre service elementary teacher)*
- expressed numerical relationship between all things
- study of numbers and variables and how they can relate to each other through various manipulations *(pre service secondary teacher)*
- study of numbers, and how they apply to the world in which we live. (*pre service secondary teacher)*
- the study of numbers
- Mathematics is a way of studying how things behave. This is a very broad answer but I don't feel that you can give a precise definition without leaving things out. I feel that mathematics by itself may not be able to accomplish very much but when used with other academic disciplines I feel it is essential

Table 5. Student Responses to "In Your Opinion, What Is Mathematics?"

THE EVOLUTION OF PROFESSIONAL DEVELOPMENT ACTIVITIES DESIGNED TO MEET THE CHANGING NEEDS OF GRADUATE STUDENT TEACHING ASSISTANTS

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ABSTRACT

The preparation to teach that graduate student teaching assistants (TAs) receive is critical for a number of reasons. TAs are responsible for a considerable portion of undergraduate instruction in the U.S. Furthermore, future mathematics faculty come from the current population of graduate students, who are likely to carry habits they develop as TAs into their careers. In addition, recently undergraduate mathematics instruction in the U.S. has experienced some changes. As a result, now TAs may be asked to teach in ways that they did not themselves experience as students (for example, using collaborative group learning). The preparation and support TAs receive, especially early on, has the potential to shape instructional experiences for a substantial number of undergraduates now and in the future, and is especially important during this time of change.

In this paper I describe how a learn-to-teach course evolved in response to TAs, needs. These TAs taught classes where students spend significant time working challenging mathematical problems in small collaborative groups. In contrast to "traditional" teaching assignments (where TAs may be expected to answer homework questions and present solutions), these TAs assisted students as they worked in groups, provided problem solving support, and led whole-class discussions. As more was learned about challenges TAs faced and difficulties they encountered, activities were designed and revised. The activities were designed to promote reflection on issues of teaching and learning. These activities included TAs viewing videotape of their classes and observing groups of students for extended periods of time as they worked on problems in collaborative groups.

Keywords: calculus instruction, teaching assistants, mathematics graduate students, teacher professional development, collaborative groupwork,

Introduction

In this paper I describe the design and revision of two professional development activities used in a course for mathematics graduate students who were teaching for the first time. First, I provide some background information about undergraduate instructional reform and the role of graduate student teaching assistants (TAs) in these instructional contexts. Next, I describe the courses the TAs were teaching and the semester-long class where the professional development activities were used. In the subsequent sections, I describe the origin of activities, the difficulties that arose when using them, and the revision that occurred as a result. In particular, I discuss classroom videotaping and observation of student groupwork as opportunities for TAs to reflect on issues of student learning. In the final section, I discuss what I believe professional developers might learn from the problems that arose and from the solutions that were devised.

Over the past two decades, there has been an increase in the attention paid to the state of undergraduate instruction in the U.S.. In particular, low enrolment and retention rates in introductory mathematics courses have been the cause of considerable concern (National Science Foundation, 1986; 1989; Seymour & Hewitt, 1994). In addition, the level of understanding that mathematics students demonstrate has not been what faculty might wish (Douglas, 1986; Steen, 1987). As part of the response to this situation, some institutions have worked to alter the content and modes of instruction in their introductory courses.

One such response has been to incorporate the use of collaborative groupwork into calculus courses. The aim is for students to engage with the material in more active and extensive ways than might be traditionally common. Utilizing this highly interactive form of instruction effectively is challenging for any teacher and may pose particular challenges for new instructors whose knowledge and skills for teaching are just beginning to develop. Indeed, these teachers are being asked to teach in ways that may differ substantially from the ways in which they were taught, and may deviate substantially from what they envisioned as their role as a teacher.

At some universities, graduate students are responsible for a considerable portion of the undergraduate instruction. Consequentially, graduate students as well as new faculty are in need of professional development and support as they learn to teach– even moreso to teach in these new ways. In addition, future faculty will come from the current pool of graduate students. Thus, current graduate students play important roles in implementing reform as well as shaping the nature of future instruction. Because of these factors, in recent years there has been an increase in the attention paid to the preparation and support that graduate students and new faculty receive for their teaching responsibilities. Little is known about mathematics graduate students and faculty in their capacity as teachers or about their experiences as they learn to teach. Although the K-12 research and professional development literature has much insight to offer into the design and use of activities to help people learn to teach, there is much we do not know about how TAs or new faculty will respond to such activities, and how their experiences will relate to the challenges they face as they learn to teach. This paper is one effort to examine these issues in the context of reform-oriented calculus courses.

The site of this work was the University of California at Berkeley. On the Berkeley campus, calculus courses for physical science and engineering majors are taught with large (200-400 student) lectures and accompanying smaller (20-30 student) discussion sections. Lectures are given by faculty and discussion sections are led by TAs, nearly all of whom are doctoral students in mathematics. In lecture, the faculty member presents ideas, solves model problems, and discusses theory and applications.

In recent years, changes were made to how time is spent during discussion sections. In the "old" discussion sections, TAs generally reviewed material from lecture, presented sample problems and solutions, and answered homework questions. By contrast, in the "new" discussion sections, the TA assists students as they work on problems in small groups. The problems are designed to be quite challenging (some similar in difficulty to the ones they might see on an exam) so that working collaboratively in groups is advantageous to the students. The students do their work at the blackboards and the TA circulates in the room and assists students. The goal is for the TA to act as a facilitator and a resource to the groups, but not to tell students directly how to solve the problems. The TA asks guiding questions, probes for deeper understanding of the ideas, and encourages students to explain and justify their solutions. The TA also holds periodic whole-class discussions about the problems and solutions. This format for running discussion sections was modeled in part on other programs that make extensive use of collaborative group work in calculus classes (Fullilove & Treisman, 1990; Treisman, 1985; Treisman, 1992).

All TAs in the mathematics department are required to take a semester-long course that provides professional development and support concurrent with their first teaching assignment. The course is organized around a series of in - and out-of-class activities. Here I will describe the evolution of two of the out-of-class activities. In one activity, TAs watch videotape of their classes. In the other activity, TAs observe in another TA's class. While these activities had met the needs of TAs in the old discussion section context, in their original incarnation, they were less effective in addressing the issues that arose for the TAs working in the new format. In response to challenges that emerged in the new context, the activities were modified to focus on issues particular to collaborative groupwork.

Videotaping and teaching consultation activity

Over the course of the semester, all new TAs are videotaped as they teach their classes. Each TA has a "teaching consultant" (an experienced TA from the math department or a graduate student in education) who tapes the class and then meets with the TA to discuss the class and to help devise strategies for improvement. This particular activity had a long history in the class for new TAs and in similar courses found elsewhere on campus. The underlying belief is that TAs can learn a great deal from watching themselves teach. Variations of this activity are frequently used with much success in pre- and in-service professional development for school teachers. For the TAs, the process was relatively unstructured. Before being taped, TAs were asked to identify several areas in which they would like feedback. After the taping, they were given some information about watching themselves on tape designed to reduce the associated anxiety, but were not given specific directions for viewing their tape.

In the context of the "old" sections, TAs needed to learn to make clear written and oral presentations. This included, but was not limited to, summarizing material, reviewing ideas, presenting solutions to problems, and discussing problem-solving techniques. TAs also needed to be able to answer students' questions effectively. In this context, the videotaping and teaching consultation activity was very effective in helping them improve their instructional practices. For instance, by watching their tape, TAs had the opportunity to see themselves from the students' perspective, to see and hear their presentations and assess the clarity, and to reflect on the quality of their answers to student's questions. In most cases, TAs naturally focused on these issues as they watched their tape and were able to make observations about their teaching that were relevant to the major goals of the old sections.

Under the new format, however, the activity was not as productive. Although TAs still reported it was very useful in helping them develop their teaching practices, it did not appear to help them focus on the set of issues central to the goals of the new sections. Certainly attention still needed to be paid to issues of clarity and presentation, but the change in format made other instructional concerns more pressing. In particular, TAs often found offering assistance and suggestions to students in lieu of providing the answer to be quite challenging. Instead of asking a guiding question to assist students, the TAs often simply gave students the answer. When students had completed a problem, instead of asking for explanations or justifications, they often looked over the work and declared it correct or pointed out errors.

These issues related to their interaction with students in groups was not what TAs paid attention to while watching the tape of their class. It became clear from the consultations with the TAs that they were not focusing on the issues the consultants felt were most essential in learning how to teach in these ways. When reflecting on their tape, some commented on the clarity of the answer they gave, but rarely raised issues surrounding their decision to provide an answer. They would frequently make observations about the accuracy of the students' solution, but rarely attended to features of the discussion of the solution.

Due to the very general directions about watching the tape and the pre-taping consultation where they identified areas of concern, TAs were not using the opportunity to gain feedback about the aspects of class that we were most foreign, new, and difficult for them. These also happened to be the aspects of class most central to the success of this model of instruction. Even TAs who demonstrated an ability to teach in ways consistent with the goals of the course did not necessarily gravitate towards being reflective about their practice in new ways.

The basic idea of the videotaping and teaching consultation activity appeared to be very valuable, but needed to be tailored to the new context. We had observed that they were not asking enough questions of their students during class, they were giving away solutions too easily, and they were not requesting justifications from their students. How might the activities be modified to scaffold TAs observations to focus on these issues more?

To address these issues, specific reflection questions were given to the TAs. The questions focused on their interactions with students, their use of questions, their request for solutions, justifications, and other issues related to groupwork:

Student Questions

a. What kind of questions are you asking your students? How often do the questions require more than a yes or no answer?

b. Are you asking your students to explain the mathematics (by asking questions such as "How did you figure that out?," "How can we know if that's true?," "What do you think we should do next?," etc.)?

c. How do you determine if students have understood your explanations or suggestions? Do you listen carefully to students' questions and comments *in their entirety* before responding?

d. Do you ask the students to clarify their question when you aren't sure what they are asking?

e. When students ask a question, are you able to ask them a question in return that points them in the right direction?

Interacting with Groups

a. Are your discussions usually with one member of the group or are most of the students involved?

b. How can you tell if the group has understood the problem?

c. How do you figure out if the group has understood you? Can you tell when the students are puzzled or confused?

d. At what point do you end your discussion with a group and how do you do that?

The hope was that with the addition of the reflection questions, TAs would attend to issues that were more closely related to the instructional goals for the sections. As it turned out, they still paid attention to issues of presentation and clarity, but some portion of their attention was now also focused on their questioning practices and other aspects of interaction with students. The observations they made in response to the questions came closer to focusing on issues of particular relevance in the new context. TAs also made the criteria they used to judge student understanding explicit. This made it possible for the teaching consultants to discuss additional strategies for probing and enriching student understanding.

In terms of their teaching practices, during subsequent videotapings, TAs were more likely to ask questions and to support student learning without telling answers to the students than they had previously. TAs appeared to ask more follow-up questions and to request more extensive justifications of solutions from their students. Having focused specifically on these issues while reflecting on their videotape appeared to promote the use of these teaching practices in ways that were more consistent with the goals of the discussion sections.

Peer observation activity

In addition to watching their own class on videotape, TAs were paired and observed each others' classes. Subsequently, they met to provide each other with feedback. Previously this activity was rather unstructured. The TAs visited, observed, took notes, and met to discuss their observations. This activity was relatively successful given the instructional goals for the old discussion sections. TAs received feedback on presentation clarity and content, and often got new ideas of how to handle particular questions or topics. In the context of the new sections, however, something interesting happened. TAs made observations about their partner's teaching (mostly focused on the same presentational or clarity issues), but often also paid attention to the groups of students who happened to be working nearby. The TAs made very interesting observations about what happened in the groups and frequently expressed surprise at the nature of the students' conversation and the work they accomplished.

Since TAs seemed surprised by what students did when the TA was not around and appeared to find these observations enlightening, a "group observation" activity was added. In this activity, TAs spend time specifically watching a group's interaction and taking notes. This activity provides one of the few opportunities TAs have to observe a group of students "in action" over an extended period of time and to see the kind of progress they are (or are not) able to make.

The peer observation activity handout was modified to include the following:

Group observation information (to be filled in during section):

a. What problem are the students working on?

b. How many people are in the group?

c. Try to figure out what the student's names are (are they written on the top of the board?) and refer to them in your notes. Knowing who said/did what will make your observations more meaningful for your partner.

Watch and listen carefully to the students. Observe only; don't involve yourself in the group's discussion! Try to figure out how they are approaching and trying to solve the problem.

d. Describe how they started the problem, what the difficulties were that they encountered and how/if those difficulties were resolved.

e. If your partner (the TA) comes over and talks with the group, describe what happens. Did the students have a question or did the GSI just approach the group? How did your partner handle the students' questions? What kinds of things did your partner say and ask? Also, describe what happens *after* your partner leaves the group.

This activity gave TAs an opportunity to focus on what can go on in groups in a way that is not possible when they are responsible for running the section themselves. They could see how different hints, suggestions, and information are and are not helpful to students. The activity also gave TAs an opportunity to learn about how students were thinking about problems and to develop a better understanding of what learning in these ways is like for their students. For example, TA were impressed by students' abilities to work through difficulties they encountered without assistance from the TA. They were also surprised by how challenging it was sometimes for students to make progress even after the TA had spoken with the group.

This activity appeared to help TAs modify their thinking and teaching practices in several ways. During subsequent videotapings and consultations, TAs seemed less likely to assume that they knew what the students were thinking based only on the written work they had produced. TAs also appeared less inclined to believe that the information they provided to students would automatically resolve confusion that students were experiencing. TAs also expressed more curiosity about how their students were learning and were more eager to come to understand how students thought about the mathematical ideas.

These changes were reflected in their teaching practices in several ways. Since TAs were not as quick to assume they knew what students were thinking based only what they wrote, TAs appeared to ask more questions and to probe more deeply into how their students were making sense of the problems. TAs were also more likely to question students after providing them with information or a hint to find out if they had understood the ideas being discussed.

Conclusions

In the case discussed here, activities that had been very effective in the context of traditional discussion section were less useful in the context of sections involving collaborative groupwork. The basic premise of the activities was still valuable, but they needed to be modified in order to be effective in helping TAs meet the challenges of teaching a class where students spent considerable time working in collaborative groups on challenging problems.

By providing specific reflection questions, the videotaping activity encouraged TAs to direct some of their attention to issues that shape the nature of the learning experiences that students have in these classes. For example, TAs were directed to pay particular attention to the questions they asked and to the nature of the interaction among students in the groups. In the case of the peer observation activity, a group observation component was added after TAs spontaneously made interesting and useful observations about what was happening in groups during class. Although TAs can learn from interacting with groups in their own classes, this activity provided them with an opportunity to observe students for an extended period of time and in a manner not possible in the midst of teaching their own sections.

Several conclusions can be drawn from these experiences. First, it appears that with fairly minimal scaffolding, it is possible to support TAs in ways that enable them to focus more extensively on substantive issues of student learning. It remains to be seen what additional support TAs would need to make similar observations on a regular basis and in "real time" in their classrooms, but the fact that it is possible to assist them in being somewhat reflective in this context should provide encouragement to those who wish to find ways to support TAs in becoming consistently reflective teachers.

Second, in the case of the group observation activity, the need for it actually arose spontaneously. Although the hope was that TAs would learn about students and how they think about mathematics from the interactions they have during class, it was possible for them to learn even more by stepping away from their role as TA and observing students in another TA's class. These and similar activities are likely to be an essential part of professional development programs that help TAs and other beginning teachers develop teaching practices that support student learning in highly interactive instructional contexts such as collaborative groupwork.

Third, there is a great deal that people responsible for support and professional development of beginning teachers can learn from observing and talking with the teachers with whom they work. Had the videotaping and consultation activity not existed, it might not have been possible to discover that TAs were failing to reflect on the groupwork component of class and needed support in doing so. If there had not been substantial discussion after the peer observation activity, the group observation component might never have been developed. The mathematics education community is most likely to meet the substantial demands of providing support for teachers who are learning to teach in innovative ways if we finds ways to truly listen to, learn from, and respond in substantive ways to the challenges these teachers face.

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PRE–SERVICE AND IN–SERVICE TEACHER OF MATHEMATICS' TRAINING IN TEACHING WITH THE USE OF COMPUTERS

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ABSTRACT

In this paper we propose a program of pre–service and in–service teacher of Mathematics' training in teaching with the use of computer software programs. The program consists of a) the presentation of the most characteristic theories of learning, teaching methods and models and models of using computers in the teaching–learning environment and b) the training of teachers in the use of computer software programs that are being chosen as appropriate to offer more than the traditional instruction.

Moreover we present the results of the application of the program in the Mathematics' Department of the University of Athens during the academic years 1999–2000 and 2000–2001. The computer software program used in the application is Mathematica® in point of the possibilities it offers in the negotiation of mathematical subjects in Secondary Mathematics' Education. This paper studies the evaluation of the use of the program by the pre–service and in–service teachers that participated in the research in point of the aims, the operation and the use of the program in the teaching–learning process of Secondary Mathematics' Education. Moreover it studies the differences between pre–service and in–service teachers' opinions and the relation between their opinions and their interest and experience in the use of computers.

Keywords

Adult learning; Computers; Constructivism; Discovery Learning; Improving classroom teaching; Media in Education; Mathematics; Postgraduate Education; Software; Teaching/learning strategies; Training; Undergraduate Education

1. Introduction

Contemporary "society of information" offers the possibility for self-education and self-training, but also requires the competence of using and making worthy of the possibilities offered by the new technologies by everyone. The generalized use of new technologies and especially computers in almost everyone's working and personal life, could not leave untouched the sensitive field of Education which is the "mirror" of society. In that context the role of the teacher is significantly modified, since the education of students which affects and provokes, is necessary to include all these elements that will make them competent to participate, act and operate in the society of today and tomorrow.

The introduction and appropriate use of computer software programs in the teaching – learning process of Mathematics is proposed by contemporary research as able to cure some of the weaknesses of traditional teaching [7], [8], [9]. In particular the use of computers seems to provoke the students' interest and attain their attention. Moreover it seems to help students develop an inquiring attitude towards mathematical concepts and ideas through the experimentation with the program, the formation and checking of their conjectures and hypotheses. Lastly it seems to help the transfer of their knowledge (concepts and ideas) in other domains, transforming it to functional knowledge.

In this paper we propose a program of pre–service and in–service teacher of Mathematics' preparation in teaching with the use of computer software programs. Moreover we present the results of the application of the program in the Mathematics Department of the University of Athens during the academic years 1999-2000 and 2000-2001. The computer software used in the application is Mathematica® [3], [12], [13], in point of the possibilities it offers in the negotiation of mathematical subjects in Secondary Education. This paper studies the evaluation of the use of the program by the pre-service and in-service teachers that participated in the research. Also the differences between pre-service and in-service teachers' opinions and the relation between their opinions and their interest and experience in the use of computers.

2. A Theoretical Framework

According to Discovery Learning, proposed by J. Bruner, the basic role of the teacher is to help and encourage his students to discover the mathematical concepts and ideas; moreover to help them develop a general attitude of exploration and experimentation towards mathematical concepts and ideas [1], [2]. According to Constructivism, based on J. Piaget' s ideas and developed by various theorists and researchers of Education, the teacher through the preparation of appropriate activities and problematic situations, should provide his students with an environment where they can construct knowledge actively, using their preexistent knowledge [10], [11].

The possibilities that the contemporary computer software programs have to offer make computers the ideal tools that Discovery Learning and Constructivism are describing in theory. Computer Aided Learning (CAL) includes all the activities via which computers contribute in the process of learning. In the emancipatory paradigm of CAL computers are used as accommodating tools that are partially engaged in the process of learning [5]. In that context, Mathematica® and other programs with similar possibilities, that does not presuppose efficient programming skills, yet have function–based structure, can be used effectively in the visualisation of concepts, in the quick and precise plotting of graphs and in difficult and complex calculations [9], [3], [10].

The teacher must adjust to the formatted new settings and learn how to make worthy of the new technological means and especially computers in his lesson. The conditions in order for him to live up to this new role [5] are:

i. To acquire a positive attitude towards the value of the new technologies

Teacher training as much as any vocational training has one significant difference from any other form of education: In vocational training the trainees are directly interested in the subject of learning, having as a basic objective to making worthy of the result in the vocational level. In that context the teachers must be motivated and convinced for the necessity and the value of the use of computers in the teaching–learning process.

ii. To learn how to organize his teaching effectively

The teacher must be aware of the theories of learning, teaching methods and models and models of using the technological means in the teaching-learning process, in order to be able to select the appropriate technological means and to introduce them in his teaching appropriately.

iii. To have as a priority his pedagogical role

The teacher, being free from everyday time wasting, tiresome tasks, is able to dedicate more time to the special difficulties of his students and help them overcome these difficulties. Also he is able to spend more time with each student, to adjust his answers to the students' individual skills, to evaluate, help, encourage and guide them appropriately.

iv. To be educated in the effective use of the means of technology and especially computers and to be trained constantly

The teacher must be aware of what is considered as the most appropriate way of using the means of technology in order to make worthy of the possibilities they have to offer so as the learning and educational results to be maximized.

3. The Methodology of Research

The research was designed for pre-service and in-service teachers of Mathematics. The research was realized in the Mathematics' Department of the University of Athens during the academic years 1999–2000 and 2000–2001. Two groups participated, a group of 75 undergraduate students of the Mathematics Department (pre–service teachers) and a group of 29 postgraduate students specialized in Mathematics Education (in–service teachers), a total of 104 teachers. A program of teacher of Mathematics' training in teaching with the use of computer software programs was designed and realized in the classrooms and computer laboratories of Mathematics' Department.

A questionnaire was designed and developed in order to evaluate the use of the computer software Mathematica® by the teachers that participated concerning the aims, the operation and the use of the program in the teaching-learning process of Secondary Mathematics Education. The questionnaire was given to the teachers after the completion of the program of training.

The data that was gathered by the encoding of the questionnaires was analyzed with the statistical programs SPSS®. The percentages are counted to the whole of the students that answered each question, given that it is not significantly lower than the whole of students that participated. The three methods used in the statistical analysis are tests of hypotheses. In particular the methods used are X^2 – Testing for homogeneity, X^2 – Testing for independency and Mann-Whitney (U) [8].

The results that arise by the statistical analysis form only conjectures about the tendencies of the students and the relations between their characteristics and not safe conclusions, since the teachers that participated in the research were not selected via one of the Sample Survey methods of Statistics.

4. The program of training

The program of training consists of two parts. The first part is theoretical and includes the presentation of:

A. The theories of Discovery Learning [1], [2] and Constructivism [10], [11]. A brief reference to other theories.

B. The classification of teaching methods and models to teacher–centered, student–centered and interactive [4]. A selection of methods and models that support the principles of the theories mentioned above presented adjusted so as to make efficient use of computers.

C. The change in the role of the teacher and the interactions that are taking place between the teacher, the student and the computer in the contemporary educational environment [5].

D. The classification of educational software to Computer Aided Learning (CAL) and systems that make use of techniques of Artificial Intelligence (AI). The paradigms of CAL: a. Computer Assisted Instruction (CAI) (CAI tutorials and Drill and Practice), b. Relevatory (Simulation), c. Conjectural (Modelling or Modelisation) and d. Emancipatory. The systems of AI: a. Expert Systems, b. Intelligent Didactic Systems and c. Intelligent Computer Assisted Instruction (ICAI) [5].

E. Propositions about the evaluation of educational software programs [6].

The second part concerns the training of teachers in the use of computer software programs in the Emancipatory model of CAL; in particular Mathematica® or Maple® [9], [3], [10]. It includes the presentation of problematic situations and activities in the teaching of:

A. The geometric quantities of functions and the change of the graph of a function according to the change in its parameters, using multiple traces on a graph and animation.

B. The limits, derivatives and integrals of a function. The use of graphs in the study of the monotony and the extrema of a function using derivatives and the geometric interpretation of the derivative of a function.

C. The definition of plane curves (cycle, parabola, ellipse, hyperbola), as cone intersections, as locus and the geometric quantities of curves.

5. The teachers that participated in the research

The in–service teachers that participated in the research (Group A) were 29, with ages from 23 to 49, with mean age 39,14 years and std. deviation 9,20 years. As for the gender 41,4 % were males and 58,6 % were females. The pre–service teachers (Group B) were 75, with ages from 20 to 25, with mean age 22,58 years and std. deviation 1,09 years. As for the gender 74,6 % were males and 25,4 % were females.

The pre–service teachers show a greater tendency to use computers at the University (76 %) than the in–service teachers (44,8 %) (X^2–Homogeneity, $X^2 = 9{,}235$, P–value = 0,002, Df = 1). The teachers posses and use computers at home, as 93,1 % of the in–service and 77,3 % of the pre–service teachers stated, but we cannot come to a safe conclusion about their homogeneity (X^2–Homogeneity, $X^2 = 3{,}483$, P–value = 0,062, Df = 1) (see Table 1).

Both in–service and pre–service teachers have experience in the use of computers (X^2–Homogeneity, $X^2 = 3{,}841$, P–value = 0,147, Df = 2); in particular 77,8 % and 83,1 % respectively use computers for more than 1 year and 55,6 % and 39,4 % respectively for more than 3 years (see figure 1).

The in–service and pre–service teachers are interested in the use of computers (X^2–Homogeneity, $X^2 = 1{,}654$, P–value = 0,198, Df = 1), as the whole of in–service teachers and 94,5 % of the pre–service teachers stated.

A series of extra questions were posed to the in–service teachers about their previous experience in Education. The statistical analysis showed that 93,1 % of them is working or has worked in Education, mainly in Secondary Education. The teachers have worked mainly in High Schools (75,9 %), Junior High Schools (69 %), Tutorial Schools (48,3 %) and private lessons (86,2 %). The teachers are experienced in educational work, as 93,1 % of them has been working as teachers more than 3 years and 62,1 % more than 10 years.

72,4 % of the in–service teachers state that the teaching approaches they use in their lesson are partly in accordance with what they consider as appropriate; only 20,7 % stated full accordance. That is supported by the statistical analysis that showed the independency of the variables of the teaching approaches they use and the approaches they consider as more appropriate (X^2–Independency, $X^2 = 6{,}694$, P–value = 0,153, Df = 4).

The in–service teachers do not use computers in their lesson as 93,1 % state. On the contrary the whole of them believe that computers would give an aid to the lesson, with 51,7 % of them in a great extent. Moreover 86,2 % believe that there is a need for the introduction of computers in their lesson.

6. Evaluation of the use of Mathematica® in the teaching-learning process

6.1 *Aims of the program*

The program is regarded as suitable to be used primarily in High School (90,4 %) and in Higher Education (84,6 %); secondarily in Junior High School (61,5 %) and in Training (55,8 %).

The program can offer more than the traditional instruction mainly in subjects of Geometry (90,4 %) and Analysis (90,4 %), secondarily Algebra (39,4 %).

6.2 *Evaluation of the operation of the program*

The program is considered to start easily by 86,2 % of in–service and 97,3 % of pre–service teachers; we cannot come to a safe conclusion about their homogeneity though (X^2–Homogeneity, $X^2 = 4{,}580$, P–value = 0,032, Df = 1).

The program is considered to be easy to use by both in–service (76 %) and pre–service teachers (69,9 %) (X^2–Homogeneity, $X^2 = 0{,}344$, P–value = 0,558, Df = 1) (see figure 2). The opinion of the teachers about whether the program is easy to use is independent to their interest (X^2–Independency, $X^2 = 1{,}719$, P–value = 0,190, Df = 1) and experience in computers' use (X^2–Independency, $X^2 = 3{,}259$, P–value = 0,196, Df = 2).

The prerequisite skills for the use of the program by the teacher are mainly knowledge of the program's commands (80,4 %) and experience in the use of computers (66,7 %); programming skills are regarded as accommodating skills (24,5 %).

The prerequisite skills for the use of the program by the students are also knowledge of the program' s commands (69,4 %) and experience in the use of computers (67,3 %). 12,2 % of the teachers state that there are not prerequisite knowledge and skills for the students.

The clarifications and explanations that are given by the program when there are mistakes in the input of commands or programs, are characterized by both in–service and pre–service teachers mainly as good (34,3 %) or adequate (34,3 %) (Mann–Whitney, U = 1032, P–value = 0,836). The help browser of the program is also regarded by both in–service and pre–service teachers mainly as good (43,6 %) or adequate (34 %) (Mann–Whitney, U = 836, P–value = 0,541).

The teachers disagree on how the teacher can learn how to operate the program, (X^2–Homogeneity, X^2 = 14,026, P–value = 0,003, Df = 3). In–service teachers believe that mainly the specialist's help is required (48,3 %), while pre–service teachers believe that either direct operation in association with the help browser (40,8 %) or the use of a manual (50,7 %) would do. The teachers seem to agree that the teacher's help is required when the students is learning how to operate the program (in–service: 75 %, pre–service: 57,7 %); we cannot however come to a safe conclusion (X^2–Homogeneity, X^2 = 6,506, P–value = 0,089, Df = 3).

6.3 *Evaluation of the use of the program in the teaching-learning process*

The whole of teachers believe that the use of the program in the lesson would provoke the students' interest for the lesson; indeed 72,4 % and 65,3 % respectively in a great extent (see Table 2). In–service and pre–service teachers are homogenous (X^2–Homogeneity, X^2 = 0,476, P–value = 0,490, Df = 1). Their opinion is independent to their interest (X^2–Independency, X^2 = 2,082, P–value = 0,149, Df = 1) and their experience in the use of computers (X^2– Independency, X^2 = 2,068, P–value = 0,356, Df = 2).

A noteworthy result is that the whole of in–service teachers and 89,3 % of pre–service students believe that the possibilities of the program would provoke the students' interest for Mathematics as a science; 48,3 % and 46,3 % respectively in a great extent. In–service and pre–service teachers are homogenous (X^2–Homogeneity, X^2 = 3,385, P–value = 0,184, Df = 2). Their opinion is dependent to their interest in the use of computers (X^2–Independency, X^2 = 11,576, P–value = 0,003, Df = 2). On the contrary it is independent to their experience in computers' use (X^2–Independency, X^2 = 5,468, P–value = 0,243, Df = 4).

93,1 % of in–service teachers and 81,7 % of pre–service students believe that the use of the program would enable the students' active participation in the lesson; 70,4 % and 55,2 % respectively in a great extent. In–service and pre–service teachers are homogenous (X^2–Homogeneity, X^2 = 3,970, P–value = 0,137, Df = 2). The opinion of teachers is independent to their interest (X^2–Independency, X^2 = 0,920, P–value = 0,631, Df = 2) and their experience in the use of computers (X^2– Independency, X^2 = 6,249, P–value = 0,181, Df = 4).

93,1 % of in–service teachers and 87,7 % of pre–service students believe that the use of the program would enable the students' self–action, exploration and experimentation; 63 % and 65,6 % respectively in a great extent. In–service and pre–service teachers are homogenous (X^2–Homogeneity, X^2 = 0,697, P–value = 0,706, Df = 2). Their opinion is independent to their interest (X^2–Independency, X^2 = 0,916, P–value = 0,633, Df = 2) and their experience in the use of computers (X^2– Independency, X^2 = 0,715, P–value = 0,949, Df = 4).

The teachers have different opinions about the teaching approaches that support the most effective conditions for the use of the program (X^2–Homogeneity, X^2 = 25,946, P–value < 0,001, Df = 3). Although both pre–service and in–service teachers support discovery learning approaches, in–service teachers present a greater percentage (100 % to 88 % respectively).

52,1 % of pre–service teachers believe that the students would be able to use the program as means of self–instruction, opposed to 13,8 % of in–service teachers (X^2–Homogeneity, X^2 = 12,499, P–value < 0,001, Df = 1).

72,4 % of in–service teachers believe that teaching with the use of the program saves time compared to the traditional teaching, opposed to 48 % of pre–service teachers (see figure 3). We cannot however come to a safe conclusion about their homogeneity (X^2–Homogeneity, $X^2 = 5{,}033$, P–value = 0,025, Df = 1).

Teachers present differences in point of the optimum distribution of students per computer (X^2–Homogeneity, $X^2 = 22{,}729$, P–value < 0,001, Df = 5). Although they both propose 2 students (72,4 % and 74,7 %) and 1 student per computer (44,8 % and 10,7 %), pre–service teachers propose also 3 students (13,4 %) and 1 computer operated by the teacher (4 %).

The whole of in–service and 94,4 % of pre–service students state that they would select to use Mathematica® in their lesson and / or propose it to others; 65,5 % and 46,3 % in specific subjects. The selection of Mathematica® by in–service teachers is independent to their beliefs about computers giving an aid to the lesson (X^2– Independency, $X^2 = 0{,}419$, P–value = 0,518, Df = 1) and the need for the introduction of computers in the educational environment (X^2– Independency, $X^2 = 0{,}495$, P–value = 0,482, Df = 1).

7. Conclusions

The teachers that participated in our research were pre–service and in–service teachers of Mathematics. They presented great interest and experience in the use of Computers. The in–service teachers also presented experience in Education with many years of work and service in many degrees and forms of Education.

The teachers proposed the use of the program primarily in High School and in Higher Education, where the students; s abilities are harmonized with the function–based structure of the program. The program can offer more than traditional teaching mainly in Geometry and Analysis, where the graphic negotiation of subjects can essentially aid in the understanding of concepts and subjects studied in general.

The program starts easily and is easy to use as both in–service and pre–service teachers stated. The teacher and the students, in order to use the program, must be familiar with the program' s commands and have some experience in the use of computers; programming skills are not required, yet can accommodate the teacher.

The clarifications and explanations, given by the program when there are mistakes in the input of commands or programs and the help browser of the program, are sufficient.

The teacher can learn how to operate the program according to in–service teachers mainly with the help of a specialist, while according to pre–service teachers via direct operation in association with the help browser or the use of a manual. The students in order to learn how to operate the program need the teacher's help.

The use of the program in the lesson can provoke the students' interest for the lesson. Moreover the possibilities of the program can provoke the students' interest for Mathematics as a science. The students that come in contact with a strict, static, inspired yet untouchable form of Mathematics usually lose their interest since they regard Mathematics as a structure that can be handled only by inspired minds.

The use of the program can also enable the students' active participation in the lesson and the students' self–action, exploration and experimentation, making the lesson of Mathematics an energetic, exploratory, collaborative, social process.

The teachers disagree about whether the students would be able to use the program as a means of self-instruction; pre–service teachers believe they would, in–service teachers they would not. If the teacher is present in the laboratory when students are using the program by themselves, the students can use the program as means of self–instruction, yet ask for the teacher's help when and if they need it.

The majority of in–service and approximately half of pre–service teachers believe that teaching with the use of the program saves time compared to traditional teaching. An important question that arises though is that even if it does not save time, should the teaching–learning benefits of the use of the program to be sacrificed in the altar of presenting one more theorem or solving one more exercise?

The students should work on computers in groups of two students or individually. If the number of computers in the laboratory is not sufficient, the groups of students could include but not exceed three students. If the laboratory cannot be used or there is a limitation of time, 1 computer operated by the teacher could be used.

The teachers are very positive towards the use of Mathematica® in their lesson. They state that they would select to use that program and / or propose it to others. Mathematica® is a powerful, promising tool, a tool in the service of educators who want to provide their students with an environment in which they are able to develop higher order and transferable skills, skills they will use in the society of today and tomorrow.

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Table 1: Evaluation of pre-existent skills of the students and views-attitudes relative to Computers and Education

		In–service teachers	Pre–service teachers
1. Are you using computers at the University?	**Yes**	44,8 %	76 %
	No	55,2 %	24 %
2. Do you have and use a computer at home?	**Yes**	93,1 %	77,3 %
	No	6,9 %	22,7 %
3. Do you find the use of computers interesting?	**Yes**	100 %	94,5 %
	No	0	5,5 %
4. Do you work or have you been working in Education?	**Yes**	93,1 %	–
	No	6,9 %	–
5. Are the teaching approaches you use at school in accordance with the teaching approaches you consider as more appropriate?	**Fully**	20,7 %	–
	Partly	72,4 %	–
	No	6,9 %	–
6. Do you use computers in your lesson?	**Yes**	6,9 %	–
	No	93,1 %	–
7. Do you think there is a need for the introduction of computers in the teaching–learning environment?	**Yes**	86,2 %	–
	No	13,8 %	–

Table 2: Evaluation of the use of the program in the teaching-learning process

		In–service teachers	Pre–service teachers
1. Do you think that the use of the program would provoke the students' interest for the lesson?	**Yes, in a great extent**	72,4 %	65,3 %
	Yes, in some extent	27,6 %	34,7 %
	No	0 %	0 %
2. Do you think that the use of the program would provoke the students' interest for Mathematics as a science?	**Yes, in a great extent**	48,3 %	41,3 %
	Yes, in some extent	51,7 %	48 %
	No	0 %	10,7 %
3. Do you think that the use of the program would allow the students' active participation in the lesson?	**Yes, in a great extent**	65,5 %	45,1 %
	Yes, in some extent	27,6 %	36,6 %
	No	6,9 %	18,3 %
4. Do you think that the use of the program would allow the students' self-action, exploration and experimentation?	**Yes, in a great extent**	58,6 %	57,5 %
	Yes, in some extent	34,5 %	30,1 %
	No	6,9 %	12,3 %

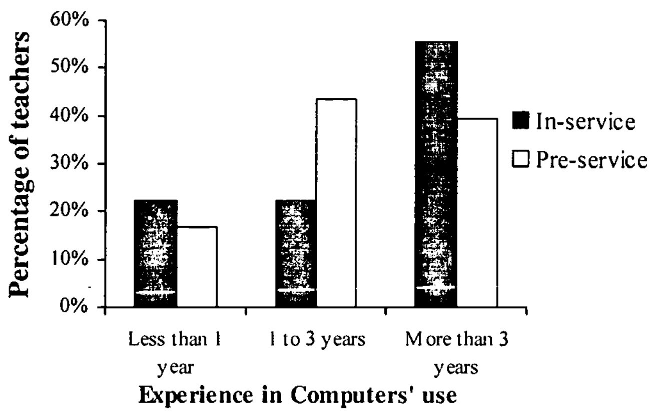

Fig.1. "For how long have you been using Computers?"

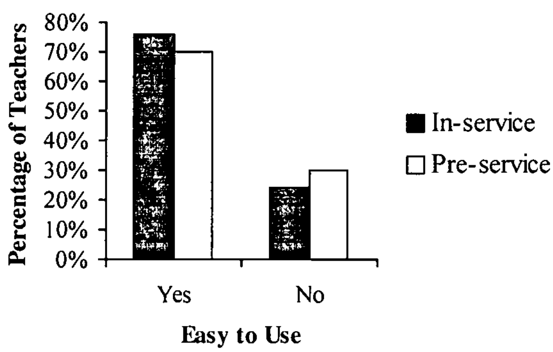

Fig.2. "Is the program easy to use?"

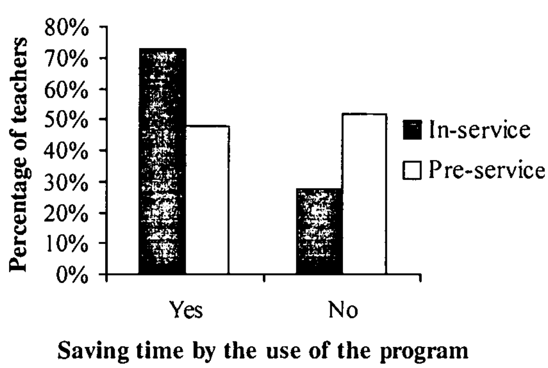

Fig.3. "Does the program save time compared to traditional teaching?"

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DEVELOPMENT OF CALCULUS CONCEPTS THROUGH A COMPUTER BASED LEARNING ENVIRONMENT

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ABSTRACT

This paper investigates the students learning of calculus, particularly the use of the definition of derivative, in undergraduate calculus course in a computer based learning environment in which Interactive Set Language (ISETL) and Derive were used. ISETL was used to help students to construct mathematical concepts on a computer, followed by the discussion held in the classroom. Derive was used to do the manipulations and to draw graphs. The study was carried out with 59 first year undergraduate mathematics and mathematics education students. An essay type test measuring students' understanding of limit and derivative was developed and administered as a pre-test and post-test. Follow-up interviews were conducted with 11 randomly selected students. The analyses of written and verbal responses to the tasks given in the test revealed well increase in the development of derivative concept. The results also showed that computer, particularly ISETL, prevented students to acquire knowledge by rote learning.

Key words: Calculus, Computer, ISETL, Derivative, Errors

1. Introduction

This study is part of a comprehensive research concerning students' learning of calculus concepts. In this paper, descriptive and qualitative results concerning the effect of an instructional treatment, based on having students make various constructions on the computer using ISETL (Dautermann, 1992) and developing manipulative skills and visualization using DERIVE (1989), followed by classroom discussion of mathematics concepts corresponding to these computer tasks, on the learning of the use of the definition of derivative are reported. There was also a certain amount of paper - and – pencil work for the students to do, both in and out of class. The results of the statistical analysis are reported in detail and discussed more fully elsewhere (Ubuz & Kırkpınar, 2000).

Studies about derivative and ideas related to it (such as tangent lines) have emphasized students' misconceptions and common errors (Amit & Vinner, 1990; Artique, 1991; Orton, 1983; Ubuz, 1996, 2001). Ubuz (2001, p.129) reported that students' common misconceptions on derivative were as follows: " (a) derivative at a point gives the function of a derivative, (b) tangent equation is the derivative function, (c) derivative at a point is the tangent equation, and (d) derivative at a point is the value of the tangent equation at that point." Ubuz also stated that students seem to think different concepts as the same. The reasons appeared to be "(a) the lack of discrimination of concepts which occur in the same context or the confusion of a concept with another concept describing a different feature of the same situation, (b) the inappropriate extension of a specific case to a general case, and (c) the lack of understanding of graphical representation."(p.133). To improve students' conceptions of calculus, there have been studies (e.g. Breindenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Schwingendorf, 1991; Dubinsky, 1997) concerning teaching and learning of mathematical concepts using ISETL since the development of the programming language SETL (Schwartz, Dewar, Dubinsky, & Schonberg, 1986). These studies have mainly focused on the constructions of mathematical knowledge in a theoretical perspective rather than students' misconceptions and common errors. A central idea of the constructivist theory is "that understandings are constructed by learners as they attempt to make sense of their experiences, each learner bringing to bear a web of prior understandings, unique with respect to content and organization" (Simon and Schifter, 1993, p.331). Within this theoretical perspective students' existing and acquired concept images were investigated.

The purpose of this study was to answer the following research questions: 1) Is there any improvement in learning calculus concepts through the computer-based learning environment?; 2) what particular errors or misconceptions are in evidence?; 3) what kind of patterns do errors and misconceptions form?; 4) are the patterns of the errors associated with the different tasks?; and 5) which of these endure over time?

2. Method

Subjects

The sample consists of 59 first year undergraduate students in four sections of Math 153 Calculus I course offered at Middle East Technical University. Students were pursuing a major either in mathematics or mathematics education. The sections were formed randomly and different teachers taught each section. Two of those teachers who taught section 1 and 2 were male and the rest were female.

Table 1 shows the numbers of students, who took the pre-test and the post-test on derivative. The students who took both the pre-test and the post-test were taken as the sample of the study.

Table 1: The sample of the study

Section	Pre-test	Post-test	Pre-test $\cap$ Post-test
1	21	25	17
2	15	17	13
3	26	18	15
4	26	15	14
Total	88	75	59

33 (%56) students of those 59 were majoring in mathematics and the rest 26 (%44) students in mathematics education. 57 (%97) of those students have not taken this Math 153 course before and 53 (%90) students have also not taken Math 100 course given prior to Math 153 course. Math 100 course is given to the students who are not able to do 35 mathematics questions out of 52 in the university entrance examination. In the sample, 20 (%34) students were female and 39 (%66) students were male.

Instrument

The test used for assessing students learning of derivative consisted of 6 questions, some of which having different tasks (altogether 32 tasks), on which students were to work individually to provide written responses. Demographic survey questions to gather personal information about each student were included at the beginning of the test. The test was given as a pre-test and post-test without prior warning. The pre-test was administered at the beginning of the semester and the post-test at the end of the semester. Each semester lasts 14 weeks. Each task in the questions were graded by one of the four categories: correct (3), partially correct (2), incorrect(1), and missing (0). The factor analysis carried out for the questions in the pre-test revealed that the test was two dimensional. The first factor was related to the graphical interpretation (GI) (questions 1, 2, 4, and 6) and the other was related to the use of the definition of derivative (DfD) (questions 3, and 5). As mentioned previously, the results related with the questions on the definition of derivative (see Appendix A) are the focus of this study.

Treatment

The study was conducted in a course (Math 153) designed to teach functions, limit, derivative of a function, graph sketching, problems of extrema, and basic theorems of differential calculus: intermediate, extreme, and mean value theorems. The instructional treatment consisted of mainly having students make various constructions on the computer using the programming language ISETL, followed by class discussion of concepts corresponding to these computer tasks. DERIVE was also used by the students for doing activities which are difficult to do by hand. For example, drawing the graph of ($\mathrm{Sin}\frac{1}{x}$). There were also exercises to be done with pencil and paper after the class. Handouts were given on how to use DERIVE and ISETL at the beginning of the course. The textbook used in the course was *Calculus, Concepts, and Computers* (Dubinsky, Schwingendorf, & Mathews, 1995). This course has been conducted for approximately last ten years as it is.

Classes met 6 class hours of a week for 50 minutes each. Two of these hours were at the computer laboratory. There were two 2-class hour sessions during the week and students had to attend only one of these sessions. Some weeks, classes met in the class instead of computer laboratory, and quiz was given each such week. In the lab, students worked individually, each with her or his own terminal. Assistants were available to answer questions, give help with syntax, and etc. There were three computer rooms available, each equipped with 20 computers.

The first week of the semester was used to form the groups of 4 students and to make the introduction for the course. Students who knew and agreed with each other, and had common free time included in the same group. Each week groups were required to complete one activity on the computer by submitting it on the disk, and to complete exercises done with pencil and paper. The group members sat together in the class, because often they had to answer the questions collectively. Every member of each group must be involved in these works as they were going to take their exams individually. Late submissions were not accepted since solutions to the assignments were discussed in class.

The main purpose of the lab sessions was to make sure that every student had at least attempted to perform certain computer tasks before coming to class. The idea was to present the students with the problems so that they could make useful mental constructions. Brief explanations of the activities together with their examples are given below:

I. Functions

1. Writing computer programs of the given different situations where the functions are given in the form of: piecewise , graph, (in)finite SMAP , table, tuple, and string. For example, see question 1 in the book called *Calculus, Concepts and Computers* (CCC) (Dubinsky et al., 1995, p.69). This question is an example of the type piecewisely defined function.
2. Interorizing the action by taking different values from the domain and evaluating them. This makes the students to think about what computer is doing when it makes those evaluations. For example, see the question 1 in the CCC.
3. Drawing the graph of given expressions to understand the function concept and to learn the graph reading.
4. Encapsulating the composition of functions by giving an ISETL code directly and then make students to give meaning to the code. For example, see question 3 in the CCC (p.80).

II. Limit

1. Understanding that the limit value exists regardless of the existence of the function value at that point. For example, question 2 in the CCC (p.132).
2. Interorizing the behaviour of a function near a specified point or at large values i.e. variable tends to infinity. For example, question 3 in the CCC (p.132).
3. Making the idea of the formal definition of the limit more concrete by writing a computer function for taking limit , right limit , left limit , limit at infinity and limit at minus infinity. For example, question 1 in the CCC (p.142).

III. Derivative

1. Encapsulating the concept of derivative by the help of writing a computer program using the concepts difference quotient and the limit. For example, question 1 in the CCC (p.191).
2. Determining the extreme values of a function by graph reading. For example, question 7 in the CCC (p.219).

In the course there were 2 midterms and one final exam. These were in the form of solving problems or proving with paper and pencil without calculator or computer. Exams also contained short questions to be solved using the computer language ISETL. Grading was as listed: Assignments (activities and exercises) 10 %, Class work (participation in class, quizzes, and attendance) 20 %, 2 midterm exams 50 %, Final exam 40 %.

3. Students' Procedures and Conceptions

The analysis of students' written and verbal responses revealed significant information regarding the nature and characteristics of students' understanding of derivative.

The distribution of the scores for the 5 tasks according to four-point scale is reported in Table 2. The scoring criteria for each task are given in Appendix B. As mentioned previously, each task in the questions was graded by one of the four categories: correct (3), partially correct (2), incorrect (1), and missing (0).

Table 2: The distribution of the number of students according to the scoring criteria

	Pre-test				Post-test			
Questions	**0**	**1**	**2**	**3**	**0**	**1**	**2**	**3**
3a	-	2(3)	-	57(97)	2(3)	3(5)	-	54(92)
3b	1(2)	7(12)	-	51(86)	2(3)	12(20)	-	45(76)
3c	9(15)	7(12)	1(2)	42(71)	3(5)	9(15)	1(2)	46(78)
5a	5(8)	7(12)	24(41)	23(39)	1(2)	5(9)	9(15)	44(75)
5b	6(10)	12(20)	19(32)	22(38)	2(3)	7(12)	8(14)	42(71)

Students' attempts to find the value of a function in test tasks 3(a) and 3(c), and the derivative of a function in test tasks 3(b), 5(a) and 5(b) resulted in a variety of erroneous procedures being used. Appendix C contains the erroneous procedures used and the number of students who applied these specific procedures for the five tasks. For the purpose of discussion, the procedures in the table in Appendix C are numbered. It is evident that most of the erroneous procedures results from inappropriate graphical and numerical association or inappropriate visualization.

Although the erroneous procedures occurred on the post-test was not due to the erroneous procedures on the pre-test, the reasons behind these procedures were more or less the same. Also the same students in the pre-test and the post-test did not make these procedures. Two of the 12 errors on task 3(b), three of the 9 errors on task 3(c), one of the 5 errors on task 5(a) and two of the 7 errors on task 5(b) in the post-test made by the same students. Three of the 3 errors on task 3(a), ten of the 12 errors on task 3(b), five of the 9 errors on task 3(c), three of the 5 errors on task

5(a) and three of the 7 errors on task 5(b) in the post-test made by the students who had given the correct answer in the pre-test. The rest of the errors resulted from the omission answers.

Following the post-testing the interviews on test questions 3 and 5 led to the disclosure of various aspects of students' conceptions regarding the use of derivative and the definition of derivative. During the interview sessions students were encouraged to give reasons for procedures they had applied and to define the definition of derivative. The interviewees gave broader array of appropriate associations when explaining the concept of derivative. There was a considerable range among students in their explanations of derivative. Here are some typical responses from the students to the question, "What is a derivative?":

The slope of a tangent line drawn to a curve at any point.(Student S)
Geometrically, the slope of a tangent line drawn to a curve at any point...the change in y over the change in x. The quotient I found the slope of a secant line. When Δx approaches to zero the secant line approaches to tangent line. As a result I can find the slope of the tangent line.(Student U)

Responses from the students in the interviews also showed that students were able to distinguish the difference between the 'derivative at a point' and 'derivative of a function'. Student S made the following remark with respect to his application of erroneous procedure 3c.4 in the post-test (see Appendix C): "*.. first I found the slope as 4/5 rather than 2/5. By mistake I had written 4/5 in the equation of tangent line...*". It is evident that this student's carelessness had come into play here. Student U found the correct answer for task 3(c) in the post-test but wrote that the formula used was the mean value theorem. During interview he expressed his opinion: "*..I think I should have used approximation. But I have done it incorrectly..*".

The interviews on test items 5(a) and 5(b) showed that even some students made some erroneous procedures in using quotient formula to find the derivative at a point of a piecewise function in the post-test they gave the correct explanations during the interview. Student E made the following remark with respect to her application of erroneous procedure 5a.2 in the post-test (see Appendix C): "*as x=3 is greater than –1, I should have used* $2x^3$*.... At for x=-1 I should have looked the right and left limit of the quotient formula and they should be equal to each other...the function must be continuous.*" Students C, S and M who gave the correct answer by using quotient formula and student U found the correct answer by differentiating for task 5(a) in the post-test gave also the correct explanation in the interview as student E. Student M gave the incorrect answer, " $\lim_{x\to -1^+} 2x^3 = -2 \qquad \lim_{x\to -1^-} -x^2+4x+3=-2$ ", for task 5(b) in the post-test but responded correctly in the interview.

4. Conclusion

The general analysis of students' performance, which participated in the study, pointed to a growth of formation and development of derivative concept from the significative increase in the number of correct answers in Pre and Post tests.

The main conclusion supported by the analyses is that the learning process in the computer context with the ISETL becomes very efficient as students work on the computer prior to the class. Student M, for example, drew attention to the point that: " *my point of view has changed from the pre-test to the post-test. At the beginning I was doing without thinking. Now I feel that I am thinking or I force myself to think. While doing homework on the computer, I become obliged to think definitions in some degree.*"

The students overwhelmingly reacted positively to the idea of using computers in a calculus class. A recognized drawback is that there is not enough time for both calculus and computers. In most cases, though, a compromise is thought possible. A significant number of students would like to expand the time spent on computers and their applications. It was observed that the use of computers served not only to facilitate and deepen the understanding of certain concepts but also produced changes in students' attitudes toward the subject.

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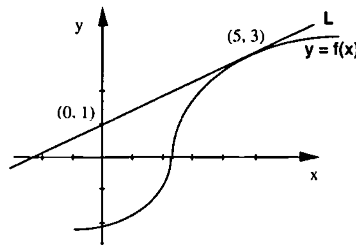
Appendix A

Test Questions

3. Line L is a tangent to the graph of $y = f(x)$ at the point (5, 3).
a) Find the value of $f(x)$ at x = 5.
b) Find the derivative of $f(x)$ at x = 5.
c) What is the value of the function $f(x)$ at x=5.08?
(Be as accurate as possible)



5. Let f be a function given by f(x) = $\begin{cases} -x^2+4x+3 & if\ x \le -1 \\ 2x^3 & if\ x \rangle -1 \end{cases}$

Use the difference quotient to find the slope of the line tangent to the graph of f at

(a) x = 3 (b) x = -1

Appendix B

The Scoring Criteria for the Tasks together with the Examples from the Students' Answers

Questions		**SCORES**	
	3	**2**	**1**
	Totally correct answer	**Partially correct answer**	**Totally incorrect answer**
3			
(a)	Correct value of f(x) at x = 5 (e.g. "{3}")	N/A	(e.g. "{5}")
(b)	Correct value for the derivative of f(x) at x =5 (e.g. "{2/5}")	N/A	(e.g. "{0}")
(c)	Correct approximate value for f(x) at x = 5.08 using quotient formula (e.g. "{3.032}")	Estimated approximate value. (e.g. "It can be near to 3, but I can not say a number")	(e.g. " f(5.08) must be a bit smaller than 3.")
5			
(a)	Correct answer for the slope of the tangent line to the graph of f at x = 3 using the quotient formula (e.g. **"{54}"**)	Finding the correct answer by using differentiation rather than the difference quotient formula	**(e.g. "{-2}")**
(b)	Correct answer for the slope of the tangent line to the graph of f at x= -1 using the quotient formula (e.g. **"{6}"**)	Finding the correct solution without using the difference quotient formula	**(e.g. "{0}")**

Appendix C

Classification and Distribution of Errors for Each Task

Error	Illustrative Example of Students' Responses	Description	Pre	Post	Both
3b.1	f'(5)=3/5	Tangent line is taken as passing through zero	0	1	0
3b.2	F'(5)=2.5	The slope formula $y=(y_2-y_1)/(x_2-x1)$ is taken as $y=(x_2-x_1)/(y_2-y_1)$	0	2	0
3b.3	$f'(5)=\frac{3-0}{5-(-2)}=\frac{3}{7}$ "tanx = ½"	Assuming that the graph is passing through (-2, 0).	4	6	1
3b.2& 3b.3	f'(5)=7/3		0	1	0
3b.4	f'(5)=3	The value of the function at a point was taken as derivative at this point	1	2	0
3b.5	Unclassified		2		
3c.1	F(5.08) must be a bit smaller than 3	Not aware of that the function is increasing	3	0	0
3c.2	F(5.08)=5	The value of x is taken as the value of the function at x=5.08	0	3	0
3c.3	F(5.08)≅2/5	The value of the derivative at a point is taken as the value of the function	3	1	0
3c.4	L = 3/5(5.08)+1= 4.015	Not aware of that the value should be quite close to 3	1	4	0
3c.5	F(5.08)=5.16/5	Unclassified	0	1	0
5a.1	$\frac{f(3+h)-f(3)}{h}=\frac{f(4)-f(3)}{1}$ $\frac{2\times 64-2\times 27}{1}=74$	Using quotient formula but taking the big h value	1	0	0
5a.2	$\lim_{h\to 0}\frac{-(x+h)^2+4(x+h)+3-(-x^2+4x+3)}{h}$ $\lim_{h\to 0}-2x-h+4=-2x+4\Rightarrow m=-2$	Incorrect function	0	4	0
5a.3	X=2 f(x)=7 X=-4 f(x)=3 M=(7-3)/(2-4)=-2 Slope of the line is equal to f'(3)=-2	The slope of the derivative function is taken as the derivative at a point	1	0	0
5a.4	M=tanθ=y/x At x=3 ⇒ y = 18 , tanθ = 18/3=6	Assuming that the tangent line passing through (0,0)	1	0	0
5a.5	Dq= f(x+h)-f(h)/h F(x)-f(x+h)/h	Incorrect quotient formula	2	1	1

Appendix C (Continued)

Error	Illustrative Example of Students' Response	Description	Pre	Post	Both
5b.1	$\frac{f(-1+h)-f(-1)}{h}=\frac{f(0)-f(-1)}{1}$ $\frac{0-(-2)}{1}=2$	Using quotient formula but taking big h value	1	0	0
5b.2	$\lim_{h\to 0^-}\frac{2(-1+h)^3+2}{h}$	Incorrect function	1	1	0
5b.3	We cannot draw a tangent line to the graph of f at x=-1 since it is not continuous at x=-1	As the function defined in parts according to the domain of the function being greater or less than −1, students thought that the function is not continous	3	1	0
5b.4	Since f does not have the same slope for neighbourhoods of −1 we have to be careful to choose close values $X_1=-1$ $f(x_1)=-2$ $X_2=0$ $f(x_2)=3$ $f'(-1)\cong 5$	The slope of the derivative function is taken as the derivative at a point	1	0	0
5b.5	$M=\tan\theta=y/x$ At $x=-1$ $y=1-4+3=0 \Rightarrow m=0$	Assuming that tangent line passing through (0, 0)	1	0	0
5b.6	Dq= f(x+h)-f(h)/h f(x)-f(x+h)/h	Incorrect quotient formula	2	1	1
5b.7	Unclassified		3	4	0

Note: Both refers to pre and post tests together.

In task 3(a) while two students in the pre-test gave incorrect answers such as 5.8 and 3.1, three students in the post-test gave incorrect answers such as 4, 5/2, and 5.

"VISUALISATION, MIND MAPS, RELAXATION,CONFIDENCE AND THE OUTSIDE SCHOOL TUTOR: A CASE STUDY

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ABSTRACT

In this paper I will outline firstly, my study, and secondly, some of the important findings and how they can be used positively in mathematics education. The study was carried out on a group of five students chosen at random from a large group of two hundred students who were surveyed using the Fennema-Sherman attitudinal scales. A case study approach was design was used to draw comparisons and evaluate the effectiveness of the methods.

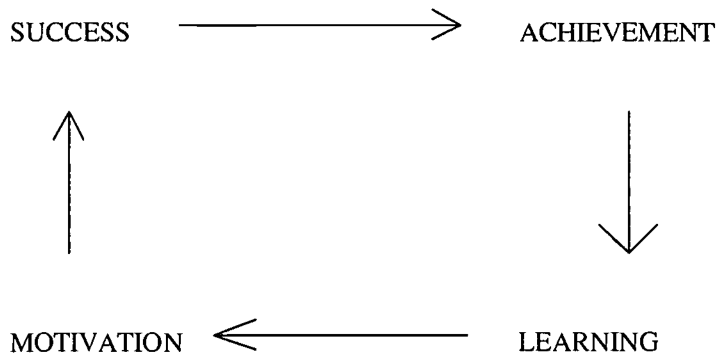
1. Background

Research suggests that as students progress through schooling, her/his perception of the difficulty of mathematics increases. Confidence is an important ingredient of success in any of life's pursuits. In an academic subject such as mathematics this is even more prevalent and therefore, the difficulty that is perceived with mathematics could be mistaken with a decline in confidence in the subject.

The study hoped to show that students improved their understanding and thus performance in mathematics with the help of the tutor, not only in an academic sense but also in building confidence generally.

It is recognized that if a person's confidence level is enhanced then not only an increased performance level will result in the particular activity, but the activity becomes more enjoyable. Academic achievement can easily be paralleled with that of a sporting venture with the teacher acting in a similar vein to the sporting coach in preparing the student: providing a good example and personal knowledge and building the self-esteem and confidence of the student so that with practice the skills will become more familiar and understanding of content will occur. Procedures such as relaxation, visualisation and mind maps are amongst tools can be used to achieve this goal.

Any discussion involving issues such as confidence requires an examination of the affective domain as a whole. The affective domain encompasses a student's feelings about a subject, the classroom environment and students as learners. Each student brings with them a set of feelings, which have an influence over their attitude and confidence level. Burton (1977) expresses the student's position in the affective domain with the following model:



It is a cyclical model because once some success is achieved learning takes place and that in turn is seen as success.

Reyes (1984) contends that confidence is the most important affective variable. It is confidence that influences student willingness to approach new material and persist with it when it becomes difficult. Reyes identified important research issues that should be noted when studying affective conditions including:

a) The nature of the variable;

b) The important factors in the development of the variable (e.g. student, teacher, peer, and classroom);

c) The long-term implications;

d) Stability;

e) Variation of different instructional and mathematical contexts;

f) Relationship to age, sex, socio-economic status;

g) Relationship to other influencing variables;

h) Age at which it can be measured reliably.

(p.73)

In this qualitative study, data triangulation was used to compare the multiple sources of data used. Using Fennema-Sherman Attitudinal Scales and others and comparing the results from each source to confirm expected results. Comparing collected data from questionnaires and statistical data from Fennema Sherman and seeing the comparisons. The data sources for the qualitative part of the study where the students responses to the Fennema - Sherman Attitudinal Profiles and the qualitative part of the study was the investigation, the observation and the questionnaires given to the students throughout the study. The following diagram (Figure 1.1) will outline the method and the types of data collection that we used in the study.

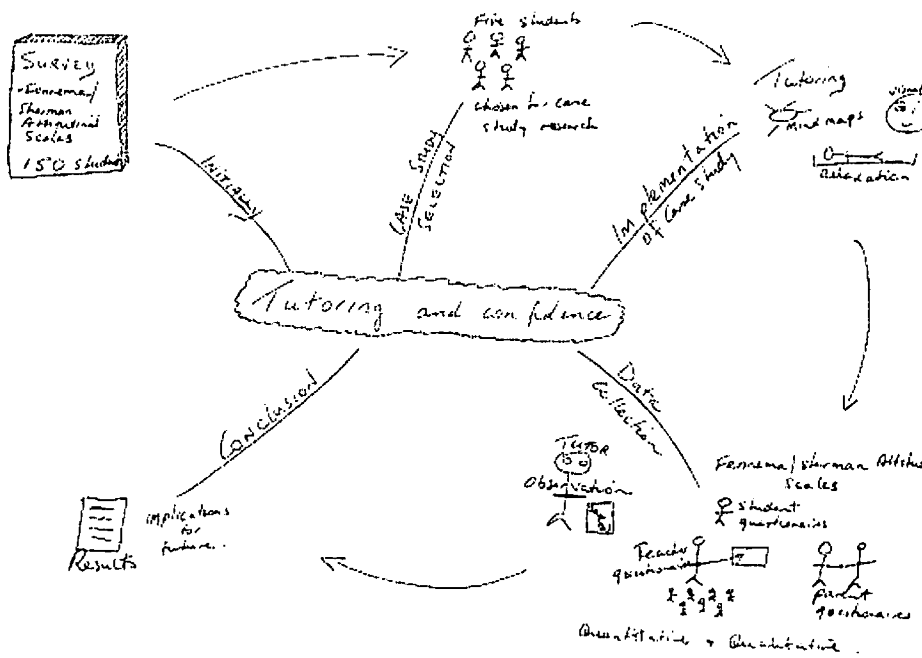


Figure1.1 Memorymap - Sources and Procedures of the study

2. Method and Procedure

2.1 Overview

The study was carried out principally on a group of 5 students picked randomly and who were at various stages of their mathematical development. A large group of High School students were initially surveyed with the Fennema- Sherman Attitudinal Scales and the results of their perception of their confidence shown in this questionnaire were tabulated. The smaller group of 5 students were then chosen from the large group and initially surveyed along with their teachers and parents. A process of tutoring was carried out over the eighteen months with student and teacher observations recorded. At the completion of the study, students, teachers and parents were again surveyed and the results tabulated. These results, together with the tutor's observations, are used to determine the findings of the study.

2.2 Design of the intervention

1. I collected the data initially by attending a number of schools in Sydney and giving the students the Fennema- Sherman Attitudinal Scales instrument.

2. A group of five students were then selected from this group. These students were offered extra tutoring to gage the effect this would have on confidence.

3. The group of five was given a permission slip and a screed of information about the tutoring process, the aim of the study and the expectations of the researcher. . Once the parents had agreed to the conditions, they were given the initial questionnaire. The students and teachers were also given the initial questionnaire and the researcher collated the answers and provided an overview of the perceived confidence level of each student using these questionnaires and the Fennema-Sherman scales that the students had filled in.

4. The tutoring was ready to begin. Each student was given a 1.5-hour session of tutoring per week in a group of 4 students (the other 3 were not in the study). The tutoring sessions took the following form:

a) 5 mins of relaxation

b) 10 minutes of initial easy general mathematics questions that the tutor knew that the student found easy and study of previous memory maps

c) The student is asked what they are doing at school at the moment and the tutor records this and prepares some questions from textbooks based on this topic, but at a very basic level to what they are doing at school. This way the students can get most of the questions correct. The tutor advances gradually until the student is up to where they are at school. The tutor always uses positive language when explaining mathematical principles to the students. The tutor would also use visualization techniques to help the student understand and remember concepts. If a school class test is to be taken in the next week the session will be finished with a "mock test" with the questions graded from very easy to the standard of study at the time. This is the main part of the tutoring session and takes about 60 minutes

d) The final 10 minutes is spent allowing the student to draw a memory map of what they have done in the session today - allowing them to consolidate the information in their minds. The students will answer a progressive questionnaire every third session.

e) Each session the tutor filled in a journal of their perceptions of the student progress in terms of their perceived confidence and any noticeable changes that have taken place in the teachers' observation. The students are welcome to add their own notes to this journal. Often the tutor would record observations on audiocassette and record student responses with this technology. This was found to be the most efficient method of recording.

This research is an example of qualitative interpretive research using multiple sources of data. There will also be some quantitative data collected and interpreted as part of the process. All formal and informal discussions with students, teachers and parents will be logged and used as an integral component of the study results.

2.3 Instruments used

1. Fennema- Sherman Mathematical Attitudinal Scales.

2.Initial student questionnaire. After the students completed the Fennema- Sherman Attitudinal Scales questioning they were given a more specific questionnaire about their confidence, progress in perception of themselves as mathematicians. This was used as a benchmark as to the student's position at the beginning of the study. Each student was asked to rank himself or herself on the perception of the confidence in mathematics. This was compared with the Panama Sherman results so that the tutor can put together a profile of the student for comparison later.

3. Initial Parent questionnaire. A similar questionnaire to that given to the student. The parents gave their perception of the student's confidence level. This is important because much of the students own decision comes from parent expectation and perception. This is committed to what was written by the students so as to set up the initial profile.

4. Initial teacher questionnaire. A third similar questionnaire was given to each of the student's teachers. The teacher gives their opinion about the student's confidence and their opinion as to the ability of the student. The teacher continually works the student and therefore can give an opinion about the student confidence. Again, this questionnaire is used to set-up the original profile.

5. Student Journal. This is an integral part of the study where the student regularly (every couple of weeks) writes out the ideas about the tutoring process, their confidence level and anything that is concerning them in mathematics. A pro forma was given to each student because they enter things more exactly when all questions are asked with a section at the end where free response was allowed. The tutor used these journals to keep a record of the progress of the students.

6. Tutor observation. Each time a student completed a journal entry, the tutor would also do a similar journal entry. The tutor would compare the entries and adjust the continuing profile accordingly. This gives an on going perspective of the progress of the student in the study through the eyes of the tutor.

7. Final student questionnaire. This is the same as the original student questionnaire. It is used to draw interesting conclusions as to how the study has affected the students' perception of the confidence during the study. This is an integral part of the final profile of the student at the tutoring will seize up an idea with the original to form a coalition this along with

8. Final parent questionnaire and final teacher questionnaire, the same as the original given to the parents and teachers.

9. Final Fennema- Sherman Mathematical Attitudinal Scales. This rounds off the study well enabling a statistical representation to be compared with the findings through the initial questionnaires, journals and observations and final questionnaires. All these together give an excellent representation of the student's pre and post study results.

3. Findings

3.1 Visualisation

The students were slow to begin this process and found it the most difficult. Visualisation had the most varying degrees of success. Edward who found it fantastic and used it everywhere, to Mark used it at sport (with help of the sports coach), and Kelly couldn't really handle it because it required her to be too quiet. Alana and Carla both had positive thoughts about the visualisation process throughout the study. By the end of the study all students were quite comfortable in using the visualisation process.

3.2 Memory maps

In comparison to the others, this process was the most successful. It is practical, colourful and allows the student to use their imagination and creativity. In a subject like mathematics the use of imagination and creativity is often not encouraged, so memory maps seemed to be positively received. All five students used memory maps, again to varying degrees, but successfully. All students in the study stated that their self- confidence had increased because of the memory maps and they could use them in other areas of their schooling. Memory maps were very easy for the students to learn to draw but sometimes they were little time consuming and provided an excuse for some students to waste time.

3.3 Relaxation

Relaxation exercises were practiced at the beginning of each session. Most of the students were self- conscious about them at the beginning of the study, but once they saw other students doing the exercises they got involved. With the exception of Mark, all other students achieved success at relaxation. Kelly used it for concentration, and even did it at home with her mother; Alana for settling her nerves in mathematics tests and Carla used it to just relax. Edward saw relaxation as part of the whole process (holistic approach) and stated it helped him as each of the other processes did.

3.4 Outside School Tutoring

The aim of the study was to discover any relationships to confidence using the above methods in an after school-tutoring situation. Much research has been done on peer tutoring and in-school based withdrawal programs, but little has been formally researched on outside tutoring.

The study showed that the students' confidence did increase markedly using the above methods in the chosen environment. I was pleased with the reports from the students, teachers and my own conclusions. Any form of extra help for students carried out in a positive way is beneficial to the students' confidence as well as helping them achieve better results in examinations. "Success breeds success." Of course the students and tutors must put a positive effort into any tutoring program.

4. Results and conclusions

The following outcomes were achieved:

a) The use of alternative methods of teaching such as relaxation, visualisation and memory maps have a positive effect on the confidence of students.

b) Outside school tutoring, in a small group situation, has a positive effect on a student's confidence in mathematics.

c) Positive affirmations and positive talk increase students' awareness of their confidence and ability in maths.

In addition, the study suggested skills and strategies that can be used the generally in the classroom to increase the students' confidence in mathematics.

Each of the processes combined in forming the basis of the study in an after school-tutoring situation. I would have expected that there would be varying reactions and successes with each of the process based on the fact that the sample of students selected was vast in ability, personality and academic willingness.

Each of the processes had a degree of success with each student. With the exception of Mark, who wasn't as receptive as the others. All the processes helped improve each of the student's self-confidence. I noted at the end of the study that a traditional teaching approach would have helped each student improve their marks in mathematics, but these processes also concentrated on using more creative parts of the student's brain and did increase their self- confidence and changed the way they approached the subject of mathematics.

The implications for mathematics generally and in the classroom are varied and interesting. The students enjoyed the different approach to the subject that historically had found difficult, boring in stereotyped by many students. Using visualisation and memory maps as part of the teacher's tool kit are very useful. The study shows that with these tools the mathematics classroom could become more dynamic and thus the confidence of the students would increase. The study also showed how important a positive approach to the teaching of mathematics is.

All students in the study reacted well to any positive approach adopted by the tutor, especially with positive self- talk and visualisation of problems and solutions in mathematics.

Relaxation also proved to be a useful tool. In all areas of education this process could be adopted to make the students feel more relaxed, positive and promote enjoyment of education in a non- threatening environment. The relaxation exercises are dfficult at first for the students, but with persistence a teacher, spending five minutes at the beginning of each lesson doing progressive relaxation with the students and combining the visualisation will reap rewards similar to those of this study.

Mathematics, being a traditional subject, has a specific approach. This study suggests not to change the face of how the subject is taught, but to introduce some new tools to influence the confidence of the students.

Outside school tutoring has been a bone of contention for classroom teachers for many years. Many teachers feel threatened by the need for an outside school tutor. This study showed that an outside school tutor could have a positive effect on the approach of a student to mathematics and also a positive effect on the confidence in the subject. Success breeds success.

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DYNAMIC GEOMETRY SOFTWARE NOT ONLY FOR SIMPLE DRAGGING

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ABSTRACT

Dragging is an integral part of Dynamic Geometry Software (DGS), but one runs into danger to determine only invariants without asking for the reasons and looking for arguments. It is shown, how this danger can be reduced by the additional mode of rearranging that makes possible a DGS with "overlay-technology". Together with a geometry based on translations, rotations and reflections, students are instructed to discover unaided ideas for visual proofs that can be extended to pure mathematical proofs.

Keywords: DGS, Experimental Mathematics, Proofs

1. Introduction

The National Council of Teacher of Mathematics and the Consortium for Mathematics recommended among others the following main-goals for the future of geometry (Mayes 2001):

(a) **Geometry as an Experimental Science:** Geometric objects and concepts should be studied more from an experimental and inductive point of view.

(b) **Geometry as a Formal Deductive System:** Local axiomatic systems which allow the student to explore, conjecture, then prove their conjectures, should replace the long sets of pre-formatted theorems.

These goals of experimenting, conjecturing and proving are higher level cognitive skills and require active student learning. Dynamic Geometry Softwares (DGS) provide interactive and dynamic learning environs. These tools reduce the computation, construction and measurement burdens so that the student can focus on the higher cognitive skills. Together with the celebrated drag mode the student can discover invariants by himself. DGS supports so the first goal mentioned above , together with functional thinking students can discover most of the traditional theorems of undergraduate geometry (Kautschitsch 1998, 2001):

Conjecturing = Finding of invariant properties.

But this kind of DGS contains the danger of restricting only to experiments and of looking only for invariants. There is no time and there are not adequate technological possibilities for answering the question: Why is there the invariant? What are the reasons? So usual DGS is a highly efficient tool for the process of conjecturing but geometry in general is not experienced as a deductive system.

2. The rearrange-mode: DGS with "overlay - technology"

Usual DGS, a mixture of dragging, measuring and calculating, misleads to a reduction in the focus on proofs. But proofs are still the corner stone of mathematics and especially geometry was and is the main field for teaching and learning proofs. Such a (direct) proof is (only) a deduction of a statement A from other statements A_i using some logical rules:

$$A_1 \wedge A_2 \wedge \ldots \wedge A_n \quad A.$$

The correctness of the proof depends not on the fact, if the statements A_i are already proved. Therefore a strong axiomatic-deduction foundation of geometry is not necessary for learning techniques of proving in school-mathematics. The main-goal of a proof for our purpose is – beside the verification of the conjecture – to demonstrate the **logical connections** between theorems. Students should be able to answer the question, why the observed invariants are valid. By the way students learn to keep on selected rules and statements, a soft skill useful for life and business. Two central questions arise:

a) How do students get the conclusion A?

b) Which statements A_i are useful for deduction?

Dragging and measuring support the first question, while the rearrange-mode, described below, supports the finding of appropriate A_i's. Such kind of DGS (for example realized in the package THALES developed at the Department of Mathematics in Klagenfurt/Austria) (Kadunz/ Kautschitsch 1993), has the following additional feature:

It allows a **breaking off constructive relations** and also a **re-establishing** of them.

This feature of breaking off constructive relations permits

- to act with the broken out objects, especially to rearrange them
- to look for "beautiful" figures by changing of position of some partial figures
- to carry out also transformations of congruence such as translation, rotation and reflection of partial figures

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- to duplicate objects and moving one for a simultaneously comparing of the initial- and final configurations ("**overlay - technology**").

By this rearrange-mode a dynamic sampling of decomposition, fitting together, complementing and matching is realized. The cooperation of the modes of measuring, calculating and rearranging offers a micro-world, that simulate a plane with movable parts that makes synthetic geometry possible. That is the main reason why with use of DGS with rearrange-mode one does not run into danger to determine only invariants by dragging without asking for arguments, but it needs a reorientation of geometry and teaching.

3. "Reorientation" of Geometry

The main-feature of DGS with rearrange-mode is the possibility to carry out congruence-transformations with partial figures. In the packages THALES there are buttons for translations, rotations and reflections on lines for interactively chosen parts of construction. Knowledge of properties of these congruence-transformations are assumed as already known. Students have a lot of experience with motions, so it is natural - especially when handling with DGS - to use properties of congruence-transformations as **visual evidences**, above all:

(V) Measure of **lengths**, **areas** and **angles**, parallelism and incidence are preserved under congruence-transformations.

Beside these visual evidences only two visual logical rules are used:

(L1) If two figures are congruent then corresponding parts are equal.

(L2) Removing equal parts of an equal figure - it remain figures, which must be equal.

By testing this programme with pupils it turned out that the following strategies were very useful:

(ST1) Complete to "beautiful" larger figures.

(ST2) Decompose in and match suitable partial figures

(ST3) Carry out the transformations consecutively several times

(ST4) Inscribe suitable subsidiary lines

All these should be done to

(ST5) Search known constellations such as congruent or similar triangles, the Side-Splitter-Theorem, the Screen Angle Theorem and so on.

The collection of (V), (L1)-(L2),(ST1)-(ST5) and the theorems, listed at the end of this section we call the "**visual encyclopaedia**". For learning the technics of proving this encyclopedia should be developed by using only the visual evidences (V), (L) and already proved theorems to prove the following ones. The development was tested twice with 15-16 years old students. Most of the suggestions are well-known in the literature, but they require pencil, paper and (dangerous) scissors and a plenty of time. We did it in one week (!) and most of the theorems were discovered unaided, certainly a merit of DGS.

The theorems of angles on parallel lines (Equality of corresponding angles, interior angles, vertical and alternative angles) and the Congruence Theorems on triangles play a leading role in the development. The theorems on angles can also be used as visual evidences, but they are direct conclusions of the properties (V) of our transformations (we use the visual evidence: Translations preserve parallelism).

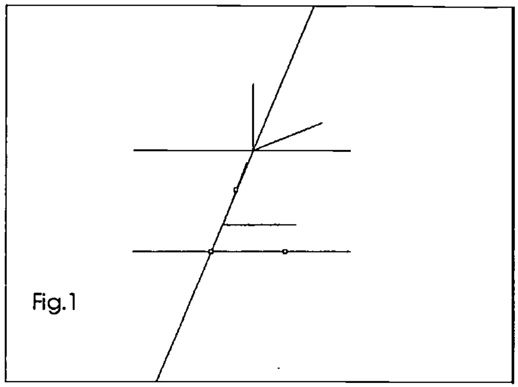

Fig.1

Strategy: Construct a second angle and move it by the overlay-technology.

From the very beginning on it is essential to give the reasons for the matching processes (e.g. in Fig. 1 the parallel lines).

It should be mentioned that a single figure can not replace the acting with DGS.

With the strategy of subsidiary line (in order to generate the above equal angles) one gets the theorems of the sum of angles in triangles and quadrilaterals and about the exterior angles.

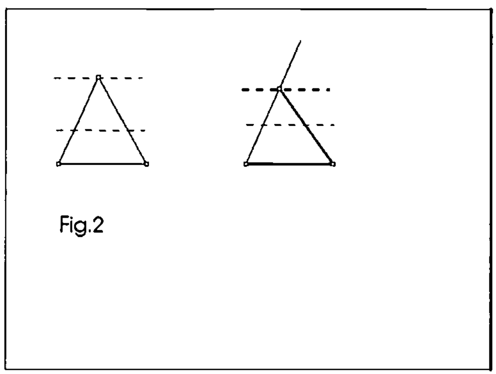

Fig.2

Strategy: Look for the known constellation "angles on parallel lines". It turned out that pupils do not discover the well known line by their own, the motion (!) of one side in the translation-mode after breaking off the relation leads pupils to the all proving subsidiary line. They were not able to imagine this line.

For the further development the Congruence Theorems on triangles are essential, for example the A.S.A. Congruence Theorem:

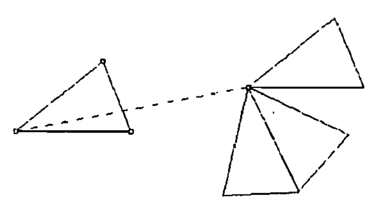

Fig.3

Strategy: Composition of translation, rotation, reflection on a line.

With the help of the S.A.S-Theorem one gets the Isosceles Triangle Theorem:

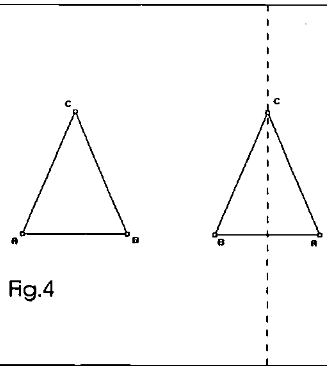

Fig.4

Strategy: Subsidiary line, reflection on the bisector of <C.

Essential: Explain the matching with the S.A.S-Theorem and use the geometric logical rule (L2). Helpful: The inscription is moved in the same way.

This theorem is used essentially in the section "angles in a circle", for example for the Cyclic Quadrilaterals Theorem and the Screen Angle Theorem (Inscribed angles that intercept the same are equal):

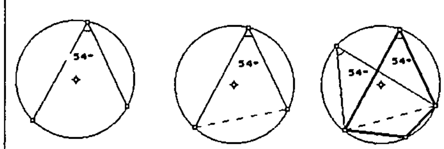

Fig.5

Strategy: Subsidiary line, completion to a cyclic quadrilateral (because it was shown before: In a cyclic quadrilateral opposite angles are supplementary).

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Completion to larger "nice" figures together with the "overlay-technology" and the geometrical logical rule (L) allow self-discovery of the usual formulas for areas and the theorems in right triangles **before** similarity.

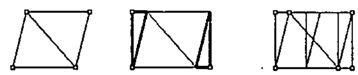


Fig.6

Strategy: Complete to a larger well-known figure such as rectangle, parallelogram and so on.

Change the position of the additional right triangles with the overlay-technology and use screen-splitting for comparing the initial and the final state:

Remove equal pieces from the equal figure and use (L2).

Such deductions of the formulas for areas without calculations are very instructive for students.

The next example concerns the theorem about the altitude to the hypotenuse.

Before knowledge of similarity it is a difficult didactical problem to discover the quadratic relationships of the sides in a right triangle. In Kautschitsch 1998 I have shown how this problem can be mastered by the drag-mode of a DGS together with the "**dependence-graph- technology**". Once the quadratic relationship is discovered one can proceed as follows:

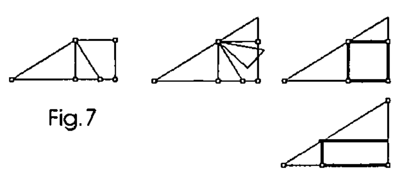


Fig.7

Strategy: Rotate the small right triangle after duplicating it (overlay- technology).

Duplicate the whole figure and change the position. Remove equal pieces and use (L2).

Again it is very important to give reasons why a right triangle is attained (vertical angles are equal, the sides of a stretched angle form a line). The resulting rectangle consists of those segments into which the altitude divides the hypotenuse.

It is well-known that the theorems on right triangles can easily be derived from similarity. In order to use only our mentioned visual evidences and already proved theorems for the developing of the theory of similarity, we use the fact about the areas of triangles that is a direct corollary of the known formula for the area of a triangle, namely:

If two triangles have equal altitudes, then the ratio of their areas is equal to the ratio of the lengths of their basis.

Naturally an excursion about ratios and proportions is necessary. Then we get easily the Side-Splitter Theorem:

If a line parallel to one side of a triangle intersects the other two sides in different points, it divides the sides in the same ratio.

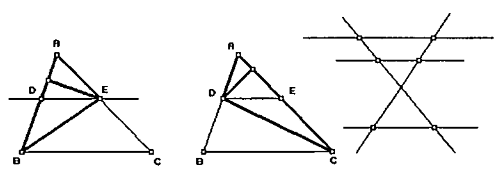
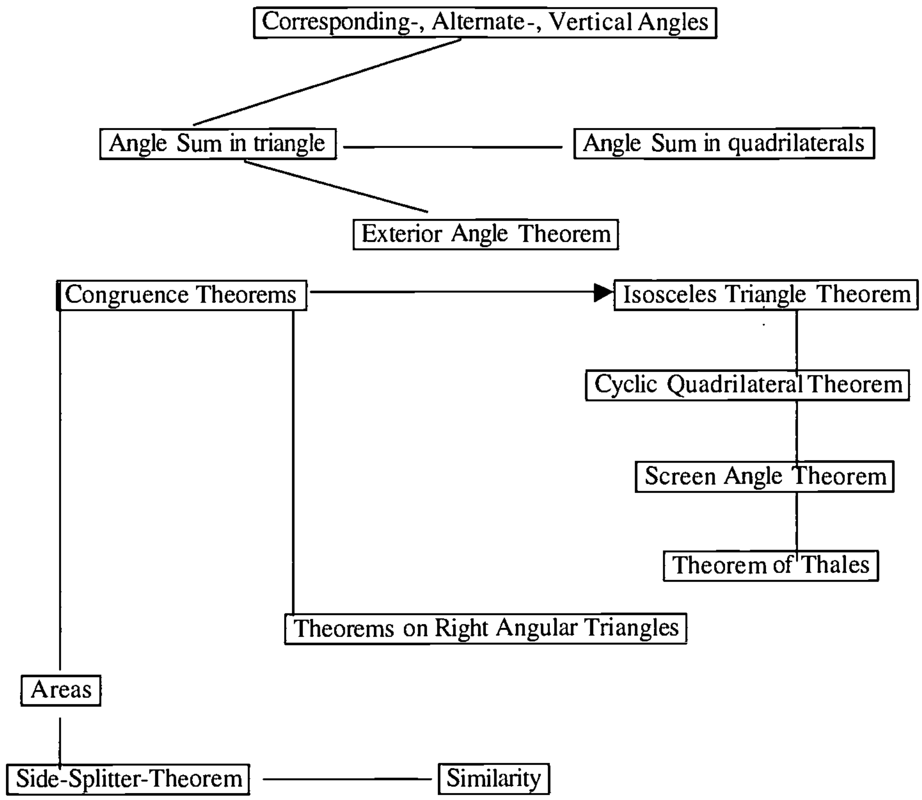


Fig.8

$$\frac{AD}{DB} = \frac{\Delta AED}{\Delta DEB} = \frac{\Delta AED}{\Delta DEC} = \frac{AE}{EC}$$

A generalization with the help of rotations generates the general "Side-Splitter-Constellation". By computation we get the usual statements about proportions in similar triangles.

Summing up we get the following connections between the theorems, which form together with (V), (L1), (L2), (ST1)-(ST5) the above mentioned "**visual encyclopaedia**":



4. Report on a Course

At the University of Klagenfurt/Austria we held two one-week courses with 15-16 years old students to test the DGS with overlaying-technology and the above described “visual encyclopaedia” concerning the following topics:

a) Are the students able to make conjectures without any help?

b) Are the students able to find proofs?

It turned out that they could discover many traditional theorems as well as exotic one's. For conjecturing especially the drag-mode with "dependence-graph"-technology was very efficient. But in fact this method led away from ideas for proving the discovered conjectures. This disadvantage could be reduced by the rearrange-mode with "overlay-technology", since this method is nearer to synthetic geometry and also rearranging can be a source for conjectures.

Example: Theorems in right angle triangles.

By measurements students “see” that the altitude to the hypotenuse of a right triangle forms two triangles that are similar to each other and to the original one.

But it is well-known that similar triangles can easily be recognized, but students have difficulties to find the corresponding sides. Next the students tried to get the Side-Splitter-Constellation (they knew that this had something to do with similar triangles). By overlay-technology they could move the triangle ΔACH. Since also inscriptions move with the triangle, they could read off the right proportion.

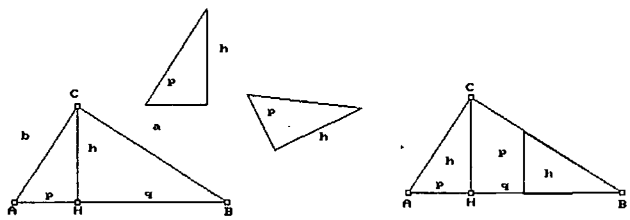


Fig.9

All students could explain the matching of the moved triangle by theorems about angles.
Example: Van Schooten's Theorem

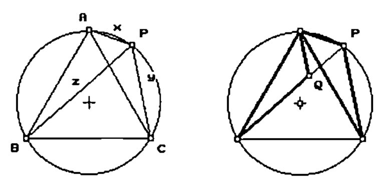


Fig.10

Given an equilateral triangle ΔABC and a point P on the circumscribed circle. What can one say about the lengths of x=PA, y=PC, z=PB?

By measuring and the drag-mode (students begin always to measure and to drag) they discovered the relationship z=x+y. They were convinced of the validity, but they wanted to know, what the reasons were for the validity.

First strategy: Since z should be x+y, x was marked off on the line z and the subsidiary line AQ was drawn.

Second strategy: Looking for congruent triangles. By measuring angles and lengths they saw, that the triangles ΔABQ and ΔAPC were congruent. Movement of a duplicated triangle showed this by matching of corresponding lines and therefore they claimed: BQ=y.

Now the difficult part remained: What are the reasons for this congruence?

Since there are only few theorems in the "visual encyclopaedia", some students concluded first by the Screen Angle Theorem the equality of the angles in B and C. By measuring they saw the equilateral triangle PQA. It took a long time, that some could conclude again by the Screen Angle Theorem that the angle <QPA measured 60°, and by the Isosceles Triangle Theorem the other angles

must then be also 60°. Therefore QA=x and again by the Screen Angle Theorem the angles <BQA and <APC measure 120°, so by the A.A.S-Theorem the triangles must be congruent and by (L1) BQ=y.

5. Conclusions

The success of this programme is based on the following facts:

a) **DGS with overlay technology** offers escape routes for students in hopeless situations into the familiar domains of transformations of figures by translations, rotations and reflections.

b) **DGS with dependence-graph-technology** shows relations that even can not be seen.

c) The "visual **encyclopaedia**" consists only of **few** theorems, two visual logical rules and five strategies. It is so a "minimal" generating set for other theorems.

d) The contents of this encyclopaedia were learned by doing on their own. This procedure helps to discover or recognize well known constellations in unfamiliar situations.

Most of this programme can be done with paper, pencil and scissors. But the use of DGS has many advantages:

(a) The constructions are precise and can be repeated quickly. Many relations can be seen directly and guide so the process of thinking. Subsidiary lines can be discovered by motions of parts of the figure. The imaginative faculty in general is too weak. DGS offers constellations that students can hardly imagine. So DGS offers imaginations outside of the head but of the same or better quality (moveable, precise, only correct imaginations).

(b) Measurements and calculations facilitate the finding of congruent or similar triangles or other equal parts.

(c) The overlay-technology permits comparison of the initial with the final state by screen-splitting and Congruence Transformation of partial figures. This is a source for conjectures that even includes ideas for proving.

(d) Congruence Transformations covers many properties, so the usual long sets of pre-formulated and pre-sequenced theorems can be replaced.

The development of this programme shows the student how a mathematical proof does work. If each matching process is explained by the chosen visual evidences and already proved theorems then no dragging for getting more examples is necessary. So pure mathematical proofs are obtained, expressed only in actions with pictures.

Visual Proving = Finding of always practicable actions with pictures.

We got the experience that students understand the proof, if they could describe the actions and gave reasons for matching. Writing down the arguments was a problem and did not increase the understanding.

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AN APPROACH FOR THE EFFECTIVE INTEGRATION OF COMPUTER ALGEBRA IN AN UNDERGRADUATE CALCULUS AND LINEAR ALGEBRA COURSE

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ABSTRACT

In this paper we describe an approach for an effective integration of computer algebra systems in an elementary calculus and linear algebra course.

In our mathematics courses at Wageningen University, an education and research centre for the biological, environmental, agrotechnical and social sciences, we have noted that students often show a lack of conceptual understanding while using computer algebra systems. A reason for this seems to be that the students do not establish a right link between the computer algebra techniques and their mental approach of mathematics. We have composed a framework that aims at establishing such a link. Because the students have developed their mathematical way of thinking in close relation with paper-and-pencil methods, this framework is based on an integration of computer algebra and paper-and-pencil techniques. We have used this framework for the set-up of an elementary calculus and linear algebra course for first year students in social sciences.

We first describe this framework, which is made up of several steps. In these steps the use of paper-and-pencil and computer algebra alternate and reinforce each other. Next we show how we worked out this approach for an example from calculus: the determination of the stationary points and extremes of functions of two variables. In this example also the graphic facilities of the computer algebra system are exploited. The last part of the example is an application on maximising the profit of a production process, both without and with constraints.

Keywords computer algebra, integration of technology, teaching scenarios, functions of two variables

1. Introduction

In our mathematics courses at Wageningen University, an education and research centre for the biological, environmental, agrotechnical and social sciences, we have noted that students often show a lack of conceptual understanding while using computer algebra systems. A reason for this seems to be that the students do not establish a right link between the computer algebra techniques and their mental approach of mathematics. In this study we describe the set-up of an elementary calculus and linear algebra course for first year university students in social sciences, in which we attempt to establish such a link in a systematic way. In this course we have integrated the use of a computer algebra environment into a more traditional course, but with special attention for the connection between both approaches. In particular, we have composed a framework that aims at an effective integration of paper-and-pencil work and computer algebra techniques. This framework is made up of several steps in which the use of paper-and-pencil and computer algebra alternate and reinforce each other.

In section 2 we describe the educational setting and the aim of the use of a computer algebra environment in this course. In section 3 we describe our framework for the integration of paper-and-pencil and computer algebra techniques. We continue with an illustration of our framework in section 4, describe some results in section 5, and complete the paper with a discussion in section 6.

2. Educational setting and aim of the use of computer algebra in the course

The course had been set up for first year university students in social sciences. Before they entered university, most of these students had taken a curriculum in upper secondary education preparing for a study in social sciences at university level. That curriculum contained mathematics courses in which the mathematics was dealt with in a realistic context, but algebraic skills such as formal manipulation were not highly developed. We note that the students had not made use of a graphing or symbolic calculator in that curriculum.

Our university course covered subjects from calculus and linear algebra. In the course applicability of the mathematics received more emphasis than its theoretical finesses. Applications relevant for the social sciences were included. Also the course aimed at conceptual insight rather than at far reaching technical skills. During a period of six weeks the students had to attend four 2-hour lessons each week. Three of these weekly lessons were given in a more traditional classroom setting without computer facilities, whereas in the other weekly lesson a computer algebra environment was available. In the more traditional lessons, alternately the teacher explained the mathematics and the students were studying the subject, for instance by making assignments. During these lessons the students just had a hand held calculator at their disposal without graphing or symbolic facilities. In the other weekly lesson computer algebra was used in combination with paper-and-pencil techniques.

In this course we did not aim at developing large skills in the use of computer algebra, or at acquiring a thorough knowledge of it. The amount of time available for the use of computer algebra (all together only six lessons of two hours) was not sufficient to achieve such goals. Instead, the aim of the use of computer algebra in this course was to support the mathematical learning process of the students. For this reason we selected an easily accessible computer algebra program (we chose Derive 5.0). Besides, the use of computer algebra gave us the opportunity to treat applications that would demand too much technical skill or too much time if dealt with without a computer.

3. A framework for the integration of computer algebra and paper-and-pencil techniques

Computer algebra can be expected to facilitate the process of gaining conceptual insight, see e.g. Heid (1988). An efficient use of computer algebra in the teaching of mathematics is not self-evident, though, see e.g. Artigue (1997), Lagrange (1999), Drijvers (2000), Drijvers and Van Herwaarden (2000). Students often show a lack of conceptual understanding while using computer algebra systems. As mentioned above, a reason for this seems to be that they do not establish a right link between the computer techniques and their mathematical way of thinking. The students have learned mathematics using paper-and-pencil methods, and their mental approach of mathematics has developed in close relation with these methods. Therefore, one can suppose that a successful internalisation of computer algebra techniques can be reached by an appropriate link with paper-and-pencil methods.

In the course we have tried to establish such a link in a systematic way. In the lessons without computer facilities we treated the subjects of the course in a more traditional (paper-and-pencil) way. In the weekly lessons with computer algebra facilities we have tried to integrate the use of computer algebra and paper-and-pencil work. We have taken the following approach, in which four steps are distinguished. First the students make an exercise with paper-and-pencil. This exercise is of a type they have already dealt with in one of the more traditional lessons, but not too elaborate or requiring too much technical skill. In the second step the students have to solve this exercise using computer algebra. To let them not get stuck in the computer manipulations at this stage, we have provided sufficient details on the required computer algebra commands; in some cases the expected computer algebra output has also been added. In the third step the students have to make some similar exercises to obtain more practice. Finally (step 4), they have to make some more difficult assignments, for example extensions of previous exercises that are too elaborate to handle without computer algebra. These assignments may also be applications from the social sciences. But, when useful, also in these assignments paper-and-pencil questions are included to achieve an appropriate link with the computer algebra work.

4. An illustration

The framework of section 3 will now be applied to an example: the determination of the stationary points and extremes of a function of two variables. In one of the more traditional lessons the students have already learned how to determine partial derivatives, stationary points, and extremes by hand. Then in the next lesson with computer algebra the function

$$F(x,y) = x^2y - 2x^2 - 2y^2 + 4y + 1$$

is considered.

Stationary points. First the students have to determine the stationary points of this function using paper-and-pencil. After calculating the partial derivatives, they are expected to factor these derivatives, if possible, and to make appropriate combinations to determine the stationary points. In this case we expect them to obtain the equations

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$$\begin{cases} F_x(x, y) = 2xy - 4x = 2x(y-2) = 0 \\ F_y(x, y) = x^2 - 4y + 4 = 0, \end{cases}$$

from which they should deduce the combinations

$$\begin{cases} x = 0 \\ x^2 - 4y + 4 = 0 \end{cases} \quad \text{and} \quad \begin{cases} y - 2 = 0 \\ x^2 - 4y + 4 = 0. \end{cases}$$

From these combinations they obtain the stationary points $(0,1)$, $(2,2)$ and $(-2,2)$. For our students one of the difficult steps is to make those right combinations. In particular, a common error is the wrong combination of $x = 0$ with $y - 2 = 0$, leading to an incorrect solution $(0,2)$. Another common error is that students do not obtain the third stationary point, because they forget that the equation $x^2 = 4$, which results from solving the second combination, has two solutions.

Next the students determine the stationary points with computer algebra. First they have to check the partial derivatives already obtained by hand. Next they have to determine the stationary points with Derive in two different ways: graphically and algebraically. For both approaches we have described in rather much detail the actions and commands they have to carry out. We thus hope to prevent that at this stage difficulties with computer manipulations would distract the students' attention from the mathematics. In the graphical approach the students make a plot of the equations $2x(y-2) = 0$ (in red) and $x^2 - 4y + 4 = 0$ (in blue). Then each point of intersection of a red graph and a blue graph represents a stationary point. In this way the students can see that the wrong combination mentioned above does not correspond with the intersection of two differently coloured graphs. Also those students that had missed the third stationary point should now become aware of this and correct their paper-and-pencil work. To determine the stationary points with computer algebra in an algebraic way we let the students solve the system of both equations with Derive's Solve > System command. As a check the expected screen output

$$[x = 0 \wedge y = 1, x = 2 \wedge y = 2, x = -2 \wedge y = 2]$$

is included in the accompanying text. Some of the students have difficulties in interpreting this notation. In fact, in this case the paper-and-pencil result appears to be helpful in explaining the computer algebra notation. Thus the paper-and-pencil work and the computer algebra method are used to reinforce each other.

Extremes. Next the students have to investigate if the stationary points are extremes. In one of the more traditional lessons they have already learned how to classify stationary points (using the determinant of the Hessian matrix). Now in this computer algebra lesson they first have to classify the stationary points of $F(x,y)$ with paper-and-pencil. They obtain that $(0,1)$ is a maximum, whereas $(2,2)$ and $(-2,2)$ are saddle points. Then they have to investigate the stationary points with computer algebra, again both algebraically and graphically. In the algebraic approach they have to check their paper-and-pencil work using Derive, and to correct it, if necessary. In the graphical approach the students have to plot level curves of the function $F(x,y)$. Such a plot yields a very instructive picture of the behaviour of the function, in particular near the maximum and the saddle points. In the accompanying text we have again provided the computer algebra command that the students can use to produce the plot:

$$\text{vector}(F(x, y) = z, z, -2, 4, 0.05),$$

which yields 121 level curves. By providing this command and the screen settings we hope that the students do not get stuck in the computer manipulations. In this case the aim of the plot is to enlarge the students' understanding of the subject and not to master this vector-command. Finally, the students check their results by plotting 3D-graphics of $F(x,y)$ in the neighbourhood of the stationary points, together with the (horizontal) tangent planes in these points. Thus the paper-and-pencil and computer algebra methods complement each other and improve the students' understanding of the subject.

An application. We note that up to here steps 1 and 2 of the framework introduced in section 3 have been applied twice, both for the determination and the classification of the stationary points. We now let the students continue with a similar exercise to obtain some more practice (step 3). Then in the final step the students have to turn their attention to an application from economics: the maximisation of the profit of a production process. We consider a firm producing a single product that is sold in two different markets. Say, x and y are the outputs in the two markets. Certain simple assumptions for the demand curves and the total cost function lead to the following profit function in the variables x and y:

$$P(x, y) = -5x^2 - 2xy - 8y^2 + 4200x + 10200y.$$

First we let the students determine the marginal profits (first order partial derivatives) for some specified values of x and y. Next they have to determine the stationary point and to investigate if it is an extreme, in particular if it is a maximum. They may answer these questions, which are of a type they have become familiar with by now, with paper-and-pencil or computer algebra, as they prefer. It turns out that most of them solve these questions using Derive. The application ends with a question that is new for the students. Suppose that the firm's production capacity is constrained by $x + y = 801$. In that case the (unconstrained) maximum, $P(300,600)$, is not attainable. The students have to plot the constraint and level curves of the profit function in one figure, making use of Derive's vector-command (see above). Of course, the next question is to find the maximum of the profit function subject to the constraint. To obtain this maximum, the variable y (or x) can be isolated from the constraint and substituted in the profit function. Maximising the resulting function of one variable then yields the solution.

5. Results

At the end of each computer algebra le sson we asked individual students for their opinion. We also interviewed a sub-population of the students at the end of the course. In general the students were positive on the set-up of the course. They remarked that

- the use of computer algebra created the possibility of checking their paper-and-pencil results; it enabled them to discover their mistakes, and it clarified the methods
- the alternation of paper-and-pencil and computer work helped them to 'keep awake'; they had to work very intensively during the computer lessons
- in some cases too much repetition of paper-and-pencil work already carried out in the lessons without computer algebra had been included in the computer lessons
- the computer commands had been described in sufficient detail; the computer work had not raised too many obstacles
- because of the link between the paper-and-pencil and computer algebra work, they had the feeling that they knew what they were doing when using the computer ('not just pushing buttons')
- for many of them the computer algebra lessons, and in particular the integration with the paper-and-pencil work, played an important and useful part in preparing for the final written exam, even though it had to be made without computer algebra.
- Specifically on the computer lesson on functions of two variables the students remarked the benefit of the computer graphics for their understanding of the subject.

We also observed the students' reactions in the weekly computer algebra lessons. We noticed that in general the students worked very intensively. In the lesson on functions of two variables many students had questions on the computer algebra approach to determine the stationary points graphically. They needed explication why the points of intersection of the red and blue graphs represented the stationary points. This created an opportunity to show, when necessary, that the wrong combination of factors leading to the incorrect solution $(0,2)$ is mistaken, indeed. It appeared that some students confused the red and blue graphs with level curves of $F(x,y)$. Specifically the plot with 121 level curves (in colour, and being drawn on the screen gradually) caught the students' attention. From their reactions and our enquiries we deduced that most of them could tell the behaviour of the stationary points (extreme, saddle points) from these level curves. The students also produced 3D-plots of the function in the neighbourhood of the stationary points, together with the horizontal tangent planes. Many of them pointed out, though, that they considered the plot with level curves to be more informative. The question to determine the maximum of the profit function subject to the constrained production level appeared to be difficult. Many students who reached this question, managed to plot the constraint and level curves of the profit function in one figure. They could also point out where in this plot the constrained maximum is attained (the point where the 'constraint line' is tangent to one of the 'iso-profit curves'). So it seemed that they had obtained a good understanding of the problem. Most of them needed a clue, though, to calculate the constrained maximum algebraically: only after the teacher's suggestion to isolate one of the variables from the constraint and to substitute the result in the profit function, they succeeded in obtaining the correct solution.

The results on the final written exam were good, but it is hard to assess the influence of our integration of paper-and-pencil and computer algebra methods. One of the assignments was to determine the stationary points of a function of two variables, similar to the exercise in section 4, and to classify them. The results on this assignment were rather good, slightly better than the results of a comparable group of students in a course before the introduction of computer algebra. But the results on a question to determine a constrained maximum were not very encouraging. Many of the students just checked some points; only 10-15% substituted the constraint into the object function, and only half of them knew how to continue. Without guidance by the teacher this assignment is apparently too difficult for the students.

Finally, the teachers, who could compare the course with the former traditional course, were unanimously positive on the set-up. It was their impression that the carefully staged interaction with computer algebra caused the students to be more conscious of their paper-and-pencil work. Comparing with computer lessons of other courses where computer algebra and paper-and-pencil work had not been integrated that systematically, the teachers remarked that the students seemed to know better what they were doing when using the computer for their calculations.

6. Discussion

We now reflect on the framework that we adopted for the integration of paper-and-pencil and computer algebra techniques. We expect that this integration is helpful for the conceptual understanding of the mathematics involved. One obvious reason is that in this set-up the students work out assignments in two different ways: with paper-and-pencil and with computer algebra. Not only that repetition, but especially the interaction between both approaches may be expected to support the mathematical insight. As an example we mention the interaction between the graphical computer algebra approach for the determination of the stationary points (as the points of intersection of red and blue graphs) and the algebraic paper-and-pencil approach. We recall that the students thus obtain insight why the wrong combination of factors (pointed out above as a common error) is incorrect, indeed. Another aspect is the checking of paper-and-pencil results with computer algebra, in combination with the correction of mistakes in the paper-and-pencil work. We note that the teacher also plays a part in this by focussing the student's attention again on his/her paper-and-pencil results. We emphasise that in steps 1 and 2 of the framework the paper-and-pencil and the computer algebra work should not raise obstacles that divert the attention from the mathematics. At this stage of the process attention should not be focussed on technical problems, but on basic concepts and techniques.

We remark that within our framework the graphic facilities of computer algebra programs can be successfully exploited. A good example is the plot of level curves, which yields an excellent picture of the behaviour of a function of two variables near an extreme or a saddle point and thus enlarges the insight in these concepts. These plots may be helpful for the students to create a 'mental picture' of these concepts. In turn, these 'mental pictures' can reinforce the paper-and-pencil approach. Also 3D-plots may be helpful, though in our example the behaviour of the function can be deduced better from the level curves, as noted by the students.

The framework for the integration of paper-and-pencil and computer algebra techniques described in this paper appears to be a good and efficient approach in our educational setting. We think, though, that also in other educational settings this systematic framework might be useful, because it aims at developing conceptual insight and concurs with the way students have learned mathematics, i.e. using paper-and-pencil. Also for more advanced subjects it may be useful. For example, it can be utilised for an extension of the example in section 4 with Lagrange's method for the determination of extremes subject to constraints.

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WHAT CAN WE KNOW ABOUT PRE-SERVICE TEACHERS' MATHEMATICAL CONTENT KNOWLEDGE THROUGH THEIR E-MAIL DISCUSSIONS WITH 6TH GRADE STUDENTS?

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ABSTRACT

In this study, my aim was to understand whether pre-service teachers (from a nontraditional mathematics classroom) have developed good understanding of fraction concepts and are able to do algebraic thinking. Furthermore, I sought to determine whether they could use their knowledge of fractions and algebra clearly and effectively in their communications with 6th grade students (also from a nontraditional mathematics classroom.)

The approach in this college mathematics class, "Learning Mathematics via Problem Solving"(Masingila, Lester, & Raymond, 2002), is very different from a traditional approach. In this class, students construct their knowledge through active involvement with challenging mathematic problems while the instructor facilitates, guides, and helps students share their own knowledge. During a semester in 2001, pre-service teachers at our university studied mathematics in groups of four, shared their mathematical ideas and thinking with the entire class, kept daily math journals, and participated in a math communication project in which they discussed mathematics problems about fractions and algebra with 6th-grade middle school students via e-mail. The instructor and I wanted to determine whether a "learning via problem solving" approach enabled our students to understand fraction concepts and engage in algebraic thinking.

In this research, I analyzed the e-mail messages/discussions written to the 6th graders by the pre- service teachers in order to understand the pre-service teachers' content knowledge. This analysis enabled me to relate the type of pre service teachers' content knowledge to how they responded to middle school students' e-mails on fractions and algebra. The findings included here are the result of a preliminary analysis of the data, other possible categories or depth of the analysis will be included in further publications.

Keywords

Pre-service Teachers, Prospective Teachers, Content Knowledge, Subject Matter Knowledge, Mathematical Communication, Mathematical Writing, Group Work, Problem Solving, Children's Thinking.

1. Introduction

Many researchers have investigated the relationship of mathematical content knowledge of teachers and their teaching. It is generally accepted that teachers have to have strong mathematical knowledge for effective teaching. As Liping Ma (1999) stated, "Teachers with profound understanding of fundamental mathematics are able to reveal and represent ideas and connections in terms of mathematics teaching and learning." The teachers who do not have the understanding of mathematical concepts cannot engage students in productive discussions or cannot recognize student understanding when it occurs (Fennema, Romberg, 1999). When teachers have a strong understanding of math, this understanding will help them choose and implement tasks that have the valuable mathematic content and the potential to motivate students (Hiebert et all, 1997). The literature about teacher's mathematical content knowledge shows parallelism with Martin A. Simon's "Teaching Cycle." According to Simon, teachers' knowledge of mathematics affects their learning goals, the plan for learning activities, the hypothesis of students' learning process, and their assessment of students' knowledge (Simon, 1995).

For pre-service teachers, as well, the mathematical content knowledge should affect their instruction by how they choose tasks, how they lead mathematical discussions or how they asses their students' knowledge. Higher education has considerable impact on pre-service teachers' mathematical content knowledge before they enter the teaching profession (NCTM, 2000). Generally, Schools of Education and Mathematics Departments at universities are responsible for the mathematical content courses in which pre-service teachers learn the mathematics they need to teach. After the content courses, pre-service teachers take method courses, in which they learn how to teach mathematics and concentrate on children's mathematical thinking.

In this study, Dr. Beatriz D'Ambrosio, my mentor and the instructor of the course, and I wanted to know more about pre-service teachers' mathematical content knowledge as they took their last content course before entering the teacher education program. The pre-service teachers of this mathematics course had a project in which they discussed and wrote about mathematical problems to their 6th-grade partners through e-mail. In this research, I investigated the e-mail messages/discussions of pre-service teachers to understand their level/type of mathematical content knowledge.

2. Context for the Study

For my graduate internship, I assisted Dr. D'Ambrosio with her semester-long college-level mathematics course. Students in this course expected to be admitted to the elementary education program after finishing this content course. In this class, they learned mathematics quite differently from traditional classes.

In this class, the "Learning via Problem Solving" method was used (Masingila, Lester, & Raymond, 2002) and group work was central to the "Learning via Problem Solving" method. By the help of our instruction and group work, it became important for pre-service teachers to "understand" other people's solutions and to rely more on their own mathematical abilities, as well as to know the importance of building new mathematical knowledge through their own efforts (e.g. asking questions for understanding in group work).

While pre-service teachers were getting used to reflecting on their thinking in their small group discussions, the instructor and I required daily mathematics journals. In these journals, we wanted them to reflect deeply on the main problems, which were discussed in the class session. Our aim was to build habits of reflective thinking while improving their mathematical content knowledge.

The assessment in this class was very comprehensive. In addition to math journals, we assigned homework problems related to the mathematical content studied during the week. The pre-service teachers had tests, which also emphasized group work. They did group projects and presentations at the end of the semester. They also wrote a paper about children's thinking based on the experiences in an e-mail discussion project with 6^{th}-grade students as a requirement of the course.

In the e-mailing project, every pre-service teacher was paired with at least one 6^{th}-grade student who sent e-mails about one problem they solved in their mathematics classroom every week. Our pre-service teachers were familiar with the problems because they had solved them by themselves as an assignment before they responded to the 6^{th} graders. These responses were about the childrens'mathematical approaches and the pre-service teachers' own approaches to the same problems.

The e-mailing project was designed for two related considerations:

The first consideration was the mission of higher education in pre-service teacher's training (NCTM, 2000). The course was designed to give strong mathematics knowledge to future teachers before they started to teach preK-12 mathematics and to prepare them to understand children's mathematical thinking in school based learning communities. The course was not a methods course but this project helped pre-service teachers to experience children's' ways of thinking in a content course. We wanted pre-service teachers to come to their own realization of the "need" for strong mathematical knowledge, through the interaction they had with 6^{th}-grade students in the project. Pre-service teachers had more real situations about what kind of students and mathematical thinking to expect in their future classrooms rather than the examples in the written resources.

The other consideration was the curiosity of the 6^{th}-grade classroom teacher. She wanted her students to have a real audience for their mathematical writing, thinking this would help them to improve their writing. She suggested a partnership project of pre-service teachers and her 6th graders in order "to make a more meaningful, real-life mathematical communication opportunity" for her students (Schoen Strabala, 2000).

At the end of the semester, while grading the pre-service teachers' papers about children's thinking according to the professor's guideline, I wanted to see the actual e-mail discussions they had with the 6^{th}-grade students. I thought that writing back and forth was a real situation for those future teachers, in which they were using their mathematical content knowledge. Pre service teachers were using mathematical knowledge in more realistic situations compared to our classroom assessments, such as, when they were reading 6^{th}-grade students' e-mails, when trying to make sense of 6th graders' mathematical explanations, when struggling with the mathematical ideas, when asking questions for clarification, and when giving feedback on the 6th graders' solutions and explaining their own approaches.

I was thinking that pre-service teachers were doing those tasks (the process of replying back and forth to 6^{th}-grade partners) depending on what they knew about mathematics and how they knew it. This experience was my starting motivation for why I did this study: "What can we know about pre-service teachers' mathematical content knowledge through their e-mail discussions with 6^{th}-grade students?"

3. Assumptions

In this study, we assumed that pre-service teachers were learning "communication" while working in groups of four people and while reflecting on their ideas in daily math journals. The group work gave the students an opportunity to ask appropriate questions in group discussions to clarify their thinking about the solutions to the same problem. The math journals helped individuals to realize how they were using mathematics. Writing demanded more effort for them than just discussing verbally; they had to think about what they wrote and why they found the written pieces meaningful and mathematical. The instructor and I were commenting on their journals in that way. Therefore, we expected them to use those communication skills during the e-mail discussion they had with 6^{th}-grade students.

Because we knew that the 6^{th}-grade teacher was using similar instructions to ours, we assumed the 6^{th}-grade students were used to thinking deeply about the problems, develop strategies and plan for the solutions. Based on this assumption, we also thought that the 6^{th}-graders would solve problems and write about them independently of their teacher, which would make rich discussion opportunities for our pre-service teachers.

4. Data and Analysis

Pre-service teachers, who took Problem Solving in Context of Teaching Mathematics (6-hour credit), posted their first message in which they introduced themselves to their assigned 6^{th}-grade partner early in the semester. In return, the 6^{th}-grade students also introduced themselves via e-mail before a face-to-face meeting. During this project, pre-service teachers visited the middle school two times, when we started (to say "hello") and when we finished at the end of the semester (to say "bye") and in each occasion, they played a mathematical game. Between these meetings, pre-service teachers and 6^{th}-grade students used a technological support service for classroom instructions for communicating via e-mailing.

There were 28 pre-service teachers. Each of them sent, between 7 to 24 reply e-mails to their 6^{th}-grade partners regarding four mathematical problems related to fractions and algebra. In this paper, the e-mail responses related to "the Fair Share Problem," one of the problems discussed in the project, will be presented as a sample analysis of the study.

Content analysis of the e-mails was used to develop categories for understanding of pre-service teachers' content knowledge. According to Fraenkel and Wallen (2000):

> Content analysis is a technique that enables researchers to study human behavior in an indirect way, through an analysis of their communications. A person's or group's conscious and unconscious beliefs, attitudes, values, and ideas often are revealed in their communications. Analysis of such communications (newspaper editorials, graffiti, musical compositions, magazine articles, advertisements, films, etc.) can tell us great deal about how human beings live (p.469).

Analyzing the e-mailing task and learning a great deal about pre service teachers' content knowledge is similar to Fraenkel and Wallen's (2000) "indirect way" of studying human behavior.

5. Categories and Findings

Preliminary findings of the analysis, and sample email messages are discussed below. Ma's (1999) and Hibert's (1986) categorization of "conceptual knowledge" and "procedural knowledge" helped me in my categorization of pre-service teachers' mathematical content knowledge.

Based on the e-mail messages, the pre-service teachers fell into three categories. The categories are conceptual knowledge, non-conceptual and "others."

The pre-service teachers who demonstrated conceptual knowledge asked good questions for understanding or making the child's explanations clear. They were able to discuss or introduce different solutions to a problem, or create related examples in their response pushing the child to think further and thus demonstrating that they themselves had thought more deeply about the problem.

The pre-service teachers who had no conceptual knowledge just accepted the child's thinking and did not try to understand child-constructed mathematics deeply. They sometimes did not recognize the child's explanation or solution as a legitimate solution since the child was not using procedures that the pre-service teachers were used to. At that times, they asked procedural questions, since they were lost in child's explanation, or they were not sure with their results after comparing them with the child's results.

In addition to the previously-defined two categories, there was a third category of pre-service teachers who totally avoided writing mathematically or even showing mathematical procedures for the solutions. This category is called "others." There were 6 out of the 28 pre-service teachers who showed a pattern of not responding to any of the children's emails related to the four mathematical problems. Since, these pre-service teachers wanted to be teachers, we thought they cared about children and childrens' thinking; they chose to be in this profession. Therefore, the explanation for not responding appears to us that they do not have enough knowledge, interest, and confidence in mathematics to show their work in this e-mailing project, which was done with 6^{th} grade children.

The categorization in the "Fair Share Problem" is used as a sample to show how the analysis was done in this study (you can find the problem and more examples of analysis in Appendix):

Category One (Conceptual Knowledge):

The pre-service teachers who demonstrated conceptual mathematical knowledge, made appropriate connections of mathematical concepts in the emails, asked clear questions, followed student's mathematical constructions, and explained their own thinking clearly in different ways. There were 4 out of 28 pre-service teachers who showed conceptual understanding in different levels of their categorization.

Email-1: (6^{th} grader's)

Posted: 2/27/2001

Now let's get down to business with the Fair Share problem. I 1st set the treasure # at 1 because it would give me the fractions. I subtracted 1/3 because that is what the 1st guy took. Next, I took away 1/6 for the 2nd guy, and I don't know why now, and that gave me 3/6 or 1/2 for the last guy.

I am confident that my fist answer is correct, but my 2nd and 3rd aren't. I know that a third of a third is a nineth. That would make 1/3 of 2/3, 2/9. You multiply because that is what of means. 2/3-2/9=4/9. Those are the only parts I had touble with. Either way the 3rd guy comes out ahead. Ask if you have any questions.

Email-1: (pre-service teacher's response)

Posted: 3/5/2001

You ended up with all of the shares being divided ok and yes the last guy ended up with more than the other two. Can you think why after the first guy took 1/3 leaving 2/3 you would say the second guy took 1/6? There were 2 shares remaining equal to 2/3 of the total. The second guy took only 1/3 of the 2/3. It might help if you look at the whole in fractions that are divisible by 3, 6 and 9.
Let me know if this is too confusing or if it helped.

Posted: 3/9/2001 7:16:08 PM

Okay lets finish 'Fair Share' first. I didn't mean to confuse you with the 3, 6, 9. Here's my thinking on the problem. We have a whole that guy 1 divides into three and then takes 1/3. Guy 2 splits the 2/3's remaining into thirds and takes 1/3 of it. How can we divide the 2/3 equally into 3/3? If I split the 2/3 into sixths there would only be 4 to split up because 2 of those sixths were taken by the first guy. I can't split 4/6 into 3 groups equally. Now I decide to split the 2/3 into ninths. Since I know the first guy took 1/3 of the ninths that is equal to 3/9. There are 6 ninths remaining to split into 3 groups (2 per group) of 2/9 each. Guy 2 took 1/3 equal to 2/9, leaving guy 3 with 4/9.
This sounds more complicated than it should. I am often not very good at explaining things so don't worry about telling me you don't understand what I'm saying!!

The child was struggling with the idea that there was something wrong with her thinking on the shares of the second and third treasurer. The pre-service teacher was asking a simple but powerful question to open it, "Can you think why after the first guy took 1/3 leaving 2/3 you would say the second guy took 1/6?" she was not restricting the child's thinking into procedure (e.g., how did you get 1/6. multiply or divide?) by her question; her question was very open.

This pre-service teacher had also an interesting thinking on the Fair Share Problem. She was different from the other thinkers on how she thought to have thirds, sixths, and ninths. Her explanation to decide what to use for dividing the whole (into 3, 6 or 9) was related to the amount of the treasure that was left for the second and third treasurer, and their shares.

In her explanation, she had some misuses related to the whole or the language linked to the fraction parts. She wrote "how can we divide the 2/3 into 3/3?" (She was considering how we could divide 2/3 into thirds, because second treasurer needed to take 1/3 of the remaining treasure-2/3 of the treasure) or "If I split the 2/3 into sixths there would only be 4 to split up because 2 of those sixths were taken by the first guy"(she thought splitting the whole treasure, not the 2/3 of the treasure, into sixths; which would give her 4 split for 2/3 of the 6 splits). However, when you follow her explanation, it is easy to understand what she meant in those sentences.

Category Two (Non-Conceptual Knowledge):

The pre-service teacher in this category could not demonstrate conceptual knowledge. They did not use mathematics in depth; generally, pre-service teachers gave their own results or compared them to the results of the 6^{th}-grade partners. When pre-service teachers asked questions in their response e-mails, the questions showed that either they were unable to follow the child's solution or they were only able to ask non-conceptual questions based on their knowledge. According to the categorization of this study, there were 12 out of 28 pre-service teachers who didn't reveal conceptual knowledge.

Email-2: (6^{th} grader's)

Posted: 2/27/2001

On the Fair Share Problem I got that the 3rd guy had the most treasure because if there were 9 the first guy took 1/3, which is 3 there would be 6 left and 1/3 of 6 is 2, so there were 4 left and the last guy took the rest. At first I didn't know that the last guy only took 1/3 of the 4, so I thought the first guy had the most, but I was wrong.

Email-2: (pre-service teacher's response)
Posted: 3/5/2001
The Fair Share Problem: What was the correct answer and how did you solve the problem?

The pre-service teacher was not able to follow the child's work. Child was already explaining how he solved the Fair Share Problem and what he got. However, since the pre-service teacher was concentrating on the results and might be on the procedural solutions; she didn't see the child's obvious solution as a satisfactory and mathematical solution. To be able to understand child's thinking, one needs to have algebraic thinking to ask further question. The child's reasoning should be questioned when he chose the number "9" to represent the treasure in his model for the solution.

Category Three (Other):

The pre-service teachers who were in this category avoided discussing mathematics. They did not demonstrate either conceptual or non-conceptual mathematical knowledge. The y did not apply the mathematics communication experiences from their own mathematics classroom to the e-mail discussions. There were 12 out of 28 pre-service teachers who fell into this categorization for this problem. 5 of them didn't respond in continuou sly for other 3-mathematics problem and 6 of them skipped this problem either they didn't have any idea about this problem and related mathematics or they were in a hurry in that particular time of the semester, while 1 of those 12 pre-service teachers was writing about everything but not about mathematics in this problem.

7. Conclusion

E-mail discussions of pre-service teachers can be used as an additional assessment tool in pre-service teachers' content courses. The pre-service teachers demonstrated different types of mathematical understanding in their e-mail messages/discussions with 6^{th}-grade students. Approximately 4 out of the 28 pre-service teachers showed conceptual understanding by exchanging and exploring mathematical knowledge deeply with their partners. Most of them (around 18 pre-service teachers out of 28) showed non-conceptual understanding of mathematics by just giving the results or asked simple questions that showed they were unable to follow the children's mathematical thinking or just agreeing on the solution to make their 6^{th} grade partners feel better or confident in mathematics. Approximately 6 of the 28 pre-service teachers did not discuss mathematics at all. In addition to tests, the real communication pieces of pre-service teachers can be used for building a hypothesis about their knowledge and improving the mathematical experiences of these students.

Based on the previous research [like Ma (1999), Ball (1988), since researchers used direct interview], we did not expect to find e-mail responses like those placed in the "others" category-avoided mathematics. These pre-service teachers' attitudes and knowledge about mathematics can be analyzed further with different research techniques.

This research raised other questions about pre-service teachers' knowledge and their future teaching and training. For example: What experiences must the teacher education program provide in order to help pre-service teachers in each category to grow? Can the pre-service teachers who avoided writing mathematically become good teachers? What experiences do non-conceptual thinkers need in order to become more open to children's thinking?

It is hoped this study will be an opening for understanding and/or assessing the mathematical content knowledge of elementary pre-service teachers with different methods. In this case, we used their real communication pieces written to children throughout the semester in a content course.

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Appendix (Additional Analysis)

The Fair Share Problem:

Three brave, but not very bright, treasure hunters recovered a small box of Spanish doubloons aboard a sunken ship. They took the coins back to their campsite. Since it was late, they decided to go to sleep and divide the treasure the next day.

One of the treasure hunters, fearing the others didn't understand mathematics well enough to give out fair shares, took 1/3 of the coins in the middle of the night and fled into the darkness.

Later that night, another treasure hunter awoke and saw that some of the coins were missing. The treasure hunter took 1/3 of the remaining coins and fled into the darkness.

The third treasure hunter awoke and was surprised to see the others gone and many of the coins were missing. Trusting that the others left a fair share, the third treasure hunter took the remaining coins and walked away whistling happily.

Which of the treasure hunters ended up with the greatest share of doubloons?

Category One (Conceptual Knowledge):

Email-4: (6th grader's)

Posted: 2/27/2001

The third man would have gotten the most coins. Say there were 33 coins toi begin with. If the first man took 1/3, there would be 22 coins left. The first man got 11 coins. If from that 22 coins, the second man took 1/3, he would have taken around seven. There would be 15 coins left. The third man took all 15, making him have the most coins.

Email-4: (pre-service teacher's response)

Posted: 2/27/2001

I do believe that your answer to the fair share problem is correct. I did the problem last night and I figured out that the last guy would get the most coins also. No matter how many coins they started out with, the last person would always get the most. Do you know why that is?

This pre-service teacher was looking at the problem by considering that the amount (the number of the coins) in the whole was not matter as long as the fraction parts were same for each case (e.g., the first treasurer always took the 1/3 of the treasure or the second treasurer always took the 1/3 of the remaining treasure). She had the ability to generalize, in which case she didn't depend on the certain number of coins. This shows different understanding; she went beyond the procedures that were valid for just one occasion (e.g., 33 coins in the treasure) and she was generalizing it conceptually.

Category Two (Conceptual Knowledge):

Email-5: (6th grader's)

Posted: 2/27/2001

I noticed that the second guy was the stupid one, because if the dubloons were to be even, #2 would have taken 1/2. Instead, he took 1/3, accidentally leaving 2/3 left for 2 more people, in which there was only one person, so getting this from the original number, the 1st guy got 1/3, the stupid one got 1/6, and the lucky guy got 1/2.

Email-5: (pre-service teacher's response)

Posted: 3/13/2001

I think your reasoning on the fair share problem was awesome, although I really had to think about the problem. When I read your reply, I was like, "Oh, yeah, that makes sense." That second one really wasn't very smart.

The pre-service teacher was thinking that the child's way was "awesome"; so what was awesome in this solution for this pre-service teacher? How could one be sure about the child's thinking without asking, "the second guy took 1/6 of what? And how can you compare his share to the first one's share and the third one's share?" The child said "the stupid [*the second treasurer*] one got 1/6, and the lucky guy got ½ and the pre-service teacher didn't asked the child about how the child got 1/6 for the second treasurer or ½for the third treasurer.

Email-6: (6th grader's)

Posted: 2/26/2001 4:39:34 PM

The paragraph says the firs person takes 1/3 of the treasure. The second person got one sixth. I know this because he took 1/3 of what was left. There was 2/3 left 1/3 of 2/3 is 1/6. The last person got the most because he got 1/2. I knowhe got 1/2 because I converted 1/3 into 6ths so I could add 1/6 and 2/6 which equals 3/6. 6/6 - 3/6= 3/6. Which when reduced equals 1/2.

Email-6: (pre-service teacher's response)

Posted: 3/5/2001 10:13:09 PM

When you said you took 1/3 of 2/3 and got 1/6, were you multiplying or dividing? How do you know this? I was just curious why you thought that way. We are also going over fractions in class right now, and it has been a long time since I have worked with fractions, and by reading your steps to solving problems I have started to remember them again,

This pre-service teacher asked a simple question to follow how the child got 1/6 when he took 1/3 of 2/3; but thinking procedurally, she was asking procedural question "were you multiplying or dividing?" Her question showed that she was not thinking algebraically because either way when one multiplies or divides 1/3 with 2/3, the result can't be make 1/6. Pre-service teacher's questions might show that she realized that the child got 1/6 of the treasure for the second person, so the child got wrong for the third person depending on what he found for the second person. However, her intent to know about used operations were not related to analyze the child's thinking, as well as the meaning of operations that the child used while getting 1/6 as his answer.

WEBWORK
Generating, Delivering, and Checking Math Homework via the Internet

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ABSTRACT

The WeBWorK system delivers homework problems to students through standard web browsers, giving them instant feedback as to whether or not their answers are correct. It has been developed and used extensively for calculus instruction and physics courses at the University of Rochester over the last six years and is and is currently in use at over 30 other universities.

WeBWorK provides an individualized problem set for each student and, as with standard homework, students are allowed to work on each assignment until the due date. When students submit an answer, WeBWorK analyzes their answer and informs them whether or not it is correct, but does not give the correct answer. Students immediately know their status. They have succeeded or they can find and correct a careless mistake, review the relevant material before attacking the problem again, or seek further help with this problem (frequently via e-mail) from friends, the TA or the instructor. With this system, nearly all of our students, after some work, complete almost all of their homework assignments 100% correctly. Our surveys indicate that they are very happy with the instant feedback and the resulting control they feel over their education.

WeBWorK's large collection of existing problems and its extensible macro framework (modeled on TeX) for posing questions *and* checking the answers, allow each instructor to ask the mathematical questions they *should* as opposed to the questions they *must* because of machine limitations.

By focusing on checking homework answers alone rather than also supplying guidance and instruction, WeBWorK plays to the strengths of computers, and avoids some of the difficulties inherent in trying to build "intelligence" into a computer program. WeBWorK collaborates well with existing educational practices such as traditional lectures, reform calculus, workshops, and expository writing.

See http://webwork.math.rochester.edu for more information.

1. Introduction

A longstanding truism of the mathematical community is that "mathematics is not a spectator sport." Proficiency in calculation and in problem solving come only through practice and that means doing your homework. But what if you are doing the homework wrong? While some students can tell whether an answer is "coming out right," many have little sense of whether their answer is reasonable or not. Once they have an answer, any answer, to a homework problem, it is up to the TA or professor to check the answer and respond before they need to think about it again. In these circumstances, homework collected, graded and returned a week later simply does not provide sufficient feedback to be of much help.

WeBWorK changes this situation dramatically by providing instantaneous feedback for the mathematics homework problems encountered in pre-calculus and calculus courses using an internet-based method for delivering these problems to the students. It provides students with instant feedback about the correctness of their answers and, in the process, creates a learning climate in which students continue to work at their homework until they "get it right."

2. Description

Each WeBWorK problem set is individualized in that each student has a different version of each problem (see, for example, the two versions of the problem presented in Figure 2). Students complete the assignment, log onto the internet and enter their answers into a web browser. The WeBWorK system responds telling them whether a specific answer (or set of answers) is correct or incorrect; students are free to try problems as many times as they wish until the due date. As they make entries, the system records the correctness of each attempted answer so that instructors may easily monitor their students' progress.

By focusing on checking homework answers alone rather than also supplying guidance and instruction, WeBWorK plays to the strengths of computers, and avoids some of the great difficulties inherent in trying to build "intelligence" into a computer program. WeBWorK allows the new technology to collaborate with existing educational practices rather than to replace them. Assigning and grading homework is more effective, more efficient and more uniform when using WeBWorK than when using traditional paper-and–pencil problem sets. This allows the TAs and instructors to devote more time to helping students with the conceptual and problem solving aspects of the course material.

In using WeBWorK at the University of Rochester, we have chosen to follow the homework paradigm in which a student is given a fixed assignment and a fixed period of time in which to do it. The computer allows us to change the traditional model in these important ways:

- We can give slightly different versions of the problems to each student and still effectively check them.
- We can efficiently give instant intermediate feedback, at any time of the day or night, as to whether a student's answer is correct or not. We do not usually give answers or hints via the computer, but the student is allowed to attempt the same question again without penalty. After the due date the correct answers, and often complete solutions to the problems, are made available over the internet.
- In addition, we can efficiently and accurately grade every homework problem of every student, even in large introductory classes; we can easily extend the due date of assignments for

individuals as appropriate without causing added bookkeeping burdens, and we can easily and efficiently monitor class performance on the homework problems.

3. Coverage and Availability

At Rochester, WeBWorK is used in our pre-calculus, first year calculus, multidimensional calculus, and differential equations courses as well as in various courses in physics and astronomy. We are just starting to experiment with using it in statistics and finite math courses. In addition, other institutions have used WeBWorK for college algebra (e.g. Howard University, University of Utah), finite mathematics (e.g. Arizona State University, Computer Science at Stony Brook), actuarial science (Georgia State University), and financial mathematics (University of Virginia). It is also being used at the high school level (e.g. Detroit Country Day School) and Ken Appel at the University of New Hampshire is experimenting with using it in an abstract algebra course. Table 1 gives the URL's for over 30 institutions currently using WeBWorK including pointers to many of the sites cited above.

WeBWorK is distributed with a collection of over 2000 problems covering pre-calculus, standard first year calculus, vector calculus, differential equations and elementary statistics.

A list of the current problem collection is available at:

http://webhost.math.rochester.edu/webworkdocs/ww/listLib?command=setsOnly .

WeBWorK is freely available to educational institutions and can be downloaded from http://webwork.math.rochester.edu. (Follow the "download WeBWorK" links.)

4. Technical comparisons

Since the inception of WeBWorK in 1996, many course management systems and gateway testing programs have been made available for use at universities and colleges. WeBWorK's greatest strength, in comparison to these other systems, lies in the variety of mathematically oriented questions it can successfully present and grade and *the ease with which this capability can be extended.* In addition to numerical answers, short answer questions, and a wide variety of matching questions, it is possible to *effectively check answers that are functions or equations.* Because of its modular design, it has been possible in the last year to add new problems containing graphs generated "on demand," vector field graphs, and problems involving complex numbers, all without changing the basic underlying WeBWorK program.

Another significant core feature of WeBWorK is the ability for instructors to specify new methods of checking the answers at the same time that they write the problem. If an instructor can specify an algorithm for checking an answer, then that algorithm can be implemented within WeBWorK. This has allowed the easy extension of the function evaluator to functions of several variables, and allowed differential equations problems to be written in which the answer is not a single equation, but a family of equations, any one of which is a correct answer to the problem.

Every medium limits the kind of questions that can be realistically asked and effectively checked, and WeBWorK, or any computer-mediated teaching method, is no exception. However, within its focused objective of checking homework answers, WeBWorK's design goal is to enable the instructor to "ask the questions they should, rather than the questions they can" to the greatest extent possible and with the least possible hassle. With this goal in mind, WeBWorK was designed so that the instructors can use it in simple ways without mastering the entire system. If their desired homework problems are similar to existing problems, then they can quickly and easily modify these template problems to achieve their goals.

At the other extreme, instructors are not blocked from making innovative and creative use of WeBWorK in line with their own educational philosophies. WeBWorK is built for extensibility, so instructors with programming experience can extend WeBWorK's functionality without rewriting the entire system from scratch. This modular design is achieved by creating macro packages that simplify the creation of each homework problem. Since it is possible to use the full power of the text processing language Perl in writing the macro packages, this provides great flexibility for experimentation and improvement without altering the basic WeBWorK framework.

The extensibility of WeBWorK, and our experience so far with the institutions currently using WeBWorK, lead us to predict rapid growth in the variety of types of mathematics questions available through WeBWorK, from traditional mathematics questions to reform calculus questions and the rapid development of WeBWorK problem sets in many other scientific fields.

5 Assessment

Having designed WeBWorK software and acquired some experience in using it effectively, we are currently working on more formal assessment activities to determine more precisely how WeBWorK affects student learning, and simultaneously to suggest improvements that would make it work even better. The WeBWorK system remains a moving target as we constantly incorporate improvements, and instructors at the University of Rochester and at other institutions add new problems on an ever-growing variety of topics.

Our goals during the initial phase of assessment were to understand student opinion of the WeBWorK and to collect information about how students engage with the system. Our assessment tools have included interviews, an electronic survey (developed at the University of Rochester from local interview responses), observations of students using WeBWorK, and the data captured by the system regarding the level of homework completion, number of attempts per answer, and level of overall correctness of final answers.

The positive responses collected during the interviews and observations and on the electronic surveys can be grouped in several key categories: students appreciate WeBWorK because it eliminates paper-based homework, it provides unlimited attempts for solving problems, it promotes continued efforts toward completing all problems, and it serves as an aid in test preparation. Survey and interview responses also indicate that WeBWorK is regarded as an encouragement for legitimate collaborative study among students since the system creates individualized problem sets. The most strongly endorsed benefit of WeBWorK, however, is the immediate feedback provided to students. For example, in our Fall 1999 survey, completed by almost 90% of the students enrolled in WeBWorK-supported courses, the strong majority of the respondents valued the rapid feedback provided by the system, as noted in the following sample comment:

The part that I like about WeBWorK is that as soon as I submit my answer I know if I got it right or wrong. I can enter the answers as many times as I want to. All the assignments are right there for me if I ever want to go back and study over those problems.

In general, students report value in WeBWorK. A Fall 2001 survey respondent explained how WeBWorK supported his/her calculus learning:

I appreciate the wide variety of material it covers....If I find that I have difficulty with the webwork then I go to the textbook or the instructor for understanding and guidance. I enjoyed webwork because it helped to show me my strengths and weaknesses as a student and where I stand with the material that has been taught.

On the other hand, some respondents report ongoing difficulty in entering answers correctly into the system (which we term "syntax problems"), and express the desire for hints or partial credit when a wrong answer is submitted:

I don't like that it's not possible to receive partial credit. Since assignments that you hand [in] are graded by a human being, you can receive partial credit and know where you went wrong in doing the problem. Sometimes with webwork problems you may have made only a tiny mistake but it's still wrong and you don't know why.

Other students found that the precision required by the system led them to further investigation and connection with the overall course:

Occasionally the syntax of complicated problems caused the answer to not come out correctly. This often made me wonder whether I was doing the problem correctly, or if I just typed it in wrong. However, this problem is remedied by the feedback option, the recitations, and general class notes. So although it was at times frustrating, it wasn't a huge problem.

In terms of student usage, we found a number of different patterns; some students print the problem sets and work them out on paper first, while others work only on the computer and scratch paper. Many begin the assignment several days in advance of its due date; others wait until a few hours before the deadline. Here at the University of Rochester, a highly residential campus, most students complete their WeBWorK assignments in their dorm rooms; a smaller number used campus computer facilities. This finding is of special interest to those who are responsible for designing and maintaining student computing resources.

Of particular interest to math faculty, however, is the information captured by WeBWorK about student persistence. Since the system records all student attempts, we have been able to document a remarkable thoroughness towards full and accurate completion of homework: nearly all students using the system here at the University of Rochester completely virtually all of their homework sets until their answers are nearly 100% correct. Student comments lead us to believe that the immediate feedback feature drives this persistence:

It is nice to have the opportunity to end up with all correct answers and not be penalized for having tried the problem ten times because it encourages me to not give up and actually LEARN how to do the problem. If it were pencil and paper homework, I would try it once, hand it in, receive a wrong answer, and then just be upset about it, but not DO anything about it.

Math has never been my strong subject and I get discouraged very easily when doing problems. I like how I can find out if I am doing my work right on WebWork. Unlike paper and book assignments, I know if my method is wrong and can try and fix it right away. This is much better than book work, which I sometimes, unknowingly, can do an entire assignment wrong and waste many hours in the process, because I may have mixed up a rule or step. Here, I know right away and can fix my problems. I now can use my time more efficiently, which is a big bonus. Plus, when I get the 100% on my problem, it is like a small victory. I don't get that encouragement with paper/book work. I also like how I can email my professor if I need to. That's nice to know I have that option.

Because WeBWorK captures so much information about students as they complete their problems sets, it is possible for us to examine possible connections among various approaches to homework and eventual performance in the course. We are seeking to analyze the data to determine if there are certain patterns which indicate that early intervention -- extra help, encourage or tutoring -- would be beneficial. Preliminary analysis indicates that requiring many attempts to answer questions in the third week correlates positively with low midterm exam scores. We are continuing to refine our analysis, but if these patterns are robust, we believe that WeBWorK will provide instructors with an easy means by which to identify students early in the semester who could benefit from intervention, such as more assistance from the instructor, teaching assistant, or tutoring center staff.

Table 1
Educational Institutions Currently Using WeBWorK

Alfred University	http://cs.alfred.edu/webwork/mat120/
Arizona State University	http://hobbes.la.asu.edu/119/
California State at Long Beach	http://bosna.natsci.csulb.edu/webwork
Cleveland State University	http://webwork2.math.rochester.edu/csu-mth181/
Columbia University	http://www.math.columbia.edu/~achter/2a/help/studentintro.html
Dartmouth College	http://www.math.dartmouth.edu/webwork
Detroit Country Day School (high school)	http://gauss.dcds.edu:5127/webwork/honalg2/
Georgia State University (actuarial science)	http://webwork2.math.rochester.edu/LifeCon/
Gustavus Adolphus College	http://www.gac.edu/oncampus/academics/mcs/webwork/
Harvard University	http://calculus.math.harvard.edu/
Howard University	http://webwork2.math.rochester.edu/howard-mth156/
Hobart and William Smith Colleges	http://math.hws.edu/webwork/math131
Indiana University	http://www.indiana.edu/~mathwz/
Johns Hopkins University	http://xena.mat.jhu.edu/webwork/
McGill University	http://msr01.teaching.math.mcgill.ca/webwork
National Chiao Tung University (Taiwan)	http://calculus.nctu.edu.tw/webwork/
Ohio State University	https://webwork.math.ohio-state.edu/
Penn State at Altoona	http://webwork.aa.psu.edu:8080/
Radford University	http://ruacad.radford.edu:8080
Rochester Institute of Technology	http://webwork2.math.rochester.edu/rit-mth305
Rochester Institute of Technology (Physics)	http://spiff.rit.edu/webwork/phys311_w2002/
Rutgers University	http://www.math.rutgers.edu/courses/135/135-f01/WWindex.html
SUNY at Stony Brook	http://webwork.ams.sunysb.edu/ams161/
Syracuse University	http://webwork.syr.edu/webwork/
Union College	http://omega.math.union.edu
University of Akron	http://golovaty.math.uakron.edu/webwork/calc2/
University of California at Irvine	http://homework.ps.uci.edu/webwork/
University of Hartford	http://zeus.hartford.edu:3142/webwork/
University of Michigan	http://instruct.math.lsa.umich.edu/classes/215/webhw/
University of Rhode Island	http://webwork.math.uri.edu
University of Rochester	http://webwork.math.rochester.edu/-
University of Toledo	http://www.utoledo.edu/~klesh/3860/
University of Utah	http://webwork.math.utah.edu:8080
University of Virginia	http://webwork.math.virginia.edu/
Victor Valley College	http://webwork2.math.rochester.edu/VVC-math26B/

Screen shots of some integration problems:

Prob. List Next

Our records show problem 1 of set 6 has not been attempted.

(1 pt) Evaluate the indefinite integral

$$\int 9\cos^3(69x)\,dx$$

Show Correct Answers

Submit Answer | Preview Answer | *Note: it is after the due date. Answers available.*

Display Mode: text, formatted text, typeset

Logout Feedback ? Help Problem Sets

Problem Set Version Number: 580664
Page produced by script: /ww/webwork/system/cgi/cgi-scripts/processProblem5.pl

(1 pt) For each of the indefinite integrals below, choose which of the following substitutions would be most helpful in evaluating the integral. Enter the appropriate letter (A,B, or C) in each blank. DO NOT EVALUATE THE INTEGRALS

A. $x = 7\tan\theta$

B. $x = 7\sin\theta$

C. $x = 7\sec\theta$

____ 1. $\int \frac{dx}{(49 - x^2)^{3/2}}$

____ 2. $\int \sqrt{x^2 - 49}\,dx$

____ 3. $\int \frac{dx}{(49 + x^2)^2}$

____ 4. $\int (x^2 - 49)^{5/2}\,dx$

____ 5. $\int \frac{x^2\,dx}{\sqrt{49 - x^2}}$

Another student would see this version of the same problems:

▲Prob. List Next ▶

Our records show problem 1 of set 6 has not been attempted.

(1 pt) Evaluate the indefinite integral.

$$\int 35\cos^2(22x)\,dx$$

☐ Show Correct Answer

Submit Answer Preview Answer Note: it is after the due date. Answers available.

Display Mode: ○ text ○ formatted-text ◉ typeset

Logout Feedback ? Help Problem Sets

Problem Set Version Number: 726751

Page produced by script: /www/webwork/system/cgi/cgi-scripts/processProblem8.pl

(2 pt) For each of the indefinite integrals below, choose which of the following substitutions would be most helpful in evaluating the integral. Enter the appropriate letter (A, B, or C) in each blank. DO NOT EVALUATE THE INTEGRALS.

A. $x = 4\tan\theta$

B. $x = 4\sin\theta$

C. $x = 4\sec\theta$

____ 1. $\int \sqrt{x^2} \quad 16\,dx$

____ 2. $\int \frac{dx}{(16 - x^2)^{3/2}}$

____ 3. $\int \frac{dx}{(16 - x^2)^2}$

____ 4. $\int x^2\sqrt{16 + x^2}\,dx$

____ 5. $\int \frac{x^2\,dx}{\sqrt{16 - x^2}}$

Figure 2

INTEGRATING TI-92/CAS IN TEACHING CONCEPS FROM CALCULUS: HOW IT EFFECTS TEACHERS' CONCEPTIONS AND PRACTICES

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Abstract: Although there are many efforts today, trials and experiments in many countries, there are no clear answers how to teach various concepts effectively in mathematics, in particular pre-calculus and calculus, by using hand-held personal technology (H-hPT) and CAS. To inform and train a group of prospective mathematics teachers (PMTs) and practicing teachers in Turkey we have attempted to organize a series of seminars and workshop on H-hPTs for the last few years. In the present study, we report our experiences at the certificate courses for a group of PMTs in Selcuk Uni-Konya, and show the sample of materials designed to teach various basic concepts in calculus. In the study, we concentrate on how the PMTs can apply their knowledge from mathematics and pedagogical courses in teaching of mathematics and use TI-92/DERIVE, share our experience how to improve the curricula by designing new instructional materials and implementing new strategies in teacher education and training.

Keywords: TI-92/Derive supported/aided teaching, Teacher education, Teaching calculus, Linearity and Proportionality, Guided discovery learning

1. Introduction

Teaching mathematics is a complex endeavor and dynamic process, and the changing role of mathematics teachers for the contemporary society requires new tasks and a rather different training (Ersoy, 1991, 1992a). In such an endeavor, teachers are important figures in changing the ways in which mathematics is taught/learned in schools and they should use cognitive tools properly for effecting teaching. Because, computer and the hand-held personal technology (H-hPT), namely graphing and advanced calculators, is profoundly changing various aspects of teaching and learning of mathematics, as well as doing mathematics, and they are considered as cognitive tools. Therefore, considerable attention must be paid to the pre-and continuing (in-service) education of teachers and the integration of such tools into mathematics/science education (Ersoy, 1992a, b). Then, we enrich teaching and learning environment, and may have chances to improve the quality of teaching mathematics/science al all level of schools. Although there are many efforts, trials and experiments in many countries, there are no clear answers how to teach various concepts effectively in mathematics, in particular pre-calculus and calculus, by using computers, H-hPT and software, e.g. computer algebra systems (CAS) today.

To inform and train a group of prospective mathematics teachers (PMTs) and practicing teachers in Turkey we have attempted to organize a serie of seminars and workshop for the last few years. One of them was hold for a group of PMTs who got their BSc degree from department of mathematics in various universities in Turkey, on August 2001 as an integral part of ongoing projects at the Middle East Technical University (METU) in Ankara which is guided and directed by the researchers (Ersoy, 2001) and of the teaching certificate for becoming high school mathematics teachers. In the present study, we report our experiences at the certificate courses for a group of PMTs, and show the sample of materials designed to teach various basic concepts in calculus. In the study, we concentrate on how the PMTs can apply their knowledge from mathematics and pedagogical courses in teaching of mathematics and use TI-92/DERIVE, share our experience how to improve the curricula by designing new instructional materials and implementing new strategies in teacher education and training. In fact, the present study is part of ongoing project about the calculator-supported/aid mathematics teaching in Turkey. In the study, we introduce some samples of instructional materials on the concept of linearity, and reflect the response of a group of PMTs about the use of TI-92/Derive as a cognitive tool. Thus, we would like to share our experience with other experts as well as improve the curricula by designing new instructional materials and implementing new strategies in teacher education and training.

2. Background

Although there are many efforts today, trials and experiments in many countries, there are no clear answers how to teach various concepts effectively in mathematics, in particular pre-calculus and calculus, by using hand-held personal technology (H-hPT) and CAS. Here a short overview about the background of the present study is given.

2.1. Teachers Teach Mathematics with Technology

The use of technology in either doing or teaching/learning mathematics has influenced and changed many aspects (e.g. Howson & Kahane, 1985). Many authors have investigated the issue and reported the influences, impacts and research the findings. The degree and aspects of the impacts depend upon several variables and various factors e.g. implemented H-hPts and software, backgrounds, views and teaching styles of teachers, etc. Any way they may lead to change in teachers' views and their use of computerized tools (Ford & Ford, 1992). Therefore, it is important to prepare teachers for the future needs of their students (Zehavi, 1996). In the training of teachers, technology is not the focus of learning but its use in teaching. Rather, it empowers teachers and students to explore mathematical concepts with real world data and simulations of real world events.

When technology is used in the way prescribed above, interdisciplinary and real world connections become a natural and powerful way for students to make sense of mathematics (Drier, Dawson, and Garofalo, 1999). Because, both CAS calculators, i.e. H-hPTs, and computers are valuable cognitive tools which enable us trivialization, visualization, experimentation and concentration (Kuzler, 2000). Therefore, there are many groups in developed and industrialized countries collaborate and come together to discuss the issues and share their experience and the instructional materials developed. Among them, T^3-US and T^3-Europe are well known, and they design technology-rich and H-hPT-supported/aided curricula and train many teachers each year. Because, technology-rich curricula can meet the demands of the new standards for more inquiry based learning and new content, and can support more sweeping change that goes far beyond what is envisioned in the NCTM standards (Tinker, 2001)

The average rate of use of information and communication technology (ICT) for instruction and of instructional materials in the education system of Turkey is less than ten percent, but the percentage is increasing gradually. However, few mathematics teachers and some instructors are trying to use ICT, implement TI-92/Derive-Cabri in teaching various mathematics topics (Ersoy, 2001). The present study is part of ongoing project in Turkey called *"T^4: Teachers Teach with Technology in Turkey"*. Activities on research and training teacher began at the METU early 1990s and continue with collaboration with other experts in Turkey and abroad, in particular US, UK, France, Austria, etc. During the last decade we have accumulated some experiences and presented our research findings in the national and international congress and symposiums. We are still doing various researches on the technology supported/ aided mathematics teaching and learning, and organizing seminars and workshops for teachers.

2.2. Design of Study and Implementation of Technology

Before the experimental study, we interviewed several PMTs first, and administered a questionnaire to get PMTs' attitudes and opinions about the advanced graphing calculators (AGC), namely TI-92 and the CAS-software Derive as cognitive tools. We found out that none of the PMTs ($n1 = 67$) had any idea about AGC and CAS; and were reluctant to learn and use the cognitive tools in learning and teaching mathematics[1]. Therefore, we decided to inform and train the PMTs, who participate in the teaching certificate courses (a special program) taken place at the Selcuk University in Konya for a couple of weeks. Thus we scheduled a new program on the operation system of TI-92 and main features of CAS-Derive and tried to find out the PMTs' needs to learn how to teach mathematics in TI-92/Derive environment.

We continuously have the impression that if a learner performs the requested tasks carefully he/she will explore the concept him/herself. Then he/she may discover a new relationship between concepts in TI-92/Derive environment and understand it deeply. After the introduction of new trends and innovations in mathematics education in the seminar and give information on the use of TI-92, we interviewed with some PMTs. We assigned five activities to the PMTs who would understand the concepts deeply and discover certain rules by answering the posed questions. The assignments were designed by the researchers in the form of worksheets but the PMTs in pairs

[1] The gathered data is the process and the results will be reported later. Here we only reflect a few results and some personal views on the issue.

would find out relations among certain concepts learned in pre-calculus or calculus before and understand the potential of the technology.

In the course, the PMTs worked in-groups of two ore three; but each had to access calculator. The activities were purely concerned with the concepts decay and growth, linearity, local linearity, limit by approximation, uniform continuity, direct proportionality, arithmetic and geometric sequences. In the end of these activities it was asked the PMTs (n = 24) to construct a concept map showing the relations among the concepts on which they worked out in detail. Finally, we requested them to prepare a lesson plan for the implementation of the modified version of the instructional materials or new ones in their own mathematics classroom later.

3. Worksheets and guidelines

3.1. Aim and General Features of Activities

Aim: The aim of the present study is two fold: training a group of PMTs and reflecting their views on the use of H-hPTs. More specifically, one of our aims is that the groups of the PMTs explore some features of the cognitive tools, namely TI-92/Derive, and discover themselves the effective ways of teaching calculus concepts in schools.

General Features of Activities: The researchers designed a set of activities by using the guided discovery approach and basic ideas of constructivism. It is important to notice that these activities help student teachers comprehend basic concepts and integrate H-hPT in teaching mathematics. A set of some activities prepared for training the PMTs is below. Although there are more activities, we present six of them here only. In the training periods, the PMTs should work in-groups, and follow the guidelines/instruction explained in each activity and answer all questions therein. Of course, depending upon the needs the materials presented here can be modified and transformed into other forms, used for some other purposes and may be in new contexts.

3.2. Activities and Guideline

Activity 1 (Linear motion): A body of mass has 4 m/s velocity and is moving along a straight line. Examine the rate of change in distance by the change in time. Explain this rate of change by means of certain concepts, e.g. linearity, direct proportionality etc.

Guideline/Instruction: Read the following statements carefully and do the operation without skipping any step.

1. Write a function for the distance (x) as a function of the velocity (v) and time (t).
2. To calculate the first-degree differences in time and distance and the rate of change of them, use TI-92 and sketch the graph of the rate as follows.

- Press Diamond+TblSet and give the initial value of time (t) as Tblstart: 0, ΔTbl:1.
- Press Diamond+[Y=] and write the distance function, rate of first degree differences in distance/ first degree differences in time and distance/time as y1=4x, y2=4(x-1), y3 = y1(x) - y2(x), y4 = y1(x)/x.
- Press Diamond+Table and see the numerical values and press Diamond +GRAPH to see the gaphs of the functions.
- Fill in the table below using numerical values from the table on the display.

X	Y=f(x)	1st degree differences in x	1st degree differences in f(x)	Rate of change
0	0			
1	4	1= -0	4= 4-0	= 4/1
2	8	1=2-1	4=8-4	=8/2
3	12	1=3-2	4=12-8	=12/3

Result 1: From the ratio at the last column of the table can you find out the that the points (x,y) are Linear or not?

Result 2: Using the ratios $f(x_i)/x_i$, $i \in N$, can you find out the that x_i and y_i are directly proportional? Why?

Result 3: If the x variable forms an arithmetic sequence then does the range values of $y = 4.x$, $x \in R$ are forms an arithmetic sequence?

Fill in the blanks in the following propositions.

- The directly proportional quantities are ....
- The directly proportional quantities form both .... and ....
- If the ratio of the first degree differences is constant for a function then it is called ....

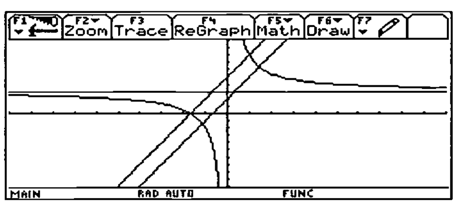
Activity 2 (Symbolic and numerical and graphical representation): Using a TI-92 calculator, study whether the set of points on a straight line are directly proportional or not.

Guideline/Instruction: Read the following instruction carefully and do the operation without skipping any step.

1. The set of the points (x,y) on the graph of the function $y = 3x + 5$, $x \in R$ are linear.
2. Test the values of x and y are proportional or not and construct a TI-92 model for the problem.

- Press Diamond+TblSet and give the initial value of time as Tblstart: 0, ΔTbl:1.
- Press Diamond+[Y=] and write the function, changes in function in time as a function and the ratio of the changes by the time as $y1 = 3x + 5$, $y2= 3(x-1) + 5$, $y3 = y1(x)-y2(x)$, $y4 = y1(x)/x$.
- Press Diamond+Table and see the numerical values and press Diamond +GRAPH to see the gaphs of the functions y1(x), y2(x), y3(x), y4(x) which are all linear.
- Fill in the table below using numerical values from the table on the display.

x	y1(x)	y2(x)	y3(x)=Δy/Δx	y4(x)=y1(x)/x
0	5	2	3	undefined
1	8	5	3	8/11
2	11	8	3	11/2
3	14	11	3	14/3
4	17	14	3	17/4



- What have you perceived from the last column of the data table for the ratio y/x?
- Do the quantities x and y are right proportional? Why?
- What is the relation between the concepts set of linear points and directly proportional quantities?

Activity 3 (Change in Population): Consider the data in Table 1 for the population of Mexico in the early 1980s. Study the changes in population increase and calculate the rate of changes by time. Using this ratio, can you explain the linearity of the population function? Do the quantities, population and year, directly proportional? Do the population data forms a geometric sequence? Study the relations among the concepts linearity, direct proportionality and geometric sequence.

Table 1. Population of Mexico, 1980-1983

Year	n	Population (p(n))	Ratio of Changes ($\Delta p/\Delta y$)	p(n)/p(n-1),n = 1,2,3,4	P(n)/n, n=0,1,2,3,4
1980	0	p(0) = 67.38		1.026	Undefined
1981	1	p(1) = 69.13	1.75	1.026	69.13
1982	2	p(2) = 70.93	1.80	1.026	35.465
1983	3	p(3) = 72.77	1.84	1.026	24.257
1984	4	p(4) = 74.66	1.89	1.026	18.665

- Calculate the first-degree differences in population and time and the rates of change in population by the change in time using TI-92 calculator and fill the forth column of the table. What have you perceived? Does the population function linear or exponential? Why?
- Calculate the ratio p(n)/p(n-1), n =1, 2, 3, 4 and fill the fifth column of the table. Write the population function p(n) related with the initial population p(0). Do the population data form a geometric sequence?
- Calculate the ratio p(n)/n, n = 0, 1, 2, 3, 4 and fill the sixth column of the table. Do the quantities p(n) and n are right proportional? Why?
- Find out the relations among the concepts linearity, direct proportionality and geometric sequence.

Note: Get data for the population of Turkey in early 1950s, 1970s and 1990s. Then find out the changes of population of Turkey in each given period and in the last 50 years.

Activity 4 (Quadratic expression): Calculate the ratio of first-degree differences for the function $f(x) = x^2$, $x \in R$ in the neighborhood of x =1 by the change in x using TI-92. Study the global linearity and local linearity of the function f(x) on R and in the neighborhood of x =1 respectively. Do the quantities x an y are directly proportional? Study the relations among the concepts global linearity, local linearity and directly proportional quantities.

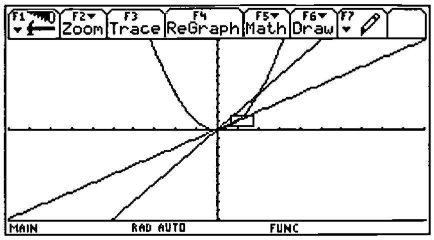
Guideline/Instruction: Read the following instruction carefully and do the operation without skipping any step.

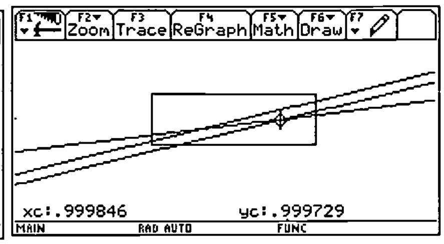
- Press Diamond+TblSet and set the initial value of x as Tblstart: 0,999, ΔTbl:0,001.
- Press Diamond+[Y=] and write the function, rate of first degree differences in f(x)/ first degree differences in x and the rate of change in f(x) by change in time as $y1= x^2$, $y2 = (x+0,001)^2$, y3 =(y1(x) - y2(x))/ 0.001, y4 = y1(x)/x.
- Press Diamond+Table and see the numerical of the functions.

X	Y1	y2	y3 = $\Delta y/\Delta x$	y4 = y1/x
0.999	0.998001	0.996004	1.997	0.999
1.000	1.000	0.998001	1.999	1.000
1.001	1.002	1.000	2.001	1.001
1.002	1.004	1.002	2.003	1.002
1.003	1.00601	1.004	2.005	1.003

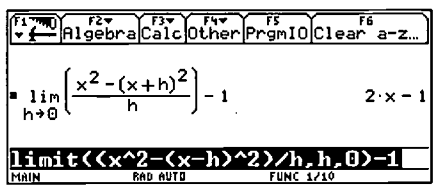
- What have you perceived from the forth column of the data table? Do the function $y = x^2$ global linear on R ?

- What have you perceived from the last column of the data table? Do the quantities x and y are directly proportional? Why?
- Press Diamond+GRAPH and display the Graph screen, lets you draw a box that defines a new viewing window, and updates the window. The display after defining Zoombox by pressing $\boxed{\text{ENTER}}$ you will see the graphs consecutively.





- What have you perceived from the Zoombox? Does the function [Y=] x^2 local linear at x = 1?
- Write a new function for the points local linear in the neighborhood of x = 1. On the HOME screen calculate the following limit.



- Do this result same with the new function? Why?
- Compare the rate of change y3 = (y1(x) - y2(x))/ 0.001 with the result of the limit process. What have you perceived?
- Construct a map showing the relations among the concepts global linearity, local linearity, right proportional quantities, first-degree differences and the limit?

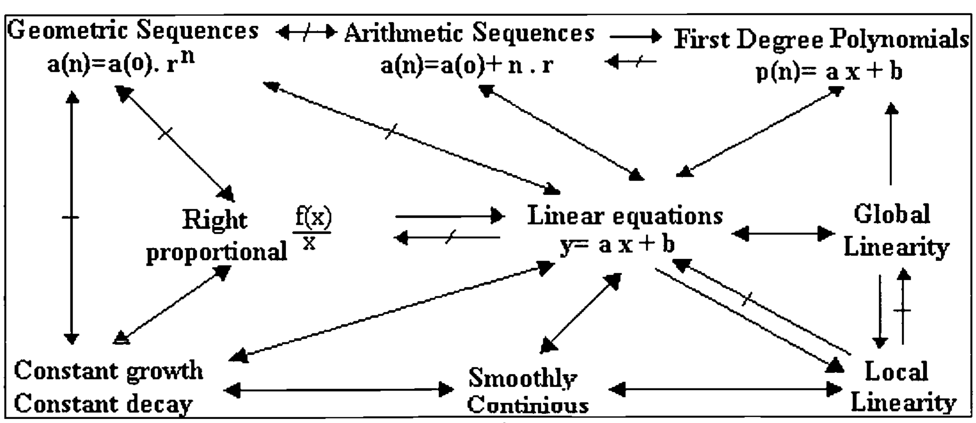


Figure 1. Concepts Maps on Linearity

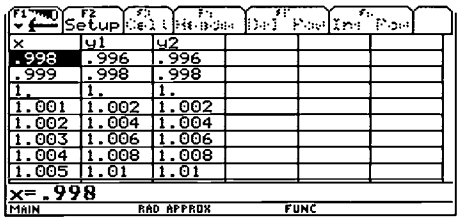
<u>Activity 5(</u>Local linearity and square root):

1. Display the MODE dialog box. For graph mode, select FUNCTION.

2. Display and clear [Y=] editor. Then define $y1 = x^2$ and press $\boxed{F2}$ 6 to see the graph of the function.
3. From the graph screen, press $\boxed{F5}$ and select A:Tangent. Set the tangent point. Either move the cursor to the point or press $\boxed{A}$ and type its x values as 1
4. Press $\boxed{ENTER}$. The tangent line is down, and its equation is displayed.
5. Repeat the process at the step 3 for the x values 2, 3, 4 etc and then fill the table below.

x	$y=x^2$	Tangent y = m x + n	Slope of Tangent	Slope of Tangent / x	y / n
0	0	y = 0 x + 0	0	0 / 0	undef
1	1	y = 2 x – 1	2	2 / 1	1 / -1
2	4	y=4x-4	4	4/2	4/-4
3	9	y=6x-9	6	6/3	9/-9
4	16	y=8x-16	8	8/4	16/-16

6. Compare the range values of the function and tangent function for the x values in the very close neighborhood of x = 1.



x	y1	y2
.998	.996	.996
.999	.998	.998
1.	1.	1.
1.001	1.002	1.002
1.002	1.004	1.004
1.003	1.006	1.006
1.004	1.008	1.008
1.005	1.01	1.01

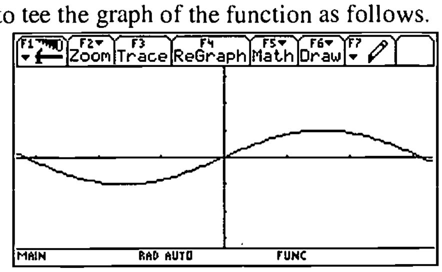
x=.998
MAIN RAD APPROX FUNC

7. Using this data table, study the following questions.
   - The tangent line approximation is $x^2 \cong 2x - 1$, for every $x = 1 \pm \Delta x$, Δx tends to zero. Why?
   - Using the above approximation we write $1 \pm \frac{\Delta x}{2} \cong \sqrt{1 \pm \Delta x}$, Δx tends to zero.
   - How can you calculate the square root of 1.00015 with the six digits after the point?
   - Calculate the square root of 3 with the error less than 0.001 using the approximation above and compare the direct result of calculator. Use the formula (error = First degree differences-slope of the tangent) to calculate the error.

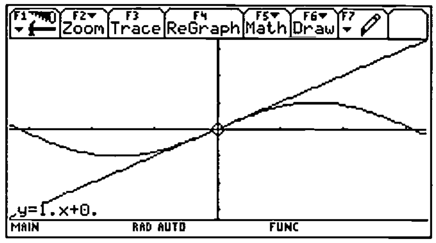
<u>**Activity 6**</u>.(Local linearity and limit process): Use TI-92 calculator and the local linearity in activity 5 to find $\lim_{x \to 0} \frac{\sin(x)}{x}$ and compare with the traditional limit process.

<u>**Guideline/Instruction:**</u> Read the following instruction carefully and do the operation without skipping any step.

1. Press Diamond+TblSet and set the initial value of x as Tblstart:-0.006, ΔTbl:0,001.
2. Press Diamond+[Y=] and define the function, $y1 = \sin(x)$ and $y2 = x$.
3. Press Diamond +GRAPH to tee the graph of the function as follows.



4. From the graph screen press [F5] and A:Tangent. Set the tangent point. Either move the cursor to the point or press $\boxed{\text{A}}$ and type its xc value as 0. Press ENTER. The tangent line is drawn, and its equation is displayed.



5. Now, calculate the range values of the function and the tangent line in the very close neighborhood of zero.

- Press Diamond+Table and see the numerical values of the functions.

x	y1	y2
-.003	-.003	-.003
-.002	-.002	-.002
-.001	-.001	-.001
0.	0.	0.
.001	.001	.001
.002	.002	.002
.003	.003	.003
.004	.004	.004

x= -.003

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- What have you perceived from the second and third column of the data table? Do the function y1 = sin(x) have local linearity near 0? For a constant change in x of 0.001, there is a nearly constant change in sin(x) of 0.001. Thus, near x = 0 the sin(x) function appear nearly linear with slope 1. So, the local liberalization of sin(x) near x=0 is $\sin(x) \approx x$. When x tends to zero, also sin(x) tends to zero. Thus, local linearity tells us that $\frac{\sin(x)}{x} \approx \frac{x}{x} = 1$.

The traditional way to find this limit is to use circle and the $\sin(x) < x < \tan(x)$ inequality. The local liberalization is more meaningful than the traditional way to find limit in functions.

4. CONCLUDING REMARKS

The available H-hPTs in the market have changed the practice of doing research in mathematics education and is profoundly changing the teaching and learning of mathematics at all levels of schools, teacher training colleges and universities. All mathematics teacher, regardless of age and experience need training on the use of H-hPTs in teaching of mathematics in almost all countries. In this process, teacher educators in the developing countries face on more problems and constraints and do not have enough funds to meet basic needs. The project on both research and training PMTs is the first experimental study in mathematics teaching in the Selcuk University, Konya and it is still going on. We presume that we have achieved our goal and work out the details by considering the following results and impressions.

The PMTs think of that the designed worksheets were very valuable for them and the H-hPTs should be used in teaching various concepts in calculus as well as other topics. Most members in the groups of the PMTs stated that their background and previous education were not suitable for to use such cognitive tools in mathematics teaching at the beginning, but it gradually changed in the end of the course. All groups constructed the concept map shown in Figure 1 and they said that they liked Activities 5 and 6 very much. The most important the PMTs' view was that the activities were more meaningful than the traditional examples thought in calculus before. Thus, it is our personal impression during the training periods that the PMTs become aware of various teaching strategies, benefits of group study, visualization and the power of TI-92/Derive, i.e. CAS calculators (Ardahan & Ersoy, 2002).

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DEVELOPING COLLEGE STUDENTS' VIEWS ON MATHEMATICAL THINKING IN A HISTORICAL APPROACH, PROBLEM-BASED CALCULUS COURSE

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Key words: history of mathematics, problem-based, views, mathematical thinking, college calculus

ABSTRACT

It has been held that heuristic training alone is not enough for developing one's mathematical thinking. One missing component is a mathematical point of view. Many educational researchers propose problem-based curricula to improve students' views of mathematical thinking. Meanwhile, scholars in different areas advocate using historical problems to attain this end. This paper reports findings regarding effects of a historical approach, problem-based curriculum to foster Taiwanese college students' views of mathematical thinking.

The present study consisted of three stages. During the initial phase, 44 engineering majors' views on mathematical thinking were tabulated by an open-ended questionnaire and follow-up interviews. Students then received an 18-week historical approach, problem-based calculus course in which mathematical concepts were problematizing to challenge their intuition-based empirical beliefs in doing mathematics. Several historical problems and handouts served to reach the goal.

Near the end of the semester, participants answered the identical questionnaire and were interviewed to pinpoint what shift their views on mathematical thinking had undergone. It was found that participants were more likely to value logical sense, creativity, and imagination in doing mathematics. Further, students were leaning toward a conservative attitude toward certainty of mathematical knowledge. Participants' focus seemingly shifted from mathematics as a product to mathematics as a process.

1. Introduction

Polya's four-phase theory sketches a blueprint for mathematical problem solving and initiates study of heuristics during the 1970s and 1980s. Contemporary studies suggest that teaching heuristics could significantly improve students' ability to employ heuristics in solving non-routine mathematical problems, yet research in this phase has long been questioned by many scholars for its limited capacity for preparing students to extrapolate the ability (Lester, 1994; Owen & Sweller, 1989; Sweller, 1990). Relevant researchers thus revisited the ultimate goals of mathematics instruction and how problem solving fits within the goals. National Council of Teachers of Mathematics (1991) defines the aims of teaching mathematics as "to help all students develop mathematical power" and "all students can learn to think mathematically" (p. 21). Learning to think mathematically means developing a mathematical point of view (Schoenfeld, 1994), a missing part in traditional training of problem solving (Schoenfeld, 1992).

On the other hand, scholars in different areas have evoked the use of historical problems in developing students' mathematical thinking (Barbin, 1996; Rickey, 1995; Siu, 1995a, 1995b; Swetz, 1995a, 1995b). The gist of this argument is that using historical problems in a classroom can benefit students in not only the affective domain but also the cognitive domain. Ernest (1998) interprets the rationale for using historical problems as indicating mathematicians in history struggled to create mathematical processes and strategies that are still valuable in learning and doing mathematics to this day.

Note that the relationship between students' views of or beliefs about doing mathematics and their learning behaviors has attracted considerable attention in recent years (Carlson, 1999; Franke & Carey, 1997; Higgins, 1997; Kloosterman and Stage, 1991; Schoenfeld, 1989). Empirical investigations suggest students who view doing mathematics as a rigid process may be more reluctant to engage in creative mathematical activities. Conversely, an active view would potentially promote an individual's desire to undertake challenging tasks (Carlson, 1999; Franke & Carey, 1997; Henningsen, & Stein, 1997; Higgins, 1997, Schoenfeld, 1989, 1992). A basic understanding of the intrinsic essence of mathematical knowledge is requisite for mathematical literacy. To reach the goal, learners need to comprehend the nature of mathematical thinking (American Association for the Advancement of Science, 1990). On the basis of empirical evidence, investigating and developing problem solvers' views of mathematical thinking are noteworthy issues to receive further attention.

2. Purpose of The Study

Though scholars in various fields have addressed the critical role that history of mathematics plays in mathematics education for years, empirical studies designed to explore the issue are rare. This research aims to investigate interrelationships between a historical approach, problem-based calculus course and Taiwanese technological college students' views of mathematical thinking, particularly regarding in what aspects and to what extent participants' views on mathematical thinking evolve during such a course.

3. Procedure

Data collection proceeded in three stages of instruction: initial, intermediate, and late. The instructor, meanwhile, was the researcher of the present study. A six-item questionnaire (developed in four stages of pilot studies) examining participants' pre-instruction views of mathematical thinking (Appendix A) was administered to 44 Taiwanese engineering-major college students and collected at the first class meeting. Students were also requested to hand in their math biography at the next class meeting, serving as auxiliary data for interpreting pre-instruction views. The questionnaire and mathematics biography were followed by several semi-structured individual interviews to validate written data and elicit more information from nine randomly selected students.

The course was scheduled generally in accordance with historical order, handouts relevant to historical knowledge assigned as supplemental materials. In class, mathematical concepts were problematizing to challenge students' intuition-based empirical beliefs in doing mathematics, comprehend the necessity of rigorizing mathematical ideas, appreciate alternative strategies for attacking identical problems. Historical problems (Appendix B), differing from ordinary exercises in nature, served as demanding tasks to motivate intrinsic thinking. All problems assigned were related to curriculum taught at the time. As answers were collected, students demonstrating elaborative thinking were invited to share their ideas on the board, followed by a whole-class discussion.

In the late instruction stage, the identical questionnaire was again conducted on all participants, followed by several one-to-one interviews validating written responses and comparing interviewees' views before and after instruction. To minimize potential bias, respondents were never informed about the purpose of study.

4. Pre-instruction Views

Data analysis began the first day of data collection. Participants' initial views on mathematical thinking were analyzed on the basis of written responses on six-item, open-ended questionnaires and transcriptions of follow-up interviews conducted with nine randomly selected interviewees. Moreover, students' past learning experiences, as told in their mathematics biographies, served as auxiliary data for interpreting initial views.

In the first item, all respondents defined mathematical thinking, aiming to profile the essence of the construct in their minds. Twenty (45%) associated mathematical thinking with ways of solving problems or deriving answers. Further, participants tended to relate solving problems to derive answers by following predetermined routes and perceived pondering on mathematics more as recalling and applying formulas. On the other hand, 12 participants (27%) referred to mathematical thinking as a process of logical thinking or reasoning; several interviewees expressing this view but confessed they had never experienced the merit.

How good a problem solver in some sense is subject to how well one copes with untried and demanding tasks. The second questionnaire item aimed at exploring how students reacted to predicaments; 15 (34%) reported that the first thing they would do is seeking outer assistance or skip it entirely. Others adopted conservative strategies to evade difficult positions by recalling formulas or similar problems, eight (18%) claiming they would think on their own before asking for help. One of the interviewees, Ming, reported he was usually persistent. When asked about his motivation, he responded:

> There is little to do with confidence. *This is what mathematics is all about* [italics added]; you have to think. ...You would feel it easy when you achieve a breakthrough in your thinking. (Ming, pre-instruction interview)

It appears that Ming demonstrated a thoughtful belief about mathematics as well as an active view on mathematical thinking, a mathematician-like disposition.

The mathematician is typically regarded as the perfect mathematical problem solver and laypersons usually conceptualize mathematicians' ways of thinking as an archetype. On the basis of this notion, participants were asked to propose how the mathematician thinks of a mathematical problem. Ten respondents (23%) considered mathematicians as generally being able to attack problems from diverse angles or apply alternative approaches. Many attributed mathematicians' ability to owning solid knowledge background, as evidenced by the following quote:

> Mathematicians' brains *must be filled with various kinds of definitions and solutions for solving problems* [italics added]. . . they are able to solve problems *by using very simple, quick and precise approaches* [italics added]. (Mong, pre-instruction questionnaire)

In contrast, four respondents cited hard thinking as critical to mathematicians' vocation. Chang plotted a vibrant mode involving activities like survey, making/testing conjecture, and verifying results, revealing the empirical aspect of mathematics.

Mathematics is typically seen as requiring creativity, yet memorization is usually viewed as the best way to learn it (Schoenfeld, 1989). It is noteworthy to scrutinize participating Taiwanese college freshmen's views on this concern. Twenty-six respondents (59%) thought problem solving in mathematics is much like a creative activity. Among them, 12 claimed that solving problems involves personal creativity because there are always various ways to do mathematics. Eleven respondents (25%) took a neutral position (both creativity and preset procedure are required for doing mathematics), and some perceived the issue as doer-dependent—creativity for experts, preset procedure for novices.

It is presumed that students must own certain impressions, adequate or inadequate, regarding mathematics after years of learning the discipline. Surprisingly, when asked to define mathematics, eight (18%) were mute on this concern. Among those responding to the item, nine (20%) associated mathematics with numbers; seven (16%) interpreted mathematics as a practical tool in daily life; five (11%) professed that mathematical results must be infallible through the ages. Contrarily, some saw mathematics from alternative windows, viewing it as fundamental to science and inextricably related to the study of reality. Moreover, participants were asked to address, at their best understanding, how mathematical knowledge developed. Thirteen (30%) considered growth of mathematics progressive and subject to human demand. On the other hand, interviewees were further asked whether mathematics could exist parallel or unrelated to human demand. They in general showed poor understanding of this issue; an appreciation of abstract thinking was seemingly lacking.

5. Post-Instruction Views

Analysis of students' post-instruction views was mainly based on written responses to post-instruction questionnaires and selected interviewees' transcriptions. Initial and late views were compared and contrasted to identify any commonality or distinction. Several short essays regarding classroom activity, written by participants, served as auxiliary data sources for interpreting professed statements.

Similarly, while responding to what mathematical thinking is, participants were more likely to associate it with the process of solving mathematical problems; 18 of them (41%) claimed mathematical thinking means figuring out a way to reach answers. Their wording, however, differed in some way. They tended to conceptualize mathematical thinking as solving problems in one's own way, multiple approaches, or peculiar ideas. In addition, participants were more likely to value logical sense in doing mathematics this time. For instance, Liu, who considered mathematical thinking merely as a route leading to answer at the outset, professed:

> Mathematical thinking could mean that attaining reasonable answers through logic of making sense and reasonable generalization. In sum, it is a process of solving problems by means of reasonable ideas and procedure. (Liu, post-instruction questionnaire)

By reasonable procedure, Liu meant evidential and meaningful facts. Several respondents also cited mathematical thinking as a way of exploring rationale of formulas and intuition alone as unreliable, suggesting justification began to loom larger in their minds.

Participants' strategies reacting to predicaments generally showed wide diversity. In addition to looking for relevant material and asking for outer assistance, 11 (compared to two at the beginning) emphasized they would try to understand a problem, identify all knowns and unknowns, then make a plan. Moreover, several participants exhibited more willingness to discuss with others, yet neither written nor oral responses manifested any significant improvement of individual persistence while doing mathematics.

During their instruction, participants witnessed several ancient mathematicians' approaches to specific problems. It is therefore noteworthy to investigate again their thought about how the mathematician thinks. Contrast of answers yielded an unchanged point of view: mathematicians are good at attacking a problem from multiple facets and diverse angles. Nonetheless, they stressed more a mathematician's imagination and creativity, less one's approach as most convenient and quickest. Shern initially proposed mathematicians tends to think by reasoning, later turned to highlight their capability of association and imagination. In interview, he took Newton and Archimedes as instances:

> Just like capability of association, many figures had discovered calculus but not specific until Newton. I consider imagination is more important is because of Archimedes. I feel he is so strange. He derived the volume of a sphere by means of lever... How did he think of it? Plus, he transferred a circle into a triangle. I feel his imagination is quite strange. (Shern, post-instruction interview)

He further labeled Archimedes' approach inaccessible when merely relying on reasoning, the cause for changing his mind. Moreover, following recognition of mathematicians' imagination, the majority of participants held that doing mathematics involves more individual creativity as opposed to following preset procedure.

An important issue in the present study is, in such a historical approach course, whether participants' epistemological belief regarding mathematics had been affected in some way. By contrasting responses, several distinctions emerged. While a majority still viewed mathematics as a fundamental subject (involving numbers, operations and logic) for exploring other disciplines, one chief difference was that no participant claimed mathematical knowledge is absolute truth. During the semester, several inaccurate mathematical conceptions in history, such as Euler's mistake on infinite series, were presented to students to demonstrate the fallible aspect of mathematical thinking. It appears students were impressed by these examples and leaning toward a conservative attitude toward certainty of mathematical knowledge. Asked about the possibility of new mathematical facts superseding old ones, no interviewees showed doubt; all defended by citing examples given in class. According to them, mathematical criteria evolve over the course of time, and validity of mathematical knowledge is constantly examined.

Calculus taught at school today is entirely credited to European mathematicians, but several concepts of integral calculus had occurred in the oriental world as well. This historical approach course also covered issues of different approaches to deriving area of a circle and volume of a sphere between ancient Chinese mathematicians (Liu Hui and Zu Chongzhi) and Archimedes. Participants were then asked to compare and contrast the different fashions between these types of mathematical thought. Most held that Chinese mathematicians tended to think in intuition, operated mathematical ideas via concrete figures, and usually demonstrated results without justification, whereas the Greek was more likely to approach a problem from unusual angles by integrating physical concepts and verify answers in a meticulous manner. In short, Chinese method is direct and intuitive rather than theory-laden; Archimedes' thinking is indirect and skillful with rigorous confirmation.

6. Summary and Discussion

The aforementioned findings suggest that, as a rule, participants initially viewed doing mathematics as a solution-oriented activity, in which mathematical thinking is degraded as fixed processes leading to final answers. Thinking of mathematics as such was interpreted as a way of recalling content; mathematicians therefore were seen as figures possessing more solid knowledge background and experiences in solving problems. The phenomenon can be explained by the mathematics biographies, revealing exam-oriented mathematics teaching in Taiwan had intensely, but distortedly, shaped recognition of mathematical thinking. Conscious reflection was lacking while engaging in mathematical activity, resulting in superficial understanding of the essence of mathematics.

After an 18-week historical approach, problem-based calculus course, students' views of mathematical thinking in particular, mathematics in general, had shifted in some ways. Though still referring to mathematical thinking as a procedure for deriving answers, post-instruction responses showed an inclination to stress the role of creativity in solving problems and necessity of involving relevant concepts of other disciplines. Such leaning, on the basis of interview transcripts, may be attributed to ancient mathematicians' imaginative approaches learned in class, demonstrating a wide range of possibilities in attacking a problem. Meanwhile, after exposure to historical mistakes, they were less likely to believe mathematical knowledge is time-independent truth and more likely to value necessity of justification. Participants' focus seemingly shifted from mathematics as a product to mathematics as a process.

Despite these above inspiring outcomes, several emergent issues merit further attention. Firstly, many participants showed more eagerness to try, whereas individual persistence in thinking on mathematics did not significantly improve. Strategy most often adopted by them was discussing with others, mostly because of the difficulty of assigned problems. Selecting moderate tasks from history, challenging but accessible, thus is a critical factor in success of a study of this type. Secondly, most participants were impressed by Archimedes' fashion of thinking, but his ideas were not viewed as applicable by most interviewees. In their minds, a good method ought to be simple, precise, and intuitive. Asked to compare Zu Chongzhi's and Archimedes' approaches to deriving volume of a sphere, eight interviewees preferred Zu's thinking; Archimedes' peculiar thought was more like models in the shop window, drawing gaze but not approach. In some sense, one chief purpose of the effort made in the present research is to foster students' appreciation of ingenuity and beauty of mathematical thinking. This finding nevertheless reveals a restriction of this study. Is this an educational challenge or a cultural issue? Cross-cultural study may help to resolve these doubts. Thirdly, several respondents showing not much difference in their professed views not only expressed a rigid view about the concerns, but also demonstrated conservative performance on the challenging tasks. They tended to approach problems in a fixed and traditional fashion. The interplay between an individual's pre- and post-instruction views and degree of consistency between one's views and behavior make noteworthy issues for further study.

Integrating history into mathematics curricula has been promulgated for decades, yet cannot be accepted without question. The present study is not an experimental design, so no cause-effect inference can be made; this is an exploratory attempt laying groundwork for further research in this respect. Fried (2001) raises several critical concerns regarding possibility of combining mathematics education and history. Best strategy for revealing the doubt is probing what history can and cannot do for mathematics education through empirical investigations. History of mathematics is by no means the prescription of mathematics education, but definitely can be a guide to it.

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Appendix A (open-ended questionnaire)

1. In your understanding, what is mathematical thinking? Please explain your answer with examples.
2. When you are stuck on an unfamiliar mathematics problem, what is your instant reaction to and strategy for this?
3. In your understanding and imagination, how do mathematicians think while solving a problem? Is there any difference between a mathematician's way of thinking and a layperson's?
4. Some hold that solving mathematical problems is a thinking activity involving personal creativity; others argue that getting correct answers requires following predetermined, known procedures. What is your opinion about this? Why? Please defend your answer with examples.
5. In your opinion, what is mathematics? What makes mathematics differ from other disciplines?
6. In your opinion, how does mathematical knowledge develop? Does the development of mathematical knowledge follow any rule? Please defend your answer with examples.

Appendix B Historical problems

1. Finding the area of a circle (Archimedes, Liu Hui, Seki Kowa)
2. The method for finding the area of a circle on Rhind Papyrus
3. Archimedes' quadrature of the parabola
4. Fibonacci sequence
5. Computing the sum of $1-1+1-1+1-1+\ldots$
6. Approaches to finding the tangent line to a curve (Descartes, Fermat, and Barrow)
7. Napier's logarithm
8. Fermat's approach to find extreme values
9. The curve of witch of Agnesi
10. The Tractrix problem
11. Finding the volume of a sphere
12. Finding the volume of a sphere inscribed in a cylinder

INFORMATION – ACTIVITIES POINT OF VIEW AS THE POSSIBLE BASIS OF HIGHER MATHEMATICS

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ABSTRACT

The course of higher mathematics in the higher educational establishment must aim at: 1) the formation of mathematical knowledge and skills to apply it; 2) the formation of methodology for solving applied problems.

The first and the second functions can be carried out in general on the basis of certain point of view and the leading role of the development of student's mentality. For the effective teaching of higher mathematics we must take into consideration the double nature of mentality, that may be formed as search of values and personal sense of life, and as information process, which is determined such as the perception, keeping and remarking of information. In the abstract meaning the remarking of information by all means is regulation and compression of it to determinate aim or the given problem, by definite language or code. For professional activity this is the selection of information from the "noise", lowering the entropy according with the interests, orientations and possibilities of the specialist's personality.

The factor of integration of the variety of different given training of methods and application for development of thinking of the future specialist concerning its double nature, can be information - activity point of view to professional training and development of the student's personality.

Its main idea is the organization of the educational- professional activity, in which the compression of the training information takes place by regulating it, by imitation of professional activity according to the subject, the nature of motives, knowledge and actions of their application.

The course of higher mathematics in the higher educational establishment must be aimed at these functions: 1) the formation of mathematical knowledge and skills to apply it; 2) the formation of methodology to solve of the applied problems.

The first and the second functions can be carried out in general with the third function on the basis of certain point of view and the leading role belongs to the function of the development of students' mentality, which we have been examining in the course of higher mathematics in more detailed way.

These aims were abovementioned from the general problems of education and the tendency of its were development. In particular, the specialists note that the majority of students are not clever enough to solve creative problems. The solution of these problems requires *quick orientation* [1, 2] in the given information and the problem situations, *organization* of the problem (defining what information can be take from the problem situation, what is required to describe it, what information and methods can be used for the solution), *planning* of solution of the problem (the definition of lacking information (self-education) for solution, the selection of the mathematical methods of solution and definition of its algorithm and also mathematical verification of it), *realization* of the algorithm of solution with the help of computer, *the control* of the process and results the new information (or results of solution), *interpretation of the obtaining result* in the language of problem situation (the decoding of the new information).

Thus, in the course of mathematics, the students must learn the basis methods of solution the problems, the their algorithms, mathematical verification of these algorithms and to use them creatively.

The knowledge of the methods of solution the problems may be defined as the methodology of solving the applied problems. For example, the subject of the calculus is the approach to the solution of the applied problems, consisting of two positions:

1. All real processes or objects "in the infinitesimal" can be take proportionally and can be classified with the help of the prime models or linear dependence.
2. The simulation of real processes or objects can be extended by means of limited transition from the "infinitesimal" to the real size.

Thanks to this approach for solving the applied problems the differential and integral calculus "were born" as well as, the operations of the differentiation and integration, the notions of derivative and integral, the methods of differentiation and integration and the mathematical verification. The students, learning the calculus as a methodology for solving the problems, receive the concept about the possible situations of application, mathematics and understanding the existence of the other points of view to the solution of the applied problems (for example, the theory of catastrophe, which studies the leap transitions in the real processes, unlike the calculus), while learning they can discover and mathematically verification their own method of solution.

The first and the second functions can be carried out in general on the basis of certain point of view and the leading role of the development of students' mentality [3], because for the creative use of different mathematical methods and their algorithms the progress of the memory [4] is needed as well as, the attention, the rate of thinking and other qualities, without which the effectivity of orientation in the information and its application for organizing and solving given problems is lowering.

How must mentality be developed? To answer this question we must take into consideration the double nature of mentality that may be examined as search of values and personal sense of life [5], and as informational process [6], which is determined as the perception, keeping and remaking of information. In the abstract meaning the remaking of information by all means is regulation and compression of it according to definite aim or the given problem, by definite language or code. For professional activity this is the selection of information from the "noise", lowering the entropy according with the interests, orientations and possibilities of the specialist's personality. It is the essence of the intellectual progress, which is limited by the back of this level of intellectual development of the students, or the discrepancy of their intellectual level of development to the demanded level for doing an educational-professional activity according to the solution of the professional problem. The new organization of mentality is formed by special process or prolonged alternation of special influences. Every stage of any process may be characterized by the new information structure of mentality, which is reflected in its qualities, because the essence of the mentality progress is the purposeful accumulation of the learning information and simultaneous regulating and structuring it according to educational aims of the professional training and the future professional activity. The results of the psychological research confirms that the process of learning in the determine conditions forms the intellectual structure, which correspond to successful cognitive and professional activity. On the other hand, the learning-professional activity presents the all the conditions for intellectual progress of future specialists. Therefore, for the forming the professional activity, the application of the knowledge of mathematics is it necessary and sufficiently to organize intellectual progress on the basis of professional direction of learning information.

Thus, the intellectual progress or the professional direction can not be take separately as they both promote to the becoming of professional activity.

Obviously, that manifestation and development of all the qualities of also thinking take place in the mode of life, in all the training, but in the special centripetal organization of the training is the most intensive and effective. The perspectives of high professional education can be seen in the individual development and discovering of the functional qualities and possibility of brain for remaking of information by means of pedagogic and realization of its possibilities.

The factor of integration of the variety of different given training methods and application for development of thinking of the future specialist, concerning its double nature, can be information – activity point of view to professional training and development of student's personality. Its main idea is the organization of the educational - professional activity, in which the compression of the training information takes place by regulating it, by imitation of the professional activity according to the subject, the nature of motives, knowledge and actions of their application.

Information-activity point of view is the unity of the information point of view, worked out in the theories of information (N. Viner, K. Shennon, S. Goldman and etc) and by psychologists (S. L. Rubenshtain, A. N. Leontiev and etc). The activity access, carried out in psychology for the formation of given characteristics presupposes to include the subject to the definite activity. The fact is that the description of activity point of view doesn't define the mechanism of students' thinking, which can be revealed and realized on the basis of information point of view.

Information point of view is the applicable means of research, which has the specific sense in the teaching of mathematics that determines the ordinary informing of the students. Such learning doesn't ensure the necessary accuracy of knowledge.

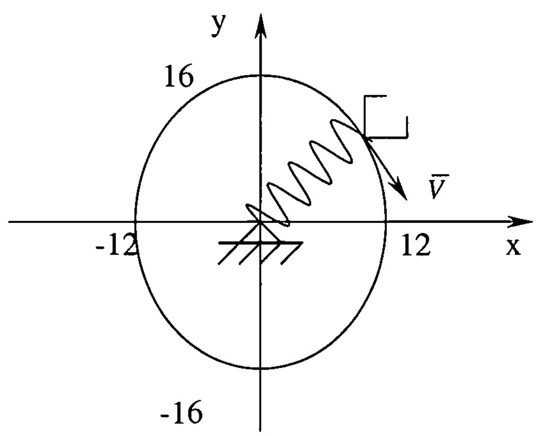
For example, the physical sense of the derivative of function often is identifying with as a derivative itself: the derivative = the speed of moving. Such distortion of the understanding of the derivative of function doesn't allow applying it in the solution of problems; the progress of the mentality in this case doesn't take place.

The progress (or its precision) of knowledge and the mentality of the students may take place by engaging the students into a special activity. This may be organized by solving the problem: Let's use the simple example avoiding the notion vector-function.

Define the modulus and the direction of the speed of the movement of the arm of the manipulator, if the trajectory of its movement is defined with the parametric function:

$$\begin{cases} x = 12\cos t, \\ y = 16\sin t. \end{cases} \quad \text{t – time}$$

Solution:



In this case the given trajectory of the movement in the variety is the ellipse:

$$\frac{x^2}{144} + \frac{y^2}{256} = 1.$$

The derivative of this implicit function y'_x is the speed of the change of the function (dependent variable) y, concerning the argument (independent variable) x, but it isn't the speed of movement, because the speed of movement is the function of time (this is the change of the curve in the unit of the time). The derivative of the parametric function of coordinates of the arm of manipulator as the function of time defines namely the module of speed of the arm manipulator movement:

$$|\bar{v}| = \sqrt{(x')^2 + (y')^2} = \sqrt{\left[(12\cos t)'\right]^2 + \left[(16\sin t)'\right]^2} = \sqrt{144\sin^2 t + 256\cos^2 t} =$$
$$= \sqrt{144\sin^2 t + 256(1 - \sin^2 t)} = \sqrt{144\sin^2 t + 256 - 256\sin^2 t} = \sqrt{256 - 112\sin^2 t}.$$

The direction of speed may be dinited by the slope of the tangent to the trajectory:

$$\frac{x^2}{144}+\frac{y^2}{256}=1:\ y=\pm 16\sqrt{1-\frac{x^2}{144}},$$

$$y'=\pm 16\cdot\frac{\frac{2x}{144}}{2\sqrt{1-\frac{x^2}{144}}}=\pm\frac{x}{9\sqrt{1-\frac{x^2}{144}}}$$ - this is the slope of the tangent to the trajectory (direction of speed) as function of coordinates x.

The solution of the similar problem depends, specifies and *orders* [7] the receiving information in accordance with the aim of the learning-professional activity during the course of study. When, as is noted, the ordering of knowledge is the new structure of the mentality, it means the transition from one stage to another that is the progress of the students' mentality.

Thus, the information point of view as the possible basis of higher mathematics is aimed at preparing of the future professional activity.

That is why one should examine the information – activity point of view in the professional training, in particular as the possible basis of higher mathematics.

This point of views may be determined by some ideas of thermodynamics [8]. The primitive informing of the students is as «Brownian» or «chaotic» component of mentality (aimless accumulation of the information). The development of mentality is as vectorial or directed component (*ordering* of information or lowering of its entropy in accordance with aims of the professional training).

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"THALIS"

A REPRESENTATION SYSTEM FOR UTILIZATION IN TEACHING AND LEARNING FRACTIONS

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ABSTRACT

The purpose of this paper is to present an attempt to promote the development of teachers' own intuitive mathematical knowledge of rational numbers (1) with the use of a new teaching tool which is named "Thalis". For the construction of this tool consideration has been taken in children's difficulties to produce adequate intuitive models to represent rational numbers and operations with them.

Currently, to teach concepts of rational numbers, traditional representation systems are utilized; some of them are not self-consistent, since they are capable of producing contradictory situations, whereas, there are others, self-consistent but over-specific (2) since they are capable of producing multiple representations of a problem's solution.

This paper recommends a new representation system for the teaching of rational numbers which has the form of a natural transformer. This system does not allow for any misconceptions since, as it will be discussed, it is a model of the field of rational numbers. Moreover, since it is a natural transformer, it permits authentic measurement activities and ratio computations in school contexts.With this new system an improvement is expected in:

1. Children's ability to experiment
2. Teachers' ability to plan constructivistic activities for the teaching of rational numbers

The paper presentation structure for Thalis will be the following:

1. Representation systems of rational numbers
2. Informal presentation of Thalis
3. Examples of Thalis use in representing some operations of rational numbers
4. Discussion about Thalis being a model of the field of rational numbers.
5. A teaching script with the use of Thalis

1

A central part of the Curricula (ages 9-13) of elementary schools in most countries focuses on the teaching of fractions. However, didactical research indicate that a significant number of students have serious difficulties in understanding the concepts of fractions and techniques of operations with fractions.

The most important proposed interpretations for these difficulties are[1]

- Rational numbers are used less than natural numbers.
- Many children find it difficult to accept a given fraction as a number and tend to view it as two whole numbers.
- Students often incorrectly attribute observed properties of operations with natural numbers to those with rational numbers.
- The many different interpretations of, and notations for, rational numbers.

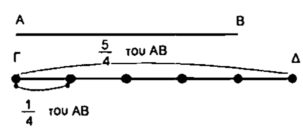
Also[5],

- The over-development of the instrumental versus the conceptual knowledge.
- The non-use of the students' intuitive knowledge.

On the other hand, we observe that textbooks in order to facilitate teaching of rational number concepts and operations, use a rich variety of representations with different roles. Some representations are used for a better visualization of the part-whole interpretation of the fraction, some others for a better visualization of addition and some for better visualization of multiplication, etc.

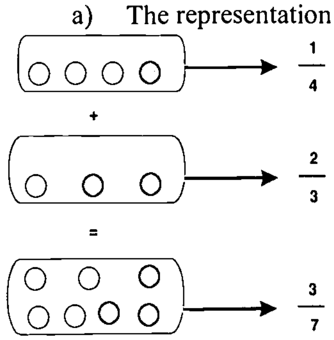
Our hypothesis is that many of the difficulties students have in understanding fractions are related to the nature and the consistency of the textbooks' representations

In Greek textbooks, most common representation systems contain:

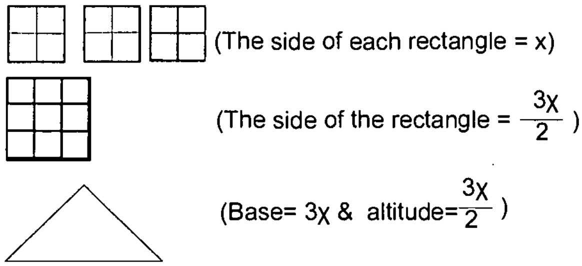
1. Representations with discrete objects E.g. 
2. Representations with two or three dimensional figures 
3. Representations with straight lines. 

These representation systems are either non self-consistent, since they are capable of producing contradictory situations, or self-consistent but over-specific since they are capable of producing multiple representations of a problem's solution. More specifically:

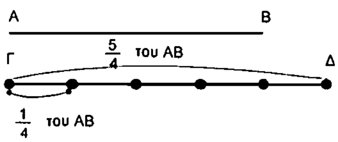
a) The representation system, which uses discrete objects, is not self-consistent since it leads to paradoxes. For instance, in order to compare $\frac{1}{4}+\frac{2}{3},\frac{1}{2}$ it is possible, by using this representation system, to elaborate the following "proof" which leads us to the paradox $\frac{1}{4}+\frac{2}{3}=\frac{3}{7}<\frac{1}{2}$



b) The representation system, which uses two or three-dimensional figures, is over-specific since it is capable of producing multiple representations of a problem's solution.



For example on the left hand figure there are some of the possible representations of the problem's "Find $\frac{9}{4}$ of the rectangle of side x" solution.



c) The left hand figure is a common representation system which shows the result $\frac{5}{4}$ *of* AB but it does not explain how to find ¼ of AB.

d) Textbooks indications are guiding students to construct only subdivisions of 2 of any manipulating aids. For example, textbooks show ways of finding the ¼ but they do not show ways of finding the 1/7 of a paper sheet.

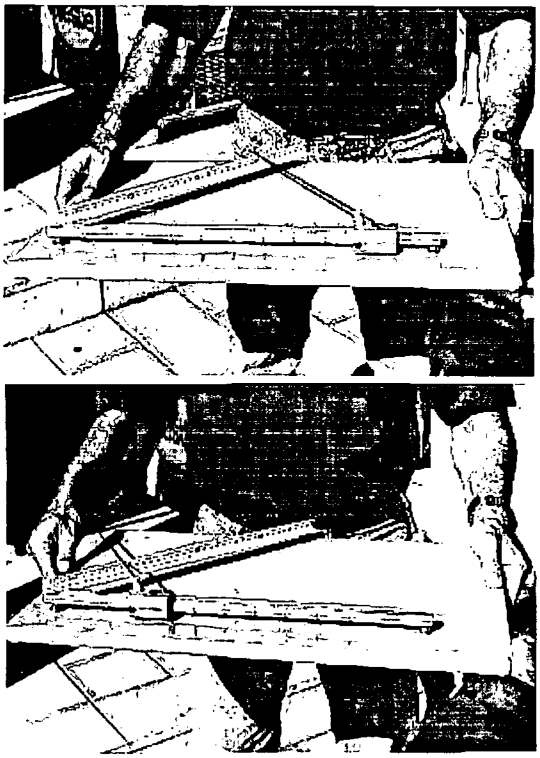
These kinds of problems, in the use of the above representation systems pose, in our opinion, a serious issue that concerns the quality of the school textbooks. It is a direct reason for the difficulties faced by the students when comprehending fractions.

2

Based on the above analysis we propose an alternative representation system named "THALIS" which has the form of a natural transformer and consists of:

1. A wooden board
2. A numbered axis (a) in the bottom of the board
3. A small wagon which can move along the axis (a)
4. A rotating needle placed on the wagon
5. A numbered axis (b) which forms an acute angle with axis (a) and has the same origin O as axis (a)

This mechanism has a function, which transforms lengths in the following way. A line segment OA is drawn on axis (b). In order to compute $\frac{p}{q}$ of OA, the wagon is placed on axis split q, the needle rotates as to indicate A, and the wagon moves to axis split p. Then the needle indicates a certain point B on axis (b). OB is the requested segment



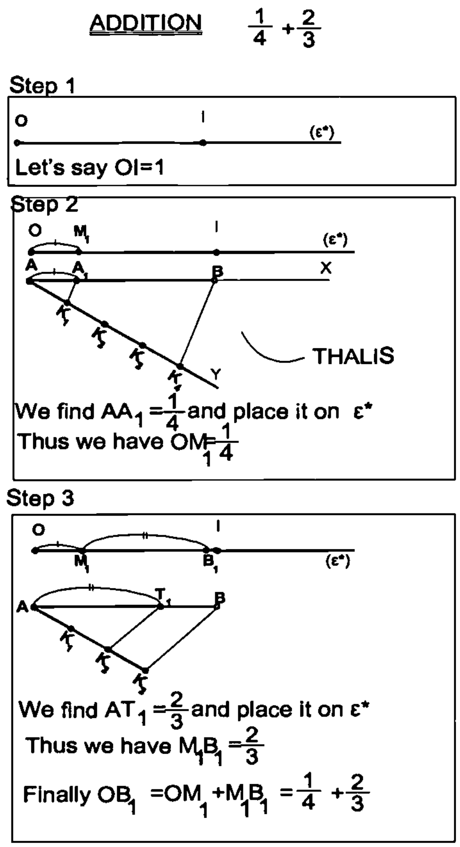
E.g.: In order to find the 3/10 of a line segment with a length of 30 cm, the wagon is placed on 10; the needle rotates until it indicates the end of the segment: i.e. 30.Then the wagon moves to 3.Now the needle shows the 3/10 of the line segment, which is a 9 cm segment.

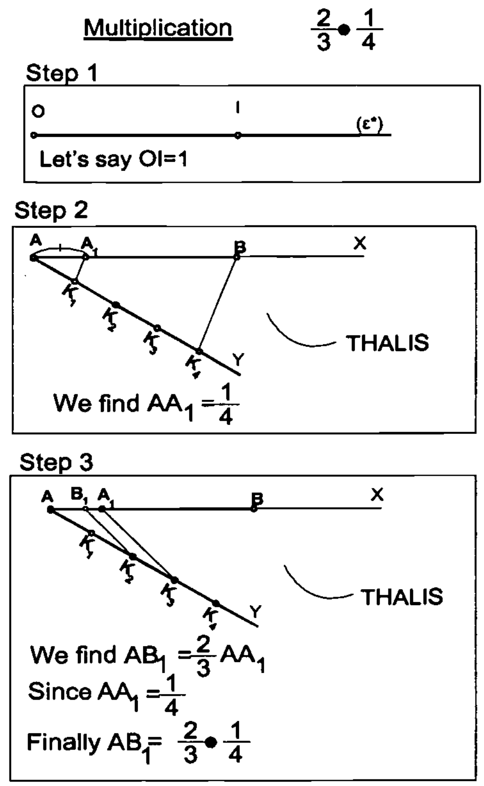
THALIS succeeds on the following:

1. It visualizes fractions as well as all of their properties and operations.
2. It is self-consistent since it is not capable of producing contradictory situations.
3. It is not over-specific since it does not produce multiple representations of a problem's solution.
4. It can easily create any subdivision of the form a/b of the usual manipulating aids (e.g.: you can easily find the 1/7 of a paper sheet)

3

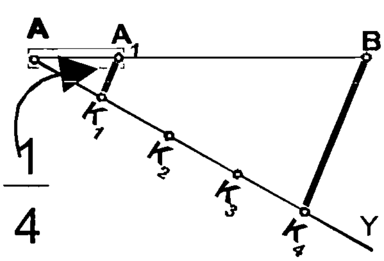
Below are some examples of the use of THALIS

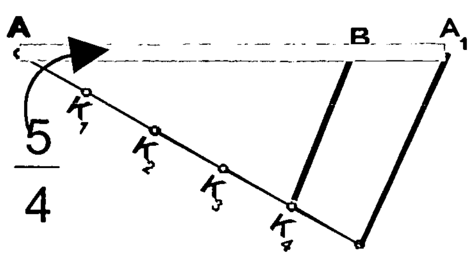




4

In order to support the scientific soundness of THALIS, we must prove that it is a mathematical model of the field of rational numbers. This way, the structures of THALIS and the field of rational numbers match completely. For that purpose, we follow the steps below. First, we find the set K of segments whose length is a rational number. The following classic method of finding the segments of length $\frac{a}{4}, a=1,2,3,4,5,...$ can also be applied for finding segments of length $\frac{a}{2},\frac{a}{3},\frac{a}{5},...,a=1,2,3,4,...$





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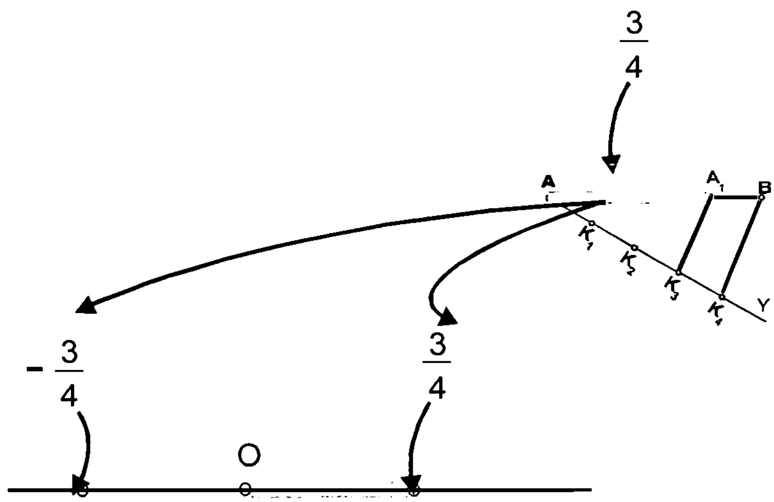
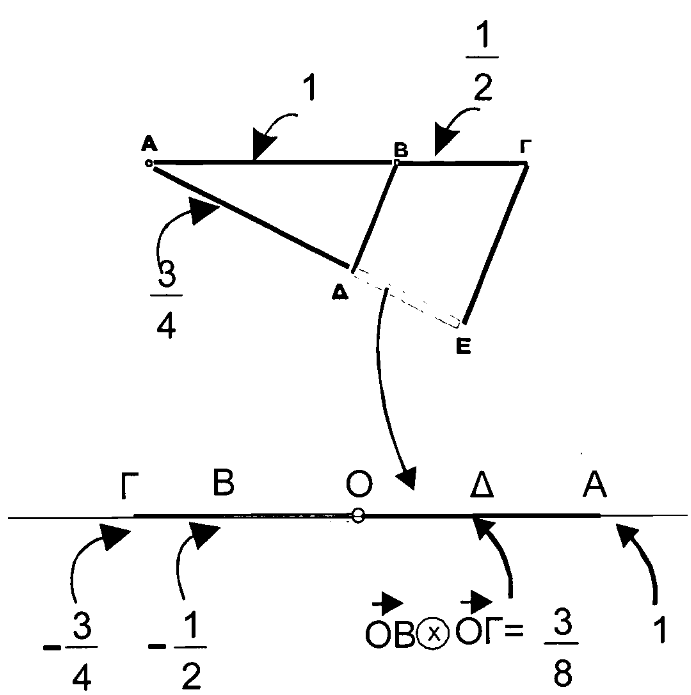
Each of the above segments creates two opposite vectors on an axis. For example, the figure below shows how the segment of length 3/4 creates two opposite vectors.



We name F, the set of all vectors of the axis that have been created in the above way. Thus, the F set contains all the vectors of an axis whose length is a rational number.

We supply the set F with the usual vector addition $\oplus$. Also, if $\vec{e}$ is the unit vector and $\vec{a}, \vec{\beta}$ are any two vectors of the F set, we define their product $\vec{a} \otimes \vec{\beta}$ as the vector $\vec{\gamma}$ with has positive direction if $\vec{a} \uparrow\uparrow \vec{\beta}$, and it has negative direction if $\vec{a} \uparrow\downarrow \vec{\beta}$ and length for which the proportion $\dfrac{|\vec{\gamma}|}{|\vec{\beta}|} = \dfrac{|\vec{a}|}{|\vec{e}|}$ is valid.



The left hand figure shows how to find the product of the vectors $\vec{OB}, \vec{O\Gamma}$ which have the same direction and $|\vec{OB}| = \frac{1}{2}$ & $|\vec{O\Gamma}| = \frac{3}{4}$

If we consider the language of the fields $L = \{+,-,\cdot,0,1\}$ and the L-structures Δ_1,Δ_2 with domain the set of rationals Q and the set F, correspondingly, then, we can easily find that all symbols of L keep their interpretation on the structures Δ_1,Δ_2. That means that there is an isomorphism $\omega : \Delta_1 \rightarrow \Delta_2$. Since the structures are isomorphic, therefore Δ_1 constitutes a model of Δ_2, which is a model of the field of rational numbers. Δ_2 is the basis of THALIS; therefore THALIS constitutes a model of the rational numbers field.

5

We propose to use THALIS to construct a composium of classroom activities aimed at promoting the students' construction of rational numbers as mathematical objects, and their construction of transformations and operations upon those objects. Put another way, the objectives of this collection are to have students first construct transformations as **"things to act with"(LEVEL 1)** and then to reconstruct them as **"things to act on"(LEVEL 2)**.[4]

More specifically:

LEVEL 1

- There is a discussion which concerns the finding of e.g.: $\frac{1}{2},\frac{1}{3},\frac{1}{4}$ of several quantities
- THALIS is introduced along with the instructions for its use
- Students experiment
- Students will be asked to think of fractions in a new way; not as a part of a whole, but as a transformation of a quantity into another.[4]

LEVEL 2

- In order to clarify that each rational number can be represented in infinite ways, the following problem may be given: "Find the $\frac{2}{3},\frac{4}{6},\frac{6}{9}$ of a thread. What do you observe?"
- Compare $\frac{1}{2},\frac{1}{3}$
- Find the products $\frac{15}{2}\cdot\frac{2}{3},\frac{2}{3}\cdot\frac{15}{2}$. What do you observe?
- Find the sum $\frac{15}{2}+\frac{2}{3}+\frac{5}{6}$

Before using THALIS in the school context, teachers must become familiarized with it, in order to comprehend the specific problems that the traditional representation systems had in the past, and subsequently feel comfortable and secure with its use.

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FOUR CRITICAL ISSUES OF APPLYING EDUCATIONAL TECHNOLOGY STANDARDS TO PROFESSIONAL DEVELOPMENT OF MATHEMATICS TEACHERS

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ABSTRACT

To have students adequately prepared for adult citizenship, computer-based technology is to be routinely used at schools and universities. To achieve this end, new approaches to teacher education are to be developed and utilized, which should be based on some suitable educational technology (ET) standards. As computers are, in general, rarely used in mathematics classrooms, such an ET-based approach, enabling these standards to be eventually widely applied, requires several issues to be kept in mind and dealt with in an adequate way. The most important among these issues are probably the following four dealing with basic indicators of ET standards, computer attitudes, software selection and a proper utilization direction, and Web-based professional development of mathematics teachers. This paper examines these four issues, offering practical solutions that may be used in the design and utilization of an ET-based professional development for mathematics teachers.

Introduction

To have students adequately prepared for adult citizenship, computer-based technology is to be routinely used at schools and universities (Pelton & Pelton, 1998). To achieve this end, new approaches to teacher education are to be developed and utilized, which should be based on some suitable educational technology standards, like those developed by *International Society for Technology in Education* (http://cnets.iste.org).

The current edition of the ISTE National Educational Technology Standards for Teachers comprises 23 indicators divided into six broad categories. They are: technology operations and concepts (2); planning and designing learning environments and experiences (5); teaching, learning, and curriculum (4); assessment and evaluation (3); productivity and professional practice (4); and social, ethical, legal, and human issues (5). These standards are connected with the ISTE Technology Foundation Standards for Students comprising 14 indicators, which are organised into the following six categories: basic operations and concepts (2); social, ethical, and human issues (3); technology productivity tools (2); technology communications tools (2); technology research tools (3); and technology problem-solving and decision-making tools (2).

Computers are, in general, rarely used in mathematics classrooms (see, for example, Manoucherhri, 1999). To have these standards eventually widely applied in mathematics education, an ET-based approach to professional development of mathematics teachers may primarily require us to keep in mind and adequately deal with the following four issues.

1. Many teachers, especially those less-experienced and not so technology-minded, may find 37 indicators of the ISTE standards quite demanding. A solution may be to base teaching practice just upon several basic indicators, still bearing in mind the broader context. What, then, may such indicators be?
2. It has been realized that computer attitudes influence not only the acceptance of computers, but also their use as professional tools or teaching/learning aids. To have computers widely used in mathematics classrooms, we should help teachers develop positive attitudes toward computers. What may a promising way to achieve this be?
3. What is the most appropriate software for the teaching/learning of mathematics? Secondary teachers may primarily base their classroom activities on a computer algebra system and a dynamic geometry environment. What should a proper utilization direction of these or other able programs be?
4. Being aware of rapid developments in educational technology, how to achieve and maintain a critical, balanced and well-designed use of computers in mathematics education? Is Web-based professional development of mathematics teachers an adequate solution? What can be achieved by its use?

The next section deals with these four issues in more detail, providing concrete answers that may be used in the design and utilization of an ET-based professional development of mathematics teachers.

Four Issues

Basic indicators of ET standards

As a part of the course *Didactics of Computer Science*, which the author has taught at the Mathematical Faculty of the Belgrade University (http://www.matf.bg.ac.yu) since the academic

2000/2001 year, future secondary school teachers of mathematics and computer science[1] are first introduced to the ISTE standards and their indicators and then asked to choose some of the indicators (up to 10) as their basic teaching directives. Mostly organized into groups of 3-4, the students work for some 45-60 minutes, after which a student from each group presents the chosen indicators. A brief summary of the students' proposals for the two academic years is given in Table 1.

Even though the list is short, this summary may be viewed as a good "iteration" towards a 10-indicator list. As an exercise, the reader may try to compile/make his/her own list of basic indicators. This exercise is particularly beneficial to those involved in pre-service and in-service professional development of mathematics teachers, especially when it focuses on issues that are subject to change. We find three reasons for such a claim. Firstly, it gives some personal meaning to the examined official proposals, the underlying reasons and assumed values of which are rarely fully explicated and therefore are not accessible to a wider public of teaching practitioners.[2] Secondly, this exercise increases the students'/teachers' motivation to reflect on their (future) profession and to apply such digested recommendations. Thirdly and finally, the exercise evidences that, contrary to typical mathematics lessons "one question - one answer", educational questions do not have unique solutions and frequently raise new questions. Thus, instead of obtaining final answers, the exercisers are becoming increasingly aware of the complexity of computer-based educational practice.

2000/2001 5 groups, 18 students listed are indicators chosen by at least three groups	2001/2002 9 groups, 33 students listed are indicators chosen by at least five groups
• Have good knowledge and skills and update them. • Use technology to increase productivity and solve problems. • Consider students' diverse backgrounds, characteristics and abilities. • Use technology to foster communication among all participants in the educational practice. • Use technology for assessment. • Develop positive attitudes toward computers.	• Stay in touch with the development of educational technology. • Use technology to foster logical thinking and creativity. • Use technology to affirm diversity. • Use technology to communicate with other colleagues, students and their parents.

Table 1. Students' proposals for basic ET indicators

Computer attitudes

As has already been underlined, computer attitudes influence both the acceptance of computers and their use as professional tools or teaching/learning assistants (see, for example, Woodrow, 1991). Computers will, therefore, be widely used in mathematics classrooms when teachers develop positive attitudes toward them, which can be achieved, at least to some extent. Having in mind that many studies have demonstrated that computer experience has a positive effect on computer attitude (see, for example, Kadijevich, 2000), positive attitudes would be developed

[1] A two-subject study group (mathematics & computer science)

[2] Consider the following issues: "Viewing curriculum reform as a technical rather than a moral and ethical process causes reformers to neglect not just basic questions but also the people who should be involved in answering them. Teachers, for example, may not be especially able to confront value dilemmas. They can be as stupid and short-sighted as anyone else. Their involvement is nonetheless essential." (Stanic & Kilpatrick, 1992, p. 415)

through proper computer activities. The author's experience with a group of first year students of geo-economics[3] suggests that extensive experiences with an able general-purpose environment such as *Microsoft Office*[4]— which coupled with *Microsoft Internet Explorer* helps teachers maintain various day-to-day activities like lesson preparation, students' administration, assessment preparation, report realization, e-mail communication, Web-site examination, etc. — may be an optimal solution to promoting positive computer attitudes. Of course, it may also be an optimal solution for teachers of other subjects, but multitasking with those *Microsoft* programs usually requires some degree of algorithmic thinking that is, because of their formal education, usually more familiar to mathematics teachers than to teachers of other subjects. Those who doubt that such a thinking is needed, since programming is not required here, may consider the author's ET indicator

> Promote/exercise thinking in terms of: (a) input and output data, (b) data that should/could be stored and queries that can be asked, and (c) modules the problem situation may be divided into

having in mind a work with *Microsoft* programs involving some text-processing, an Internet search, a spread-sheet handling and a database management (the purpose of which is producing a Web or slides-based presentation, for example).

Software selection and a proper utilization direction

Despite the fact that a mass of computer-based environments are available now at the educational market, it seems that less than 10 percent of this total may be given an "A grade" for quality (Neill & Neill, 1993). This figure may not be so discouraging as regards software for mathematics education, but it does raise the question of most appropriate software for the teaching/learning of mathematics. Although this question can be answered in many ways favouring various learning environments (especially in primary and middle grades), the author's experience as a mathematics teacher at a Gymnasium (a high school) suggests that secondary teachers may primarily base their classroom activities on a computer algebra system (CAS) and a dynamic geometry environment (DGE). Having in mind software cost, the availability and suitability of the accompanying literature on classroom activities as well as research findings, a good choice may be to use *DERIVE* and *CABRI Geometry* - two able products of the *Texas Instruments* company, whose demo versions can be downloaded from the TI Web-site (http://education.ti.com/parent/product/csw.html). It is true that one may question the educational value of CASs and DGEs because of some CASs' conceptual and procedural shortcomings (Kadijevich, 2002) as well as the fact that DGEs' drag-mode changes the traditional status of points and lines requiring new styles of reasoning (Hölzl, 1996), but their use does enable us to create and exploit learning environments that are more meaningful and thought-provoking than traditional ones. Note that a CAS such as *Scientific Notebook* produced by *MacKichan Software* (http://www.mackichan.com/) may be a suitable solution for those wishing to apply technology in the assessment process.

Other teachers and researchers may propose other able learning/teaching environments. But, regardless of which able learning environment is being used, students should not only improve their procedural and conceptual mathematical knowledge but also establish links between the two.

[3] By using a sample of 8 students whose computer attitudes were assessed by Selwyn's (1997) computer attitude scale translated into the Serbian language, it was found that almost within a month, after five 90-minute sessions with MS Office's programs Word, Excel and Power Point, the subjects' computer attitudes increased from 75.6 to 82.5 points (out of 105 points), which was a significant improvement ($t = 3.26, p = .014$; the Wilcoxon test: $Z = 2.52$, $p < .05$). The alpha reliability of the applied measure was acceptable (.84 before the treatment and .85 after it). Details of this pilot study can be found in Kadijevich (2002a).

[4] See http://microsoft.com/uk/education/, for example.

These links have rarely been studied and accomplished so far despite their high educational relevance (Kadijevich & Haapasalo, 2001). This seems to be quite a challenging aim in case of CAS or DGE.

Web-based professional development of mathematics teachers

Being aware of progress in educational technology, we find that a critical, balanced and well-designed use of computers in mathematics education requires a Web-based professional development for mathematics teachers to be utilized along with the traditional one. This claim is based upon the outcome of a recent project regarding such a Web-based development. This project was aimed to promote the *NCTM Professional Standards for Teaching Mathematics* (http://www.nctm.org/standards/), including but not focusing on technology. The project evidenced the following benefit to teachers: "consistent opportunities for reflection and sharing; a shortened cycle for training, implementation and evaluation; and teacher empowerment through direct access to information" (Shotsberger, 1999; p. 49). It is therefore important that mathematical faculties and professional organizations of mathematics teachers *also* support this form of professional development and maintain some appropriate Web sites focusing on technology-based mathematics education. These sites – the content of which may elaborate on the reported project (http://instruct.cms.uncwil.edu/) promoting the above-mentioned ISTE and NCTM standards – should, among others, critically inform their visitors of some programs, their usage and suitable classroom activities utilizing them. The usage of each program should be explained in form of a tutorial (see those placed at http://www.bcschools.net/staff/home.html or http://www.fgcu.edu/support/office2000/), which, within a few hours, enables a productive and successful practical work provoking further own experiences.

CODA

It seems that, even when computers are available, mathematics teachers rarely use them in their educational practice because they do not have (enough) knowledge and skills related to what and how can be achieved by using these tools (Manoucherhri, 1999). To change the present practice, we need to innovate, promptly yet thoughtfully, *both* pre-service and in-service professional development for mathematics teachers taking into account the four issues discussed above. In doing so, we should not forget that one's learning results from a complex interplay among his/her cognitive, metacognitive and affective domains (see, for example, Schoenfeld, 1985), the last of which, based upon mathematics and computer attitudes, determines the global context where cognition (say ET-based mathematics teaching/learning) takes place monitored and controlled by metacognition (say ET and other learning/teaching standards).

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DIDACTICAL CLASSIFICATION OF PROBABILITY PROBLEMS LINKED WITH THEIR FORMULATION

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ABSTRACT

Our research aims at the study of the relationships between problems involved in the teaching of mathematics at school and those encountered both during the historical development of mathematics and in educational research.

We study the common elements and features of the formulation of these problems as well as the way they influence each other, aiming at the improvement of mathematical problems used in classroom contexts. This improvement is related to their content and the way they are presented, so that they will have epistemological account and be related as well as be improved by research results. The interrelations that will be presented focus on Probabilistic Problems and their teaching to 5-11 year - old children.

Before coming to these interrelations, we initially collected problems, which were found: 1) in governmental school books different for each level as well as in several published books, 2) in published articles and conference proceedings (problems which have been used in researches), 3) in history, philosophy and epistemology books.

Moreover, after registering the features of their formulation, we created categories and sub-categories, in which each problem was incorporated. Resulting data were statistically analyzed by Factor Analysis methods in order to classify the problems and obtain the appropriate taxonomy. Research results and conclusions will constitute educational material for teacher training because, although Probability Theory is a very important and socially useful branch of mathematics, it has been observed that there are many difficulties in their learning as well as teaching process.

Keywords: Probability problem's formulation, primary school (5-11 year – old), educational material, teacher training.

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1. Introduction

Elements of Probability Theory have recently been included in the curriculum of primary education because this area is considered a very important and socially useful branch of mathematics education. However, it has been observed that there are many controversies and difficulties as far as the comprehension and acquisition of this theory are concerned as well as its teaching process. These controversies mainly derive **a)** from the confusion caused by the parallel and not critical adoption of more than one philosophical theories (classical, frequentist, subjective) and **b)** from the fact that probability acquisition activates a "practical" and "common sense" framework rather than a typical and very abstract approach. Therefore, when a typical and abstract approach is "incumbent" or "implied", the teaching process becomes incomprehensible and creates difficulties in the theory acquisition.

Much research done relate, both to the difficulty of understanding the concept of probability and to the ways these difficulties can be dealt with, but their results do not seem to have influenced education.

Our point of view focuses on the influence the probability problems formulation has on the learning and teaching activities. We have formed the assumption that the mathematical problems used for research purposes differ from the problems we find in schoolbooks. For this reason, we assume that research explanations and interpretations are not sufficient and this is the reason why research results cannot improve the teaching process.

Historically, there is a view claiming that the probability theory has derived from "problem-games". This view has resulted in the study of these "famous" probability problems as well as of new ones. Our first finding was that historical problems could not be found in the context of the teaching of the probability theory in the primary school.

The purpose of this paper is to study we have studied the relationship between probability problems involved in the teaching of mathematics in primary school (5-11 year olds) and those encountered both during the historical development of mathematics and in educational research, focusing on the study and analysis of their formulation. Our aim is to create educational material for the improvement of the teaching process and also for the better understanding of the probability theory.

The formulations of probability problems have some special particularities compared to the formulation of the problems of other mathematics branches.

For example, with regard to formulation **kind**, apart from simple formulations, which we also find in other mathematics branches, we have complex formulations in which many materials are combined with many concepts at the same time. More specifically, we can observe in the same formulation the use of dice, coin and spinner related to the probability of an event, sample space, graphical representation and modeling.

As regards **form**, the formulations can be either verbal or mixed (verbal and iconic), but, in this case, they are diverse in several ways depending on whether they depict the material, a drawing, a table, a probability scale or a graphical representation.

The **material** in probability problems formulation has its own special role. In such kind of formulations we have a large variety of material, often with hidden information presupposing student's "knowledge". For example, if the material is a "dice", most of the times it is taken for granted that the child knows what is a "dice" (cube with numbers 1,2,3,4,5,6). This also happens with lotto as well as with other materials.

The **solving process**, apart from the formulations that request a unique answer, is usually complicated and needs prediction, experimentation, interpretation and comparison of the results.

As regards the suggested **way of solving**, the probabilistic formulations cause the collaboration in groups for problem solving because by nature the probability theory, due to the experimentation involved, requires comparison of results for its better approach.

Due to the fact that there are several "schools" of probability, each with a somewhat different interpretation, we find several approaches within the formulations. One of the interpretations is the classical one, according to which the probability of an event is simply the ratio of the number of alternatives favorable to that event to the total number of equally - likely alternatives. Another one is the frequentist interpretation, which defines probability in terms of the limiting relative frequency of occurrence of an event in an infinite or near infinite, number of trials. This interpretation is applied to events that are composed of non-equally-likely alternatives. Beyond the above mentioned, in many formulations we notice the request for an **experiment** in events with equally - likely alternatives, a fact that creates confusion.

Another particularity is the existence or lack of the sample space, as well as the way it appears (if it is found verbally or iconic, if it does not appear, if it is requested, or if it does not exist). This means that the way it is presented makes problem solving easy or difficult.

What becomes clear from all the above is the different nature of the formulation of probability problems, compared to these of other mathematics branches. In this way, difficulties are caused both to students and teachers: the former need to work in different ways; the latter have to apply a different evaluation method. Using Factor Analysis methods we study the relationships between problems involved in the teaching of mathematics at school and those encountered both during the historical development of mathematics and in educational research.

2. Methodology

We collected 282 probability problem formulations, from Greek and Cypriot schoolbooks, from English and American published books, from research articles and educational activities, which refer to 5-11 year-old children. We also collected historical and philosophical formulations from history books.

Then, we wrote down their basic characteristics and we created variables and categories, which are based on these characteristics. These variables related to:

1) the kind of the text that includes the formulation
2) the book publication date or the original date for historical and philosophical formulations
3) the formulation level
4) the formulation country of origin
5) the kind of formulation
6) how clear the formulation is
7) the formulation form
8) the formulation number data
9) the formulation sample space
10) the existence of a solved example
11) the formulation material
12) the formulation solving process
13) the formulation solution
14) the formulation experimental trials

15) the formulation probability concept

Next, we coded each formulation in a category. The resulting data was statistically analyzed through Factor Analysis methods, in order to classify the problems and obtain the appropriate taxonomy.

3. Results

Factor Analysis pointed to two major taxonomy formulation criteria corresponding to the first two factors.

The first factor mainly represents the diversification between the published for the first time in USA formulations and these published in the other countries with the characteristics that accompany each case.

VARIABLES	CATEGORIES	
	Factor's 1 positive side	**Factor's 1 negative side**
Country	USA	Other Countries
Kind	Complex	Simple
Form	Mixed (Verbal and Iconic)	Verbal
Solving Process	Experimental	Simple Answer
Way of Solving	Teamwork	Individual
Trials	Existent	Not Existent

The second factor mainly represents the diversification between old and more recent formulations. The table below describes this diversification:

VARIABLES	CATEGORIES	
	2^{nd} factor positive side	**2^{nd} factor negative side**
Date	Historical	After 1979
Kind of text	Book, Article	School book
Level	1-6	1,2,3,4,5,6
Country	France, Italy	Cyprus

On the level of factor 1 and factor 2 below (figure 1) we notice the categories-projection configuration, which could lead to four groups. Each group contains different formulation characteristics.

More specifically group I contains formulations from English and Israelian research tests, group II from recent Cypriot schoolbooks, group III French and Italian historical formulations and finally group IV formulations from the USA.

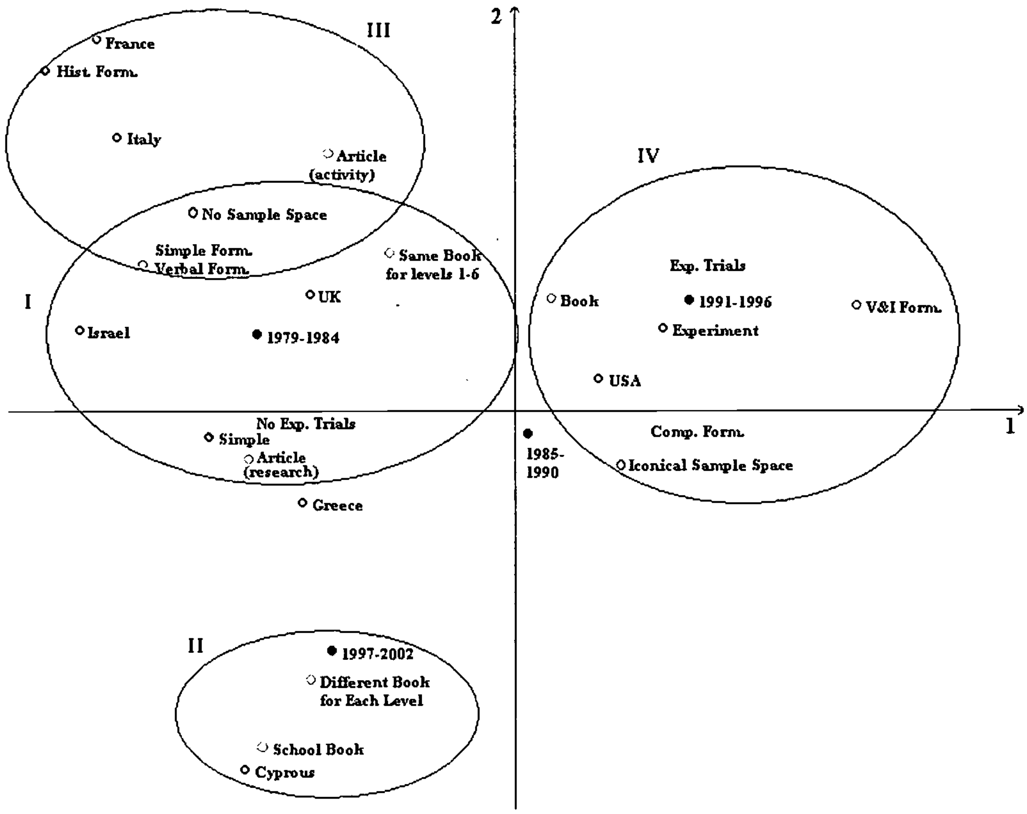

Figure 1: Categories -projection configuration

The hierarchical cluster analysis below (figure 2) confirms the above results. In particular, cluster 4 (group IV in diagram) is clearly separated from the others. Cluster 3 (group III in diagram) follows, which differs from clusters 1 and 2. Finally, there are clusters 2 (group II in diagram) and 1 (group I in diagram).

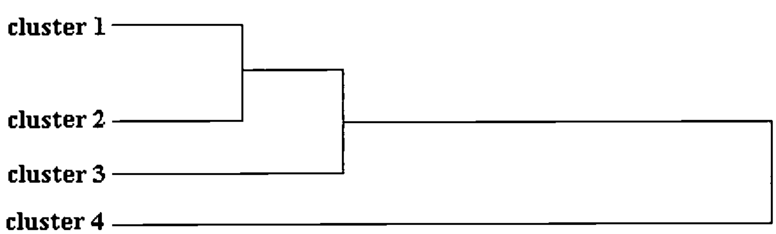

Figure 2: Cluster analysis dentrogram

4. Conclusion

After analyzing the correlations emerging from the characteristics of probability formulation, we noticed that our initial assumption, that problems used for research purpose differ from the problems we find in schoolbooks, was confirmed. This might be a reason why we assume that research results cannot improve the teaching process.

Also, the "famous" historical problems considered as the foundation of probability theory, which, as a framework, would lead children to reinvent and rediscover the concepts and the real difficulties of this theory, are not presented at all in probability school-formulations.

The simple and verbal school-formulations requesting a unique answer from the student prevail, while, according to the particularities we mentioned in our introduction, the probability problem solving requires the depiction of material, experimentation and teamwork for better results.

The above conclusions (as well as others that will come up from further data analysis) will be utilized for the construction of educational material, especially for problem activities, for the improvement of teaching, for the facilitation of understanding the probability concept by primary school students as well as a research context for teacher training.

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USING INTELLIGENT ALGORITHMS TO GUIDE A BEST SOLUTION EXPLANATION MODEL FOR AN INTELLIGENT TUTORING SYSTEM IN ALGEBRA MANIPULATION

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ABSTRACT

The aim of *MathWeb* is to augment students' learning processes in algebra manipulation. Previous research has shown that the system can provide more effective learning experience than using traditional methods. In this paper, we describe the architecture of *MathWeb II,* which is based on a set of new algorithms including a model tracing reasoning mechanism and term rewrite technique to support students' manipulation skill in linear equations. One of the most important aspects of the *MathWeb II* is to provide "optimal solution" explanation to inform the student how to solve linear equations in more effective ways which can help the student to have a better understanding about their learning process. The paper starts with a brief description of the *MathWeb II* system's architecture. This will be followed by a detailed presentation of the organisation of the best solution explanation model for linear equations. Finally, the paper will draw some general conclusions and present a description of some further work.

1. Introduction

Many researchers have agreed on the benefits of using such systems to improve the student's learning process in mathematics [9]. However, the student may answer the algebra question correctly but in a tortuous, inefficient and complex manner. As a result, the overall student performance will be reduced due to the time used to consider for processing unnecessary calculations. Experience has shown that students will gain a better understanding of manipulation skills if they are exposed to the considered "ideal" problem solution. This has formed our motivation to develop appropriate support in order to study and model this observation. Therefore it is necessary to teach the student how to calculate the algebra question in a more efficient way.

This paper describes the theory behind the development of an intelligent algebra tutoring system (*MathWeb II*), which can be used to improve the student's learning process in linear equations. An overview of the *MathWeb II* is given containing the functional facilities of the system. The system has been developed for the purpose of examining the student answer step by step and provides "optimal solution" explanation to inform the student how to solve different linear equations in a better way when the student answer is correct but it is not the best solution. In order to provide "optimal solution" explanation, a generative approach is developed based on a set of new algorithms with the previously developed model tracing reasoning mechanism [3] and term rewrite technique [7]. Two types of "optimal solution" explanations are provided for improving the learning process in solving different linear equations. The first type shows the student how to find a best solution step while the second type is to show all the best solution steps for solving the whole equation.

2. The Architecture of MathWeb II

MathWeb II is an intelligent tutoring system for algebra manipulation. The system's logic and operation is based on the already developed term rewrite technique [2], [7] and model tracing reasoning mechanisms [3], [8]. The purpose of the new *MathWeb II* system is to provide the best solution explanation in order to improve the student's algebra manipulation skill with a 'learning by doing' environment. The current system capability is limited to polynomials and linear equations. As shown in figure 1, the system's architecture consists of a set of expanded components. These include a user-interface, a best solution explanation model and a student model including diagnostic [2] and performance model [8].

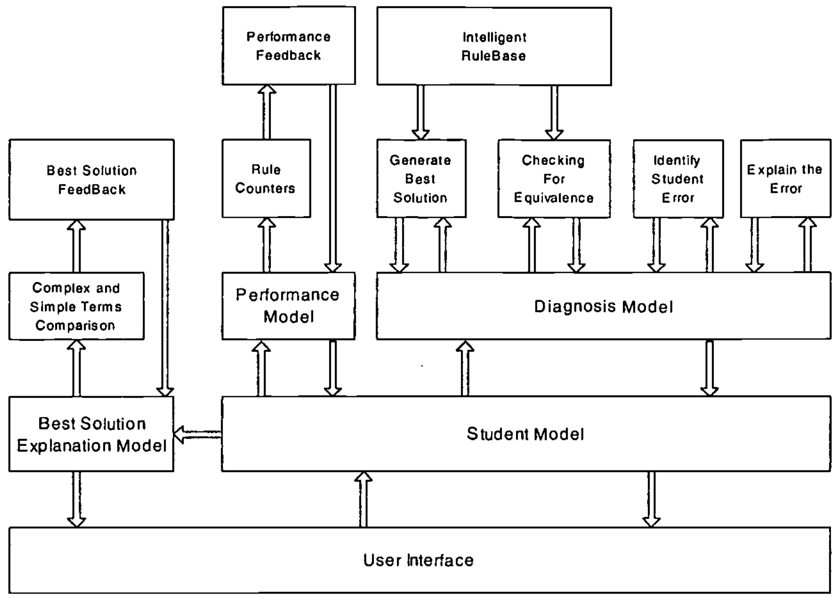

Figure 1: The MathWeb II system configuration

2.1 User Interface

When the answer is inputted by the student, an interface component is required which presents the question to the student, provides an input tool for the student to enter various algebra answers, recognises the student's response, and return the feedback message. The user interface component also has a function to recognise a typing error, which gives a message to urge the student to re-input the expression.

2.2 Student Model

A student model can be separated into two sub-components: diagnosis and performance model [3]. The diagnosis model can be described as the process of getting information concerning the student behaviour. The performance model can be described as a set of data structures to record the data generated by the diagnostic module.

2.2.1 Diagnosis Model

The purpose of the diagnosis model is to analyse the student answer, identify any student error and provide a suitable explanation according to the student response. This can be done by transferring the subject knowledge into many different sets of rewrite rules, which transfer a term (polynomials and linear equations in this case) to another equivalent expression. The rewrite rules include not only the correct rewrite rules, but also include other rewrite rules which are organized into several sets, namely transparent rules, mal-rules, and linear equation rules.

The first set is the correct rewrite rules include two types of rewrite rules, regular rewrite rules, and conditional rewrite rules. The conditional rewrite rules arise from the fact that some mathematical laws are not universally valid, such as $nx = m \rightarrow x = m/n$, which is only valid when n is non-zero. The second set of rewrite rules is the set of transparent rules, which can be defined as basic algebra rules that should be well known by the students. The third set of rewrite rules is the linear equation rules, which can be used in the process of solving integer linear equations with one unknown. The final set of rewrite rules is consists of incorrect rewrite rules (mal-rules), which can be used to express the errors that students make [2]. Table 1 presents part of a set of rewrite rules, which can be used to simplify polynomial and linear equation with one unknown to a solved form, specifically for the algebra domain of integer polynomials and linear equation.

Correct Rules	
Rules	**Semantics**
$A+A \longrightarrow 2*A$	*Addition of unknowns*
$A*(B+C) \longrightarrow A*B+A*C$	*The distributive law*

Transparent Rules	
Rules	**Semantics**
$0+A \longrightarrow A$	*Adding zero to unknown*
$+(A) \longrightarrow A$	*Positive sign*

Linear Equation Rules	
Rules	**Semantics**
$(M*X=N) \longrightarrow (X=N/M)$ where $M \neq 0$	*Dividing by the coefficient of the unknown (Isolation)*
$M*X \pm N = 0 \longrightarrow M*X = \mp N$	*Add- subtract of the unknown*

Table 1: Example of rewrite rules

If the student inputs an incorrect answer, then mal-rules can be used to express the errors made. For example, if the problem is to solve a linear equation 4+4*(x-1) = 2 and the student may enter a step 4+4x–1 = 2. After the comparison process using rewriting and evaluation techniques (described in section 4), we know that the student answer 4+4x–1 = 2 is incorrect as it is not equivalent to 4+4x–4 = 2. In order to find out the type of student error, the mal-rule A * B $\longrightarrow$ B is applied to generate an incorrect system answer 4 + 4x + **4*-1** = 2 $\longrightarrow$ 4 + 4x + -1 = 2 which is equivalent to the student answer. Then we know this student may have a problem in using the distributive law.

2.2.2 Performance Model

The idea of the performance model is to use a set of rule counters to store what types of errors are made by the student during exercises. As a result, the system will generate performance feedback. The performance feedback will not only contain the result of student performance, but also a detailed explanation of their performance (why the student made these errors).

In order to provide accurate performance data, the performance model will use rule counters to store the numbers of different rewrite rules used within each step in the student's solution. There are two different types of rule counter, for logical and non-logical errors in the performance model. The rule counters of non-logical errors will also contain a set of sub-counters to identify the incorrect operator used to simplify an integer polynomial or a linear equation. The performance model uses the diagnosis model to identify, as well as to locate the student errors [8]. The following is an example to show how this model analyses the student performance.

Mal-Rule	Modify System Output	Student Answer
A * B -> B	4 + 4 * x + 4 * -1 = 2	4 + 4x – 1 = 2

Performance Counters	Logical		Non - Logical					
	Left Distributive	Right Distributive	Addition +, -, *, /	Subtraction +, -, *, /	Multiplication +, -, *, /	Division +, -, *, /	Sign
	1	0	0, 0, 0, 0	0, 0, 0, 0	0, 0, 0, 0	0, 0, 0, 0	0

Table 2: Identify student error using mal-rule for left distribute law

From the result of the rule counter, we can identify the status of the student performance clearly. In this case, the performance model will have some information to suggest that the student has a misunderstanding of the left distributive law "A * (B + C) → A * B + A * C".

3. Best Solution Generation

In order to generate the best solution for a particular linear equation, we propose a new algorithm (optimal solving for linear equations) with our developed rewrite rules [2][7] and model tracing reasoning approach [3][8] to generate a best solution with a minimum number of reasonable steps of simplification for the linear equation.

3.1 Polynomial Optimal Tree

The idea of the polynomial optimal tree is to generate a best solution for each step simplification using the minimum number of steps to achieve the final correct answer. This can be done by building a problem solving strategy with the use of a set of rewrite rules [6][7]. The structure of the problem solving strategy is based on a binary tree format. This algorithm divides an equation into sub terms, using the priorities of the operations. For example, to generate a problem solving strategy for expanding a polynomial 2 * (x + 137 – 131), the system will analyse the polynomial structure and then divide it into sub polynomials based on the priority for each operator.

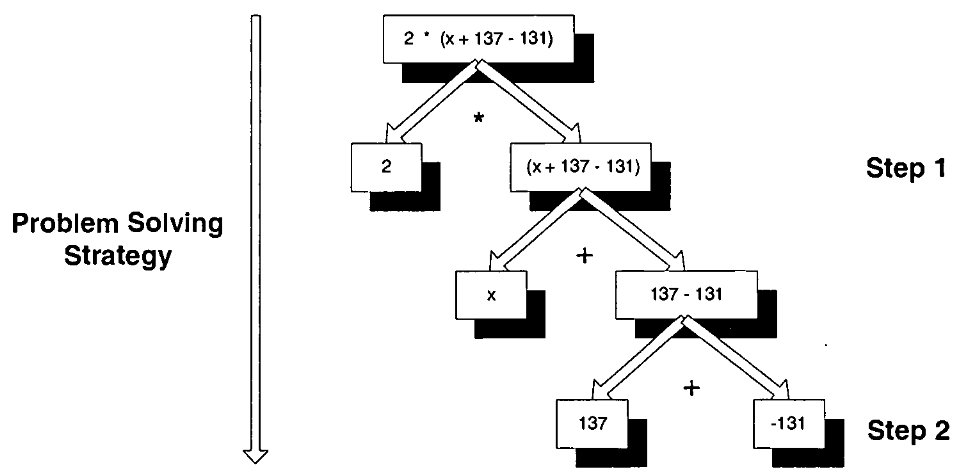

Figure 2: Generate problem solving strategy for a polynomial

From the problem solving strategy, there is an "optimal" solution generated for this particular polynomial. The best solution is represented in a set of ordered steps. Each ordered step is identified as simplifying the smallest sub polynomial. Therefore, we can manipulate any polynomials with rewrite rules in a best way by following these ordered steps.

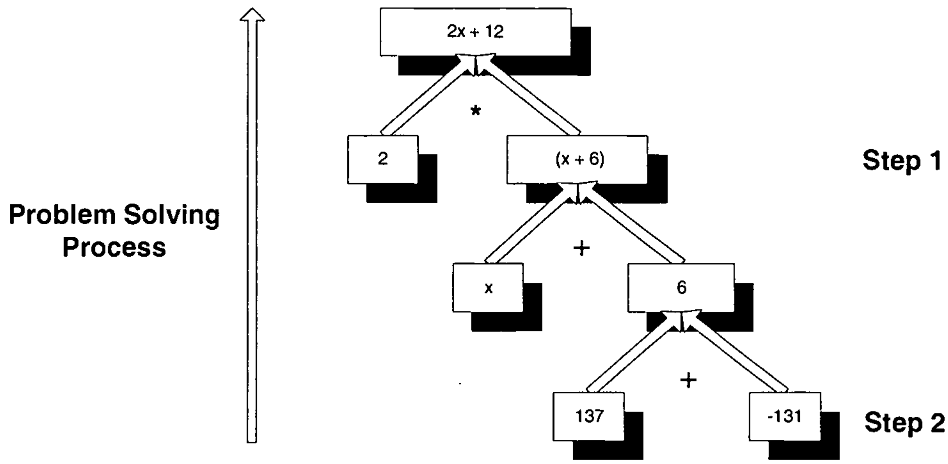

Figure 3: Optimal problem solving process

3.2 Optimal Solving For Linear Equation

The purpose of this algorithm is to provide an "optimal" solving strategy for linear equations, which can be used with our rewrite rules and polynomial optimal tree in order to simplify different linear equations with one unknown in a best way. This algorithm will first generate the problem solving strategies (polynomial optimal tree) for the polynomials on both the left and right hand sides of a linear equation. The polynomials (on both left and right sides) will be simplified based on the problem solving strategy with rewrite rules to obtain the simpler form. After the polynomials are simplified on both the left and right sides, the linear equation rewrite rules are applied to move the unknowns to one side and move the numerical terms on the other side. Finally, it will apply associated calculations to the numerical terms to obtain the final answer for the unknown. This can be done with the following steps.

Step 1: Use polynomial optimal tree algorithm to generate problem solving strategies for the polynomials on both the left and right sides.

Step 2: Apply polynomial rewrite rules to these polynomials to obtain the fully expanded forms.

Step 3: Use linear equation rewrite rules to move a term to another side.

Step 4: Apply polynomial rewrite rules to the modified polynomials. If there is not a final answer for the unknown x then go to step 3 otherwise stop process

For example, to solve a linear equation x – 4 + 2 = 2 * (3-1), it will divide the linear equation into two polynomials x – 4 + 2 (the polynomial on left-hand side) and 2 * (3-1) (the polynomial on right-hand side). Then it simplifies these polynomials to obtain the expanded forms as x – 2 and 4. After that it moves the unknown to one side to form the equation into the final structure x – 2 = 4 -> x = 4 + 2. Finally calculate the value for the unknown x as 6.

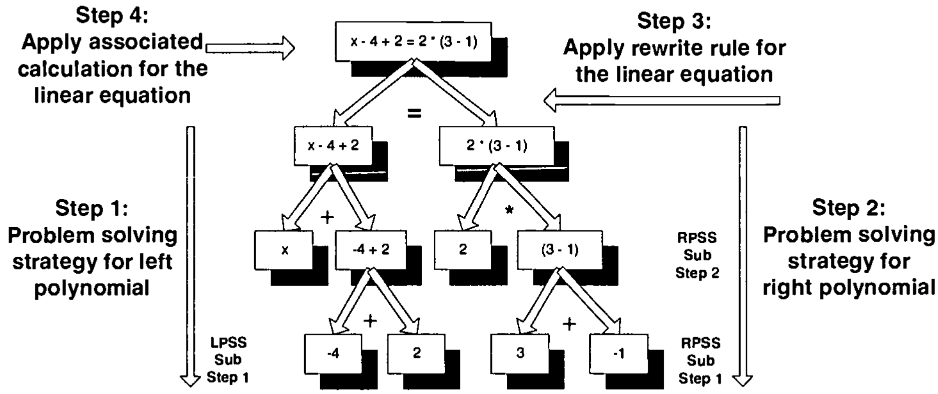

Figure 4: Generate optimal solving strategy for a linear equation

In the above "optimal" solving strategy for the linear equation, it will execute the steps within the Left Problem Solving Strategy (LPSS) first and then process the Right Problem Solving Strategy (RPSS).

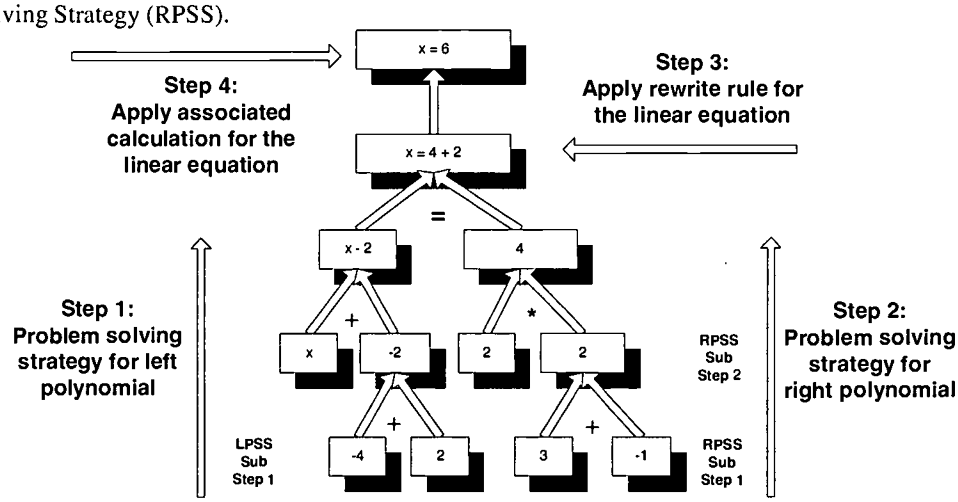

Figure 5: Optimal problem solving process for a linear equation

4. Checking For Equivalence (Rewriting and Evaluation)

We also need to check the correctness of each simplification within each step in the student's solution for a linear equation. This can be done through two stages of validation process as rewrite technique [7] and evaluation [4]. The first stage of the validation process is to apply rewrite rules to the linear equation in order to obtain the value for the unknown. After that, this value is used to evaluate the student answer for checking the equivalence. For example, a question is to solve a linear equation 3x - 4 = 2x + 4 and the next step student's answer is 3x = 2x + 4 - 4. In this case, a set of rewrite is applied to the "optimal" solving for linear equation (described in section 3) to obtain the value for the unknown x as 8.

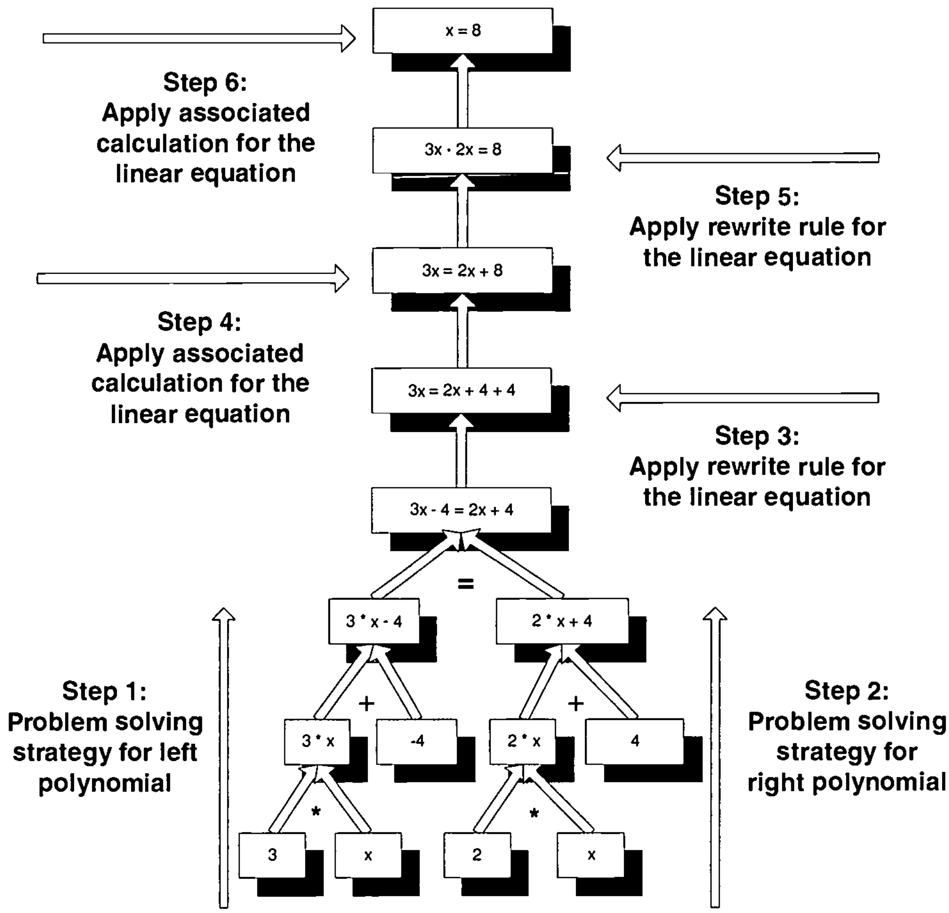

Figure 6: Calculate the value for unknown x with optimal solving strategy for linear equation

After we have obtained the value for the unknown x, the evaluation process is undertaken for checking the equivalence of student and correct answers. This process can be done, by evaluating the student answer with a correct substitution of the unknown, so that the values are examined on both sides of the student's answer. For the above example, the student answer is incorrect because the values are different 3x = 2x + 4 – 4 -> 24 = 16 on both sides after the evaluation.

5. Best Solution Explanation Model

In this section, we purpose a new best solution explanation model to provide the best solution explanation for linear equation. The purpose of best solution explanation model is to improve the student's algebra manipulation skills with a 'learning by doing' environment [5]. We believe to obtain the simpler form for sub polynomial first is always the best way to solve a linear equation. The best solution explanation can be generated based on the "optimal" solving strategy for linear equation described in section 3. For example, if the problem is to simplify a linear equation 2 * (x + 137 – 131) = 4 + 1 –3 and the student decides to simplify the left polynomial 2 * (x + 137 – 131) first. Then the best solution explanation model will check the correct student answer with the problem solving strategy for this polynomial to identify whether it is a best solution or not. If the student input for next step is 2x + 274 – 262, then the best solution explanation model will inform the student that the best solution is to first simplify 137 – 131 before expanding 2 * (x + 6). On the other hand, if the student decides to simplify the right polynomial 4 + 1 – 3 and the student answer for next step is 4 – 2, then the best solution explanation model will inform the student that the best solution is to first calculate 4 + 1 before subtracting 5 - 3. The best solution explanation model will also inform the student with a best solution explanation when the student move a polynomial to another side as 2 * (x + 137 – 131) –3 –1 + 3 = 0 before these polynomials in the simpler form on both side. This can be done through the following steps. Suppose that the previous step was the equation pL(x) = pR(x), and the student enters the step sL(x) = sR(x).

Step 1: Apply polynomial sorting to form both the correct and student answers in the same format in order to identify the student action. If either sL(x) is different than pL(x) or sR(x) is different than pR(x) then go to step 3. If sL(x) and sR(x) are both different than pL(x) and pR(x) then go to step 2.

Step 2: If the both sL(x) and sR(x) are not in the simpler form, then go to step 3. Otherwise stop process as the student answer is a best solution.

Step 3: Calculate the minimum number of steps to achieve the normal for the both correct and identified student answers. If the number of student steps is greater than the number of correct steps, then it is not a best solution step and stop the process, otherwise go to step 4.

Step 4: If the number of student steps is less than the number of correct steps, then it is a best solution step and stop process otherwise go to step 5.

Step 5: If the number of student steps is equal to the number of correct steps, then compare the identified student answer and correct answer. If they have the equivalent structure then it is a best solution step and stop process otherwise it is not a best solution step and stop process.

The best solution explanation model can also generate a best solution explanation to show the "optimal ways" for solving the whole linear equation. For example, to simplify the left polynomial of a linear equation 2 * (x + 137 – 131) = 4 +1 – 3, the first step is to simplify 137 – 131 and the second step is to expand the simplified polynomial 2 * (x + 6) to obtain the simpler form 2x + 12. On the other hand, to simplify the right polynomial of a linear equation 2 * (x + 137 – 131) = 4 +1 – 3, the first step is to calculate 4 + 1 and then the second step is to calculate 5 –3 to obtain the simpler form 2. After the polynomials are in the simpler form on both sides, then the best solution explanation model will inform the student that the next step is to move the value 12 to another side as 2x + 12 = 2 -> 2x = 2 – 12. After that the next "optimal" step is to calculate 2 – 12 and then move 2 to another side as x = -10 / 2. The final step is to calculate –10 / 2 to obtain the final answer –5 for the unknown x as x = -5.

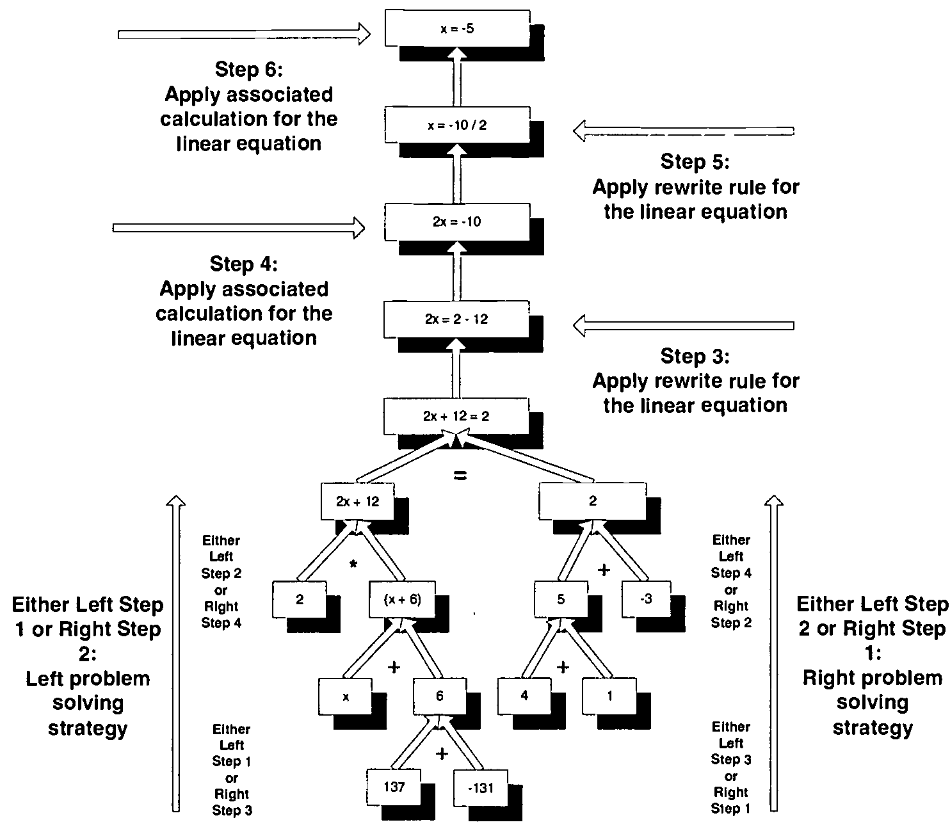

Figure 7: Generate best solution explanation for a linear equation

By informing the student how to simplify the linear equation in an "optimal" way, we believe that the student's manipulation skill can be improved in a better way for manipulating different linear equations.

6. Conclusions

Many researches have agreed on the benefits of using intelligent tutoring systems that it can improve the student's learning process in mathematics. The *MathWeb II* is an intelligent tutoring system, which provides "optimal solution" explanations with a 'learning by doing' environment in order to improve the student's manipulation skill in linear equation. This paper describes the theory behind the development of an intelligent algebra tutoring system (*MathWeb II*). As an overview of the system architecture is given containing the functionality for each model. A set of new generative approaches is also developed to dynamically generate correct answer (optimal solving for linear equation) for different linear equations and provide "optimal solution" explanation in the student's learning process. The idea of the best solution explanation model is to calculate the number of steps to achieve the normal form and analyse the polynomial structure in order to identify whether the correct student answer is a reasonable best solution or not. If the student answer is not a best solution then the best solution explanation model will inform the student how to simplify the next "optimal" step.

The best solution explanation model also provides a best solution explanation for solving the whole linear equation.

As a result of our previous research [1], we believe that the *MathWeb II* can be used to provide more effective learning than doing the same exercise using pencil and paper on your own. However, we still need to prove our system to ensure that the *MathWeb II* will achieve its aim and objective. Therefore, a system implementation is required with a validation study, so that student's manipulation skill can be examined to see the potential effects on student's understanding about their learning process. The experiments will take place in local schools with the student evaluation so that a number of users will use the system to see the affect of using such a system on the students' manipulation skill in linear equation.

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USING THE WEB TO ENHANCE STUDENT LEARNING

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ABSTRACT

When mathematics interacts with technology, the possibilities of usage are vast. Technology can affect the way mathematics is done. It can also have a profound effect on the way that we as academics manage our teaching, carry out assessment and interact with students. The web provides exciting opportunities which can significantly enhance the quality of the learning and support which our students experience, whilst still maintaining the personal contact so necessary for a complete education.

We present here some of our recent experiences in implementing a range of initiatives concerning the use of the web with mathematics undergraduates, as part of curriculum innovation involving the integrated use of technology both for doing and for learning mathematics. The particular unit reported here involves the explicit critical study of mathematical technology, at first year undergraduate level:

Matters arising have included:

- Full material support for each unit provided on the web.
- Communication networks (lists, discussion groups, etc)
- Automatic monitoring of student activity on the material of the unit
- The need for a new approach to assessment

This final point warrants further discussion. Traditional approaches to assessment of mathematical activities most frequently involve an examination, with a pass mark of typically 40%, but little other feedback available to the student. This would be entirely inappropriate in this context because of the wide variety of skills to be assessed. We had adopted a novel approach in which the students may score up to 1000 points. Some marks are available for particular activities such as evaluating a piece of software, but students also accumulate a small proportion of marks week by week by completing a continuously-monitored online learning diary. Thus as part of the approach, students acquire a marked profile of their range of skills and experience, and automatically receive a high degree of feedback on their progress.

Keywords: TECHNOLOGY, CURRICULA INNOVATIONS, INNOVATIVE TEACHING METHODS

Introduction

As academics in higher education, we are faced with the problem of trying to deliver high quality courses with ever-diminishing resources. Direct contact time with students is limited, and we must find innovative ways both of using that time and of exploiting technology to support those students at other times. Motivation is also an issue - students will only really commit themselves to a topic if it is likely to be assessed. This tends to lead to a surface learning approach. By changing the assessment pattern we can, perhaps, go some way towards a deeper learning approach, at the same time giving the students essential skills which are valued in particular by employers world wide (Challis and Gretton 1999 and 1997, Davis 1991, Gretton and Challis 1997). In this presentation, we wish to report some preliminary results of our experiences using a variety of web-related support tools with a first-year group of undergraduate students.

Using the Web to Provide Basic Resources

At the most basic level, the web can be used to provide a library for course materials. In the Mathematics group at Sheffield Hallam University (SHU), we use the web to provide a user-friendly front-end, giving students one-click access to these materials (see Figure 1). For ease of staff use, we also offer an FTP service – this is made available to staff by means of a drive mapping through Network Neighbourhood so all teaching staff have to do is drag and drop whichever files they wish to make available. For students, the FTP site can be accessed either by a web browser or by means of an FTP client at any time and place (see for example, Figure 2). Freeware versions of all software that may be useful is provided to new undergraduates on their own individual customised CD.

Communicating with Students

Class contact time is being eroded owing to resource cut backs so it is essential that students are able to get some help when they need it. To this end, we have set up several systems, which help. Firstly, we have web-based discussion forums. These use the freeware PHP-based *Phorum* software (Phorum 2002), and although they are easy to use, our experience is that students are reluctant to utilize such systems. Despite encouragement over several years, usage remains very low, and such messages as are posted tend to be frivolous. We believe that systems, which require the *active participation* of students, are used far less than those which are inherently passive. E-mail, for example, is still the most widely-used form of staff/student electronic communication, and for that reason the second system we have implemented makes use of e-mail discussion lists. We have set up a list for each year of the course, and this works well both for staff-student communication as well as student-student communication. Staff regularly uses this as the primary mechanism for the distribution of course and unit-related messages. For student-staff communication, person-to-person e-mail is still the most successful approach.

Students are increasingly using cell phones to communicate, particularly with each other, and text messaging is becoming pervasive. We are developing an integrated SMS messaging system, which will allow staff to send text messages to students – this will be useful when contact is required urgently. ("Why have you missed your lecture?"!)

A third support system which has been implemented is the Frequently-Asked-Questions page for each unit, also referred to as the knowledge base. The idea is for staff to post articles, which address students' most, asked questions – hopefully this will save both staff and student time if any other student subsequently needs to find the answer to the same question. This is in the embryonic stage, since staff need to post these messages and their time is limited, but the structure is in place.

Monitoring Student Work

Students on the SHU Mathematics degree programme take a unit *Mathematical Technology*, which spans both semesters of year 1. In this unit, students develop expertise in web page creation and web site design, computer programming and the use of spreadsheets, computer algebra software and hand-held devices for mathematics. Also they begin to develop their reflective skills, an essential part of the critical skill of improving one's own learning and performance. To start this process we have adopted a multi-pronged approach.

Firstly, students must begin to develop a web-based portfolio of their work. This is partly in response to the UK Quality Assurance Agency (QAA) plans (QAA 2002) for students to keep *progress files* but mainly because it is an essential part of their professional development. The intention is that during their progress on their undergraduate degree (3 or 4 years) each student will accumulate an on-line collection of their work, together with necessary text and annotation, in a conveniently accessible format. In this unit, they begin by creating an on-line resumé (which is regularly updated) and create a separate page or pages for each module they take on the course. Each student is given password-protected web space and is expected to provide a suitable means of navigating their on-line portfolio.

The advantages of this are many – for students, they are learning skills which they can immediately see will be useful, and make them more employable; for potential employers, a readily-accessible summary of the students' work is instantly available and for staff, it is easy to see exactly how much progress each student has made.

The second 'prong', intended to develop students' planning and reflective skills, is an on-line logbook. Each week the students write a few sentences about each module they are currently studying, saying what has gone well, what has not gone well, and also what plans and steps they intend to take to deal with any problems that have arisen. The advantage for the students (apart from developing reflective skills) is that they are encouraged to confront problems and explicitly commit strategies for solving them. For staff of course, it is most enlightening to see what these problems and strategies are. It is also valuable to get continuous feedback on the progress of each module as seen from the students' perspective. The is far more informative than the usual staff-student meetings each semester, since these often raise problems too late and tend to reflect the views of a small vocal minority of the student cohort. By contrast, staff reading the student logbooks see the views of each and every student continuously.

Inevitably, students will not do this each week unless there is some tangible reward. Staff provide encouragement and point out the advantages of keeping the logbook up to date, as well as generating an incentive by incorporating completion of the logbook into the assessment schedule (see Assessment later). To streamline this process, a series of web programs has been written to provide a secure framework within which the students can manage their on-line logbook. These are illustrated in Figures 3a-d. Students log in, and once authenticated can move freely through the system. All relevant student details are stored in a central database so that upon log-in the system knows which year the student is in and hence which modules should be made available.

The logbook contains a number of entries, one each week (ideally!) for each unit (see for example Figure 3a). Students can add new entries (Fig 3b), edit or delete previous entries (Fig 3c) – but only those up to seven days old, change their e-mail details, request new passwords and request that their login details be e-mailed to them.

A facility has been provided to give staff an overview of the whole set of logbooks, so it is easy to see who is up to date and who has fallen behind (Figure 3d). Each student's name is a hyperlink to their individual logbook, so staff can readily view these.

The majority of students at present have engaged with this very well – some exceptionally so. After a slow start, approximately two-thirds of the group are completing the logbooks weekly, as required, and a few stubbornly resist all attempts to get them to join the party! However the mark sheet for the unit rolls on week by week and students do not like seeing zero marks!

Assessment

Much of the learning taking place in this unit is formative. Students are learning techniques, ideas, methods of approach and reflection and organizational skills as well as knowledge. Therefore, and because students need the incentive, our approach to assessment is different from the 'traditional' approach. We aim to award a small element of assessment for each part of the work students are expected to do. Altogether 1000 points are awarded during the year. This is broken down as follows:

Web Logbook (200 marks)

- Weekly updates of the logbook (95 points). Up to 5 points are given for updating the logbook each week from week 6 to 24 (the start is delayed to accustom the student to the assessment practice)

The balance of the marks will be awarded summatively on the basis of:

- Breadth and depth of commentary (35 marks),
- Evidence of reflection, and of taking appropriate action as necessary (30 marks)
- Content, including the description of specific tasks to be carried out and the development of action plans and target-setting, in response to problems that occur (40 marks).

Web Portfolio (200 marks)

End of week 9. 60 marks awarded for the resumé, and the implementation, content and structure of the portfolio.

End of Semester 1. 70 marks awarded for the further development of the resumé, and again the implementation, content and structure of the portfolio. For the latter, more sophistication, and broader and deeper content is expected.

End of Semester 2. 70 marks awarded for the final development of the resumé, to include updates and refinements, and the completion of the portfolio with respect to level 1 modules.

Credit will be awarded for implementation of recommended guidelines for use of HTML as indicated in the handout, and for providing a suitable justification of deviations from this. All parts of the portfolio must be accessible from the main home page.

There are also three other 'standard' coursework assignments spread evenly through the year, which will develop mathematical skills in relation to available technology, each worth 200 marks.

Following each element of assessment, including the weekly logbook entries, marks are entered by staff into a spreadsheet. A program has been written to export the data into a customized web page, linked to the main page for this module, so that the students can always see a current view of their accumulated assessment (Figure 4). Following each major assessment point for the logbook and portfolio, students receive detailed feedback via e-mail. This is achieved by another customized computer program – staff type the comments into an ordinary text file and the program (armed with a list of students' e-mail addresses) mails the comments to each person. This approach has the benefit of being both personal and fast, and by its very nature a copy of all comments is retained by staff for subsequent presentation to external examiners.

Conclusion

In the early days of dedicated mathematical technology Waits and Demana (1995) made a statement that "professional development blending mathematics curriculum reform with appropriate use of technology should be a top priority for educators in the next five years". The present authors have observed that what is also needed is a change in assessment practice. But what has happened? Seven years on from that statement, technology is widely used, although in many cases not in an integrated way, but changes in assessment practice are slow to arise: the test or examination continues to dominate. If educational results are quantified only by passing tests and examinations, students become ensnared in superficial learning habits, and it is actually not possible to assess the full range of learning which we claim to encourage, including those skills valued by employers (Davis 1991).

In this paper we have addressed some of these issues. To mark a module out of 100 is arbitrary, and the concept of having 1000 marks to give enables a whole range of skills to be assessed. More importantly, in the assessment-driven, strategic learning environment in which we live this gives us more carrots to dangle in front of our students. Human nature and student economics being what they are, most students need to be pressed to act or react, and do so if tempted or pressured by credit. However this absorbs more staff effort, but minimising the impact of that is one of the tangible outcomes we have achieved using the Web.

The topic of assessment is something that many of us regard with irritation rather than interest, but we have found it does raise more heated discussion than many other topics, because of its crucial importance in the learning process. We have reported on an attempt to broaden assessment style to embrace skills that are necessary and demanded of students by future employers. Normal examination type of assessment cannot do this. The examination still has a role, but is not sufficient by itself, and the nature of the assessment must remain under review to make sure that it addresses the outcomes of learning and the skills for the future. In following through the work on this module the students have on the web their work, evidence of planning, reflection and improving their essential skills and performance, evidence of a multitude of IT skills, and a catalogue of feedback and marks!

In closing then, we say that technology is not a "threat to every university"(Daily Telegraph, 1999), but in the current example provides a significant enhancement of our students' experience by giving an opportunity to address and assess the full range of valuable skills.

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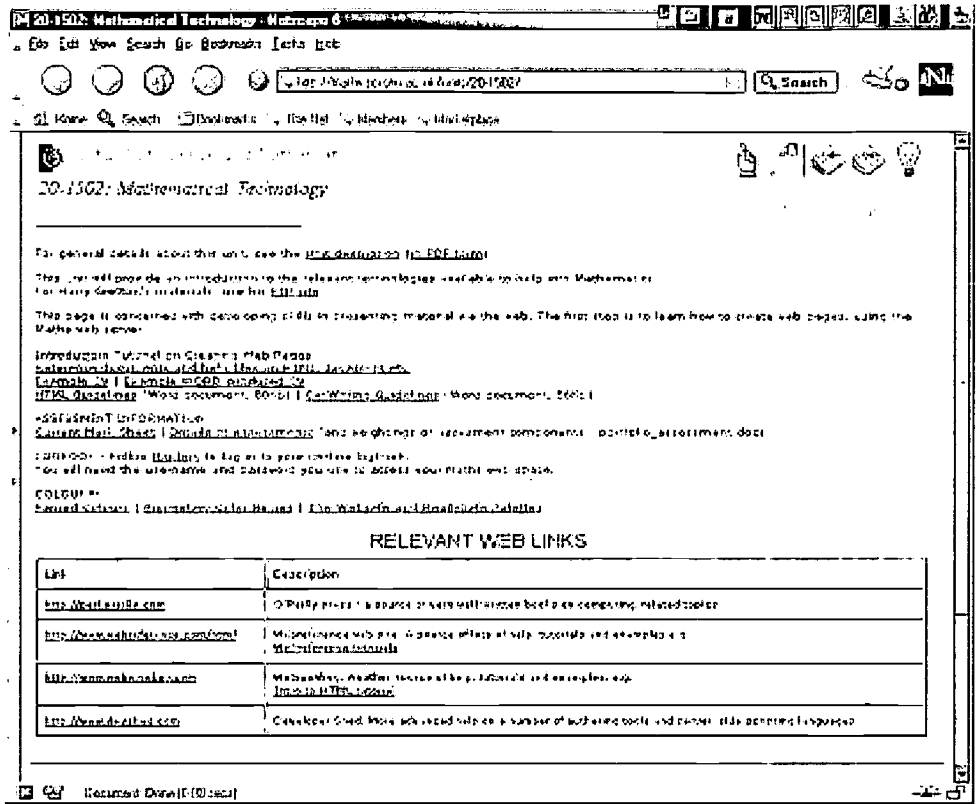

Figure 1 : The *Mathematical Technology* home page

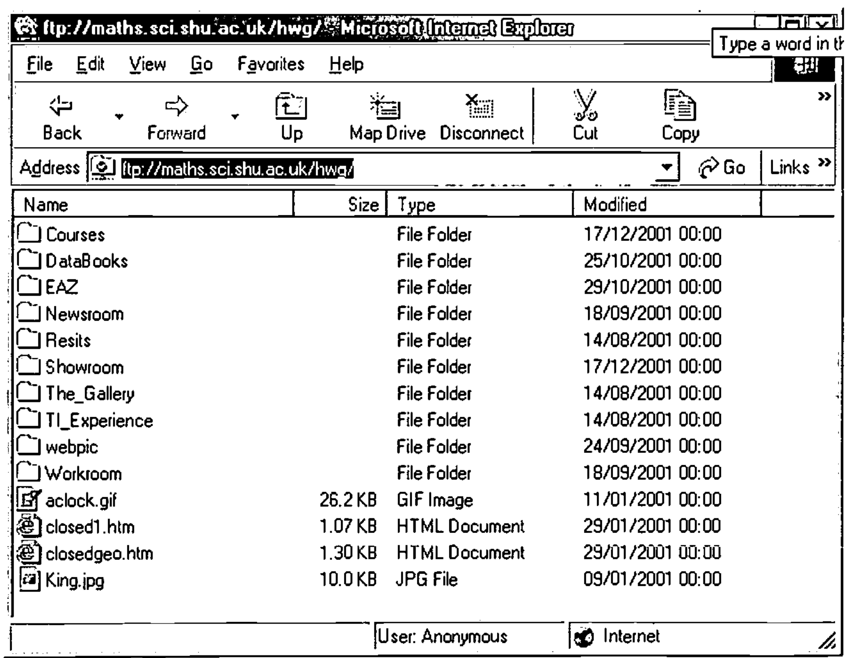

Figure 2: An example of the SHUMaths FTP site

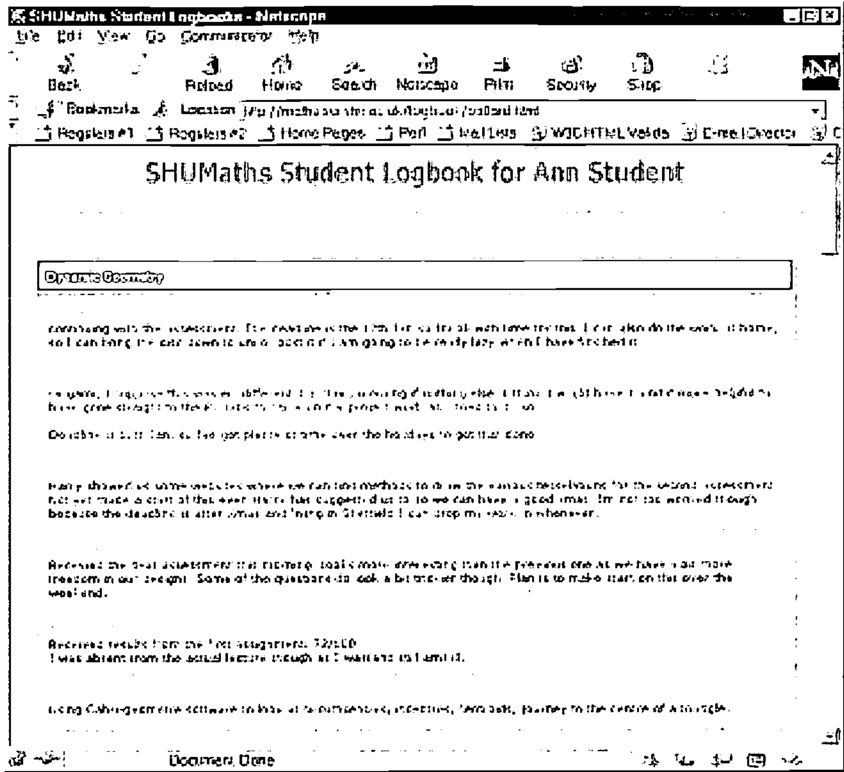

Figure 3a: An example of part of a student on-line LogBook

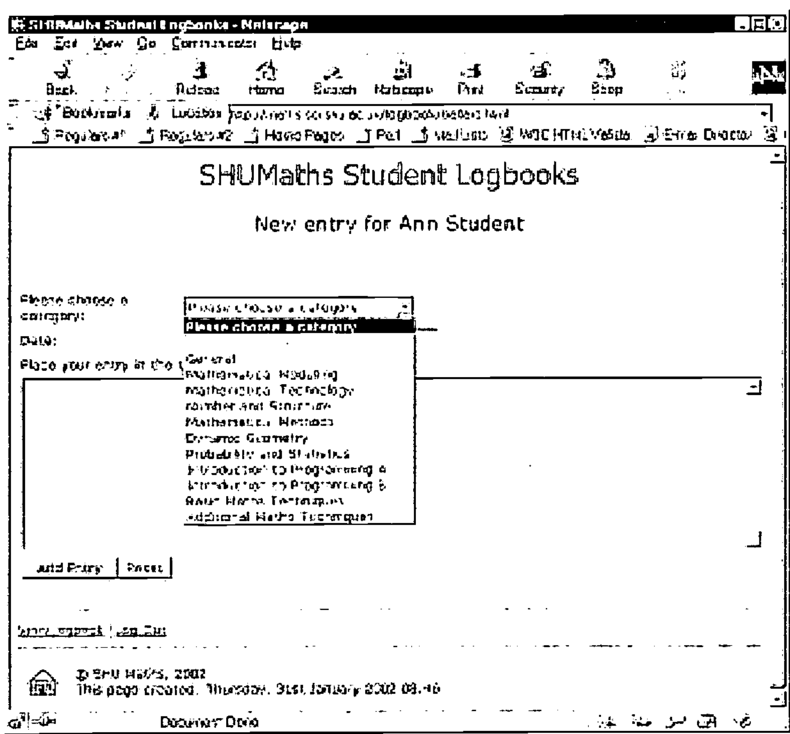

Figure 3b: The form used by students to add a new entry to their logbook

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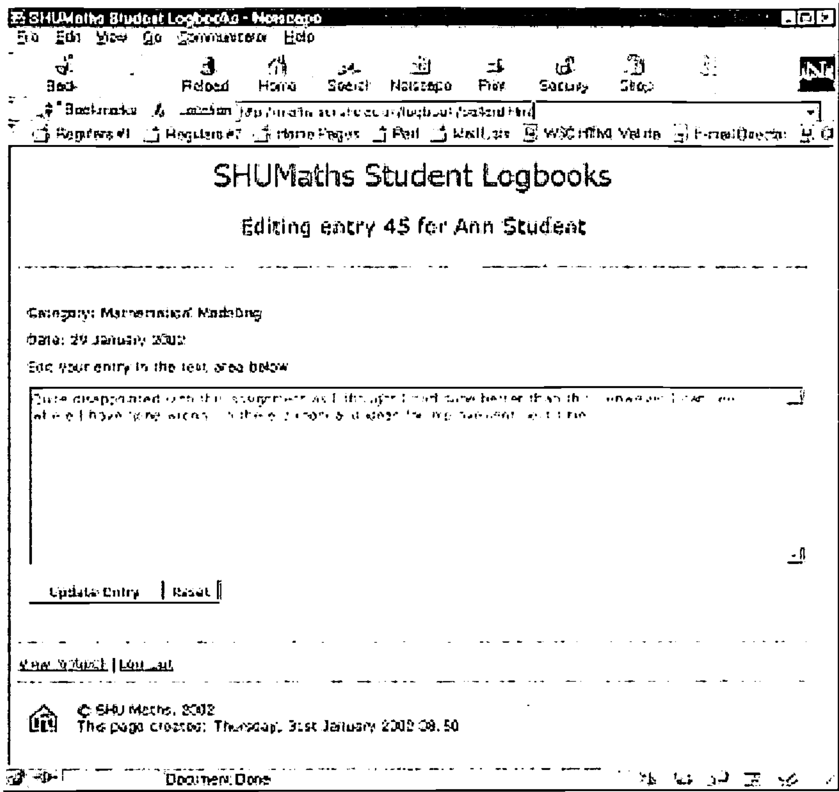

Figure 3c: The form used by students to edit an entry in their logbook

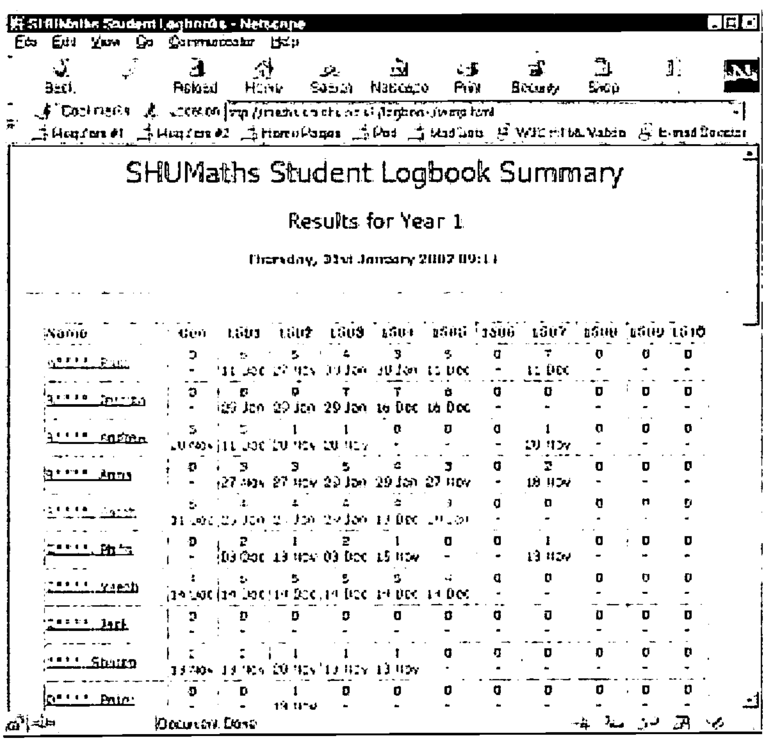

Figure 3d: The staff summary page

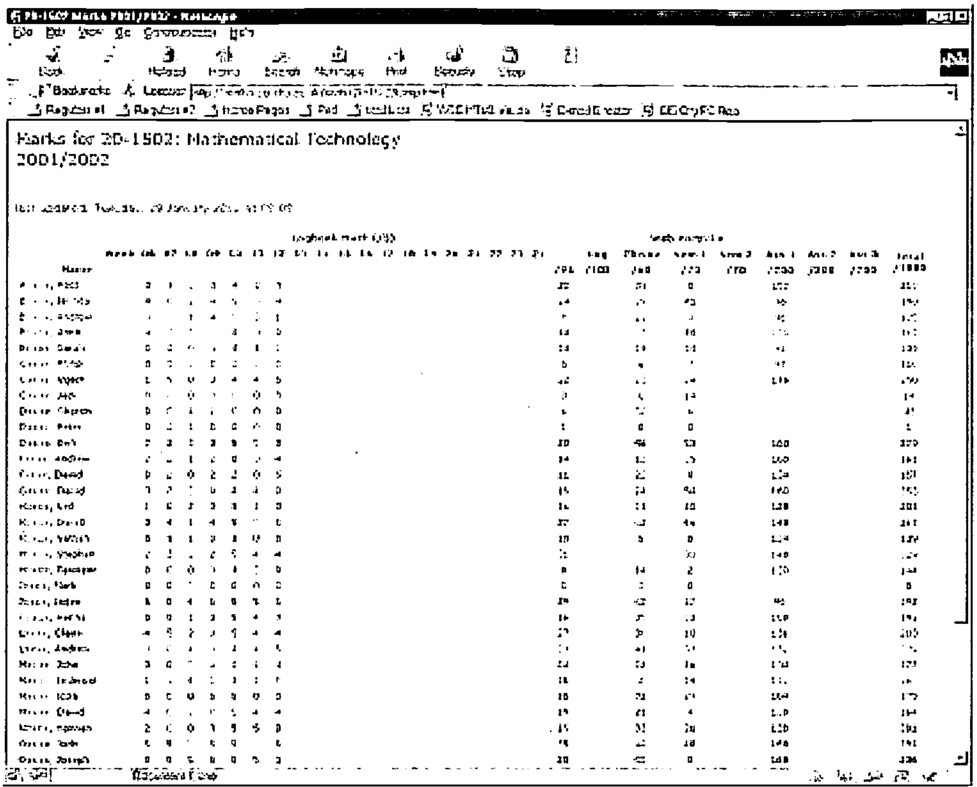

Figure 4: The marks sheet as seen on the web

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A GENETIC APPROACH TO AXIOMATICS

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ABSTRACT

The genetic method is often regarded as a counter-current to the New Math and its exaggeration of formal and axiomatic mathematics. As a consequence axiomatics has been considerably reduced (nearly deleted) in school mathematics, whereas university mathematics is mostly still presented in a rigid deductive way. This discrepancy leads many freshmen to considerable difficulties, as we all know.

In this paper I will propose a synthesis between genetic and axiomatic method. In particular the axiomatic method is not only a method but also an interesting and very important subject of teaching and research itself: a milestone in the development of mathematics (Euclid), its philosophical background (Aristoteles), its purpose (Zenon), its consequences (construction with compass and ruler). Axiomatics as a model for representation of topics of mathematics (and other sciences) up to now, axiomatics as a destination of a process, not a starting point.

Examples of how to cope with axiomatics at school and at university will be discussed in this paper.

1. Preliminary Remarks

In the fifties/ sixties of the last century one of the main goal of the New Math was to reduce the gap between mathematics at school and at universities. School mathematics tried to imitate university mathematics, which was dominated in those days by the Bourbaki style. This led

- to an overemphasis of the axiomatic method
- to a high level of abstraction and formalization
- to the disdain of heuristic approaches.

Soon educators had to learn that the New Math was doomed to failure. As a counter-current to the New Math the genetic method was rediscovered. As a consequence, intuitive approaches and heuristic methods were esteemed again. Another consequence was the elimination of contents introduced by the New Math a few years ago. With some of these eliminations I agree; other eliminations I regret. In my opinion, the proponents of the counter-current failed to notice the fact that not necessarily the contents or goals were wrong, but the result of these contents together with the traditional methods of teaching and assessment. Furthermore, I am convinced that most of the abstract concepts have been placed too early in the curriculum (e.g. the concept of group in 7^{th} grade!) and caused therefore difficulties. If these concepts would have been introduced in higher classes the students had have a better chance to get a grip on them.

Abstract concepts and axiomatics usually are starting points at the university level (in books, papers and lectures), but even at this level students have difficulties to cope with them. The history of mathematics shows that these are rather final goals resp. final steps of a (sometimes long) scientific development.

Whenever axiomatics is (or was) used (at the university or at school in the New Math period resp.) I miss(ed) a discussion ABOUT this method: What is the advantage of this method? What was the reason to invent (develop) this method? What are the problems which can be treated better with this method than without it?

2. The invention of axiomatics

In ancient Greece, among others the Pythagoreans made no mean contribution to mathematics, not only their famous theorem, but also their philosophy and their theory of music, which supported their conviction that all in the universe is ordered by ratios of natural numbers. When they became aware of the problem of incommensurability they tried to apply their methods to infinity, too. Zeno showed with his famous paradoxes that these temptations may cause difficulties: Does a line consist of (indivisible) points (atoms)? Do we get these points when we bisect the line infinitely often? Can we make up a line out of points? (For more details see Boyer 1959, 23f, Kirk et al 1983, section 327-329, Struik 1967, 44; see also: Aristotle: From the Metaphysics and Physics, in Calinger 1995, 85-90.) Mathematicians of this period encountered difficulties in answering these questions. This showed the incompleteness of mathematical argumentation and produced a deep crisis of mathematics. Such inabilities led the Greek mathematicians to look for a consolidated basis of mathematics.

The answer to this question was (based on the philosophy of Plato and Aristotle) the method of axiomatics, which we can find in Euclid's Elements: As long as the postulates and the axioms are accepted and the deductions are correct, noboby can contradict the result. This gave a feeling of certainty to the mathematicians in the discussion with critics like Zeno.

These postulates became famous centuries ago by the slogan, "construction by compasses and straight-edge". All one can construct by these "Euclidean instruments" is also deducible from the postulates of Euclid, and leads therefore to undeniable results.

The construction with compasses and straight-edge became famous especially in connection with the three classical problems of the antiquity: the duplication of the cube, the trisection of an angle, and the quadrature of the circle. (Kronfellner 2000)

Although we can find Zeno´s paradoxes in textbooks, I did not see in the student´s material their role in the history of mathematics, especially in connection with the invention of the method of axiomatics, until now.

A similar role like Zeno played Bishop Berkely with his criticism of the faulty foundation of the early calculus. (Eves 1976, 446) This can be regarded as the motivation to look for solid foundation of the calculus, a problem which needed more than one century to be solved.

3. The axiomatic characterization of the real numbers

Most of the theorems in real analysis (such as the intermediate value theorem and many others) can easily be illustrated and confirmed. For an exact proof one needs an axiomatic basis of the real numbers, which guarantees the completeness of R.

In my opinion, it is not necessary to teach the exact proofs in school. But the students should know that arguments based on graphical illustrations do not fulfil the demand on exactness which is usual (and necessary) in higher mathematics. The example of the ancient Greeks mentioned above should underline this necessity. This fact can also be illustrated by an anecdote of the german mathematician Richard Dedekind (1831-1916). When he had to prepare a lecture for freshmen at the Zurich Polytechnikum he wanted to facilitate the conclusions by avoiding arguments based on illustrations. So he came to the insight that he is missing an axiomatic basis of the real numbers. To this end, he developed the famous Dedekind cut.

4. Minimizing the axiom system

Already in ancient Greece the mathematician tried to minimize Euclid´s axiom system. The famous fifth postulate – the parallel postulate – seems not to be a proper postulate, but rather looks like a theorem. For many centuries mathematicians tried to prove this "theorem", that is, to deduce it from the other postulates. The solution that there cannot be found such a proof led to the invention of Noneuclidean geometry by Janos Bolyai.

5. Linear equations and the concept of group

For a simple genetic (but not historic) reconstruction of the development of the concept of group suitable for classroom teaching one may pose the questions:

- What do we need to be able to solve an equation like x+a=b?
- In which sets (structures) of mathematical objects is it possible to solve such an equation (with solutions within this set)?

The analysis of the solution ($a, b \in M$)

$x + a = b$ $\qquad$ $\exists e \in M\ \forall a \in M\ \ a + e = e + a = a$ and

$\forall a \in M\ \exists a^* \in M: a + a^* = e$

$(x + a) + a^* = b + a^*$

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$$x + (a + a^*) = b + a^*$$
$$x + e = b + a^*$$
$$x = b + a^* \quad (\in M)$$

shows that one needs exactly the axioms of a group. On the other hand, in every group it is possible to solve such an equation.

In some sense, these usual axioms of a group are also a counterexample to the usual temptation to minimize (or generalize) a system of axioms. In particular there exists the possibility of using the more general demands only of the existence of a left unit and left inverse elements (or right-... respectively) and to prove that these elements fulfil the conditions of right units and right inverse elements, too. In spite of this possibility, most authors demand only (for the sake of simplicity) a neutral element and inverse elements (the same elements for both sides). On the other hand they usually do not demand uniqueness in the axioms; this is proved as a theorem.

6. Once more: axiom or theorem?

What is the difference between a theorem and an axiom? Can an axiom be proved?

From my students I have to learn that such questions are not trivial! They are usually unfamiliar with these concepts. To explain the difference I use the following example:

I start teaching linear inequalities based on the axioms:

$a<b \Rightarrow a+c<b+c$

$a<b$ and $c>0 \Rightarrow ac<bc$

$a<b$ and $b<c \Rightarrow a<c$

prove further rules and apply these axioms and rules to problems. At the end of the chapter I ask the students whether we can prove $a<b \Rightarrow a+c<b+c$. I repeat that - by definition! - we cannot prove an axiom. But we can build up a new "theory" (equivalent to the previous one) based on other axioms:

$a, b > 0 \Rightarrow a + b > 0$

$a, b > 0 \Rightarrow ab > 0$

$a<b$ and $b<c \Rightarrow a<c$

Within this new system, it is possible to prove these laws, which we used as axioms in the previous system, easily as theorems. (Kronfellner/Peschek 1995, 66, problem 19137; for an extended version see Kronfellner/Peschek 1981, 139-141.)

7. Final remark

I do not completely agree with G. H. Hardy´s words, "Greek mathematics is ´permanent´,more permanent even than Greek literature. Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not." (Calinger 1995, 1) But these words should underline once more the importance of ancient Greek mathematics and mathematicians; and it should be our duty to use every opportunity to teach our students about their cognition, in particular the axiomatic method, which influenced not only the mathematics but also other sciences up to now.

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RACING TO KEEP UP

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ABSTRACT

The preparation of secondary mathematics teachers for today's technology rich classroom environment is a continually evolving process. Mathematics teachers are expected to demonstrate the ability to incorporate a variety of instructional strategies and technological tools as well as multiple assessment techniques in their teaching. Classroom technology options have expanded from the once innovative graphing calculators and data-collection devices to include more all-inclusive software packages, graphics, video clips, digital images, and more. The confluence of an increased number of technology options, stronger technological background of today's students, and the expectation that mathematics teachers demonstrate content knowledge as well as the ability to incorporate a variety of instructional strategies, technological tools, and multiple assessment techniques in their teaching finds teacher preparation institutions constantly updating their programs. This paper examines the current state of technology preparation of pre-service teachers and presents one university's approach for updating the technological readiness of pre-service secondary mathematics teachers. This update includes a description of how varied technological tools are employed in developing and assessing mathematical understanding.

Key words: teacher education, technology, assessment

1. Introduction

Preparing secondary mathematics teachers for today's technology rich classroom environment is a continually evolving process. Classroom technology options have expanded from the commonly used graphing calculators and data-collection devices to include more all-inclusive software packages, graphics, video clips, digital images, and more. The confluence of an increased number of technology options, stronger technological background of today's students, and the expectation that mathematics teachers demonstrate content knowledge as well as the ability to incorporate a variety of instructional strategies, technological tools, and multiple assessment techniques in their teaching finds teacher preparation institutions scrambling to update their programs. This paper examines the current state of technology preparation of pre-service teachers and presents one university's approach for updating the technological readiness of pre-service secondary mathematics teachers. This update includes a description of how varied technological tools are employed in developing and assessing mathematical understanding.

2. Status of Pre-service Teachers' Technology Preparation

Technology has profoundly affected how people live and work in today's global and digital economy. It has changed what students need to know and be able to do in order to be successful. Tapscott (1998) asserts that today's students are "growing up digital." Unprecedented access to information and ideas across real-time, web-based, interactive media has spurred societal changes in ways that previous technologies have not. According to Ruskoff (1996), students are natives to cyberspace; the rest of us are immigrants. Despite this characterization of students as technology-savvy, most pre-service teachers know very little about effective use of technology in education. Students have access to computers and technology skill development courses, but they have little experience with the application of technology in teaching and learning.

Student learning is enhanced by technology utilization in the following ways: (a) real-world contexts; (b) connections to outside experts; (c) visualization and analysis tools; (d) scaffolds for problem solving; and (e) opportunities for feedback, reflection, and revision (Bransford, 1999). With the emergence of new technological tools, many teacher preparation programs emphasize the active engagement of students in learning and doing mathematics through the use of real-world contexts. Modeling and solving problems based on real-world situations is more accessible as a result of e-mail contact with outside experts, as well as the computational and graphical capabilities that technology provides. The process of developing a model for the problem situation, obtaining feedback, revising the model, and reflecting on the process and product is less cumbersome when technology is employed. It is technology that offers many powerful tools for constructing a mathematical foundation that supports how children and adults learn and do mathematics (Dunham & Dick, 1994; Sheets, 1993; Rojano, 1996).

A number of surveys and reports from the late 1990's conclude that although teacher training programs have increased technology utilization, technology is not well integrated into the college classroom (National Council for Accreditation of Teacher Education, 1997; Persichitte, Tharp, & Caffarella, 1997; President's Committee of Advisors on Science and Technology, 1997). Moursand and Bielefeldt (1999) report that although technology skills of college faculty are comparable to the technology skills of their students, most faculty do not model the use of instructional technology in their teaching. Students' exposure to coursework utilizing technology is generally not tied to curriculum, instructional methods, field experience, or practice teaching. Numerous studies indicate that the instructional methods employed when teaching a pre-service teacher are important factors in shaping the pre-service teacher's instructional delivery and assessment practices (Rahal & Melvin, 1998; Raymond, 1997; Stanford, 1998; Wilcox, Schram, Lappan, & Lanier, 1991). Teachers teach as they are taught. Despite this, pre-service teacher education, is not adequately preparing educators to work in a 21st century technology-enriched classroom.

The disparity between the actual versus the desirable technology-based instructional skills possessed by recent teacher education graduates is often large. Focusing attention on understanding and minimizing this disparity, the 1998 Milken/International Society for Technology in Education (ISTE) survey on instructional technology in teacher education identifies four essential components for the instructional technology preparation of new teachers: (a) facilities for students and teachers, including Internet access, classroom arrangement, numbers and technical features of computers, technical support, and continuing funding; (b) integration of technology in learning, including faculty modeling of instructional technology usage, project-based learning and problem-solving situations, computer-assisted instruction, and experiences in varied classroom technology configurations; (c) student ability to use applications including word processing, e-mail, web browsers, and electronic grade books; and (d) field experience opportunities where instructional technology is available and actually used and with supervisors and master teachers who can model and advise on classroom technology use (Bielefeldt, 2001). Released two years later, the ISTE National Educational Technology Standards for Teachers (ISTE NETS) provide teacher education programs in the United States with comprehensive guidelines for assessing the technological preparation of pre-service teachers (ISTE, 2000). The teacher education program accreditation process offers the greatest prospect for assessing the full impact of these standards.

3. A Course for Updating Pre-service Teachers' Technological Readiness

At North Georgia College & State University (NGCSU) a major component of the effort to address the technological preparation of pre-service secondary mathematics is a course entitled Technology in Mathematics. This course combines technology-related content, pedagogy, and assessment. Students receive direct instruction on graphing calculators that includes the TI-83 Plus, TI-89, and TI-92 Plus; data-collection devices including the Calculator-Based Ranger (CBR), Calculator-Based Laboratory (CBL), and related probes; and software such as *Geometer's Sketchpad* (*GSP*), *Cabri*, *Fathom*, *TI-InterActive*, and *Excel*. However, the primary emphasis is on the demonstration of pedagogically sound instructional and assessment techniques. Modeling the appropriate use of technological tools in the learning and doing of mathematics is imperative as pre-service teachers experience the teaching behaviors that teacher education programs seek to develop in them.

The format of class sessions includes dialogue, hands-on activities, student presentations, and reflection on practice. The dialogue portion provides students an opportunity to discuss their perspective on the implementation of classroom activities and ideas for improvement based upon experiences with peer presentations. The hands-on activities portion includes instruction in the areas of content, technology utilization, and the assessment of student learning. The presentation portion is led by the pre-service teachers and includes content-related activities self-chosen from a list of options. Cooperative and collaborative learning activities utilizing graphing calculators, CBR, and CBL provide a structured basis for group projects and student-led technology presentations. The reflection on practice portion includes self-assessments and reflective logs that are designed to engage pre-service teachers in a critical analysis of their teaching performance and their selection of instructional strategies, materials, and assessment alternatives.

Instruction on specific technology is couched in the context of mathematics concepts that are appropriate for use in the secondary mathematics curriculum. Although students enrolled in this course are typically in or near their last year of college, their recollection of much of secondary mathematics is sketchy. Consequently, learning to use the technology serves as a vehicle for reinforcing, and in some cases developing, mathematics concepts that they are expected to know and be able to convey when they begin their teaching internship. Even more critical is the performance of students on a nationally administered content knowledge test, the PRAXIS 2. This test also requires students to demonstrate proficiency with using a graphing calculator and is required of all prospective secondary mathematics teachers before they can acquire a teaching certificate.

Graphing calculators are used in this course for exploring concepts of number theory, data representation and analysis, probability, discrete mathematics, rates of change, and functions. Used in conjunction with data-collection devices, graphing calculators provide media through which problem solving and connections between mathematics and other disciplines, as recommended in the *Principles and Standards for School Mathematics* (NCTM, 2000) are solidified. In particular, the CBL with its microphone, voltage sensor, and temperature and light probes has breathed fresh life into how previously learned mathematics concepts connect with other subjects and how these connections can be described. Students perform experiments both inside and outside of class that involve the investigation of coefficients of friction, sound waves, pendulum motion, heating and cooling models, and light intensity relationships. They appear genuinely surprised and pleased to discover that mathematics actually relates to everyday situations. Although the "Walk the Line" CBR activity that engages students in trying to match the graph of their walk with a given graph is commonly used in mathematics classrooms spanning many grade levels, it is the first time that a significant number of these students have been forced to think about slope in terms of a physical phenomena. An even more enlightening activity for them is using the CBR to investigate the calculus of motion, specifically the derivative and definite integral. Students capture a walk that incorporates both forward and backward motion, relate the resulting curve to velocity, and then determine the area under the curve they walked. Through such hands-on experiences, pre-service teachers are convinced that the use of technology is an effective means of explaining and predicting real-world phenomena.

On-line data sites supplying real-world data that can be represented, analyzed, and interpreted provide another opportunity for engaging pre-service teachers in explaining and predicting real-world phenomena. *TI-InterActive* has proven to be a valuable software tool for developing and reinforcing algebra, data analysis, precalculus, and calculus concepts. Because it includes a web browser, computer algebra system, spreadsheet, lists, graphs, word processor, and data collection transfer capabilities in an integrated package, students find it very easy to use. Data obtained from student activities incorporating the CBR, CBL or graphing calculators such as the TI-83 Plus, TI-89, or TI-92 Plus is easily downloaded to *TI-InterActive*. Once the data is in a list, a graphical representation is created and a regression analysis completed.

Fathom is particularly well-suited and effective for developing and reinforcing statistical concepts. Students use its simulation capabilities for conducting experiments and then analyze their results. One of the greatest obstacles that exist with students' background in statistics is their lack of true understanding of statistical concepts. Although well-versed in formulating hypotheses and conducting statistical tests, they have minimal understanding of the underlying concepts. Herein illustrates another example of where technology as an instructional tool is effective in developing deeper and more thorough conceptual understanding.

This course employs dynamic geometry programs to enhance students' experience with two- and three-dimensional geometry. Pre-service teachers complete activities that incorporate interactive geometry software such as *Geometer's Sketchpad* (*GSP*) and the TI-92's *Cabri* or *GSP* for investigating geometric concepts, making and validating conjectures, and writing paragraph proofs or justifications. Interactive geometry projects are especially effective and serve several purposes: (a) to increase student understanding of geometric concepts, (b) to actively engage students in the learning process, (c) to illustrate how the van Hiele levels of geometric thinking apply to students of all ages, and (d) to promote student enjoyment of mathematics. For pre-service secondary mathematics teachers, it is the experience of designing, selecting, implementing, and reflecting on technology-based activities that has been most effective in transferring the responsibility for learning from the instructor to the student.

Continual updating of this course is necessary as new technologies emerge. Most recently, the digital camera and the digital video recorder have been added to this course's technology arsenal. Students use digital cameras and video cameras to capture commonly occurring items and situations such as light bulbs, the path of water rising from a fountain, airplane propellers, flowers, shells, and the roofline of a building. These captured images are then used to determine an equation in function, polar, or parametric mode that models the given situation. Using this equation, concepts such as area, volume, and surface area are explored. Mathematics makes more sense and is easier to apply when connections with existing knowledge are made (NCTM, 2000). Technology facilitates the process of making connections and provides a vehicle for accessing previously inaccessible real-world applications of mathematics.

4. Assessment and Technology

Personal experience confirms the value of utilizing several formative and summative assessment techniques including presentations, reflective logs, group and individual projects, peer and self-evaluation, writing prompts, journals, and portfolios in teacher preparation programs. A key component of assessment development is the design of tasks that enable students to use and demonstrate a broad range of abilities. Activities, discussions, and student presentations are structured in a way that incrementally builds a foundation from which informed decisions relative to the selection of technological tools and developmentally appropriate instructional and assessment activities that support how children and adults learn and do mathematics can be made.

Technology-based presentations are conducted by individual students as well as by groups of students. Upon completion, student presenters prepare a one-page reflective summary describing what went well and what could be improved in future presentations. In addition, each presenter completes a self-evaluation based on a prepared rubric. Participants complete a rubric-based peer evaluation and the results are shared with each presenter.

Journals are an enlightening component of the assessment process. They provide feedback about the students' understandings, require students to explain concepts and thought processes, foster creativity and confidence, and supply a venue for students to reflect on their own learning.

The information gleaned from students' journal entries offers insight into exactly how and what students know and are able to do. Journals are a valuable tool for illuminating our practice.

Portfolios offer a means for students to self-assess their learning, to integrate what they have learned in the course, to document their intellectual growth, and to experience a process they may wish to use when they become teachers. Portfolios are also helpful to faculty in providing another source of direct evidence for what pre-service teachers know and can do. Barton (1993) identifies several strengths that portfolio usage offers teacher education programs: (a) empowerment, the shifting of ownership of learning from faculty to student; (b) collaboration, allowing students to engage in ongoing discussions about content with peers and teachers; (c) integration, making connections between theory and practice; (d) explicitness, focusing on the specific purpose of the portfolio; (e) authenticity, linking included artifacts with classroom practice; and (f) critical thinking, reflecting on change and growth over time. Portfolios are widely used in teacher education programs as a means of bringing together curriculum, instruction, and assessment. Students and teachers develop a shared understanding of what constitutes quality work. Portfolio usage leads to classrooms that are student-centered rather than teacher-centered, chiefly because students accept more responsibility for their education

Digital portfolios are quickly becoming the preferred portfolio type. The advantage of the digital portfolio lies in the broad range of technological competencies possessed by the pre-service teacher that can be captured and showcased. Possible artifacts include video and sound clips of a pre-service teacher leading a class activity, specific mathematics software and calculator proficiency demonstrations, the creation of a web page or electronic presentation, and the incorporation of digital images in a variety of media.

The culminating assessment for pre-service teachers in the Technology in Mathematics course is the creation of a digital teaching portfolio. This portfolio showcases students' proficiency with incorporating multiple technologies as instructional and assessment tools in the mathematics content area. Although the principal purpose that digital portfolios serve in the Technology in Mathematics course is evaluative, pre-service teachers report using their digital portfolio as a presentation medium when interviewing for a teaching position. A compact disc with the interviewee's documented technology skill set has become a valuable and authentic means of showcasing achievements, proficiency, and the capability to use technology to support lifelong professional development.

5. Conclusion

Mathematics teacher preparation programs face numerous challenges as they update their programs to reflect the emergence of new technologies. The teachers we prepare must have adequate mathematical and technological knowledge to provide appropriate support for today's K-12 students. Three areas for concentrated effort on the part of teacher preparation units are technological currency, instructional delivery methods, and assessment. Ensuring that the most recent tools are available for student use is a daunting and time-consuming process for faculty. Becoming an expert with each new technology is certainly not as important as modeling our willingness to be a life-long learner in search of the most effective instructional tools for the mathematics classroom. Active engagement of students in learning and doing mathematics through the use of real-world contexts is now easily achieved via technology and must become a larger and more pragmatic focus for teacher preparation efforts. Faculty must first think about students' learning in terms of actively involving them in investigating and making sense of mathematics.

The development of conceptual understanding in pre-service teachers is crucial if they in turn are to develop understanding in their students. Coupling this instructional responsibility with the need for pre-service teachers to be knowledgeable of a broad range of assessment options further complicates the process. Technology provides many unique capabilities for supporting varied forms of instruction and assessment. Increased infusion of integrated and interactive software packages, digital images, graphics, and video clips offer expanded potential for collecting information about students' performance of complex tasks and for their selection of work samples. It is important to remember that the challenges we face in preparing secondary mathematics teachers for tomorrow's technology rich classroom environment are not necessarily negative, but rather opportunities for meaningful growth on the part of faculty and pre-service teachers.

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HOT AND ABSTRACT:
Emotion and learning undergraduate mathematics

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ABSTRACT

Undergraduate mathematics students' affective responses to their studies have been collected from interviews, questionnaires and observations as part of a three-year longitudinal study of a cohort of mathematics students at two UK universities and from other opportunities from working with undergraduates and post-graduates.

The central point of this report is that emotion has a significant part to play in learning mathematics at this level. Far from mathematics being cold and abstract it is hot and abstract!

Affect has been classified into the three subdomains of belief, attitude and emotion (McLeod 1992). Attention here is on emotion, the least researched of these subdomains in undergraduate mathematics education. Reasons for the lack of attention in this area are attributed to the elusive task of tracking others' emotions as well as the abstract nature of mathematics with its concomitant 'cold' image.

This image belies the strong feelings expressed by or observed among mathematics students or recent graduates, and frustration is more prevalent than joy. Students mostly attribute their original choice of mathematics as a specialist subject to enjoyment. Enjoyment is highly correlated with skill. When these students become unable to understand the mathematics presented, frustration, fear or bitterness often arise. What role does the mathematics lecturer have in harnessing their emotion to pull them through to success? For emotional engagement, rather than just a good attitude or compatible beliefs, is the real key to desire to learn something which is abstract.

The report will be in three parts: firstly, a brief outline of some relevant literature will be given; then, secondly, a selection of data will be presented which will be interpreted to indicate the presence and importance of emotion in learning undergraduate mathematics; and finally, attention will be dawn to the role of mathematics lecturers.

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Introduction

In the UK young people generally choose a specialist subject to study at university or higher education around the age of 17 and, after recent expansion of higher education, more than 30% of this cohort participate in post-school education. However, the proportion choosing to study mathematics as a main subject at university is falling: from about 2.5% in 1986, to 1.5% in 1999 (Higher Education Statistics Agency, 1999). This paper, stimulated by this evidence of decreased participation, is part of on-going research into the particular challenges and rewards of being an undergraduate in mathematics. The focus in this report is on trying to track ways emotion impinges on learning mathematics as an undergraduate. The issue of 'emotion' was not an a priori target for the research just mentioned, but has arisen from a grounded theorising from data which includes interview transcripts, questionnaires and discussion with students (within and outwith the project).

What is special about mathematics? Don't young adults tend generally to feel an unpredictable mixture of being excited and threatened or insecure when they enter the university? After all there is considerable investment in them from society as well as from family and school whatever subject they are studying. What *is* special about mathematics is that beginning university mathematics is invariably presented as an abstract subject, without fuzziness or debateable results, which is assessed through individuals' timed exam performance. Such assessment arrangements are personal and adrenalin-producing yet the assessment's mathematics does not express any personal view. There is nothing to hide behind in mathematics: no experiment, no interpretation of evidence, no comparison of criticisms. The students are relatively more exposed – intellectually and emotionally – than in other subjects.

This paper firstly presents a brief review of literature on affect relevant to this study then offers a selection of data which indicates the significance of emotion for undergraduate mathematicians, after drawing out the centrality of emotion in learning this abstract subject, finally I briefly consider the significance of the role of mathematics lecturers/university teachers in the emotional and intellectual development of mathematics undergraduates.

From mathematics education literature

There has been considerable research on attitudes and beliefs in mathematics education (see Osborne *et. al.*, 1997, for a review), but Leron and Hazzan (1997) observe that there has been a "strong emphasis on cognitive aspects, and consequent neglect of affective and social factors" (ibid. p266). Indeed, McLeod (1992) having classified of affect into an ordered set: 'beliefs, attitudes and emotions' where beliefs were stable but less intense and emotions less stable but more intense, also remarked that emotions have not been a major area for research in mathematics education (ibid. p582) and attributes this lack of attention to the relative instability of emotion.

The principal sources of work on emotion in mathematics have focused generally on negative emotions like anxiety (gender studies, e.g. Fennema 1996) or panic (Buxton, 1981) though Celia Hoyles' (1982) survey of pupils' feelings about learning mathematics included some positive as well as negative replies. Laurie Buxton's seminal work showed through his case studies that generally successful adults could have a debilitating fear of mathematics, which could not be easily dispelled. He designed and ran 'mathematical therapy' for his subjects to enable them to re-establish the confidence essential for actually engaging in mathematical activity. Buxton's remark:

"reason is powered by emotion or, more often, hampered by it" (ibid.:3) is also relevant to undergraduates.

More recently, Jeff Evans has published his long term study on 'Adults Mathematical Thinking and Emotions' (confirming my thinking that mathematics produces 'hot' responses). Evans' study focussed mostly on anxiety, with adults of all ages and on quite a basic level of mathematics, nevertheless, the theorising he offers can be applied to the undergraduate context too. The main differences are the wider range of emotion which undergraduates exhibit (with pleasurable emotions more significant), the fact that they are all young adults (17-23 years old) and that the mathematics is 'abstract'. The core of Evans' theory is that "affect and emotion [are] inseparable from thinking, including mathematical thinking." (*ibid.* :228) and extends Buxton's work. Emotion does not necessarily 'interfere' with mathematical thinking – although panic-driven blocks do occur – rather "emotion [is seen] in terms of *charges of feeling attached to* ideas and thus related to the cognitive" (*ibid.,* italics in original: 230).

Evans' theorising furthermore employs the notion of 'practice'. This is a sociological term which signifies a set of customs, language, values, interests, tools, etc. held by a 'community' defined in turn by these customs, language, values, etc. Wenger (1998) gives a thorough working of this concept and relates a person's practice(s) to their identity (and so their feelings or emotions). Examples of 'practices' include those of being a nurse or a skate boarder or a yogi. With Evans' subjects, their identity was not essentially bound up with mathematics – as our undergraduates' identity inevitably is - so he is able to consider which practices were called upon by the subjects when they are engaged in mathematics. When considering mathematics undergraduates, the situation is more subtle, for one of the things which develops – or arguably should develop - over the period of undergraduate study, is a connection with the mathematical community and an increased sense of oneself as a mathematician. In the undergraduate context, students' investment in the practice of mathematics holds personal significance for them and is related to their other attitudes and beliefs about their course of study. Mathematics undergraduates' attitudes and beliefs have been studied by Kathryn Crawford and colleagues in Australia (e.g. 1994) showing that a fragmented notion of mathematics tends to correlate with a more superficial learning style.

This question of learning style also impinges in our discussion on emotion and learning mathematics. John Mason (1989), noting that the etymology of the word 'abstract' is to 'draw away', associates the "extremely brief moment" (*ibid.* p2) when the mathematician/student draws away from the particular to the general as the experience of abstraction. Students habituated into a superficial style of learning dare not 'draw back' - or 'abstract' – they focus on their need for rules to pass exams. A link between 'hot and abstract' can be found in this notion of Mason's that abstracting is a "delicate shift of attention" (*ibid.*) which brings together personal and mathematical processes: the undergraduate person wrestles with the subject matter of mathematics.

Now 'hot' is a metaphor for visceral energy, felt by a person; while 'abstract' in mathematics connotes ethereal, rational, person-independent generality. Why has there been, in western culture, a prevalent image of a universal or a generality as 'cold'? Could not the image of a universal have been 'like the sun' giving life and heat? Jere Confrey's answer to this question comes from her delving into prior usages of the word 'abstraction', (Confrey, 1995): the medieval priesthood associated abstraction with being "free from sin" and of course sin is of the body and hell fire! Confrey proposes to characterise abstraction by recognising: "1) a genuine dialectic between practical activity and sign use; 2) the value of multiple forms of representation; and 3) the role of action in the act of abstracting." (*ibid.* p40). Her notion of abstraction thus ties in with both Evans' 'practice' (1) and Mason's 'doing' (3).

Undergraduates' emotion and their learning of Mathematics

As mentioned already, the results of this report are not obtained from an explicit search for emotional response, rather it is from a detailed reading of interview transcripts and field notes that these issues have emerged. As students were not generally prompted for an emotional response so, recognising that our emotions are often subliminal and not available for conscious reflection, their emotion often comes out in a change of tone or juxtaposition of ideas. The quotations chosen are representative or illustrative.

Success, pleasure and belonging

We found that students are attracted to mathematics principally because of their prior success in the subject or because of their pleasure in engaging in mathematical activity, and these are linked. The following are extracts from interviews after one semester at the university:

Lucy: ... I've always liked doing maths. I've just really enjoyed it, I could do it ... I don't know, it's more enjoyment than anything else.

Stephen: Knew when you got out on good maths degree you'd usually be earning quite a lot. ...Well I wouldn't have done it if I didn't, you know, enjoy maths in some way.

Here we see that doing and enjoying go hand in hand, even when there are other motivations. While we expect initial euphoria to be tempered by the reality of course demands, we can see from the following extracts from Robert's first interview how 'up and down' he is in himself at the end of his first semester and how dependent this mood is on his ability to do the maths:

Robert: [of a lecturer] He just went through it really quickly. I don't know there was just - there was no time in between writing it down and listening to him, and trying to understand. ... I dunno cos you don't know what to do, you think, well what's the use cos you can't do anything. I'll just have to go and read it on my own and see, so.

Later on Robert says of the applied maths module:

it was a bit of a struggle and then towards the end of the term, it began to make sense and that was good.

Interviewer: Can you give me any specific examples?

Robert: I dunno. I can't think of any, I just remember enjoying it because I could grasp it, you know.

Despite this brief pleasure, still further on in the interview he says:

Robert: ... I just sort of, you know getting myself in a rut of not doing enough work and then doing badly and getting rubbish marks, I was thinking I shouldn't really be here, but I dunno, I just sort of, my mum, I said to my mum that I wasn't really enjoying it ...

Robert is on edge emotionally: he is frustrated that he doesn't understand the lectures, feels he ought to be self motivated and get the books out, is dependent on a feeling of understanding for some pleasure, and seeks nurture and advice from his mother. Another student who expresses vividly his feelings about 'doing it' and pleasure is Raj:

Raj: I've not had the, satisfaction of getting it and I miss that cos I'm falling so far behind and you know in like the A levels and in other maths when you can do it like you can kind of things get like satisfaction and so you just like get stuff like that click, clicks into place quite easily, and cos I wasn't getting that and I wasn't making an effort and I knew and I was like depressed with all that an all kind of things ...I wasn't enjoying maths and kind of thing, I'd always liked maths and was happy doing it and cos I know what makes me happy and I have to be on top of my work for me to be happy kind of thing...

Raj is one of the few students in his university's cohort who is of Asian (Indian sub-continent) ethnic background and whose parents are in working class occupations. Raj attributes some of his problem as due to a disjunction between communities. Earlier in the interview he says:

Raj: Really have to start work. It was probably the worst start I could have hoped for, but to be honest I couldn't realise, what it was going to be like cos I really hadn't a clue what was involved in going to university, hadn't a clue when I applied and then I came here and it was like the biggest eye opener I could imagine, just being here, nobody told me, nobody knew and I never knew what to ask, kind of thing cos I never really thought much about it and nobody I knew, knew much cos people at my school don't go to university, my mum and dad's never been so nobody could tell me. I can look at what I've done and that was a mistake, and now I know I've got to catch up,...

These two students illustrate how identities are bound up in different ways with their feelings about their position at the university and their ability to learn and do mathematics. Success and pleasure and belonging are closely linked.

Mathematical security

While people the world over may have different conceptions of mathematics, there is a wide consensus that early university mathematics becomes more symbolic, employs more algebra and proves its results more than mathematics experienced at school. The respect for young children's own methods of calculating which we see reflected in projects like the National Numeracy Strategy (DfEE, 1999) does not seem to have an analogy at undergraduate level. At university it seems there are less negotiable and individual ways to get results, despite the considerable debate about standards of proof (see, for example, Thurston, 1995). However, it precisely the non-negotiable aspects of mathematics which some undergraduates find enhances their feelings of security about what they are studying. The following extract from a student who spent a year at studying law expresses this sense of security:

Janusz: I feel more in control of maths in a way, I know what I have to learn, I know when I've got it right and I think I understand the structure of the course and what bits go together. I never felt that with Law, it just all felt so immense and it was like I was always doing little bits of things and I never had any idea where it would lead or how big the thing was that it was part of. I just never felt I had any control and I don't know if it was just me or if that is what Law is like, I don't know but maths is different and I'm enjoying it much more. It feels smaller somehow, or perhaps it's just how it is at the moment but I do know that maths as a subject is immense but maybe it's the way it's taught, that makes it seem as if you can manage it, do it and get it right. I don't know but it's alright so far.

'Getting it right' confirms and satisfies. Other students also remarked about the insecurity of doing an arts subject where they believed essay marks were more subject to the assessor's interpretation than their mathematics problems would be.

Role Mathematicians

Mathematics does not speak for itself at this level of undergraduate study. New abstract material requires mental accommodation – rather than assimilation of related ideas (Skemp, op. cit.). Invariably, undergraduates have to struggle in learning such new mathematical content and the ability to struggle involves energy and desire for knowing. Where does this energy come from? Fear sometimes. Sometimes the energy comes from association with practices which are favourable (e.g., talking about mathematics, working mathematical problems, etc.) and the associated projection of meaning along a chain of ideas driven by a positive emotional call (e.g. respectively: I like him and he likes maths, I can complete this problem and feel satisfaction, etc.) (Evans, op. cit.). In particular, the lecturers are important to the students. Students show considerable insight when they praise or criticise their lecturers' efforts. For example, Oliver compares his Dynamics lecturer – who uses paper aeroplanes - with others he has had:

> He puts some, what's the word, if you, kind of enthusiasm for the subject, some kind of er, he just enjoys it ... He's interested about what we're doing and it comes across. Which is good, a very good thing. I think some of the lecturers can be a bit bored with it, and if they're bored with it how are we supposed to be interested, d'you know what I mean. Cos some of it's hard and you have to get the enthusiasm and interest to try and understand it but when they seem bored you can think, this isn't, this just isn't worth the effort and you might just make the effort for the exams or something but you don't really enjoy it cos it all just comes to be a big effort to pass the exams or something.

In another related paper, (Rodd, 2002), I claim that one reason that students go to lectures is that there is a chance of having their imaginations stimulated by abstract mathematics presented by an inspiring lecturer. This resonates with Oliver's observation that when mathematics is 'hard' an extra surge of energy is required to 'go for' the understanding and this energy is sought from the lecturers, the 'guardians' of the required knowledge with whom the young adult neophytes may associate and may see as role models. (Clearly, there are gender issues here given the dearth of female university mathematicians). However, looking at Robert's early perceptions we can see that the rejection he feels from the lecturers may contribute to his feelings of alienation (see above) about the course:

Robert: Real teachers explain things so you can understand and real teachers help you. Like the teachers I had at school, they helped you but here you've just got to help yourself 'cos they're too busy doing their lecture and writing on the board, they don't even look at you. That's something I've learned and I never thought it would be like that. Some are good but some, doing the lecture is all that seems to matter to them. They come in, some of them just come in and start, like they can't wait to get going and if you've not got your pen out and that, you've missed the start. They don't speak to you or anything, they just start like they've never been away, like it's just a continuation of where they left off. That's hard to get used to. I don't like it, I don't know why they do it.

Are mathematics lecturers teachers? This undergraduate seems to think they should be. In the UK academics are given a university post by virtue of their successful mathematical research, yet their job includes teaching undergraduates, many of whom go through periods of being quite unsure about being a mathematics undergraduate.

In summary

While emotions are illusive and difficult to track, there is enough information from our undergraduates to tell us that indeed their feelings are very important and intimately bound with their learning mathematics at the university. This paper has not emphasised negative emotions like anxiety– though we have evidence of these feelings existing – but has considered the importance of pleasure, and consequences of its lack, as well as the satisfaction that mathematical completion can bring and the energising role of an enthusiastic lecturer.

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A STUDY ABOUT THE DEVELOPMENT OF KNOWLEDGE in fifth to eighth graders, subject to the same didactical intervention involving ordering relations

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ABSTRACT

The authors present, in this article, the results of a research on the knowledge about relations of 5th to 8th graders (10 to 14 years old). The study was conducted in three phases: a pre-test, a class and a post-test among 4 groups of students from different grades totaling 64 subjects of a school in the state of São Paulo, Brazil. The results showed that students used the relation "come before than" *meaning* only "come immediately before than" (restricted conception) and the ordering relation "not come after than" meaning "come before than" instead of "come before than or at the same time as". The authors analyzed the effect of a didactical intervention (the same for all grades under study) aimed at reaching the learning of effective ordering procedures as well as the overcoming of a restricted conception of the relation "come before than" besides the acquisition of the mathematical conception of the ordering relation "not come after than". In this didactical intervention, the students' production was under questioning based on the Didactical Situations Theory by Brousseau (1997).

The results show that the acquisition of a broader conception of these relations depended on the didactical intervention.

The fact that the problems diagnosed in the pre-test disregarded the grade fomented the development of a similar study among students of higher levels.

1. Introduction

Igliori and Maranhão (2000) checked, in a late study, that fifth graders restricted the meanings of the relations "come before than" and "not come after than" in solving problems. For these students, "come before than" meant "come immediately before than" and, in the ordering relation "not come after than" (equivalent to come before than or at the same time as), they did not admit "come at the same time as" as an ordering. The authors also checked if it was possible to see an improvement in the knowledge of students subject to a didactic intervention based on the Didactical Situations Theory by Guy Brousseau (1997). To this end, questions about enunciations of the following kind were proposed:

"A teacher wanted to know the order of arrival of her students. They informed her but **she could not figure out the exact order of their arrival**. Give the possible orders of arrival according to the statements students gave her.

Maria said that she came to school before Eni. Eni said that she came before Bia. Rita could not remember the arrival of her mates, but she was sure she came after Eni."

In this study, the order of presentation of characters in the problems' enunciation was not questioned as a possible didactic variable (which influences the students' performance). Besides that, it was restricted to fifth graders. Bearing in mind that establishing relations is adamant for the learning of mathematics in the various teaching levels, the authors elaborated the present study, which widened the former, based on the following questions:

a. Did fifth to eighth graders (10 to14) have the same problems diagnosed in the late research? And if they had, was the evolution of a restricted conception to a broader one different for each grade for the same didactical intervention?

b. Was the order of characters' presentation in the problems' enunciations a didactic variable?

2. Theoretical Framework

This research was developed according to the principles of Brousseau's Didactical Situations Theory (1997). In this theory, a problem regarded as a source of learning must lead the student to a reflection, which involves him in an action phase. So, the action phases are understood as researching ones, aiming at the knowledge of a mathematical object. In a dialectical process, this phase is followed by a formulation phase, which is regarded as one of explanation of conceptions by the students, usually provoked by an action phase. In this process, the validation phase provides a confrontation of conceptions explained by the students, either through debates among themselves or through questioning by teachers/researchers. The teachers/researchers must propitiate an atmosphere in the classroom that activates the dialectical process, boosting the formulation and validation phases, aiming at knowledge evolution. The problems must be conceived so that the student has the knowledge to solve it, at least in part, and that some mathematical knowledge is crucial for the complete solution. So, the problems proposed to the students in this study were conceived in such a way as to benefit both goals, that is: the use of their cultural background and the acquisition of mathematical notions. Therefore, the problems were derived from their daily routine.

3. Methodology

The research was conducted among 4 groups of 5^{th} to 8^{th} graders (10 to 14 years old), totaling 64 students, from a school in the state of São Paulo, Brazil. We had 13 students in 5^{th} grade, 23 in 6^{th}, 17 in 7^{th} and 15 in $8^{th\ grade}$. There were 3 application sessions dated a week apart always conducted by the same researcher and followed by the same observers, based in the theoretical framework.

In the 1^{st} session, the students solved the problems individually with paper and pencil. In the 2^{nd} session, a debate was promoted in each group about the students' outcome based on their answers in the previous session (made available to students). The discussions were held in two steps. In the first one, the teacher/researcher discussed effective strategies for ordering, such as the use of an arrow corresponding to the ordering relation "come before somebody" (for instance, if the group decided that "before somebody" should correspond to the positioning "on the left", an arrow was drawn pointing to the left and above it was written "before somebody"). Besides that, any positioning was checked, for each ordered character, against the problem's enunciation. In the second step, the teacher/researcher conducted discussions with the students, aiming at the acquisition of the mathematical conception of relations. In the 3^{rd} session, the students solved the problems, whose descriptions had been altered only by changing the name of the characters.

Data was obtained from the answer sheets filled out by students and from observers' notes who were present in all sessions.

We used 4 problems divided in two categories.

The 1^{st}, made up of problems 1 and 2, allowed the analysis of the order of characters 'presentation in the text as an influence over the students' performance. They were also used in class aiming at the development of effective strategies for ordering.

The 2^{nd}, composed of problems 3 and 4, allowed the analysis of the meaning attributed to the relations. These were also used in the classroom to lead students to an analysis of a possible multitude of correct answers.

3.1. The problems

A teacher wanted to know the order of arrival of her students. They informed her and **she could determine their exact order of their arrival**. Give the possible orders of arrival according to the statements students gave her.

1. Antonio said he came to school before Marcos. Marcos said he came before Sueli. Sueli said she came before Débora. With this description, can we determine the order in which they might have arrived? If this is possible, write it down.
2. Tadeu said he came to school before Elaine. Elaine said she came before Vera. Otávio said he came before Tadeu. Based on this description, can we determine the order in which they might have arrived? If this is possible, write it down.

A teacher wanted to know the order of arrival of her students. They informed her but **she could not figure out the exact order of their arrival**. Give the possible orders of arrival according to the statements students gave her.

3. Sérgio said he came to school beforc Carla. Carla said she came before Júlia. Ronaldo said he did not remember about the other colleagues, but he was sure he did not come after Carla.

a. Based on this description, can we determine the order in which they might have arrived? If this is possible, write it down.

b. According to this description, can you conclude that there is only one possibility for

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their order of arrival? If not, indicate one or more possible orders.

4. Sandra said she came to school before Vicente. Fátima said she came before Sandra. Roseli said she did not remember about the other colleagues, but he was sure he did not come after Sandra.

a. Based on this description, can we determine the order in which they might have arrived? If this is possible, write it down.

b. According to this description, can you conclude that there is only one possibility for their order of arrival? If not, indicate one or more possible orders.

4. Results and Analysis

The first two problems allowed the students to present just one correct answer. The same situation applied to questions 3a and 4a. Questions 3b and 4b allowed them to present up to three correct answers different from those presented in 3a and 4a.

The data obtained from the written answers for the tests (or problems) of sessions 1 and 3, were coded and organized in tables (Figures 1, 3, 5, 7).

The codes used were the following:

0 – supplied incorrect ordering or left questions blank;

1 – scored in problems 1e 2 and in questions 3a and 4a, or presented a correct answer for questions 3b e 4b different from the one for 3a and 4a;

2 - presented two correct answers for questions 3b and 4b different from the ones for 3a and 4a;

3 - presented three correct answers for questions 3b and 4b different from the ones for 3a and 4a.

That what was regarded as relevant from the statistical analysis to evaluate the effect of the didactic intervention was also organized in tables (Figures 2, 4, 6, 8).

	Results of 1st in 5th grade						Results of 3rd session inn 5th grade					
	Question						Question					
Code	1	2	3a	3b	4a	4b	1	2	3a	3b	4a	4b
0	0%	0%	30.77%	53.85%	23.08%	46.15%	0%	8%	31%	31%	23%	15%
1	100%	100%	69.23%	30.77%	76.92%	30.77%	100%	92%	69%	15%	77%	15%
2				15.38%		23.08%				23%		23%
3				0.00%		0.00%				31%		46%

Figure 1

Statistical analysis of experiment in 5th grade	$t_{crít}$	-3.18988	
	d.f.	12	
	p-value	<0.005	highly significant result

Figure 2

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Results of 1st session in 6th grade							Results of 3rd session in 6th grade					
Question							Question					
Code	1	2	3a	3b	4a	4b	1	2	3a	3b	4a	4b
0	4.35%	13.04%	60.87%	60.87%	43.48%	47.83%	0.00%	8.70%	30.43%	21.74%	30.43%	26.09%
1	95.65%	86.96%	39.13%	34.78%	56.52%	39.13%	100.0%	91.30%	69.57%	30.43%	69.57%	17.39%
2				4.35%		13.04%				26.09%		21.74%
3				0.00%		0.00%				21.74%		34.78%

Figure 3

Statistical analysis of experiment in 6th grade	t_{crit}	-6.95646	
	d.f.	22	
	p-value	<0.001	highly significant result

Figure 4

Results of 1st session in 7th grade							Results of 3rd session in 7th grade					
Question							Question					
Code	1	2	3a	3b	4a	4b	1	2	3a	3b	4a	4b
0	5.56%	0.00%	16.67%	27.78%	22.22%	22.22%	0.00%	5.88%	0.00%	0.00%	0.00%	5.88%
1	94.44%	100.00 %	83.33%	66.66%	77.78%	66.67%	100.0%	94.12%	100.00 %	17.65%	100.0%	11.77%
2				5.56%		11.11%				64.70%		58.82%
3				0.00%		0.00%				17.65%		23.53%

Figure 5

Statistical analysis of experiment in 7th grade	t_{crit}	-6.97555	
	d.f.	16	
	p-value	<0.001	highly significant result

Figure 6

Results of 1st session in 8th grade							Results of 3rd session in 8th grade					
Question							Question					
Code	1a	2	3a	3b	4a	4b	1	2	3a	3b	4a	4b
0	6.25%	12.50%	31.25%	50.00%	37.50%	43.75%	0.00%	0.00%	6.67%	20.00%	0.00%	6.67%
1	93.75%	87.50%	68.75%	43.75%	62.50%	50.00%	100.0%	100.00 %	93.33%	13.33%	100.0%	33.33%
2				6.25%		6.25%				40.00%		33.33%
3				0.00%		0.00%				26.67%		26.67%

Figure 7

Statistical analysis of experiment in 8th grade	t_{crit}	-7.09407	
	d.f.	14	
	p-value	<0.001	highly significant result

Figure 8

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5. Conclusions

There was no significant students' performance gap for problems 1 and 2 when the order of presentation of characters in the text was changed both for the pre-test and for the post-test. This leads to the conclusion that it is not a didactic variable.

The relevant values to interpret the effects of a didactical intervention are: $t_{crít}$. d.f.. and p-value.

The value of $t_{crít}$ stands for the standardized scoring for the statistics under study. For all the instances, the values for $t_{crít}$ were regarded highly significant. This means that there actually was a statistical shift in the students standard answer in favor of a better comprehension of the broader meaning of the relation "come before than" and of the ordering relation "not come before than". This alteration is attributed to the didactic intervention.

The statistical analysis shows that the result for 6^{th}, 7^{th} and 8^{th} grades (p-value < 0.001) was better than for 5^{th} grade (p-value < 0.005), in the present sample.

In 7^{th} grade there was the least deviation, that is, the improvement for each student was very much alike in spite of the likeliness of experimental conditions to the other grades. This can be attributed to various factors such as students' and teachers' practices in grades before 7^{th}. This will be investigated through an analysis of teachers' resumes and interviews with the teachers and coordinators of the researched school.

It should be highlighted that the pre-test results (1^{st} session) show no difference in the standard for answers among the grades as regards the non-mathematical conception of the investigated relations. This is based on figures 1, 3, 5 and 7 that show 0% of students bearing code 3 for all grades for the answers to questions 3b e 4b in the 1^{st} session. This code, for these questions, is what strikes the difference between those students who have a mathematical conception to those who do not. This proves that the existent difficulties disregard the grade and this fact foments the authors to carry the same study among higher-level students.

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DEVELOPING A PEDAGOGIC DISCOURSE IN THE TEACHING OF UNDERGRADUATE MATHEMATICS: On Tutors' Uses of Generic Examples and Other Techniques

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ABSTRACT

This paper reports from a research project at Oxford in the UK that focused (a) on university mathematics teachers' conceptualisations of first-year undergraduate teaching related to observation of their teaching; and (b) on issues relating the conceptualisations to mathematics as a discipline. This research builds on a qualitative study of learning difficulties of first year undergraduates in their encounter with the abstractions of advanced mathematics within a tutorial-based pedagogy. Six tutors' responses to and interpretations of such difficulties were studied in semi-structured interviews conducted during an 8-week university term and following minimally-participant observation of their tutorials.

We describe a 4-stage spectrum of pedagogical development (SPD) that emerged from the analysis of the tutors'

1. conceptualisations of the students' difficulties;
2. descriptive accounts of the strategies they employ in order to facilitate their students' overcoming of these difficulties; and,
3. self-evaluative reflective accounts regarding their teaching practices.

We then exemplify the third and fourth stages of SPD with regard to (2) through a discussion of characteristic examples from the interview data. In these stages the tutors' strategies begin to resemble less a traditional induction process and more a process of facilitating the students' construction of mathematical meaning. In our discussion we employ tools from sociocultural, enactivist and constructivist theories on the teaching and learning of mathematics. In particular, the data used here exemplify certain tutor strategies such as: *encouraging the students' use of* ***rich and evocative verbal descriptions*** *of mathematical concepts, properties and relationships; using* ***generic examples*** *and offering* ***genetic decompositions*** *to create and reinforce concept images of newly introduced concepts; highlighting the* ***transferability*** *of a* ***technique*** *rather than dwelling on mastering its execution; employing* ***empathetic methods*** *(pretend ignorance of sophisticated methods) to achieve consideration of students' needs.*

Overall we propose SPD as a useful pedagogic descriptor of undergraduate mathematics teaching.

1. Rationale and Theoretical Perspectives

In the UK and other countries, in recent years there has been a number of changes that have affected the teaching of mathematics at university level: the number of students attending university has increased while the number of students opting for mathematically-oriented studies is decreasing (Holton et al 2001); recruitment of good mathematics graduates to mathematics teaching is at an all-time low; profound changes have taken place in secondary education pedagogy and curriculum; the gap between secondary and tertiary mathematics education regarding teaching approaches has increased substantially (LMS, 1995); the rapid development of information technology has affected educational practice in the use of computers and calculators in mathematics instruction; finally there has been an increasing demand from universities to be accountable to society regarding, in particular, the quality of their teaching. Moreover, despite the response so far to these changes being mostly towards modifications of the university mathematics curriculum to adjust to the skills of the new intake of students (Kahn & Hoyles, 1997), there is an emerging realisation that reform should be focusing on teaching (Jalling & Carlsson, 1995). The above imply that there is a need for a revision of the underlying principles as well as the practices with regard to the teaching of mathematics at university level (HEFCE, 1996) and that this revision may need to go beyond the extensive, curriculum-based literature in this area, mainly in North America, focusing on central topics such as Calculus (e.g. Ganter, 2000) and Linear Algebra (e.g. Leron & Dubinsky, 1995). Further, and given the often strained relationships between mathematicians in mathematics departments and their colleagues in mathematics education, research that builds the foundations of collaboration between university mathematics teachers and mathematics educators is crucial and, given the pressure currently exercised on universities regarding the need for a scrutiny of their teaching practices, timely. The research project we draw on in this paper aimed at contributing in this area.

Given this state of affairs pedagogical research involving the undergraduate mathematics teacher is limited (e.g. see (Burton and Morgan, 2000). Indeed the research on teacher thinking processes that has informed our study is largely located in the secondary sector (e.g. Brown and McIntyre 1993; Jaworski 1994). In the words of Brown and McIntyre this influence can be described as 'making sense of teaching from the perspective of teachers themselves'; 'how they construe and evaluate their own teaching, how they make judgements, and why in their own understanding, they choose to act in particular ways in specific circumstances to achieve their successes' (p1). This theoretical perspective is relatively new (until the 50s the focus was mostly on the *didactics* of particular *topics* and in the 50s and 60s teachers' classroom *actions* also attracted research focus; it was in the 70s that researchers realised the necessity also for a systematic study of teachers' *thinking*). Since then several models that attempt to describe teachers' thinking processes have emerged (see for example Brown and McIntyre (1993) and Morine-Dershimer (1990) for relevant reviews).

Often however these attempts suggest 'deficit' models of teaching: in interviews, for example, expert teachers tend to focus on atypical situations of their teaching perhaps because they perceive most of their classroom actions as so ordinary and so obvious as not to merit any comment. As a result, the researchers' attention too tends to be directed mostly towards the *problems* (rather than *achievements* of their craftsmanship) which teachers experience and which they often choose to discuss. This 'deficit' model of teaching is unsatisfactory: innovation needs to take account of what is already being done in classrooms. Moreover no evaluation of teaching can be valid in the absence of extensive and systematic observation of actual teaching and of knowledge on how teachers conceptualise their own teaching. The study we report here seeks to explore the professional *craft* knowledge of undergraduate mathematics teachers - allowing, hopefully, some space for the 'bonus' of what Brown and McIntyre (1993) call 'teachers' flashes of artistic genius'.

Fundamentally, this research has tried to gain insights into the undergraduate mathematics teacher thinking processes through the complementary lenses of the following three theoretical perspectives (for more detail see (Nardi, Jaworski and Hegedus, submitted):

- Sociocultural theory, particular its enculturative dimension in which participants in a social community are seen to be drawn into the language and practices of the community and to develop knowledge through communication and practice (e.g., Vygotsky, 1962; Lerman, 1996; Wenger, 1998);
- Constructivist theory, particularly its account of individual sense-making of experience, and related cognitive models and structures that describe and explain the construction of knowledge (e.g., Cobb, 1996; Confrey, 2000)
- Enactivist theory, particularly its aspect of codetermination, in which living beings and their environment are seen to stand in relation to each other through mutual specification or codetermination (e.g., Dawson, 1999; Varela et al, 1991; Kieren, 1995)

We now briefly introduce the methodology of the *Undergraduate Mathematics Teaching Project* (UMTP) – for more detail see (Jaworski, Nardi and Hegedus, submitted).

2. UMTP and the Spectrum of Pedagogical Development

The Undergraduate Mathematics Teaching Project is a one-year qualitative study funded by the Economic and Social Research Council in the UK and was motivated by an earlier study of undergraduate tutorials (Nardi, 1996) which indicated the richness of the tutorial context in learning *and* teaching incidents. Participants were six experienced mathematicians who acted as tutors to first year undergraduates. Data collection took place over one university term (8 weeks - a third of the academic year) with one member of the research team observing one or two tutorials (each of one hour) for each tutor per week, and conducting one half-hour interview per week related to the tutorial(s). Thus, data consisted of about 75 hours of audio-recordings from tutorials, plus associated field notes, plus 45 audio-recorded interviews each of 30-45 minutes, transcribed fully.

The questions for the semi-structured interviews were directly related to instances from the observed tutorials and to the theoretical perspectives of the researchers. The analysis of the interview data, drawing from data-grounded theory techniques (Glaser & Strauss, 1967), was initiated by the construction of interview protocols: factual summaries of the interview contents. Two levels of coding were undertaken, one mathematical focused and one pedagogically focused. The most commonly occurring pedagogical codes were found to be:

REC STU PRO	Recognition of and reaction to students' problems, needs and abilities.
TUT OBJ STU LEA	Tutor's objectives for students' learning.
TUT MATH STR STU	Identifying mathematical strategies for students.
DIFF TUT HELP STU	Tutor's difficulties in deciding on an approach to help students overcome difficulty.

Alongside the coding process, 82 significant episodes were extracted from the data, approximately two from each week for each tutor. These episodes were set against the most

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commonly occurring codes and a subset of 32 episodes was found in which these codes were most evident. Further analysis of the 32 episodes was undertaken and presented in tabulated format that included: details on the episode such as its duration and position on the recording, name of tutor, mathematical content and associated codes; a brief description of the episode content; the fully-transcribed part of the interview that constituted the episode; and an analytical account. Scrutiny of the analytical accounts across the 32 episodes led to identification of themes which in turn led to what we call a 'spectrum of pedagogic awareness or development' which is the focus of this paper.

The 4-stage *Spectrum of Pedagogical Development* (SPD). The *spectrum of pedagogic awareness*, or *development* emerging from this research sought to capture aspects of tutors' pedagogical thinking as expressed through their articulation of teaching issues in the interviews. There seemed to be a number of levels of awareness which were captured under four headings, forming a progression or spectrum as follows:

I. Naive and Dismissive: acknowledging ignorance of pedagogy; recognition of student difficulties with little reasoned attention to their origin or to teaching approaches that might enable students to overcome difficulty.

II. Intuitive and Questioning: involving implicit and hard to articulate but identifiable pedagogic thinking; recognition of student's difficulties with intuition into their resolution, and questioning of what approaches might help students.

III. Reflective and Analytic: including evidence of awareness in starting to articulate pedagogic approaches and of reflection enabling making strategies explicit; clearer recognition of teaching issues related to students' difficulties and analysis of possibilities in addressing them.

IV. Confident and Articulate: involving considered and developed pedagogic approaches designed to address recognised issues; recognition and articulation of students' difficulties with certain well-worked-out teaching strategies for addressing them; recognition of issues and critiquing of practice.

We use the term 'spectrum' to indicate a sense of continuum, with sharp points. Episodes might fit neatly into a category but, more typically, characteristics would shade between categories. We also need to emphasise that these are not categories of teacher or tutor. They reflect particular teaching events or approaches: different tutors exhibited different characteristics at different times. The nature of the research, in asking tutors about their teaching, encouraged (or maybe even required) tutors to reflect on their teaching. Research has shown that such encouragement leads to teachers taking a more questioning, enquiring and articulate attitude to their teaching (Jaworski, 1994). We recognise, therefore, that the pedagogic articulation and development we report are to some extent outcomes of the research itself.

3. Exemplification and discussion of SPD Stages III and IV

Analysis, discussion and exemplification of the data was arranged along three strands that have emerged from the typically recurring codes in the 32 episodes as follows: REC STU PRO underpins Strand 1 (the tutors' conceptualisations of the students' difficulties); TUT OBJ STU LEA and TUT MATH STR STU underpin Strand 2 (the tutors' descriptive accounts of their practices with regard to these difficulties) and DIFF TUT HELP STU underpins Strand 3 (the tutors' self-reflective accounts regarding these practices). Here, through two characteristic examples, we exemplify the third and fourth stages of SPD along Strand 2 (for a more panoramic presentation see Nardi, Jaworksi and Hegedus (submitted) where we present 12 characteristic examples, four within each strand and where also each example is preceded by a brief description of the larger pool of examples from which it has been drawn). Note: we indicate where a part of a sentence in the transcript has been omitted with [...].

Strand 2 The tutors' descriptive accounts of their practices

If formalisation and abstraction are respectively the driving force and the aim of official mathematical communication, materialised on the basis of a number of conventions that are characteristic of the formal mathematical culture, then their adoption is synonymous with a learner's advanced mathematical enculturation (Sierpinska 1994). This process of enculturation may take place with varying degrees of responsibility and ownership between the tutor and the learner. Stages I-IV are described here in terms of these degrees and exemplify tutor practices within Stages III and IV.

At Stage I the tutors perceive their role as being in charge of enculturation. In Hall's terms (Sierpinska 1994) the learner's mathematical enculturation is seen as taking place at the 'informal level': through the accumulation of mathematical experience shared with the expert, the tutor, and through appropriation of the expert's cultural practices. These cultural practices constitute the new-to-the-students habitat of mathematics. The incidents here suggest that the tutors, while recognising the students' difficulty with adopting these practices, appear apprehensive or unaware about the role of teaching in overcoming this difficulty.

At Stage II, the attempts at the enculturation exemplified in the incidents at Stage I, are more focused and more organically informed by the students' needs. In most of the incidents here, the tutors elaborate students' difficulties and employ this elaboration to justify their pedagogical strategies. These strategies include: facilitating the students' resorting to the familiar, previously established knowledge; disentangling students' misconceptions through exposition of correct definitions; enculturating students into the importance and uses of formal mathematical notation and language; enculturating students into the importance and necessity of formal mathematical proof; demonstrating and developing an arsenal of techniques to be used in establishing formal mathematical arguments, e.g. in the context of convergence of sequences and series; suggesting mathematical arguments which optimise the ones suggested by the students; highlighting the epistemological significance of newly introduced concepts, e.g. the concept of coset. Engaging the students in this enculturation process is implied in the tutors' intentions but enacted only to a limited extent.

At Stage III the attempts at the enculturation exemplified in the incidents at Stages I and II, begin to resemble more a process of facilitating the students' construction of mathematical meaning than an induction process. The tutors here openly consider the students' learning and this consideration informs directly their pedagogical practice. The strategies suggested by the tutors here include: disentangling misconceptions through thorough scrutiny of the students' written responses; supporting the construction of mathematical meaning via highlighting the usefulness of verbally describing concepts, properties, relationships etc while remaining alert to what does not carry across from language to mathematics; establishing the importance (necessity and relevance) of formal mathematical reasoning (various ad hoc practices are suggested); coping with the students' reluctance to apply formal definitions (various ad hoc practices are suggested); encouraging the identification of patterns; strengthening students' perseverance on solving a problem by contrasting (under)evaluations of their own work and their actual progress on the problem as well as by providing problem-solving 'tips' (various tips are suggested); determining content of the tutorial on a carefully balanced combination of pragmatic, pedagogical, epistemological and cognitive grounds; using generic examples to create and enrich concept images of newly introduced concepts.

Example 2.III (Strand 2, Stage III): Using generic examples to create and enrich concept images of newly introduced concepts. Amongst the most discussed strategies that the tutors use in order to assist their students' concept image construction (Vinner & Tall 1981) is the use of examples that embody the essential features of the newly introduced concepts. This has been observed to be a central function of the majority of tutorials as opposed to the more definition oriented, condensed character of the lectures. For example: in the following extract the tutor discusses the role of generic examples in the context of newly introduced topological concepts such as open and closed set of a metric space:

> Tutor: ...as a tutor you're in a position where *[pause]* you know what the relevant examples are which spell out every pitfall and [...] you want to present them with an example which contains all the *[pause]* relevant, um, features and, and phenomena. So you don't want to give an example and say this is your typical open set or something, 'cause it might give them loads of prep- misconceptions about things and so, but *[in this case]* it was a good opportunity to do that. The fact they asked me about metric spaces gave, gave me a chance to explain, you know, the difference between an open and not, um, sorry, not-open and closed and, and er *[pause]* to see why it's not a crazy thing to think of, you know, the closed interval zero one as being open in itself [...] And, but it's actually very important [...] to show that the zero, one closed is open *[pause]* Doesn't look very open if you sit in $\boldsymbol{R}^2$. [...] it's just, it's just a feature of the space you're working in. I think that's, that's the only problem they'll have in metric spaces. I think that's the standard problem that all undergraduates have is, they always, they always have, they carry this baggage with them like in every other subject, you're trying to remove the baggage and make them *[pause]* think in the way you want them to think. And the baggage they carry into metric spaces, the intuition, the trick's there, it's the ambient space, they all work in the bloody ambient space!

In the above passage on learning-as-construction, the tutor explores the incessant state of conflict and accommodation his students' concept images appear to be in. In particular the chosen example from Topology incorporates significant linguistic and geometric elements that are known to exert strong influence on students' understanding of new topological concepts (Dubinsky & Lewin 1986). The account is significantly strengthened by certain vivid metaphorical associations – such as 'doesn't look very open if you sit in $\boldsymbol{R}^2$' which seems to allude to a physical embodiment of mathematical ideas (an area of investigation which is currently under vigorous development in works such as Lakoff & Nunez 2000).

At Stage IV the tutors' pedagogical strategies are strongly determined by their intention to engage the students with their own learning and make them active participants in the construction of new mathematical meaning. These strategies include: facilitating the students' construction of new concepts; facilitating the students' acceptance and enactment of formal mathematical proof; enabling students to disentangle misconceptions; suggesting mathematical arguments which optimise the ones suggested by the students; highlighting the transferability of a technique rather than dwelling on mastering its execution; enabling students to overcome the inefficiency of a compartmentalised view of mathematics; devolving responsibility for learning; employing empathetic methods (pretend ignorance of sophisticated methods) to achieve consideration of students' needs (see Jaworski, Nardi & Hegedus 1999 for further elaboration); offering genetic decompositions (Dubinsky & Lewin 1986) of new mathematical concepts.

Example 2.IV (Strand 2, Stage IV): Overcoming the inefficiency of a compartmentalised view of mathematics. Having observed their students' attempts at problem solving often being severely curtailed by the compartmentalisation of the university mathematics course in deceptively distinct topics (in our data the tutor quoted in this Example elaborated on the potentially damaging effect of this compartmentalisation) the tutors perceive the overcoming of this inefficiency as a major part of their role. In another Example (under Strand 1, Stage III), the tutor, engrossed by her students' convoluted attempts at a question, where a substitution from Analysis would have provided a one-line answer, she referred to the possibility of seeing parts of Probability Theory in conjunction with parts of Analysis, under the wider umbrella of Measure Theory. In the following extract she expands on her role to alter this compartmentalising attitude:

> Tutor: ... the analogy of integration is interesting because I always try to convince them that summation and integration are really the same thing. Because they are, it's just Measure Theory. Um, but it makes life much easier if they can think of sums as integrals and so I do tend to try to do the two together. *[explains the details of doing so in the particular Probability question]* And we'll come back to it when they have to do it again. And they will see this again. This is something that comes up all the time but they've now got the idea and they can worry about it a bit. [...] I mean constantly trying to do these links.

This statement epitomises one of the main characteristics of this tutor's teaching practice, the necessity to make links between mathematics in other courses and links within a single course itself. Later in the interview, in a part omitted here, she exemplifies the methods she employs to pursue this objective: she uses what she calls 'leading diagrams'. The tutor's main aim in making links is to develop a mathematical awareness that enables the student to fit all the pieces of the jigsaw (Analysis, Probability etc.) together.

4. SPD as a useful pedagogic descriptor of undergraduate mathematics teaching

The evidence from UMTP supports *the value of reflection within practice* (e.g. Schon 1987; Brown and McIntyre 1993). In general the tutors' response regarding the significance of this observation-interviewing process as a means of triggering immediate and long term valuable reflection was overwhelmingly positive. Evidence of this was extracted from the parts of the interviews coded as SIGN Q (Tutor highlights a significant event from this week's tutorials) and UMTP METH (The tutor makes an evaluative comment regarding the observation-interviewing procedure of UMTP) – see Nardi, Jaworski and Hegedus (submitted) for a detailed exemplification). This evidence suggests that the explicit intentions of UMTP to engage the tutors in a non-deficit discourse on their pedagogical practice were being conveyed, at least as the data collection period was evolving. The majority of the comments were reflective / pro-active. Can we see then in this self-reflective, pro-active process - implemented in the context of UMTP as a part of the research process - the seeds of a pedagogy for undergraduate mathematics teaching? Can we see, in other words, an undergraduate mathematics teacher's development as *the route from Stage I to IV of the Spectrum of Pedagogical Development?*

As UMTP explored the pedagogical practices of the tutors from an explicit non-deficit point of view, the fruits of this exploration were remarkable. The tutors' perceptions of student learning and practices could often be embedded in the findings of current cognitive and educational research (see examples in (Nardi, Jaworski and Hegedus, submitted)). This embedding could also be made with regard to the pedagogical strategies employed or suggested by the tutors (such as the use of generic examples in Example 2.III). We wish to propose that the relationship between reflection-in-practice (as provided here by the tutors) and the findings of educational theory (the concept-based research works in the area of advanced mathematical thinking and the sociocultural, enactivist and constructivist theories that were the lenses through which learning and teaching were explored in UMTP) can be a strong one.

What UMTP provided was a context in which educational theory could emerge from a close observation of practice but also a context in which tutors' practice could be informed by an intensive exercise of self-reflection. The claim here is not that in the tutors' struggle to express their perceptions of pedagogic issues (with all the ums, ers, and repetitions) the articulated insights and issues have not been thought about by educators or researchers, but that these are genuine insights for tutors who have given little thought to pedagogy previously. This was an opportunity for the inception and growth of pedagogic ideas – not as a revelation to the mathematics education community – but in demonstration of an evolving growth of awareness of mathematicians and tutors about pedagogy. We might thus suggest that what we report here are insights into how tutors just begin to be reflective on their practice, and how discussions with educators can facilitate this process. So we have here not only findings from a research project, but indications of a way ahead in encouraging pedagogic growth in teacher/tutor development in university mathematics teaching.

Having focused intensively on the elements of effective practice in the tutors' teaching, our findings directly point at the potential of the above outlined dialectic relationship. More action-oriented research in this area is needed in order to substantiate this potential.

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"COOLING-OFF": THE PHENOMENON OF A PROBLEMATIC TRANSITION FROM SCHOOL TO UNIVERSITY.

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ABSTRACT

This paper will investigate the transition from School to University focussed specifically on mathematics. It will explain how students negotiate their response to the changes in the dynamics of the teaching and learning milieu. In particular, it will consider an important new viewpoint on the well-documented cognitive difficulties that first-year Mathematics Undergraduates encounter: their developing loss of interest in mathematics which we call the "cooling-off" phenomenon.

The paper is focussed on a study based on a close qualitative observation of 12 students who were followed from the last year of school through their first year at a prestigious mathematics department at a UK university. The data illustrate the development of the attitudinal profile of the students and the persistence of their beliefs about the nature of mathematics. We will consider how these persistent beliefs influence their 'didactical contact' (their view of their role and the teacher's role in the teaching/learning process). Comparing extracts of interviews from both school and university will highlight some of the subsequent difficulties in students' abilities to engage with learning and doing advanced mathematics. We will develop a theory which links the characteristics of the "cooling-off" phenomenon which, we hope, will simplify our understanding of some of the affective aspects of the transition to advanced mathematical thinking.

The paper will finally propose ways in which the mathematical community can diagnose the symptoms of "cooling-off" phenomenon and embark upon an adjustment of the mathematical courses in order to deal with it.

KEYWORDS: TERTIARY EDUCATION, ATTITUDES

Introduction

> "University mathematics is a lot like trying strong cheese-really difficult to swallow until you get used to it. Even then though it still tastes odd." (*First year undergraduate in mathematics*)

The above comment is representative of the views expressed by many first year undergraduates in mathematics we have come across (and there is no reason to believe that it does not represent views of many mathematics students more generally).

The issue of the transition from school to university in the case of mathematics has been of great concern in the field of mathematics education (Hoyles *et al*, 2001; LMS (1995); Robert & Schwarzenberger, 1991) by focusing mainly on the epistemological and cognitive difficulties first year undergraduates face upon their entrance to a Mathematics Department. As Tall (1991) argues "the formal presentation of material to students in university mathematics courses [...] involves obstacles that make the pathway very difficult for them to travel successfully". Bibby (1985) refers to that difficult pathway for students as a critical disorientation experienced by them due to the content of Analysis "in the sense that it regards as problematic what the student has taken for granted hitherto" (p. 48) and due to the rigour and formality in the style of teaching and learning.

However, as Sigel (1982) notes, cognition and affect are embedded in the same schemas and should be treated as equal components of the schema formed by one's experiences. The same could be argued in the case of the transition from school to university and as the research of Meyer & Eley (1999) reports, negative affective dispositions towards mathematics, could predispose students not to apply more elaborative learning processes.

Nevertheless the current studies concerning the transition from school to university mathematics provide only a general observation and description of the affective difficulties first year undergraduates face, without examining in depth the further consequences of students' difficulties with advanced mathematics setting and environment. Cooper's (1990) reinterpretation of Clark's (1960) theory of "cooling-out" reveals that students show a general tendency to lose their interest in mathematics after their transition to university, but his findings do not explore the nature of the development of this phenomenon.

The evidence from our research not only supports Cooper's theory but also elaborates on the characteristics and dimensions of what we are calling the "cooling-off" phenomenon: students' developing loss of interest in mathematics due to a combination of cognitive and affective factors with a focus on the persistence of their mathematical beliefs. In the case students' behaviour is characterised by more intense characteristics, the "cooling-**off**" phenomenon appears to have even more serious consequences in the students' academic performance and develops into what we are calling the "cooling-**out**" phenomenon.

The data presented in this paper will illustrate the developmental and affective nature of the "cooling-off" phenomenon by outlining the attitudinal profile of two students representative of the "problematic" student behaviours through the transition from school to university.

METHODS OF DATA COLLECTION

The paper is focussed on a study based on a close qualitative observation of 12 students who were followed from the last year of school through their first year at a prestigious mathematics department at a leading UK university.

Semi-structured interviews were conducted with the students while they were in their last year at school and another four series of interviews were conducted with them during their first year at the university. In addition the gathered data were triangulated though students' responses in attitudinal questionnaires and the attendance of their university supervision classes. The data that will be presented here are extracted from the first interview at school and the second one at the university in the middle of the first term, in order to include a fair range of the attitudinal development of students. At the end of the school interview, a mathematical task was given to the interviewees, adapted from Mason, Burton & Stacey (1982): "A four digit palindrome is always exactly divisible by 11. Is that true?". The mathematical task for the second interview at the university was: "How many natural numbers satisfy the inequality $3^n \leq n^3 + 1$?".

The Student Types

The documented lack of a universal definition for attitudes (Kulm, 1980; Triandis, 1971) makes their measurement very difficult and therefore hinders the formation of a student's attitudinal profile. In our research, a student's "attitudinal profile" consists of their beliefs about the nature of mathematics, their beliefs about the teaching and learning of it and their previous experiences with it.

The analysis of our data suggests that there are two "problematic" student behaviours according how students deal with university mathematics, the degree to which the "cooling-off" phenomenon can be observed and the gravity of its consequences for students' further development on their mathematics course. The different student types are: the students who are expressing signs of the "cooling-off" phenomenon and the ones who are expressing signs of the "cooling-out" phenomenon. In the following paragraphs we will describe the characteristics of each one of these student types by presenting extracts from interviews with two representative of the above categories students: Kenneth and Katherine during two different interview times, once at school and once at the university.

"Cooling-off" type: The case of Kenneth

The main characteristic of the "cooling-off" types is that these students start with a quite positive attitude towards mathematics while they are at school as this is expressed through positive feelings about their interaction with school mathematics and through positive, although rather restricted beliefs, about the nature of it.

The preferences of these students concerning mathematical topics are also very "systematic" because they favour exercises and topics where "working towards a definite answer" is the key goal, as Kenneth said in the attitudinal questionnaire distributed to students at school. The "cooling-off" types also have an inclination towards mathematical problems where they can count on the "security" of a known topic or method or even in the teacher's guidance.

I: Why do you think that Further Maths is more difficult?

K: Uhm, it's just like different topics, like Complex Numbers and things like that which...well take a bit more time. 'Cause things like, with the Statistics it's really straight, it follows on from Normal Maths that is pretty easy, but uhm, the Pure Maths it's just like...a bit different. It doesn't really follow up from what we've done before.

The way Kenneth initially attacked the mathematical problem was through the testing of numerical examples in order to find a pattern that worked. His initial approach did not include any formalisation and it was until further on during the solving of the exercise that he realised the need for a mathematical proof but could not proceed with it. Only after being prompted for the general representation of a four-digit palindrome he managed to produce an algebraic formula for it (yxxy = ya +xd, where a = 1001 and d = 110) and justify his answer. His reflections after the end of the mathematical task were very indicative of his dependence on known methods:

K: Uhm...probably difficult to know where to start because basically I didn't know where to start at all! And 'cause we haven't done anything like that at A-level, uhm...

When the "cooling-off" students enter university they are confronted with a teaching, learning and working environment that is not only different from the one experienced so far but also from the one they know how to function in successfully. The experienced mismatch between their beliefs about the nature of mathematics and its rigorous university character soon makes them lose their interest in mathematics and develop a negative attitude towards it.

I: Uhm, so uhm, could you tell me how do you find Maths at the moment? In what stage you are...

K: Uhm, I'm finding it quite difficult, uhm mainly because it's really quite a lot different to A-Level Maths, and it's not like where you, 'cause at A-Level Maths you just like got told a method of doing something and then you just had to apply that to different questions whereas here it's more like sort of, I don't know proofs and stuff like that. [...]There are quite a lot of things, especially in Analysis they were like, you had to prove these things, but quite a lot of them, I mean you could just look at, and say that looks true, whereas...so you weren't proving anything.

The first signs of the "cooling-off" phenomenon make their appearance from the first week of the course with their most intense expression around Week 4 of a UK university, meaning slightly before the middle of the term when the students can no longer cope with both the advanced content of mathematics and their loss of interest in it. But what differentiates the "cooling-off" types from the other student types found in this research is that they gradually manage to adapt to the new environment after the "peak" point of their "cooling-off" route, their attitudes towards the course and university mathematics start to "warm-on" and they put more effort in adjusting their beliefs to the new status. The following passage from Kenneth's interview could show this:

I: Right. You said it got better after some time, could you, sort of,..two questions, could you define more or less when, could you tell me from what point and on it became better, and better in what sense?

K: Uhm, I think sort of from the start of last week, start of week 5. I'm still not finding it sort of easy, but, especially with the Analysis booklet, I sort of understood it more and, now I'm getting used to like how you structure the proofs and how you write them. Getting more used to doing things like that, so that's got a bit better now.

Once again Kenneth approached the mathematical task given to him at the end of the interview by trying out some numerical examples. He was quite willing to proceed with the solution of the exercise and he recognised himself the need for a mathematical proof after his informal justification of the result, although he finally gave only a verbal proof. When Kenneth was asked to reflect upon the mathematical problem after solving it he said:

K: I don't think it was too difficult because just by putting numbers in that's like a way you could start it, so it's not like you looked at it and you didn't have any idea about how to go about it, so just by playing around with it you can get some ideas.

We believe that Kenneth's mathematical reaction during the solving of the exercise was indicative of his regained interest in engaging himself with mathematics in particular and of his "warming-on" behaviour in general.

"Cooling-out" type: The case of Katherine

The characteristics of the "cooling-out" types are very similar to the "cooling-off" ones at least when the students are still at the school environment. The starting attitude is also very positive and these types of students also enjoy doing mathematics. The difference at this point is at the focus of attention of the students. Although their preferences concerning mathematical topics include topics where the exercises have "an exact answer to be found at the end", as Katherine responded in the school attitudinal questionnaire, it is the "convenience" of mathematics that attracts them the most by emphasising on correct answers obtained in a short time-span, as the extract below indicates:

KP: I'd always liked maths. I think I just, uhm during the lower school and GCSEs I just got sick of writing essays! And I preferred the scientific approach, just an answer and sort of short explanation answer rather than 3 pages essay!

The students who belong to this category usually seek for the teacher's guidance in the solving of exercises instead of counting on independent work and like working towards definite answers through applying an already acquired technique.

The way Katherine initially approached the mathematical task was by specialisation through the use of numerical examples. However she was not feeling confident at all about her chosen technique and when she reached an (incorrect) result could not prove it "mathematically" as she said. Reflecting back upon the mathematical problem Katherine justified her mathematical behaviour during solving it:

I: How did you find this exercise?

KP: Uhm, it's quite interesting. I didn't have a clue where to start.

The university environment of instruction and learning has very critical consequences for "cooling-out" students. The "cooling-out" students find themselves in a situation where their beliefs about the nature of mathematics and the setting of exercises are no longer appropriate for their adjustment to university mathematics. That has as a result the students' diversion of attitudes towards negative scales and the subsequent appearance of "cooling-off" signs. What then differentiates these students' behaviour from the previous' category one is the intensity of these signs and their persistence, which make their performance in the university setting almost unbearable. The following extract from the interview with Katherine is representative of the "cooling-out" behaviour:

I: Uhm, I've only got one big question to ask you which is how do you find Maths until now?

KP: Incredibly difficult! Uhm, yeah, uhm, I'm finding it so hard! Uhm, all of the courses. With the exception of Statistics. Stats Lab 1 I can do because basically we haven't done anything yet, that I didn't cover in my A-Level. And, I'm finding that really easy, which is good because I know that there is one lecture which I can go to where you can go "yeah I know what that is", but all the others are a nightmare!

The first signs of "cooling-off" are revealed from the very beginning of the first academic term but they also continue to be present even after the middle of the term with their intensity remaining at high levels. The "cooling-off" behaviour blocks students from adapting to university mathematics and also creates a feeling of personal disappointment, which in its turn prevents them even more from making an effort to adapt to it. As Katherine says:

I: Right, so overall how do you find maths, apart from difficult, as you said before?

KP: (*hesitates*) Well I don't really know what else to say, it's just very difficult! I just feel so lost, with all of it at the moment.

The same attitude could be manifested during Katherine's attempt to solve the mathematical task given to her. Her initial reaction to it was: "Oh my Goodness! I don't know where to start." Katherine not only was lacking significantly in confidence but also was not willing to proceed with her starting strategy for the task: "the only thing I can think of is logs, but I don't really know how you would use them." Her lack of interest to work with the mathematical problem was evident throughout the whole time dedicated for its solution and her reflections upon it illustrate it very successfully:

I: How do you find this one? What do you think of this exercise?

KP: It's horrible; I just don't know where to start. I really don't know where to start.

Katherine's mathematical behaviour was very representative of the "cooling-out" types regarding her lack of motivation and even denial to engage herself with the task.

Generalisation

The close observation of all twelve students during the realisation of this study confirms the similarity in the behavioural route of the ones who belong to the same category as Kenneth through a gradual "cooling-off" and then a "warming-on" period. There were 6 students out of the 12 who showed signs of "cooling-off" behaviour and 4 students who had developed a "cooling-out" behaviour similar to Katherine's, which was present even during the last interview with them at the third term of the university. Finally only the remaining 2 students appeared to be the ones who managed to make a smooth transition to university, without any signs of "cooling-off", not only because their starting beliefs about mathematics were more in line with the university setting but also because they managed to quickly modify the ones that weren't working during the very first week of the term.

From "Cooling-Off" to "Cooling-Out"

The above illustrations of the two student types throughout their passage from school to the middle of first term at the university demonstrate the development of their attitudinal profile from positive scales to relatively negative ones and the consequences this development has in their further mathematical performance and way of approaching mathematics in general.

In the case of Kenneth, a "cooling-off" type, his positive attitude towards mathematics while he was still at school was a product of his confidence and successful performance in working with it and his matching beliefs with the actual school setting. However, the carriage of his system of beliefs to the university created substantial difficulties in his engagement with the advanced mathematics setting. His beliefs about the didactical contact, the learning process, the setting of the mathematical exercises and the nature of the mathematical concepts could no longer be in accordance with the university teaching and learning style, and the working and assessment requirements. Kenneth's reaction to this mismatch of his beliefs caused him to increasingly develop a negative attitude towards university mathematics and gradually lose his interest in it, which was the indication of his "cooling-off" behaviour. Nevertheless, Kenneth soon found ways to "recover" from his "cooling-off" disposition and showed signs of "warming-on" by readjusting some of his beliefs and modifying his negative attitude. His mathematical behaviour, which was intimately linked to his attitudes towards mathematics, was also altered during the "warming-on" period, including a more willing approach of the mathematical problem given and an adaptation to university techniques and levels of formality.

In the case of Katherine, the gap between her beliefs about mathematics while she was still at school and the experienced situation in the university caused her to firstly express signs of "cooling-off" behaviour from the very first days at the university. Her expectations of university mathematics not only didn't match the reality but also were causing her a cognitive and attitudinal block for her adjustment to university. However, Katherine's attitude towards mathematics underwent a serious break and turned towards a very negative preoccupation towards university mathematics causing her a complete loss of interest in it. It was at this point that Katherine passed from the "cooling-off" behaviour to the "cooling-out" one, a gradual switch that was accompanied by her denial to alter her beliefs and her subsequent attitude towards university mathematics and mathematics in general. Her mathematical behaviour was also indicative of her "cooling-out" disposition, since Katherine had expressed no sign of any interest in the mathematical problems given to her, almost refusing to attempt a solution from her fear of failure and her lack of confidence in her mathematical abilities.

However, the trends found in the twelve students who participated in this research illustrate a phenomenon rather that quantify categories. Students' documented transition from school to university is a problematic one and the phenomenon of "cooling-off" is one of its affective consequences. Before students come to university, their attitudes towards mathematics are very positive, especially in the case of students applying for a mathematics degree, and these attitudes are shaped by their beliefs about the nature of mathematics and its school teaching and learning approaches. When these students enter university some of these beliefs might no longer be appropriate for the advanced mathematics environment causing the students cognitive and affective difficulties during their adjustment to the university setting. The students who experience a mismatching of their beliefs with the actual university situations encountered are very likely to demonstrate a negative attitude towards university mathematics in particular and mathematics in general, which can result in its turn in a malfunctioning in the university environment. The students then undergo a gradual loss of interest in mathematics, which is manifested as the "cooling-off" behaviour. It is then up to the students themselves to either recover from that phase or give up and lose not only their interest in mathematics but even their interest in the course and develop a "cooling-out" behaviour, which is very intense in its signs, very difficult to change and might even result in a drop-out from the course.

The questions that immediately come into mind are: how can we detect the symptoms of the "cooling-out" phenomenon and how can we prevent it from reaching the "cooling-out" stage? Since this research has taken place in one of the leading UK universities for mathematics with very supportive tutorials and supervision classes, we could assert that it is representative of a phenomenon present in most universities. The prevention of the appearance of these phenomena should start in school, perhaps by integrating a mathematical problem solving module or topic from an advanced point of view. At the university level, a revision of the first term's courses in order to include a systematic introduction to the university style of working should be considered. The carefully planned run of support tutorials could also be very beneficial not only for the students themselves but for the trained tutors who could trace the possible signs of a "cooling-off" behaviour and prevent it from being an obstacle for the students' academic performance and attitudes.

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THE MEDIATIONAL EFFECTS OF TEXTS AND TECHNOLOGY IN TEACHER PREPARATION

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ABSTRACT

Teaching at school and at tertiary level may be characterised by three types of practice: direct instruction, mediation and facilitation. The mark of an expert teacher is revealed by his or her ability to move between these various modes of practice, in response to the immediate needs of the students within a classroom, lecture hall or learning environment. This wisdom in practice comes from a clear understanding of the nature of these teaching practices and an awareness of the potential of each to produce the desired responses from the recipient students. This paper suggests that this wisdom or awareness in practice may be developed through learning programmes in which novice and experienced teachers experience and reflect on different modes of teaching practice, centred on their subject discipline, within an interactive learning environment. This environment extends the traditional modes of learning (guidance by an expert) by including mediated learning with written texts and interactive technology. Furthermore reflective practice is also built in as a necessary component for learning. This reflection refers to the learning of the subject discipline and to the teaching of that discipline.

This paper reports on a case study in which prospective teachers were asked to investigate a geometric problem, to reflect on the methods used to construct a solution to the problem and to produce a written formal report based on these actions and reflections. Particular attention is paid to the participants' use of written texts and computer technology in their resolutions to the geometric problems and as a consequence to their recognition of these resources in the report.

The result of this investigation raises questions as to the effectiveness of mediational resources in supporting mathematical progress and stimulating creativity and independence in the classroom. It also suggests that the introduction of resources of research into the learning environment may temper intrusive or inappropriate intervention by the expert.

KEYWORDS: Geometry, mediation, resources, pedagogical reasoning, teacher education.

1. Introduction

This paper explores the awarenesses of a group of prospective mathematics teachers who were part of an interactive learning course aimed at stimulating the process of pedagogical reasoning (Shulman, 1987) in geometry and its teaching. The course can be seen as a multi-perspective programme designed for future teachers of mathematics that focuses simultaneously on mathematical practice, on the resources that promote mathematical practice and, through reflection, on pedagogical practice. As such the course fills the gap between traditional mathematics courses at a tertiary level and pre-service teacher education courses (Hockman, 2000).

In this course it was hoped that awarenesses–in and on–practice would emerge through the students' reflective discourses and through the creative teaching units they produced as a course requirement. These discourses were analysed to reveal:

- the resources used or not used in solving a mathematical problem (action),
- the awarenesses or lack of awareness of the roles played by the resources in solving the problem (comprehension),
- the degree of transformation of learning strategies into teaching strategies as evident in the written reports (transformation).

The problem episode, under consideration in this case study, formed part of a sequence of problems within the course. The sequence was designed as a process to evoke awareness-in-counsel in mathematics and awareness-in-discipline in mathematics teaching (Mason, 1998). The study examined the activities and discourses of the students as they, firstly, solved a geometrical problem, and secondly reflected on solving the problem in a formal written report.

2. Scaffolding and Fading

Collins, Brown and Newman (1989) and others introduce the concept of scaffolding and fading in teaching and learning. This concept is widely used to refer to the ebb and flow of teacherly support or mediation during teaching practice. The aim of the scaffolding is to induct novice practitioners into the practice of the discipline. Once the novice practitioner is able to complete the task, or shows awareness-in-action in the discipline (Mason, 1998), the teacherly support fades away. The result is the attainment of independent working strategies by the student. Hence scaffolding and fading usually refer to the direct interaction between student and teacher, or to social discourse within the classroom environment. In this case study the process of direct and indirect teacher intervention is augmented by a broadening of the social dialogue in the classroom to include all the participants; through the accessing of traditional mathematical experience in written texts and visual mediums; and through explorations. The resources incorporated in this augmentation are referred to as resources of research.

The processes of action, comprehension and transformation in pedagogical reasoning are set in motion through the social dialogue in the classroom. That is, the teacher directs attention away from him/herself by encouraging the use of the resources of research. Teacherly guidance or intervention fades as these resources support the students in their work. In turn the students begin to use the resources of research spontaneously in place of expert guidance.

From Mason (1998) it appears that awareness–in–action in mathematics confirms the ability to do or to practice mathematics. Awareness–in–discipline (awareness of practice) in mathematics confirms the awareness of the process of doing mathematics and is a generalisation of practice. These processes include not only the heuristic strategies suggested by Pòlya (1945) and Schoenfeld (1985) and deductive skills of conjecture and proof, (Hersh, 1998), but also resources (source of ideas) that may promote the transfer of ideas from context to context. I suggest that these resources may be found in the category of research. That is, through reading and writing, exploration and social dialogue (de Villiers, 1996; Henderson, 1990; Wood, 1997).

The transformation of mediation marks a change from student teacher dependence on direct teacher mediation or scaffolding during the work process to a situation where such intervention has faded and has been replaced by independent work strategies using the available resources. That is, there is transformation in mediation starting from direct instruction by the coach, to indirect mediation through the resources, and then to direct and independent use of resources.

3. The Method

3.1 The participant sample

The case study involved 15 students in their third year of mathematics study. Most but not all the students were prospective mathematics teachers. The students worked in small groups of 3-4 throughout the course. These groups are named A, B, C and D. Teaching in the course ranged from standard lectures, to coaching in an interactive environment, to mediation in a computer laboratory. Thus the environment supported various types of teaching and learning, yet the central focus of learning was through an interactive environment or *reflective practicum* (Schön, 1987) using the resources of research.

3.2 The study design

This study focuses on a particular problem–solving exercise during the course. The exercises in the course were sequenced in a way to allow the resources of research to become operational through direct instruction, and then to be integrated as working strategies through the mediational prompting of the teacher educator. The final problem-solving exercise in this sequence set the scene for the resources to be assimilated and then utilised as independent learning and teaching strategies by the participants. The problem–solving exercise under consideration falls midway in the sequence of problems, aiming at the integration of the resources of research into the working strategies of the participating students.

3.3 The problem: construction of touching circles

Construct a heritage village: The village is to have three round huts, to accommodate the chief, his wives and his children. In addition two enclosures have to be built.

(i) A circular enclosure that must touch the outer section of each hut to separate the village from the fields

(ii) A circular enclosure that must touch each hut on the inner sides to demarcate the cooking and entertaining boma of the compound.

Apollonius' problem of tangent circles asks if it is possible to construct, with ruler and compasses, a circle tangent to three given, non-concentric, non-coaxial circles. Apollonius' problem has both algebraic and geometric solutions. While the algebraic solution involves solutions to simultaneous equations, a geometric solution uses tangency together with circle inversion.

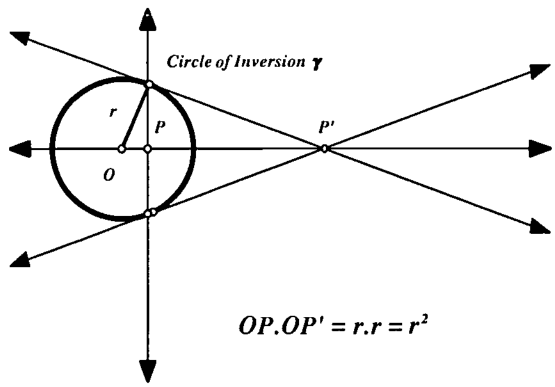

Figure 1: Circle of Inversion γ centre O and radius r, with inverse points P and P'

Although the problem is quite difficult to solve, its statement is easily understood. The exercise was completed in a workshop over a period of four hours. The students had access to computer technology, including *Geometer's Sketchpad*, course notes, source material (Courant, (1948); Coxeter H. S. M. (1967), Johnson, R. A. (1960), Cadwell, J. H. (1966), Eves, H. (1976)) dealing with Apollonius and his problem; compasses and graph paper.

3.4 Data Collection

Each of the groups submitted a written report at the end of the period. Rough sketches and working documents were collected. The students recorded their feelings about this experience in journals and I kept a record of my impressions of the episode. The topic of Apollonius' circles was referred to during the interviews with the students at the end of the course. These comments were recorded. The audio and video recordings were made during the work period. These recordings form the primary source of data in this episode with the other data complementing and supporting claims.

4. Reflections on Action

"*Although the problem was not as easy and intuitive as we thought at first, and an extra push and some material was required to get into the right track, the experience was good. It was good to be able to see how all our knowledge from the course thus far could be called into action. As expressed previously, the trouble I have had with maths content in the past is that it involves too many theorems, too much memory work and too little application and integration of all the work.*

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The exercise also called into action skills. Having to work in a limited time to digest information, solve a problem, express the solution, co -ordinate a group to prepare a document entailing all concepts, and ensure that each person understands each stage of the solving process. It was probably the most worthwhile life experience maths has ever created" (PH: 27/8/98).

The placing of the problem in the novel context of the construction of a heritage village appeared to have caught the imagination of many of the students. Many of the students had never seen mathematics used in a meaningful way or thought about the real contribution they might make as mathematicians in the future. The experience appeared to have heightened their awareness of the potential each one had to make a difference in life. The 'real' application of work that they were engaged in made their endeavours all the more worthwhile. While the application of the ideas of tangency and inversion to this new context in some cases needed considerable support from me, this intervention in no way diminished the feelings of satisfaction recorded by the students.

The difficulty of the task forced the students to engage with the printed texts. The initial reaction to these texts differed from group to group. Group A, who had previously used text material as a strategy to solve problems, quickly adapted to using the texts. They quickly picked up on the references to inversion and were able to integrate this new idea with their knowledge of tangency and inversion formed during the course. I certainly was called on during this period to confirm these connections and later to suggest ways of making the problem slightly easier but I did not dominate the activities.

In contrast to the action of group A, the rest of the class (groups B, C and D) took a long while to come to terms with the text materials. They felt that the text should have provided a clear and complete solution to the problem, not just suggestions and oblique remarks. They also felt that using the text was a negative reflection of them as it highlighted the fact that they were not able to solve the problem themselves.

Action and exploration dominated the whole problem-solving episode. These activities ranged from rough drawing, to paper, compass and ruler constructions, to the use of *Geometer's Sketchpad* to validate and check hypotheses. Group B only rejected a suggested solution after exploring it with *Geometer's Sketchpad* and Group A attempted to use the program to create a more accurate diagram and plan for their construction. Groups C and D used only compass and ruler constructions for explorations.

The circumstance of working in a *reflective practicum* within a limited time frame to complete a difficult problem forced the students to become dependent on each other. This social interaction dominated the transcripts and was a central issue in the remarks made by the students after the problem-solving episode. All the students were aware of the role that they played in solving the problem, valued the teamwork and understood the nature of my interventions. More importantly they also realised that a great deal of effort was needed to communicate their ideas succinctly and logically. The quality of 'group work' and of 'sharing the load' contributed to the students growing awareness of the nature of social discourse and its contribution to problem resolution.

Rigorous proof dominates mathematics studies at a tertiary level and has also had a negative effect on attitudes to mathematics at school level. The problem that the students investigated during this episode could have been stated as follows: "*Use the principle of inversion to prove that it is always possible to construct a circle tangential to 3 non-coaxial circles*". This statement would have hidden the rich history of the problem, de-contextualised the situation, and robbed the students of seeing how mathematics finds its way into real life situations. Furthermore the very request of 'proving' would have had a deflating and negative impact on the behaviour of the students. The students' idea of what constituted a rigorous geometrical proof had something to do with the 'given – required to prove – proof' format that is inculcated as part of a standard geometry course at school, and this experience did not encourage the possibility of different solutions to one problem. In the case of Apollonius' tangency problem, the students proved the theorem by using their existing knowledge, drawing inspiration and ideas from traditional mathematical experience, exploring various options, and working together as a unit within their groups. Each of the students participated in the writing of these proofs, perhaps the first they had ever attempted to construct, as opposed to copy and learn. They could validate or back-up each statement with a reason or a reference. In this way they were initiated into the community of practice of working mathematicians.

Teacher intervention remained a significant feature of the social discourse within the reflective practicum. However, as the period progressed changes occurred in the relationship between myself, as coach, and the students. I noted various factors with respect to my interventions, which influenced the end result of the problem-solving activity.

On the one hand, my interventions of direct instruction and mediation did not appear to be completely intrusive and the students not only completed the exercise but also thoroughly enjoyed the experience. They integrated their texts, experience, and explorations and worked as a group to complete the problem. In addition they felt empowered by the experience recognising that they could use the tools of research to supplement teacher mediation.

On the other hand, it became apparent that the authority of the teacher could be replaced and challenged by the students when empowered by the tools of research. This was particularly evident when a solution was proposed that deviated from the train of thought that I, as the coach, had envisioned. My conception of the problem was founded on my own solution. This particular method of solving the problem coloured my interventions in the class. I believed the students needed to be explicitly aware of the degenerate cases of the problem and hence I spent a substantial portion of time drawing the students into recalling and listing these cases. This effort was certainly beneficial to the problem-solving activity of group A. This group of students produced a solution that mirrored mine and used the approach of expanding the given circles to touch at the centre of inversion. The degree to which I supported their progress was limited and this scaffolding was successfully transferred to the independent agents research.

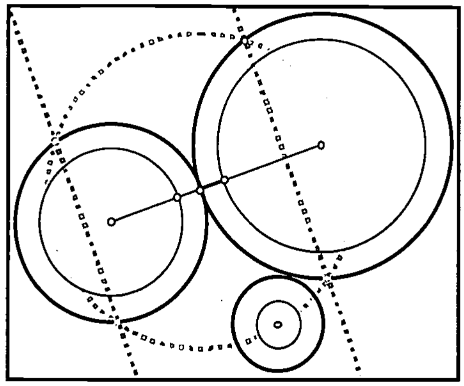

Figure 2: Inversion circle centred on the point of tangency of expanded circles and passing though the centre of the third circle.

The situation was not as clear-cut in group B. They chose to shrink the circles down to the case of a point and two circles. Although this certainly is a degenerate case of Apollonius' problems, its solution does not depend on solving this degenerate case, but rather on the construction of mutual parallel lines to disjoint circles. Using this approach made the time spent on discussing the degenerate cases redundant and may even have been obstructive and confusing. This mediation may have unintentionally shifted the attention of the students, and aborted a reflective moment that may yet have borne fruits.

As a result my support and mediation in the efforts of group B extended over a long period. Initially we appeared to talk at cross- purposes. The students worked hard to make meaning of my vague and cryptic suggestions. However as the period progressed there was a change in this dynamic. Firstly the students did not accept my rejection of their constructions as a solution to the problem, and moved to the computer to validate or refute their suggestion. As a result the refutation was through their explorations. Secondly, they then initiated a solution to the problem that I had not completely explored and was an innovation of the solutions suggested in the texts. In this case I had to work hard to make meaning of their suggestions. This reversal in roles, due to a quality of un-preparedness and un-seeness on my part, allowed for a community of practice to emerge between group B and myself. This period of interaction was very different to the un-certainty the students expressed during the previous interventions during the session. In this latter period, I believe my role shifted from instructor/mediator to joining in the general social discourse of the group discussions. These discussions were mediated by the texts and the explorations that the students themselves had made. The reports made by the students and their journal reflections confirmed these conclusions.

In the case of groups C and D I was much more forthright in my suggestions, and gave clearer directions to the students. As a result their progress was smooth, but closely monitored by me. In these latter cases, unlike the situation in group A and group B, I believe very little comprehension and transformation occurred.

5. Concluding Statement

This case study investigated the integration of resources of research into the working strategies of the participating students. The evidence shows that groups A and B adapted to using the social environment, the text materials and the available technology appropriately and creatively. They showed an awareness of the advantages of extending their resource base and thoroughly enjoyed their newfound independence. I claim that in these students mediational resources supported mathematical progress most effectively.

Group D showed slower progress, adapting well to the text material but still reticent to explore using the technology. They preferred to continue to use the compass and straightedge methods to construct their solution. Yet they made a point of noting that technology is not always available and should be used only with circumspection. I believe that these students used the resources of research in a meaningful way, showing awareness of the potential of extending their source of ideas to include research.

Group C however continued to rely on the guidance of the coach. They waited to be told how to proceed. In this case I believe that the resources of research were not fully integrated into the work strategies and remained at the operational level throughout the exercise.

The problem-solving exercise had been chosen for the specific reason that there were many ways to find the solution. Yet I was surprised at how my own conception of the problem coloured my conduct in the classroom. Close examination of the transcripts made me aware that, where possible, the expert must enter into the discourse of the students. The expert must attempt to remain open-mined in response to suggestions made by novices and aware of the dangers of prematurely aborting their reflective moments.

In conclusion, the case study confirmed the integrated use of, social dialogue, texts and exploratory devices as work strategies in many of the participants. These resources enhanced the independence of the participants in the learning experience. There was also a growing awareness of the potential of research to stimulate learning. Finally there was an indication that teacher intervention can be invasive. It is proposed that balanced co-operation and mutual support during classroom activity may be created by allowing the students to moderate their own ideas and progress, in their own time, with the resources of research. Future research may investigate this further.

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A PROFILE OF FIRST-YEAR STUDENTS' LEARNING PREFERENCES AND STUDY ORIENTATION IN MATHEMATICS

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ABSTRACT

This paper reports on action research activities during 2000-2001, involving first-year engineering students in an extended study programme of the School of Engineering at the University of Pretoria. Students in the participating group were enrolled for a support course aimed at facilitating the fundamental concepts underpinning a study in calculus as well as complementing the development of personal, academic, communication and information skills. The thinking style preferences of three groups of students taking a first course in calculus were assessed and the study orientation in mathematics of the participating group was determined. The possible effects of thinking preferences and study orientation on performance in a first course in calculus were assessed. Analysis of the thinking style preferences of the students indicates a diversity representing an array of preferences distributed across all four quadrants as measured by the Herrmann Brain Dominance Instrument and differences between the thinking style preferences of science students and engineering students were also found. Analysis of data obtained from the Study Orientation Questionnaire in Mathematics shows that students of the participating group entered tertiary education with mathematics anxiety and a history of inadequate study environments. In this paper it is envisaged that freshman mathematics students can seemingly benefit from a learning facilitation strategy for mathematics that endorses a student-centred and a brain-based approach. Such a strategy is aimed at developing the mathematics potential of the learners, fostering awareness of thinking style preferences and improving study orientation in mathematics.

Keywords

Mathematics education; whole brain learning facilitation; thinking styles; learning styles; study orientation in mathematics.

Background

In 1994 the Five-year Study Programme was introduced in the School of Engineering at the University of Pretoria. This programme extends the minimum four years of engineering study to five years in that the first two years of the Four-year Programme are spread over five years. The purpose of the five-year Programme is to create an opportunity for students who have the potential to become engineers but who are academically at risk because of their educational background. Students involved in the five-year Programme are given extensive academic support in their first year engineering courses through a tutoring system that is administered by the different departments and conducted mainly by senior students.

In spite of this support, some of these students are still at risk on account of the varying levels of educational competency in South African schools. For these students an additional two-semester credit-bearing support course, Professional Orientation (JPO), is presented during the first year of study in the School of Engineering. The course comprises a mathematics component, the development of personal skills, academic skills, skills in information technology, communication skills and writing skills needed for engineering study. In the first semester the main focus is on the mathematics component. The mathematics component of the support course is done independently from the main stream calculus course (presented by the mathematics department) and in addition to it. The aim of the mathematics activities in the support course is twofold. This first objective is to ensure that students thoroughly understand two-dimensional functions, their properties and graphs and the second is that students gain insight into their own thinking and learning preferences (regarding mathematics) and their study orientation in mathematics.

The first objective is met through a learning facilitation strategy in which computer graphing technology is used to visualize and explore the graphs of two-dimensional functions in an active learning environment (Carr & Steyn 1998; Greybe, Steyn & Carr 1998). Our previous research projects at the University of Pretoria have indicated that these activities endorse individualized instruction as well as co-operative learning and involve extensive communication in mathematics (both orally and written) (Steyn 1998; Steyn, Carr & De Boer 1999; Steyn & Maree 2002).

Teaching and learning facilitation principles

One of the main aims of the mentioned support course (JPO) is the development of each student's mathematical potential in order for him or her to pursue engineering studies successfully. Overall the educational activities in this course are viewed as "contributive learning" (Steyn 1998) in the sense that faculty and students are participants in a dynamic process in which teaching and learning are improved through the contribution of both faculty and students to each other's learning. This learning is not confined to (mathematical) subject content and can be diverse including aspects of student learning as well as successes and pitfalls of instructional activities.

Students arrive at tertiary institutions with established thinking style preferences and ensuing learning styles that influence all cognitive activities and consequently also conceptualisation of mathematical content (Felder 1993). Lecturers have established ways of thinking and so teaching styles interact with learning styles to encourage or discourage students depending on a match or mismatch of styles (Felder 1993). In order to accommodate individual students' diverse thinking style preferences and to encourage the utilisation of their less preferred competencies, the teaching learning strategy in the JPO course can be regarded as a "four-quadrant whole brain approach". This approach is based on ongoing research since the 1970s on the functioning of the human brain that indicated that specialised cognitive functions could be associated with different parts of the brain. For approximately 90% of the population logical, analytical, quantitative and fact-based knowledge is located in the left brain hemisphere whereas the right brain hemisphere predominantly supports and co-ordinates intuition, emotion, spatial perception and kinaesthetic feelings. In the case of the other 10% of the people the location of these functions is transposed.

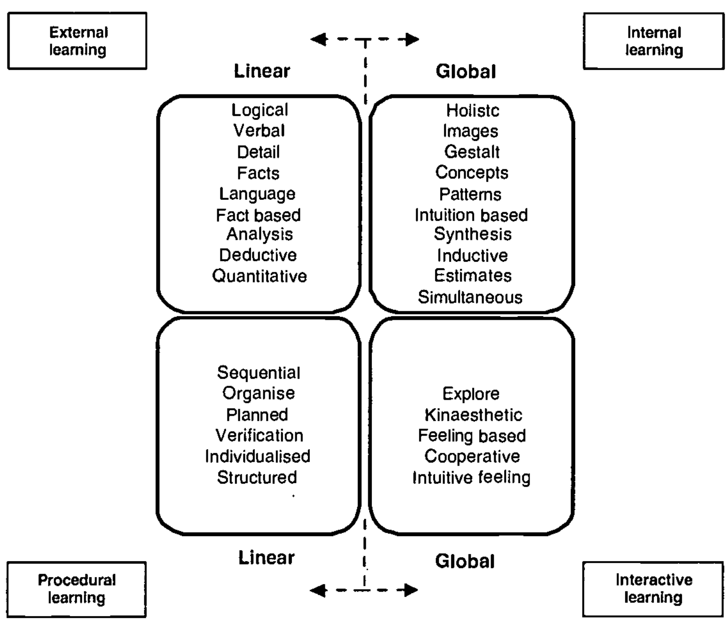
Herrmann's four-quadrant whole brain model

Herrmann (1995) combined this knowledge with how the brain is physiologically organised in order to develop a four-quadrant whole brain model. Figure 1 illustrates an adaptation of Herrmann's model that also includes the following four modes (Lumsdaine & Lumsdaine 1995) that describe student learning:

- *External learning* is related to learning through listening (lectures) and reading of textbooks, scientific literature, etc.
- *Internal learning* is related to learning through insight, understanding concepts holistically and intuitively, synthesis of data and personalising content into context.
- *Interactive learning* comes from experience, hands-on activities, discussion and feedback.
- *Procedural learning* is characterised by a methodical approach, practice, repetition and testing.

If learning activities in mathematics are structured to include different modes of student learning (implying different thinking and learning preferences), a whole brain approach is followed and competence in mastering concepts is fostered. Furthermore, functioning in any professional capacity requires working well in all thinking style modes (Felder 1996).

Figure 1 **A four-quadrant whole brain approach to teaching and learning facilitation**



The Herrmann Brain Dominance profiles in Figure 2 are examples from the study reported here and illustrate the tilt when a strong preference for the thinking mode associated with a specific quadrant is dominant. A preference for the A-quadrant (upper left quadrant in Figure 2A) means that a person favours activities that involve critical, logical, analytical and fact-based information. Individuals with a B-quadrant preference (lower left quadrant in Figure 2B) favour organized, planned and detailed information. A preference for the C-quadrant (lower right quadrant in Figure 2C) indicates favouring information that is interpersonal, feeling based and involves emotion. A preference for the D-quadrant (upper right quadrant in Figure 2D) is mainly characterized by a visual, holistic and conceptual approach in thinking.

Figure 2 Individual profiles showing thinking preferences according to the HBDI

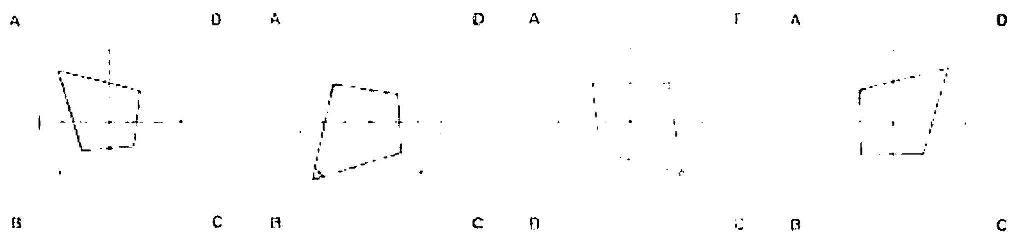


Figure 2A
Profile showing an A-quadrant thinking preference

Figure 2B
Profile showing a B-quadrant thinking preference

Figure 2C
Profile showing a C-quadrant thinking preference

Figure 2D
Profile showing a D-quadrant thinking preference

The diagrams in Figure 2 show the distribution of the individual profiles for each of the groups. The diagrams in Figure 2A and in Figure 2B both illustrate dominance in the distribution of profiles in the upper left A-quadrant. The diagram in Figure 2C illustrates dominance in the distribution of profiles in the lower left B-quadrant.

In addition to the four-quadrant whole brain principle, active learning (in mathematics) is viewed as a further core pedagogical principle in the support course. In this regard active learning involves activities that engage students in *doing* something instead of only observing what can or should be done.

During 2000-2001 this developmental approach, based amongst others, on the principles of whole brain learning facilitation and active learning, was structured as an action research study that included the determining of the students' thinking style preferences and their study orientation in mathematics. In the following sections aspects of the study are discussed

Research project

The action research activities reported in this paper formed part of course activities and the students were never regarded as merely 'research objects'. Therefore references are to 'participating' students and 'other' students where the participating students represent those on the support course (JPO) in the School of Engineering.

Aim

During 2000 the Herrmann Brain Dominance Instrument (HBDI) (Herrmann 1995) was used to provide students with insight into their own thinking preferences and to measure the preferred thinking styles of the students. During 2000 and 2001 the Study Orientation Questionnaire in Mathematics (SOM) (Maree 1997) as well as the Study Orientation Questionnaire in Mathematics Tertiary (SOMT) (Steyn 2002) were used to determine the students' study orientation in mathematics and to investigate whether either of the SOM or SOMT is a significant predictor of performance in mathematics.

Null hypotheses

The null hypotheses that were to be investigated by this study, were the following:

$H_0 1$:There is no difference between the arithmetic means of the scores of the students on the support course (JPO) and a group of first-year civil engineering students participating in the four-year programme for the quadrants of the HBDI.

$H_0 2$ There is no difference between the arithmetic means of the scores of first-year engineering students on a support course and first-year science students on a support course for the quadrants of the HBDI.

$H_0 3$ There is no difference between scores in the different fields of the SOM and students' marks in mathematics.

$H_0 4$ There is no difference between scores in the different fields of the SOMT and students' marks in mathematics.

Instruments

The HBDI

The HBDI is an assessment tool comprising a survey of 120 questions that quantifies relative preference for thinking modes based on the hypothesized task-specialized functioning of the physical brain. A thinking preference profile, compiled from scores on an inventory, is displayed on a four-quadrant grid. The higher a score in a quadrant, the stronger the preference for the thinking style related to that quadrant.

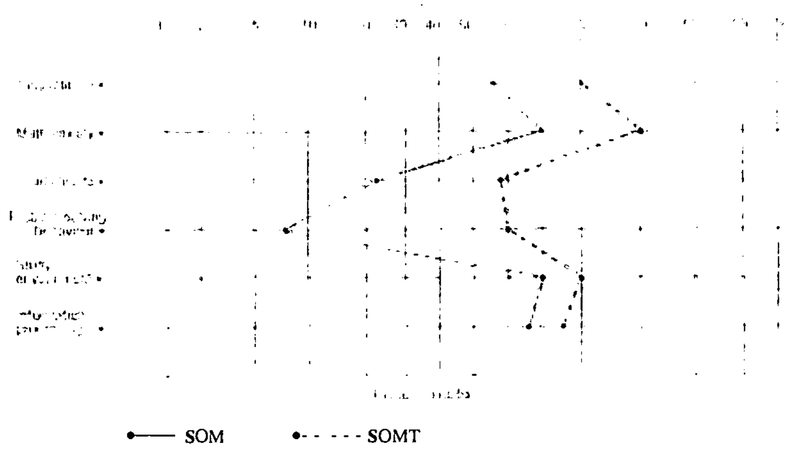
The SOM and the SOMT

The SOM and SOMT both comprise six fields including 92 statements that relate to how individuals feel or act regarding aspects of their achievement in mathematics. The SOM was developed in the mid-1990s for high school students but the scope of the questions is also applicable to first-year tertiary students. In the SOMT the terminology was adapted to represent a tertiary environment. These changes do not affect the scoring of the instrument. The six fields of the SOM and SOMT can be summarized as follows:

- *Study attitude* deals with feelings (subjective but also objective experiences) and attitudes towards mathematics that are manifested consistently and that affect students' motivation, expectation and interest with regard to mathematics.
- *Mathematics anxiety* concerns an 'uncomfortable' feeling when such anxiety manifests itself in aimless behaviour (like excessive sweating, scrapping of correct answers and an inability to formulate mathematics concepts).
- *Study habits* addresses the displaying of acquired, consistent and effective study methods.
- *Problem-solving behaviour* in mathematics includes cognitive and meta-cognitive strategies in mathematics.
- *Study environment* includes factors relating to the social, physical and experience environment.
- *Information processing* reflects on general and specific learning, summarizing and reading strategies, critical thinking and understanding strategies such as optimal use of sketches, tables and diagrams.

Answers to the SOM and SOMT can be converted to percentile ranks after which profiles (as in Figure 3) can be drawn. Any shift (regarding any of the fields) to the right indicates a more favourable aspect of a learner's study orientation. Figure 3 is an example of the results of the SOM and SOMT of a student in the study. In this case the SOMT profile shows an overall improvement towards a more favourable study orientation compared with the student's SOM profile. It should be noted that a high percentile rank for 'mathematics anxiety' indicates that a learner is less anxious. For example, the SOM profile in Figure 3 indicates that the learner is less anxious than 70% of the relevant population.

Figure 3 **Example of a SOM and a SOMT profile**



Participants

The research relating to the thinking style preferences using the HBDI involved 101 students. Of these students, 33 were taking the engineering support course, 30 were first-year civil engineering students on the four-year programme and 38 were first-year science students on a support course in the BSc extended programme in the Faculty of Science. The data relating to the HBDI of the latter group were determined in a research project in the Faculty of Science during 1999 (De Boer & Steyn 1999).

The research regarding the students' study orientation in mathematics involved only the students enrolled for the support course (JPO). In the year 2000, 30 students completed the SOM and 26 completed the SOMT. In 2001, 38 students completed the SOM and 24 completed the SOMT.

Method

Students did the HBDI towards the second half of the first semester. In both the 2000 and 2001 studies the SOM was done four weeks after the start of the academic year. The 2000 students did the SOMT in the middle of their second year and the 2001 group did it at the start of the second semester in the first year. Results according to all the instruments were given to the students individually and feedback explaining the instruments and results in general was given to the groups.

Limitations of the study

This was a limited, local study, and the findings reported in this article have limited generalisation value; they do, however, have naturalistic generalisation value (Cohen, Manion & Morrison 2000). Furthermore, owing to limited resources, the study was carried out on a small sample of students.

Ethical considerations

Written permission for administering the HBDI, the SOM and SOMT was obtained from the School of Engineering and the Faculty of Science. In all cases the use of the instruments as part of course activities was transparent and clearly conveyed to all the students who participated. The research was thus carried out with the full consent of all participants and stakeholders.

Results

In Table 1 the number of students per quadrant of preference is given.

Table 1 **Number of students with thinking preferences per quadrant**

	A	B	C	D
JPO students	17	5	5	6
Civil engineering students	18	3	6	3
Science students	9	18	7	4

In Figure 4 the distribution of thinking style preferences per group is indicated.

Figure 4 **Distribution of thinking style preferences per group**

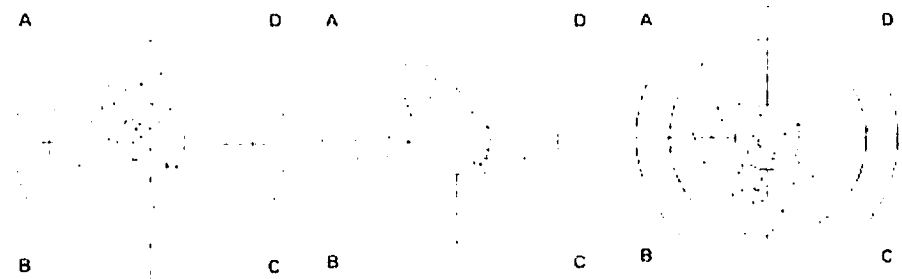


Figure 4A
Dominance map of the distribution of profiles for the JPO students in 2000

Figure 4B
Dominance map of the distribution of profiles for the civil engineering students in 2000

Figure 4C
Dominance map of the distribution of profiles for the science students in 1999

The two-sample non-parametric Wilcoxon Rank Sum Test (normal approximation) was used to compare the arithmetic mean score values between the different groups for each of the four quadrants of the HBDI. Table 2 shows the arithmetic mean ($\bar{x}$), standard deviation (s), Z-value and p-value regarding the quadrants of the HBDI for the JPO and civil engineering students and Table 3 indicates the same data for the engineering students on a support course and science students on a support course.

Table 2 **Wilcoxon scores for JPO and civil engineering students and the quadrants of the HBDI**

	JPO group (N=33)		**Civil engineering group (N=30)**		
HBDI	Arithmetic mean $\bar{x}$	Standard deviation s	Arithmetic mean $\bar{x}$	Standard deviation s	P
A-quadrant	82.06	16.89	83.66	20.70	0.5489
B-quadrant	70.45	13.59	76.03	15.25	0.1194
C-quadrant	64.75	17.44	55.03	22.03	0.0200#
D-quadrant	73.06	17.41	76.46	17.59	0.6103

# indicates a p-value that is significant on the 5% level.

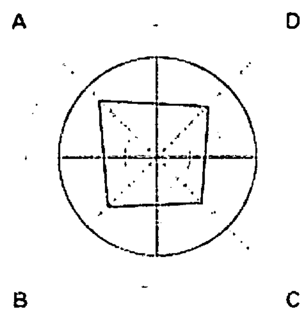
Table 3 **Wilcoxon scores for JPO and science students and the quadrants of the HBDI**

	JPO group (N=33)		Science students (N=38)		
HBDI	Arithmetic mean $\bar{x}$	Standard deviation s	Arithmetic Mean $\bar{x}$	Standard deviation s	P
A-quadrant	82.06	16.89	69.78	18.70	0.0066#
B-quadrant	70.45	13.59	83.86	16.68	0.0003#
C-quadrant	64.75	17.44	71.15	21.42	0.1964
D-quadrant	73.06	17.41	63.34	19.99	0.0180#

# indicates a p-value that is significant on the 5% level

Figure 5 illustrates the average Herrmann Brain Dominance profile of the engineering students on the support course.

Figure 5 **Average Herrmann Brain Dominance profile for the 2000 group enrolled for the support course**



Quadrant	A	B	C	D
Profile score	82	70	65	74

In Table 4 the results of a step-wise regression analysis taking the fields of the SOM as independent variables and the performance in the first semester calculus course as dependent variable are indicated for the 2000 and 2001 groups. In Table 5 the similar data are reflected with regard to the fields of the SOMT and performance in the first semester calculus course.

Table 4 **Step-wise regression model for the SOM and mathematics performance for 2000 and 2001**

Fields of the SOM	**Parameter estimate**	**Partial coefficient of determination R^2**	**Model/Cumulative coefficient of determination R^2**	**P**
Participants of 2000 (N=30):				
Information processing (IP)	0.2528	0.3918	0.3918	0.0002*
Problem-solving behaviour (PSB)	0.1428	0.0772	0.4689	0.0579#
Mathematics performance = y_1 = 34.44 + 0.25 IP + 0.14 PSB				
Participants of 2001 (N=38):				
Mathematics anxiety (MA)	0.2397	0.2515	0.2515	0.0013*
Mathematics performance = y_2 = 45.65 + 0.25 MA				

* Significant on a 5% level
\# Significant on a 10% level

Table 5 Step-wise regression model for the SOMT and mathematics performance for 2000 and 2001

Fields of the SOMT	Parameter estimate	Partial coefficient of determination R^2	Model/Cumulative coefficient of determination R^2	P
Participants of 2000 (N=26):				
Problem solving behaviour (PSB)	0.2691	0.4241	0.4241	0.0003*
Mathematics performance = y_3 = 42.16 + 0.26 PSB				
Participants of 2001 (N=24):				
Study attitude (SA)	0.4243	0.2102	0.2102	0.0049*
Study habits (SH)	-0.2947	0.0853	0.2955	0.0539*
Study environment (SE)	0.2915	0.1653	0.4609	0.0037*
Mathematics anxiety (MA)	-0.1370	0.0528	0.5137	0.0761**
Mathematics performance = y_4 = 38.45 + 0.42 SA – 0.29 SH + 0.29 SE - 0.13 MA				

* Significant on a 5% level
** Significant on a 10% level

Discussion

Thinking style preferences

Regarding hypothesis H_01, it follows from Table 2 that in quadrant C, the means for the JPO students differ significantly from the means for the civil engineering students. However, no inference can be made regarding the means for the A-, B- and D-quadrants. Regarding hypothesis H_02, it follows from Table 3 that in the A, B and D-quadrant, the means for the engineering students on the support course differ significantly from the means for the science students on their support course. In this case no inference can be made regarding the means for the C-quadrant.

The distribution of preferences indicates that in this study the students do not favour C-quadrant thinking that is, for instance, also associated with a preference for co-operative learning. Furthermore, Figure 5 illustrates the average Herrmann Brain Dominance profile of the engineering students on the support course which distinctly indicates that the preferences of the group, when combined, result in a profile that almost represents a generic whole brain profile with strong thinking preferences in all four quadrants (Herrmann 1995).

Table 1 shows that the majority of engineering students has thinking style preferences associated with the A-quadrant. This is in accordance with research that engineers (engineering students) typically favour A-quadrant thinking (Herrmann 1995; Lumsdaine & Lumsdaine 1995). On the other hand, the majority of science students on a support course have thinking preferences associated with the B-quadrant. Existing thinking preferences inevitably influence students' learning preferences.

Study orientation in mathematics

Regarding hypotheses H_03 and H_04, it follows from Table 4 and Table 5 that most of the fields of the SOM and SOMT, although not simultaneously, can be regarded as significant predictors (on a 5% level) of performance in mathematics.

Conclusion

Although research has undoubtedly indicated that peer group learning works well, it seems as if students need to be trained to work in groups and the classroom structured to foster interactivity. It seems as if there is a significant difference in distribution of thinking style preferences for the engineering students on a support course and science students on a support course whereas the distribution of thinking style preferences for both the groups of the engineering students is more similar. The fact that the preferences of the group, when combined, result in a profile that almost represents a generic whole brain profile with strong thinking preferences distributed across all four quadrants of the Herrmann model, endorses the necessity to structure learning facilitation of mathematics not only to accommodate different thinking styles but also to develop less preferred thinking modes.

As far as the SOM and the SOMT are concerned, it is clear that lecturers will be able to use the results of these tests to help students improve their study orientation in mathematics and consequently realise their mathematics potential at a higher level. Students can, *inter alia*, be helped to become acquainted with the basic principles of executive study in mathematics, as well as the important role of study conditions, including motivation and background factors, in academic success.

In summary, it can be stated that the combined use of the above-mentioned instruments with science and engineering students at first-year level appears to be a potentially useful strategy to facilitate optimal achievements in Mathematics.

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INTRODUCING EXPERIMENTS INTO A FIRST COURSE IN CALCULUS

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ABSTRACT

It has become widely accepted that mathematical software can contribute significantly to the learning and understanding of Mathematics. In particular, the visualization capabilities of software packages and Computer Algebra Systems, students can explore function behaviour and phenomena that would be impossible without the use of computers. A Mathematics instructor has a wide choice of software tools to consider for use in undergraduate courses. Yet, the problem remains how to construct interesting problems that would challenge the student and where the technology is an important tool assisting in the exploration, yet allowing one to reflect, analyse, modify one's thinking until the appropriate conclusion is reached.

In this paper, I give examples of challenging problems within the conceptual reach and understanding of Calculus students. These problems were given in the fall 2001 to students taking a first course in Calculus. A characteristic of these examples is that without technology it may be difficult for students to do the analysis and to obtain the answer, yet the technology and its visualization capabilities provide the student with a mechanism for experimentation and testing, allowing them to modify their hypothesis and their thinking to lead them to a solution. Second, in these problem tasks, there is not one correct answer and the answer can be given to different degrees of generalization allowing students to go as far as they can in their analysis. In these assignments, a written component was added so that the students can reflect on their own thinking. They were required to do a write up showing the steps used in their analysis and an explanation of why the conclusion they arrived at is a valid one. Technologies used in these experiments are the TI-89/92 calculator and the dynamic software "*Autograph*".

Keywords: Calculus Reform, Technology, Visualization

Introduction

It has become widely accepted that mathematical software can contribute significantly to the learning and understanding of Mathematics. In particular, the visualization capabilities of software packages and Computer Algebra Systems allow students to explore function behaviour and phenomena that would be impossible without the use of computers. A Mathematics instructor has a wide choice of software tools to consider for use in undergraduate courses. Yet, the problem remains how to construct interesting problems that would challenge the student and where the technology is an important tool assisting in the exploration, allowing one to reflect, analyse, modify one's thinking until an appropriate conclusion is reached.

In this paper, I present examples of examples of the use of technology that was done with students taking a first course in Calculus in the fall 2001. A characteristic of these examples is that without technology it may be difficult for students to do the analysis and to obtain the answer, yet the technology and its visualization capabilities provide the student with a mechanism for experimentation and testing, allowing them to modify their hypothesis and their thinking to lead them to a solution. In these assignments, a written component was added so that the students can reflect on their own thinking. They were required to do a write up showing the steps used in their analysis and an explanation of why the conclusion they arrived at is a valid one. The technologies that were utilized were the TI-89/92 calculator and the dynamic software *"Autograph"*. The reason these were used was the ease with which the calculators could be brought into the classroom in addition that some students already had them. The *"Autograph"* was used because it was simple to learn and provided powerful animation and a numerical component.

Background of this experience

The students in this class came from quite a varied background. Some of the students come from Lebanese schools but who either did not pass the official exams in the math and science sections or some who may have been in a literary and humanities sections and therefore did not do enough mathematics to enter at the sophomore level to major in Engineering or in Computer Science. A large number of students come from Arab countries and may have studied in Arabic but generally have a poor background in Mathematics. There are two sections of the course and one section was set up to be an experimental section where the teaching would be non traditional with a strong emphasis on the visual and numerical aspects. The textbook used for the course was *"Calculus, from Graphical, Numerical and symbolic Points of View"* by Ostebee and Zorn [4].

The rationale for this experimental section was that we were dissatisfied with what students were learning after two years of Calculus, as have been experienced by many math educators everywhere. Students come out not having any conceptual understanding of the concepts in Calculus and after two years have almost forgotten everything as I witness this in my Numerical Analysis course. Even though in our teaching, we tried to emphasize the visual as well as the conceptual, and we assigned projects using mathematical software, the students seem not to place the same importance to the topics done in class and would be satisfied if they were able to do the routine problems specified in the textbook. The large number of routine exercises as well as what they were used to in high school seemed to define to them what "mathematics" was important. Even though we added *"Mathematica"* projects, these were done in a mechanical way and little or no benefit was gained. Thus we were coming to the conclusion that you cannot change student's perception of what is the mathematics that is important, and therefore their attitudes to the subject as long as we use a traditional textbook. As Porzio [4] concludes: "in revising the curriculum it is not sufficient to tag on technology on the topics covered, but to emphasize equally the various representations in the study of various concepts and to design problems that will reflect this philosophy, and that will engage the student in moving between the various representations in their solution." At the same time, we were reluctant to make changes for all of the courses without first having an experience of what it would entail and how students and the departments for whom Calculus is a service course would react to this change.

A large number of handouts were given to the students and a number of projects were assigned to be done in groups of two, where a written component was required. The other section was taught in a traditional way. I will not go into the details of how the course was conducted but will focus on the experiments that were done using the TI 89/92 calculators and *"Autograph"*.

The Experiments:

1. Experiment 1: Studying the graph of a polynomial of degree 4

The graphs for this experiment were generated using *"Mathematica"*, however the students were expected to do this using the TI 89/92 calculator or any other graphic calculator available to them. Students could also borrow graphic calculators from me. The graph of a polynomial of degree 4 having four distinct roots and opening upwards was given along with its equation. The defining equation was $f(x) = 3(x-2)(x-1)(2x+1)(x-3)$ and they were given that it expands to $6x^4 - 33x^3 + 48x^2 - 3x - 18$. The graph as given is shown in the following diagram, and no scale was indicated in the diagram:

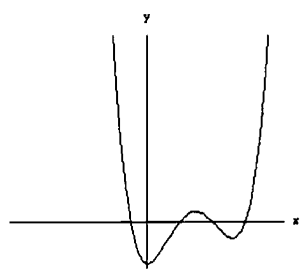

Figure 1

Following that they were a number of graphs of 4^{th} degree polynomials and they were asked to use experimentation with the graphic calculator in order to obtain an equation yielding a graph which has similar properties as the ones indicated in the given graphs. They were also asked to describe the process by means of which they were able to obtain their answer, including those that did not yield the needed graph. So, here it was important for the students to realize that experimentation is valid and that it was important for them to reflect on their thought processes.

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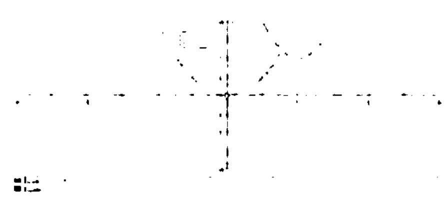
Figure 2

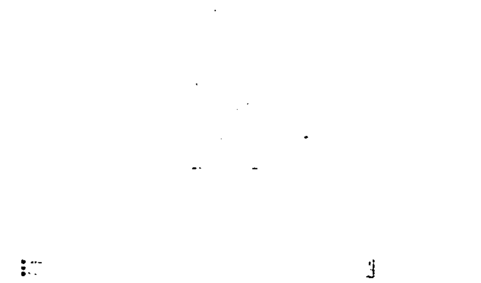
At first the equation of the given polynomial was not given in the factored form, and the students had difficulty in trying to figure out how to go about experimenting with graphing different equations. What also made it more difficult for them was to manage the window defining the part of the graph needed. When the hint was given, that the equation had four distinct roots and could be written in its factored form, they were then able to determine equations whose graphs were given. They chose a scale on the x-axis, found the zeros of the polynomial and thus were able to define the polynomial. They had to discover that if the tangent line was horizontal, then it meant that the root was a multiple root. Finally, where the required graph did not intersect the x-axis, they were able to take a similar graph, and take its defining equation and add an increment to shift it up or down as was needed.

2- Experiment 2- Transformations

This experiment involves the study of how transformations affect the graph of a function, and *"Autograph"* was used for that purpose. In class, we studied the effects of linear transformations on functions, and several handouts and homework exercises were given for that purpose. The students were given a function and its graph as well as the graphs of other functions obtained by applying a transformation on that function. The objective of the exercise was to determine the transformations and hence the equations defining these graphs. Initially, the graph of $y=x^2$ and that of equation 4 were given in class (figure 3 (i)) as an exercise, but students were unable to find the answer, so the intermediary graphs (figure 3 (ii)) were given to help the students how to determine the equation of the graph.



(i)



(ii)

Figure 3

This exercise was instructive in that it showed the students that in case one cannot find the answer from what is given, one can introduce intermediary elements that will helping bridging the gap from the given to reach the required result.

3. Experiment 3- The Derivative

The purpose of this experiment was to help the students understand the concept of the derivative as the limit of the difference quotient of the function as well as the tangent line to a function at a given point. They were asked to use "*Autograph*" in order to plot the graph of y= x^2 and to consider what happens when x=1. They were asked to zoom in at the point (1,1), and to

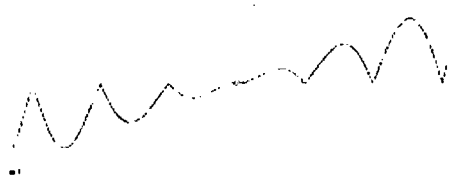


Figure 4

make their observations. The idea that the function is differentiable at a point if it is locally straight is an important idea (see Tall [6]) that is much easier for students to grasp than the concept of the limit. In fact they can see that if they zoom at any point, the graph will be locally straight. However, when they are asked to plot the graph of the function y= x $|\sin x|$, they will be able to see that at the point (0,0), the graph is locally straight, whereas at (π, 0), the function is not locally straight and hence not differentiable at that point.

4. Experiment 4- The Derivative as a limit

The next experiment was designed to help the students learn the concept of the derivative at a point as the limit of the difference quotient. Here they were asked to draw the graph of a function for example y= x^2, and select two points (1,1) and another point close by, and to draw the gradient by selecting the two points, then select to the gradient as is shown in figure 5

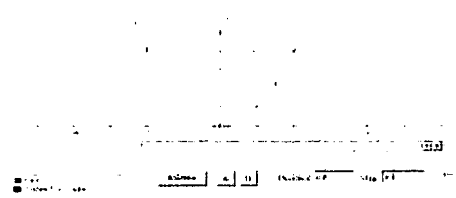


Figure 5

The picture can be animated and at the bottom of the graph is shown the value of x, y, the gradient, the change in x and y as well as the equation of the secant line. Students were asked

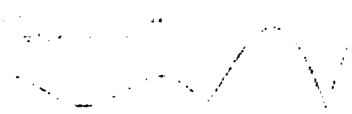


Figure 6

to make a table of these values and to write their observations. This exercise is repeated but with another graph namely that of y= x |sin x|, and are asked to repeat this points at which the function is differentiable and points where the function is not differentiable.

4. Experiment 5- A family of functions

In this project they were supposed to study the family of functions $x^2 + a\,x$, the derivative function as well as the tangent line at x=0. "*Autograph*" allows a user to enter a family of functions and initially the parameter is set to 1. The project is provided in the appendix, and students are expected to see how the graph changes as a varies. They can either do manual animation or an automatic animation or have simultaneously the graphs drawn at the same time. They were asked to draw the graph of the tangent line at (0,0) a common point to the family of graphs. Here, they had to figure out what the equation of the tangent line is and enter its equation. In another part they had to draw the graph of the derivative of the function and to draw the family. Following is the output:

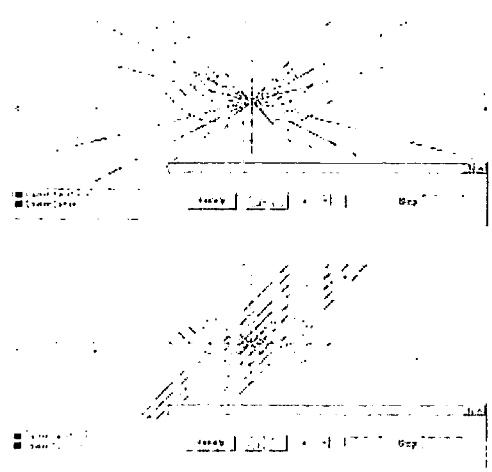


Figure 7

Further elaborations on this exercise are indicated in the appendix. It is clear that students who did these exercises developed a strong geometrical meaning to the concept of the derivative.

Conclusion

From this experience, it is clear to me that the students got engaged in their projects done with the TI 89/92 calculators and with "*Autograph*" and gained a better and deeper understanding of concepts studied in the course. This is contrary to previous experiences I had in introducing technology in other math courses such as Calculus III. I believe that the reasons are: first that the entire course emphasized the multiple representations. This was done during class time, homework assignments as well as questions on tests. The algebraic part of Calculus, namely obtaining the formulae for the derivatives was only done in the last three weeks of the semester. The second reason was that the calculators were often used in class and many of them also used them at home. So they were familiar with them. On the other hand, "*Autograph*" was easy to learn and had transparent syntax. I believe that this experiment was successful but it required a lot of work and also determination as the students were unhappy with the approach at the beginning and wanted to do mathematics the way they were used to in high school.

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Appendix
Sample
Calculus I
Using Autograph
Plotting a family of functions

This handout explains how to use the *Autograph* software to plot a family of functions. This project is similar to the one demonstrated in class. This project will be done in groups of 2 using the "Autograph" software, which is available in the Computer Center.

Consider the family of quadratics whose equation is given by $y= x^2+ a\,x$, where a is a parameter. Define a function f given by $f(x)= x^2 + a\,x$. We would like to study their behaviour as a varies by using Autograph software.

Questions:

Q1 What do you notice about the graph? How many roots are there? What is the axis of symmetry of the graph?

Q2 Now zoom in around the point (0,0)! Continue zooming around that point? What do you notice about the graph around the point (0,0)? Can you read the slope of the tangent line around that point? What is its equation?

Step 3- We will now show what happens to the graph as the value of a is varied. Go to the **Constant controller** button, which is on the item before last on the second tool bar. Open that and you will see that you can vary the value of a. Choose the step to be 0.5. Now go with the forward arrow to allow a to take values 1.5, 2, 2.5, 3, 3.5. Now go backwards to allow a to decrease until a = -3

Q3 Now describe what you have observed?

Step 5- Now go back to the **Constant Controller** dialog and choose **Family** and then **Family Plot**. Now the graphs will be plotted simultaneously.

Q4 Can you identify which function belongs to which curve? You may want to print the graphs! You can do that by going to file and the press the Print button

Step 5- Now delete all equations by going to the equation button and select the Delete all equations, so that we can start with a fresh page. Now we enter equation1 to be f(x) and we want to enter another equation namely that of its derivative. Enter for equation the derivative of x^2 + ax. Now we will go over step 3, 4 and 5 to see how the function and its derivative at the same time.

Q6 Can you find where the derivative is equal to zero and hence where the function has a minimum value. Can you read the minimum value. Make a table of the values of a, the function f(x), the value of x where the derivative is 0 and the minimum values of the function.

Step 5- Now we want to draw the tangent line at a given point we start with (0,0). Now go to the **equation** button and select the **Delete all** button to get a fresh start. Enter equation1 to be y = f(x) and for equation2, the equation of the tangent line at (0,0). What is the equation of the tangent line? The derivative at x=0 is a, therefore the equation of the tangent line y = a x. Repeat steps3, 4 and 5 and describe what is happening with the tangent line at the point (0,0)?

What if we want the tangent line when x= 1? What is the value of y? What is the derivative at x=1? What is the equation of the tangent line at x=1?

THE MATHEMATICS BRIDGING COURSE AT THE UNIVERSITY OF SOUTH AUSTRALIA

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ABSTRACT

The Division of Information Technology, Engineering and the Environment at the University of South Australia runs a Bridging Program with courses in Mathematics, Physics, Chemistry and Communication. The goal is to provide an alternative pathway for prospective students to gain access to a science or engineering degree program. The author has been the coordinator of the mathematics component in the Bridging Program for several years. The innovative methods that have been devised to try and fill some of the gaps in the students' background will be canvassed. Traditionally, computer software in mathematical education has been primarily used for problem solving utilising such packages as Matlab and Maple. However, spreadsheets are a remarkably capable tool for both 'doing' and illustrating mathematics. Their almost realtime graph alteration and their recursive capabilities make them ideal for illustration of mathematical concepts . This capability will be demonstrated using examples from this and other courses for which the author is responsible. I will also present an assessment of how well the bridging course has prepared the successful students for the degree programs they have subsequently undertaken.

Keywords: Bridging Course, Spreadsheet Teaching Tools, Computer Aided Illustration of Mathematics

1 Introduction

The Division of Information Technology, Engineering and the Environment at the University of South Australia runs a Bridging Program with courses in Mathematics, Physics, Chemistry and Communication. The term 'bridging' usually conjures up the concept of a group of students already enrolled in a university degree obtaining some aid in a discipline to fill in the gaps in their background knowledge. At the University of South Australia the goal is to provide an alternative pathway for prospective students to gain access to a science or engineering degree program.

There are various methods of entry into a university degree program in South Australia. The most usual form is through direct entry following secondary school. Under this approach, prospective students list preferences for up to five programs. They undertake a publicly examined set of subjects and are accepted for the highest of their preferences for which their aggregated results reach the cut-off score. This score is calculated through determining how low a score will allow the program to fill its quota of students, inspecting the list of students applying for the program. Alternatively, after a period of two years has elapsed since the student has left secondary school, they may sit an adult entry set of tests to determine if they can qualify to enter university. The level of their results will determine which program they can enter, with the cut-off scores again reflecting the level of demand of the program.

The Bridging Program offers the only method of entry which also provides the students with an opportunity to enhance their skills as well. They study full-time for one semester or part time for two semesters. It is designed for people who either have a gap in their science background or have a comprehensive background but at some time in the past. The author has been the coordinator of the mathematics component in the Bridging Program for several years. The primary goal has not only been to determine the suitability of the students for entry, but also to maximize the chance of success for those who do qualify in this manner.

Innovative teaching methods have been devised to enhance students' understanding of mathematics. These have revolved around the use of spreadsheets to illustrate mathematical concepts. Spreadsheet based design has been chosen because of its what-if capabilities, the virtually instantaneous response to alteration of parameters, its graphical capabilities and its inherent use of recursion which proves eminently suitable for many mathematical applications. Some of these capabilities will be illustrated.

Additionally, it is vital for the successful operation of such a program to evaluate whether or not it is fulfilling its purpose. Thus two measures of the success of the program will be presented. One is a simple measure of how well the graduates of the program perform in their chosen degree program - do they complete the degree or look as though they will (the analysis includes people who will not have had time to fulfill all requirements). There is also a measure which relates specifically to their mathematical performance. Once again, it is a simple measure - is their result in mathematics in the bridging program related to their results in mathematics courses undertaken during their degree?

2 Spreadsheet Tools

Dubinsky [1] outlined six methods to enliven the mathematics curriculum by

- aiding students in visualisation,
- dispensing with much routine symbolic manipulation,
- dealing with larger, more realistic problems,
- providing an environment which encourages exploration,
- making use of animation wherever possible and
- providing an environment for constructive development in mathematics learning.

The author has developed spreadsheet tools over a number of years in order to satisfy at least some of these goals. York and Arganbright [2] contend that 'the spreadsheet provides us with a format that closely parallels the way we think about mathematics. At the same time, it provides students with a creative tool for conducting What-if? explorations.' These explorations can be animated to an extent because of the automatic recalculation of graphs. A number of researchers have used spreadsheets to teach advanced concepts using a problem-based approach. For instance, Mays *et al* [3] developed 'five student-centred projects that examine important problems in the fields of Mechanical Design, Dynamics of Machines, Fluid Dynamics and Thermodynamics.' De Mestre [4] uses Excel to find numerical solutions to differential equations, matrix inversions for solving systems of linear equations and to check integration via numerical integration. Das and Hadi [5] use the Solver option in Excel to solve optimisation problems. York and Arganbright [2] discuss the modeling of growth and harvesting on spreadsheets.

The author began with designing tools to cover a range of topics in the preliminary stages of calculus and linear algebra instruction. Additionally, there is some mathematical modelling, usually also embodying some further explanation of some basic principles. For example, there is some basic predator-prey modelling. Using spreadsheets to develop simple numerical solutions to the equations using Euler's Method results in a powerful tool because of the recursive and graphical features. Stepping forward in time is afforded simply by the dragging of the appropriate formulae. Changes to parameters such as relative initial values of predator versus prey results in the instant graphical visualisation of the effects of the changes. Inherent in this topic is a reinforcement to the student of the formal definition of the derivative as the limiting value of a sequence of slopes of secants.

A summary of the material with a few comments will give an idea of the scope of the practicals. They begin with comprehensive examination of the meaning of parameters of functions. In this exercise, students are presented with a graph of a function and asked to alter one or more of its parameters. The example given in Figures (1) and (2) shows what results when they alter both amplitude and phase of the standard sine function. They are able to see the results of each separate alteration virtually instantaneously and of course repeat it as many times as they feel necessary in order to understand the effect. The graph is designed to retain the original configuration for comparison. In the next class they perform similar tasks in Matlab in order to familiarise themselves with it.

The remaining calculus topics include limits, Euler's Method as stated above, Newton's Method, logarithms and exponentials through exploring pH and Newton's Law of

Cooling, and finish with Riemann Sums. The last one is particularly helpful as the visual effect of increasing the number of sub-intervals is quite dramatic. In the elementary linear algebra presented in the bridging course, spreadsheets are found to be particularly useful in helping the understanding of Gaussian elimination and the algebra of matrices. In the exercise, the students are asked to construct formulae which replicate the standard Gaussian elimination procedures and copy them tableau to tableau until a stage is reached where they can determine the type of solution set possible. This is done for a set of three equations in three unknowns and they can then use this template and substitute in different coefficient matrices to check their results for other example questions. Another exercise has the students performing many examples of matrix addition, multiplication, inversion and so on. They learn about the size of the result of matrix multiplications very quickly since the spreadsheet method of this procedure requires them to highlight the region for the result before performing the operation. Also, they receive error messages if they try to invert a matrix for which there is no inverse, so they learn to check the value of the determinant beforehand.

It is worth mentioning in passing that even though it was not used in this course, a particularly useful example of the spreadsheet utilisation is in introductory infinite series discussions. Students can be sceptical when it is proved that $\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}$ converges for $\alpha > 1$ and diverges for $\alpha \leq 1$. However, using a spreadsheet, one can quickly calculate partial sums including thousands of terms and graph the sums as a function of the number of terms and easily alter the value of α. This illustrates, rather than proves the result. In essence, this delineates exactly the purpose of this use of spreadsheets. It could be referred to as a re-naming of CAI as computer aided illustration rather than computer aided instruction. It is used not to solve mathematical problems, but rather to illustrate mathematical concepts to improve understanding.

The spreadsheet capabilities have been utilised in various other courses as well to great benefit. A first year course in mathematical modelling uses the population modeling attributes mentioned above, curve fitting using least squares methods and Markov modelling. A time series and forecasting course has utilised the curve fitting capabilities, Markov modelling, spectral analysis (complex analysis is possible in spreadsheets) and the Visual Basic programming module used for constructing autocorrelation and partial-autocorrelation functions and parameter estimation. Some of these obviously are also performed using more time series specific software, but as mentioned above, it is often the visual aspects of a spreadsheet package that are most useful for illustrating concepts. Table 1 gives a concise description of the mathematics concepts that have been illustrated in various courses and the attributes of spreadsheets which have made this medium appropriate.

Concept	Attribute(s) of Spreadsheets
Effects of change of parameters of graphs	Automatic recalculation Dynamic alteration of graphs
Gaussian reduction	Matrix configuration Automatic recalculation
Limits of sequences and functions	Relative referencing Automatic recalculation Recursion
Sums of Series	Recursion Graphics Automatic recalculation
Fundamental Theorem of Calculus	Graphics
Growth Models	Recursion Graphics
Linear programming	Solver
Matrix Analysis	Matrix arithmetic
Spectral Analysis	Complex arithmetic
Time Series	Visual Basic Programming
Optimisation	Solver

Table 1: Mathematical concepts illustrated using spreadsheets

3 Evaluation

There are two aspects to the evaluation that have been performed. One is an evaluation of how the use of the spreadsheet based practicals has been viewed by the students - do they see them as a useful learning tool? The other part of the evaluation is the estimation of the bridging program as an entry vehicle to the university and specifically, is the mathematics component of it helpful to the students in their future mathematical studies?

3.1 The Practicals

The University of South Australia requires staff to elicit an evaluation of every course from students as part of its quality assurance program. There are a set of core questions but also it is possible for lecturers to add questions if they wish to elicit information about particular aspects of a course to ascertain their value to the students. Over the five years that the course has been offered since the adoption of the practicals, the statement 'I found the practical sessions using Excel helped my understanding of mathematical concepts.', rated on a scale of 1 (strongly disagree) to 5 (strongly agree), has been used to test the students' opinion of this aspect of the course. The results have been consistently good, with averages for different groups ranging from 3.3 to 3.7 out of 5. Of significance also is the very low frequency of 'disagree' or 'strongly disagree'. Obviously, the results have to be taken with caution since it is not compulsory to fill in the form. There is another factor which can be viewed in various ways. Since this is an entry program, often a number of students who find that they are not coping will drop out of the program before the end and before they would have filled in the form. So therefore, one might say that the results are skewed by being predominately from those who have a higher probability of passing. On the other hand, they may also be the students who got the most out of the course, being the ones who expended the most effort to avail themselves of the material available. In summary, one is justified in claiming some measure of success for this particular instrument.

3.2 Measures of the Subsequent Success of Graduates of the Program

The program is designed to try and determine if candidates are suitable for coping with a degree program at a university. Thus the ultimate measure of success of the bridging program is the number of graduates who go on to complete a degree program. Obviously, there will be students who would have completed a degree program no matter how they were able to obtain entry. Specifically, there is the set of tests for adult entry alluded to previously. However, there is anecdotal evidence to suggest that at least a certain number of the students believe that having undergone a semester's work, rather than simply passed an entry requirement, has better equipped them for success in a degree program. It should be said that there is a natural lag in the results. Thus, when viewing students' subsequent performance, it was decided to count as a success students who appeared to be well on their road to a degree. Given this criterion, it was determined that of 78 students who have successfully completed the bridging program and gone on to attempt further study at the University of South Australia, 56 can be classified as successful, slightly over 70 per cent. There were actually more than 78 successful in this time, but 10 have not taken up study in this university, so it is possible they have begun studies in another one. Also, no students from 2001 have been included because of course they have not begun their university studies at the time of writing. It is worthy of noting that three of the successful students have completed Honours degrees.

The other measure of success that was investigated is how well the students' results in the Bridging Mathematics Course can be used as an indicator of their subsequent performance in university level mathematics. What has been calculated is the average of their subsequent mathematics results and then these have been regressed on their Bridging Mathematics scores. Figure 3 gives the data and the line of best fit. The correlation between the two sets of results is $r = 0.494, p < 0.01$.

There are a number of features of this regression analysis which are noteworthy. There are a number of outliers, and it is usual to question whether these should be included in the analysis. In this situation though, there is no reason to discard them. They are in the main indicative of students who managed to obtain passing results in the bridging program, but were not up to the task of a degree program. Another aspect is that the slope of the regression line is $m = 0.62$, indicating that if one were using the relationship to predict performance in mathematics courses in a degree program, there would be a systematic reduction from the results in bridging mathematics. Ideally, one would hope that there would be a slope of unity, but there are a couple of reasons why one wouldn't realistically expect that. One is that the assessment procedure for bridging mathematics is designed to favour a learning procedure more than that which would be present in the degree mathematics courses. In this, there is a higher emphasis on assignments, and less on examinations. The other main reason could be that the students in their degree programs are taking mathematics as supplementary to their main focus of study and thus a simple passing grade is sufficient. One would expect them to focus more on their main areas of study. It is encouraging though that there is a significant relationship between the two sets of results.

4 Conclusion

The effectiveness of the Bridging Program in Science and Engineering has been demonstrated through two measures. The success rate of students when they go on to degree programs has been more than 70 per cent. Additionally, the mathematics segment of the bridging program has proven to be a good indicator of success in mathematics courses in the degree program.

It has also been shown that study materials developed for this course involving explaining mathematical concepts using Excel spreadsheets have been rated as successful by the students. The particular aspects of spreadsheets that have been useful have been described, as well as how tools have been also developed for other mathematics courses. As stated previously, the spreadsheet formulations are used primarily to improve understanding of mathematical concepts through illustration, rather than for the solution of problems.

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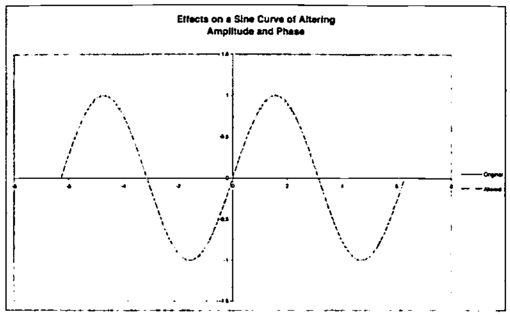

Figure 1: Original Configuration

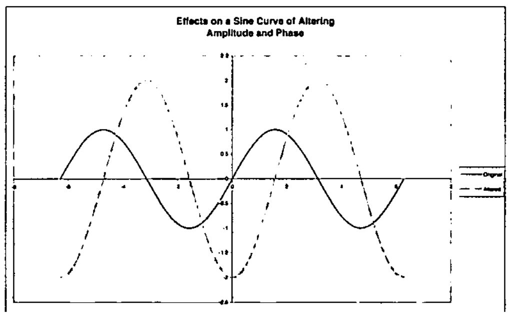

Figure 2: Altered Configuration

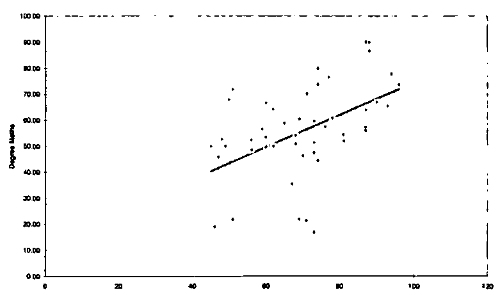

Figure 3: Mathematics Results as a Function of Bridging Mathematics Results

USE OF THE COMPUTER IN MATHEMATIC TEACHING FOR ENGINEERS: A POWERFUL CALCULATOR?

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ABSTRACT

Since a few years ago, the use of the computer as a tool to support teaching has been extended in University, especially in practical subjects. However, we should think about the following question: are computers under-used in mathematical teaching?

Nowadays, computers are being used in university teaching as great powerful calculators, but not as tools which help to carry out a substantial change in Mathematics teaching. Their use as a tool which helps to encourage the mathematical creativity of the students has not been extended yet. In most cases they are used as calculators, since they are used as a tool for calculation with numbers, although they are also used for algebraic manipulations, representation of curves, etc. All these uses are good to complement or simplify the traditional method of teaching, but they do not constitute an important improvement within Mathematics teaching.

This is partly due to the fact that we –the teachers– have been educated within an education system in which we have never received any training on this matter. Therefore we have not taken on the computer culture necessary to be able to prepare activities in order that our students receive the adequate mathematical backgrounds for their professional future that is ahead of them.

In short, the challenge we must face up in the future is overcoming this situation in order to use computers as tools for increasing mathematical creativity. As this is not an easy task, it is advisable that, at least, exercises with computer should be included in every subject related to Mathematics, and especially in courses for undergraduate students in Engineering.

In this paper, we present the experience carried out in courses for undergraduate students in Technical Telecommunication Engineering, using some kind of innovative exercises with computer. More concretely, we focus on the exercises of integration in several variables developed within the subject *Vectorial Analysis and Differential Equations*. We will end with the obtained conclusions and with the corresponding references.

1. Introduction

Since a few years ago, the use of the computer as a tool to support teaching has been extended in the University, especially in practical subjects. However, we should think about the following question: are computers under-used in mathematical teaching?

The answer is clearly affirmative. In most cases they are used as calculators, since they are used as tools for calculation with numbers, although they are also used for algebraic manipulations, representation of curves, etc.

All these uses are good to complement or simplify the traditional method of teaching, but they do not constitute an important improvement within Mathematics teaching.

This is partly due to the fact that we –the teachers– have been educated within an education system in which we have never received any training on this matter. Therefore we have not taken on the computer culture necessary to be able to prepare activities in order that our students receive the adequate mathematical backgrounds for their professional future that is ahead of them.

Nevertheless, an important improvement has been achieved: up to a few years ago, pupils of secondary education were taught, for instance, how to use log tables. Now, this is not explained to them anymore; instead, they learn how to operate with a computer or a calculator. Without any doubt, the present procedure is faster, but we must not make the mistake of thinking that we have improved the way of explaining the meaning and use of logarithms. The improvement we referred to above lies in the fact that this use is already considered as normal and it is not seen as an extraordinary thing or a new experience. That is, mathematicians have contributed with their work to the creation of tools, which are being used by non-mathematician professionals, up to the point of being fundamental in their work.

In short, the challenge we must face up in the future is overcoming this situation in order to use computers as tools for increasing mathematical creativity.

As this is not an easy task, it is advisable that, at least, exercises with computer should be included in every subject related to Mathematics, and especially in courses for undergraduate students in Engineering. Thus, it will be possible that future professionals be certainly capable of elaborating programs, which change substantially the way of teaching Mathematics.

The fact that the student could carry out this kind of exercises with any mathematical software will benefit him not only for the Mathematics subject in question, but he will also be able to use it in other subjects which need to make calculations of a certain complexity. Moreover, he will be ready to face up the resolution of problems that may occur in his professional future.

In this paper, we present the experience carried out in courses for undergraduate students in Technical Telecommunication Engineering, using some kind of innovative exercises with computer. More concretely, we focus on the exercises of integration in several variables developed within the subject *Vectorial Analysis and Differential Equations*.

2. Software and work setting

The choice of the program to be used is one of the most important matters of the entire process. Among the great amount of mathematical software now available on the market, we have chosen the program DERIVE® for several reasons:

1. First of all, this software is, from our point of view, easier to use than other mathematical programs which are "more powerful" as it operates with a very simple syntax.

2. Due to what we have set out in the previous point, the student is capable of starting to solve problems by using the program in a short period of time, since basic functions and operations are available in several menus.
3. It needs few requirements, with regard either to memory and physical space, when it comes to installing it.

In the following we will use the term *practical* for those exercises developed in a computer laboratory using DERIVE®.

The practicals are performed in a laboratory fitted with 30 units, with a maximum of two students per computer, and they are carried out in every Mathematics subject of the degree course.

The distribution of the practicals is as follows:

1. A first practical, which is two hours and a half long, to provide the student with the basic notions about how to use the program. This practical is carried out during the first weeks of the first four-month period, and it is aimed at students of the first year.
2. A specific practical for each subject where typical problems of such subjects are cleared out. This practical, which is two hours long, is carried out during the last days of the corresponding academic year in order to cover as much syllabus as possible. Another practical with particular contents can be fixed in the middle of the course, if the teacher thinks it would be convenient. Thus, for instance, it is usual, in the subject *Vectorial Analysis and Differential Equations*, to fix a practical on vectorial analysis and another one on differential equations, whenever the availability of time permits it.

Each specific practical consists of three different parts:

1. In the first part, theoretical-practical aspects which are to be developed in the practical are pointed out and DERIVE® own functions or macros to be created to solve eventual future problems are indicated.
2. The second part consists of examples of application of the concepts referred above, which will be solved during the course of the practical. Within these examples, and depending on the subject, some of the macros needed for the resolution of problems are elaborated.
3. In the third part, the students can solve a list of proposed problems in order that they reinforce the knowledge acquired during the practical.

The two hours of each specific practical are distributed as follows: one hour and a half for the first two parts and half an hour for the third part.

These practicals are carried out in a guided way, that is, by means of the teacher's explanations, so that, in order to obtain a better assimilation of the introduced contents, the teacher can make the appropriate comments. It is important to point out that the development of the practicals is not reduced to the mere execution of the application examples, but that each example is useful to remind the student the theoretical-practical aspects seen in the conventional lectures. Thus, these practicals serve also as a review of the subject.

3. Innovative aspect of the practicals

All that has been commented before would fit in the development of classical practicals for Mathematics subjects. We now go on by presenting our contribution, which consists of the elaboration of innovative practicals insofar as the student participates actively in their creation. We emphasize that, in these practicals, apart from solving typical problems of the subject in question, the students elaborate macros in order to solve such problems. This elaboration of macros constitutes the principal innovative aspect of the practicals and requires the student to have an exhaustive knowledge of the subject.

Thus, for instance, for the elaboration of a macro which proves if a differential is an exact one, the student will need to know which one is the condition for this differential to be exact. Whereas for the elaboration of macros to calculate triple integrals he will have to take into account the following elements: the function to be integrated, the system of coordinates and the three variables of integration with their corresponding limits of integration. Besides, as the order of integration is important, he will have to take it into account when it comes to elaborating such macros. Obviously, the fact that the student himself is the one who elaborates the macros has a very positive influence when it comes to applying the macros in order to solve concrete examples.

So, with this kind of practicals the student does not just solve problems but he creates the macros to solve them as well. Our aim is that the use of computers will not further be reduced only to its most classical and usual application (that is, making calculations as if it were a powerful calculator), but that the computer is also used as a tool that encourages mathematical creativity.

4. Development of the practical about vectorial analysis

By way of example, now we develop the practical carried out as part of the subject *Vectorial Analysis and Differential Equations* about vectorial analysis. Note here that the original language has been conserved in the names of the macros.

Practical with DERIVE
Vectorial Analysis and Differential Equations
Technical Telecommunication Engineering

First of all, for the correct developing of the practical, it is necessary to load the file ANALVEC.MTH (use the option **Load – Utility file**). This file contents the definition of some commands to solve the exercises.

Gamma and Beta functions. Scalar and vector fields.

- *Gamma function*
  - Syntax: GAMMA(value)
  - Example: gamma(7/2) to calculate $\Gamma\left(\frac{7}{2}\right)$
- *Beta function*
  - Syntax: BETA(value1,value2)
  - Example: beta(3/2,5) to calculate $\beta\left(\frac{3}{2}, 5\right)$
- *Gradient*
  - Syntax: GRADIENTE(scalar field)
  - Example: gradiente(x^2+y^2+z^2) to calculate the gradient of the scalar field $x^2 + y^2 + z^2$
- *Divergence*
  - Syntax: DIVERGENCIA(comp1,comp2,comp3)
  - Example: divergencia(x^3y,2xzy,z^2) to calculate the divergence of the vector field $(x^3y, 2xzy, z^2)$
- *Curl*
  - Syntax: ROTACIONAL(comp1,comp2,comp3)

- Example: rotacional(x^3y,2xzy,z^2) to calculate the curl of the vector field $(x^3y,2xzy,z^2)$

- *Laplacian*
  - Syntax: LAPLACIANO(scalar field)
  - Example: laplaciano(x^2+y^2+z^2) to calculate the laplacian of the scalar field $x^2+y^2+z^2$

Line integrals

- *Exact differential in R^2*
  - Syntax: DIFERENCIALEXACTA2(comp1,comp2)
  - Example: diferencialexacta2(y^2,2xy) to check if $y^2\,dx + 2xy\,dy$ is an exact differential
- *Exact differential in R^3*
  - Syntax: DIFERENCIALEXACTA3(comp1,comp2,comp3)
  - Example: diferencialexacta3(x+z,-(y+z),x-y) to check if $(x+z)\,dx - (y+z)\,dy + (x-y)\,dz$ is an exact differential
- *Potential function in R^2*
  - Syntax: POTENCIAL2(comp1,comp2)
  - Example: potencial2(y^2,2xy) to calculate the potential function of $y^2\,dx+2xy\,dy$. If the differential is not an exact one, the macro answers "this is not an exact differential"
- *Potential function in R^3*
  - Syntax: POTENCIAL3(comp1,comp2,comp3)
  - Example: potencial3(x+z,-(y+z),x-y) to calculate the potential function of $(x+z)\,dx - (y+z)\,dy + (x-y)\,dz$. If the differential is not an exact one, the macro answers "this is not an exact differential"
- *Line integral of non-exact differentials in R^2*
  - Syntax: LINEAPARAMETRICA2(comp1,comp2,cur1,cur2,a,b)
  - Example: lineaparametrica2(xy^4,x^2y^3,t^3,t,0,1) to calculate the line integral of (xy^4,x^2y^3) along the curve $y^3 = x$, from $(0,0)$ to $(1,1)$
- *Line integral of non-exact differentials in R^3*
  - Syntax: LINEAPARAMETRICA3(comp1,comp2,comp3,cur1,cur2,cur3,a,b)
  - Example: lineaparametrica3(3x^2+6y,-14yz,20xz^2,t,sqrt(t),t^(1/3),0,1) to calculate the line integral of $(3x^2+6y,-14yz,20xz^2)$ along the curve $x = t$, $y^2 = t$, $z^3 = t$, from $(0,0,0)$ to $(1,1,1)$

Double and triple integrals

- *Double integration in cartesian coordinates*
  - Syntax: DOBLE(function,var1,lim1,lim2,var2,lim3,lim4)
  - Example: doble(xy,y,0,x+1,x,0,2) to integrate the function $f(x,y) = xy$ in the region bounded by $x = 2$, $y = x+1$, $y = 0$ and $x = 0$
- *Double integration in polar coordinates*
  - Syntax: DOBLEPOLAR(function,r,r1,r2,theta,theta1,theta2)
  - Example: doblepolar(x^2y^2/(x^2+y^2),r,0,1,theta,0,pi) to integrate the function $f(x,y) = \dfrac{x^2y^2}{x^2+y^2}$ in the region bounded by $x^2+y^2 = 1$ with $y \geq 0$
- *Triple integration in cartesian coordinates*
  - Syntax: TRIPLE(function,var1,lim1,lim2,var2,lim3,lim4,var3,lim5,lim6)
  - Example: triple(x+yz,z,0,5-x-y,y,0,5-x,x,0,5) to integrate the function $f(x,y,z) = x+yz$ in the solid bounded by $x+y+z = 5$, $x = 0$, $y = 0$ and $z = 0$

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- *Triple integration in cylindrical coordinates*
  - Syntax: TRIPLECILINDRICA(function,z,z1,z2,r,r1,r2,theta,theta1,theta2)
  - Example: triplecilindrica(sqrt(x^2+y^2),z,r,1,r,0,1,theta,0,2pi) to integrate the function $f(x,y,z) = \sqrt{x^2+y^2}$ in the solid bounded by $z = 0$, $z = 1$ and $z^2 \geq x^2+y^2$
- *Triple integration in spherical coordinates*
  - Syntax: TRIPLEESFERICA(function,r,r1,r2,theta,theta1,theta2,alpha,alpha1,alpha2)
  - Example: tripleesferica(x+y+z,r,0,2,theta,0,2pi,alpha,0,pi/2) to integrate the function $f(x,y,z) = x+y+z$ in the solid bounded by $x^2 + y^2 + z^2 = 4$ with $z \geq 0$

Surface integrals. Gauss' theorem

- *Surface area in polar coordinates*
  - Syntax: AREASUPERFICIERXYPOLAR(z surface,r,r1,r2,theta,theta1,theta2)
  - Example: areasuperficierxypolar(sqrt(x^2+y^2),r,0,1,theta,0,2pi) to calculate the area of the part of the surface $z^2 = x^2 + y^2$ inside of $z = 2 - x^2 - y^2$
- *Unit normal vector to an explicit surface*
  - Syntax: COSENO1(explicit surface)
  - Example: coseno1(x^2+y^2) to find an unit normal vector to $z = x^2 + y^2$
- *Unit normal vector to an implicit surface*
  - Syntax: COSENO2(implicit surface)
  - Example: coseno2(x^2+y^2+z^2-4) to find an unit normal vector to $x^2 + y^2 + z^2 = 4$
- *Flux of a vector field. Double integral in polar coordinates*
  - Syntax: FLUJORXYPOLAR(comp1,comp2,comp3,z surface,r,r1,r2,theta,theta1,theta2)
  - Example: flujorxypolar(x,2y,x+z,x^2+y^2,r,0,4,theta,0,2pi) to calculate the flux of $(x,2y,x+z)$ over the part of the surface of the paraboloid $z = x^2 + y^2$ for which $0 \leq z \leq 16$
- *Gauss' theorem. Triple integral in cylindrical coordinates*
  - Syntax: FLUJOGAUSSCILINDRICA(com1,com2,com3,z,z1,z2,r,r1,r2,theta,theta1,theta2)
  - Example: flujogausscilindrica(x,2y,3z,z,r,2,r,0,2,theta,0,2pi) to calculate the flux of $(x,2y,3z)$ over the closed surface bounded by the cone $z^2 = x^2 + y^2$ and the planes $z = 0$, $z = 2$

Examples

1. Build the macro BETA.
2. Evaluate $\Gamma\left(\frac{9}{2}\right)$ and $\beta\left(\frac{7}{2}, 9\right)$.
3. Build the macros GRADIENTE and LAPLACIANO.
4. Given the scalar field $f = 2x^2y\text{-}xz^3$, calculate its gradient and laplacian.
5. Given the vector field $F = (xz,-y^2,2x^2y)$, calculate its divergence and curl.
6. Build the macros DIFERENCIALEXACTA2 and POTENCIAL2.
7. Find, when possible, the potential function of:
   a. $(xy^2 + x + 1)$ dx + $(x^2y - 2)$ dy
   b. $(yz+y+z)$ dx + $(xz+x+z)$ dy + $(xy+x+y+2z)$ dz
8. Calculate the line integral of (xy^2+x+1, x^2y-2) along any path from $(1,2)$ to $(-2,5)$.
9. Calculatc the line integral of $(yz+y+z,xz+x+z,xy+x+y+2z)$ along the segment that joins $(1,2,3)$ with $(-2,7,3)$.
10. Build the macro LINEAPARAMETRICA2.
11. Calculate the line integral of $(xy,2x)$ along the ellipse $x^2 + \frac{y^2}{4} = 1$.
12. Build the macro DOBLE.

13. Integrate the function $f(x,y) = x^2 + y^5$ within the rectangle with vertices $(0,0)$, $(2,0)$, $(2,1)$ and $(0,1)$.
14. Calculate the area of the circumference $x^2 + y^2 = 4$.
15. Build the macros TRIPLE and TRIPLECILINDRICA.
16. Integrate the function $f(x,y,z) = x + yz$ in the solid bounded by $x+y+z = 5$, $x = 0$, $y = 0$ and $z = 0$.
17. Integrate the function $f(x,y,z) = \sqrt{x^2+y^2}$ in the solid bounded by $z = 0$, $z = 1$ and $z^2 \geq x^2+y^2$.
18. Calculate the volume of the sphere $x^2 + y^2 + z^2 = a^2$.
19. Build the macro AREASUPERFICIERXYPOLAR.
20. Calculate the area of the portion of the sphere $x^2 + y^2 + z^2 = 16$ outside of the paraboloid $x^2 + y^2 + z = 16$.
21. Build the macro COSENO1.
22. Calculate an unit normal vector of the following surfaces:
    a. $z = x^2 + y^2$.
    b. $x^2 + y^2 + z^2 = 4$.
23. Build the macro FLUJORXYPOLAR.
24. Calculate, using two different methods, the flux of the vector field (x,y,z) over the closed surface bounded by $x^2+y^2 = 4z$ and $z = 4$.

Exercises

1. Calculate $\Gamma\left(\frac{7}{2}\right)$, $\Gamma(6)$, $\beta(5,6)$ and $\beta\left(\frac{5}{2},\frac{7}{2}\right)$.
2. Given the scalar field $f = 2x^2y^3 - xy^2z^5$, calculate its gradient and laplacian.
3. Build the macros DIVERGENCIA and ROTACIONAL.
4. Given the vector field $F = (xyz, -y^2z, 2x^2y+z)$, calculate its divergence and curl.
5. Build the macros DIFERENCIALEXACTA3 and POTENCIAL3.
6. Find, when possible, the potential function of:
    a. $(x+y+z)\,dx - (y+z)\,dy + (x-y)\,dz$
    b. $ye^{xy+z}\,dx + xe^{xy+z}\,dy + e^{xy+z}\,dz$
    c. $(ye^{xy}+x)\,dx + (xe^{xy}+3y)\,dy$
    d. $(2xy+y^2)\,dx + (x^2+y^3)\,dy$
7. Calculate the line integral of $(ye^{xy+z}, xe^{xy+z}, e^{xy+z})$ along any path from $(1,2,-3)$ to $(-2,5,11)$.
8. Calculate the line integral $(ye^{xy}+x, xe^{xy}+3y)$ along the segment that joins $(1,2)$ with $(-2,9)$.
9. Build the macro LINEAPARAMETRICA3.
10. Let $A = (3x^2+6y, -14yz, 20xz^2)$. Evaluate $\int_C A\,dr$ where C is the path from $(0,0,0)$ to $(1,1,1)$ given by:
    a. The curve $x = t$, $y^2 = t$, $z^3 = t$.
    b. The segment that joins both points.
11. Calculate the line integral of $(xy, x+3)$ along the ellipse $x^2 + \frac{y^2}{4} = 1$.
12. Integrate the function $f(x,y) = x^2+y^5$ in the region bounded by $y = x^2$ and $y = 2 - x^2$.
13. Build the macro DOBLEPOLAR.
14. Calculate the area of the region bounded by $x^2+y^2 = 2ax$, $y = x$ and $y = 0$.

15. Integrate the function $f(x,y,z) = x+yz$ within the solid bounded by the planes $x = 0$, $x = 2$, $y = 0$, $y = 1$, $z = 1$ and $z = 3$.
16. Build the macro TRIPLEESFERICA.
17. Calculate the volume of the solid bounded below by the cone $z = +\sqrt{x^2+y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 9$ in both cylindrical and spherical coordinates.
18. Calculate the volume of the solid bounded by the cylinder $x^2 + y^2 - bx = 0$ and the cone $x^2 - z^2 = -y^2$.
19. Calculate the area of the part of sphere $z^2 = x^2 + y^2$ inside the paraboloid $z = 2 - x^2 - y^2$ with $z \geq 0$.
20. Build the macro COSENO2.
21. Calculate an unit normal vector of the following surfaces:
    a. $z = 2 - x^2 - y^2$.
    b. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z^2}{c^2}$.
22. Build the macro FLUJOGAUSSCILINDRICA.
23. Calculate, using two different methods, the flux of the vector field $(2z,x,y^2)$ over the closed surface bounded by $z = 4 - x^2 - y^2$ and $z = 0$.

5. Conclusions

After having carried out this kind of practicals for the last four years, we have been able to verify advantageous results either for students and teachers. Among other aspects, we can emphasize the following:

1. The student is provided with a powerful tool for the resolution of problems that, besides, can be used to verify the results obtained in the exercises he does.
2. The student feels more motivated towards the subject because of the chance of working with the computer to solve problems that occur to him.
3. This motivation leads to a better preparation of the subject by the student, what, at the same time, entails that classes can be given more easily since the student is more prepared and receptive.
4. Before carrying out the specific practicals of each subject, the student is reminded that in such practicals some examples and exercises concerning the whole content of the subject will be solved, so that he must get conveniently prepared prior to them. This, together with the fact that during the practical itself the most important theoretical-practical concepts of the subject are reminded, makes the student be better prepared when it comes to facing up the final exam.
5. As the attendance at the practicals is voluntary and students must register previously on a list, the students who attend these lectures are those who feel really motivated towards the subject and towards the carrying out of these practicals, what has a very positive influence on their progress.

Finally we would like to insist on the innovative aspect of this kind of practicals that gives rise to a double consideration. On the one hand, these practicals require something more than the mere fact of using the computer as a tool for calculation, because, if the student wants to solve a problem with the macros that have been created, he has to set it out previously. On the other hand, the elaboration of the macros by the student requires him to have some knowledge of programming and some mathematical reasoning. Therefore, with this kind of practicals, we show that computers are more than powerful calculators.

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FRAMEWORK FOR INSTRUCTION AND ASSESSMENT ON ELEMENTARY INFERENTIAL STATISTICS THINKING[1]

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ABSTRACT

The main objective in this paper is to describe a framework to characterize and assess the learning of elementary statistical inference. The key constructs of the framework are: populations and samples and their relationships; inferential process; sample sizes; sampling types and biases.

To refine and validate this scheme we have taken data from a sample of 49 secondary students sample using a questionnaire with 12 items in three different contexts: concrete, narrative and numeric. Theoretical analysis on the results obtained in this first research phase has permitted us to establish the key constructs described below and determine levels in them. Moreover this has allowed us to determine the students' conceptions about the inference process and their perceptions about sampling possible biases and their sources.

The framework is a theoretical contribution to the knowledge of the inferential statistical thinking domain and for planning teaching in the area.

Keywords: inferential statistics, theoretical framework, secondary level.

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1. Introduction

One of the characteristic features of the current society is the enormous technological development that has been applied for the social and economic improvement of the citizens. In this technological society information and communication play key roles and education should provide the citizens with the necessary elements to develop within. The access to information, the use of data, data analysis and the taking of informed decisions in uncertain situations, the understanding and the capacity of criticism of the information provided by the media, etc., form part of the new formative necessities of citizens in the current world. As an answer to these new social necessities the educational systems have introduced reforms in the curricula that affect statistical education in many countries and at all teaching levels, for example, MEC(1990), Junta de Andalucía (1992; 1994; 1997), NCTM(2000). One of the novelties of the reforms in Spain has been the introduction of statistical inference in the curricula for the compulsory teaching level (ESO, 12-16 years old) and the Bachillerato (16-18 years old). Parallel to this, the introduction of more and new statistical contents, and at more elementary teaching levels each time, outlines a bigger necessity of further research on the learning of these contents and their throughout the student's schooling years. Although we already have some results of research carried out in this respect in the field of data analysis and of probability, this field can be considered mainly, as emergent and developing (Shaughnessy, 1992; Mokros and Russell, 1995; Gal and Garfield, 1997; Batanero and cols., 1994; Jones and cols., 2000). In the field of statistical inference the research works carried out are even more scarce (Watson, 2000; Jacobs, 1996; Moreno and Vallecillos, 2001; Vallecillos, 1998; in print). Jones and cols. (2000) propose a framework to characterize the children's statistical thinking based on the cognitive development model described by Biggs and Collis (1991). In our work we have tried to develop a similar framework for the case of statistical inference thinking, so finally we can have an applicable general framework for elementary, descriptive and inferential statistics. To do that, on a review of previous research works and based on our own researching experience on the topic, we have built an initial theoretical framework to evaluate the learning of statistical inference in secondary education students. Then, we have elaborated a questionnaire that 49 students of this level have completed and we have analyzed the results obtained. Finally, by incorporating the obtained information, we have refined the initial framework and we have elaborated the conclusions of this phase of the study.

This theoretical framework of analysis developed to evaluate the learning of the basic statistical inference has been validated with secondary level students but it can be used to plan teaching of the topic and to evaluate the learning of the students in introductory courses at the university level too.

2. Theoretical Framework

Teachers need a good knowledge about how students understand statistical concepts and how they engage in solving problems. Students exhibit statistical thinking over the different school levels and develop in time. So the framework is situated in a general cognitive development model (Bigg and Collis, 1982; 1991). These authors describe three levels of observed learning outcome:

1. Unistructural responses, those taking in to consideration only one aspect of the concept or task considered;

2.Multistructural responses, those in which several aspects of the concept or task are considered but not all, and

3.Relational responses, those in which all aspects are considered and integrated exhibiting an integrated understanding and a meaningful learning.

Situated in this general cognitive model (Bigg and Collis, 1982; 1991), Jones and cols. (2000) formulate a framework to characterize children's statistical thinking. They define four constructs, describing, organizing, representing and analyzing and interpreting data. Within each one of these constructs they establish four thinking levels representing a continuum from idiosyncratic to analytic reasoning. Results of the study, authors say, confirm that children's statistical thinking can be described according to the framework. Our initial framework for inferential statistical thinking is also situated in the general cognitive model (Bigg and Collis, 1982; 1991) and is like Jones and cols. (2000) framework with four construct and four thinking levels within each one. Nevertheless, we consider two related aspects for determining construct and levels in the framework: the statistical content and the result of the questionnaire filled in by students. In the initial framework we have determined the constructs and level in statistical content based; afterwards we have considered the students' responses to the questionnaire too in order to establish constructs and levels in them in the inferential statistical framework. We have established four constructs, population and samples and their relationships (PS), inferential process (IP), sample sizes (SS) and sampling types and biases (ST), and four thinking levels in each one.

3. Method

3.1. Aims of this research

The objectives of this paper are mainly three: a) to develop an initial framework to characterize and assess the learning of basic statistical inference; b) to elaborate a questionnaire to asses statistical inference learning at secondary level; c) to test framework in order to get the first objective and validate and refine it with the questionnaire results.

3.2. The constructs

We seek to describe and to fix, in the first place, the elements and key concepts of statistical inference for the basic training of the students at introductory teaching levels. To do that we will use the expression "construct" that is used in the field of Psychology to describe complex phenomenon such as the personality, motivation, etc., of difficult definition. For us each "construct" represents a category of concepts all of them under only one epigraph in which they can be described. We believe that the description of the samples, the populations of the ones that have been extracted and their relationships; the questions related with size, the selection methods and the possible sources of biases in the selection of samples are important conceptual nuclei that are in the basis of learning in statistical inference that can be described in these terms. All these elements have already been recognized previously as such by teachers, researchers or curricular documents. Our proposal includes a novelty: we have included as a differentiated construct the one that we have called "Inferential process" because we find that it deserves a special mention. Indeed, the students sometimes do not distinguish well between population and sample and they are not conscious, therefore, that the conclusions obtained in the study of a sample are not those that we need, and it is necessary to make them aware that a generalization under the conditions of the study is carried out and therefore subject to certain limitations and to the possibility of error. Other times the students do not admit the generalization possibility and they only believe in the carrying out of census and so it is necessary to make them reflect about the impossibility of these in certain situations, such as destructive tests or with unbroachable temporary or economic costs. We describe the key constructs below:

A) Populations and samples and their relationships

We try to understand the ideas of the students about the sample and population concepts as well as the relationship between them. These concepts are intuitively used in many environments of daily life, outside the school environment. Concepts such as the variability and sample representativeness have a great incidence in many aspects of social life. Kahneman and cols. (1982) have investigated thoroughly on these aspects and find that people reason using heuristics that lead them to erroneous conclusions most times. Among secondary level students the presence of thinking heuristic has also been detected (Rubin and cols., 1991; Moreno and Vallecillos, 2001). In another order of things, we are also interested in discovering if the scheme 'part-everything' used in the teaching of contents of numerical type such as the fractions and rational numbers, is also used in this context and how it is used.

B) Inferential process

We try to understand how the students conceive the process that allows them to describe the population on the basis of the information obtained from the observation of one of its samples. To do that we have determined the students' conceptions (Artigue, 1990) about the process, such as theoretical models built which supposedly guide the students' answers.

C) Sample sizes

In order to get a good learning relative to the sample concept it is necessary to keep two aspects that are essential in it in mind: the sensitization of the students about the importance of the sample size and the appreciation of the same when judgements are emitted or decisions are made based on samples. The works of Kahneman and their colleagues determined the "law of the small numbers" as a very widely believed among the population, even among people with statistical training. This belief is part of the representativeness heuristic leading people to believe that the samples, even the very small ones, always reproduce the population's characteristics from which they proceed, showing an insensitivity towards the size of the sample (Kahneman and cols., 1982).

D) Sampling types and biases

The sampling based on the randomization of the statistical units provides the representative samples of the populations under study. In this section we consider two aspects basic for the good teaching of the topic: the sensitization of the students about the importance of the randomization in the selection of the samples used as well as about the presence of biases in any other case and of the derived pernicious effects of the use of biased samples.

3.3. The inferential statistical thinking framework

In Table 1 we described it.

Table 1: The inferential statistical thinking framework

Construct	Level 1: Idiosyncratic	Level 2: Transitional	Level 3: Quantitative	Level 4: Analytical
Populations and samples and their relationships (PS)	Usual population concept Usual sample concept Neither identifies population nor sample Confuses population and sample	Statistical population concept Population of discrete type Identifies population or sample in concrete context	Statistical population concept Population of discrete/continuous type Identifies population and sample in certain contexts only	Statistical population concept Sample space concept Identifies and poses in relation to population and sample in all contexts
Inferential process (IP)	Subjective criteria Previous conception	Subjective criteria and/or numeric with errors Deterministic conception	Numeric criteria with informal expression Identity conception	Numeric criteria and formal expr ession Correct conception
Sample sizes (SS)	Sample size characteristic recognizing Sample size insensitivity	Sample size characteristic importance recognizing Recognizes sample size interest in some context	Sample size and estimation relationship Recognizes sample size and/or put in relationship with estimation in numeric contexts	Sample size and estimation relationship in all contexts Sample size sensitivity in all contexts and in relation with characteristic estimated
Sampling types and biases (ST)	Sampling concept Different sampling types possibility recognizing Type sampling insensitivity Biases insensitivity	Sampling methods Randomization Different sampling types recognizing Recognizes the biases possibility	Random sampling types Recognizes different random sampling types Recognizes the biases possibility	Sampling method and characteristics estimation Recognizes the more adequate sampling type Biases sensitivity

3.4. Participants

Participants are 49 secondary students from two Spanish high schools distributed in two different courses. 30 students from 3° de ESO (14-15 years old) without any previous statistics information and 19 from COU (17-18 years old). COU is the last course at secondary level and these students had some statistical knowledge.

3.5. Questionnaire

The questionnaire was made up of two different parts with 12 questions each one about elementary inference concepts. Items are presented in three contexts, concrete, narrative and numeric. We include two different items, one of Part I and one of Part II of the questionnaire for readers illustration. The complete version of the questionnaire may be obtained from authors on request.

Item I.1. We have a bag with 100 balls of the colors red and green. We want to study the number of balls of each color. To do that we take 25 balls from the bag and we observe that 14 of them are red and 11 are green. Write:

a) The set objects we are studying:

b) The sample observed:

Item II.3. The town council is starting a campaign for explain to the citizens what they may do when they need to get rid of old furniture. They want to know if the instructions have been clear and understandable. The population of the city is 300.000 people and so they decide to ask 2000 adult citizens about their opinion. They are asked in small and big quarters, some male and some female, some old and some young people, some who live in flats and who live family houses and so on. They think that they have a varied group of people. They are 73% of these people say that the given instructions are clear and the 27% say not.

¿What can you say to the town council about the percentage of adults in the whole city who think that the given instruction are clear?:

a) 50% because probably half of the people think the instructions are clear and the other half think they are not.

b) 73% because the adults asked gave a general idea about the results as if the whole population were asked.

c) I can't say anything because the result of the inquiry could have been anything.

d) I can't say anything because I can't ask all the adults in the city.

e) Were .... because....................................

3.6. Procedure

Third course ESO's students filled in Part I questionnaire in one 60 minute session and Part II in another 60 minute session. In some questions in Part I of the questionnaire the researcher intervened for concrete material handling required or to explain to students what they are being asked. Then they fill in the questionnaire individually. COU students use only a 60 minute session to individually fill in both parts of the questionnaire.

3.7. Results

A) Populations and samples and their relationships (PS): a lot of students have not identified the sample and population studied correctly, although there are notable differences in correspondent items success percentages in the different contexts. The higher age group (COU) have got better global results than the ESO group and in the numerical context. About two thirds of the ESO students can not identify either population or sample while in COU only a fifth of them cannot do so.

B) Inferential process (IP): we have grouped students' responses under three headings characterizing each one determined conceptions. They are summarised below:

C1) Correct conception: the inference process is a chance ruled process and can not permit the precise population characteristics determine on the basis of the information obtained from one of its samples.

C2) Identity conception: the inference process permits to us describe the population with characteristics identical to the one of its samples.

C3) Previous conception: the population has characteristics described by previous ideas and not for the ones observed in the extracted sample.

C4) Deterministic conception: the population can only be described by doing a census and not by studying samples extracted from it.

In this category we have found very great differences between contexts: not all conceptions appear in each context, e. g., in narrative context the previous conception do not appear and the deterministic conception only appear in the narrative contexts.

C) Sample sizes (SS): in the lower age group (ESO) about 50% of the students do not take in to consideration the sample size and in the COU group the success percentage is a little better but only a quarter of all the students relate the sample size and the population characteristic estimation.

D) Sampling types and biases: most of all the students recognizes the different sampling types and most of the higher age group students, the different kinds of random sampling too, e. g., simple versus stratified sampling.

4. The revised inferential statistical thinking framework

This actual inferential statistical thinking framework needs to be tested in an other field: the instructional field. The students who have participated in the research have been taken in their natural classes and without any special preparation to do that. In order to prove the inferential statistical thinking framework for instruction usefulness we are designing didactic resources to use in secondary classrooms and based on the four constructs described previously. We need to take data about teaching and learning in the classroom functioning of the framework. Selected students will be interviewed afterwards and the inferential statistical thinking framework will be profoundly and globally revised. That will be the second research phase.

5. Conclusions

In this paper we have presented an initial inferential statistical framework for instruction and assessing secondary student learning of the same. We have in synthesis described the four constructs and the four levels within each one that the scheme constitutes. We have tested it with secondary students from two different courses in Spain. With the results obtained from the questionnaire filled in by them we have revised and completed the inferential statistical framework that we have describe before. As a first general conclusion we have experimented several difficulties in two different areas mainly, of a theoretical and of a didactical nature. In the theoretical area, to determine the essential theoretical aspects, concepts or constructs that are basic and essential and so it is necessary to include them it in any general elementary curriculum for statistical education for all citizens in order to make peoples aware and be able to take informed decisions. In the didactic area, once the adequate curriculum content has been determined, how do the students get the bests results?. The inferential statistical framework in our actual personal contribution to these problems. This research is now completing its instructional slope, developing classroom resources for testing it and for a global revision of the inferential statistical framework.

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THE INTEGRATION OF INTERACTIVE EXCEL TUTORIALS INTO A BRIDGING (PRE-CALCULUS) COURSE

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ABSTRACT

Inadequate mathematical skills and understanding act as a barrier to students wishing to study a variety of courses at university. At the University of Cape Town a first year course called "Effective Numeracy" is offered to such students, with the objective of supporting their study of other subjects and preparing them for mathematics courses in later years. Addressing the problem of the lack of mathematical and quantitative reasoning skills in these students is very challenging, and calls for the use of various techniques and approaches.

Excel workbooks coded with VBA have been found to be a very effective environment for creating interactive tutorials that students can use for self-paced study. The Excel tutorials constitute one third of the course (in terms of time and credit), and are very firmly integrated into the overall curriculum of the course. Although there are slight variations in timing of delivery (because the class is divided into three groups), the content of any tutorial session consolidates and enriches material covered in the classroom within the same week.

A large part of the curriculum is devoted to pre-calculus, focussing particularly on the understanding of the function concept and the idea of slope. The design of the tutorials includes a custom-built "graphing device" which can be incorporated into any Excel workbook at every point that it is required for the execution of the exercises. This means that a student can easily produce the graph of a function without leaving the context of the tutorial and interrupting the interactive "conversation" of the exercise.

This paper reports on our experiences in implementing this multimedia intervention and in attempting to answer questions about how the students experience these tutorials.

1. Introduction

At the University of Cape Town (UCT), it is recognised that many students have inadequate quantitative literacy and mathematical skills to enable them to cope with their chosen course of study (Brink, 2001). Apart from the fact that the traditional approach to teaching mathematics in schools does not develop sufficient levels of quantitative literacy (as discussed by Hughes-Hallett, 2001), in South Africa there is a legacy of educational disadvantage that still affects the majority of the population. Under Apartheid, there was an explicit policy of denying black students access to mathematical and scientific knowledge, and it will take many years to reverse the effects that this has had on the education system.

For disadvantaged students who are studying economics-related subjects in the Humanities Faculty, the Mathematics Department at UCT provides a first-year course entitled "Effective Numeracy", which has the objective of increasing students' level of quantitative literacy, supporting their studies in the rest of their programme and preparing them for mathematics and statistics courses in later years. The philosophy and development of this course over the last five years is outlined in papers by Brink (2001) and Frith and Prince (2001). One of the most important principles in the design of the course is to create a non-threatening co-operative environment for learning, that will allow students to develop confidence in their ability to succeed, as many of these students have a high degree of "maths-anxiety". The very wide range of ability and prior experience, particularly in the use of computers, within the classes, poses a significant challenge to the realisation of this objective.

Currently the course consists of two semesters with slightly different objectives. In the first semester the emphasis is on developing quantitative literacy through the use of context-based applications. The second semester of the course content is closer to a more traditional pre-calculus (bridging) course.

One third of this course throughout the year (in terms of classroom time and credits) is conducted in the computer laboratory through the medium of Excel-based interactive tutorials, which are tightly integrated into the curriculum. In the first semester there is an emphasis on learning to use Excel in "real-life" contexts to perform data analysis and to represent quantitative information graphically. In the second semester the Excel tutorials are intended to support the learning of mathematical concepts, in particular, the idea of functions, graphs and gradients. This paper focuses on the use of computer tutorials in this part of the course.

2. Excel-based interactive tutorials

In designing the Effective Numeracy course, we have assumed that using computer-based tutorials in the "bridging" component of the course will enhance the students' understanding of the mathematics concepts. There are numerous reasons to believe that properly-used computer tutorials can add value to a mathematics course, which were comprehensively reviewed by Kaput (1993). The most obvious advantages that are exploited in our course are the computational and graphical abilities of Excel, which allow more examples to be done by the student and allow the use of more realistic values in these examples. The Excel environment facilitates the understanding and representation of functions in the four different ways that the course emphasises: with a formula, with a table of values, graphically and verbally (The "Rule of Four"of Hughes-Hallett, Gleason, McCallum, et al., 1998).

The computer also makes it possible to illustrate certain concepts and processes graphically in a dynamic manner, which is difficult to achieve on paper or a blackboard, which we believe assists the students to develop the ability to produce their own mental images in situations where they are helpful. Obviously we also believe that there is an advantage in approximating the "conversation" of an individual tutorial situation where each student receives immediate feedback as they work through a computer tutorial at their own pace.

There is a great deal of experience of and knowledge about using Spreadsheets to enhance learning in mathematics, some of which can be accessed through a website at Vienna University (Neuwirth, 2001). Some practitioners have also used extensive Visual Basic for Applications (VBA) code to create Excel-based tools and environments that can be used to enhance learning (for example Carr, 2000). Our approach is to make use of VBA code within an Excel workbook, to program self-contained "tutorial-simulations" (Laurillard, 1993) that can be used semi-independently by the students without the support of any other materials.

Thus a typical tutorial will consist of several electronic Excel worksheets, one stating objectives, some containing the interactive presentation of relevant mathematics content, and others comprising examples and exercises, many of which make use of a custom-built "graphics tool". This allows the student to produce the graph of a function (as they would with a separate graphics package), but at the relevant place within the worksheet and without interrupting the interactive conversation that comprises the tutorial.

Some of the advantages of using Excel as an authoring environment are that the students are familiar with it, that all the functionality of Excel remains accessible to the students throughout the tutorial, and that Excel is so commonly used that the tutorials become extremely portable. In addition it is very easy in this environment to change the content, for instance to modify an example or introduce different data. It is also relatively easy to program fairly sophisticated interactions and animations using the in-built VBA macro language, and to record students answers to a database for automated processing.

3. Integration of tutorials into the course

Every week the students spend one 2-hour session in the computer laboratory, and two 2-hour periods in the workshop-lecture environment. The actual mathematical material dealt with in the computer tutorial for any particular week is always covered in the classroom within that same week, and the lecturers concerned are encouraged to make the links between the laboratory and the classroom materials explicit.

There is a critical relationship between assessment practices and the nature of student learning (Luckett and Sutherland, 2000) which means that planning the assessment structure and practices is an integral part of the curriculum design for computer interventions as well. The continuous assessment process built into the design of the entire course (which has both a formative and a summative purpose) is also applied to the computer component. To assist with this process a system has been developed whereby students' responses to questions in the tutorials and assessment can be recorded automatically to a database, where they can be processed to produce feedback to individual students about their misconceptions, and to the tutors about the class's performance in general. This system also can be used to automate some of the burden of marking that the regular evaluation schedule generates.

Every third week there is a "computer evaluation" in which students complete a computer tutorial that is submitted for marking and counts towards their class record. As pointed out by

Laurillard (1993), use of computer learning material "must be integrated with other methods, the teacher must build on the work done and follow it through, and most important, the work students do on the material must be assessed." One third of the final examination for this course is also conducted in the computer laboratory through the medium of an Excel workbook similar to the tutorials, but without feedback.

In section 2 above, the ability of the computer-based tutorial to provide the student with immediate feedback (and a "conversational" tutorial environment), is stressed as one of the main motivations for using this type of intervention. In the Excel tutorials, in almost every case where a student provides and answer to a question, they will immediately receive explanatory feedback (see Figure 1). Ideally this feedback would be "adaptive", allowing the tasks presented to the student to be tailored to their particular needs, as manifested by their performance on questions in the tutorial (Laurillard, 1993). This level of interactivity is not built into our Excel workbooks, but is made available through the presence of the lecturers and tutors in the laboratory sessions. All students perform the same tasks on the computer, (although they can control the order), and more advanced optional tasks are also made available. Students who need additional support, can get almost immediate assistance from a tutor, of whom there are 3 present in each class of less than 30 students. It is not our intention that these tutorials be used for independent study, but rather to support the learning of material that is also dealt with in workshop-lectures.

As mentioned before, the Effective Numeracy course concentrates on quantitative literacy in the first semester and on pre-calculus in the second. During this semester, students explore the properties of different functions and use the graphical capabilities of Excel to solve "word-problems" graphically. Rather than draw up a table of values for every function they wish to represent, or change to another environment such as a graphics calculator, they are provided with a pre-programmed "graphics tool", which allows them to create up to five graphs simultaneously merely by typing in the formulae. A separate instance of this tool is placed at the appropriate place in the tutorial, wherever it is needed, so that producing a graph can be done in a manner that creates the least distraction to the student's train of thought while working through the tutorial. An example of an instance of the Graphics tool in context is shown in figure 1.

A further advantage of the "in-situ" graphics tool, is that, when marking a student's work, the lecturer can see the graph(s) that the student actually plotted. This allows for partial credit being given for partly correct work, and allows the lecturer to develop an understanding of students' difficulties and misconceptions. If students were using a graphics package or calculator "on the side", one would only have access to their final answers for evaluation purposes.

4. Students' response to the tutorials

There are several questions we would like to be able to answer about the way students' respond to the computer tutorials used in this part of the course, and to which we have so far received partial answers, which will be outlined below:

1. *Do the students believe the tutorials helped them to understand the mathematics content of the course?*

The students in the Effective Numeracy course every year, complete a comprehensive course evaluation questionnaire at the end of the first and the second semester of the course. In addition a randomly selected sample of students are interviewed about their reactions to the course at the end of the year.

The course evaluation results obtained in 2001 are representative. In response to the questions on forms filled in by about three quarters of the class at the end of the 2001 course, 70% of the respondents claimed to have found the explanations contained in the tutorials useful, while 82% were positive about the usefulness of the feedback in improving their understanding. These results are illustrated in Figure 2. The aspect of the tutorials that many students "particularly liked" was the automated drawing of graphs. Others mentioned the feedback, the visualisation of concepts, and the fact that they were "doing maths" on computers.

In 2001, extensive interviews were carried out with 11 students, chosen at random, in which one of the questions asked was: "Did the laboratory tutorials contribute to your understanding of the maths?". These were some of representative responses:

- "Yes, definitely. The class and labs helped each other. For example the labs helped with understanding graphing and the class dealt with the equations, which were needed in order to do the labs and so on. Both helped each other."
- "Yes, if you don't understand in class, it comes together in the labs. The feedback is very useful. It shows how the lecturer would answer the same questions... (The labs) really brought about a greater understanding. What we did in class we would get immediate practice in the labs."
- "Labs were very visual. You can actually see how things work. In class you have to imagine for yourself what things look like – in the labs you can see it clearly before your very eyes, it is easier to understand. They are helpful, they show more than you see in class – especially the graphics package (I explored more than the tut told me to)"
- "Ja, it did, but mostly to computer skills... Labs gave direction for solving word sums, helped for doing similar ones again later."
- "The good thing about the labs is that even if you don't want to ask questions, you still get feedback. There are lots of exercises and real-life examples. They helped you through the problems and helped you to understand how to integrate concepts."
- "Yes, they provide reinforcement to what was done in class, step-by-step. I liked learning how to use the computer."

Seven of the eleven students felt strongly that the computer tutorials had helped them to understand the mathematics content of the course, and there were also seven who remarked on the interdependence of the laboratory and classroom material. However there was a tendency to see the laboratories as reinforcing work done in the classroom, rather than the other way around. This is consistent with the observation that students prefer to see new ideas first in the classroom. Two of the students specifically referred to the feedback as being helpful with learning the mathematics and two remarked on the usefulness of the step-by-step approach utilised in the tutorials for solving "word sums". It is interesting that one of the students saw the visual (graphical) nature of the tutorials as making the greatest contribution to her understanding.

2. *Do the tutorials have any effect on students' attitudes to learning mathematics?*

This is a difficult question to answer because of the close integration of the tutorials into the course. Any recorded attitude changes cannot be ascribed to any particular component of the course. However there are indications from the course evaluation results, that there was a general improvement in students' feelings of confidence in doing mathematics (and in using computers). These results are illustrated in Figure 3. However, to study attitude changes more thoroughly, will involve the use of properly validated scales, such as those described by Cretchley et al (2000).

3. *Does the timing of the delivery of a computer tutorial relative to the related classroom sessions have an effect on the learning?*

Since there are three lecture groups who attend the laboratories on different days of the week, there is variation in the experience of the students in different groups as far as the timing of the computer tutorials is concerned. Some students will encounter a new idea for the first time in the laboratory and then have it reinforced in the classroom, while others see the same idea for the first time in the classroom, (and then have it reinforced in the laboratory).

Our observations and students' comments indicate that some students in our course are more comfortable when they encounter new knowledge on paper first, and then apply that knowledge in the computer tutorials. It is possible that it is easier to assimilate new ideas (and transfer them to other contexts) when they are first introduced in a medium that is familiar.

An attempt was made to gain insight into whether the timing of delivery of the computer tutorials has an effect on the learning, by performing pre- and post-tests on the function concept in the class where the concept was first introduced in the laboratory and in the classes where it was first encountered in the classroom. The intention was to explore whether it is possible to observe differences in the effectiveness of the learning between the classes who experienced the different learning environments in a different order.

The results were inconclusive, but provided a great deal of insight into the requirements for the design of such an experiment, (which we intend to repeat in the second semester of 2002). The most noteworthy result was that the medium in which the pre-test was delivered had a very significant effect on the students' performance. Since the students were assigned to the different classes at random, and there was no significant difference in their performance throughout the year, it was justified to assume that the three classes should all achieve similar results for the same questions in a pre-test conducted in all classes at the same time.

It was found however that the class who performed the pre-test in the medium of an Excel tutorial performed significantly better. This could be because the students were more inclined to engage seriously with the pre-test presented to them as part of an Excel tutorial than as a paper-based test. This effect highlights the need to present pre- and post-tests to all groups in exactly the same way (even if the questions are identical in content) and also to ensure that the students engage with the questions in a manner that truly reflects their knowledge.

4. *How do the students interact with a typical tutorial? Which features contribute best to their learning?*

This is a very broad question and should be the subject of a whole investigation in its own right. However, close observations of students interacting with the tutorial dealing with functions and their graphs yielded some very useful insights into the learning processes that take place while a student is engaged in working through a computer tutorial. These insights were sufficient to convince us of the need to conduct more extensive observations more frequently. These should lead to a much greater understanding of the nature of students' misconceptions, which types of activities in the tutorials lead to better understandings, and consequently how to design better tutorials that are more effective in meeting the students' needs .

5. The way forward

The results of observations, course evaluation questionnaires and student interviews indicate that in general the use of interactive Excel-based tutorials for supporting the learning of mathematics concepts is well-received by the students in the Effective Numeracy course. They appear to have a positive opinion of the value of the computer tutorials in contributing to their learning.

The incorporation of a custom-designed graphics tool that can be included within an Excel workbook wherever it is needed has enhanced the design of the tutorials used in the pre-calculus part of the course, and a system for recording student responses to a database has allowed for some of the assessment to be automated. We will continue to refine both these initiatives.

As the ability to learn mathematics is strongly influenced by affective factors, especially confidence and anxiety, we explicitly try to address these factors in our curriculum design. Thus, we plan to carry out a more systematic study of the effect of the course, and the computer component in particular, on students' attitudes towards and feelings about learning mathematics and using computers.

A study of the effect of the timing of the delivery of the computer tutorials (relative to the classroom materials) will be continued in 2002, making use of pre- and post-testing of students' understanding of concepts, and detailed observations of students interacting with the computer tutorials. These observations will provide further insights that will inform the design of computer tutorials that more effectively meet the needs of students in the Effective Numeracy and other courses in the future.

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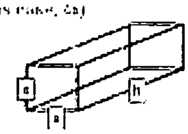
Example 1

The most economical shape for a parcel

To take advantage of a special postal rate, it is a rule that the combined length and girth of a parcel must not be more than 72cm. You wish to design a box that can be used to send things at this rate, but want it to contain the biggest possible volume. Your box will be rectangular with a square cross-section.

(Girth is the "measurement around the middle" in other words the perimeter of the cross-section, in this case, 4x)



The problem:

What is the maximum volume that a box with a square section can have, if the sum of its length and its girth is 72cm? What must its dimensions be, to get this volume?

What are you expected to maximise?

volume

You want the maximum volume

Write a formula for the volume (V) of the box in terms of x and h:

x^2h

V = h.x^2

Now use the given information to express volume in terms of one variable only [illegible]

[illegible]

Length + girth = h + 4x = 72, so h = 72 - 4x, so V = (72 - 4x)x^2 = 72x^2 - 4x^3

What kind of polynomial is this expression?

cubic

The highest power of x is 3, so it is a cubic

What is the minimum value that x can have?

0

x must be bigger than 0

What is the maximum value that x can have?

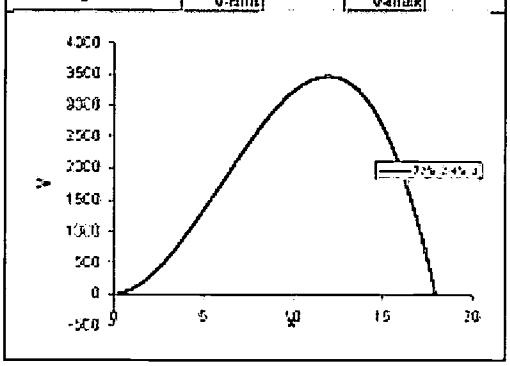
18

If the length was 0 then the girth would be 72, so x would be 72/4, which is 18

Now use the graphics package to draw a graph of the function (volume as a function of x)

Remember to restrict the x-values to the domain you worked out above

Graph	Independent Variable	x	Left	0
	Dependent Variable	V	Right	18
Function: V =	72*x^2-4*x^3			
Viewing Window	h.min		h.max	
	v.min		v.max	



From the graph, estimate the maximum volume the box can have?

3455

3456 cubic centimetres (to the nearest whole number)

From the graph, estimate the value of x where the volume is a maximum?

12

12 cm

What will the length of the parcel (h) be when the volume is a maximum?

72-48 = 24

From above, h = 72 - 4x, so if x is 12 then h = 72 - 4(12) = 24 cm

Figure 1: An instance of the "graphics tool" in the context of an exercise on the graph of a function (with student responses filled in). Note that the feedback is all-visible in this view, but only becomes visible to students after they have attempted an answer.

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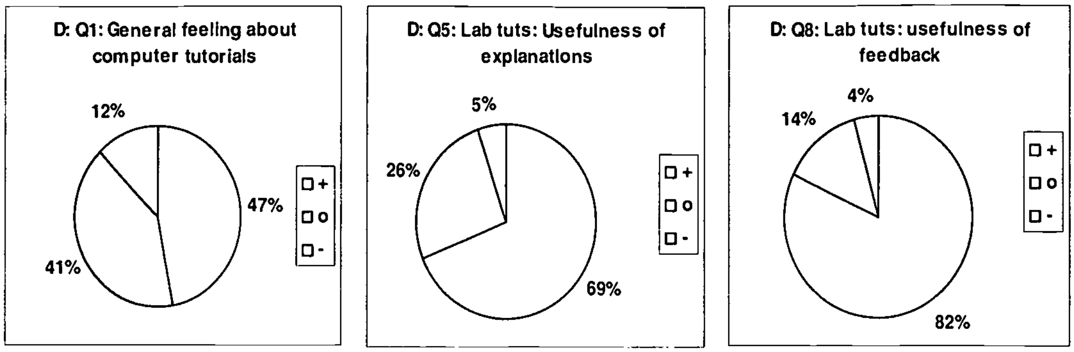

Figure 2: Summary of responses to selected course evaluation questions about usefulness of computer tutorials in the Effective Numeracy course. For every question, students were required to choose either a positive (+), a neutral (0) or a negative response (-).

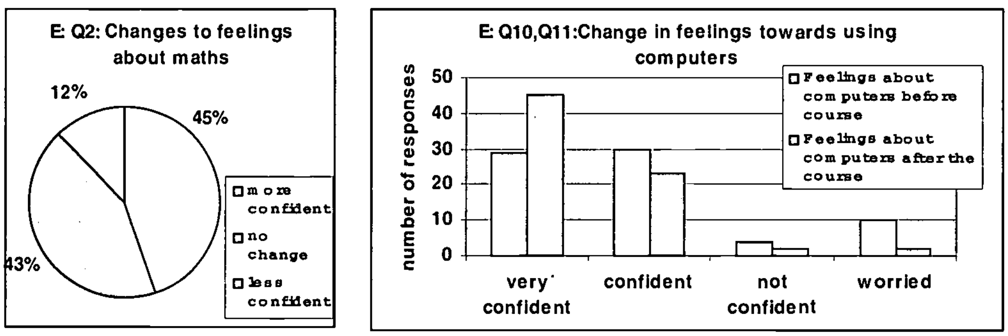

Figure 3: Summary of responses to course evaluation questions about changes to feelings of confidence with mathematics and computers.

ACTIVITY MATHEMATICS

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ABSTRACT

Mathematics is something that *people do*. In the ages before the recent rapid developments in technology, this activity called "doing mathematics" has been restricted to those who happened to be able to master a variety of artificial, mechanical, formal processes. The necessary conditions for one to be a candidate to practise mathematics have included for instance mastery of the mindless symbolic process of manipulation of formulae, and possession of the magician's box of techniques such as symbolic integration.

Now though, technology allows freedom for many more people to benefit from being able to "do mathematics", and for others to benefit from the results of that. Doing mathematics has always been much more than just being able to carry out manipulations on paper. It is now easier to perceive it and to present it to people at large as a broader activity which enables one to gain insight into the world, encompassing a rich combination of communication between reality both internal and external, words, pictures, and numbers, and a formalised language. Thus the idea of thinking logically and analytically in order to make human sense of the world can receive more emphasis than the repetitive practice of mechanical skills. In propagating this wider view, mathematics becomes more obviously a "people" activity.

We present in this paper some of our own recent experience of positively developing courses, for students of mathematics and others, to incorporate developing technology. Packages involved include for example Mathsoft Studyworks, TI-Interactive and Cabri Geometry, but the important issue is not precisely which packages we currently use, but how we have changed what we now perceive as "doing mathematics" now that rapidly changing technology is here to stay.

Keywords: TECHNOLOGY AND MATHEMATICS, CURRICULA INNOVATIONS,
INNOVATIVE TEACHING METHODS

1. Introduction

Before the recent rapid advances in technology, "doing mathematics" was restricted to those who could master a variety of artificial, mechanical, formal processes; what some have called mindless symbolic manipulation of formulae. The mathematician's magic box of symbolic techniques, such as integration, has been an essential tool for a candidate actually to practise mathematics. Technology now allows freedom for many more people to benefit from being able to "do" mathematics. There has always been much more to doing mathematics than just being able to carry out the manipulations, but it can now be viewed more obviously as a broader activity allowing insight into the world, through its structures for communication between reality, words, pictures, and numbers, and a formalised language. Mathematics thus becomes more obviously a "people" activity.

Here we present some recent examples from our practice of incorporating a variety of technology and software into our teaching of what we now perceive as doing mathematics, now that rapidly changing technology is here to stay.

2. Going for a *SONG*

To many people, "mathematics" is practically synonymous with mental arithmetic and algebraic manipulation. To emphasise our point that these aspects form only a part of the mathematical way of way of looking at the world, we encourage our students to "go for a ***SONG***" (Challis and Gretton 1999 and 1997). We encourage them to approach a mathematical concept from a variety of directions, by combining *S*ymbolic, *O*ral, *N*umeric and *G*raphic approaches, and thus hope they will acquire a richer understanding. The pervasive presence of mathematical technology is the major factor in allowing, prompting, and perhaps even demanding this broader approach.

'O' is for Oral

Let's consider the collection of symbolic manipulation, mathematicians' tricks and technical jargon to be a language. This language is useful to us as mathematicians, in that it is precise and concise and often helps us to 'do the sums'. But it isn't widely spoken. Most problems are communicated in some other language, and even if we can find a solution to the problem, we usually need to communicate it in the same language as the problem was posed, to those who are not fluent in 'our' mathematical language.

A typical process of solving a problem is to specify it in some formal, possibly symbolic way (translate from English, say, to our mathematical language); use the tricks of our language to find a solution; and then justify our solution to the person who originally posed the problem (translate back from the mathematical language into the original language). A mathematician (and therefore a mathematics student) does not only need to be fluent in the mathematical language, but also be able to translate into the original language.

Consider the problem:

> *Think of a number. Multiply by 3. Add 8 more than the original number. Divide by 4. Subtract the original number. Does everyone get the same answer or does it depend on the number you started with?*

Using the TI Interactive™! package we can consider specific examples, say of 4 and 5 as our original number:

$$\frac{4\cdot 3+(4+8)}{4}-4 \qquad 2 \qquad \frac{5\cdot 3+(5+8)}{4}-5 \qquad 2$$

This suggests that we will get 2 whatever we start with. Most mathematicians would prefer a generalised approach, ie let our original number be x:

$$\frac{(x\cdot 3)+(x+8)}{4}-x \qquad 2$$

Thus we see the process always gives the same answer 2.

This solution is relatively easy for someone with some algebraic experience, but what if it had to be explained to the general user who had never come across algebra? Here we come across a significant problem in doing mathematics: explaining to someone who does not have our expertise. So what do we do? One possibility is to resort to convincing. Enumeration of many possibilities is one way of doing this, even though as mathematicians we know this is not equivalent to a proof (in fact it can be a good opportunity to make that point!) This can be made slightly less exhausting if we use a spreadsheet or a calculator:

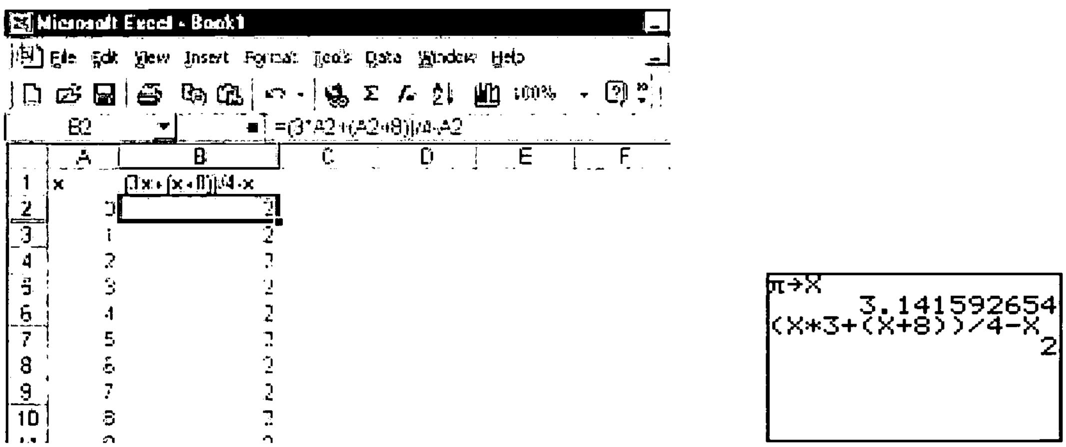


Figure 1

Since we start with words it is interesting to try to find a convincing argument which has *no* algebra in the solution. To try to convince your audience with just words is not easy, but it can be a useful ingredient of convincing a non-mathematician, and a thought-provoking activity for our students *and* us! Of course we may also note that *convincing* someone that something is the case is not the same as *explaining* to them precisely *why* it works - or even dare we say, proving it. It is something of a challenge to do this without symbols!

'G' is for graphic

This process of convincing audiences can perhaps be facilitated by pictures. Mathematicians are used to the notion of drawing graphs from some data or function, but graphics can come in many more formats than this. They could be a video of a process, an analogue output from a system, a picture in digital format, and so on. A student can collect, view or even extract their own data from 'pictures' thereby gaining ownership and highly interactive engagement with the problem.

For example, a student with a TI calculator, a CBL™ (Calculator-based Laboratory) and a light measuring device, let loose collecting information on light bulbs, computer screens, calculator, or mobile phones, can provide a plethora of real data for the construction or validation of mathematical models. Figure 2 shows such data from a computer monitor and a room light.

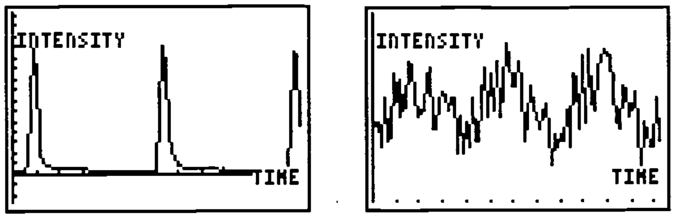


Figure 2 Light intensity data from a computer monitor (left) and room light (right)

Understanding the significance of the pictures can certainly be seen as a mathematical process, regardless of whether we think graphically (what kind of shape is it?), symbolically (what function does it represent?) or numerically (what discrete data does it show?).

Extracting numerical data from graphics is not restricted to graphs. Video evidence (home or otherwise) provides another opportunity for students to use graphics and to extract data from an experiment that they cannot easily reproduce in reality. Consider video of a motorcycle crash (Fig. 3).

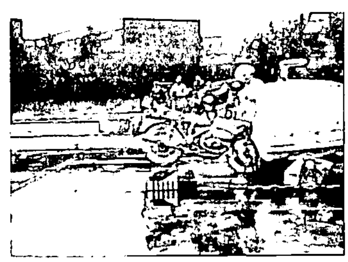


Figure 3 (Produced in Multimedia *Motion*, ©Cambridge Science Media 2000)

The student proceeds with the active investigation by collecting personal data from the video of the crash thereby ensuring ownership of the problem.

	Motorcycle crash 1			
	Dataset 1		Dataset 2	
t/s	x/m	y/m	x/m	y/m
0	0.699	2.007	0.135	2.563
0.03	1.134	2.014	0.556	2.578
0.06	1.532	2.022	0.954	2.593
0.09	1.953	2.029	1.36	2.593
0.12	2.366	2.044	1.773	2.608

Table 1(Produced in Multimedia *Motion*, ©Cambridge Science Media 2000)

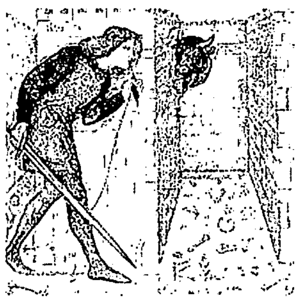
Thus pictures lead to numbers, which leads us to...

'N' is for *Numeric*

Increasingly in mathematics (and the rest of the world!) we are dealing with discrete numeric data – whether collected from some experiment or extracted from some video. Even the solutions of classical calculus problems are often sought from numerical methods. (We might eventually ask, if calculus is derived from letting small differences tend to zero, but solutions are sought by approximating over discrete differences, can we miss out the nasty business of infinitesimal changes altogether? But perhaps that is an argument for another time.)

Think of the major developments in mathematics over the past decade: mobile phones, money, chaos, rendering images etc. They all have their roots firmly planted in discrete mathematics. We even view continuous processes in discrete packages - the eye scans at a rate of one per 1/30 of a second.

In fact, Crete's ancient guest, the Minotaur, is reducible to a collection of numbers defining a bitmap image now (see Figure 4).

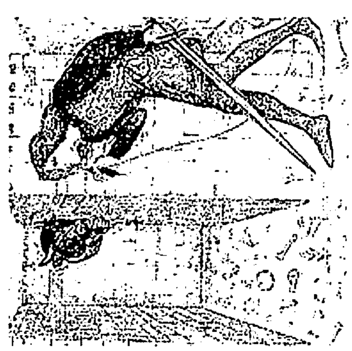


M =

	1	2	3	4	5	6	7	8	9	10
1	185	197	194	175	191	196	204	198	203	204
2	202	181	197	210	210	192	191	192	197	211
3	181	219	204	191	186	205	204	200	201	198
4	142	181	174	202	211	206	206	205	203	212
5	119	148	167	162	201	203	208	207	203	202
6	111	183	233	173	173	200	187	200	198	201
7	188	198	218	180	81	106	145	195	191	171
8	181	193	208	204	75	128	203	166	215	187
9	184	168	202	207	188	143	202	202	154	121
10	207	201	198	187	210	185	168	221	18	145

Figure 4 Picture from Island-of-Crete.net (2001)

These pictures (numbers) can be easily "moved" by standard matrix manipulations (see Figure 5). Whilst we learnt about matrix transformation by looking at a moving a unit square on a graph, our students can look at much more interesting problems.



M^T =

	1	2	3	4	5	6	7	8	9	10
1	185	202	181	142	119	111	188	181	184	207
2	197	181	219	181	148	183	198	193	168	201
3	194	197	204	174	167	233	218	208	202	198
4	175	210	191	202	162	173	180	204	207	187
5	191	210	186	211	201	173	81	75	188	210
6	196	192	205	206	203	200	106	128	143	185
7	204	191	204	206	208	187	145	203	202	168
8	198	192	200	205	207	200	195	166	202	221
9	203	197	201	203	203	198	191	215	154	18
10	204	211	198	212	202	201	171	187	121	145

Figure 5 Picture from Island-of-Crete.net (2001)

We have argued so far then that mathematical thinking resides in pictures, numbers and words. Finally being 'proper' mathematicians we must necessarily turn to symbolic processes.

'S' is for Symbolic

Whilst many professional mathematicians might recognise our wide view of mathematics, this is not always reflected in mathematics teaching at universities, where what we teach, and indeed particularly what we assess is often firmly rooted in the symbolic. While we do not wish to underplay the importance of the symbolic, this can contribute to the distorted view amongst the public at large about what mathematicians do.

It is easy for *mathematicians* to use precise, compact, symbolic notation. This specialised language enables 'simple' solutions of problems. To those fluent in this language, it is easier and more efficient to write down an equation or some shorthand mathematical expression than to express the idea in words. Unfortunately it is commonplace also to assume or hope that any student (or broader) audience has equal facility with this language, and often to complain when they do not. A series of reports in the UK (London Mathematical Society 1995, Engineering Council 2000) has indeed concentrated on and identified what are perceived as increasing shortcomings in this respect in UK students newly arriving at university.

We do not doubt that students need to understand symbolic mathematics, but there is a difference between recognising the meaning of something and being able to perform *the full range* of repetitive, algorithmic mindless symbolic manipulation. Concentrating on manipulation, which many people view as 'proper' mathematics, obscures the richness of the subject, and is unnecessary when computer algebra software (CAS) is so readily and cheaply available. A major task for researchers and developers over the next few years is to find out how much of the "pencil and paper" manipulations a student needs to be fluent in, before being able to use CAS with complete comfort. For example how many integration techniques must a student of engineering be able to use before being able to then trust Maple or Derive to give the answer to one they cannot do? Exactly what constitutes competence in algebra now?

Once the CAS can be used comfortably, benefits accrue. For example on a TI-89 (Texas Instruments 2002(2)) the solution of differential equations becomes simple, leaving time to think and reflect, to justify the process, and to interpret the answers (Figure 6).

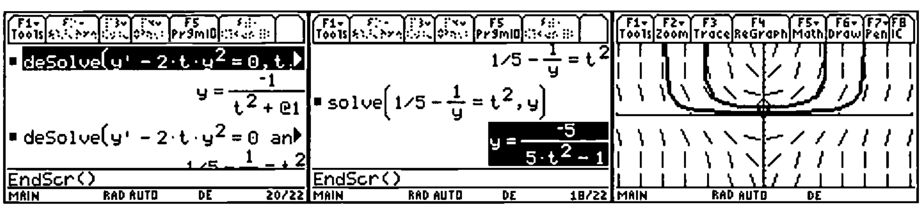


Figure 6

In some ways of using CAS, for example using the Script facility on say a TI-89, the CAS becomes part of the communication strategy. This is illustrated in Figure 6 for the Newton-Raphson process.

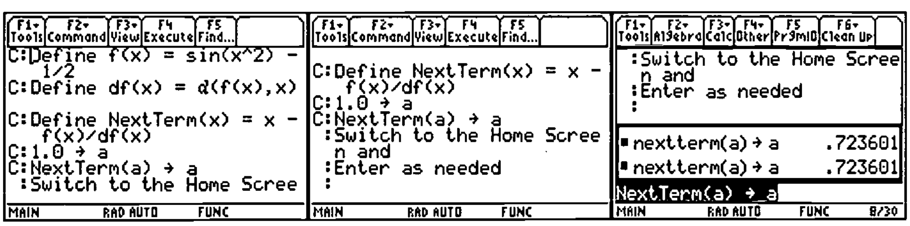
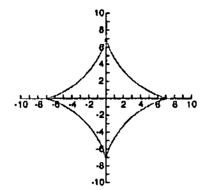


Figure 6

Technology which deals with *SONG*

Integrated software packages now appearing reflect our views here. One example is TI-Interactive™ (Texas Instruments (1)). This PC package provides an integrated environment, which includes CAS, word processor, spreadsheet, graph plotter, and a variety of links, to the internet, hand-held machines, and other students. It thus can be used to address the range of activities described above. One interactive process sheet, in this case the "magnetic bottle" function, is shown in Figure 8.



a 7.4

Cartesian equation: $x^{2/3} + y^{2/3} = a^{2/3}$ Parametric equation: $x = a\cos^3 t, \quad y = a\sin^3 t$

Figure 8 TI-Interactive™ output

The user can make the connection with reality through the package in a variety of ways. Figure 9 shows the wealth of data available for modelling from the associated web site. Alternatively one can collect personal data using a data logger such as CBL™ with the appropriate probe or a CBR™, or gather data from less accessible places using a package such as Motion (Cambridge Science Media 2000).

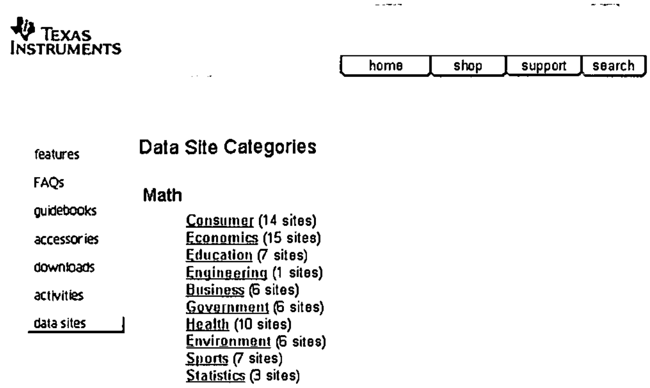


Figure 9

Discussion and conclusion

"We must not train people for our past but for their future" (Jones 2000)

The view of what constitutes 'mathematics' varies according to whom you ask. To mathematically uninformed people it is almost exclusively symbolic manipulation; a little like equating geography with English because one studies one using the other. Mathematics students might add that it includes learning what some people long dead did a long time ago, and being able to imitate them. Professional mathematicians might choose to define it as being about logical and analytical thinking processes, the role of rigorous proof, and so on. Some would choose to label what we have described here as science rather than mathematics, despite the fact that the kind of activities we have described are a core part of many (especially, perhaps, applied) mathematicians' work.

Before the advent of widespread technological tools for doing mathematics, it could be argued that anyone wishing to 'do' mathematics needed many years of training in mathematical language and history before being able to be let loose on the exciting tasks of formulating problems, interpreting results and relating mathematics to the real world. Hence the widespread experience that you really only started *doing* mathematics if you stuck at it until doctoral level. In this paper we have reported examples from our experience of how using technological tools can help in making mathematics a subject which all students can *do* for themselves. In a way, technology is allowing students to engage in mathematical processes, within the constraints of their knowledge, in a similar way to professional mathematicians. The issue of how fluent one really has to be in what range of symbolic processes, or how much one can depend in that respect on technology, is a critical one.

In the end, mathematics is wider than almost any one-sentence definition one could give. We certainly recognise the importance of our mathematical history and the role of analytical processes and rigorous proof, but mathematical ideas and concepts arise and are generated from our experience of the world. We believe that the kind of activities we describe in this paper fall within the wide range of what constitutes mathematical activity, and is a valid and useful part of a student's mathematical education. Technology gives us an opportunity to re-balance mathematical education to include these broader aspects, although the extent of that balancing must be the subject of much debate yet.

The educational processes described here are proving useful at various levels of mathematics. They enhance the development of concepts, and motivate students to engage in the subject. Interest in the world around them, perhaps through practical examples from sport, music, nature or the environment, can be used as an engaging vehicle. The process works both ways: using real problems enhances the development of mathematical concepts, *and* the mathematical ideas and language can be used to help make sense of reality.

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USING TECHNOLOGY TO INTEGRATE CONSTRUCTIVISM AND VISUALISATION IN MATHEMATICS EDUCATION

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ABSTRACT

This paper provides a discussion of the pros and cons of instructivism and constructivism in the mathematics classroom, and endeavours to show why the latter is a preferable methodology to the former when considering the effective use of technology to enhance visualisation.

The adoption of a constructivist approach to the teaching and learning of Mathematics has highlighted a shift from teacher dominance. Visually stimulating computer environments can allow students to become immersed in their own knowledge construction. However, it is not a trivial matter how to utilise this considerable technological capability most effectively for educational benefit, emphasising the importance of a teaching and learning methodology.

It is necessary to encourage more exploratory approaches to learning, where students can be the initiators and controllers of their own learning. There is much empirical evidence that this approach significantly improves the understanding of higher order concepts.

Knowledge is built up from personal experiences, and making these experiences more dynamic will assist in the development of cognitive structures. Computer-based attractive environments with visually compelling displays, together with facilities for interaction, can provide the setting for more dynamic, powerful experiences. These environments are filled with stimuli, which encourage rich constructions, by students. The integration of constructivism and visualisation can encourage the reformulation of conceptual structures and the development of higher order skills.

Having reviewed and examined the effectiveness of previous work by authors such as Tall, Dubinsky, von Glasersfeld, etc., and different constructivist perspectives, consideration is given to the best way to employ constructivism in teaching and learning with computer-based visualisation. The effectiveness of this approach is evaluated, and students' experiences are discussed in terms of the enhancement of mathematical skills via the constructive use of visual software.

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1. Introduction

For completeness, this section provides an overview of instructivist and constructivist approaches to teaching and learning in the mathematics classroom, and endeavours to explain why the latter is a preferable methodology to the former as the use of technology in teaching increases.

Instructivism reflects the traditional hierarchical view of mathematical study, where instructive representations are finely tuned to a particular purpose (O'Reilly et al., 1997). Students who are subjected to this instructivist approach have to learn to discriminate between contexts in order to appreciate when one finely tuned representation is needed as opposed to another, which is clearly a non-trivial process.

The instructivist, or behaviourist, approach is to pre-plan a curriculum by breaking down a subject area (usually seen as a finite body of knowledge) into assumed component parts, and then sequencing these parts into a hierarchy ranging from simple to more complex (Fosnot, 1996). Instructivism assumes that listening to explanations from teachers will result in learning. Learners are viewed as passive, and educators spend their time developing a sequenced, well-structured curriculum and determining how they will assess, motivate, and evaluate the learner. The learner is expected to progress in a continuous, linear fashion as long as clear communication and appropriate reinforcement are provided.

Schifter sums up the instructivist way of thinking in the following - *The teacher shows the students procedures for getting right answers and then monitors them as they reproduce those procedures. To ask a question without having previously shown how to answer it is actually considered 'unfair'* (Schifter, 1996).

As a result of schools taking an instructivist approach to teaching, it was reported almost a decade ago that students could not apply their knowledge to unknown problem solving situations (Honebein et al., 1993). This is unfortunately still an issue that needs addressing today with teachers using technology. A different type of learning activity is required, i.e. constructivism. Here the concern is not mastery in a test of procedural skills, but rather the ability to function successfully in unknown problem solving situations. The focus here is to be able to take the knowledge gleaned from local tasks and apply it globally (Honebein et al., 1993). The learning activity has a purpose that goes beyond simply demonstrating mastery of the local tasks; the purpose for a learning activity is driven by the global underlying concepts. It is therefore not the ability to recall information that educators should be interested in, but instead the ability to apply knowledge and skills in different problem based environments. The constructivist approach, therefore, concentrates on a holistic view of learning mathematics, and focuses on deep understanding and strategies, rather than facts and rote memorisation (Honebein et al., 1993; Fosnot, 1996).

The fundamental principle of constructivism is that learning is very much a constructive activity that the students themselves have to carry out. From this point of view, then, the task of the educator is not to dispense knowledge but to provide students with opportunities and incentives to build it up (von Glasersfeld, 1995, 1996).

Lerman has described how Piaget's constructivist perspective is that the individual is responsible for his thinking and his knowledge, and is the central element in meaning-making, whereas Vygotsky attempted to develop a fully cultural psychology, placing communication and social life at the centre of meaning-making (Lerman, 1996a), where the individual can construct knowledge facilitated by a teacher or more able peers.

The zone of proximal development (Scardamalia and Bereiter, 1991; Lerman, 1996b) is the area in which the student can perform tasks successfully, but only with some assistance. The student therefore works in a constructivist manner, inside an instructional domain. Vygotsky defines the zone of proximal development as the gap between what a child can do on her or his own and what she or he can do with, for example, a teacher. The learning activity constitutes the zone of proximal development; it is actually the difference in activity between 'with or without' the teacher. The teacher is there to guide, and to share in evaluating their progress.

Strategic questioning, known as the Socratic method, is used to facilitate the construction of a target concept, working within the students' zone of proximal development (Rowlands et al., 1997). Rowlands et al. explain that this method of strategic questioning challenges (and hopefully removes) misconceptions, and facilitates the construction of knowledge. The key is to ask qualitative questions that lead the student to reach the target concept without it actually being given by the teacher. Consistent with the Vygotskian perspective, these questions provide hurdles to overcome in order to develop cognitive growth, yet which also serve as props or hints to facilitate the process. The teacher must use questions that challenge students to think according to the properties of the target concept. Rowlands et al. discuss how intuitive concepts stand at one end of the zone of proximal development, and the target concept stands at the other - strategic questions stand in between and facilitate the progression from the former to the latter.

2. Constructivism in Relation to Educational Technology

The constructivist use of technology allows the opportunity to change the nature of the material to be taught and learnt from routine-based to discovery-based activities. Knowledge, as discussed in the previous section, is built up from personal experiences, and making these experiences more dynamic will assist in the development of cognitive structures (see for example Tall, 2000, 2001). Computer-based environments with visually compelling displays, together with facilities for interaction, can provide the setting for more dynamic, powerful experiences. These environments are filled with stimuli, which encourage rich constructions, by students (Nelson, 2000). Graphic representations, coupled with social interactions, are seen as leading to the development of an individual's knowledge, and are seen as leading to the adaptation of concepts (von Glasersfeld, 1996).

The authors have observed, via classroom experiences, that 16-19 year old students find it difficult to answer questions about concepts that have been placed in contexts separate from their immediate concrete experiences. The constructivist use of the computer is a more powerful means of providing the student with vivid experiences in order to convert the concrete into the abstract more successfully (Dubinsky, 1991). This can in turn provide students with the appropriate mental structures that can be called upon to utilise conceptual knowledge in unknown situations (Honebein et al., 1993). The activities that are carried out in a computer environment provide meaningful experiences for learners that help them transfer skills and knowledge to other problem solving activities and subject domains.

While engaged in mathematical activity, students construct images (Wheatley and Brown, 1994). When they 're-present' their image at a later date, they are operating from the image that they originally constructed. The nature and quality of the image will influence the re-presentation, hence the importance of quality mathematical software for image generation. This act of re-presentation is a complex one. Piaget has shown that the image constructed may undergo change over time without any intervention - the original image-making process supported by appropriate software is therefore vital. Activities that encourage the construction of images can greatly enhance mathematics learning. Students who naturally use images in their thinking easily make sense of novel mathematics tasks while students who are not good visualisers often do not (see for example Habre, 2001). It would be desirable to develop learning activities that promote the development of image-making skills for all students.

Powerful, multiple representation software can be used to encourage the learner to construct meaning for different representations and their interrelations. The relationship between representations lies at the heart of much mathematics (O'Reilly et al., 1997). Multiple representation software can demonstrate these links explicitly. Within such software, constructive changes in one representation trigger automatic changes in another. Thus, for example, a change in algebraic representation of a function should immediately promote a corresponding change in the graph. A learning tool cannot be used constructively, however, unless the students are genuinely in control.

An illustration of the zone of proximal development is where the teacher takes on the role of facilitator in the construction of knowledge (rather than a giver of knowledge) by providing props and hints to develop students' cognitive framework. The teacher aids the learner in accomplishing the activity, not by doing the task for the learner or giving the learner the correct answers, but by providing guidance that require learners to formulate their own solution to the problem (Honebein et al., 1993). Strategic questioning is employed by asking probing questions which act as a catalyst to get students to reach the desired goal, without taking away the ownership of the task. In this manner, students can eventually arrive at a required level of understanding for themselves, which is not only advantageous in terms of the learning process, but also increases satisfaction and boosts confidence.

3. Examples of Constructive Mathematical Software and their Use

Teaching mathematics from a constructivist perspective involves the provision of activities designed to encourage and facilitate the constructive process. This can be achieved readily nowadays by employing visually compelling mathematical software such as Autograph (www.autograph-math.com), Cabri Geometry (www-cabri.imag.fr), or a Computer Algebra System such as Derive (www.derive.com), with which students can explore mathematics. These packages have various features which facilitate a constructive approach to learning mathematics. Autograph allows the user to 'grab and move' graphs, lines, and points on screen whilst observing changes in parameters, and vice versa. Cabri-Geometre encourages the user to drag points around the screen whilst observing the effects of such changes on geometric shapes. Derive, with its multiple representation capabilities, allows the user to switch easily between numeric, symbolic and visual representations of information. These examples of software that can enhance constructive learning can be used effectively to encourage 'what if ' situations for students to explore.

Strategic questions need to accompany the use of technology. For example, an instructive question concerning functions might be to find turning points, asymptotes, etc., and then, as an afterthought, to plot the graph. An example of a constructive question, however, could be to consider some function, f(x), and then determine what happens when a particular symbol or parameter within the expression is altered; the students would then be encouraged to explore and investigate. The constructivist philosophy thus invites students to find answers for themselves.

In order to establish any practical evidence of enhanced mathematical skills of students having experienced a constructive approach to learning, a research project was set up to assess the effectiveness of the constructive employment of computer-based visualisation. To develop students' conceptual understanding of the relationship between graphical and symbolic forms, a piece of bespoke mathematical software was written entitled 'Graphs of Functions: A Constructivist Approach'. The controlled study involved 16-19 year old students prior to entering undergraduate mathematics degree courses. The control group contained students who had been taught 'functions and graphs' by traditional instructivist methods, and the experimental group contained students who had learnt 'functions and graphs' via the interactive software (for further details of the experiment see Malabar, 2002).

Whilst using the software, the students were given a series of function graphs of polynomials, trigonometric functions, exponentials, etc., as well as combinations of these basic functions. The task was to determine, via constructive explorations, the correct symbolic form of the function. In this manner, students could build up their conceptual understanding of the links between algebraic and pictorial representations as a result of both successful and unsuccessful conjectures and evaluations. The teaching style adopted was the Socratic method of strategic questioning as described in Section 1. Working within the students' zone of proximal development, props and hints were used to challenge misconceptions and lead the student to the construction of the target concept.

The students in the experimental group felt that they owned the problem, which they felt compelled to resolve. This philosophy provided an organising role and a purpose for learning. When they were faced with contradictions to their own conjectures, it was up to them to find resolution. The activities were concerned with exploration and debate; there was not a finished body of knowledge to be accepted, accumulated, and reproduced. Instead of concentrating on technique and strategy, this approach helped the students to develop an attitude of inquiry toward the learning of mathematics.

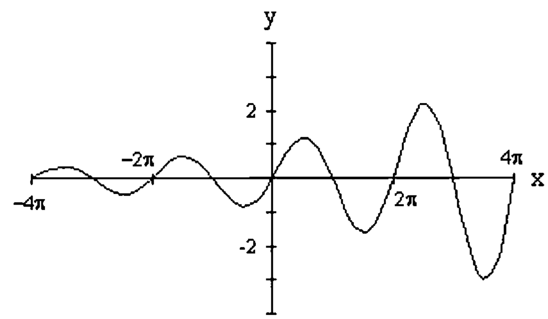
The constructive use of this software provided students with vivid experiences in order to convert the concrete into the abstract more successfully, and encouraged them to construct meaning for different representations and how they are related.

4. Evaluation

In order to help evaluate the effectiveness of our constructivist approach in terms of students' skills, we can refer to a taxonomy known as the MATH taxonomy (Mathematical Assessment Task Hierarchy). The MATH taxonomy (Smith et al., 1996) describes a hierarchy of skills ranging from lower order skills, such as factual knowledge and the ability to follow procedures, to higher order skills such as the ability to interpret, conjecture and evaluate, as in the table below:

Group A	Group B	Group C
Factual Knowledge	Information transfer	Justifying and Interpreting
Comprehension	Application in new Situations	Implications, conjectures and comparisons
Routine use of Procedures		Evaluation

It was conjectured that the constructive process had enabled students to develop more Group C skills, whereas students undergoing the instructivist treatment were mainly limited in skills to those of Group A. The evidence suggested that this was indeed the case, but moreover there was evidence to suggest that *linkages between the skill groups were more pronounced, creating a more holistic view of mathematics*. This is best summarised by considering a typical posed question (although only one example, it is indicative of the findings in general. A detailed statistical analysis of the above experiment can be found in Malabar, 2002):



The above graph is a graphical representation of which of the following functions ?

A $y = \sin(x) + e^{0.1x}$ **B** $y = \sin(x) + e^{-0.1x}$ **C** $y = \sin(x)\, e^{0.1x}$
D $y = \sin(x)\, e^{-0.1x}$ **E** all of the above

The above question assesses whether or not the students have been able to take the knowledge gleaned from local tasks and apply it globally. When faced with a graph, which was the result of a combination of functions, the group who were subjected to an instructivist approach struggled to find the correct solution, whereas the constructivist group used their knowledge relating to other families of graphs to arrive at the correct function. The group that learnt constructively had a more holistic view of the topic and were therefore not fazed by the nature of the task, i.e. to employ their conceptual knowledge of combining familiar, specific functions (and the effect on the graph) to an unfamiliar (but similar) situation. The instructivist group's sequential style, however, hindered their progress as they could not see any other way around their limited, linear methods.

The constructivist group had done some work with the bespoke software concerning combining different functions, and so this could clearly have helped in solving the above problem. They were more successful as they had the ability to combine functions and understand the effect this would have on the graph, irrespective of the specific functions studied. Their whole approach to learning equipped them with better strategies for problem solving. The richness of global thinking proved beneficial as they could check their answers by more than one approach.

The instructivist group had not studied combinations of functions explicitly, and were struggling to match this question to any prior experience. They did not have a 'recipe' or 'template' to solve such problems, and therefore had a very limited solution strategy. The problem could be solved in an instructivist manner, e.g. to methodically eliminate possible answers by considering values of x where the graph cuts the x-axis, then considering the substitution of different values of x into e^x and e^{-x}, etc., but the instructivist group did not seem to have the necessary problem solving skills to tackle it, even in an instructivist way.

It would appear that the constructivist group had a greater mathematical skills set with more flexibility in moving between the different skills when applying them. The instructivist group tended to see things only that had been explicitly taught, as the goals were specified by the teacher and success was determined by the teacher. As a consequence, students often operate mindlessly in this type of environment, simply following rules without any critical evaluation, and hence without a clear understanding of the reason for the rules (Honebein et al., 1993).

This example illustrates that understanding needs to be independent of the specific examples used. For example, the bespoke teaching software looked at investigations specific to certain functions, but the newly acquired conceptual structures could be applied to *any* function. It is through the learning of concepts separate from the immediate and the concrete that cognitive structures are built (Vygotsky, 1962).

5. Discussion

This paper has produced evidence of some positive and practical findings for the benefits of a constructivist approach to teaching with technology and the use of visualisation, and there is some evidence that a constructivist approach to learning can broaden a student's skills base. However, as a result of this and other experiments, important questions have surfaced that require further research:

- Can any **generic** conclusions be derived?
    - Are the outcomes limited to certain age groups? e.g. is an instructivist approach necessary before a constructivist approach takes over?
    - Are the outcomes limited to particular subject domains? e.g. will a constructivist approach to teaching develop better ideas of formal proof?

- Do traditional **assessment** methods favour an instructivist approach and hence limit constructivist activities?
    - Which methods of assessment effectively document genuine learning?
    - Should technology be used in examinations to measure abilities in conceptual understanding?

- How do we take into account psychological and motivational factors when using a constructivist approach?
    - Is learning via a constructivist approach more 'fun'?, and does it lead to increased motivation for *all* students?

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IFORS tutORial PROJECT

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ABSTRACT

Everyone knows that the Internet in general, and the World Wide Web in particular, provides new exciting tools for the development and usage of teaching/learning resources. Unfortunately, it is also a fact of life that the development of educationally rich web-based resources is not easy and can be expensive. One way to alleviate this difficulty is through cooperation and professional societies can play a major role in initiating and coordinating such joint projects.

In this paper we describe a project called tutORial that was initiated by the International Federation of Operational Research Societies (IFORS) in 1999. The goal of this project is to provide a framework for an international collaboration in the development of educationally rich tutorial models for standard Operations Research (OR) and Management Science (MS) subjects.

The project will be officially launched at the IFORS 2002 conference (July 8-12, 2002, Edinburgh, UK) but its web site is already open for preview (www.ifors.org/tutorial/). The site currently features more than twenty highly interactive modules covering topics from areas such as linear algebra, discrete mathematics, linear programming, integer programming and dynamic programming. OR/MS students are currently using it worldwide.

The goal is to expand this collection over time with contributions from OR/MS professionals and organizations worldwide. Details concerning preparation of contributions to the project can be found at the project's web site. All you need in order to use these modules is access to the Internet and a web browser. These modules are accessible free of charge.

In this discussion we give a very broad overview of the project and explain how its modules can be incorporated in undergraduate applied mathematics courses. The presentation at the conference will also feature a live demonstration of some of these modules.

Key words: math education, on line tutorials, operations research, management science, IFORS, tutORial.

1. Introduction

In this paper we take a guided tour of the IFORS tutORial project and discuss matters related to the educational resources it provides and how they can be used in actual and virtual classroom. This project will be launched officially in July 2002 during the IFORS 2002 conference, but its web site (www.ifors.org/tutorial/) has already been open for review for more that a year. Readers interested in more details about the project are invited to visit the site.

As we all know, the World Wide Web has already established itself as an extremely important technology for the development and delivery of educational resources. Unfortunately, it is also a fact of life that the development of educationally rich web-based resources is not easy and can be expensive. Thus, there is plenty of scope for co-operation in this area and professional societies can play a major role in initiating and co-ordinating such co-operative projects.

In this paper we report on a project called tutORial that was initiated by the International Federation of Operational Research Societies (IFORS) in 1999. The goal of this project is to provide a framework for an international collaboration in the development of educationally rich tutorial models for standard Operations Research (OR) and Management Science (MS) subjects.

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The paper is structured as follows: In Section 2 we explain the basic philosophy underlying the project. The reason why the focus is on tutorials is explained in Section 3. Then in Section 4 we examine some of the educational aspects of the project, including the "Solve Button" dilemma. Section 5 briefly looks at the technology we have used so far in the development of the modules and Section 6 explain the basic organisational structure of the project including copyright and intellectual property issues. Section 7 lists the modules currently available at the project's web site and Section 8 briefly describes how they can be used. Section 9 reflects on the Operational Research aspects of the project. Some preliminary conclusions are drawn in Section 10.

2. Basic Philosophy

The basic principle guiding the development of this project is: "Keep it simple, mate!" It reflects two important facts about the project. Firstly, this is a very low budget project so there are no financial resources for the development of complex and elaborate tools. Secondly, the aim of the project is to serve the global OR/MS community and therefore the user interface must be friendly.

So the basic assumption is that anyone connected to the Internet via a standard WWW browsers should be able to use the modules. Another consequence of this principle is that each module is essentially a self-contained, stand-alone object. The implication is that each such module can be easily incorporated within an external courseware, and if necessary be slightly modified to suit specific needs of students and/or lecturers.

The rules for participating in the projects are also very simple: the modules are available for use free of charge.

3. Why tutorials?

The modules developed in this project are not designed to replace traditional books and lectures. They are viewed as supplements to conventional educational resources rather than as replacements. It is envisaged therefore that persons using these modules do not start from scratch. Rather, they are already familiar with the subject and wish to use the module to practice what they have learned elsewhere.

More importantly, the modules are designed to provide the students with an *interactive* facility for experimenting with methods and algorithms, including *immediate feedback* on their performance.

We adopted this approach because this kind of support is ideal for web-based implementations and serves well the international OR/MS community.

In short, the idea is to use the WWW not merely as a delivery system of static material such as lecture notes and assignment/solution sheets, but as a framework for providing students *interactive learning facilities*. In this regard the modules are tutorials rather than lectures oriented.

4. Educational matters

Our main concern is to provide what we call 'educationally rich facilities'. By this we mean that the modules are not designed merely to provide answers to questions. Rather, they are designed to enable students to practice what they learn in class or read in books and to obtain immediate feedback on their performance. For example, suppose that the math topic under consideration is 'systems of linear equations'. Then we are not interested in providing the students a facility capable of merely solving systems of linear equations. Rather, we want a facility that will enable the students to experiment (step by step) with the methods taught in class for solving such problems. For example, such facilities should enable students to experiment with row operations and use these operations to solve systems of linear equations with immediate feedback on the student's performance.

We have found that this kind of facilities is very useful in dealing with two common types of help sought by our students:

- "I obviously did something wrong here, but I do not know what/where!?"
- "I got the correct final answer but I am not sure whether the process is OK?!"

We have also noticed that students who experiment with such modules tend to be better prepared for the formal tutorial sessions so that there is much less need to spend time on rudimentary matters during these sessions.

The incorporation of such modules in OR/MS courseware pose the following dilemma: educational speaking, is it a good idea to 'automate' to the traditional (manual) drill-drill-drill approach? Should we let students attempt to solve problems on their own? Isn't this what learning is all about? Aren't we depriving students from experiencing a fundamental and essential ingredient by providing them sophisticated electronic problem solving tools?

These are of course legitimate questions that must be carefully addressed by lecturers using problem-solving tools. They are in essence the same as those raised many years ago with regard to the use of pocket calculators.

It should be noted, however, that the incorporation of such tools in math courseware does not have to be at the expense of traditional teaching methods and processes. The fact that students have access to an educationally rich interactive module on a topic such as say systems of linear equations does not necessarily means that students are denied the joy of solving such problem hand. Nor are they necessarily disadvantaged by an exposure to such tools.

5. Technology

The technology we have used so far is 'standard" so that users do not have to purchase any special software/hardware. A computer equipped with a recent version of one of the commercial web browsers is all that is needed to use the modules via the WWW. Needless to say, producing mathematical text for the WWW is still not as straightforward as it should be. The new math language for the WWW (MathML) may ultimately resolve this issue (Tittle [1998]).

A more annoying aspect of the technology is that it is not truly platform independent. Therefore special attention must be given to differences between operating systems (eg. Mac, Unix, Windows) and browsers (eg. Netscape Communicator, Microsoft Internet Explorer). The international nature of the project makes this issue especially important, as there is basically no control on the software/hardware used by the visitors to the site.

6. Organisation

The project is organised in a simple manner. All the modules are open to the general public free of charge. Copyright and intellectual property issues are handled in a straightforward manner: contributors retain complete control on their contributions and are free to withdraw their contributions anytime.

The project as a whole, as well as individual modules, are being incorporated in the IFORS On Line Encyclopaedia (www.ifors.org/ioe/). The modules will provide exciting facilities for live experimentation with methods and techniques discussed in the encyclopaedia.

7. Content

The web site of the project currently contains more then twenty modules dealing with various OR/MS topics. The choice of topics was not the result of a deliberate analysis, but rather a reflection of the basic nature of the project: modules are contributed by OR/MS groups worldwide. In any case, the current list is as follows:

- Linear Algebra:
    - Row operations
    - Matrix Inverse
    - Linear Equations
- Linear Programming::
    - A number of Simplex Modules
    - Dual Problems
- Dynamic Programming:
    - Shortest Path Problem
    - Travelling Salesman Problem
    - A number of Knapsack Problems
    - A number of Counterfeit Coin Problem
    - Critical Path Problem
    - Dijkstra's Algorithms
    - Towers of Hanoi

- Prince's Pub Problem
  - Chained Matrix Product
- Integer Programming:
  - N-Queen Problem
  - 8 Easy Pieces
  - Knapsack Problem
  - Gomory's Cutting Plan
- Simulation:
  - A Random Number Testers
  - A number of Queueing System Generators

The University of Malta contributed the Simulation modules. The University of Melbourne contributed all other modules.

Additional modules are currently being developed and will appear on the web site soon. In particular, in view of the special and important role that games play in mathematical education, a directory dedicated to OR/MS oriented games is now being created.

8. User's guide

As indicated above, the modules are organised in a 'stand alone' fashion so there is no global environment to deal with on the part of users. The modules are listed according to topics and you simple surf to the module of interest.

The most important thing to remember when using the modules is that they are not designed to replace lectures and/or books. In particular, it is assumed that students have basic knowledge of the topic before they use the relevant module.

Guidelines for Lecturers: Math convention, notation and terminology are not uniformly 'standard' in all areas of operations research and management science. Thus, if you use a tutORial module in your class make sure that the students are comfortable with the notation and terminology used in the module.

Guidelines for students: The modules were not designed to facilitate easy production of solution to homework assignments. While it is perfectly OK to use the modules to check results derived manually by you, it is important that you do not become heavily dependent on them. In particular, it is very unlikely that you'll have access to these modules during exams! In short, use the modules mainly to check that you know how do solve problems on your own and to identify things that you do not do properly.

For obvious reasons, we cannot offer on-line help on the math content of the modules. We do offer, though, help with regard to the user interface of the modules.

9. Conclusions

The Internet in general and the WWW in particular are already used extensively in the delivery and development of educational resources. The ability to create educationally rich on line interactive modules for math topics poses the math community plenty of opportunities as well as major challenges.

However, the development of such modules is beyond the means of most individuals and departments. Therefore, professional societies can play an important role in co-ordinating projects whose aim is to coordinate and share the development of such resources for specialised math topics.

IFORS tutORial project serves as an example of such an initiative. We shall be delighted to share our experience with other professional societies.

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OBJECT TEACHING OF GRAPH ALGORITHMS

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ABSTRACT

The base concepts and theorems of the Graph Theory and related Graph Algorithms are taught in the context of the subject Discrete Mathematics at our university, the University of Hradec Králové, Czech Republic. The Graph Theory is a wonderful, practical discipline, often as little as puzzles. A good understanding of graph algorithms develops logical thinking in students, therefore we focus properly on these problems. When we explain algorithms we put emphasis on mutual relations between individual algorithms. When students make sense of the concepts tree and spanning tree we start to speak about the well-known optimisation problem, the minimum spanning tree problem. We show them three classical algorithms (Borůvka's, Jarník's and Kruskal's algorithms) and also for comparison one dual algorithm (dual Kruskal's algorithm). We describe all methods as an edge colouring process. On the base of Jarník's solution of the mentioned problem we continue our lectures with descriptions of other algorithms. We show the relationship of the Jarník's method to Dijkstra's algorithm for finding the shortest path. We speak about Breadth-First Search and Depth-First Search algorithms based on Jarník's algorithm. And on the base of these searching algorithms we discuss several other graph algorithms.

In this article we will present a theoretical background and at the conference we will introduce visual presentations in the Delphi environment, which we use in the lectures as a very nice complement for illustration of all above-mentioned algorithms.

KEYWORDS: Graph Theory, Graph Algorithms, Minimum Spanning Tree Problem, Dijkstra's algorithm for finding the shortest path, Breadth-First Search and Depth-First Search.

Introduction

The subject Discrete Mathematics taught at our university gives quite a large space for explanations of several graph algorithms. We are sure that a good understanding of graph algorithms greatly develops logical thinking of students. It is evident that our students first must be familiar with base concepts and theorems of Graph Theory. Then we can start to introduce interesting and practical algorithms on graphs.

We always explain individual algorithms with the help of mutual relations among them. On the one hand there are many algorithms solving one problem and on the other hand we can get algorithms solving other problems using various modifications of only one algorithm. For students it is easier to understand the problems and to remember the main idea of algorithms when they can see mutual relationships among described algorithms.

In the article, as in our lectures, the well-known Minimum Spanning Tree Problem will be first discussed more deeply. We meet our students with the history of the problem, introduce three classical algorithms and explain the basic differences among them. We also mention at least one dual algorithm solving the problem. Then on the base of Jarník's solution of the minimum spanning tree problem we illustrate the known Dijkstra's algorithm for finding the shortest path and Breadth-First-Search and Depth-First-Search algorithms. All methods are described as an edge colouring process. We are sure that exactly this way of description greatly increases understanding of algorithms.

Using description of algorithms as an edge colouring process enables object teaching not only with chalk and blackboard but also with help of new modern technology. We are very happy that our faculty has good, modern equipment and that there are several students there who are able and enthusiastically willing to prepare nice multimedia programs. At the conference we intend to introduce a multimedia program prepared in the Borland Delphi environment where several graph algorithms are visualized.

The minimum spanning tree problem

Some historical facts

Too important mathematicians, Vojtěch Jarník and Otakar Borůvka, were born in Czech Republic about one hundred years ago.

At the end of 1925 Otakar Borůvka met Jindřich Saxel, an employee of West Moravia power-stations, (Moravia is part of Czech Republic). It was during the electrification of south and west Moravia and Borůvka was asked for help in solving the problem Saxel was just working on. The challenge was to how and through which places to design the connection of several tens of municipalities in Moravia region so that the solution was as short and consequently as low-cost as possible. Otakar Borůvka not only correctly stated this problem but also solved it. His technical solution is mentioned in the article *Příspěvek k řešení otázky ekonomické stavby elektrovodních sítí* (Contribution to the solution of a problem of economic construction of electricity power networks) [1] and mathematical background he gave in the article *O jistém problému minimálním* (On a minimum problem) [2].

There did not exist suitable mathematical terminology in this area of mathematics at that time and thus the proof of the correctness of the solution was rather complicated. Vojtěch Jarník, another Czech mathematician, was aware of the complexity and importance of this problem. He wrote the article named *O jistém problému minimálním* with subtitle (*From the letter to Mr.*

Boruvka) [3]. In this article Jarník offered another and easier method of creating the demanded construction.

Both Czech mathematicians preceded their fellow mathematicians by a quarter of a century. The enormous interest about this problem, which is considered to be one of the best-known optimisation problems, broke out with unusual vigour again in after 1950 and that time was connected with the application of computers. That time Borůvka's and Jarník's method was discovered independently several times more.

The third solution of the problem different from the previous ones invented Joseph B. Kruskal in 1956 in his work *On the shortest spanning tree of a graph and the travelling salesman problem* [4]. The following fragment from the letter of J. B. Kruskal brings near the situation related to the birth of this problem [5]:

"It happened at Princeton, in old Fine Hall, just outside the tea-room. I don't remember when, but it was probably a few months after June 1954. Someone handed me two pages of very flimsy paper stapled together. He told me it was "floating around the math department".

Two pages were typewritten, carbon copy, and in German. They plunged right in to mathematics, and described a result about graphs, a subject which appealed to me. I didn't understand it very well at first reading, just got the general idea. I never found out who did the typing or why.

At the end, the document described itself as the German-language abstract of a 1926 paper by Otakar Borůvka.

The abstract described a method for constructing the shortest spanning subtree of a graph whose edges have known lengths, and from this method trivially derived the corollary that the shortest spanning tree is unique if no two of the lengths are equal. For me, and it appears for almost everyone else, the interest of the paper was the method of construction, not the corollary.

In one way, the method of construction was very elegant. In another way, however, it was unnecessarily complicated. A goal, which has always been important to me, is to find simpler ways to describe complicated ideas, and that is all I tried to do here. I simplified the construction down to its essence, but it seems to me that the idea of Professor Borůvka's method is still in my version.

After reaching this simplification, I started wondering whether it was worth publication. Fortunately someone advised me to go ahead, and many years passed before another of my publications became as well-known as this simple one."

Also Kruskal's algorithm has been discovered independently several times. The survey of the works devoted to the minimum spanning tree problem until 1985 is given in the article by R. L. Graham a P. Hell: *On the History of the Minimum Spanning Tree Problem* [6] and this historical paper is followed up in articles [7], [8], [9].

In spite of the fact that the minimum spanning tree problem was solved it remained in the centre of attention of many specialists. Their effort has been to invent the quickest and most sophisticated algorithm of the Minimum Spanning Tree Problem not only for common graphs but also for special classes of graphs or solving problems of gaining minimum spanning tree that satisfies additional conditions.

Three classical solutions of the Minimum Spanning Tree Problem

Now let us look at the modern formulation of the problem and modern description of all above-mentioned best-known solutions (Borůvka's, Jarník's and Kruskal's algorithms).

The Minimum Spanning Tree Problem

Given a connected graph $G = (V, E)$ having n vertices and m edges. For each edge e let $w(e)$ be a real weight of the edge e. Our task is to find a minimum spanning tree of the graph G.

In Borůvka's algorithm we will in addition presume that any two different edges have different weights. This condition does not restrict the universality of the problem (for example we can list all edges and in the case that two edges are equal weights the first on our list we consider as the bigger one).

Borůvka's algorithm

Initially all edges of the graph G are uncoloured and let each vertex of the graph G be a blue tree (we suppose a blue forest which consists of n blue trees).

Repeat the following colouring step until there is only one blue tree, the minimum spanning tree:

COLORING STEP: For each blue tree T select the minimum-weight uncoloured edge incident to T (i.e. edge having one vertex in T and the other not). Then colour blue all selected edges.

Jarník's algorithm

Initially all edges of the graph G are uncoloured. Choose any single vertex and suppose it to be a blue tree.

At each of $(n - 1)$ steps colour blue the minimum-weight uncoloured edge having one vertex in the blue tree and the other not. (In case, there are more such edges, choose any of them.)

The algorithm finishes by gaining a blue spanning tree, the minimum spanning tree of the graph G.

Kruskal's algorithm

Initially all edges of the graph G are uncoloured. Order the edges in non-decreasing order by weight. Let each vertex of the graph G be a blue tree.

At each of m steps decide about colouring exactly one edge if it is coloured by blue colour or not. The edges are examined in a sequence defined by above-mentioned ordering. The chosen edge is coloured blue if and only if it doesn't form a circle with the other blue edges (i.e. in case that both vertices do not belong to the same blue tree).

The algorithm is finished when $(n-1)$ edges are coloured blue. Blue edges form a minimum spanning tree of the graph G.

If we consider the weight of edge as its length then ***the basic difference between these three algorithms can be characterized as follows***:

Kruskal's algorithm connects the two nearest blue trees in one blue tree at each step in which one edge is coloured blue.

Jarník's algorithm at each step spreads the only blue tree, which contains the initial vertex by the nearest vertex.

In Borůvka's algorithm at each step the union of all the blue trees being the nearest one another is performed.

Students know that each spanning tree in a connected graph with n vertices can be found not only by including $(n-1)$ edges that don't form a circle but also in the dual way; it means by consecutive removing edges from circles until there is no circle in a graph. Thus we introduce to our students to some dual algorithms for finding a minimum spanning tree too, as e.g. Kruskal's dual algorithm.

Kruskal's dual algorithm

Initially all edges of the graph G are uncoloured.

Order the edges in non-increasing order by weight. Let each vertex of the graph G be a blue tree.

At each of the m steps decide about colouring exactly one edge if it is coloured by red colour or not. The edges are examined according the above-mentioned order. The edge will be coloured red if and only if the edge belongs to some circle, which does not have red coloured edge.

The algorithm is finished when $(m-n+1)$ edges are coloured red. Remaining $(n-1)$ edges form a minimum spanning tree of the graph G.

Modifications of Jarník's Algorithm

From Jarník to Dijkstra

Given the graph G (figure 1).

Jarník's algorithm for gaining the minimum spanning tree supposing vertex a to be a blue tree can be illustrated as it is shown on figure 2. By each vertex there is a window with 6 parts corresponding to steps of algorithms. At each of 6 steps we write into the corresponding part of all windows, which belong to the vertex that doesn't lie in the blue tree, *the actual information describing the nearest distance between the vertex and the blue tree* (the sign ∞ means that the vertex isn't connected to the blue tree in the given step). Among all these vertices we find the nearest one and we spread the blue tree by this nearest vertex (we colour blue the corresponding edge). Finally we get the minimum spanning tree containing 6 blue edges.

In the similar way we can illustrate the known Dijkstra's algorithm for finding the shortest path from the given vertex u to the other vertices in a connected graph with non-negative weights of edges. The only difference is that in each step we write *the actual information describing the nearest distance between the vertex and the initial vertex u* (figure 3).

From Jarník to Breadth-First Search and Depth-First Search

Given a connected graph with all edges having the same weight (e.g. weight $w(e) = 1$ for each edge e) and let us trace the Jarník's algorithm for gaining the minimum spanning tree on this graph. We see that at each step *an arbitrary edge*, having one vertex in the blue tree and the other not, is coloured blue. Jarník's algorithm works on the given graph in the same way as on a graph without weighted edges. A consecutive adding of vertices (at each step we spread the blue tree by one vertex, the end-vertex of an exactly blue coloured edge) we can understand as a consecutive search of them. To get either Breadth-First Search or Depth-First Search algorithm (for consecutive searching of all vertices) we simply modify Jarník's algorithm in the following way.

Breadth-First Search: At each step we choose from the uncoloured edges, having one vertex in the blue tree and the other not, such *an edge having the end-vertex being added to the blue tree as the first* of all in blue tree lied end-vertices belonging to the mentioned uncoloured edges.

Depth-First Search: At each step we choose from the uncoloured edges, having one vertex in the blue tree and the other not, such *an edge having the end-vertex being added to the blue tree as the last* of all in blue tree lied end-vertices belonging to the mentioned uncoloured edges.

Conclusion

In the article we have outlined one possible way of object teaching concerning graph algorithms. It is always very useful to present more solutions (if they exist) of the given problem to be more thoroughly understood. Moreover, it is also useful to use a modification of the already known algorithm by explanation of a solution of the given problem.

In conclusion let us mention that the description of all explained methods as an edge colouring process is really very welcomed and favoured by our students.

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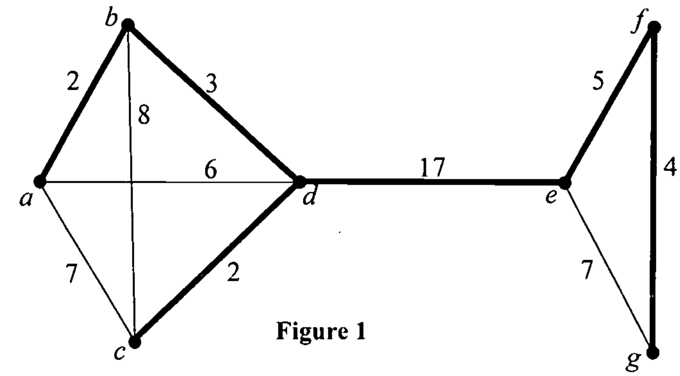

Figure 1

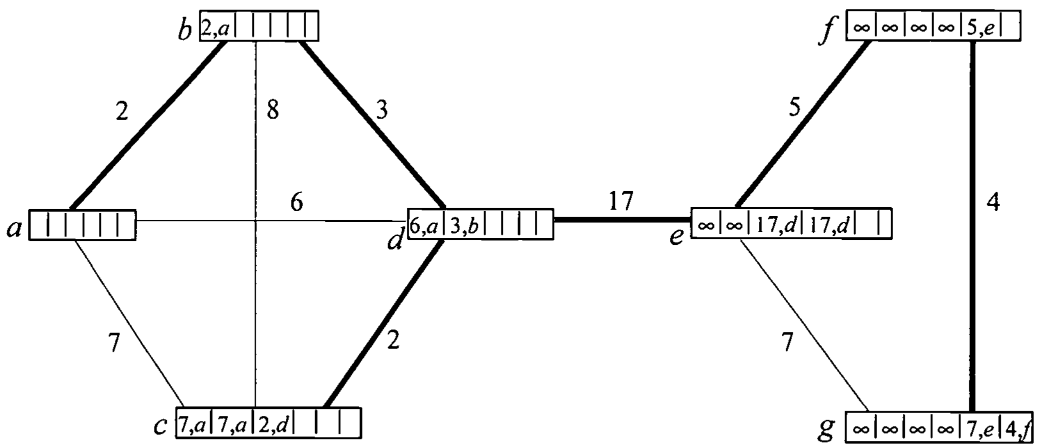

Figure 2

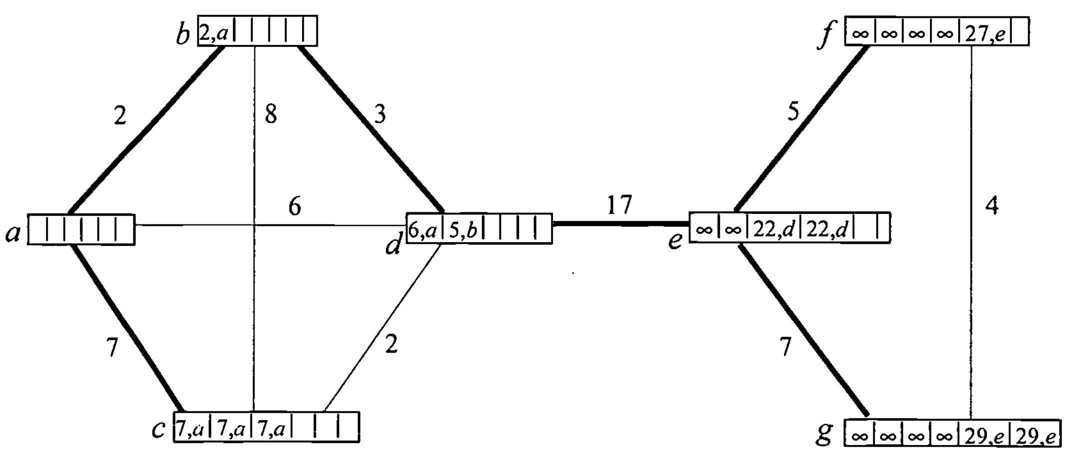

Figure 3

THE FRONTAL COMPETITIVE APPROACH TO TEACHING COMPUTATIONAL MATHEMATICS

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ABSTRACT

The paper develops a general methodological framework for teaching Mathematics so that some minimum learning outcomes are achieved for all students and at the same time student-dependent learning outcomes due to individual creativity and effort are also possible.

Although the theoretical ideas and pivotal concepts behind the paper, such as motivation, feedback, reinforcement and others, are well known, specific implementation of these ideas may present a real challenge to practical teachers. The paper contains a comprehensive detailed view of one aspect of differentiated teaching: behavior modification. This aspect is considered as a prime necessity in circumstances when Mathematics is being taught to a large student audience. The methodological conditions and technological practices necessary to implement such an approach are discussed. Examples in the context of teaching numerical methods of Linear Algebra and other related courses are given.

Key words: Mathematics Education in Universities, Creativity in Mathematics Education, Mathematics Competitions in Mathematics Education.

1 Introduction

As Russian mathematician Yakov Tsypkin wrote jokingly in the preface to one of his works (Tzypkin 1970), reading mathematical books results in the three levels of knowledge. The first level means that a reader has understood the author's argumentation. The second level means that the reader has become capable of reproducing the author's arguments. And the third level means that the reader has acquired a capacity to refute the author's argumentation.

This joke reflects the fact that mathematics as a subject has a specific feature that it can not simply be put in memory. It means that a mathematics student can not stop at the first level of understanding. He needs to reach at least the second level. To do this the student has to pass all the information through his mind by solving a large number of tasks independently, thus as we say "adjust his head and hand". But even this is not enough, because as Hungarian mathematician Alfred Renyi said (Renyi 1967), "who learns the solution without understanding the matter can not use it properly". Independence, critical approach and creativeness – these are the third level features and only such knowledge has real value when learning mathematics.

Of late years, Russian mathematics community has been really feeling the need for novel teaching methods to stir the students to greater activity. One would expect a wealth of methods to choose from and apply. But upon examining methods that are practically being used, one finds a lack of appealing and interesting approaches that would create and hold students' interest and make them continue to study.

Here it is pertinent to note that mathematics has another feature. Mathematics may be defined as "chamber" science by the nature and mathematicians are often told to be "piece-goods". However many universities traditionally practice teaching to a large student audience. For example, large audience-oriented teaching has become the trait of Russian universities to the extent that the large audience have been assigned a special term,"a stream".

It is almost evident that the great size of the stream stands on the way, – it erects obstacles to awaken and hold students' interest in learning mathematics. It is a handicap to independent thinking, as many of students get used to "flow over the stream" and prefer to be as ordinary as their classmates. How can teacher transform this obstacle into advantage? How can he encourage students' independence? How can we help them to understand their outstanding abilities? And finally, how can we prove that mathematics is a live, beautiful science but not a collection of incontestable proofs, unquestionable facts and irrefutable arguments?

Fortunately, mathematics itself often prompts us how to achieve these goals. One of such methods termed the Frontal Competitive Approach (FCA) is discussed below. We consider it mainly in circumstances when the subject is being taught to a large student audience where students' behavior modification is a prime necessity.

2 Main idea of FCA

It may be explained by the definition itself. "Frontal" means general, involving all students to meet one common goal. "Competitive" means opportunity for a success due to individual's creative and non-standard solutions or actions. To implement FCA we need to do the following: (1) Organize creative environment. (2) Encourage students' creative potential. (3) Give start to students' instinct of competition. (4) Ensure transparency of assessment.

Let us examine these components in detail in the context of teaching Computational Mathematics.

2.1 Creative environment

Most of the existing educational materials on Computational Mathematics provide main theoretical data and sometimes theoretical instructions on how to program a numerical method or algorithm. However, it seems to be inadequate to the end. We believe that the true understanding of a numerical method may be achieved if: (a) a student completes assignments related to a challenging programming project; (b) each project results in practical use of that particular method assigned for the student; (c) the student conducts a set of extensive computational experiments with the program he developed independently; and finally (d) frontal rating of the projects is carried out by the teacher together with the students.

Programming in itself is beneficial for student due to a number of reasons. First, it provides an opportunity to understand and learn a numerical method "from inside". This is quite different from utilizing ready-made software and significant for any creative professional. Second, it improves student's computer proficiency, as it requires keen programming. And finally, it develops general analytic and solution seeking performance and implants practical skill to attack and solve computationally oriented problems.

To organize creative environment while teaching numerical methods to a large audience requires to make programming assignments as varied as possible in terms of the methods' algorithmic significance rather than their initial data. However, the number of variations on every method is usually limited. In these conditions, finding as many as possible versions of every numerical method becomes a matter of great methodological importance for each teacher.

Organizing creative environment means, also, that we should evaluate any laboratory programming project as a single study objective which possesses all the features of a completed software product. Among them are modular structure, convenient interface, efficient utilizing computer resources (memory and time), and possibility to implement a wide plan of computational experiments. This differs definitely from a widely used technique when the students work on one and the same ready-made software when they only enter their initial data and wait passively for a result. The approach we apply makes them perform valuable creative operations, stimulates each student's competitiveness, prevents cheating and helps to improve overall class performance.

A classic example of how to find as many as possible variant forms of a numerical method is the topic "Elimination and Matrix Inversion Methods". First of all, the teacher should systematize a set of Gauss and Gauss-Jordan elimination specific characteristics. They are: (1) direction of elimination of unknowns, (2) mode of access to the matrix entries, (3) mode of updating the active sub-matrix, (4) pivoting strategy etc. (Ortega 1988). Then independence of these characteristics will result in a significant number of different variants of assignments on the same topic being studied.

Over the course on many years, our work is focused on the possible ways of applying FCA to teaching numerical methods in Linear Algebra, Least Squares, Optimal

Filtering, Optimal Control, Linear Programming and Nonlinear Optimization. As a result, we recommend that teachers use textbooks, that offer a good choice of various project assignments. The first one contains: Topic 1 - Elimination and Matrix Inversion Methods 26 assignments in total, Topic 2 - Sparse System Solution 48 assignments in total, Topic 3 - Cholesky Decomposition 40 assignments in total, Topic 4 - Orthogonal Transformations 28 assignments in total, Topic 5 - Simultaneous Least Squares 28 assignments in total, Topic 6 - Sequential Least Squares (Semoushin & Kulikov 2000), and Optimal Filtering 25 assignments in total. The second one contains: Topic 7 - Simplex Method 70 assignments in total, and Topic 8 - Nonlinear Optimization 30 assignments in total(Semoushin 1999).

2.2 Student's creative potential

Lectures usually prove one variant of a numerical method in a certain topic. For example, we prove: LU-factorization theorem, LU-factorization theorem with the choice of pivots, LU-factorization algorithm replacing the original matrix by factors L and U, and so on. However, practically always for each proven theorem or algorithm there exists a dual variant. In our case the dual variants are those with $UL-$factorization (LDU and UDL factorizations are also possible). Therefore it will be expedient if each assignment contains formulation and proof of the theorem/algorithm for the assigned variant, which has to be made independently. Thus students are trained to understand subject of mathematics in a wider sense, their creative abilities and potential become more active and may be well evaluated especially for the gifted students.

2.3 Instinct of competition

As a rule, students of mathematical departments are very eager to gain and demonstrate professional knowledge of computers and modern programming technology. They are not interested in "hanging about" the initial level of computer proficiency. More to it, they express definitely their wish to show their skills for creating "outstanding" software. FCA ideally supports this instinct of competition. Indeed, multiplicity of assignment variants fortunately comprises two features of the variants: their resemblance and difference. Due to the resemblance between variants students' projects can be compared, and due to the difference they are of individual nature. Teachers applying FCA note surprising cases when a student who has got already a credit on the project, for example in sparse system solution, continues upgrading the software by changing access mode to matrix entries in order to achieve faster operation. Sometimes students arrange a kind of competition between them, in whose software has better interface or faster operation. Sometimes in holidays we hold a presentation of the best programs developed by the students. At first we required that a software should be written in Pascal but now we accept usage of various tools: Visual Basic, Delphi, Builder C++ etc.

2.4 Transparency of assessment

Teacher's role in FCA application is very important. Besides all above mentioned, a teacher should put forward a precise and definite system of requirements and evaluation criteria for students' projects. A student should know definitely what mark and for what work quantity and quality he will get. Starting his work student chooses by himself the level of assessment he initially pretends for. System of assignments should be designed in such a way that allows each student to move independently from one assessment level to another according to his own work. For example, the system of assignments on simplex method (Semoushin 1999) contains 70 different variants divided into three groups dependent on their complexity: basic level (20 variants), advanced level (30 variants), and higher level (20 variants). The marks are given accordingly: sufficient, good, and excellent. This transparency of assessment have a notable effect on the students' activity.

3 Technological summary of FCA

Some general tools indicative of the FCA are the following.

1. *Creative environment.* A broad assortment of assignments and tasks is offered to students together with the clearly differentiated scenarios of their accomplishment.

2. *Goal setting.* A clear formulation of both short-term and long-term goals is offered: (1) marked improvement of programming skills together with the deeper understanding a particular numerical method, and (2) profound understanding the subject of Computational Mathematics and ability to attack computationally oriented problems.

3. *Challenge.* The environment including non-trivial assignments and tasks with increasing levels of difficulty, challenges the student to keep self-independent working.

4. *Student-controlled navigation.* Putting the locus of control in the hands of the student has a great psychological effect: even the weak students put in a claim for higher grades and try to move to the upper level of difficulty while choosing the assignment.

5. *Competition.* Competition by "playing" against others appears to be naturally embedded in the teaching process because the above tools are in excellent agreement with this human instinct.

6. *Rewards.* Rewards, such as higher grades or "automatic" credit for the course, may be offered as students show an obvious increment in skill and success.

4 Some empirical data

Since 1988 the standard approach (SA) in teaching Computational Mathematics and Optimization Methods was used at two Ulyanovsk universities. The SA meant that, while studying Computational Linear Algebra, students were offered to fulfill only one laboratory programming project per semester in order to be allowed to take examinations, and this project was on Elimination and Matrix Inversion Methods without essential differences between separate project assignments. The courses of Numerical Methods and Optimization Methods included some problems solution using the Dialog Computing System (DCS) developed in Mathematics Institute of Byelorussia Academy of Sciences for the purpose of teaching and learning Applied Mathematics (Fourunzhiev, 1988). The DCS contained a set of subroutines. The student had to send his/her commands to the system and wait for the system requests and messages thus organizing a dialog during the work. In 1993 we switched to the FCA at Lomonosov Moscow State

Table 1: Percent student distribution under the two teaching approaches conditions

Grading system		NoW under SA conditions		NoW under FCA conditions	
Grade	NoW[1]	attempted	fulfilled	attempted	fulfilled
Excellent	3	12	4	40	24
Good	2	36	12	44	48
Satisfactory	1	42	56	16	24
Failure	0	0	28	0	4

[1]Number of works.

University Branch in Ulyanovsk (transformed into Ulyanovsk State University in 1996) and in 1995 at Ulyanovsk State University of Technology.

Over the four years of the SA usage and then the nine years of the FCA application we collected our observations, so this time can be considered as the length of our teaching experiment intended to analyze performance differences between the two teaching approaches, SA and FCA.

Usually the researcher has a number of techniques to help gather and make sense out of the data collected. However in education practice, there is a lack of definite, standard metrics for teaching performance assessment. All the data should be treated as subjective in some way or another as human participation is inherent in several stages of the education experiment. Nevertheless, subjective data from human observations and judgements can be considered objective and valid as performance measures if the observations are verified and judgements are derived from what is purported to be measured.

To obtain the "big-picture" of what is good and bad and why in the two approaches, we used the following student performance measures: (M1) Amount of work attempted at the beginning of semester, (M2) Amount of work fulfilled by the end of semester, (M3) The week when a student begins to work at full power, and (M4) The week when a student defends the first work fulfilled. A large amount of data from teacher observations and student questionnaire responses were collected, summarized and averaged for two core requirement (compulsory) courses: Computational Linear Algebra (Topics 1 through 4) and Optimization Methods (Topics 7 through 8), and also for two major requirement (elective) courses: Recursive Least Squares and Optimal Filtering Algorithms (Topics 5 through 6), see Section 2.1 for topic numbering. For Computational Linear Algebra, Table 1 shows the averaged percent of students who attempted and fulfilled three, two, one or zero laboratory work assignments under the SA and FCA approaches, and Table 2 shows the above mentioned "averaged" student performance measures.

These results indicate convincingly that FCA is superior in effectiveness to SA as better stimulating students' interest in Mathematics. This is also reflected in the fact that the students who choose to be enrolled for Recursive Least Squares and Optimal Filtering Algorithms have increased in number after completion the course of CLA. Analogously, having completed the Linear Programming case study project under FCA conditions, the majority of students express their desire to fulfill the Nonlinear Optimization case study project independently, i.e. only on the basis of going into the

Table 2: "Averaged" student performance measures (Topics 1 to 4)

Measures	M1	M2	M3	M4
SA	1.50	0.92	10	16
FCA	2.24	1.92	3	10

recommended literature. We explain this by the increased students' self-reliance and confidence in their ability to make sense of new material without assistance.

5 Conclusion

Designing efficient education process in a large audience is complicated and time consuming. This paper has touched on a few basic teaching tools to exploit when motivating students to learn Mathematics. Called FCA and applied in different Russian universities, this approach has proved that students have a generally positive response to it. Individual, self-dependent work within a large audience encourages students' sound competition, desire for creative solutions and better performance indices.

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THE ROLE OF THE TEACHER IN A TECHNOLOGY BASED CALCULUS CLASS

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ABSTRACT

Although the focus of this paper is on the method I use in teaching a multivariable calculus course, based on Calculus and Mathematica (C&M) of Davis, Porta and Uhl, at the University of South Carolina Aiken, the issues I discuss and the teaching techniques I describe may be relevant to many other technology based math courses. In particular, I discuss how the objectives of the course are set and why I favor C&M over the traditional approach, how I try to alleviate problems related to student weaknesses in computational skills, lack of interest or commitment, and lack of familiarity with Mathematica, and how I evaluate the success of the course.

The method I use differs substantially from the method suggested by the authors of C&M. In the presentation I will discuss, in some detail, the following four basic elements of my approach. (a) Introduction. Each chapter is introduced by a lecture that is very close to the C&M approach, using both hand calculations and Mathematica code. The students also receive a handout containing a complete list of the new concepts and some brief explanations. (b) Tutoring. The students are required to complete, at least part of each homework assignment, in class, usually in teams of two students. This allows the students to discuss problems among themselves and to ask for the instructor's help when needed. (c) Feedback. The students are required to study the answers to the homework problems they miss. (d) Constructive testing. The tests are used as diagnostic tools of student deficiencies. Students who fail a test are required to study further and retest.

This method is quite demanding on the instructor's time. It seems to be quite beneficial for the students, however, because it helps reduce considerably their frustration, excite their interest, and improve their conceptual understanding of calculus.

1. Introduction

In 1994, our department, the Department of Mathematical Sciences at the University of South Carolina Aiken, at the recommendation of Dr. Robert Phillips, then a senior member of our faculty, adopted the use of *Calculus and Mathematica* (C&M) (Davis, Porta and Uhl 1994) in all of our calculus classes. Very quickly, however, this approach stirred up an intense controversy among our faculty that continues to this day. While some of us are extremely pleased with C&M, others dislike either the approach C&M takes or the way it is received by their students. Some have abandoned C&M and are using other texts.

Six years ago I started using C&M in my vector calculus class. Compared to the traditional approach, I find that C&M can be much more beneficial to the students. Its greatest strength is that C&M can help the students develop an intuitive understanding of vector fields, of the operators grad, div and curl, of line and surface integrals, and of the meaning and usefulness of Green's, the divergence and Stokes's theorems. In this respect, the traditional approach is not very successful. Though the traditional approach is supposed to be a rather formal approach, based at least to some extent on theorem proving, in reality the emphasis is almost exclusively on symbolic manipulations. Most exercises are on such manipulations and few, if any, require a deeper conceptual understanding. Furthermore, many important topics, such as line integrals, Green's theorem and surface integrals, are given such a brief treatment that the students are unable to develop any useful understanding of them, while quite often the divergence and Stokes's theorems are completely left out.

I believe that the intuitive understanding of the concepts and theorems of vector calculus that the C&M students can develop will help them considerably in their future studies. Students majoring in mathematics will be better prepared to understand and appreciate the rigorous and complete theorem proving development of the theory when they are introduced to it. Science or engineering majors will be able to develop early a more complete understanding of abstract physical theories. I have the rather unusual opportunity to compare the results produced by C&M and by the traditional approach, because I also teach the calculus-based physics classes. As I discuss elsewhere (Kapranidis 1998) I find that, compared to the students of the traditional calculus courses, the C&M students are considerably better prepared to understand the theory of physics and in particular the theory of electricity and magnetism. For these reasons, the main objective I set for my vector calculus class is to help the students develop an intuitive understanding of calculus.

Despite its potential, the C&M approach does not always produce the desired results. Soon after I started using C&M in my class, I realized that the teaching method recommended by the authors of C&M did not work well for many of my students. The authors recommend that no introductory lectures should be given and that the students should not read the book before they experience a new concept on the computer screen using the C&M electronic text. This approach works well for some students, but many other students find the way the C&M text develops new calculus concepts very difficult to follow.

Certainly, part of this problem can be attributed to the students' lack of interest or discipline, weaknesses in their math background, unfamiliarity with Mathematica and the limited amount of time they allow for their study. It is not, however, an exclusive problem of the students who are unprepared, uninterested or have poor study habits. It also affects many of the rather typical students of the American colleges, who have adequate preparation and ability, and make a genuine effort to learn.

In my opinion, the difficulties experienced by these students are strongly related to the nature of calculus and of the learning process itself. I believe that these problems seem to be more noticeable when the C&M approach is used, because the emphasis of this approach is on the concepts of calculus rather than on symbolic manipulations that are easier to master. Further, I believe that, to alleviate these problems and still take advantage of the C&M approach, the teacher must assume a much more active role in the students' learning process than that required of a teacher in a traditional calculus class.

After experimenting with different ideas, I developed a special teaching method that I use in my vector calculus class. The basic text I use is C&M. I also use some support materials that I have developed. My approach has substantial differences from the approach suggested by the authors of C&M. I feel, however, that it takes care of some of the problems encountered by many teachers who use C&M in their calculus classes. This approach is not particular to vector calculus, and can be adapted and used in any technology based math class in which the emphasis is on mathematical concepts.

2. The Teaching Method

All of our calculus classes meet six hours a week, though they are four-credit-hour classes. In this respect, calculus classes are similar to our typical classes that have a lab component, such as physics or chemistry classes. The Mathematica based calculus classes meet in a classroom currently equipped with 20 Windows based PC's in which Mathematica 4.01 is installed. One of these computers is equipped with an LCD projector.

The course is organized in the following way. The material is divided into four units. Specifically, the first three units consist of three chapters each, while the last unit consists of two chapters of the C&M text. We complete one chapter per week. For each chapter, the students have to do a homework assignment consisting of several problems. The assignments are collected and graded promptly. At the end of each unit, the students take a test. At the end of the semester, the students also take a comprehensive final exam. The final grades are calculated as follows: Homework: 40%, four tests: 40%, final exam: 20%.

My teaching method is characterized by four basic elements for which I use the terms (a) introduction, (b) tutoring, (c) feedback, and (d) constructive testing.

a. Introduction. This is probably the main element that makes my approach substantially different than the approach suggested by the authors of C&M. Contrary to their recommendation, for each new chapter I give a thorough introductory lecture. These lectures are precisely structured. In the first part of the lecture the students are given a general orientation. That is, I define the objectives of the chapter, the particular context of the subject, and the connections of the new concepts to the previous knowledge of the students. In the second part, the new concepts are introduced. Though I always stay very close to the C&M text in terms of the subject matter, I introduce the new concepts in a way that is completely independent of Mathematica, and then I work my way down to the particulars of the C&M text. Finally, in the third part of the lecture, the actual C&M text is projected onto a screen and we discuss and analyze it in detail. During this time, the C&M approach is compared and contrasted with the previous introduction of the same concepts. Along the way, we also discuss the Mathematica code and we often use the power of Mathematica to explore cases and possibilities that are not covered in the C&M text or to explore in depth questions that the students may come up with. Each C&M chapter is divided in three sections named "Basics", "Tutorials" and "Give It a Try". In the introductory lecture for each chapter I try to cover the "Basics" as completely as possible, and also some selected parts of the "Tutorials".

To help the students focus better, both during the lecture and later in their study, for every new chapter I prepare a handout with a summary of the new concepts and the definitions, theorems and formulae the students are expected to learn.

b. Tutoring. After the introductory lecture the students, usually in groups of two, are required to first study the "Basics" and "Tutorials" parts of the chapter, and then work on their homework assignment, which consists almost exclusively of problems from the "Give It a Try" section of the C&M text. Occasionally the assignment may include problems requiring pencil and paper symbolic manipulations. On the average, they have about one hour of class time for the "Basics" and the "Tutorials" and two hours for the homework assignment. Though this is not enough time to complete the assignment, it gives them adequate opportunity to ask questions, receive personal tutoring when it is needed, and at least make sure that they have a good idea on how to approach the problems.

c. Feedback. Each group must turn in one completed homework assignment per chapter for grading. The assignments are graded promptly and returned to the groups. At this time the solutions to the problems in the assignment are provided, and the groups are required to study the solutions, especially of any problems they missed. The students are also advised, on a personal basis, on what deficiencies their errors may indicate and further studying is recommended. Occasionally, the groups are allowed to resubmit corrected solutions and recover some of the missed points.

d. Constructive testing. Testing is used not only as a means for grading the students but, most importantly, as yet another opportunity for them to learn. The whole testing process is organized in the following way.

Soon after a unit is completed, we have a review session to help the students prepare for the test. The students are given a new handout containing a summary of the concepts in that unit and examples of problems similar (but not identical) to the problems on the test. The tests are pencil and paper tests and do not involve the use of Mathematica.

Each test is graded right away and each student is personally given an evaluation of their performance. When deficiencies are detected, the students are given one week to further study and retest. Typically, if they score less than 70% they must retake a complete test, otherwise they have to take only a partial test, that is focused on the types of questions they had difficulties with on the first test.

At the end of the semester, we also have a two hour session in which we review all of the material in the course, in preparation for the final exam. The final exam is comprehensive. No retesting is allowed for the final exam.

3. Discussion and Conclusions

The method I use in my vector calculus class is the result of my efforts to address some serious problems that quickly became apparent when I first started using C&M in my class. All these problems seemed to stem out of the fact that the students had great difficulty in understanding the new concepts by studying the C&M text on their own. Feeling the pressure from the fast approaching homework deadlines, they would start working on the homework problems before they had a clear sense of what exactly they were supposed to do. Their way of dealing with this problem was to look over the examples in the text and to try to imitate and adapt them, by trial and error, so as to produce "solutions" to the homework problems. These "solutions" were most often totally meaningless, while their effort was routinely frustrated by errors in their Mathematica code.

Using an introductory lecture for each chapter seems to be an effective way to address this problem. The benefits of the introductory lectures are two-fold. First, the students get quickly oriented and focused on the subject. This increases the rate at which they learn and reduces substantially their studying time. Second, by using the actual C&M text in the lecture I have the opportunity to decipher the Mathematica code and help the students learn how to use the software correctly. This works quite well, though there are some students who can never get completely over some previous bad experiences with Mathematica. In general, however, as the semester progresses the ability of the students to use Mathematica correctly rapidly improves and soon Mathematica is not a problem any more. This is also true for students who have no previous exposure to Mathematica.

Students are not expected to understand the new concepts completely by the end of an introductory lecture. What is important however is that the students understand enough so that they feel well oriented and able to continue to study on their own.

The most intense learning happens during the time when the students working in groups study the C&M text and do the homework problems. For this reason, having at least part of these activities take place in class gives me the opportunity to provide some individual tutoring or to give to the whole class general instructions on how to approach some of the more challenging problems. I also have the opportunity to assess the level of understanding of the students by discussing with them the approaches they take in solving the homework problems. This allows me to develop a better sense of the strengths and weaknesses of my students, gives me the opportunity to suggest to them ways by which they may overcome any deficiencies they have, and helps my overall teaching to become more focused and effective. At the same time, the personal attention the students receive further decreases their level of anxiety and frustration and improves their learning.

Another very important element of my approach is homework feedback. I find this element to be especially important when the C&M approach is used, because the solutions of the C&M homework problems often require verbal explanations, descriptions or justifications rather than predominantly symbolic manipulations. Students are not accustomed to this type of problems, and initially the overall performance of the class is not very good. Requiring the students to study the correct solutions to the homework problems, especially those that they miss, helps them improve their understanding of the material, their ability to make logically consistent arguments, and the way in which they express themselves. It is very important that this feedback takes place as soon as possible, before the students forget their own work. Thus, the homework should be graded promptly and returned to the students as soon as possible.

The effect of homework feedback is quite remarkable. The class average of the first homework assignment is usually low, in the 70-75% range. Typically, by the forth assignment the class performance becomes quite good and remains high for the rest of the semester. During this time class average for the homework is in general above 90%, while quite a few groups achieve 100%.

Finally, testing is used in a constructive way in an effort to achieve three important goals. The first goal is to give the opportunity to the students not only to review the material of a substantial part of the course but, most importantly, to see how the elements we study in different chapters fit together to form a larger picture. This is accomplished in the two hour review session that we have before each of the four tests.

The second goal is to use tests as diagnostic tools of the deficiencies that the students may have. For this reason, the questions in a test should be carefully chosen and, if possible, they must cover all important topics studied in a unit.

The third goal is to use the tests as an incentive for the students to further study and improve their understanding in areas where they were found to have some problems. To achieve this, the students who miss some questions in a test are given the opportunity to further study and retest, for a substantial fraction of the points they missed the first time. This approach has another important result. It reduces test anxiety because the students know that they will have a chance to improve their initial result.

The method I use in my vector calculus class requires from the teacher considerable amounts of time outside the classroom for preparing handouts, detailed homework keys, multiple tests and also for retesting. Furthermore, teaching is more intense, especially during the time when personal tutoring takes place. The class must also be very well organized. If the time available for a unit is exceeded, other units will not be allowed enough time for all the required activities to take place. However, when the schedule of the class is closely followed, there is adequate time for all the units and for reviewing and testing. If the class is too large, having a well qualified teaching assistant during the tutoring phase may be necessary. Our department provides graders for all calculus classes. This is very useful because it makes it possible to have the homework assignments graded promptly. I choose to grade the tests personally, however, because this allows me to develop a better feeling of the level of class performance and helps me adjust my teaching.

Despite the high demands that my approach makes on the teacher's time, I believe it is worthwhile because it reduces considerably the level of frustration of the students, and improves their learning by making it possible for them to take fuller advantage of the C&M approach. For the last three years that I have used my method in the way I describe here, the drop rate because of problems that students had either with the C&M approach or Mathematica has been practically reduced to zero. When I first started using C&M, about 20% of the students who started the class would either drop it or fail it. Some students still fail the class. These are always very weak students who are certainly not prepared to take vector calculus in any format. There is also a percentage of the students, probably as high as 10%, who manage to pass the class though their skills are not as high as I would like them to be. I find this somewhat disturbing, but it is by no means a problem unique to this class. The percentage of the students who are very successful with this method and make an A in the class typically is in the range of 30-40%, while I believe that all the students develop a deeper understanding of calculus concepts compared to the students in the traditional classes.

I would certainly like to stress here that C&M deserves full credit for the teaching methodology I use in my class. My approach simply provides the support many students need to enable themselves to follow C&M and benefit from it. Though I have developed this method specifically for my vector calculus class and I have used it exclusively with C&M as the basic text, I feel that it could be applicable in other classes. I believe though, that it is not suitable for traditional classes. Without the C&M electronic text the class time could not be used as efficiently and there would not be time for all the other activities to take place in class. Thus, the method can only be used in technology based classes for which some text and software analogous to C&M are available.

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ANALYZING FUNCTIONS' BEHAVIOUR IN A COMPUTATIONAL ENVIRONMENT

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ABSTRACT

This article reports the research conducted with first year Calculus students. During the last five years, the authors have been investigating whether exploring functions in a computer environment would improve students' performance. This is a research report on the "Analysis of the Behavior of Functions," with the computer as the main instrument in this methodology. This tool must be used bearing in mind certain criteria. The teacher must control not only the content, but also the software used. In the first attempt, the activities were carried out in the computer laboratory right after the discussion of the subject in a theoretical class, and students insisted on presenting "exact" results. Not satisfied with this type of behavior, the researchers decided it would be important to change students' attitudes. How could this be encouraged? By using, for example, an instrument of analysis, such as the effects of the "didactic contract." Later, new activities were prepared in an attempt to promote the computer → theory → computer dynamic. This dynamic proved to be highly efficient in fostering behavior that was active, critical, investigative and more independent of the teacher.

Keywords: Behavior of function – Graph – Image – Didactic Contract – Computer environment

Introduction

Over the last six years, the authors have investigated whether the exploration of each of the topics of differential and integral calculus in a computer environment helps improve students' performance in this discipline. To this end, they prepared activities which were applied and, after revision, reworked. This is the report on the research conducted into "Analysis of the Behavior of Functions."

At first, the class plan was: theoretical considerations followed by computer-based activities. As this dynamic did not prove to be satisfactory in achieving the objectives proposed, we decided to change the focus. Our group believes that effective learning occurs when the situations proposed provide a reciprocal exchange, thus favoring the construction of knowledge. As such, the following teaching procedure was used: *computer* $\rightarrow$ *theory* $\rightarrow$ *computer.* This new approach was put in practice, breaking a didactic contract (Brousseau, 1986) in the process.

The main tool in this methodology is the computer. Especially in the study of the functions, it enables one to show that image plays a partial role in realization: the graphs allow one to see the function. If the resources of this tool are fully exploited, students may construe basic elements for forming concepts regarding differential and integral calculus. Nevertheless, the group believes that the computer cannot substitute theoretical classes, much less the teacher, but is an ally in making conjectures, testing hypotheses and validating student results.

Description of the Study

At the beginning of the course, the students study the behavior of affine, quadratic, cubic, exponential, logarithmic, and trigonometric functions. New tools are necessary to extend the study: limits and derivatives. In educational books, algebraic expressions are generally used to determine the items necessary for preparing graphs of functions.

According to the Theory of Conceptual Fields (Vergnaud, 1990), it is important to offer a variety of situations for students to identify the invariants of a particular concept. As such, it may be desirable for students to be able to recognize and identify the following elements in graphs: domain, image, symmetry, parity, critical points, maximum and minimum values, inflection points, tangent lines, asymptotes, behavior close to infinity and points where the function is not defined.

For these purposes, the group posed itself the following question:

"How can analysis of the behavior of functions be conducted in a computer environment?"

Two activities were initially prepared to be worked on in the computer laboratory, directly after a discussion of the subject in a theoretical class.

For the first activity, functions were chosen which allowed one to observe a variety of behaviors, the objective being to have the students recognize the critical points ($f'(x) = 0$ or $f'(x)$ does not exist) in the graph of the function, in addition to the sign of the derivative, in order to determine both ends of the function. The students were also asked to decide whether the graph had asymptotes. With the software used, students were able to draw the tangent to the curve at each point; analyzing the angle formed by the tangent line and the axis of the abscissae, and were able to decide on the sign of the angular coefficient and, consequently, the derivative at this point.

The first activity is presented below.

I. Graph Sketching

For each of the functions below.

(a) Plot the graph using a curve sketcher.

(b) Identify the points where $f'(x) = 0$ occurs.

(c) Looking at the graph, identify the x values for which $f'(x) > 0$ occurs.

(d) Looking at the graph, identify the x values for which $f'(x) < 0$ occurs.

(e) Calculate algebraically the points for which the derivative is zero.

(f) Compare the answers obtained in (b) with those obtained in (e): what happened?

(g) Looking at the graph, check whether the function has: local maximums, absolute maximum, local minimums, absolute minimum, inflections, asymptotes, points where the function is not derivable.

1. $f(x) = x^4 - 2x^2$ 2. $f(x) = \dfrac{x^2 - x + 1}{x^2}$ 3. $f(x) = \dfrac{x^2}{x^2 - x - 2}$

4. $f(x) = xe^{-2x}$ 5. $f(x) = \sqrt[3]{x^2 - x^3}$ 6. $f(x) = e^{\frac{1}{x}}$ 7. $f(x) = e^{-x^2}$.

Most students answered the question satisfactorily, apparently just through observing the graph, as requested. Some carried out algebraic calculations, especially to determine the ends, using the sign of the derivative.

It was noted that the last item involved many concepts simultaneously, which is not advisable from a pedagogical perspective.

In the second activity, seven functions were chosen and for each one, questions were selected which were more appropriate to their graphs.

Although the objective was to interpret the graphs, many students presented algebraic calculations and about half of them calculated the limits for determining the asymptotes.

The second activity is presented below.

II. Interpretation of the graphs

1. (a) Use a *graph sketcher* to plot the graph for: $f(x) = \exp(1/x)$.

(b) Give the algebraic expression of each asymptote.

2. (a) Use a *graph sketcher* to plot the graph for: $f(x) = xe^{-2x}$.

(b) Give the coordinates of the maximum points and maximum values.

(c) Give the coordinates of the inflection points.

3.(a) Use a *graph sketcher* to plot the graph for: $f(x) = \sqrt[3]{x^2 - x^3}$.

(b) Give the coordinates (x_0, y_0) of the minimum point. What is the value of $f'(x_0)$?

(c) Write the equations of the tangent lines at the critical points.

4. (a) Use a *graph sketcher* to plot the graph for: $f(x) = x^4 - 2x^2$.

(b) Does the graph of f present any symmetry? If so, what type?

(c) Is the function even? Odd? Neither? Why?

(d) Give the angular coefficient of the tangent lines to the graph of f at the abscissa points x = -1, x = 0 e x = 1.

(e) Give the image of f.

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5. (a) Use a *graph sketcher* to plot the graph for: $f(x) = \frac{x^2}{x^2 - x - 2}$.
(b) Give the domain of f.
(c) Give the algebraic expressions of the asymptotes.
(d) Give the coordinates of the maximum and minimum points.

6. (a) Use a *graph sketcher* to plot the graph for: $f(x) = e^{-x^2}$.
(b) At which points is the function negative?
(c) What is the behavior of f when close to $+\infty$ and to $-\infty$?
(d) Give the algebraic expressions of the asymptotes.
(e) Give the image of f.

7. (a) Use a *graph sketcher* to plot the graph for: $f(x) = (x^2 - x + 1)/x^2$.
(b) What is the behavior of f when close to $+\infty, -\infty$ and $x = 0$?
(c) Give the maximum value of function f.
(d) Give the minimum value of function f.

Upon analyzing the outcome of the activity, we realized that the choice of function in question 2 did not to make it easier to view the inflection points, making it difficult to answer. This may have led many students to study the first and the second derivatives.

In an overall analysis of the two student activities, one may observe the students' difficulty in realizing that the current "didactic contract" had been broken: in the interpretation of a graph, approximate answers are expected, however, there was an insistence on presenting exact results, resorting to calculations based on the algebraic expression of the function.

According to Brousseau, the "didactic contract" is a set of behaviors that each of the participants of a teaching/learning relationship expects from the other in terms of mathematical knowledge. One "clause" of the Contract, very engrained in students' minds, is that every mathematical problem has only one solution, known by the teacher beforehand and, to discover it, the student must find, in the details of the problem, the best means of attaining the solution.

It is importance to debunk this notion so the students may identify the invariants suggested by Vergnaud in is his Theory of the Conceptual Fields:

"How does one foster this change in perception?"

Firstly, by exploring graphs of functions before theoretical considerations; secondly, by making deep changes in the structure of the activities. With the first change, we encouraged the breaking of another "clause" of the didactic contract, that is, the students were only allowed to answer the questions after the theory had been explained by the teacher. The second change was a result of the first and was also due to the large number of items involved. It was decided that they should be explored separately, and in the following year, this resulted in the preparation of five new activities to be applied instead of the two previous ones.

These activities contain leading and open-ended questions in an attempt to provide an opportunity for students to work autonomously. In addition, at the end of each one, the theoretical result concerning the concepts studied is included.

The objective of the first is to relate the sign of the first derivative to the increase of the function. Two functions were used, a polynomial and a rational function. For the graph of the first, the students were asked to: choose three points to draw a tangent line - one of them with a positive angular coefficient, another with a negative angular coefficient and a third one with an angular coefficient value of zero -, identify and algebraically calculate the points at which f '(x) =

0, identify the intervals where f '(x) > 0 (f '(x) < 0), and relate them to the increase (decrease) of the function.

Activities 1

I.

1. Use the graph sketcher to obtain the graph of the function $f(x) = x^4 - 2x^2$.
2. Draw a tangent line to the graph of f with a positive angular coefficient. Give the coordinates of the point of tangency. What is the angular coefficient of this line?
3. Draw a tangent line to the graph of f with a negative angular coefficient. Give the coordinates of the point of tangency. What is the angular coefficient of this line?
4. Draw a tangent line to the graph of f with an angular coefficient value of zero. Give the coordinates of the point of tangency. What is the value of the derivative of function f at the abscissa of this point?
5. At abscissa point x = 1/2, is the angular coefficient value of the tangent line positive, negative or zero?
6. Looking at the graph, identify all of the x values for which $f'(x) = 0$.
7. Calculate these values algebraically.
8. Looking at the graph, identify the intervals for which $f'(x) > 0$.
9. In these intervals, is function f increasing or decreasing?
10. Looking at the graph, identify the intervals for which $f'(x) < 0$.
11. In these intervals, is function f increasing or decreasing?

For the graph in the second activity, students were asked to establish this relationship, though without drawing the tangent lines.

II.

1. Use the graph sketcher to obtain the graph of the function $f(x) = (2x^2 - x + 1)/x^2$.
2. Looking at the graph, identify the intervals in which the derivative has a positive sign.
3. In these intervals, is function f increasing or decreasing?
4. Looking at the graph, identify the intervals in which the derivative has a negative sign.
5. In these intervals, is function f increasing or decreasing?
6. Identify the abscissa points x where $f'(x) = 0$ occurs.
7. Calculate algebraically the x values in which the derivative is zero.
8. In these two examples, what relationship do you see between the increase of a function and the sign of its derivative?

The objective of the second activity is to identify the maximum points through the increasing/decreasing of the function. Three functions were chosen: a rational function with R domain and an absolute maximum, a polynomial function with a relative maximum and a modular function with a relative maximum at points in which the derivative does not exist. Students were initially asked to identify the highest value of the first function and then study its increase/decrease close to this point. For the second function, students had to invert the process, i.e., choose an interval in which the function showed an increase followed by a decrease, then find its relative maximum point. The procedure for the last function was identical to that of the second, except that there was no derivative at the maximum point. For the three functions, students were asked to find the value of the derivative at the maximum point. After studying these functions, students had to describe and test an algebraic method to determine the relative maximum points of a function (if there were any). At the end of the activity, four statements were presented to the students, who had to decide whether they were true or false. In the final discussion, the content of the activity (minimum point) was also institutionalized.

The activity is set out below.

I. 1. Sketch the graph of the function $f(x) = x/(1+x^2)$. D = Im =
2. What is the greatest value of f ?
3. For which value of x does it occur?
4. What is the value of f′ for this x ?
5. Study the increase/decrease of f close to this x value.
6. What is the sign of f′ close to this point?

II. 1. Sketch the graph of the function $f(x) = x^3 - 3x^2 + 1$. D = Im =
2. Choose an interval in which the function displays an increase followed by a decrease.
3. What is the greatest value of f in this interval?
4. For which value of x does it occur?
5. What is the value of f′ in this x?
6. What is the sign of f′ in the interval chosen?
7. Is the number found in question 3 the highest value of the function?

III. 1. Sketch the graph of the function $f(x) = \left||x-2|-3\right|$. D = Im =
2. Choose an interval in which the function shows an increase followed by a decrease.
3. What is the greatest value of f in this interval?
4. For which value of x does this occur?
5. What is the value of f ′ in this x?
6. What is the sign of f ′ in the chosen interval?
7. Is the number found in question 3 the highest value of the function?

III. In each of the examples studied, you identified a f(x) number which was the highest value of f close to x.

This f(x) is called the *local (or relative) maximum of f*; and the corresponding x *local (or relative) maximum point of f*.

Algebraically, how would you determine the local maximum points of a function f (if there are any)?

V. Apply this procedure to the function $f(x) = 4x^3 + 15x^2 + 12x + 5$.

VI. Decide whether the following statements are true or false and justify your answers.
1. If c is a local maximum point, then $f'(c) = 0$.
2. If $f'(c) = 0$, then c is a local maximum point.
3. If f is increasing to the left of c and decreasing to the right side of c, then f(c) is a local maximum of f.
4. If $f'(c)$ does not exist, then c may be a point of local maximum of f.

The third activity presents two polynomial functions. The students were asked to relate the concavity of the graph of the function to the sign of its second derivative. To do so, they had to

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apply the results obtained in the previous activities regarding the study of a function to the derivative function.

For institutionalization purposes, at the end of the activity, the following theoretical result was presented:

If f is the derivable function up to the second order in]a, b[; then:

a) if f" (x) > 0 for every x in]a, b[, then the graph of f is concave upwards in]a, b[;

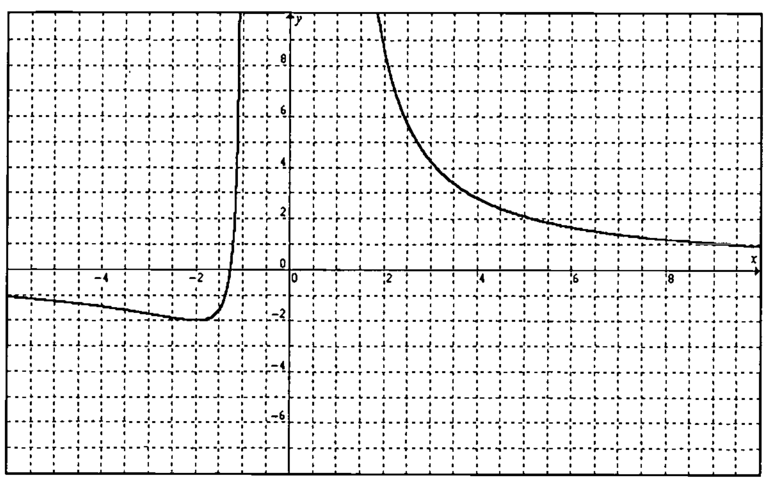
b) if f" (x) < 0 for every x in]a, b[, then the graph of f is concave downwards in]a, b[.

Observation: the point at which the graph of a function changes its concavity is called the *inflection point*.

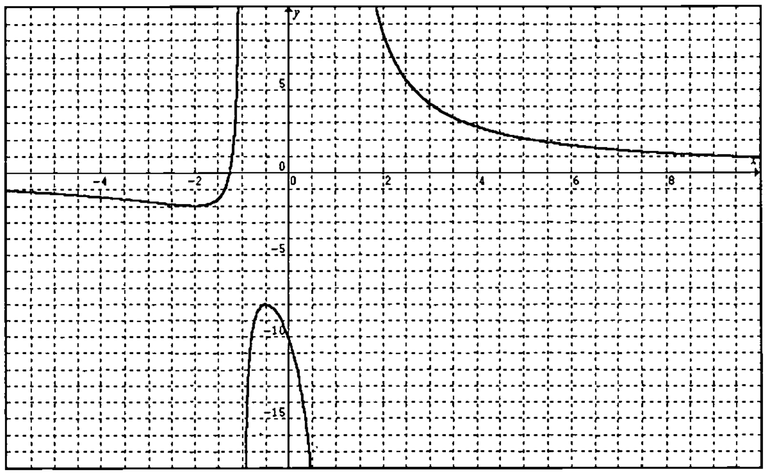
The fourth activity was a reformulation of the second activity from the previous year. The last function, whose graph did not meet the requirements of the study, was eliminated and some items of the other functions were removed and instructions rewritten.

In the fifth activity, the function $f(x) = (8x + 10)/(x^2 - 1)$ was chosen so that one branch of its graph would not appear on the screen (unless certain changes were made to the axes scale, which was not requested) in order to observe whether the students were able to critically analyze the answers obtained. The first intention was for them to use this screen to identify the characteristics of the function and its graph. They were then asked to recognize the same characteristics, however, using an algebraic expression. Finally, they were asked to compare the results obtained in both screens: graph and algebraic. These formulations perplexed the students who observed a clear contradiction between the computer results and the theoretical results obtained via the algebraic calculations.

Below is the graph presented on the computer without any alteration of scale.



After some alterations of scale, we obtained the following graph.



Some students exchanged the "dubious" results for the "correct" results. The dubious results were considered less legitimate. When it is the teacher's word against the computer, the teacher is correct (the effect of the didactic contract); but when it is the student against the computer....

The second function of this activity was chosen so that the students, using the algebraic expression (with the computer turned off), could answer questions regarding its behavior. The students were only allowed to turn on the computer after this study, to see the answers.

The student protocols provided a wealth of information regarding both aspects of the didactic contract and the limitations of the software used.

As regards the didactic contract, the following categories were identified:

- The students who did the algebraic calculations from the outset, showing that they had not noticed the breaking of the contract.
- The students who initially worked with the graphs, as requested, but who, after the algebraic calculations returned to the previously given answers and "corrected" them. This attitude shows that the clause of the didactic contract, according to which the answers obtained through algebra are the true ones, is the strongest.
- The students who worked with the graphs and algebra, who noticed that the answers were conflicting but were not surprised. In this case, they felt satisfied for having fulfilled their part of the contract by answering the questions proposed by the teacher, regardless of the mathematical knowledge involved (for example the fact that the function had two different domains).
- The students who worked with the graphs and algebra, who noticed that the answers were conflicting and tried to discover the reason for this conflict, seeking mechanisms which would allow them to view all of the "branches" of the graph. This last category includes the students who noticed the teacher's breaking of the contract.

Many students found ways of overcoming the limitations of the software used either by changing the scale or the "zoom" command. The opportunity was taken to reinforce the discussion with the students about the advantages, limitations and "dangers" of using a computer tools in the teaching-learning process.

Conclusion

When introducing the subject by exposing students to theory, the teacher may intend to "transmit knowledge." Our group believes that knowledge cannot be transmitted, rather, it is construed by the student. Teachers contribute to this process when they create learning environments which allow students to become more active, critical, independent of teachers, and more inquisitive.

The computer laboratory proved to be an excellent environment for the development of these characteristics in students and the activities prepared for analyzing the behavior of functions proposed a breaking of the conventions of the didactic contract (open-ended questions, conjectures, ...), which, in conjunction with constant renegotiation, prompted a change in attitude. For example, some students used the graph of the derivative function to obtain information about the increases, decreases and end points of the primitive function, despite the fact that the focus suggested for this was the tangent line, showing an independent attitude.

The application of these activities within the proposed dynamic allowed the concepts necessary for analyzing the behavior of functions to be construed. Consequently, producing a sketch of the graph of a function was no longer a "magical" feat performed by the teacher.

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AN ENGINEERING BRIDGING COURSE - SUCCESS OR FAILURE ?

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KEYWORDS: Approximate, Bolzano Theorem, Bridging, Educationally disadvantaged, estimate, guess, Interval Bisection, Piece-wise defined functions

ABSTRACT

The aim of the study was to evaluate the success of a one-year undergraduate bridging course in Engineering (PBS) offered to educationally disadvantaged students, with special emphasis on the role of mathematics in addressing and overcoming some of the problems encountered by Engineering students. These problems include the inability of relating classroom examples to the real world, and the impotence of students of making approximations and estimates in the absence of calculators.

The study briefly describes the aims and structure of the course which has a two fold purpose: to teach students how to think, how to use common sense, to guess, estimate and approximate, and how to translate real-life problems into mathematics, and to provide the framework and basics of the first year Engineering mathematics course. Examples of first year university mathematics problems that teach students how to think clearly and creatively are cited

The mathematical performance of this group was compared with the performance of a large control group of Engineering students in each of the three years of study. This was replicated for four groups of PBS students A statistical comparison of the groups show that there is a significance difference in the average mark and the pass rate at first year level. In the second and third year, there are no significant differences in the groups, implying that the advantage of the bridging class has been carried through to further other years of study.

Details of the subsequent career of a number of highly successful graduates have been included as well as comments and reflections of some of the students who have graduated since the inception of this program in 1986 .Universally they consider this was the best year of their lives where they learned many skills which were never taught again in their studies.

Introduction

This is an empirical study on teaching mathematics to educationally disadvantaged students who are selected for a one year undergraduate bridging course The course was designed to prepare these students for a career in Engineering. From a theoretical point of view, it addresses the issue that all students, particularly at university level, should be taught how to guess, to estimate, approximate and how to construct mathematical models of the real world. The study aims to evaluate the success in mathematics of these bridging students, by statistically comparing their results during their undergraduate years with the results of those engineering students in the same class who were not privileged to be exposed to the special attention the bridging students received in their pre-university year. The initial hypothesis is that exposure to thinking and problem solving skills offered in this one year bridging course impacts on future mathematics success.

Background

The University of the Witwatersrand Pre University Bursary Scheme (PBS) is a one year course offered to students who wish to pursue a degree in engineering. The programme was initiated in 1986, and since its inception 32 companies have sponsored more than 700 PBS students. More than 60 % of these have graduated with a degree in Engineering. In exchange for five years of free tuition and residence fees , the students are obliged to work for their sponsoring company after they graduate.

Students are offered lectures and tutorials in Mechanics, Graphics, Chemistry, Physics, Mathematics and Computing. There is also a course in Communication, where students are tutored in written and spoken skills An important facet of the programme is the personal development of the students. Students are given a Toastmasters course and have public speaking nights for the first part of the year. In addition there are life and study skills workshops, afternoon field trips to factories, laboratories and mines. Students are also required to spend three weeks per year on site work with their sponsoring companies where they are exposed to the engineering environment. Due to space logistics, the maximum number of students that can be accommodated is 60, but the numbers fluctuate dependent on the number of students selected by various large South African corporations such as Eskom, Sasol, Anglo Gold, Anglo Platinum, etc.

I have been lecturing the mathematics component to these students since 1996. Thus the contents have been more or less stable since that date. The slight variation in content each year is dependent on time constraints and the calibre of the students. Since class participation is encouraged, the interaction may initiate extra topics.

Records have been kept of their progress throughout their undergraduate years , and if possible after graduation. The statistical analysis of their results in mathematics has been done from 1997 (1996 PBS group) to the end of 2001.

Mathematics

The syllabus was not only designed to address some of the academic problems that students experience in mathematics but also to teach students to relate classroom examples to real life problems, to approximate and estimate in the absence of computers, to use common sense and previous experience to solve new problems.

Mathematics can be used to develop certain skills such as how to think, by translating real life problems into mathematics . This can be done very successfully by studying functions and their applications. In the first week of lectures, students are exposed to with word problems based on piece-wise defined functions.

Here is an example:

The monthly water tariffs of the Durban Metro for household use are calculated as follows:

There is no charge for the first 6 kilo litres; for the next 24 kl, the charge is R2.53 per kl. If a household exceeds 30 kl in a month, each additional kilo litre is charged at R5.06. Express this information as a mathematical function, and sketch the graph.

An area of mathematics which is glossed over or completely neglected at first year level is the Bolzano Theorem and the Interval Bisection Method, which unlike Newton's Approximation do not require calculus. At a later stage these two methods are compared. The students are often intimidated by the choice of an initial estimate of a root, and also by knowing how many iterations to perform. This is a very good exercise in estimation and approximation.

Linear Interpolation is no longer needed at school since calculators have replaced tables, but this is a simple technique which can be included, at any level, in the study of straight lines The process can be easily explained graphically, and trains the mind to explore different ways of tackling problems.

Another technique I use to encourage mathematical insight is "guesstimation", i.e guessing the answer before starting a problem. There are many situations where this can be successfully applied, sometimes with a great deal of fun.. Students find guessing the answer extremely intimidating, and they take some time to appreciate the value this has in promoting precision and accuracy of calculation. I think this is an important skill that has been lost over the years due to the ready availability of calculators. But this is especially pertinent for calculations using calculators. Guessing beforehand, even if it is a ball park figure, trains a student in the importance of accuracy and precision. The recognition and identification of errors when they occur is an important skill for everyone, but especially for engineers

A universal problem area for most students is the introduction of radians. Being able to anticipate the answer to a trigonometric calculation on a calculator guides them in knowing when they are in the incorrect mode. They need a yardstick (such as $\sin 30^{0} = 0.5$) with which to compare the value they obtained

An oft neglected area is checking results when the problem is solved. I ask students to examine the solution to a problem and decide if it is feasible A simple example is to sketch a polynomial curve and compare its shape with the sign of the leading coefficient.. It is very easy to check solutions of simultaneous equations and also trigonometric equations by substitution.

Syllabus

The framework of first year mathematics is presented by exposing the students to radians, to new trigonometric identities and their use in solving trigonometric equations. The students are drilled in basic differentiation and integration skills and if time permits, we study the Binomial Theorem and Pascal's Triangle and some matrix algebra with special reference to applications. The spade work for the first year mathematics course is done so that the students in the following year can concentrate on the harder applications. The purpose is not to complete the whole first year course; there must still be enough in the following year to challenge and excite them mathematically. Hopefully this exposure to thinking and problem solving skills impacts on their future mathematics education

Statistical Analysis

I started teaching mathematics to the PBS students in July, 1996, so this study is based only on those students with whom I have had personal contact. The engineering degree extends over 4 years and a mathematics is studied for the first three years only..

The study was based on the following groups of students:

PBS year	1st year Maths MATH 180	2nd year Maths MATH 280	3rd year Maths MATH 380	Graduation
1996	1997	1998	1999	2000
1997	1998	1999	2000	2001
1998	1999	2000	2001	
1999	2000	2001		
2000	2001			

In each year, a comparison is made between the mathematics marks of the PBS students and the marks of the whole group (WG) excluding the PBS students. Two-sample z-tests are used to determine whether there are any significant differences in the average final mark of the two groups, and two-sample tests on proportions are used to investigate whether there are any differences in the pass rate of the two groups.

The Results

From the tables (Appendix I) and the bar charts (Appendix 2) , the following conclusions can be reached.:

At first year level, the mean mark and the pass rate in all years was higher for the PBS students then for the whole group

- ☐ the differences in pass rates are significant in every year at the 5 % level
- ☐ the differences in mean marks are significant at the 1 % level in 1998 and 2001, and significant at the 5 % level in 1997 and 1999.
- ☐ there is no significant difference in the groups in 2000.

At second year level, the pass rate for PBS was higher in 1999 and 2000, and the mean mark was higher in 1998

- ☐ there is no significant difference between PBS and WG for either mean mark or pass rate

At third year level, the PBS pass rate and mean mark was higher in 1999 and 2001

- ☐ the difference in mean mark is significantly different at the 5 % level in 2000
- ☐ there is no significant difference in the mean mark nor in the pass rate in the other years

Conclusions

As would be expected, the mathematics marks of PBS students are better than the marks of the other students in their first year. PBS students have a distinct advantage as much of the work is familiar to them. The pass rates are also significantly higher.

In the second and third year, the mean mark and pass rate are not significantly different. From this we can conclude that although these students were academically disadvantaged when started the PBS course, by the second year there is no difference between the PBS students group and the mainstream students. The pre-university year has enabled these students to overcome their deficiencies and their academic achievement could be favourably compared with the remaining students who had been accepted directly from school based as a result of their matriculation success.

Success stories

Of the 313 PBS alumni who could by now have graduated, 150 have done so in Engineering and 51 have done so in other disciplines, many at other universities. A study at the University has shown that about 60 % of students who have completed the PBS course have graduated in contrast to about 28 % of previously disadvantaged students directly entering the mainstream courses. All the students interviewed have unanimously said that the programme was the best thing that happened to them.

To quote one successful student, who did the PBS course in 1988: "....The programme did a lot more than teach us about engineering.; it taught us how to think, it taught us to be problem solvers and to be creative at the same time.... We didn't just learn facts, we learnt how to learn. "

After graduating this student completed an MBA at the Sloan School of Management in the USA , and is presently employed at Rio Tinto in London, in charge of its world wide coal mining investments.

Four university engineering graduates have established a black empowerment engineering consultancy firm. Three of these students are PBS alumni. Two of them graduated with degrees in mechanical engineering, and the other has a degree in electrical engineering. In addition, two of these three PBS alumni have completed graduate diplomas in industrial engineering and the other is presently studying technology management.

There are many other successful alumni amongst whom are :

a Metallurgist at De Beers, a Senior Inspector for the Department of Mineral & Energy Affairs

the Chief Director at the Department of Mineral and Energy

In addition, a heart-warming success story is that of Jacob Modise, Chief Operating Officer of Johnnic Holdings who was a graduate of the Anglo American Corporation's Cadet scheme, a precursor of the Pre-University Bursary Scheme, which was offered to both Commerce and Engineering students in the early days of the programme. He was orphaned at nine years of age, was one of 10 Matric students selected for the cadet scheme, and qualified as a chartered accountant at 22 years of age. When asked in an interview what was the biggest ever opportunity he was offered in his life, he said it was his selection for the cadet scheme.

Further information about the PBS course will be found at the following websites
http://www.wits.ac.za/pbs/companies.htm
http://www.asec.org/prism/mayjune/html/global.html

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Appendix I

Tests on difference of PBS and Whole Group

PBS Year			$p-value$	Significant ?
1996		Math 180	p = 0.00269<0.01 **	****Sig at 1 % level**
	Proportions	Math 280	p = 0.38	Not Sig. at 5% level
		Math 380	p = 0.567	Not Sig. at 5% level
		Math 180	$\boxed{p = 0.0378 < 0.05}$*	**Sig. at 5% level**
	Means	Math 280	p = 0.17	Not Sig. at 5% level
		Math 380	P = 0.11	Not Sig. at 5% level
1997		Math 180	$\boxed{p = 0169 < 0.05}$*	**Sig. at 5 % level**
	Proportions	Math 280	p = 0.47	Not Sig. at 5% level
		Math 380	p = 0.21	Not Sig. at 5% level
		Math 180	p = 0.0097 < 0.01 **	**Sig. at 1 % level**
	Means	Math 280	p = 0.7698	Not Sig. at 5% level
		Math 380	$\boxed{p = 0.02 < 0.05}$ *	**Sig. at 5 % level**
1998		Math 180	$\boxed{p = 0.0149 < 0.05}$*	**Sig. at 5 % level**
	Proportions	Math 280	p = 0.23	Not Sig. at 5% level
		Math 380	$\boxed{p = 0.1032}$	Not Sig at 5 % level
		Math 180	$\boxed{p = 0.0459 < 0.05}$*	**Sig. at 5% level**
	Means	Math 280	p = 0.66	Not Sig. at 5% level
		Math 380	p = 0.1559	Not Sig at 5 % level
1999		Math 180	$\boxed{p = 0.0332 < 0.05}$*	**Sig. at 5% level**
	Proportions	Math 280	p = 0.8107	Not Sig at 5 % level
		Math 380		
		Math 180	p = 0.08	Not Sig. at 5% level
	Means	Math 280	$\boxed{p = 0.534}$	Not Sig at 5 % level
		Math 380		
2000		Math 180	p = 0.010 < 0.01 **	**Sig. at 1 % level**
	Proportions	Math 280		
		Math 380		
		Math 180	p = 0.0016 < 0.01 **	**Sig. at 1% level**
	Means	Math 280		
		Math 380		

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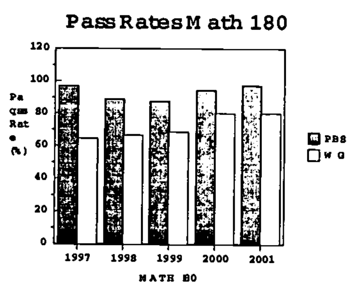
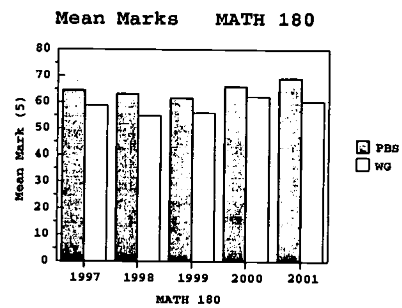
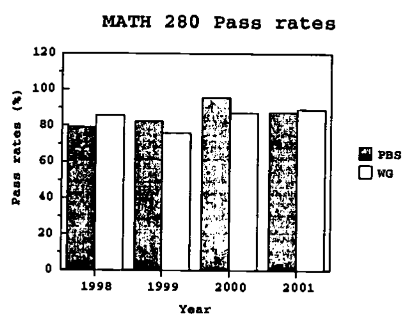
Appendix II

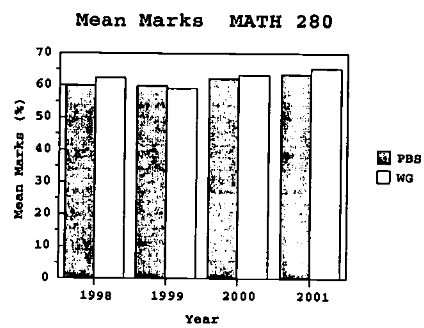
MATH 180 1997 - 2001



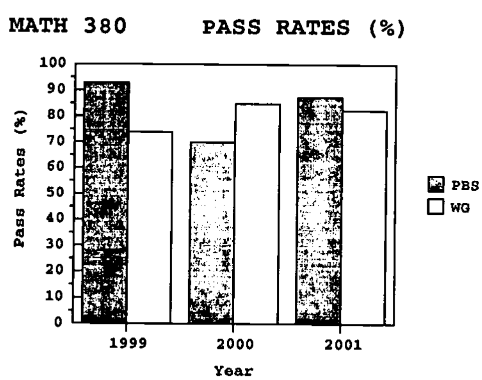


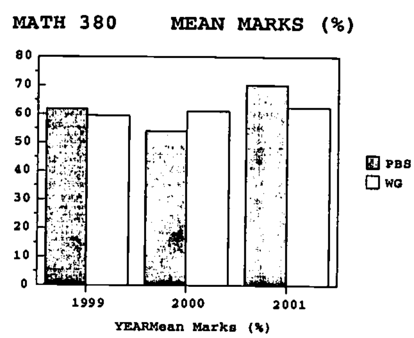
MATH 280 1998 - 2001





MATH 380 1999 - 2001





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PREPARING TEACHERS FOR A NEW CHALLENGE: TEACHING CALCULUS CONCEPTS IN MIDDLE GRADES

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Keywords: visual calculus, middle school mathematics, teacher preparation, conceptual learning, early development of advanced concepts, visualization.

ABSTRACT:

The main focus of this paper is to discuss possibilities of teaching and developing students' conceptual understanding of advanced Calculus principles in middle grades. Authors explore conditions in teacher preparation for the successful teaching of a "Visual Calculus" course (integrated 3-D Geometry and multi-variable Calculus concepts of differentiation, integration and optimization) in a middle school with a culturally diverse student population. Basic assumption is that conceptual learning leads procedural development (L. Vygotsky, V. Davydov, R. Skemp, etc.). The main distinction of the "Visual Calculus" course is its orientation toward method of ascending from general to specific, multiple connections with science and technology, as well as multiple representations with focus on the power of cognitive visualization in the development of students' conceptual understanding of advanced Calculus ideas. Final research destination of the project is the measurement of an impact that early conceptual development of students' advanced mathematics principles has on students' progress in Calculus at the high school and college level. Current stage of the project is focused on the middle grades teachers' perception of early development of Calculus concepts, relationship between teachers' content and pedagogy knowledge as well as their readiness and confidence to teach Calculus concepts in middle grades.

Key Assumptions

The project is based on the following key assumptions about learning and teaching:

- Conceptual learning leads development of cognitive acquisition of formal procedural operations. Vygotsky claims that development of advanced concepts might start much earlier and it depends on learning, on how you can create a successful learning environment to develop this concepts.
- The development of students' procedural Calculus skills is a derivative of students' conceptual understanding of big Calculus ideas and principles. Thus, development of students' conceptual understanding of Calculus principles should start earlier in the middle school and should be achieved by ascending from general to specific, from big idea to specific procedure (ascending from multivariable Calculus concepts to single-variable principles). We also believe that cognitive-visual conceptualization through the use of modeling and technology will play a powerful role in early learning of Calculus principles.
- Importance of learning through teaching approach in teacher preparation: students learn what they have to teach. Traditional teacher education programs and student teaching experiences do not provide enough time for pre-service teachers to teach mathematics in actual classrooms. This limited experience in mathematics teaching reinforces the low confidence level of most teacher education students in their ability to understand and teach these subjects. Calculus is a subject that few teachers have studied and the very word evokes massive mathematics anxiety from most teacher education candidates. The field-based experience based on learning through teaching approach provides university students an opportunity for immediate application of their knowledge and skills in actual classroom settings in a real public school environment with feedback from university teams and public school teachers. At the same time, the team teaching of mathematics content, methods, and pedagogy classes between faculty in Colleges of Education and Science helps pre-service teachers to integrate Calculus concepts with its active application to teaching and learning.

Current Research in Calculus Teaching and Learning

Last two decades Calculus is at the forefront of research and curriculum reforms in mathematics education. Majority of research in Calculus learning have been done at the level of undergraduate education and some at the high school level. Researchers observed that students enter calculus courses with a primitive understanding of concepts of function, change, continuity, etc. (Tall, D., 1996, Ferrini-Mundy, J., & Lauten, D., 1993). They also noted that students have cognitive difficulties in coordinating function concept in algebraic and graphical representations which is critical in constructing a foundation for fundamental calculus ideas (Schnepp, M., & Nemirovsky, R., 2001). Other research concentrates on different approaches to teaching calculus principles: comparison study on technique-oriented approach vs. conceptual and infinitesimal approaches of learning calculus shows that different approaches have different impact on students' language use and sources of conviction (Frid, S., 1994).

Researches also determined that cognitive obstacles to the learning of calculus arise in at least two different ways – one related to linguistic/representational aspects and the other related to intuitions. Given that so many of our algebra and calculus courses are immersed in symbolic manipulation, often at the expense of understanding, it is not surprising that linguistic/ representational factors give rise to cognitive obstacles. Also, since learners basically want to understand and make sense of what they are being asked to learn, the intuitions that students bring to bear on the concept of calculus often play a crucial role in the appropriate construction of those concepts. Researches "propose that a potentially useful framework in which to embed considerations of cognitive obstacles lies in the framework of Krutetskian cognitive processes of reversibility, flexibility, and generalization" (Norman, A., & Prichard, M., 1994, p. 76).

There is an emerging importance of making connections between different representations (concrete, visual-spatial, numeric, graphical, algebraic, etc.) in helping students' to learn calculus concepts. One of the guiding principles of Harvard Consortium Calculus text is the "Rule of Three", "which says that wherever possible topics should be taught graphically and numerically, as well as analytically. The aim is to produce a course where the three points of view are balanced, and where students see each major idea from several angles" (Hughes-Hallett, D., 1990, p. 121).

One of the most significant points which come from the analyses of research in Calculus learning is that there should be more emphasis placed on conceptual learning using multiple representations and connections before students immerse into symbolic manipulations. In order to build a rich conceptual foundation for successful learning of Calculus at the high school and college level there should be a lot of preparatory work done at the early years of schooling. "Calculus needs to be studied across many years of school, from early grades onward, much as a subject like geometry should be studied" (Kaput, J., 1994, p. 132).

"Visual Calculus" Course Content Design

In contrast to previous remarkable attempts in early development of advanced Calculus concepts (e.g., SimCalc project, CoVis (Scientific Visualization) project) which basically considered development of single-variable Calculus concepts, we start teaching Calculus principles from general multi-variable to single-variable concepts: from generic 3-D surface to arbitrary 2-D curves and then to specific elementary curves (linear, quadratic, exponential, etc.), from tangent plane to tangent line (including concept of gradient), from general infinitesimal methods to procedural calculations of derivative and integral, etc.

V. Davydov (1990) first examined the effectiveness of the method of ascending from general to concrete by teaching algebra concepts to elementary school students in early 1970's in Russia. We consider an application of pedagogy ascending from big, general idea to specific procedure as a methodological tool for designing a middle school "Visual Calculus" intervention course and a supplementary teacher education course. The main purposes of this course are development of middle school students' conceptual understanding of Calculus principles and a supplemental Calculus module for teacher education students.

In teaching multi-variable Calculus concepts we use one of the advantages of local Greater El Paso landscape – mountains (a natural model of generic arbitrary 3-D surface). In parallel with this we introduce basic 3-D Geometry concepts (3-D coordinate system, projections of 3-D objects, sections of arbitrary 3-D surface, etc.) to middle school students. We consider multi-variable Calculus as mathematically natural way to introduce 3-D Geometry concepts.

One possible extension of the project is a development of an inquiry-based "Mathematics of a Mountain" initiative for elementary school students. Hiking on the local Franklin mountains, El Paso, TX or skiing on the mountains of Ruidoso, NM will help students to understand the meaning behind the general 3-D concepts of slope (steepness of a mountain), tangent plane, tangent line, gradient (vector of maximum steepness), points of relative maximum and minimum, saddle points, etc. We consider field trips to mountains as a part of developing students' learning experiences in understanding basic multi-variable Calculus principles. Afterwards students visualize multi-variable Calculus concepts using 3-D arbitrary mountain models (made from play-dough or other materials), constructing contour diagrams, cross-sections, etc.

We provide a thorough visual hands-on introduction to three-dimensional geometry including two-dimensional surfaces in three dimensions. Students go back and forth between the three-dimensional models and surfaces and the two-dimensional representations. Planes, and their slopes, are studied as a special case, and as a transition to studying directional derivatives and gradients. We also introduce multivariable integration by finding volumes of actual three-dimensional objects, by repeated slicing. We model instruction of concepts for the middle grades teachers as well as teacher education students and they in turn teach all strategies in the actual classrooms.

Research Methodology and Professional Development Activities

Research is taking place in conjunction with ongoing NSF funded PETE (Partnership for Excellence in Teacher Education) program at UTEP, with its emphasis on field-based intervention for improvement of pre-service math and science teachers preparation. Clinical quasi-experimental design is focused on the relationship of pre-service teachers' content and method knowledge in math and upper elementary and/or middle school students' achievement in "Visual Calculus" and regular mathematics classes.

During the summer and fall-2001 we were piloting "Visual Calculus: Early development of students' advanced mathematics concepts" experimental class at UTEP in the form of professional development seminar/workshop. In summer we had 15 pre-and-in-service upper elementary and secondary teachers involved into the workshop. Some of the teachers have taken Calculus courses (up to Calculus-III), and some of them – have no Calculus experience at all. We have formed heterogeneous groups in order to involve them into discussions at multi-level Calculus learning experiences and help them to understand basic multivariable Calculus concepts. Each group worked on particular concept of Multivariable and Single variable Calculus: differentiation, optimization, and integration. In this workshop we use Harvard Consortium Calculus text (Hughes-Hallett, D., Gleason, A., et al.) and supplementary materials. During the summer session each group came up with a set of conceptual tools (activities, hands-on manipulations with physical models, technology based illustrations, etc.) which from UTEP students' perspective would be appropriate to teach to the middle school students.

The distinctive feature of the fall-2001 session of the workshop is that it reflects multi-tiered teaching experiment design (Lesh, R. & Kelly, A., 2000). We have a group of 3-4 researchers (with background and expertise in mathematics, mathematics education, cognition, engineering), the group of 15 pre- and in-service teachers with different level of Calculus experiences, and a group of 4-5 multiage students (from upper elementary, middle and junior high schools without any experience in learning Calculus) in one classroom during each seminar sessions (table 1). Each group of teachers have a chance to teach the activities, developed in summer session, to the multiage group of students with main emphasis on conceptual understanding of particular Calculus principle ascending from general multivariable idea to specific single variable case. After the teaching episode researchers, teachers, and students participate in discussion on how the teaching impacted the students' understanding of the concept.

Table 1. Multi-tiered teaching experiment

Tier 3. The Researcher Level	Researchers develop Visual Calculus conceptual model to make sense of pre-and in-service teachers' and middle school students Calculus learning activities. Researchers reveal their interpretations as they create conceptual tools and learning situations for teachers and students, and also as they describe and predict teachers' and students' behavior in increasingly complex mathematics teaching and learning environment.
Tier 2. The Teacher Level	Teachers through the study, summer workshops, and professional development seminars learn and design shared conceptual tools (activities, hands-on manipulations with physical models, technology based illustrations, assessment instruments, etc.) in order to help middle school students to develop advanced Calculus concepts. As teachers describe and predict students' behaviors, they construct and refine models to make sense of students' learning activities.
Tier 1. The Student Level	Students work on a series of conceptual model-eliciting activities/projects in which the major goals include further refining of models that reveal how students are interpreting and learning advanced Calculus concepts.

We plan to start first pilot experiment of teaching "Visual Calculus" supplementary course at middle school in fall 2002. The university content, method, and pedagogy classes for 4-8 concentration pre-service mathematics teachers will be team taught in a local middle school. The university students will participate in visual calculus projects and then be responsible for teaching the same projects to the middle school students. Lesson study method and video analysis are going to be major tools for qualitative assessment of teaching behaviors. It provides formative evaluative feedback, which guides teachers in their conceptualization of effective teaching practices. A sample of teachers participating in the project will be videotaped throughout their progress in teaching "Visual Calculus" course. In addition, paired problem solving interviews will supplement the documentation and assessment of the teacher's understanding of content, methodology, and pedagogy (different patterns of interaction).

Middle Grades Teachers' Perceptions of Learning and Teaching Calculus

Preliminary outcomes of the pilot multi-tiered teaching experiment show positive changes in teachers' perception of the early development of advanced math concepts as well as their readiness to teach Calculus concepts in middle grades. After completing summer and fall 2001 "Visual Calculus" professional development seminar, we asked teachers to evaluate each given statement below (table 2) based on the following scale:

1 - "Strongly Disagree",
2 - "Disagree",
3 - "Neither Agree nor Disagree",
4 - "Agree",
5 - "Strongly Agree".

Table 2. Middle grades teachers' perceptions of early development of Calculus concepts

##	Statement	1	2	3	4	5
1.	I had no Calculus experience before the "Visual Calculus" seminar	31%	23%	0	38%	8%
2.	Before the "Visual Calculus" seminar my overall attitude toward learning and teaching of Calculus was negative	15%	31%	23%	23%	8%
3.	Emphasis on procedures helps students to understand advanced Calculus ideas	15%	31%	15%	23%	15%
4.	Visualization is an effective approach in learning Calculus concepts	0	0	0	46%	46%
5.	Learning Multivariable Calculus concepts first helps me to better understand Single variable concepts	0	0	46%	38%	15%
6.	It is possible to develop students' Calculus concepts early in the middle school	0	0	0	54%	46%
7.	Principle "Conceptual leads procedural" underlines the main distinction between traditional and innovative way of teaching and learning Calculus	0	0	0	62%	38%
8.	Graphing skills play an important role in conceptual learning of Calculus	0	0	8%	38%	46%
9.	Discussion and reflection on micro-teaching of Calculus activities help me to understand how kids learn Calculus concepts	0	0	0	46%	46%
10.	Local landscape (mountains) and real life applications are good sources to introduce Calculus concepts to middle school students	0	0	0	38%	62%

11.	My confidence in *learning* of Calculus after the seminar is low	69%	23%	8%	0	0
12.	My confidence in *teaching* of Calculus concepts after the seminar is low	38%	46%	0	15%	0
13.	My overall attitude to learning and teaching of Calculus concepts after the seminar is positive	0	0	0	46%	54%

The main indicator of teachers readiness to teach Calculus concepts to middle school students is the answer to the question #6 (table 2): all the participants of "Visual Calculus" professional development seminar believe that it is possible to develop students' Calculus concepts early in the middle school. Another promising indicator is that if at the beginning of the seminar 31% of participants had a negative attitude toward teaching and learning of Calculus, by the end of the seminar all of them had positive attitude.

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INTEGRATING SYMMETRIES OF POLYHEDRA IN THE CONTEXT OF REAL SPACE EXPERIENCE: STUDENTS' WORK ON A GEOMETRICAL THEME

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Keywords: Thematic Approach, Project Work, Symmetry, Polyhedra, Mathematics in Context, Art and Mathematics.

ABSTRACT

Polyhedra and their Symmetries have appeared since Antiquity in diverse studies of philosophy, art, science and mathematics (especially geometry), but a lack of discussion of this context in some depth is apparent in Mathematics Education. This paper discusses related project work on the theme «Polyhedra and Symmetry» in the University classroom. Students' work is analyzed according to the didactical intentions, the project issues proposed and students' own choices and thinking processes. A case study is also included, by taking into account the particular students' «background» and «foreground», which helps in interpreting the students' choices.

1. Backround

1.1 Geometrical "Order" in Art and Nature; Symmetry of polyhedra.

There is a vast literature on mathematical "order" in Art and Nature, from Antiquity and Renaissance to our epoch. The themes are rich and diverse, but most of them reveal a mystic ideology and a Platonic or Pythagorean conception of the world. Even in the *Bauhaus*, the famous modern art school in the Democratic Republic of Weimar (1919-1932), the mystic and idealistic tendencies were renewed by some of the "Masters of Form" who taught there, as for example by the Swiss Johannes Itten, an Expressionist painter and unconventional art teacher with mystic beliefs, who was an opponent of every materialist interpretation of the world[1]. However, the "Masters of Form" (sculptors and painters) were not the only teachers at the *Bauhaus*. Walter Gropius, the famous architect who founded the School, had conceived a radically new kind of art education, in which the fine arts and the crafts would not be considered as two fundamentally different activities but as two varieties of the same thing. Therefore Gropius also appointed the "Workshop Masters" in order to equip students with manual skills and technical knowledge, while the "Masters of Form" were to stimulate the students' minds and encourage creativity[2].

If Geometry is considered as a theoretical ingredient of some creative activities in Art, Science and Technology, then a lack of a similar vision to that of the *Bauhaus* is apparent in Mathematics Education. Teachers and textbook's writers sometimes generally refer to the virtues of geometrical thought as introducing "order" in natural contexts of experience, but such general verbal descriptions usually fail to encourage any creativity in the students. Instead of discussing concrete and non-trivial examples, which could attract the students' interest and illustrate the power and applicability of geometrical methods in particular contexts, university textbooks usually contain an excessive amount of formal definitions and technical proofs. "Order" is then restricted to Pure Mathematics as an abstract and separate subject closed in itself.

On the other hand few «popular» books, in our opinion, manage to penetrate both (modern) Geometry and Art (or Nature) in such a way, that their deep mutual relationships are made apparent to the reader. One of these books –and by no means an "easy" one – is H. Weyl's *Symmetry*[3]. In this book Hermann Weyl discusses symmetry as an idea, which in his words «was essential through the ages in human efforts to understand beauty and order». After a first chapter devoted to bilateral symmetry, the book introduces the concept of a group of transformations as a key mathematical idea suitable for the study of the general notion of symmetry. More specifically, given a figure F in space (for example: a regular polygon or polyhedron) the one-to-one transformations of the space onto itself which leave F invariant form a group depending on F, $\tilde{A}=\tilde{A}(F)$ and this group describes exactly the "type of symmetry" possessed by F. Groups of rotations and translations are discussed as the most important examples, which are then applied to polygonal or other shapes of 2-dimensional ornaments and to polyhedra and natural crystals in 3 dimensions. Thus, to search for all *symmetries* of a given polyhedron P means to try to determine the group $\tilde{A}(P)$.

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[1] Whitfort, 1991, p. 52
[2] ibid, pp, 47-48
[3] Weyl, 1964, p. 52

1.2 "Themes" and Project Work in the University Classroom.

A "*thematic approach*" to Mathematics Education is described by Skovsmose (1994, pp.59-90). This approach has been adopted since 1994 in Aalborg University, Denmark, where the curriculum is organized in several "themes" (normally covering a semester) in such way that increased knowledge and cognition can be obtained progressively during the educational process. A corresponding innovation in teaching practice is *project work*, which must provide students with special professional skills (F. Kjersdam et all, 1994).

Our approach is a little different. Although we adopt the idea of "theme" as central, we conceive it as an autonomous subject of scientific and didactical discourse rather than a curriculum unit (that is why we shall not deal with the curriculum in general in this paper).

A "theme", in our approach, is an open-ended problem situation in a real context or field of experience[4], or a generic example[5], which becomes the subject of a scientific discussion in the university classroom, starting from historical roots and leading up to the present situation of research. A "theme" as above must be general and rich enough to generate several questions (or aspects), sub-problems and issues (or topics), which may be interesting to students. These questions may become the subject of separate projects, which can be progressively integrated into the central "theme".

2. Analysis of the educational process

We now proceed to analyze project work as an actual educational process in the university classroom. Our *theme* will constantly be «Polyhedra and Symmetry» with two main aspects to be covered, namely (i) *Symmetry in Art and Nature from Antiquity to Modern Times* and (ii) *Symmetries of Polygons and Polyhedra*. Our analysis is based on empirical data (participant observation, interviews and examples selected from the students' work). The whole process of students' work consists of the following general phases:

- Starting from students' own experience
- Stating a problem as an open task
- Working (usually in groups)
- Presenting the results
- Discussion and evaluation

We are going to analyze the project work specifically analysis, according to the aspects proposed, the intentions of the supervisor and the students' own choices and thinking processes. The projects actually carried out in the classroom were titled as follows (we shall refer to them as «Project I», «Project II» and «Project III» respectively):

I. Symmetry in Art and Nature from Antiquity to modern times-with emphasis on regular polyhedra.

II. Symmetry of flat figures and polyhedra.

III. The symmetry group of the cube.

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[4]In the sense of Boero, 1989. See also Patronis, 1997, for a revision of the notion of context in Mathematics Education

[5]As e.g. it is the case with Bernoulli Trials for Probability Theory, or polyhedra for Topological and Combinatorial Topology.

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These titles, which more or less cover the two aspects mentioned above, were proposed to students within an optional course on «Transformational Geometry and its Teaching» which in is offered in the mathematics department of Patras University. Several students were involved in these projects, usually in groups of two (or more).

The intentions of the supervisor of these projects were different from each case to another and accordingly different were also his suggestions to students. Project I was to cover a vast bibliography area, which had a risk of "being lost" within the subjects and the variety of the approach. However, the supervisor proposed this project to students with an actual interest in history of art and its interrelations with the developments of mathematics-especially geometry. He intended to make these students *to search for the relevant literature* and think, moreover, in their own terms[6]. On the other hand, the supervisor mentioned no bibliographic references for Projects II and III and his general suggestion to students was to try to "experience" the symmetry transformations of figures; thus *to search the subject directly by themselves* instead of searching the bibliography. Moreover there is an evident difference between Project II and Project III, namely that the scope of the former is wider and less special than that of the latter.

As a result, the supervisor's didactical choices had a remarkable effect on most students' work and especially on their thinking processes. Students participating in Projects II and III, who used no bibliography and followed the supervisor's suggestions, proceeded inductively in their own investigation, by numbering of cases of figures or of symmetries of one and the same figure. Students carrying out Project II generally remained into a "static" and particular conception of symmetry. They did not discover rotations and they treated (axial or central) symmetry as an internal displacement of points within the figures. On the other hand, a group of four students who together carried out Project III developed a more "dynamic" conception of symmetry in space. This group, as well as another student carrying out Project I, has visualized rotation of polyhedra around axes as a movement in (real 3-dimensional) space. In the case of Project I, however, many books and Internet sites have been used in addition to students' own investigation, but this additional information was not always relevant (see the case study in Section 3 below). For an analytic description of students' thinking aids and processes, the producing of drawings, the style of presentation and the character of knowledge finally obtained see Table1.

3. Students' backround and foreground: a case study

Knowledge of students' «background» and «foreground» is indispensable for a better interpretation of students' choices as related to students' own overall views and plans in their life.

We borrow from Skovsmose (1994) the concepts of (a person's) «background» and «foreground» as a framework of analys is and interpretation of students' intentions and choices. Choosing a project and choosing a particular style of study and exposition as well depends on a set of dispositions of a person. This set can be divided into the person's «background» and «foreground»:

[6] H. Weyl's *Symmetry* (Weyl, 1964) was the only book that was *named* as "important" by the supervisor. Two or three other titles were mentioned as free to the choice of students (among them Ghyka, 1971 and some sources for Platonic Solids).

«A *background* can be interpreted as that socially constructed network of relationships and meanings, which belong to the history of the person (...). But the background is not the only source of intentions. Equally important is the *foreground* of the person. By this expression I refer to the possibilities, which the social situation makes available for the individual to perceive as his or her possibilities. It is not open to me to have the (realistic) intention of being the next president of Mexico. It is not part of my *foreground* and only if I were a madman would I produce intentions of this kind»[7].

Hermes and Orpheus

We shall focus on *Hermes* and *Orpheus* (pseudonyms of the students who together have carried out Project I) as a case study. Hermes and Orpheus were together, at that time, in the last year of mathematics in Patras University. Orpheus was mostly interested in the art aspect of the project, since in parallel to mathematics he studies music. Hermes' background was different: being an expert in the computer and passionate user of the Internet, had never thought of art in relation to mathematics before this project. By «searching» in the web and the department's library as well, Hermes and Orpheus had gathered the following list of books (written at the beginning of Hermes' diary):

(1) «Is God a Geometer?», by M. Golubitsky and Ian Steward.
(2) «Symmetry», by H. Weyl.
(3) Plato's «ÔßìáéïÒ».
(4) Encyclopedia Britannica, 1999.
(5) Encyclopedia Encarta, 1999.

Only book (2) of this list had been suggested by the supervisor of the project. Not all of the books were relevant and both students (especially Hermes) had a difficulty in grasping what was essential to the theme of the project. The students' mathematical background -as Hermes himself admitted when interviewed- was rather poor for a good mathematical understanding. It seems that "popular" literature as in the above list cannot help in this direction, with the exception of Weyl's book, which probably the students did not read very carefully.

However, some mathematical definitions and classifications (as e.g. that of symmetry groups) are found in Hermes' diary, together with aesthetic questions such as the possibility of determining "beauty" by means of (mathematical) symmetry. Does a beautiful thing need to be symmetrical? Conversely, is any symmetrical thing necessarily beautiful? Hermes was a beginner in both Transformational Geometry and the History of Art, therefore his approach of these matters was rather absolute and naï ve, but he finally tended to answer both above questions in the negative.

In any case, philosophical questions concerning Platonism and the "way of existence" of mathematical structures in Art and Nature have not been discussed in the project, and this is largely due to a lack of historical and philosophical background in both students. Although their final essay has superficially an "historical" structure, Hermes' and Orpheus' views of their theme have not yet been critically developed. Hermes only seems to have gained some knowledge of regular polyhedra and their symmetries, but this knowledge is mainly integrated into a *visual-empirical* context and not into a *critical-historical* or a *theoretical-mathematical* one.

Will Hermes and Orpheus "meet" this theme in their life again? Such a possibility seems now distant to both of them, which explains why there was no continuation in their enquiry. Today, a year after doing this project, Hermes is not so enthusiastic as at the beginning of the project work. Hermes

[7] Skovsmose, 1994, p. 179. Our emphasis.

is a graduate student in the department but without a specific direction (hesitating between Computer Science and Mathematics Education), while Orpheus has postponed his graduate studies until he finishes his service in the Army. At least Orpheus has (until now) kept his enthusiasm for the art -and-mathematics issues of the theme. He now studies the same theme in relation to modern physics and cosmology, which (in his words) «introduces Pythagorean harmony to the universe once again».

4. Concluding Remarks

The character (and status) of knowledge finally obtained by the students seems to depend on many more factors than simply the intentions and suggestions of the supervisor. Project work clearly differs from usual teaching-learning processes. It is not easily adopted and understood by students, who need a continuous care and encouragement. Especially within such a traditional institution as usually a Greek mathematics department looks like, the status of project-generated knowledge seems to be low compared to technical mathematical knowledge obtained by usual rote learning. The knowledge obtained in the process of project work generally seems "unfinished", incomplete and insecure to students' own eyes. A girl described her experience with Project II as «exploring something unclear, uncertain, perhaps varying, but anyway unknown». Yet, in the words of the same student, «it is the object itself that, without doubt, challenges and fascinates». It is through this meaningful experience that the students gradually gain real self-confidence in their learning of (modern) mathematics.

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TABLE 1

	"MATHEMATICAL" ASPECT		"HISTORICAL" ASPECT
	"Dynamic" conception Projects III+I	"Static" conception Project II	Project I
Use of Bibliography	No use of bibliography	No use of bibliography	Many book titles and Internet sites
Thinking Aids and Processes	Using Visualization of "movement" of figures in space; empirical verifications and (some) calculations.	Making "direct" conjectures from the (static) drawings with empirical verifications and explanations.	Using visualization; also metaphors and comparisons of various examples from art and life.
Style of Presentation	"Inductive" (numbering of cases) and semi-formal	"Inductive" (numbering of cases) - not formalized at all	Narrative
Geometrical Drawings	By hand or computer graphics	By hand or photocopy	Use of computer graphics
Character of Knowledge obtained	Empirical towards an abstraction	Empirical but remaining particular; trying "direct" generalizations	Contextual

USING MATLAB TO SOLVE A CLASSIFICATION PROBLEM IN FINITE RINGS

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ABSTRACT

Although most linear algebra problems can be solved using a number of software packages, in our judgment MATLAB (MATrix LABoratory) is the most suitable package. MATLAB is a versatile and powerful, yet user - friendly software package designed to handle wide - ranging problems involving matrix computations and linear algebra concepts. MATLAB incorporates professionally developed quality computer routines for linear algebra computations.

In this paper, we make use of elements from MATLAB to devise a programme that helps in determining the structure and classification, up to isomorphism, of a naturally arising class of finite associative local rings.

We demonstrate this in the case where the finite local ring has a finite residue field K of characteristic p, although our results apply in fact over any field K.

1 Introduction

The use of computer technology has been widely discussed as having the potential to radically change higher education.

The ways in which information is put to work today seem almost countless, and computers are continually assuming a larger role in preparing this information suitably for the needs of teachers and students. Problem solving is an important example.

For computers to play a part in problem solving, it is necessary that communication be established between them and their users. In presenting a problem for a solution, the user still has a substantial role to play. It is still not possible to address the machine in one's own language, but it is fairly easy to learn and use programming language that resembles it and the effort to improve that resemblance is relentless. Computers are gradually being taught to accept more and more of the communications burden.

MATLAB (MATrix LABoratory) is a high-performance interactive software package for scientific and engineering numeric computations. MATLAB integrates numerical analysis, matrix computation, signal processing, and graphics in an easy-to-use environment where problems and solutions are expressed just as they are written mathematically - *without traditional programming.*

In this paper, we make use of elements from MATLAB to devise a programme that helps in determining the structure and classification, up to isomorphism, of a naturally arising class of finite associative local rings. In particular, we consider rings of the form

$$R = K \oplus J$$

in which $K = F_q$, a finite field of $q = p^r$ elements, (with p a prime, r a positive integer) and the Jacobson radical J is such that $J^3 = (0)$ and $J^2 \neq (0)$.

2 A Problem in Finite Rings

In investigating the structure of finite associative local rings, one is led to consider such a ring of the form

$$R = K \oplus J$$

in which $K = F_q$, a finite field of $q = p^r$ elements, and the Jacobson radical J is such that $J^3 = (0)$ and J/J^2 is two-dimensional and J^2 is three-dimensional over $R/J = K$.

Rings with $J^3 = (0)$ and $J^2 \neq (0)$ form an object of study (e.g. Chikunji 1999), the case $J^2 = (0)$ having long been settled (e.g. Corbas 1969, 1970).

If

$$J = Kx_1 \oplus Kx_2 \oplus J^2$$

and

$$J^2 = Ky_1 \oplus Ky_2 \oplus Ky_3,$$

then we may write

$$x_i x_j = \alpha_{ij} y_1 + \beta_{ij} y_2 + \gamma_{ij} y_3,$$

with α_{ij}, β_{ij}, $\gamma_{ij} \in K$, and these four products span J^2. The ring structure is now determined by the triple of 2×2 matrices $A = (\alpha_{ij})$, $B = (\beta_{ij})$, and $C = (\gamma_{ij})$, which are linearly independent over K and any triple of linearly independent matrices defines such a ring.

In Chikunji 2002, on the basis of computational calculations, we conjectured that there are 5 isomorphism classes for $p = 2$ and when p is odd, the number of isomorphism classes of such rings is $p+4$. We further conjectured that exactly one of these rings is commutative, for every prime p.

In this paper, we extend the above results to all finite fields F_q, where $q = p^r$.

If $(x_1', x_2', y_1', y_2', y_3')$ is a new basis of J with corresponding matrices A', B', C', then x_1', x_2' are linear combinations of x_1, x_2, y_1, y_2, y_3. Since $J^3 = (0)$, we may assume that the coefficients of y_1, y_2, y_3 are zero and write

$$x_i' = p_{1i}x_1 + p_{2i}x_2,$$

so that $P = (p_{ij})$ is the transition matrix from the basis $(\bar{x}_1, \bar{x}_2)$ of J/J^2 to the basis $(\bar{x}_1', \bar{x}_2')$.

Equally, let $Q = (q_{ij})$ be the transition matrix from the basis (y_1, y_2, y_3) to (y_1', y_2', y_3').

If we now calculate $x_i'x_j'$ and compare coefficients of y_i, we obtain equations which, in matrix form, are

$$\begin{aligned} P^tAP &= q_{11}A' + q_{12}B' + q_{13}C' \\ P^tBP &= q_{21}A' + q_{22}B' + q_{23}C' \\ P^tCP &= q_{31}A' + q_{32}B' + q_{33}C', \end{aligned}$$

where P^t is the transpose of the matrix P.

Evidentily, the problem of classifying our rings up to isomorphism amounts to that of classifying triples of linearly independent matrices under the above relation of *equivalence*, P and Q being arbitrary invertible matrices, and it is this problem of linear algebra that the paper is devoted to illustrate using elements of MATLAB.

If $< A, B, C >$ is a subspace of $M_2(K)$ spanned by A, B and C, we may equally speak of $< A, B, C >$ and $< A', B', C' >$ being "congruent" via P. Also, if $\mathcal{X}$ is the set of all triples (A, B, C), then $GL_2(K)$ acts on the right of $\mathcal{X}$ by

$$(A, B, C) \cdot P = (P^tAP, P^tBP, P^tCP)$$

and on the left by

$$Q \cdot (A, B, C) = (q_{11}A + q_{12}B + q_{13}C, q_{21}A + q_{22}B + q_{23}C, q_{31}A + q_{32}B + q_{33}C),$$

where $Q = (q_{ij})$.

These two actions are permutable and define a (left) action of $G = GL_2 \times GL_3$ on $\mathcal{X}$:

$$(P, Q) \cdot (A, B, C) = Q \cdot (A, B, C) \cdot P^{-1}.$$

By restriction, G acts on the subset Y consisting of triples with A, B, C linearly independent. This amounts to studying the congruence action (via P) of GL_2 on the subset $\mathcal{Y}$ of 3-dimensional subspaces of $M_2(K)$, Q just representing a change of basis in a given subspace. In the same way, the whole action of G on $\mathcal{X}$ may be reinterpreted as an action of GL_3 on the subset X of subspaces of dimension ≤ 3. The two triples in the same $G-$orbit will be called *equivalent*.

The complex nature of this problem prompts us to look for ways of finding the number of non-isomorphic classes.

3 Problem Analysis

With all this superlative hardware and software in place, how much is left for the user to do? As already noted, there is a programming language to be learned, which in this case is MATLAB, to complete the closing of the communications gap. But before programming can begin, the problem to be solved needs preparation. In spite of their impressive capabilities, computers still have to be told exactly what to do, in a step-by-step fashion. This process of satisfactorily achieving the required level of detail is called *Problem Analysis*, and it is the user's responsibility. It is by no means an easy assignment.

4 A MATLAB Programme With Elements From The Field $K = F_3$

In this section, we devise a programme that illustrates the use of MATLAB to solve the problem of §2. We illustrate this for the case where the ring R is of characteristic $p = 3$ and the residue field R/J is isomorphic to F_3.

In our programme, the invertible matrices P and Q given in §2 are denoted by the matrices M and N, respectively.

```
function jo(a)
global A
global B
global C
T=[ ];
for i=1:12
  if a >= 2*3^(12 - i)  T(i) = 2;  a = a - 2*3^(12 - i);
   elseif a >= 3^(12 - i)  T(i) = 1; a = a - 3^(12 - i);
   else T(i) = 0;
  end
end
A = [T(1:2); T(3:4)];
B = [T(5:6); T(7:8)];
C = [T(9:10); T(11:12)];

function joh(a)
global M
T = [ ];
for i = 1:4
  if a >= 2*3^(4 - i)   T(i) = 2; a = a - 2*3^(4 - i);
    elseif a >= 3^(4 - i)  T(i) = 1;  a = a - 3^(4 - i);
     else  T(i) = 0;
   end
end
M = [T(1:2); T(3:4)];

function john(a)
global N
T = [ ];
for i = 1:9
  if a >= 2*3^(9 - i)   T(i) = 2;  a = a - 2*3^(9 - i);
   elseif  a >= 3^(9 - i)  T(i) = 1;  a = a - 3^(9 - i);
```

```
   else  T(i) = 0;
  end
end
N = [T(1:3); T(4:6); T(7:9)];

function ph(A, B, C)
global a
a = 3^11*A(1, 1)+3^10*A(1, 2)+3^9*A(2, 1)+3^8*A(2, 2)+
       3^7*B(1, 1)+3^6*B(1, 2)+3^5*B(2, 1)+3^4*B(2, 2)+
         3^3*C(1, 1)+3^2*C(1, 2)+3*C(2, 1)+C(2, 2);

x = [1:3^12 - 1];
global x

global x
for i = 1:6560   x(i) = 0; end
global A;
global B;
global C;
global M;
global N;
global a;
for i = 6560:3^12 - 1
  jo(i);
    if A == zeros(2)  x(i) = 0;
      if B == zeros(2)  x(i) = 0;
       if C == zeros(2)  x(i) = 0;
         if rem( (A + B), 3) == 0 x(i) = 0;
         if rem( (A + C), 3) == 0 x(i) = 0;
       if rem( (B + C), 3) == 0 x(i) = 0;
      if rem( (A + 2*B), 3) == 0 x(i) = 0;
    if rem( (A + 2*C), 3) == 0 x(i) = 0;
  if rem( (B + 2*C), 3) == 0 x(i) = 0;
if rem( (A + B + C), 3) == 0 x(i) = 0;
  if rem( (A + B + 2*C), 3) == 0 x(i) = 0;
    if rem( (A + 2*B + C), 3) == 0 x(i) = 0;
      if rem( (A + 2*B + 2*C), 3) == 0 x(i) = 0;
      end
    end
  end
end
  end
   end
    end
      end
       end
       end
      end
     end
   end
end
for k = 6561:3^12 - 1
 if x(k) ~= 0
  jo(k);
    for i = 1:80
      joh(i);
        if rem( det(M), 3) ~= 0
```

```
          for j = 1:19682
            john(j);
              if rem( det(N), 3) ~= 0
                X = M * A * M';
                Y = M * B * M';
                Z = M * C * M';
                F = N(1, 1)*X + N(1, 2)*Y + N(1, 3)*Z;
                G = N(2, 1)*X + N(2, 2)*Y + N(2, 3)*Z;
                H = N(3, 1)*X + N(3, 2)*Y + N(3, 3)*Z;
                J = rem(F, 3);
                K = rem(G, 3);
                L = rem(H, 3);
                ph(J, K, L);
                if a ~= k  x(i) = 0;
                end
              end
           end
         end
     end
  end
end

global x;
n = 0;
  for i = 6561:3^12 - 1
    if  x(i) ~= 0
      n = n + 1;
       jo(i)
       A
       B
       C
     end
  end
n
```

After running the above programme, we obtain the following triples of matrices representing 7 non-isomorphic classes of the rings of §2. Of the 7 sets of matrices, there is only one triple of symmetric matrices which represents the class of commutative rings.

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};$$

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};$$

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix};$$

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};$$

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix};$$

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This programme may be modified several times to obtain results over other finite fields F_q, where $q = p^r$, p a prime and r a positive integer.

We may now state the following result based on our computational calculations using MATLAB programmes.

4.1 Theorem *For the rings of §2, there are 5 isomorphism classes for $p = 2$ and when p is odd, the number of isomorphism classes of such rings is $p^r + 4$. Furthermore, exactly one of these rings is commutative, for every prime p.*

5 Conclusion

MATLAB is an interactive system whose basic element is a matrix that does not require dimensioning. This allows one to solve many numerical problems in a fraction of the time it would take to write a programme in a language such as Fortran, Basic or C. Furthermore, as may be seen from the above problem, solutions are expressed in MATLAB almost exactly as they are written mathematically.

In university environments, it has become the standard instructional tool for introductory courses in applied linear algebra, as well as advanced courses in other areas. Just like in trying to find a solution to the above classification problem in finite rings, MATLAB can be used for research and to solve practical engineering and mathematical problems.

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IMPROVED COMPUTER SOFTWARE FOR THE TEACHING OF ORDINARY DIFFERENTIAL EQUATIONS:
The ODEToolkit in the Classroom Setting

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ABSTRACT

We discuss a collection of commonly-accepted goals for future education in the field of ordinary differential equations, especially as involves the uses of technology in the classroom. We single out several of these as particularly in need of further emphasis, and discuss methods for improvement. Specifically, we introduce the ODEToolkit, a free, online differential equations solver designed specifically to help a professor incorporate these paradigms into an introductory course.

1 Introduction

Ordinary differential equations are an extraordinarily useful tool in mathematics. Yet for many students their first experience with them in a classroom setting is a rote application of previously learned concepts. This can lead the student to develop the opinion that ODEs—and possibly advanced math in general—is a mundane and formulaic subject.

All too often instruction in ODEs takes the form of rote memorization. Students are given several formulae and told to place an ODE into the proper category and apply the formula for that category. A classic example of this is the integrating factor approach to solving linear first order ordinary differential equations. Students are given a brief explanation of where this comes from and then have to use it before they truly understand why it works.

This is a problem. Students should not be forcibly dragged through material that is interesting on its own merits. The challenge of teaching ODEs is to make these merits visible. If professors spend more time explaining the *why* instead of emphasizing the memorization of *how*, students will come away from their courses with a much deeper understanding of the mathematics involved.

Consequently we advocate a new approach to teaching differential equations; one that will emphasize the *why* without neglecting the importance of the *how*. This approach is based on the current revolution in teaching brought on by the advent of the personal computer. We take this one step further by taking advantage of the capabilities of the Internet. Specifically we introduce the ODEToolkit, a free online ordinary differential equations solver. This new software makes it easy for professors to implement our approach to teaching differential equations in their classrooms. The remainder of this paper will outline several current theories in teaching ODEs, examine how our approach expands upon these ideas, and how our software performs under these paradigms.

2 Our Approach

As we have mentioned earlier, our approach is based on the commonly accepted idea that differential equations should be much more than mere memorization and application. Many experts have spent a great deal of time designing better and more effective methods of teaching ODEs. As such we will spend the first portion of this section outlining what we believe the most important pieces of these ideas are, and how we have integrated them into our approach. After this, we will present our improvements to these methods.

2.1 Common Ideas

Ever since personal computers have become powerful enough to be useful for mathematics, a great deal of research has gone into using these tools effectively in the classroom. While there are many ideas on how to do this, most of the methods have a few things in common. We have attempted to integrate these common features into our approach. The most important idea is that computers can not replace conventional teaching. They can only augment it. We are not advocating the replacement of classical teaching styles with computer learning. However, when used appropriately, the computer can be a very effective aid in the classroom. One common way this is done is to emphasize a computer modeling approach to teaching ODEs. Instead of having students solve arbitrary equations, they are taught to set up a model of a real life situation and solve the resulting ODE.

For example, consider the case of Jimmy. Jimmy is a second year undergraduate student taking ODEs. He has just learned about first order linear equations and is doing exercises to improve his abilities. Classically he would be given a list of equations to solve, using the same technique over and over until he had mastered it. This is an effective way to learn how to perform a task, but a very ineffective way to teach understanding. Also, it is quite boring. A better way of teaching this concept would be to give Jimmy a real-life situation and ask him to answer real-life questions. A good example can be found in Cooper and LoFaro's paper *Differential Equations on the Internet*. Their example of salmon migration uses the latest data available over the Internet to model the number of young salmon that can migrate past dams on the Snake and Columbia rivers. This is a very interesting problem, and as such would engage Jimmy's mind in a much deeper fashion than simply solving arbitrary equations.

The idea of using modeling to teach differential equations is by no means new. However, it illustrates one method of improving a class by focusing on methods other than the classic "apply the formula" technique.

Another important concept for teaching ODEs is visualization. One way to get a good grasp of how equations behave and what the solutions are is to use a visual approach. Computers are an excellent tool for doing this. Using computer software a student can instantly view the myriad solution curves to a given differential equation. What better way to examine the existence and uniqueness theorems of ODEs than to actually draw the solution curves and examine how they never meet yet still hit every point? This is a very powerful technique, yet without a computer it is virtually impossible to use effectively.

The last commonly acknowledged advantage to using computers that we will speak of is the use of numerical solvers. Without a computer with a numerical solver, models must be limited to those which lead to equations with nice, clean, analytical solutions. Unfortunately, these models tend to be very artificial. By using numerical solvers we remove this restraint and allow models to become much more complex and often more accurate. This in turn leads to a better understanding of the usefulness of ODEs in solving real life problems, something that we have already argued is very important.

2.2 Our improvements

Although these measures are certainly steps in the right direction, we feel that there are aspects of teaching differential equations that need significantly more emphasis than are usually accorded to them. Consequently, we have designed the ODEToolkit to take advantage of the computer revolution to display these properties. The points we wish to emphasize will be detailed in the remainder of this section, after which we will address their application in the ODEToolkit and its potential use in teaching undergraduates.

First and foremost, we feel that the cornerstone to a good education in differential equations is interactivity. It is worth noting that it can certainly be a useful exercise to draw out by hand various solutions to a differential equation. This does not compare, however, to the insight gained by a student from instantly receiving feedback on how her choice of parameters or initial conditions affects the solution curves. It is when a student has at her fingertips the ability to quickly and easily manipulate the relevant data, and only then, that a student can understand how important every aspect of a differential equation (be it its linearity, its initial conditions, its order, etc.) is to its solutions. An example of this is shown in Figure 1. In this example we illustrate the ability to modify parameters quickly and easily, allowing students to view the effects of their changes on the solution curves.

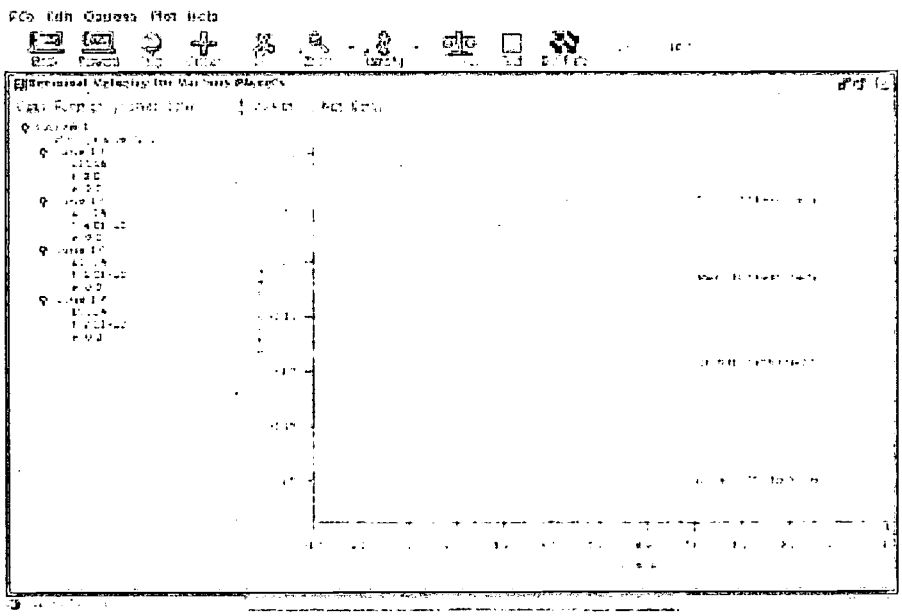

Figure 1: Normalized terminal velocities on various planets, neglecting changes in atmosphere.

A second topic of interest, which goes hand in hand with the previous one, is that of visualization. The ability to interact real-time with solution curves is an important aspect of this, but there are other facets as well. The ability to view solutions from every angle is not only desirable, but often necessary for a deeper understanding of a particular concept. As an example, let us return to the uniqueness theorem of ODEs, as applied to a system of differential equations. As seen in Figure 1 it is quite possible to have solution curves to a system of differential equations appear to intersect on the x-y plane, leading to possible confusion. It is only after viewing the curve from a higher dimensional viewpoint that the student sees the truth; that the two solution curves are actually quite distinct. This is demonstrated in Figure 2.

Another useful teaching tool is to use famous examples. There are literally hundreds of ODEs that are commonly used in real-life applications, and examining these in detail can be very enlightening. Because of this, we feel that an early introduction to a variety of these classic ODEs can drastically improve a student's enjoyment and understanding of the subject. As such, including a library of as many useful ODEs as possible is a very helpful addition to any computer software package. A good example of this is the Autocatalator reaction, a model of an oscillating chemical reaction as seen in Figure 3.

A final aspect in which we feel improvement is necessary is an ever-present problem facing those who wish to integrate computers into the classroom. Specifically, it

LINEAR TRANSFORMATIONS AND EIGENVECTORS WITH CABRI II VIA MAPLE V.

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ABSTRACT

Teaching of Linear Algebra to beginners raises many cognitive problems related to the three thinking modes intertwined: geometric, computational (with matrices) and algebraic (Symbolic). We first study linear transformations in $\mathbf{R}^2$ and $\mathbf{R}^3$ in the Maple V environment. Here the mode is only symbolic and computational. To bring the geometric mode, students can be shown animations programmed with Maple; this will improve their geometric understanding, but during such animations Maple takes the role of a moviemaker and prevents students from participating as actors. Then we use the Cabri microworld where Maple animations can be rendered with its two functions "Locus" and "Animation". However in this micro-world, students can produce easily their own movies, change transformations, vectors and run their own explorations. We claim that students performing with Cabri will enhance their geometric and as well conceptual understanding and will link better the three thinking modes in Linear Algebra.

Topics of Linear Algebra chosen in this presentation are: linearity of transformations and the search of eigenvalues and eigenvectors in $\mathbf{R}^2$ and $\mathbf{R}^3$.

Keywords: Linear Transformations, Eigen-vectors and values, Pedagogical scenarios.

1. Introduction.

What is a linear transformation? What are their eigen-vectors and values?

The usual definitions found in any textbook of Linear Algebra are:

A linear transformation T defined on a vector space V has to satisfy the following conditions:

(i) $T(cv) = cT(v)$ *for any scalar c and vector v*

(ii) $T(v+w) = T(v) + T(w)$ *for any pair v,w of vectors.*

An eigenvector of the linear transformation T is a non-zero vector v that satisfies to the property: $T(v) = cv$ *and the scalar c is then called the eigenvalue associated with the eigenvector v of T.*

Which mode of interpretation did our students choose to grasp these definitions? First, the word 'transformation' has a metaphorical meaning; it indicates a change and even a movement in the space. Then how are understood the three conditions in the definitions above? Geometrically?, arithmetically? or algebraically?

These three modes of interpretation were observed and analyzed by several authors (cf. A. Defence, T. Dreyfus, J. Hillel, A. Sierpinska & S. Khatcherian).

2. Experience with Maple V

Our pedagogical scenarios for teaching the concept of linear transformations that we have been using since 10 years will be presented first. Our students being first exposed heavily to the algebra of vectors and matrices, we use matrices as prototypes of linear transformations. The two conditions *(i) & (ii)* are easily accepted as coming from the properties of the algebra of matrices:

$$(i')\ A\,.\,(cu) = c\,(A.u)$$

$$(ii')\ A\,.\,(u+v) = A\,.\,u + A\,.\,v$$

During a 2hour workshop, ten years ago with grid papers and pencils, now in a computer lab with a CAS such as Derive first and then Maple, students are given a set of 2x2-matrices together with a 2xn- matrix representing a closed polygon with n vertices, one of which is the origin (0,0); this polygon has few right angles and pairs of parallel sides.

Students are asked to plot first the initial polygon, evaluate its area, look at its orientation when following the order in the matrix; then for each given matrix they repeat the same task: plot, area, orientation of the new polygon equal to the image under the transformation studied; they are to collect all observations into a big tableau with initial entries equal to the given 2x2-matrices; those observations are about the preservation of parallelism and the origin for each linear transformation, preservation of right angles only for symmetries, homotheties and rotations, change in areas and orientation depending on the determinant of the matrix of the transformation. Finally they are requested to write their own matrix with determinant = 0 and find out what happens in such a case to the image of a closed polygon. Students can easily see that the 2 column vectors of their matrix span the line of projection onto which the image has collapsed.

After such a workshop, during regular class time we can discuss with more comprehension on the geometric role of the conditions*: (i) & (ii):*

- The image of any line through the origin is again such a line.
- The image of any pair of parallel lines is again a pair of parallel lines.
- The image of any closed polygon is again a closed polygon or a closed interval in the case of a singular matrix.

Finally the role of the matrix as a code for a geometric transformation is clarified by exhibiting the initial basis (u1, u2) (standard basis is commonly used here) and the image pair (v1, v2) = (T(u1),T(v1)) that is also a basis if the transformation is not singular, i.e. is invertible or equivalently of determinant not equal to 0.

Then during a second class, a new lesson (created in fall 2001) constructed with Maple on eigenvectors and eigenvalues was presented. Using a CAS like Derive or Maple helped me to diminish the ambiguity in the students' minds between the arithmetic and geometric modes of representation of a linear transformation. The confusion for some students between the matrix multiplication and the scalar multiplication was alleviated thanks to different commands used in Maple.

We went back to all transformations exhibited during the first class, searching for eigenvectors and then their eigenvalues. In a geometrical context, it was easy to make the observations:

- The directions of lines preserved by the transformation give eigenvectors.
- Eigenvalues measure the ratios of the of the two collinear vectors lengths.

The situation of rotations with no eigenvectors was geometrically clear for students but then we had to stress the relation with the irreducible (over **R**) characteristics polynomial $x^2 + 1$.

A new matrix K=[[1,1],[4,1]] encoding a linear transformation was introduced. The directions of the eigenvectors are rather easy: the two symmetric lines y= -2x and y = 2x associated with the eigenvalues: 3 and –1. Then we run the experience with a symmetric matrix L= [[2,1],[1,2]] that gives the two orthogonal lines of eigenvectors: y = -x (eigenvalue=1) and y = x (eigenvalue=3). Finally we dealt with a regular symmetric markovian matrix [[0.3, 0.7],[0.7,0.3]].

Using Maple we obtain easily the eigen-vectors and -values by running an animation. The initial variable vector w = [cos(t),sin(t)] is chosen on the unit circle, then the algebraic calculation of T(w) is performed and at the same time the image vector T(w) is plotted. The animation will be executed with the parameter t running over the circle and can be interrupted as soon as w and T(w) appear collinear. In the case of a markovian matrix T, we ask Maple to evaluate the sequence of iterates T^n of T and its limit. The limit matrix consisting of two identical column vectors is to be compared with the eigenvector of T associated with its dominant eigenvalue 1.

Pictures with Maple

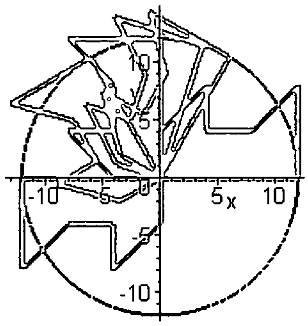


Fig. 1

Different Rotations of the same Polygon

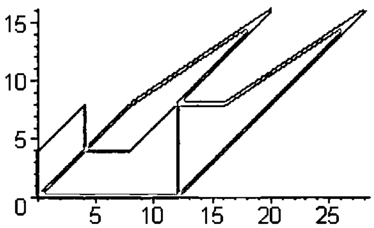


Fig. 2

Shear Transformation encoded

More Pictures with Maple

Case of $K = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ with the eigenvalues 3 and -1.

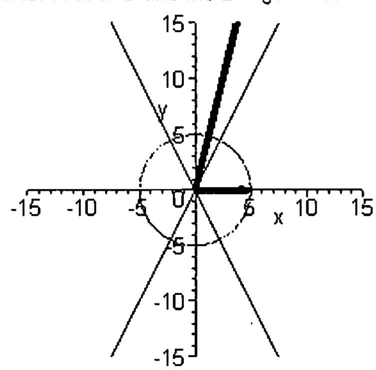


Fig. 3

Case where $L = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, with the eigenvalues 6 and 2.

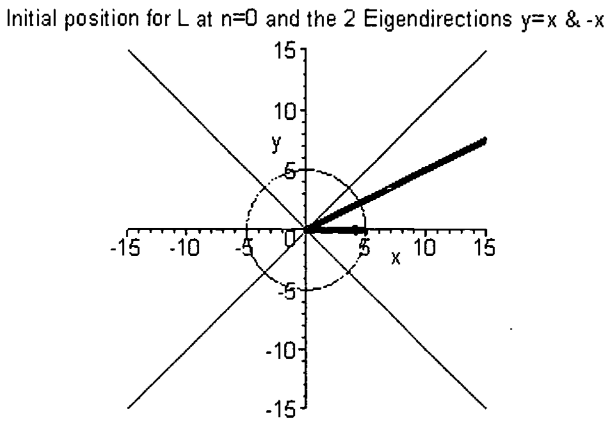


Fig. 4

Scenarios with Cabri II

Since the Fall semester 2001 we have been able to use CABRI II at Dawson College, during my Linear Algebra classes. We adapted to the Cabri environment, the pedagogical scenario described in the previous paragraph.

During the computer lab the macro-construction of a linear transformation depending on the origin O, an initial basis (u1, u2), the image pair (v1, v2) is given for help but not shown to students; only later during regular class time the macro construction will be explained and the importance of the conditions*: (i) & (ii)* will be stressed out.

We give again to students, as during the Maple exercise, the same set of 2x2-matrices encoding well-known linear transformations. Using the standard set of axes and associated grid of Cabri, students may draw their own closed polygon having O as one vertex, with parallel sides and few right angles. Then they point one vector w onto this polygon. To be able to use the Cabri "macro" of linear transformation, they will choose the standard basis for (u1, u2) and then draw two vectors v1 and v2 originating from O. They should be instructed that the pair (v1, v2) represents the image pair (T (u1), T(u2)) that happens to be the two column vectors of the 2x2-matrix representing T. Now the superb " locus" function of Cabri will trace the whole image of the polygon under the transformation T. How to change the transformation T? With Cabri it is a very easy task, as the student just needs to move the 2 vectors v1 and v2. For each matrix given during the Maple task, here we just need to put the vectors v1 and v2 into the positions of the column vectors of the matrix. Then as in a dream, the previous locus changes simultaneously to the new image polygon. We ask the students to use the animation function for two reasons:

- To observe the simultaneous moves of w onto the initial polygon and of T(w) on the image. Parallelism, right angles and orientation could be analyzed during this animation. This could not be done with Maple.

- Given a parametric family of linear transformations, as rotations in the last Maple experiment, we can animate the pair (v1, v2) together with the family of images of the polygon.

It seems that with this new scenario with Cabri, we should gain in clarity for the geometric relationships to observe.

Now we are we going to explain our second lesson on eigen-vectors and –values in this new environment of Cabri. Are we going to gain for the students, more clarity with Cabri than with Maple?

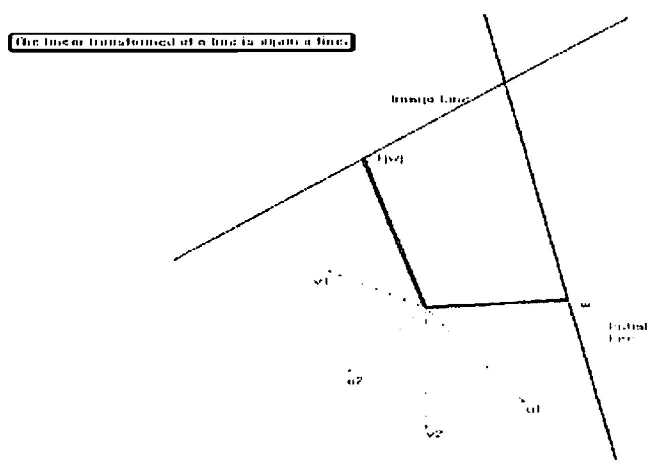
During the animation of the vector w running onto the initial polygon P while its image T(w) traces the image of P, students can be asked to be attentive at the instant when vector and image are collinear. We may also drop the polygon scenario and instead start with a circle as initial object;

We shall fix the vector w pointing to that initial circle and explore the movement of T(w); during the animation, we shall stop whenever the 2 vectors are collinear, ask Cabri to give their coordinates and to calculate their ratios in order to obtain the associated eigenvalue. Later we analyze with students the shape of the locus of T(w) by posing the following questions: Are the two axes of symmetry of the locus given by the lines of eigenvectors? If it is not always the case, for which type of matrix does it happen? To meet this instance we added to our study the symmetric matrix Q associated to the quadratic form $x^2 + y^2 + xy$.

Finally we looked at a markovian matrix T; Cabri can plot the orbit of iterates T^n (w) for any initial markovian vector w; The sequence will converge to the eigenvector associated with the dominant eigenvalue 1 and will be found on one of the line of eigenvectors.

We may conclude now that in this Cabri micro-world, students can change transformations, vectors and run their own explorations. We claim that students performing with Cabri will enhance both geometric and conceptual understanding and will be able to link the three thinking modes of Linear Algebra.

Cabri Picture Fig. 5



All following Illustrations are Cabri pictures:

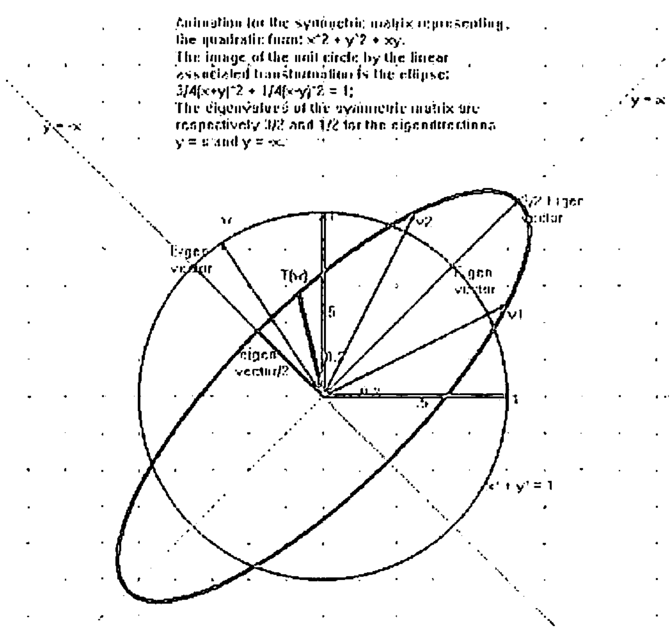


Fig. 6

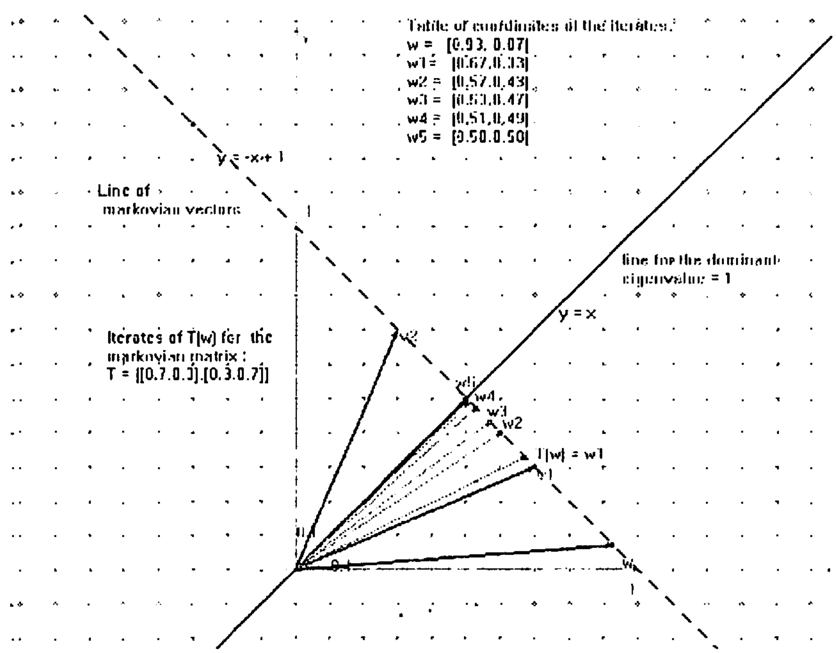


Fig. 7 : Iterates of w for a markovian transformation T converge to the line y=x.

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>
> **Byers, B.,** Concordia University, Montreal, *Working with ambiguity in linear algebra.*
>
> **Lay David,** University of Maryland, College Park, USA. Plenary Conference: *Recent Advances in Teaching Linear Algebra.*
>
> **Oktac Asuman,** Concordia University, Montreal, *Linear Algebra: Is it possible at a distance?*
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> **Norman D.,** Queen's University, Kingston, *Teaching linear algebra independence via unique representation.*
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NUMERICAL ALGORITHMS - ENHANCING PRESENTATION WHILE MAINTAINING RIGOUR IN INTRODUCTORY COURSES.
A minimalist approach to course modernisation

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ABSTRACT

Recent academic developments which include a changing student profile, the focus of contemporary research, and social trends have combined to pose significant challenges in first year undergraduate mathematics courses. Issues which arise include the content level, the need to maintain rigour, addressing the needs of specialist and non-specialist students, the need to equip students with useful, applicable, techniques, and our desire to present a picture of the important problems and directions in modern mathematics and its beauty and excitement.

Calculus reform has made a significant impact but more needs to be done. There are large areas of research in which the computer has the role of an experimental tool. The use of software packages is widespread. This has produced the need for something akin to an instinct which can identify the correct or incorrect functioning of a package or black box. To acquire this instinct some knowledge and experience of the behaviour of numerical algorithms is needed. As a consequence the way in which calculus is taught needs to be changed. It also needs to change because the computer has caused major changes in the theoretical directions of mathematics.

These influences can be used to enhance courses whose content contains the essential foundations of the subject. The foundations will not change, but investigations of numerical algorithms, for example, can pose the same fundamental questions that are to be found in texts dating back a century at least. Well founded approximation methods provide exact rigorous statements. Numerical experimentation can provide insight. There is no need to present a grab bag of computations whose output is of doubtful validity.

This presentation will briefly review the knowledge levels of entering students; it will describe some important applications and it will attempt to show how some of the challenges can be met.

Keywords: Algorithms, Numerical Analysis, Calculus, Linear Algebra, Biomathematics

1. Introduction

The pace of change in universities and other tertiary institutions closely follows that of the world at large. Scientific discoveries, changing social structures, and reforms of education all combine to exert a strong influence on the practice of mathematics whether it be in the workplace, in school and college or at the cutting edge of research.

In the context of teaching, Mathematics needs to respond to these changes. What the response should be is a topic which needs careful consideration and how a perceived response should take place is a second important issue. It is the author's view that the mathematics community in Australia and, perhaps, worldwide, has been slow to recognise that change has taken place. Many who feel the need for mathematics to support their work share this view. At the author's university the biological scientists are eager for the mathematicians to lend support to their research (arguably the most important in the 21st century) and they realise that this support cannot be sustained without a strong undergraduate degree.

The focus of this meeting is undergraduate teaching and, since many students take a first year course and may not proceed with further mathematics, suggestions for changes will be in this context. Apart from a couple of sentences in conclusion, the proposals here are modest, necessarily so against a background of inaction imposed by those who resist change.

A review and assessment of the background of the entrants to first year courses will be made initially. This will be followed by an overview of major developments in discipline areas which draw on mathematics and mathematicians as a resource. Instances of responses to these stimuli will be described and inferences regarding curriculum content will be drawn from these observations. Finally, in the last section it will be argued by example that the important values in a traditional curriculum can be maintained while presenting a more forward looking account of the subject.

2. First year undergraduate entrants

Whereas, in times hitherto, high school curricula consisted of a small number of subjects, the need for a breadth of choice to address students with diverse talents and abilities has led to a proliferation of options. The effect has been that mathematics occupies a significantly smaller fraction of school activity. In Queensland, Australia, in the past, many university matriculants would study two mathematics subjects, amounting to 200 hours each year, however no more than one of these subjects is needed for university entrance so that the time spent on mathematics has been reduced by 50% in many cases. Statistics from the Queensland Board of Secondary School Studies web site at http://www.qbssss.edu.au/statisticsandpublications/statistics/Subject_stats.html show that the percentage of students taking both Mathematics B and C as a percentage of those taking only Mathematics B fell from 29% in 1992 to 20% in 2001. Data for 2000 and 1999 suggests that these numbers have now stabilised.

It has to be recognised too that the depth of knowledge of the subje ct has reduced in other ways. Informal tests at the University of Queensland carried out on first year entrants on their knowledge of content indicated a substantial fall over the period 1973-1990 (Belward and Pemberton 1996). Now the content is problem driven, thus more time is spent on problem solving rather than on an accumulation of knowledge strengthening and technique. There are compensations however, the syllabi now require students to have some proficiency with graphics calculators or computers and they may have assessment instruments which are take home projects of two to three weeks duration. More details are available at the Queensland Board of Senior Secondary School Studies site http://www.qbssss.edu.au/Curriculum/subjectguides/MathsB.html.

3. Mathematics and Research

There was once a time when a mathematics researcher only needed to look at the journals whose titles bore mathematics connotations. At least it appeared so. Today there are many subject areas where a large amount of content is concerned with mathematics. Many IEEE journals (see http://shop.ieee.org/store/Overviews/periodicals.asp#list) contain large amounts of mathematics and there are many such journals where mathematics is a major element of the work reported. The problems under review may be engineering problems but removal of the mathematics would most often remove the content of the problem. Operations Research is another rich source of mathematics. The Simplex method for linear programming has had immense success and its development continues (Zakeri, Philpott, A. and Ryan 2000). As computing power has increased problems such as the travelling salesman problem can be solved for larger numbers of variables and the search for efficient algorithms has been pivotal in the development of stochastic algorithms and evolutionary algorithms. Much work in these areas has been done by computer scientists. Many problems in financial mathematics have a large stochastic component. The introduction of a stochastic element is necessary to model the behaviour of many investment instruments. This, concurrent with present computing power, has been a catalyst for widespread interest in the solution of stochastic differential equations (Kuechler and Platen 2000).

Utilising the biological analogies drawn by the developers of by genetic and other evolutionary optimisation algorithms we find ourselves lead into life science itself. The sequencing of DNA is a problem which has been confronted by several different approaches leading to a large variety of optimisation algorithms. There is no doubt that bio-physics, bio-mathematics and bio-informatics are all manifestations of the current surge in research in the life sciences whose major problems have been said to offer the largest intellectual challenges of the 21st century. Recent numbers of the journal *Bioinformatics* raise this matter regularly. In a recent editorial (Pearson, 2001) the view is expressed that "Genome biology presents a different scale, whose promise will not be fulfilled without an infrastructure of well-trained researchers in Bioinformatics, Computational Biology and Biomathematics ... ".

4. Some responses to research stimuli

Research institutions and universities worldwide have taken steps to meet the challenge of updating and focussing their research on the rapidly developing areas noted in the previous section. Specialist groups, centres and semi-autonomous institutes and commercial organisations have been set up to deal with these problems. This permits an initial response, however in order that this be sustained an increase is needed in the number of graduates. If the research programmes are successful, large numbers of graduates with expertise in the appropriate area will be needed.

At the University of Queensland federal funding has been made available for six emerging researchers to set up a computational biology program. The outcome will be a degree course which will be the life science counterpart to an engineering degree. The University is a participant in the Queensland Parallel Supercomputing Foundation which has an educational programme wherein materials will be made available through the internet as support materials in advanced undergraduate and master's programmes in high performance computing and visualisation. On a smaller scale the university has an honours programme in Financial Mathematics as a joint operation between the departments of commerce and mathematics.

5. The course for mathematics

On the evidence presented in the previous sections it is imperative that the content of first year mathematics change if some sort of mismatch between current content and expected outcome is to be avoided. Educational programmes must reform in the way that research programmes keep abreast of modern developments. While research in both pure and applied mathematics has gone ahead at pace, teaching methods and course content have progressed only a little. Admittedly the evidence is anecdotal, but all too often colleagues (respected and successful) from other disciplines who are users of mathematics tell me that "I didn't learn a single useful thing in my maths courses".

The task is not easy; many competing pressures are apparent. Besides those noted already, departments have reduced resources due to budgetary pressures and those same forces have resulted in classes being merged into one large single stream. Often course structures prevent potential users from completing more than first year mathematics.

Another source of pressure is the duty to remain true to ones subject; in other words to maintain integrity. This means that there are certain fundamental concepts which are important and cannot be rejected, the sorts of ideas which are essential in a degree majoring in Mathematics.

Finally there is the need to put on courses addressing the interests both of those who want to specialise in the subject and those who will use mathematics to support their chosen discipline. Mathematics is useful, but it is also exciting and beautiful. Thus it is crucial that an effort be made to show those who want to specialise in mathematics a glimpse of the modern ideas. On the other hand, believing as we do that all intellectual investigation can benefit with the application of mathematics we have to convince students that in their chosen subjects mathematics will be important and may provide important benefits to them.

6. Examples of enhancement of first year courses

In this section a minimalist approach to some reform is presented. Without rewriting a typical calculus or linear algebra syllabus it is possible to address the problem of "... not learning a single useful thing ..." while meeting the needs of both the application oriented and potential specialist students. This is possible because scientific computation uses methods whose validity relies on the analytical and algebraic foundations of the subject. Thus the material can take on a more modern appearance without loss of integrity.

Approximation theory includes application of the functional analysis to numerical approximation schemes. It is worth emphasising that approximation theory is exact, in the sense that error bounds are given on the results of calculations and therefore careful choices of numerical algorithms can be validated by these error bounds. Therefore many good numerical techniques can enjoy standards of rigour on a par with those of calculus and other areas of mathematics.

Each of the topics below appear in many first year courses. All of them merit a thorough treatment. Despite their algorithmic character a "cook book" presentation should be avoided.

Solution of nonlinear equations

In the absence of an analytical expression for the roots of a nonlinear equation a sequence of values is desired converging to an x satisfying the equation $f(x) = 0$.

Newton's method appears almost always in first year calculus books. Sometimes it is portrayed as a single step method, but in a practical situation a sequence $\{\alpha_n\}$ is generated by the rule

$$\alpha_n = \alpha_{n-1} - \frac{f(\alpha_{n-1})}{f'(\alpha_{n-1})}$$

This works very efficiently when it converges, but to be able to give bounds on the solution we require values of x which bracket a sign change in f and the knowledge that f is continuous on the interval defined by the bracket. A robust root finding program must find values, which bracket a solution, particularly when the algorithm is a component inside some other routine. For these reasons the humble bisection method, wherein intervals bracketing the root are halved repeatedly until some tolerance level is reached, should not be ignored. It might then be possible to introduce Brents algorithm which implements the secant method safeguarded by the bisection method.

Here we immediately introduce the notions of convergence and continuity without the need for contrived examples. The idea of fixed point iteration is introduced whose convergence analysis can be used as a vehicle for introducing the mean value theorem, see (Belward 1999) for an example and further details.

Numerical integration

An enunciation of Simpson's rule or the trapezoidal rule without some discussion of the error committed by their application is poor mathematics. Further to simply derive the formulae by fitting parabolas or evaluating the areas of trapezia will not equip the class with the methodology which will enable them to deal with more difficult situations later. An integration rule should be presented as the integration of an approximant to the integrand, thus we approximate $f(x)$ by $a(x)$ with an error $e(x)$. Then we integrate the relation

$$f(x) = a(x) + e(x)$$

and obtain

$$\int_a^b f(x)dx = \int_b^b a(x)dx + \int_a^b e(x)dx = \text{ quadrature approximation } + \text{ error}$$

Presenting the method in this way enables more advanced methods (e.g. product integration rules) to be understood and provides a methodology to deal with singularities.

It is also important that some numerical experience be obtained. For appropriately smooth integrands the error committed is proportional to $1/n^2$ and $1/n^4$, respectively, for the trapezoidal rule and Simpson's rule, n being the number of steps. A feeling for convergence rates may then be inculcated and experimental curiosity aroused by application to functions with and without the smoothness assumptions of the error formulae.

number of subintervals	exp(x)	sin(100x)	$x^{1/2}$
4	3.7013e-005	-2.6068e-001	-1.0140e-002
8	2.3262e-006	-2.6068e-001	-3.5874e-003
16	1.4559e-007	-2.6068e-001	-1.2685e-003
32	9.1027e-009	8.5074e-002	-4.4848e-004
64	5.6897e-010	6.3366e-005	-1.5856e-004
128	3.5562e-011	3.0707e-006	-5.6061e-005
256	2.2224e-012	1.8138e-007	-1.9820e-005
512	1.3789e-013	1.1181e-008	-7.0076e-006
Exact value	1.71828 ...	1.3768e-003	.666666...
Remarks	Because the function is so smooth the convergence rate is attained immediately	Until enough points are chosen results are bad, then rapid convergence is observed	Moderate accuracy from the start, given the number of points but the convergence rate never improves

Table 1. Errors in Simpson's rule for the integral of three functions on (0,1) against the number of intervals.

The data in table 1 summarises the results which may be obtained by experimenting with 3 functions for which we know the value of the integral on (0, 1) and therefore the accuracy for each example. Insight into these results can be gained by plotting the integrands and superimposing the quadrature points. Sketching the piecewise parabolic pieces which are used in Simpson's rule then reveals that until enough points are taken to follow the integrand the results will be unreliable. The square root function example, however, shows that understanding its results cannot be deduced from the graphical information. One needs the error formula to see why the convergence is slower.

It is not often that bounds on the accuracy of numerical integration schemes are available. It is therefore useful to note a nice example wherein upper and lower bounds on a numerical integration scheme can be found (Hughes-Hallett, Gleason, et al. 1998).

Linear algebra

There is little doubt that the solution of linear equations is one of the most useful techniques, which can be given to users of mathematics. Furthermore this both necessitates and motivates the study of linear vector space theory ; it is fully exploited in the approach of Strang in his book on linear algebra (Strang 1998). By attempting to interpolate data with unsuitable choices of basis functions one can give simple examples of systems with one, none and an infinity of solutions. (Fit a quadratic to 3 data points at –1, 0, 1. Then use 1, x^2 and x^4 as a basis with data values that are different and then the same at -1 and 1, see (Belward 1999) for further details)

Here again we have used a practical problem to introduce important abstract ideas. An important point is that one should not stray too far from the problem source. Although solving linear systems by pencil and paper using augmented matrices looks attractive, in using the algorithm students tend to forget what problem they are solving. Preference should therefore be given to the idea of always carrying equivalent sets of equations. In this way absurdities introduced by arithmetic error are more easily detected.

Using equation sets rather than matrices carries over to the simplex algorithm where many students have no idea what is happening when a tableau is processed. In comparison the dictionary method of Chvatal (Chvatal 1983) carries the constraints and an explicit expression for the objective function expressed in terms of the non-basic variables, whereupon the choices of entering and leaving variables, not to mention the optimality criterion, are made clear. Once again by keeping to the practical problem one has both algorithmic efficiency and rigour.

7. Conclusion

This article has attempted to show that the pace of change in universities and research institutions is considerable and that, as a consequence, undergraduate mathematics, particularly in first year, needs considerable adjustment. It may not be possible to make wholesale changes to large first year courses where enrolments are measured in hundreds and sometimes thousands. Nevertheless a good teacher can enliven the most turgid material and a good choice of examples can also provide considerable insight.

The author has adopted this minimalist approach. Students have found the material stimulating, they have realised that there is interesting mathematics beyond calculus and linear algebra. Interest in Scientific Computation is steadily increasing. In the past students often did not meet these ideas until their second year. With the extra emphasis put on this material in first year they are now informed of attractive opportunities in visualisation, plant architecture informatics, and genomics. Through the medium of this brief introduction they are able to make more informed choices when planning their later year studies.

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STATISTICAL DATA ANALYSIS COURSE VIA THE MATLAB WEB SERVER

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ABSTRACT

Internet based courses are at present a quite common tool of learning. While still the most of them consist just of the static texts located on web pages, the objective of many new instruction systems is to make the learning process more dynamic and interactive in comparison with a mere reading the textbook or listening the lessons. Moreover, it is a feature of mathematics that it can hardly be studied by mere memorizing the texts. That is why we searched, when preparing a series of internet courses on mathematical statistics, data analysis and quality control, for an environment enabling such an interaction and supporting the preparation and use of numerical and graphical procedures. Finally we decided to utilise the Matlab Web Server. In this environment, the author can combine the text with Matlab computational algorithms and graphical tools, the user can work with them without having its own Matlab installation. The system thus consists from text files, the Matlab programs and the procedures controlling the interface, input and output connection between the web pages accessed by the user and the Matlab algorithms (prepared by the author or used directly from Matlab toolboxes). In such a way, a student is provided simultaneously with relevant information, the examples, and graphs, he can enter his own data, too, and is offered the tests checking his knowledge. In the present contribution the system will be described and examples of its use presented.

Key words: Mathematical statistics, tutoring system, Matlab Web Server.

1. Web based tutoring systems

While the first web pages were constructed about 10 years ago, the internet is today used quite commonly and the area of possible applications is growing tremendously. As the main benefit is the information sharing and processing, one of remarkably developing fields are the internet courses and (e-) learning. These procedures or systems can offer a large amount of information in properly selected and structured form, but they also gather the information on users. Some of systems are able not only to recommend the path through the lessons, but also to control it and adapt on the basis of user's skill and progress (ITS – Intelligent Tutoring Systems). These systems contain elements of AI, in order to classify the user (User Modelling) and to control his path, including the recognition and remediation of insufficiently mastered lessons.

Thus, the construction of an ITS needs the intensive effort of a whole team of specialists. One example of such a system is the Net-Coach (www.net-coach.de), which is offered also as an "empty" system to be filled with an appropriate information, contents. On the other side, the most of (up to now) available applications deal with topics like a language course, basic courses of work with PC or programming, information of an "encyclopedic" nature, or drivers' tests (e.g. www.neuralgen.cz), though even these applications contain often certain elements of UM and adaptation methods (on the other hand, it should be said here that the 'guided adaptive tour' through the lessons was already the feature of so called "programmed textbooks" many years ago).

The development of tutoring systems in the field of mathematics is complicated by the need of simultaneous usage of text, computation, and graphics. It seems that the relatively direct way can lead from a professional mathematical software package, especially when it already has its Web version, so that the large portion of technical work has already been done. It 'remains' to make it more didactive, to change the help to tutoring texts, to add the procedures of control and adaptation (at least recommending the path through the lessons) and to prepare also the tests checking the knowledge of an user. Let us mention here the statistical system developed from the XploRe statistical package (www.md-stat.com).

We decided to use the computational and graphical environment of Matlab and its connection with internet via the Matlab Web Server. The advantage of Matlab consists in the possibility to prepare own algorithms and use them as a part of Matlab library. The programming is easy, so that Matlab is convenient for active formulation of algorithms by students. On the other side, the implementation of MWS has also disadvantages (for authors of tutoring system), because the interface of all parts (sharing the information, input and output transfers) is not so straightforward as we expected.

The applications of MWS combine Matlab m-files, graphics, and HTML texts, resp. PHP and Java scripts. The programming of each such application consists of several parts:

1. Development of HTML files enabling the transfer of input data and parameters of programmed procedure. As a rule, it is through a frame on client's display.
2. Development of Matlab m-file, which, except that it solves required computational or graphical problem, reads the data from the input HTML frame, and prepares the output field.
3. Then, other HTML procedure transfers the output to the output page or window visible on client's display. The graphs are in .jpg format and are called by the parameter – figure's name.

2. Actual contents of the course

The objective of the system is to provide:

- The tutoring text, including the theory and formulas, and practical recommendations.

- Illustrative examples, figures. The numerical examples are generated randomly. The user, in some cases, is challenged to solve the example independently and to check the correctness of his approach afterwards.

- The examples, exercises testing the level of knowledge of the user.

- Statistical data analysis "calculator" which can process even the data from the file on user's computer.

- Finally, a set of Matlab algorithms, developed by authors (and, naturally, the standard Matlab procedures and tool-boxes which are a part of Matlab installation).

At present, the material covers several chapters of basic courses on probability, mathematical statistics and quality control. Namely

1. Distribution of random variables. The most frequent types of distribution (both discrete and continuous) are presented both mathematically and graphically, their typical use is demonstrated on simple examples.
2. Explorative data analysis (EDA) methods. This part deals with basic empirical characteristics of dstribution of given data, their computation and graphical presentation with the aid of several types of plots.
3. Statistical tests of hypotheses. Again, the main methods of both parametric and nonparametric tests are recalled, its methodology explained and illustrated on examples.
4. Regression analysis presents the material on linear and nonlinear models, the least squares methods and nonparametric smoothing approaches.
5. Control charts provide the motivation, methodology, and examples of Shewhart charts for the mean value and variance, and also the EWMA smoothing method.
6. Methods of quality control in textile industry The theme follows from the specialisation of the Faculty of Textile Engineering of TU Liberec and provides the basic approaches to textile metrology and quality control.
7. The section provisionally named 'Matlab-Web' contains a selection of examples to all topics.

The orientation in the system is provided by the menu of themes and sub-themes. The main page is shown on Figure 1. Thus, the user can either select its own theme (sequence of themes) or he can follow the recommended sequence (which actually corresponds to the order of menu bars) and links to examples in the text of textbook chapters.

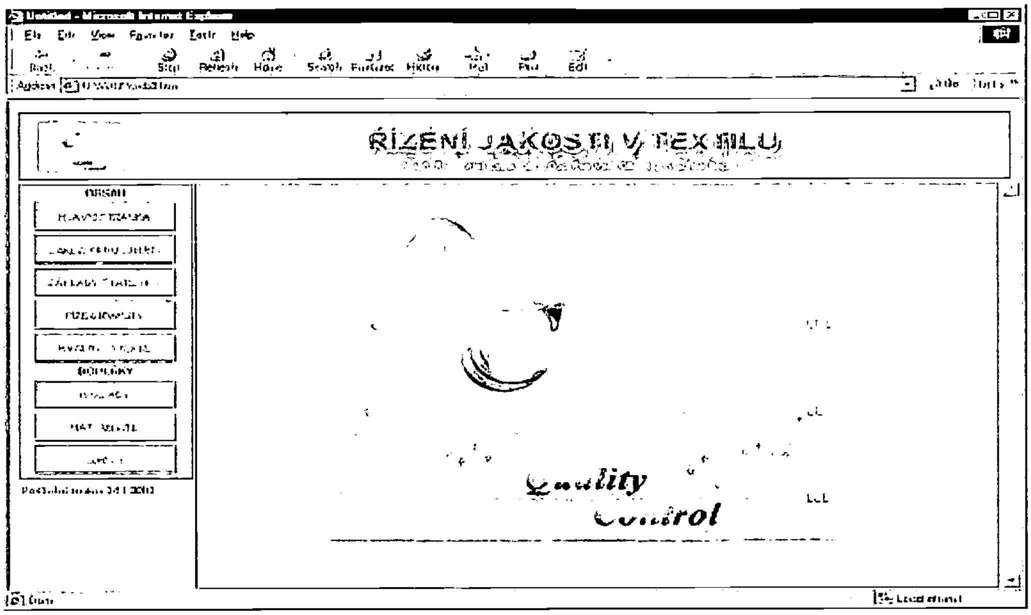

Figure 1. The main panel

3. Example – control charts

Let us assume that we are interested in a brief tutorial on control charts. First, we can follow the menu bars and select tutoring text under Quality Control / Control Charts. There, a student obtains information (description, definition, purpose, probabilistic background and way of utilisation) of basic charts, among them the Shewhart's control charts for the mean (X-bar), for the standard deviation (S-bar), and also exponentially weighted moving average (EWMA) diagram with one-step delay. Then, the link leads to the computational example. The user is first asked to enter data – and he has four possibilities: He can write the data directly to the corresponding line in the input frame on display, he can use Matlab random generator and generate the data through the MWS, further, he can select one of demonstration data files prepared on the server, and he can also enter his own data from his computer. In the last case the user should be aware (and he is informed in tutoring text) that the form of data should correspond to the procedure – for instance the control charts work with a matrix ($n \times k$, say) of n groups à k observations. Finally, the user is asked to select, in another window, the type of chart (e.g. EWMA together with smoothing parameter). Then the charts are computed and displayed graphically (see Figure 2).

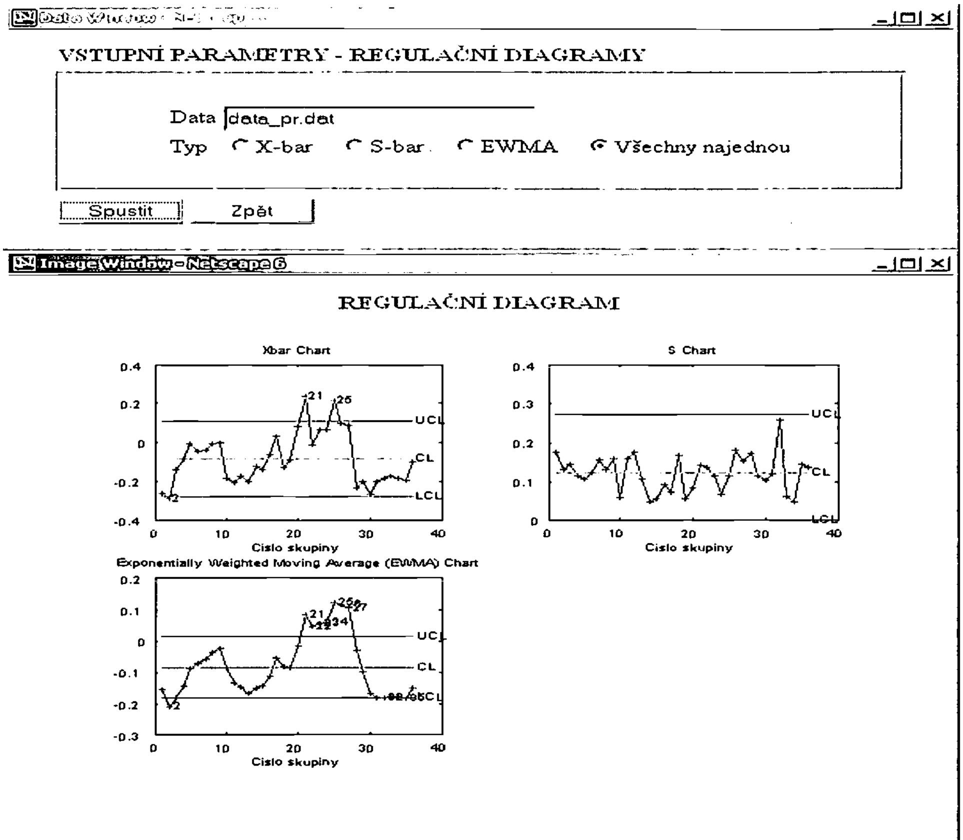

Figure 2. An example – graphs of control charts

3.1 An example of seminar exercise

Let us assume that the theme of the exercise is the polynomial regression, including the selection of optimal model. The teacher provides the instructions, the data are artificial, prepared in advance and stored on the server (for students home work) or they can be generated by students, again, according to an instruction, for example:

1. Generate the data (e.g. a complicated sinusoidal function with Gaussian random noise).

The following instructions are:

2. Plot the data, with the aid of polynomial regression procedure.
3. Select the degree of polynomial.
4. Evaluate the regression model, analyse the significance of its parameters. The procedure computes regression parameters, corresponding 95 % confidence intervals, residual variance, and also Schwarz's BIC criterion penalising the models with too high degree. Students should already know the meaning of all these variables and parameters. The next step is:
5. Change the degree of the model and repeat the analysis. Compare the results, with special attention to the values of BIC criterion.

… etc.

Finally, the procedure evaluates the model once more, omitting statistically non-significant parameters (and model components). The student is expected to perform the whole exercise, to write a report, and, eventually, to demonstrate and comment the process and results of solution to the others. Naturally, he is encouraged to utilise the relevant parts of chapter 'Regression' from the text stored on MWS. He can also be recommended to draw an additional information from other sources, textbooks.

4. Conclusion

As it has already been said, the development of applications using the Matlab Web Server is technically demanding and time consuming, not speaking about high requirements of programming skill. Moreover, the Web text languages, as a rule, do not support the writing of mathematical symbols and formulas (this is the problem of MS Word, too, at least for people using the "mathematician-friendly" environment of LaTex). Nevertheless, the MWS applications offer many advantages, the essential being the access to Matlab-procedures through the Web browser.

That is why the main purpose of our system is to provide the students the possibility of effective home-learning, because they can gather both theoretical and practical knowledge simultaneously, from the same source, and in proportion convenient to each individual student. Further, the students will use the system for the preparation of their seminar exercises and reports. The system is originally intended for the students of textile engineering in Liberec, it uses Czech language. Simultaneously with system growing (other parts of statistical methodology will be attached soon) its texts will be translated to English, in order to be available also to students of international University Nisa, founded recently in Liberec region. The actual version of the system is accessible through the links from the address: http://147.230.129.170/ales/ucebnice_v3010/.

Acknowledgement: The development of the MWS application has been supported by the project FRVŠ 1010/2001, the work continues at present under project MSM 245 100 303, both granted by the MŠMT ÈR.

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COLLEGE ALGEBRA IN CONTEXT:
Redefining the College Algebra Experience

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ABSTRACT

We have developed a redefined college algebra course, which uses an informal approach, is application driven, is technology-based, and uses real data problems to motivate the skills and concepts of the course. Each major topic contains real data examples and problems, and extended application projects that can be solved by students working collaboratively. Students can take advantage of available technology to solve applied problems that are drawn from real life situations. The students use technology, including graphing calculators, Excel, and Derive, to observe patterns and reach conclusions inductively, to check answers of solved problems, to study function types, and to create models for use in the solution of problems.

The course provides the skills and concepts of college algebra in a setting that includes applications from business, economics, biology, and the social sciences. The course was designed to provide the required college algebra skills for students in the Business major, in the Hotel, Restaurant, Tourism Administration major, and for majors in the biological, marine, and social sciences. The real life applications included are the result of collaborations with faculty in Business, in Hotel, Restaurant, Tourism Administration, and in Biology Departments in three Universities.

Each mathematical topic of the course is introduced informally with a motivational example that presents a real life setting for that topic. The problem in this example is then solved as the skills needed for its solution are being developed or after the necessary skills have been developed. Some applications provide the models for the data and have students solve related problems, while others require students to develop the models before solving the problems. For some topics, students work in small groups to solve extended application problems and to provide a written report on the results and implications of their study. For other topics, students find appropriate real data in the literature or on the internet, develop a model that fits that data, and use the model to solve problems. Most of the examples and exercises in the course are applied problems.

Introduction

Most of the students taking College Algebra are not math or science majors, and although they are quite intelligent, many are just not interested in mathematics. If we give them compelling applications, they will see that there is some reason for the mathematics to exist, they will be more interested, and they might even come to like math in this context. These people are as likely to be leaders in our communities and country as are people with science and engineering degrees, and so they need the reasoning skills and problem solving skills just as much as science majors do.

Those of us teaching college algebra realize that students taking this type of course will not likely become math majors or enter careers that are heavily dependent on mathematics. However, they will likely have a career that requires reading for comprehension, problem solving skills, and the ability to analyze and interpret. Thus we emphasize real problems rather than mathematical theory for its own sake.

As mathematicians, we sometimes think that the value we should impart is the logic involved in a rigid outline with proofs of theorems. But for students whose future is not in a mathematical field, there is a much more interesting, challenging and useful approach to mathematics. Thus we sought to offer a wide range of applications, to keep the interest of all students and show that algebra is useful and necessary to solve real problems. Thus our mission was to design a course where algebra skills were not the goals of the course but rather the tools for the attainment of more far-reaching goals.

College algebra in context

With this mission in mind, we developed an algebra course based on real life applications from business, economics, biology, and the social sciences in a setting that connects mathematical content with the real world. The course can best be described as using a transitional approach, because it has most of the positive attributes of reform algebra without sacrificing the wide variety of algebra topics included in a traditional graphing approach to college algebra. Data analysis, modeling, and technology are woven into the course so that the approach is refreshing and interesting to the students. The course provides the algebraic skills and concepts for a core course, or for the future study of calculus, in an informal, less threatening, and more meaningful setting. The course was designed to provide the required algebra skills for students in the business and economics majors and in majors in the biological and social sciences. In fact, this course provides the algebraic background needed for success in all majors other than the physical sciences and engineering. It is designed so that students can solve meaningful applications and see how mathematics relates to their future.

We feel that keeping the skills and concepts in the context of applications gives the course a new perspective for students who were successful in high school algebra and for those who were not. Students respond better to this approach and see the necessity of learning algebra skills to solve these problems. The course attempts to relate algebra to everyday life with examples and exercises relating to real-life math problems. The quality and quantity of the examples and exercises is the strength of our approach to college algebra.

We have made an effort to have real applications for every algebra skill and concept introduced in the text. The goal is to provide students with a real sense of the relevance of algebra to the real world. To this end, we concentrate on real problems as opposed to contrived applications.

We have sought a good balance of skill-building and application exercises, but we think the purpose of this course is applying algebra to the real world. We use Skills Check exercises to warm students up for the exercises that follow. This gives the students a chance to get familiar with the concepts and skills and to gain confidence.

After the skills checks, the exercises are all applied (real or realistic), most with source references. We ease students into word problems by giving similar problems with equations given. We also give exercises that break problem solution into steps. Later exercises involve critical thinking, complete solution, and interpretation. Where reasonable, problems involving integer cases are used at the beginning of the exercise sets, with real data problems requiring technology later in the set.

Those skills that are prerequisite for the course are included in an Algebra Toolbox, which provides a "just-in-time" review for the chapter in which they are needed. These topics can be left for the students to review, or can be included in the day's lecture. We prefer this to teaching or ignoring prerequisite topics in a Chapter 0 or an appendix. The Toolbox for each chapter includes the intermediate topics needed for that specific chapter.

The level of our course is appropriate for students taking a terminal course or for students going on to a non-science calculus course that uses some technology. In particular, it is appropriate preparation for a business calculus course. A slightly different approach to the topics we have prepared could also be used in a modeling course. The materials we have developed could be used in a course aimed towards business preparation or one aimed towards modeling.

In some cases, we have students develop their own models from the data, after deciding which function best models the data. Models are created from real data and problems are then solved using the models. We have students consider first, second, third, and fourth differences, and constant rates of change to determine which model may be most appropriate for a set of real data. We have included a large number of problems that ask the student to interpret, analyze, and make predictions from real data models.

Mathematical concepts can be introduced informally with technology rather than with more formal methods. The students can use graphing calculators to observe patterns and to reach conclusions inductively, to check answers of solved problems, to study function types, and to solve equations graphically and numerically. Technology can be used to develop equations that model real data, and the equations can be used to reach conclusions about the data and to solve problems about the data.

We emphasize graphing calculator use, but as an aid, not as a substitute for analytical methods. We frequently solve problems with both analytical methods and technology. The calculator or other graphing utility is integrated seamlessly into the discussion, but students are required to use analytical as well as technological approaches to problem solving. And when applicable, students are shown why the analytical method is easier than graphical or numerical methods of solving a problem. In some cases technology and analytical methods are combined to solve problems. That is, there are cases where technology is used to assist in analytical solutions, and cases where analytic methods are used to assist with graphing (for example, finding the vertex of a parabola to help set the window of a graphing utility).

Collaboration is encouraged throughout the course. Students are encouraged to work in teams just as they might in the workplace after graduation. The Extended applications are especially designed for collaboration.

A few of the examples of applications used in the course follow.

Real Data Applications

EXAMPLE 1: SALARIES OF U. S. COLLEGE PROFESSORS

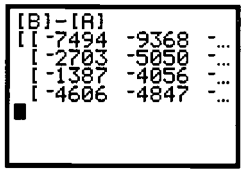
Students are asked to find data from a number of resources and to reach conclusions about salaries of male and female college faculty teaching at different levels in different types of institutions. Students were asked to discuss the relationship between male and female professors' salaries, using information from the table below.

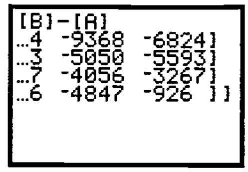
Salaries of College Professors, 1998-99
Source: American Association of University Professors

	MEN			WOMEN		
	Type of Institution			Type of Institution		
TEACHING LEVEL	Public	Private/ Independent	Church-related	Public	Private/ Independent	Church-related
Doctoral level						
Professor	$80,379	$99,979	$84,796	$72,885	$90,611	$77,972
Associate	57,653	65,843	60,059	54,322	61,956	56,180
Assistant	48,647	57,296	50,009	45,203	52,521	46,427
Master's level						
Professor	64,414	70,643	66,151	61,711	65,593	60,588
Associate	51,812	54,260	52,634	49,615	51,273	48,189
Assistant	42,673	44,511	42,317	41,189	43,002	40,312
General 4-year						
Professor	58,432	68,145	52,945	57,045	64,089	49,678
Associate	48,643	51,044	43,412	46,808	49,202	41,791
Assistant	40,625	41,551	36,534	39,245	40,634	36,017
2-year				52,461	40,252	35,496
Professor	57,067	45,099	36,422	44,835	35,513	34,609
Associate	48,321	40,515	36,359	39,561	34,219	29,774
Assistant	41,515	35,715	30,342			

The students created two matrices, containing the salaries of male and female Professors, respectively, in each type of school and at each level of instruction.

Subtracting the matrix of male professor salaries from the matrix female professor salaries shows that female salaries are lower than male salaries in every category. Looking at other comparisons gives the same result. The consistency of this shortfall led students to conclude that gender bias was present in educational institutions.

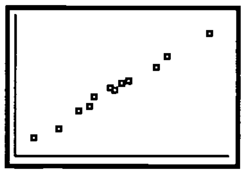


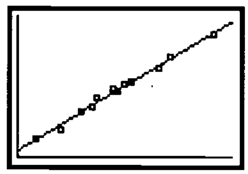


This group of students also attempted to find the relationship between salaries of the male and female professors.

The scatter plot of the data shows that there is a linear relationship between the male and female salaries, and linear regression gives the function that gives female professors' salaries as a function of male professors' salaries.

$$y = 0.8823x + 3020.51$$





Students in this class who were also taking Sociology 101 discussed gender bias in that class using the conclusions of this study.

EXAMPLE 2: ROOM PRICE AND OCCUPANCY RATE

Students who are majors in Hotel, Restaurant, and Tourism Administration who were taking both a Tourism course and College Algebra were asked to collect data that investigates the relationship between room pricing and occupancy rate at resort hotels on Hilton Head Island during the off-season. We knew that if occupancy was related to daily room price and if the resort hotels had accurately made a connection between room price and occupancy, they could set the room price so that occupancy would remain nearly constant in the off-season.

Students in the HRTA course collected the data in Table 1 below, which gives the room price and occupancy rate for 10 resort hotels during the off-seasons.

TABLE 1

Daily Room Price	Occupancy Rate (%)	Daily Room Price	Occupancy Rate (%)
110	67	119	65
120	47	79	50
100	60	89	52
120	40	69	75
115	38	39	65
75	55	39	45
65	50	45	50
70	52	62	47
44	39	67	47
38	21	36	27

As the table shows, the relationship between room price and occupancy is not as linear as might be expected; there is wide variation in the distribution, especially for rooms costing more than $100 per day. Students in the HRTA course did not reach any useful conclusions about pricing from looking at the table and scatter plot, so they sent the data to the College Algebra class.

Observing the shape of the graph over the interval under consideration led the students to use a quadratic function to model the data. The resulting model was

$y = -0.007327x^2 + 1.2865x + 0.6871$, where x is the room price and y is the occupancy rate.

They found the price (value of x) that gives the maximum value for the occupancy rate by finding the x-coordinate of the vertex of this parabola.

$$x = \frac{-b}{2a} = \frac{-1.2865}{2(-0.007327)} = 87.79$$

The maximum occupancy for this group of hotels is 57%, when the room price is $87.79 if this model is accurate for the group.

The conclusions from the college algebra course were given to the HRTA class for their consideration. They eventually raised the question of whether maximizing occupancy would necessarily maximize revenue or profit. So they suggested that students in the algebra class determine the price, if it existed, that would maximize revenue. The occupancy rate was then converted into an occupancy and multiplied by the corresponding price to get points relating room price and revenue. When this data resulted in a cubic model, it was decided that the maximum could be found graphically or that it could be found in a calculus course that many of the students were taking next semester. They graphically found that the revenue was maximized if the price was $114.25.

EXAMPLE 3: U.S. KNOWLEDGE WORKERS

WORKING WOMEN (January, 1997) states that the ratio of male to female knowledge workers-engineers, scientists, technicians, professionals, and senior managers-was 3 to one in 1983. The following table, which gives number (in millions) of male and female knowledge workers from 1983 to 1997, shows how that ratio is changing.

Year	Female Knowledge Workers(millions)	Male Knowledge Workers (millions)
1983	11.0	15.4
1984	11.6	15.9
1985	12.3	16.3
1986	12.9	16.7
1987	13.6	16.8
1988	14.3	17.6
1989	15.3	18.1
1990	15.9	18.6
1991	16.1	18.4
1992	16.7	18.6
1993	17.3	18.7
1994	18.0	19.0
1995	18.5	19.8
1996	19.0	19.6
1997	19.5	19.8

Source: *Working Woman,* January 1997

To compare how the growth in the number of female knowledge workers compares with that of male knowledge workers, the students entered the data above in lists, with the number of years from 1980 in L1, the number of female knowledge workers in L2, and the number of male workers in L3.

L1	L2	L3 1
12	16.7	18.6
13	17.3	18.7
14	18	19
15	18.5	19.8
16	19	19.6
17	19.5	19.8
	------	------

L1(16) =

The difference "male minus female" is found by using the formula "L3 – L2" after DIFF=. Note that the calculator operates as a spreadsheet if the formula is in quotes and that the values will not change if the formula is not in quotes.

L2	L3	DIFF 4
11	15.4	4.4
11.6	15.9	4.3
12.3	16.3	4
12.9	16.7	3.8
13.6	16.8	3.2
14.3	17.6	3.3
15.3	18.1	2.8

DIFF="L3–L2"

The largest difference occurs in 1983 and the differences are, for the most part, getting smaller. Because the differences are getting smaller as 1997 approaches, it appears that the number of female workers will soon equal the number of male workers.

L2	L3	DIFF 2
11	15.4	4.4
11.6	15.9	4.3
12.3	16.3	4
12.9	16.7	3.8
13.6	16.8	3.2
14.3	17.6	3.3
15.3	18.1	2.8

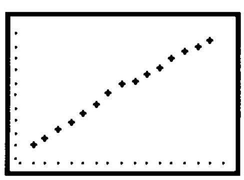
L2(1)=11

L2	L3	DIFF 4
16.1	18.4	2.3
16.7	18.6	1.9
17.3	18.7	1.4
18	19	1
18.5	19.8	1.3
19	19.6	.6
19.5	19.8	.3

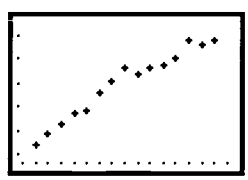
DIFF(15)=.3

To find the year in which the number of female knowledge workers equals the number of male knowledge workers, the students found functions that model the number of female knowledge workers and the number of male knowledge workers and used INTERSECT to find when the numbers are equal.

Females

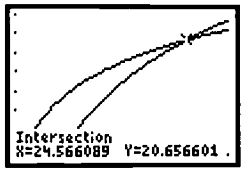


Males



$F(x) = 4.26958 + 5.11876 \ln x$ $\quad$ $M(x) = 12.11515 + 2.67076 \ln x$

The graphs of y = F(x) and y = M(x) intersect at 24.57. This indicates that the number of female knowledge workers will pass the number of male knowledge workers in 2004.

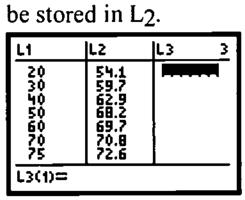


EXAMPLE 4: EXPECTED LIFE SPAN

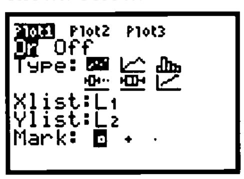
The following table shows the expected life span at birth of people born in certain years in the United States. The following steps compare different models for this data.

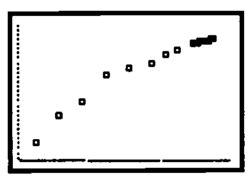
Birth Year	*Life Span (in Years)*	*Birth Year*	*Life Span (in Years)*
1920	54.1	1988	74.9
1930	59.7	1989	75.1
1940	62.9	1990	75.4
1950	68.2	1991	75.5
1960	69.7	1992	75.5
1970	70.8	1993	75.5
1975	72.6	1994	75.7
1980	73.7	1995	75.8
1987	75.0	1996	76.1

Data in the Year column can be realigned to represent the number of years since 1900, and this data can be stored in L_1. The life span data can be stored in L_2.



A scatter plot of the data is shown below.

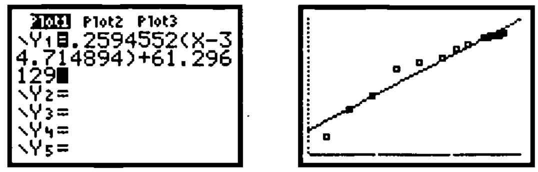




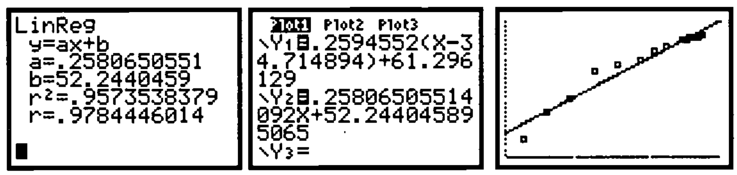
A piece of spaghetti can be used to estimate a line that is the best fit. The free-moving cursor can be used to find two points "under" this line. Example: (34.714894, 61.296129) and (84.195745, 74.134194) are two points on the visual fit line. Finding the slope of this line is used to write its equation.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{74.134194 - 61.296129}{84.195745 - 34.714894} \approx 0.2594552$$

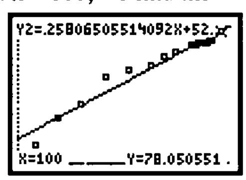
y = 0.2594552 (x − 34.714894) + 61.296129)



The built-in linear regression feature of the calculator can be used to model the linear function that is the best fit for the data.

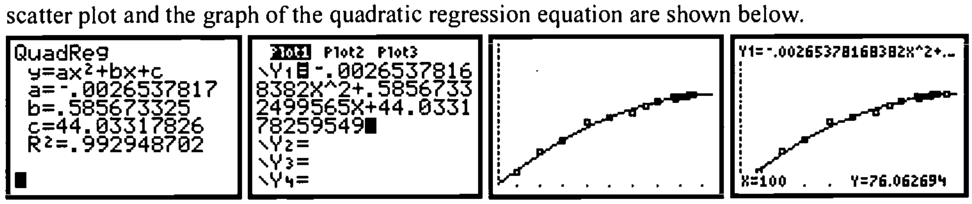


Assuming that the model applies in the year 2000, the life span of people born in the year 2000 can be predicted. If we evaluate the function at x = 100, we find the life span to be 78.



A **quadratic function** can also be used to model the life span data.

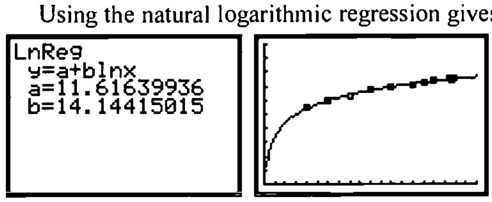
The quadratic function that is the best fit for the data is $y = -0.002654x^2 + 0.58567x + 44.03318$. The scatter plot and the graph of the quadratic regression equation are shown below.



This model can be used to predict the life span of people born in the year 2000. Evaluating the function at x = 100, we find the life span to be 76.

A **logarithmic function** can also be used to model the life span data.

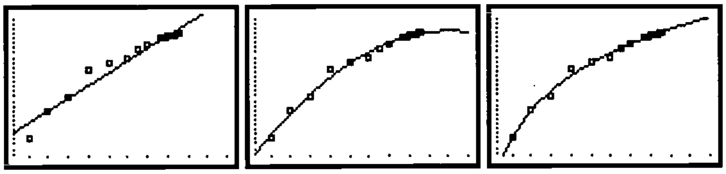
Using the natural logarithmic regression gives the equation y = 11.6164 + 14.441 ln x.



This model appears better for values of x after 20, that is, after 1920. (Distances from plotted points to each model can be checked to see which model is better.)

Evaluating this logarithmic function at x = 100 gives the predicted life span of people born in the year 2000 to be 78.1.

Recent data indicates that the expected life span for people born in the U.S. in 2000 is 76.7, so it appears that the quadratic model is the best predictor of life expectancy for the near future. The graphs below show the linear regression line, the graph of the quadratic regression equation, and the graph of the logarithmic equation, respectively, for the years through 2020.



EXAMPLE 5: TAXATION

Modeling doesn't just mean using a computer or calculator to get a regression curve from data points. It also means creating equations from available information. Consider the following real problem that requires either the creation of linear models or iteration methods for its solution.

Federal income tax allows a deduction for any state income tax paid during the year. In addition, the state of Alabama allows a deduction from its state income tax for any federal income tax paid during the year. The federal corporate income tax rate is equivalent to a flat rate of 34% for taxable income between $335,000 and $10,000,000, and the Alabama rate is 5% of the taxable state income.

Suppose both the Alabama and federal taxable income for a corporation is $1,000,000 before either tax is paid. Because each tax is deductible on the other return, the taxable income will differ for the state and federal taxes. One procedure often used by tax accountants to find the tax due in this and similar situations are called iteration and are described by the first five steps below. A second method is the direct method, which requires us to create mathematical models from the given information.

Iteration Method:

1. We first make an estimate of the federal taxes due by assuming that no state tax is due. The estimate of the federal taxes is 0.34(1,000,000) = $340,000.

2. Based on this federal tax, we can then estimate that the taxable state income is $1,000,000 - $340,000 = $666,000 giving a state tax 0.05(660,000) = $33,000.

3. Deducting this estimated state tax from the federal taxable income gives $967,000 as the adjusted federal taxable income. The federal tax on this income is 0.34(967,000) = $328,780. The state taxable income is now $1,000,000 – $328,780 = $671,220, with the state tax of 0.05(671,220) = $33,561.

4. Repeating step (3) gives

Fed Taxable Income	Federal Tax	State Taxable Income	State Tax
966,439.00	328,589.26	671,410.74	33,570.54
966,429.46	328,586.02	671,413.98	33,570.70
966,429.30	328,585.96	671,414.04	33,570.70

5. The state tax remains unchanged at $33,570.70 in the last iteration above, so this amount will not change again. And thus the federal tax will remain unchanged at $328,585.96.

Direct Solution

To find the tax due each government directly, we can create two linear equations that describe this tax situation and solve the system with graphical methods or matrix methods.

1. Let x = the federal tax owed and y = the state tax owed. Then the federal tax is given by

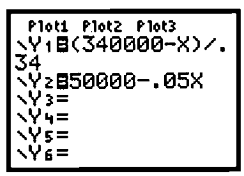
$$x = 0.34(1,000,000 - y)$$

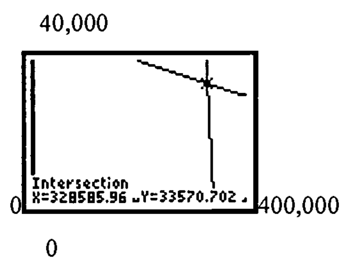
and the state tax is given by

$$y = 0.05(1,000,000 - x)$$

Solving both of these equations for y permits us to graph them on the same axes and to find the simultaneous solution graphically. The equations are

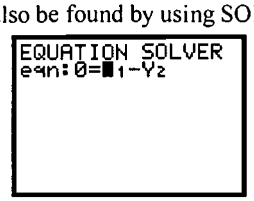
$$y = \frac{340,000 - x}{0.34} \quad \text{and} \quad y = 50,000 - 0.05x.$$

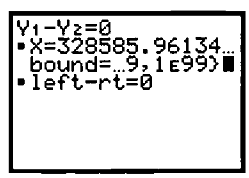




The intersect feature gives federal tax of $328,585.96 and state tax $33,570.70.

This solution can also be found by using SOLVER.

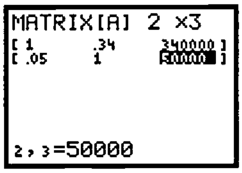


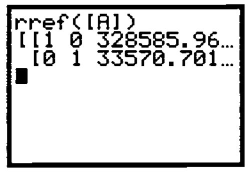


These equations can also be written in general form and solved simultaneously with matrices. The system is

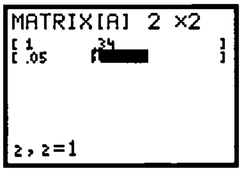
$$\begin{cases} x + 0.34y = 340,000 \\ 0.05x + y = 50,000 \end{cases}$$

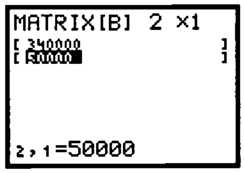
Using row reduction of the augmented matrix gives the same solution.

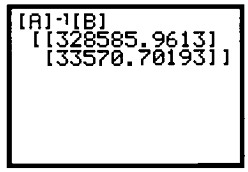




Using inverse matrices also gives the same solution.







Conclusion

Students enrolled in this course initially thought that the course would be very difficult because most of their work was with "word problems." However, they adjusted quickly, and in general recognized that the "real problems" were much more interesting than the skills check problems. Although not every algebra topic was included in the course, the level of performance on algebra skills tests was not significantly different than with traditional tests, and students had increased confidence in their math and problem solving abilities.

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LANGUAGE AS A COMMUNICATIVE AND INTERPRETIVE TOOL IN MATHEMATICAL PROBLEM SOLVING

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ABSTRACT

Discourse analysis of transcribed protocols is a very interesting field of educational research. In mathematics education, cooperative problem solving has proved to be an effective way to promote mathematical discourse. Using linguistic analysis techniques and symbolic interactionism as our theoretical framework we analysed the language that students used while they worked in pairs to solve a geometrical problem. Our analysis revealed how language was used as a communicative and interpretive tool by the participants. The use of everyday or quasi-mathematical language didn't seem to affect the establishment of shared meanings. Each student adopted a specific role during the interaction and used language to maintain it. The whole process was governed by social and sociomathematical norms which were constituted before or during the interaction.

KEYWORDS: Discourse analysis, language, problem solving, symbolic interactionism, interaction, social norms, sociomathematical norms.

1. Introduction

Verbalisation is a major component of the thinking process. Within this view, great effort is made by all educators to engage their classes in some sort of discussion. Depending on the teacher's scope and the practical limitations involved, different kinds of discussion are practised: whole class discussion, teacher-student discussion, discussion between groups or discussion within a group. Discussion is a component of mathematical discourse which is defined as "the ways of representing, thinking, talking, agreeing and disagreeing" (NCTM, 1991). One of the most effective types of discourse is the one that is produced in small groups of students engaged in what Pirie and Schwarzenberger (1988) define as mathematical discussion: "It is purposeful talk on a mathematical subject in which there are genuine pupil contributions and interaction". This kind of discussion has proved effective in all levels of education, from primary school (Pirie and Schwarzenberger, 1988) to university (Yackel, 2001). The term effective means that it either promotes students' learning and socio-cognitive development (César, 1999) or that it contributes to gaining equal-status interaction or positive intergroup relations (Cohen, 1994).

2. Theoretical framework

Symbolic interactionism is a social psychological theory developed from the work of Cooley and Mead in the early part of the twentieth century. The actual name of the theory comes from Blumer, one of Mead's students. According to Blumer (1969), this theory is based on three core principles: **meaning**, **language** and **thought**. These principles lead to conclusions about the creation of a person's self and the socialisation into a larger community (Griffin, 1997). The basic assumptions of the theory according to Mead (1934), Blumer (1969) and Yackel (2001) are:

a) People act toward things on the basis of the meanings that the things have for them.

b) Meanings are not intrinsic in things, they have to be defined before they have human reality.

c) Everything that people act upon or that has an impact upon them must go through the process of subjective meaning.

d) The meaning of a thing for a person grows out of the ways in which other persons act toward the person with regard to the thing.

e) Meaning is not merely individual and subjective, but it derives from and arises out of the social interaction.

f) Meanings are handled and modified through the interpretive process used by the person in dealing with the things he encounters.

g) Human action is created in the process of interpreting meanings.

Language gives humans a means by which to negotiate meaning through symbols. Human communication is made possible through the use of symbols (symbolic interaction). Although symbols seem to be a fixed entity, people use them in a shifting, flexible and creative manner. That process of adjustment and change involves individual interactions and larger scale features such as norms and order. Finally, thought modifies each individual's interpretation of symbols; thought based on language is a mental conversation that requires role taking or adopting different points of view.

Another notion which we found very useful in our research is **"face"** as described by Goffman (1967). "Face" is the image of the self presented; it is what the others see or consider having been expressed by the "actor". Both the actions of the "actor", and the perception and view of others, establish whether or not "face" is maintained. We analysed the kind of language that students used to maintain "face" and tried to identify some characteristics of that "face".

Discourse analysis in our research was based on the notion of the cooperative principle for the exchange of information as suggested by Grice (1989) and on the notion of context as something which is not just given as such in interaction, but it is made available in the course of it.

According to Slembrouck (2001) the cooperative principle is based on the assumption that language users tacitly agree to cooperate by making their contributions to the talk as it is required by its current stage or the direction into which it develops. Adherence to this principle entails that talkers simultaneously observe four maxims:

a) quality, i.e. make your contribution truthful and sincere,

b) quantity, i.e. provide sufficient information,

c) manner, i.e. make your contribution brief, present it in an orderly fashion and avoid ambiguities,

d) relation, i.e. make your contribution a relevant one.

There are various conditions under which these maxims may be violated or infringed upon. Our analysis did not intend to convey if these maxims were enforced or not; our aim was to observe how language was transformed by the students in order to be consistent with these maxims and if the attempt was successful or not.

The view of the context was very useful in our attempt to create a dynamic analysis, in which we sometimes had to "go back in time" in the text in order to justify our contentions.

3. Methodology

The aim of our qualitative research was "subjectively and empathetically to know the perspectives of the participants" (Jacob, 1988). The subjective character of qualitative analyses is stressed by many researchers. Lemke (1998) states quite lucidly that "It is not always possible to say what a particular choice or move means, but you can say what it might mean... Even the participants in a discourse may disagree about the rhetorical meanings of particular features, or change their minds in retrospect or with additional information."

The exploratory process of data reduction was mainly based on the assumption that everything the subjects said made sense in some ways. We intended to look for anything and everything of interest (Mitchell 2001) using different types of analyses; this is not unusual in qualitative research and an interesting example of that practice is the multiple analysis approach as described by Dekker, Wood and Elshout-Mohr (2001).

The subjects of our research were twenty undergraduate students from the Department of Primary Education of the University of Ioannina. Their ages varied from 18 to 21 years and they were asked to choose the person that they would like to work with. Thus, ten pairs were formed, seven "girl-girl" pairs and three "girl-boy" pairs. The type of task that we would assign to the students was a challenge for us. We noticed that the vast majority of research in linguistic area involves algebra problems because they allow many interpretations and solving strategies, and because they can be easily altered to produce variations of the original problem. Thus, we thought that an Euclidean geometry problem would be a challenging alternative to that tradition. The only instruction given to the students was that they should work together to solve the problem and that they should verbalise every thought they make.

The first observation we made was that the students used everyday or quasi-mathematical language in many cases. We categorised these cases according to what the purpose of that language seemed to be and then we studied the effect of it on the common understanding that the students were supposed to create.

In addition, we noticed that the students tried to use mathematical justification in many cases, although they were not clearly asked to. This was attributed to the establishment of sociomathematical norms. According to Yackel (2001) the norm is a collective notion which describes the expectations and obligations that are constituted in the classroom. Thus, our concern was to identify these norms in every episode, and then compare the findings of all the protocols.

The third and most interesting observation was that there was a clear distinction between the "roles" that were acted by each one of the two students in every couple. We studied how language was used to achieve that.

These three observations helped us to shape our research questions:

a) Does the use of everyday or quasi-mathematical language affect the common understanding of the participants?

b) Were there any observable social and sociomathematical norms constituted in the interaction?

c) How is language used by the participants to reveal their roles in the interaction?

4. A sample analysis

The following text is made of excerpts from the actual solution process followed by a pair of female students when they were given the problem: "Given a right triangle ABC and D be a point on the side BC. Let DE, DZ perpendicular to AB, AC retrospectively. Draw the line segment EZ and locate the position of D so as the length of EZ to be minimal". The two students are marked as "A" and "B" and the researcher as "K". Researcher's notes are in brackets[1].

Once the two students had read the problem, they made figure 1 (see Appendix). Then the following discussion took place:

10. B: What are we going to do now?

11. A: Now, I guess, we have to find the position of D on BC, so that we can find the minimum possible distance of ZE, of EZ. The straight line is the shortest way, isn't it? What if we draw a vertical line?

12. B. Vertical to which?

13. A: To ZE from D?

14. A: Shall I try putting D in a lower position?

15. B: On another figure?

16. A: On the same one.

(A draws D′E′, D′Z′ and compares ZE with Z′E′).

17. B: Is it shorter now?

18. A: It's nearly the same though.

19. A: If we put it in a higher position, would it be larger?

20. B: How?

(A suggests a point near C).

21. A. What if we put D here? It would be larger.

22. B. No.

The two students continue measuring ZE's length for various positions of D.

(A draws KP, KL).

31. A: It's larger now.

32. A: What if we draw a vertical line from D to ZE?

[1] The original text of the problem and of the transcriptions of students' spoken interactions is in Greek.

(B reads the last sentence of the problem).

33. B: Maybe it's somewhere here, in the middle, because this one and the other distance that we found earlier is greater compared to that one.

34. B: If we put D in the middle of BC, wouldn't that distance be smaller?

35. A: What if we draw something like a square? And put D at its center where its diagonals intersect? Do you understand what I'm saying?

36. B: Yes.

(A draws ABIC, but figure 1 has become too complicated, so the researcher suggests that they should draw a new figure. So A draws figure 2).

39. A: Shall I draw a square again?

40. B: Yes.

(B points at D in figure 2)

41. B: This has the minimum distance.

42. A: But we don't know it.

43. B: And those we had made before, below and over the middle, had the same distance, isn't that so?

44. A: We have now made a small square and one of its diagonals which is EZ is the minimum distance.

45. A: We took the center of the triangle's hypotenuse. Why did we take the center?

46. A: What if we put numbers at the sides?

47. B: This must be it.

The students felt that they were at a dead end, so the researcher decided to intervene. He explained to the students that the given triangle is not isosceles and that, in case it was isosceles, the middle of its hypotenuse would be the point that they were looking for. The students then drew figure 3 and compared the length of EZ for various positions of point D.

89. A: What if we use a formula?

90. B: Which formula?

91. K: Which formulas do you know?

92. A: We don't remember any.

93. K: You don't have to use any formula.

94. B: Do we have to prove it too? Can it be proved?

According to our research questions, the analysis of the protocol consists of three parts:

a) The use of ordinary and quasi-mathematical language is obvious throughout the whole excerpt. Examples of that use are the words "way" (11), "lower" (14), "higher" (19), and the expressions "like a square" (35), "small square" (44), "center of the triangle's hypotenuse" (45), "put numbers at the sides" (46).

The word "way" is included in the expression "the straight line is the shortest way" which is very common in Greek education although it is not considered formal mathematical language. (This is an example of contextual analysis, where the word is analysed with respect to its surrounding context).

The words "lower", "higher" and the expressions "small square", "put numbers at the sides" are examples of the enforcement of Grice's maxims for the exchange of information. A mathematician would replace the word "lower" with the expression "towards point B" or "near point B" (see figure 1). But the enforcement of the maxim of manner made student A use the least words possible without violating the maxim of quality: everybody present at the interaction understood what A was talking about. The same is the case with the other words and expressions.

The expression "like a square" reveals a shared meaning that is constructed in the course of the interaction. The students had drawn an isosceles triangle, although this was not the case, and that fact is revealed in 35. One might argue that figure 1 gave enough evidence for that; but once the word "square" is uttered, the notion of the isosceles triangle is constituted. In 39 student A says "Shall I draw a square again?" and the word "again" is what Gumperz (1999) calls a contextualisation cue, that is a word the role of which is to connect the sign (in our case the square) with its context.

b) An interesting feature of our research was to find out if this setting will constitute any social or sociomathematical norms. Previous research in this field was primarily conducted in classroom settings, over a long period with continuous assistance and guidance by the teacher. Our protocols provided some stimulating examples of social and sociomathematical norms. In this particular episode we can find two such cases. The first is in 35 where student A, after having expressed her proposal, asks student B: "Do you understand what I'm saying?". We can say that she accepts that she has to explain, if necessary, her proposal to student B, which constitutes a social norm. The second case is in 94 where student B asks the researcher: "Do we have to prove it too? Can it be proved?". Here, the norm of what is mathematically efficient doesn't seem so stable; the student wonders when she asks if it can be proved, but on the other hand her question means that she is aware of what mathematical proof is about.

c) Finally, we face the most interesting phenomenon in our research: the role playing. In all the protocols that we analysed, each student played a distinct role which was evident during the whole discussion. In the particular episode, student A played the role of the initiator – the one who proposes tasks or procedures – and the way she did it was through questions. Consider these: "Shall I try putting D in a lower position?" (14), "If we put it in a higher position, would it be larger?" (19), "What if we put D here?" (21), "What if we draw a vertical line from D to ZE?" (32), "What if we draw something like a square? And put D at its center where its diagonals intersect?" (35), "What if we put numbers at the sides?" (46), "What if we use a formula?" (89). Using a symbolic interactionist perspective, student A interpreted the original instruction "to verbalise every thought they make" by taking the role of the initiator. Student B on the other hand, interpreted her fellow student's questions as proposed solution strategies, but her reactions were different in each case. She either chose to accept the proposal (see the next lines of 14, 19, 35), reject it (see the next line of 21) or ignore it (see the next lines of 32, 46, 89).

The cross-examination of the protocols confirmed that the use of ordinary and quasi-mathematical language exceeded the use of formal mathematical language. The students were reluctant to use sophisticated mathematical expressions, but this did not prevent them from creating and handling the shared meanings that were necessary to deal with the problem. The cross-examination also helped us to clarify the social and sociomathematical norms that were constituted. Examples of these norms are that each student had to justify her thinking, listen to her partner and try to make sense of her thinking, and realise the need for mathematical justification whenever it was possible. Finally, we noticed that there was little difference between the roles that were acted by the students in all episodes. In every single pair, one of the participants proposed new ideas and the other one evaluated them. Each student, using language, tried to maintain her "face" throughout the interaction. In very few circumstances this role playing was reversed.

5. Concluding remarks

The linguistic analysis of the protocols has shown that despite the fact that students used everyday or quasi-mathematical language, they had no difficulty in creating shared meanings and understandings. This creation was not uncontrolled; social and sociomathematical norms defined the behavior – namely the language – of the participants. These norms were the same as the ones which were observed in other studies, but the interesting part was that in our case there was no previous preparation or assistance during the process. The analysis was also based on the symbolic interactionist perspective using elements from interactional sociolinguistics. From this point of view, we noticed that each participant interpreted differently the norms that were established before or during the interaction; this resulted in a role playing which proved to be extremely important for the flow of the interaction because it helped the participants to handle the discourse and cooperate in it.

Multiple analysis approach proved to be very helpful in our attempt to clarify some aspects of the interaction that takes place in cooperative problem solving. But it also raised some new questions: Is there any way for the teacher to establish all the necessary norms from the beginning of the interaction? What is the evolution of these norms in time? What makes one student adopt a specific role? Beyond these questions we can see the most important implication of our analysis: the mathematics educators must be very careful in handling the language that they and their students use in problem solving. Making the right move at the right moment can help the teacher achieve the expected cooperation and make the interaction effective.

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APPENDIX

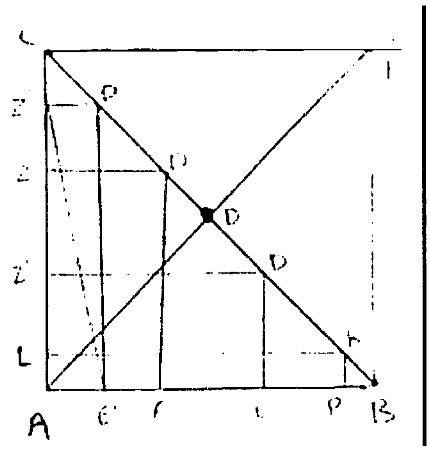

Figure 1

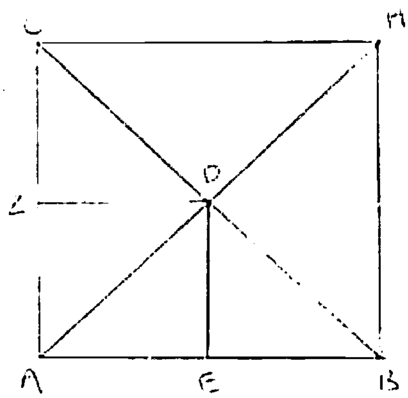

Figure 2

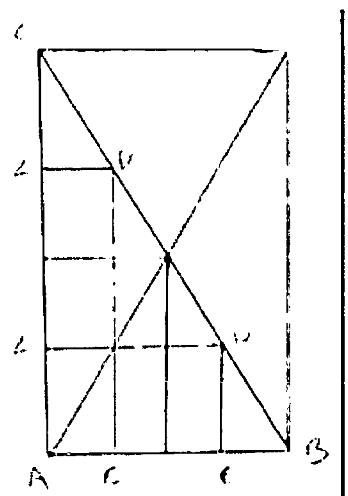

Figure 3

CRYPTOGRAPHY AND STATISTICS: A DIDACTICAL PROJECT

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ABSTRACT

Cryptography is a stimulating way to introduce and consolidate ideas in statistics, computational linguistics, combinatorics and modular arithmetic. Two of the authors have been carrying out didactical experiences starting back in 1989 at a primary school level, without any special technology. A game is set up which involves cryptographers and cryptanalysts. Simple substitution ciphers are broken by building letter frequency hystograms by parallel work, so as to achieve what is being felt as statistical significance. Pupils quickly discover Markov models and the slight non-stationarity of the linguistic process. We have initiated a new round of experiments at a different level of age, 14-16, and technology. We take advantage of computer software to deepen our analysis of cipher systems and Markov models. The friendly (and cheap) technology of graphing calculators is used to analyse perfect and pseudo-perfect ciphers and to discuss the elusive notion of randomness.

Keywords: cryptography, statistics, mathematics education, maximum likelihood, Markov processes, randomness.

1. Why cryptography?

A blunt answer to the question posed in the title might be: because cryptography is a charming and rewarding way to introduce into the classroom subjects of traditional or less traditional mathematics, algebra, modular arithmetic, computational linguistics, combinatorics, algorithms, and more specific to our point, statistical estimations and statistical tests. It is not just a matter of gratifying one's audience for fun's sake: a solid reason to use cryptography is its being an effective response to the decrease of logical abilities which has been observed in students entering university. Many ways out have been suggested, based e.g. on algebra or Euclidean geometry (cf Mammana and Villani 1998): the approach taken is then the traditional approach of axiomatic deductive theories, which, unfortunately, is not always appropriate or exciting from the point of view of pre-college students, especially when they are too young, or when they are more technical-oriented. Instead, cryptography stimulates the problem-solving skills of the pupils and enhances their argumentative abilities, in a way, which is directly linked to the "soft" logic of natural languages; cryptography is (perceived as) a *game*, but it is a motivating and sophisticated logical game! Actually, our experimentation has shown that cryptography may be introduced in classrooms at a very early stage, even at a primary school level (cf Section 3; cf also Zuccheri 1992, Leder et al. 2001); kids spontaneously formulate conjectures and develop arguments to prove or disprove them. An additional point in favour of cryptography (cf Section 4) is that it is ideally suited to make clear the advantage of an empirical approach to mathematics, pursued in a *math laboratory*, where one can use both hand calculators and full scale computers, according to the case; actually the computations involved can be quite lengthy with paper and pencil only, or even infeasible. Last, we think that, nowadays, cryptography by itself should be part of everybody's culture. In the age of the Internet and of the dramatic privacy and security problems it poses, one should understand the difference between trivial tactical aids like passwords, and professional strategic systems, as are DES (Data Encryption Standard) and RSA (so called from the names of its inventors, Rivest, Shamir and Adleman). Security is no longer a prerogative of the secret services.

Our team includes two persons active in cryptographic research and in mathematics education research (A.Sgarro and L. Zuccheri, respectively) and two teachers in charge of the class project of Section 4 (M. Borelli and A. Fioretto).

2. Ideas of cryptography, from the Bible to the web

This instant history of cryptography is used to introduce some of its basic notion; observe, however, that historical hints can be presented in the classroom to add some flavour to the technical material of Sections 3 and 4 (the standard reference to the history of cryptography is still Kahn 1967; cf also Sgarro 1989; as a reference to modern cryptography we suggest e.g. Schneier 1994).

Cryptography (i.e. *secret writing*, in old Greek) is nowadays felt as a part of computer science, and also as a part of our daily life, used as it is to protect the privacy of on-line transactions: and yet, it has always been with us. The simplest and possibly the oldest type of cipher, called a *simple substitution cipher*, appears already in the Bible. When such a cipher system is used, a permutation of the alphabet is chosen to be the *key* of the cipher; in practice one has two matched orderings of the alphabet, the normal ordering and a permuted ordering. The *clear text* is enciphered by substituting each letter as specified by the key. Breaking such a system is quite easy when the cipher text (the *cryptogram*) is long enough: in practice 25 to 30 letters will do. How to accomplish all this is masterly explained in *The golden bug*, a tale written by Edgar Allan Poe. In a natural language letters have a typical occurrence frequency (E appears 12% of an English clear text, say); this typical frequency is "inherited" by the corresponding cipher text letter, and so, after a few trials and some semantic aid, the cryptogram can be broken. The underlying method, called *maximum likelihood*, is typical of statistics, and was well known in the old Arab world: to this end the Aristotelian philosopher Al Kindi had prepared an accurate statistical description of the Arab language, obtained by sampling part of the Qur'an. Long forgotten in Europe, cryptography was re-born in Italy during Renaissance, but the lessons of the Arabs had been learnt, and it was well understood that a good cipher system should be able to "cheat" statistics. One of the ways out which were adopted was *polyalphabetic ciphers*. Initially supported by theoreticians rather than practitioners, these cipher systems took the lead in the 19th century, their implementation being now obtained by use of mechanical devices, so as to get rid of the synchronisation problems which had marred polyalphabetic ciphers in the age of paper-and-pencil cryptography. In a polyalphabetic cipher *several* permutations are selected (their number is called the *period of the cipher*), and they are used *in turns* according to a fixed scheme. As a rule, each permutation is very simply a *rotation* of the alphabet, and so it is completely specified by the substitute of clear letter A (if A is substituted by D, say, then B is substituted by E, C by F, and finally Z by C; observe that one is simply making sum modulo 3, as soon as one thinks of the letters as numbers: A=0, B=1, C=2, D=3, etc; sum is performed letter by letter, with no carry-over). This way, the very same clear text letter is enciphered by different permutations, and so has different substitutes in the cipher text. In the Renaissance, two concentric wheels were used to implement a polyalphabetic cipher; sometimes the cipher alphabet was a fancy one. Later, electro-mechanical machines, based on a cute system of rotors, allowed one to obtain extremely long periods, so safeguarding the cipher text from the cryptanalytic techniques, which were developed at the end of the 19th century by Friedrich Kasiski, a German officer. Such machines were still in use during the Second World War: an example is the notorious Enigma, adopted by the Germans and broken by the allied secret services. Substitution and polyalphabetic ciphers still exist nowadays in "extreme" forms. As for substitution ciphers, they are part of *composed* ciphers, as is DES, the *Data Encryption Standard* widely used in commercial cryptography. In a composed cipher the clear text is enciphered many times in series, and in different ways: in DES one alternates substitutions and *transpositions*, i.e. anagrams. The "asymptotic" version of a polyalphabetic cipher is called a *one-time pad*: in it the key is a totally random and potentially infinite sequence, which is summed to the clear text. Usually, the random sequence is binary and the sum is bit-by-bit sum modulo 2 (1+1=0, no carry-over), the clear text being itself binary, because it has been preliminarily encoded by means of ASCII, say. (Note that ASCII is *not* a cipher, but simply a transcription code, widely used by computer people to convert information to binary). Already the late Claude Shannon had shown that *the one-time pad is perfect*, i.e. provably unbreakable. In practice, the one-time pad needs too much key material, and so genuinely random sequences are replaced by more convenient pseudo-random sequences (cf Section 4). Unfortunately, a "pseudo-perfect" cipher is no longer unbreakable; actually, this is possibly the only example when the standard software used to generate random digits has proved to be sorely insufficient. Nowadays, besides commercial ciphers as DES, or sophisticated pseudo-random ciphers as used by the militaries, *public-key cryptography* has entered the lists. This is a revolutionary approach, which is based on the theory of algorithmic complexity; for example, the intolerable complexity of factoring integers is made good use of by a cipher system called RSA, which is widely used to safely transmit DES keys along the web.

3. Cryptography in primary schools: an exciting didactical experience

Our experimentation has been carried out during several years. It began back in 1989, in schools of North-East Italy, with approximately 300 pupils aged 7 to 10, and has continued up to the present date, due to the enthusiastic response of pupils and school teachers alike; cryptography is experienced as an exciting *game* (cf Zuccheri 1992 and Leder et al. 2001). Work in the classrooms consists of *cryptographic* and *cryptanalytic* activities (building and breaking ciphers, respectively), based on secret messages sent by encryptors (cryptographers) and intercepted by decryptors (cryptanalysts). We use substitution ciphers; cf Section 2; cryptanalysts end up "re-inventing" statistical inference, and in particular the principle of *maximum likelihood*; the basic underlying notion is *relative frequency*. The tools used are limited to zero-level technology, i.e. to paper and pencil. Initially, we use rotation ciphers only, and keep for simplicity the spacing between words; pupils readily find out cute "tricks", semantic rather than statistical, to guess the key (*rotation ciphers* are the simplest form of substitution ciphers, since the only permutations allowed are those obtained by rotating the alphabet; cf Section 2). Encryptors soon perceive that they should make life harder to decryptors. One moves to general substitution ciphers. Permutations easily memorised are based on a secret motto: one writes down the motto by dropping repeated letters, and adds all the lacking letters in the reversed order (so SMALL IS BEAUTIFUL becomes SMALIBEUTFZWV...DC); however, pupils prefer to use special alphabets for the encrypted text, e.g. the numbers from 1 to 26, and so the permutation has to be written down (an unwise policy, actually...). After a short training, we take out word spacing; pupils work on encrypted texts of approximately 300 characters each. To break the cryptogram, they begin by counting the letter frequencies of a clear text of approximately 1000 words and build up hystograms by "parallel work", so as to achieve what is being felt as "statistical significance". Pupils compare their results with standard tables of frequencies. Now they are ready to successfully apply maximum likelihood, aided by their semantic competence. Pupils go as far as discovering some basics of Markov models (e.g., in Italian, letter Q is always followed by letter U, except in the unruly word SOQQUADRO, which incidentally means, disorder, "unruliness"); they quickly realise that the linguistic stochastic process is a slightly non-stationary, especially at the incipits. Polialphabetic or homophonic substitutions can be pointed out as clever tricks to "cheat" statistical cryptanalysis (in a *homophonic cipher* the cipher text alphabet is made up of many fancy letters, 50, say - fancy letters, incidentally, can be fun in themselves - and each clear letter is given *many* possible substitutes; this way the frequency of each clear-text letter is "spread" in the cryptogram among its possible substitutes, homophonic ciphers were in use up to the age of Napoleon). Pupils construct their own enciphering devices, rotating wheels and sliding rules (the latter are quite easy to make out of cardboard paper: one writes once the alphabet on the fixed strip, and twice on the sliding strip; in a way, one "linearises" the rotating wheels). On the way, the teacher has a chance to illustrate notions as one-to-one mappings, inverse mappings and modular arithmetic.

4. From paper and pencil to calculators and computers.

In school year 2001-2002 we have extended our experimentation to a different range of age, 14 to 16-year old students attending a technical school. Two classrooms have been involved; in the first, we are simply extending and deepening the material of Section 3.

• Substitution ciphers

This part of the project makes use of full-scale computers provided with standard software to re-take the ciphers of Section 3. Histograms as in Section 3 can be now be built in a more sophisticated way; statistical significance and converging of relative frequencies to their "asymptotic values", i.e. to *probabilities*, can be made quite explicit by the support of graphics. Actually, one can construct typical frequency tables also for couples and triples, sampling large texts already available in the computer memory (our tables are arrays for single letters, matrices for couples, and dynamic lists for triples: actually, most triples are never encountered in a natural language text). These tables can be used to simulate "statistical" Italian (or English) of the 1st, 2nd and 3rd order. In the latter case one produces a sequence as ALLESTRORAMIA...; even such a short chunk contains genuine Italian words, as ESTRO (= gad-fly, and also: inventive whim), and ORA (= hour). These "texts" are meaningless, but one can soon discriminate between English and Italian, say. Application of the principle of maximum likelihood by itself leads to phoney Italian (or English) of this type, the final touch pertaining to semantics. On the way, the teacher has the chance to introduce some combinatorics; e.g. the number of keys that are available in a simple substitution cipher (the number of ways one can permute the natural alphabet) is a nice way to introduce factorials and the factorial growth.

The experimentation in the second classroom is more taxing. This is done in co-operation with the Association for the Didactic with Technology, the Italian branch of T^3, Teachers Teaching with Technologies: the friendly technology of graphing calculators can help the teacher to set up a sophisticated *math laboratory* in the classroom, for a wide range of school levels up to university, in a cheap and handy way. Such math laboratories have been introduced at an undergraduate level (cf Invernizzi et al. 2000). In particular, the program covers Monte Carlo methods and simulations by means of random digits: this is directly linked to the present project, in which cryptography is used to teach and consolidate statistical notions as are randomness and testing; we take advantage of the powerful tools for manipulating data lists, which are available in graphing calculators.

• (Pseudo)-random digits

The idea is to simulate a binary one-time pad; cf Section 2. Cryptographic theory teaches us the following: if the binary key sequence is genuinely random - is obtained by tossing a fair coin - so is the cipher text sequence, and, what is more surprising, the resulting cipher cannot be broken: the latter statement is a rigorous theorem, not just wishful thinking! Unfortunately, generating long random sequences is extremely inconvenient, and so one is tempted to resort to convenient *pseudo*-random sequences, generated by the calculator (or by the computer), as normally done in similar cases. Since graphing calculators essentially perform operations on numbers, it is better to use a "numerical alphabet", rather than using the natural one: so, the clear-text must be preliminarily encoded, e.g. by ASCII, a standard code, which, we stress it, has nothing to do with secrecy. To this end we have developed computer software which converts normal texts to a numerical form, and which can be used by the students to feed the encoded (but not yet enciphered) text to the graphing calculator, so as to form a clear text list. Random binary digits or, rather, *pseudo*-random binary digits, can be generated by the RANDOM function of the graphing calculator (suitably modified), so as to form a further list, which will contain the bits of the key sequence. The students encipher the message by summing the two lists; one uses bit-by-bit sum modulo 2 (without carry-over), i.e. *xor* logical sum. The output sequence (i.e. the cryptogram) is itself random-looking, like the key sequence; however, this "feeling" should be put to test.

• Statistical tests

One has to find a way for testing randomness. This can be done in a naive fashion by checking the occurrences of 0's and 1's in the list, or the occurrences of couples (00, 01, 10, 11), or the occurrences of triples (000, ... , 111). All this is easily accomplished on the calculator, by running a suitable cycle over the tested sequence. At a more sophisticated level, one can use the χ^2 test (goodness-of-fit), which is available on the calculators we are using. This way, one shows that the key sequence and the cipher-text sequence are indistinguishable from genuine coin-flipping sequences, at least from the point of view of statistical tests (these sort of statistical checks for randomness are generally considered to be enough in the general context of simulations, cf Knuth 1981). This concludes the technical work in the classroom; however, the teacher provides a "historical" addendum, to show that cryptography is special indeed: in cryptography one should never overlook the difference between a genuine coin-flipping sequence and a random-looking sequence generated by a cute *deterministic* algorithm like the one implemented in the calculator, even when this algorithm is considered to be quite good in the general context of simulation, since it has proved to be able to "cheat" standard randomness tests. Actually, cryptographers have proved that ciphers like ours, which rely on standard deterministic algorithms to generate the random-looking key, are quite insecure, at least from the very severe point of view of *strategic* cryptography. More specifically, they are extremely weak against attacks of a special type, when the cryptanalyst gets hold of some clear text matched with the corresponding cipher text (the clear text might be his own, e.g. because he was permitted to operate the enciphering machine for a short while); this is enough to reconstruct the key-generating function, and so to impersonate the legitimate user indefinitely; cf Schneier 1994, or Sgarro 1986. Good pseudo-random ciphers require generation programs, which are extremely sophisticated, and are sometimes classified military material.

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HOW TO FIND THE INTERNAL ANGLE OF A REGULAR POLYGON: STRATEGIES OF PRE-SERVICE TEACHERS

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ABSTRACT

The task of finding a regular polygon internal angle can be explored by students from middle school to college and beyond. This task can be investigated in many different ways from which it is possible to learn more about various properties of basic geometrical shapes such as triangles, quadrangles, and regular polygons. The study presented here is aimed at characterizing the solution strategies of pre-service teachers that were asked to find the internal angle of a given regular polygon. In this paper we describe the different solution strategies given by the pre-service teachers and discuss the contribution of the whole class discussion to the learning process. Since the different solution strategies all shared some common features, we suggest that this task could promote mathematical generalization.

Introduction

Finding a regular polygon internal angle is an issue to explore for students from middle school to college and beyond. Various studies investigated different aspects of regular polygons such as: connections between the number of polygon sides, angles and area (Battista, M., 1985; Waters, W. M., Jr.,1987; Killgrove, R. B. and Koster, D.W., 1991); Connections between a regular polygon sides' length and the length of its diagonals (Tzamir, Tirosh and Stavi, 1997); Construction of regular polygons, and their internal angles aided by a ruler and caliper (Austin and Austin, 1979; Benson and Borrkovitz, 1982); Construction of different regular polygons by joining squares, corner to corner (Muscat, 1992); Connection between regular polygon and its central angle (Happs and Mansfield, 1992); Connection between the n-sided regular polygon area inscribed in a circle and the circle' area, as n approaches to infinity (Kich, 1979), and so on.

Researchers examined also possible connections between the number of polygon sides and the value of its related internal angle. Troccolo (1987) presented a method for accurately constructing regular polygons with a given specified side length. This method is based on the idea of inscribing the regular polygon in a circle, dividing it into triangles and finding the base angle of each central triangle.

While engaging in activities connected to mathematical definitions, Borasi (1992) used the task of finding the internal angle of a regular pentagon inscribed in a circle. The students were given two hints related to circle properties (i.e., equal radii, central angle of 360^0). Using these given hints, the students divided the pentagon into five congruent isosceles triangles, first found the central angle, then the pentagon internal angle. In another activity related to the connection between polygon sides and its internal angles, Borasi (ibid.) asked students to define a polygon. Their definitions were based on the theorem "In an n-sided polygon, the sum of the interior angles measures $(n-2)*180^0$" (p. 45).

The present study focuses on a variety of strategies given by pre-service teachers while trying to find the internal angle of a regular pentagon, and to generalize it to n-sided regular polygon.

The study

The aim of the current study was to characterize strategies of pre-service teachers that were asked to find the internal angle of a given regular polygon.

Forty-two pre-service teachers participated in the study. The participants took part in a two-hour workshop dealing with regular polygons. Each student was given two tasks:

(i) What is the internal angle of a regular pentagon?

(ii) What is the internal angle of an n-sided regular polygon?

During the first part of the workshop, the participants were asked to reply to the above questions individually. In the second part of the workshop there was a full class discussion, based on the written reports.

Findings

First task – individual work

In order to find the internal angle of a regular pentagon, nine different strategies were raised by the participants.

Strategy (a): using the fact that a regular pentagon can be inscribed in a circle whose center is the center of gravity of the pentagon, many participants divided the regular pentagon into five isosceles triangles (as shown in Fig. 1 below). They found β ($\beta = 360^0/5$), and calculated the regular pentagon angle α based on the isosceles triangle ($\beta + \alpha/2 + \alpha/2 = 180^0 \Rightarrow \alpha = 108^0$).

Strategy (b): the regular pentagon was divided into three triangles by drawing two diagonals from one of the pentagon vertex (Fig. 2). The pentagon angle α was found by calculating the sum of the three triangles ($5*\alpha = 3*180^0 \Rightarrow \alpha = 108^0$).

Strategy (c): the internal angle α was found by using the fact that the external angle of a regular pentagon is $360^0/5$, and hence the internal angle α equals 180^0 - $360^0/5$.

Strategy (d): the internal angle α was found by using the formula $\alpha = [180^0(n-2)]/n$ and by assigning n = 5, α was found.

Strategy (e): the internal angle α was found by dividing the regular pentagon into five triangles and connecting the pentagon vertices to an added reference point (H) inside the pentagon (Fig. 3). The pentagon angle α was found by calculating the difference between the sum of the five triangles and the inner angle H ($5*180^0 - 360^0 = 5*\alpha \Rightarrow \alpha = 108^0$).

Strategy (f): dividing the regular pentagon into a quadrangle and a triangle by drawing one diagonal from one of the pentagon vertex as shown in Fig. 4. The pentagon angle α was found by calculating the sum of the quadrangle and triangle angles ($5*\alpha = 360^0 + 180^0 \Rightarrow \alpha = 108^0$).

Strategy (g): this strategy involved dividing the pentagon into two quadrangles (as shown in Fig. 5). The pentagon angle α by calculating the difference between the sum of the two quadrangle's angles and the straight angle k ($2*360^0 - 180^0 = 5*\alpha \Rightarrow \alpha = 108^0$).

Strategy (h): division of the pentagon to two overlapping trapezes (ABCD and DEAB as shown in Fig. 6). The pentagon angle α was found by calculating the difference between the sum of the two trapeze angles and overlapped triangle (ADB) ($2*360^0 - 180^0 = 5*\alpha \Rightarrow \alpha = 108^0$).

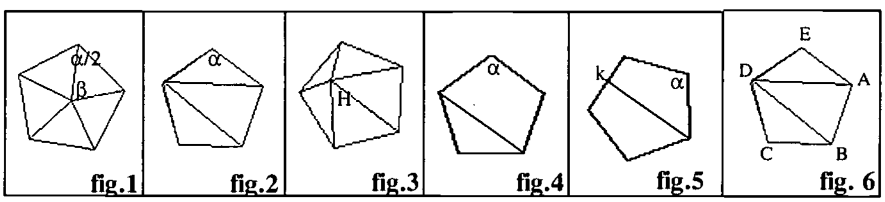

fig.1 **fig.2** **fig.3** **fig.4** **fig.5** **fig. 6**

Strategy (i): By listing the known factors (as shown in Table 1), and using the arithmetical series properties' the internal angle α was found.

The regular shape	sum of angles	Inner angle
Equilateral triangle	180^0	60^0
Square	360^0	90^0
Pentagon	*540^0*	*108^0*

Table 1: strategy (i)

Table 2 illustrates the distribution of the strategies used by the pre-service teachers while solving the first task.

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strategy	*(a)*	*(b)*	*(c)*	*(d)*	*(e)*	*(f)*	*(g)*	*(h)*	*(i)*	***Total***
Num. of students	17	12	1	5	1	6	1	1	1	**45***

Table 2: Distribution of strategies for the first task

Second task – individual work

The participants used four different strategies to find the internal angle of an n-sided regular polygon.

Strategy (1): using the fact that a regular polygon can be inscribed in a circle whose center is the polygon's center of gravity, the internal angle of n-sided regular polygon was expressed by finding the central angle β ($\beta = 360^0/n$), and calculating the polygon's angle α using the isosceles triangle ($\beta + \alpha/2 + \alpha/2 = 180^0 \Rightarrow \alpha = 180^0 - 360^0/n$) (Figure 7).

Strategy (2): the regular polygon was divided into n-2 triangles by drawing n-3 diagonals from one of the polygon vertex (Fig. 8). The polygon angle α was expressed by the equation $n*\alpha = (n-2)*180^0$ (meaning, $\alpha = 180^0(n-2)/n$).

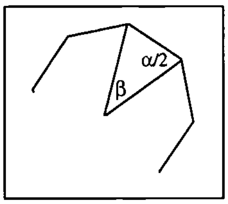

Figure 7

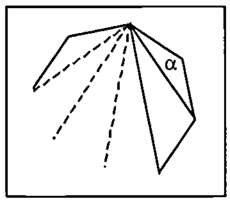

Figure 8

Strategy (3): using the fact that the external angle of a regular polygon is $360^0/n$, and hence the internal angle α equals $180^0 - 360^0/n$.

Strategy (4): stating the formula $\alpha = [180^0(n-2)]/n$ in order to express α.

Table 3 illustrates the distribution of the strategies used by the pre-service teachers while solving the second task.

Strategy	*(1)*	*(2)*	*(3)*	*(4)*	***Total***
Num. of students	25	7	1	9	**42**

Table 3: distribution of strategies for the second task

Classroom discussion

At the beginning of the classroom discussion each participant presented the strategies he used to find the pentagon internal angle. When introducing strategy (e), in which a reference point H was constructed inside the pentagon, one of the participants proposed checking a case in which H falls on one of the pentagon sides (Figure 9). The participants were asked to explore this particular case and they arrived at the solution that the pentagon angle α can be found by calculating the difference between the sum of the angles of four triangles and the straight angle H ($4*180^0 - 180^0 = 5*\alpha \Rightarrow \alpha = 108^0$).

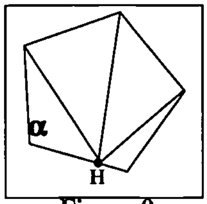

Figure 9

While discussing the former case, two questions were raised: could the internal angle of a regular pentagon be found if the reference point H falls outside the pentagon? Would it give the same solution regardless of the location of H, or would there be a different solution for each location? At this stage the participants were asked to work with the *'Geometry Inventor'* computerized program in order to explore the different cases.

Investigations within the environment led to the conclusion that this case includes five different sub-cases (Fig. 10):

(1) H falls in area A.

(2) H falls in area B.

(3) H falls in area C.

(4) H falls on one of the segments DI or EI.

(5) H falls on one of the rays IG or IF.

(6) H falls exactly on I.

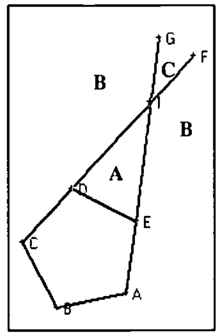

Figure 10

Figure 10.1 below describes the first sub-case. In this case the internal angle α of a regular pentagon can be found by calculating the difference between the sum of the angles of the four triangles (HDC, HCB, HBA, and HAE) and the angles of the triangle that partially overlaps them (HDE) as follows: $4*180^0 - 180^0 = 5*\alpha \Rightarrow \alpha = 108^0$.

Figure 10.2 describes the second sub-case. In this sub-case the internal angle α of a regular pentagon can be found by calculating the sum of the two triangles angles (HDC and HCB) and the angles of the concave quadrangle (HBAE) minus the angles of the triangle that partially overlaps them (HDE) as follows: $2*180^0 + 360^0 - 180^0 = 5*\alpha \Rightarrow \alpha = 108^0$.

Figure 10.3 describes the third sub-case. When the internal angle α of a regular pentagon can be found by calculating the sum of the two concave quadrangle angles (HDCB and HEAB) minus the angles of the triangle that partially overlaps them (HDE) as follows: $2*360^0 - 180^0 = 5*\alpha \Rightarrow \alpha = 108^0$.

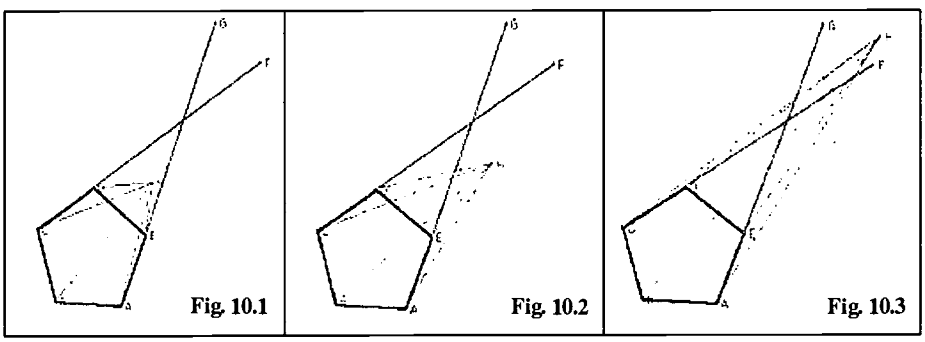

Fig. 10.1 Fig. 10.2 Fig. 10.3

Figure 10.4 describes the forth sub-case. In this instance the internal angle α of a regular pentagon can be found by calculating the sum of the angles of the three triangle's (HCB, HBA and HAE). One can see that $3*180^0 = 4\alpha + \angle DHE + \angle HED$, but since $\alpha = \angle CDE = \angle DHE + \angle HED$ ($\angle CDE$ is an external angle to triangle DHE), $3*180^0 = 5*\alpha \Rightarrow \alpha = 108^0$.

The fifth sub-case is described in Figure 10.5 where the internal angle α of a regular pentagon can be found by calculating the sum of the triangle and the concave quadrangle angles (HCB and HBAE).

Then, $180^0 + 360^0 = 4\alpha + \angle DHE+\angle HED$, yet again $\alpha = \angle CDE = \angle DHE+\angle HED$ and hence $180^0 + 360^0 = 5*\alpha \Rightarrow \alpha = 108^0$.

In case H falls on I (the intersection of DF and EG as shown in Fig. 10.6) two triangles are formed (HAB, HCB) $\angle A+\angle B+\angle C+\angle DHE = 360^0$. Assigning $\angle DHE = 2\alpha-180^0$ in the above equation will result in $5\alpha - 180^0 = 360^0$ from which $\alpha = 108^0$.

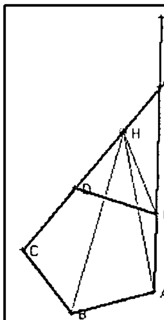

Fig. 10.4

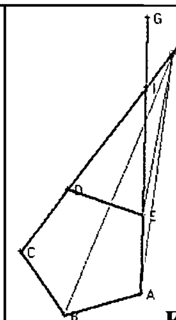

Fig. 10.5

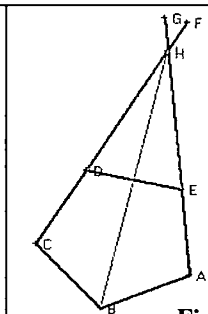

Fig. 10.6

As was mentioned earlier, in the second task, (finding the internal angle of an n-sided regular polygon) most of the participants used strategy (a) to solve the general case, while the others used strategies (b), (c) or (d). Classroom discussion yielded generalizations for the other strategies.

None of the participants used strategy (e) in the general case. In the classroom discussion the participants concluded that this strategy could also be generalized. They discovered that the n-sided regular polygon could be divided into n triangles by adding a reference point (H) inside the polygon (Fig. 11). The internal angle α can be found by calculating the difference between the sum of the n triangles and the inner angle H as follows: $n*180^0 = n*\alpha + 360^0 \Rightarrow \alpha = 180^0 - 360^0/n$.

The attempts to generalize strategy (f) caused some confusion. Some of the participants thought that the polygon should be divided into one quadrangle and the remainder into triangles (option A). Others argued that for n>5 partition of the polygon could be into as many quadrangles as possible and the remainder into triangles (option B). In Figure 12 we can see an example of a hexagon divided into one quadrangle and two triangles (option A), or into two quadrangles (option B).

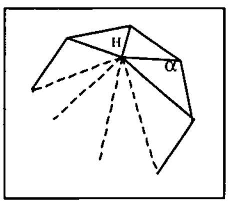

Figure 11

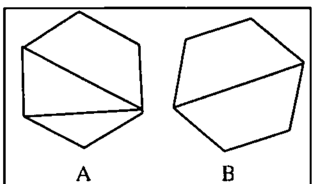

Figure 12

Generalization of option A to an n-sided regular polygon yielded a division into one quadrangle and n-4 triangles. In this case, the sum of the internal angles of the polygon will be $360^0 + (n-4)*180^0$ and hence $\alpha = 180^0(n-2)/n$.

Generalization of option B to an n-sided regular polygon resulted in two instances: (1) the number of the polygon vertices (i.e., n) is even; (2) the number of the polygon vertices is odd. In the first instance the polygon can be divided into (n-2)/2 quadrangles, which yielded $360^0*(n-2)/2 = n\alpha \Rightarrow \alpha = 180^0(n-2)/n$. The second instance was difficult to investigate. After checking a few examples, the conclusion was that in this instance the partition of the polygon will include one triangle and (n-3)/2

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quadrangles. The suitable equation is $360^0*(n-3)/2 + 180^0 = n\alpha \Rightarrow \alpha = 180^0(n-2)/n$ (notice that since every quadrangle can be divided into two triangles, there could be many other possible polygon partitions).

As shown in table 1, when calculating the intemal angle of the regular pentagon, most of the participants did not use the known formula: $180^0(n-2)/n$ but divided the pentagon into different simple geometric shapes (quadrangle, triangle, etc) and used their properties in order to solve the task. Although the formula for the sum of a polygon's internal angles ($180^0*(n-2)$) is known and relatively not complex, yet, the participants preferred to use other strategies rather than those based on it. One possible reason is that since they had to find the angle of a regular polygon, they were looking for strategies connected to symmetrical shapes. Most of the participants divided the pentagon into three triangles or divided the regular pentagon into five isosceles triangles, using the property that a regular pentagon can be inscribed in a circle, the center of which is the center of gravity of the pentagon. The tendency to divide a polygon into triangles is one of the common heuristics among students solving geometry problems (Borasi, 1992), since one of the first geometric shapes they are introduced in school is the triangle. A major part of geometry lessons is dedicated to the learning about the properties of triangles and quadrangles and most of the proofs presented use triangles' congruence. This might be the reason they try to use the triangle's properties, when available, for solving geometrical problems. Another possible reason for the surprising outcome is that there was a minor use of the formula for the sum of a polygon's angles is the loose connection between algebra and geometry. The participants preferred to use concepts within geometry to solve a geometric problem rather than to use algebraic formulae.

Discussion

The problem presented in this study is an example of a simple task with various solution strategies. Analyzing the different solutions can prompt mathematical generalization as well as many other desirable learning situations in the spirit of the NCTM Standards (2000). Analysis of the emerging solution strategies shows that there are two different kinds of generalizations in this task, the combination of which provides a powerful tool for the learning process. The first kind of generalization is to infer from a specific regular polygon (pentagon) to an n-sided regular polygon. The second kind of generalization is embedded in the participants' emerging strategies.

The regular pentagon divides the surface into three parts: the area inside the pentagon, the pentagon side and the area outside the pentagon. Looking carefully at the different emerging solution strategies shows that most strategies include the addition of a reference point H and its connection to the pentagon's vertices (Figure 13).

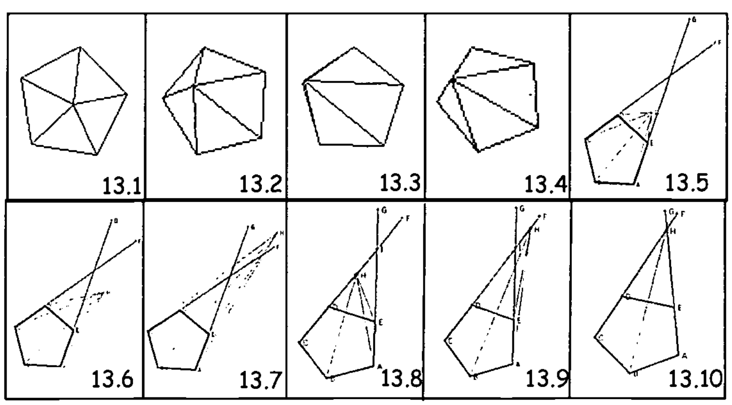

Figure 13

Analysis of the emerging solution strategies led to two main categories:

(1) The addition of a reference point.

(2) The partition of the polygon to simple geometric shapes (triangle, quadrangle)

The first category can be divided into three sub-categories by referring to the location of the reference point: (1.1) adding a reference point inside the pentagon; (1.2) adding a reference point on the pentagon edge; (1.3) adding a reference point inside the pentagon (Diagram1).

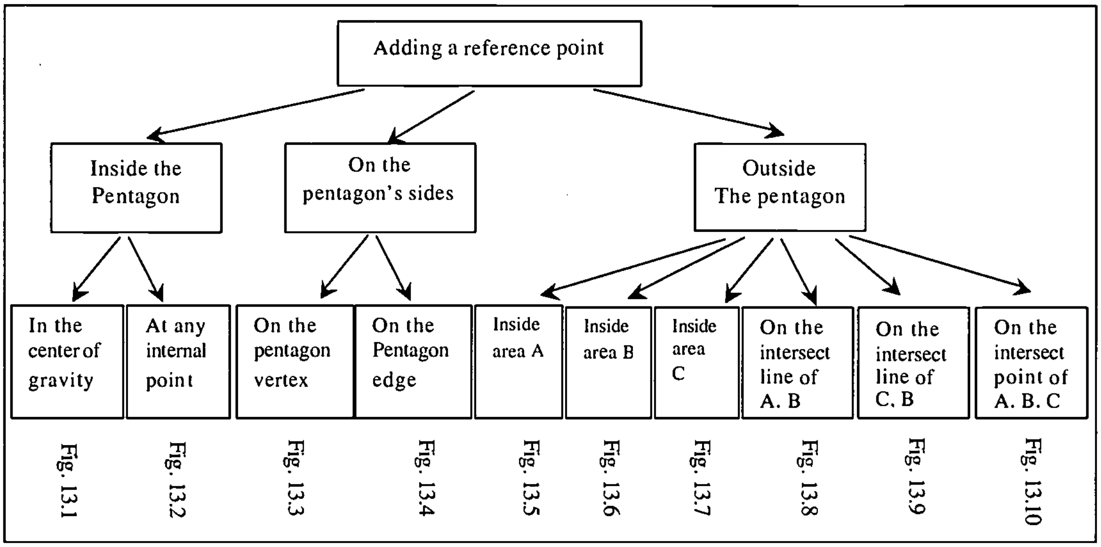

Diagram 1

The classroom discussion resulted in many more strategies. The exchange of ideas between the participants during the classroom discussion and their interaction enabled them to think of new directions for possible solution strategies. Within the classroom discussion a "mutual entity" evolved, and eventually yielded new interesting strategies. According to the socio- cultural approach the group plays an important role in an individual's learning process (Cobb, Wood & Yackel, 1993). The group encourages an individual to reflect on his thinking process and as a result he can develop and deepen his understanding (Cobb et al., 1997). Within a group, the individual can achieve more than he could if he were working by himself (Voigt, 1994).

While looking for new solution strategies, pre-service teachers felt that they had to use previous knowledge like: the sum of angles in a triangle; the connection between the triangle's external angle and its inner angles; etc. The engagement of pre-service teachers in this kind of activity could promote their awareness to the educational importance of problems with various strategy solutions. Much research was carried out regarding the educational value of making connections between different mathematical issues or concepts (NCTM, 2000; Even, 1990; Hiebert & Carpenter, 1992; Cornu & Dubinsky, 1989; Coxford, 1995; Reimer & Reimer, 1995). According to Noss, Healy and Hoyles (1997), "Mathematical meanings derive from connections: intra-mathematical connections, which link new mathematical knowledge with old, shaping it into a part of the mathematical system" (p.203).

The forgoing discussion has demonstrated main point of the activity: creating the need to apply previous knowledge in performing a new task. We hope that the pre-service teachers will take this point with them and use it in their own teaching.

This paper describes an example of a simple task that can be solved in many different ways. Individual work, reflection on its outcomes, and classroom discussion can lead to more sophisticated ways of solution strategies. Discussing those strategies could promote mathematical generalization.

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ARE STUDENTS ABLE TO TRANSFER MATHEMATICAL KNOWLEDGE?

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ABSTRACT

The ability to use mathematics in other disciplines is generally expected of all science and engineering students. Anecdotal evidence suggests that many students lack this ability. While there is a substantial body of research dealing with the transfer of training, and the transfer of mathematical skills to problem solving in everyday life, there is very little relating to the transfer of mathematics to other scientific disciplines.

This paper reports on the development and trialing of an instrument which can be used to research the ability of students to transfer mathematical skills and knowledge to other disciplines. The instrument consists of mathematical problems set in various contexts. All the problems involve exponential and logarithmic functions, and are based on scenarios from physics, microbiology and computer science. In each case, any discipline-specific knowledge required to solve the problem is given, so that all the problems can be solved with mathematical knowledge only. The problems were initially written by a physicist, a microbiologist and a computer scientist. The instrument has been trialed with 47 first year science students at the University of Sydney. Performance on the instrument has been correlated against final high school marks, first year university results, and subjects studied. These results are presented.

The paper also discusses some of the interesting issues which arose from the collaboration of a mathematician with academics from three other scientific disciplines. For example, differences in the ways the physicist, the microbiologist and the computer scientist used mathematics were apparent. Also, their use of mathematics was often quite imprecise. Such issues have important implications for the teaching and learning of mathematics, both as a subject in its own right and within other disciplines.

1 Introduction

Science and engineering degrees typically require students to study mathematics as a subject in its own right, with the expectation that students will be able to use the skills and knowledge acquired from their mathematics courses in other disciplines. Typically, also, lecturers in engineering and scientific disciplines complain that students are unable to apply mathematics in context. Clearly, the question of whether or not students are able to "transfer" their mathematics is an important one.

Psychologists use the term "transfer of training" to refer to "knowledge, skills and attitudes able to be transferred from training sessions to the work context, and from one job or task to another" (Hesketh 1997), and there is a substantial body of research on the topic. There is also a considerable amount of research on the extent to which mathematical skills transfer to problem solving in real world situations (Buckingham 1997, Carraher et al 1985, Lemire 1988, Sun 1995), and various papers which assume that students have a problem transferring mathematics. Gill (1999a and 1999b), for example, has studied the problems students of physics and engineering have with mathematics. Jackman et al (2001) report on a project involving assessment tasks designed to improve the ability of students to apply mathematics in context. Woolnough (2000) similarly describes a program in physics "designed to help students build effective links between mathematical equations ..... and the real world". There is little work, however, which specifically addresses the question of whether or not university students are able to transfer mathematical skills and knowledge.

The aim of the project discussed in this paper was to investigate the extent to which students are able to transfer mathematical skills to those disciplines represented by the project team members. The project team consisted of a mathematician (the author), a physicist, a microbiologist and a computer scientist. An instrument to test transferability was developed. The instrument consists of mathematical problems set in various scenarios. In the following sections the development of the instrument is described, and some results from an initial trial are presented.

2 Developing the instrument

Our original intention was to develop the questions which comprise the instrument around a topic taught in first year mathematics, and used in first year physics and computer science, and in microbiology. It was a little surprising to find that at the University of Sydney there is apparently no such topic. The questions are therefore based on logarithms and exponential functions, topics which are taught at high school in New South Wales.

Some purely mathematical questions were written by the author, and the other members of the project team wrote questions set in the context of their particular discipline. Their brief was to write problems which contained enough discipline specific information so that the problems could be solved using mathematical knowledge only, without any previous knowledge of the particular discipline. This proved to be rather a difficult task. The first draft of the instrument included some explanations which were not entirely comprehensible to those of us who had not written the questions. The computer scientist wrote a question, based on Big-Oh notation, which was almost totally incomprehensible to those unfamiliar with the notation. It is clearly difficult for academics not to make certain assumptions, relating to their discipline, when writing background information. Of further concern to the author was the imprecise way in which the other scientists tended to use mathematics. The original questions included variables defined incorrectly, and some rather imprecise descriptions of mathematical concepts. (For example, one question included the statement: "On a logarithmic scale, the number of photons approximate negative slopes.")

The problems were rewritten several times before all the researchers were satisfied. Some post-graduate and higher year undergraduate students were then asked to attempt the questions, and provide feedback with regard to clarity of the questions, perceived difficulty, and length of time taken to complete the questions. Five physics post-graduate students, one microbiology post-graduate student, one undergraduate microbiology student, one mathematics and computer science graduate, and one mathematics honours student agreed to do so. The feedback we received was extremely useful, and the questions were further refined in the light of the students' comments. We had expected that these students would be able to complete the questions without difficulty, and so were surprised to find that most of them were unable to successfully complete all the problems. The computer science questions proved particularly difficult for some of the students who had not studied computer science. The mathematics questions, which were of high school standard, were completed successfully only by the mathematics students. The instrument was revised further in response to this feedback.

The current version of the instrument consists of a physics problem based on exponential decay of the number of photons in a photon beam, a microbiology problem based on killing bacteria, a computer science problem based on Big-Oh notation and four straightforward mathematics questions. Where possible, the questions have a similar structure, allowing us to test the application of a particular skill in different contexts. We were able to achieve this with the physics and microbiology questions. The computer science question is quite different from the others, and may well be deleted from the instrument in future versions. The following extracts from the instrument illustrate some parallel questions.

Physics question

Consider a beam of photons with identical energies all travelling in the same direction, head-on into a particular medium. The number of photons which survive as the beam passes through the medium decreases exponentially. The distance over which the number of photons is halved is called the half-thickness of the medium. Let N be the number of photons which have survived at a distance x into the medium, and let g be the half-thickness.

1. If $N(x) = N_0 \times 2^{-kx}$, where N_0 is the initial number of photons, and k is a positive constant, express k in terms of g.

2. Suppose a medium is 10 mm thick, with a half-thickness of 0.5 mm, and that 10^{10} photons enter the medium head-on.

   Draw a graph of $\log N$ against x, with a scale marked on the axes.

Microbiology question

The bacterium *Staphylococcus aureus* ("golden staph") found in poultry stuffing is killed by heat. After a quantity of poultry stuffing has been heated to 62°C, the cell concentration of the golden staph bacteria decreases exponentially. The Decimal Reduction Time at 62°C, D_{62}, is the length of time required for the cell concentration to decrease to $1/10^{\text{th}}$ of its original value. Let N be the cell concentration of the bacteria at time t minutes after the stuffing has been heated to 62°C.

1. If $N(t) = N_0 \times 10^{-kt}$, where N_0 is the initial cell concentration and k is a positive constant, express k in terms of D_{62}.
2. For golden staph, the decimal reduction time at 62°C, D_{62}, is 8 minutes. Draw a graph of $\log N$ against t if the initial concentration is 10^5 cells/g.

Mathematics question

1. If $P = 5e^{kt}$ and $P = 10$ when $t = 3$, find k.
2. If $y = 4e^{-0.1x}$, draw a graph of $\ln y$ against x, for $0 \leq x \leq 10$.

3 Trial of the instrument

In Semester 2 2001, forty-seven first year students attempted the questions. The students were volunteers, and were paid a small amount for their participation. There were 30 science students, 16 engineering students and one arts student. Each student was given a version of the instrument with the physics, microbiology and computer science problems collated in random order, and the mathematics problems at the end. They were given 40 minutes to attempt the problems in the order in which they appeared, and then asked to attempt the mathematics questions. Each student's work has been marked, and scores for each of the questions recorded.

In the following table the students have been grouped according to the subjects they had studied in Semester 1 2001. Only those subjects of interest to us are included. For each group, the table gives an average score (out of 10) for the mathematics question, and an average score (out of 10) for the computer science, microbiology and physics questions. The latter is labelled "Transfer mark".

	No. students	*Math mark*	*Transfer mark*
Maths only	1	6.0	1.5
Maths + Chemistry + Computer science	1	6.0	3.0
Maths + Chemistry + Physics	3	4.7	2.4
Maths + Computer science	7	5.6	2.7
Maths + Computer science + Physics	7	7.3	4.6
Maths + Chemistry	8	6.5	3.0
Maths + Biology + Chemistry	10	6.8	3.8
Maths + Biology + Chemistry + Physics	10	6.9	3.9

Given the size of the sample, and the fact that the students were self-selected, it would be unwise to attempt to draw too many conclusions from these results. For example, the highest-scoring group (Maths + Computer science + Physics) in the above table contained two of the three top-scoring students, but also two of the lowest-scoring. Nevertheless, some interesting observations can be made.

Firstly, almost all the students had university entrance scores, and final high school mathematics marks, very much higher than average. The mathematics questions in the instrument are of high school standard. Performance on these questions was, therefore, surprisingly bad. The fact that performance on the "transfer" questions was even worse supports the widely-held view that students have difficulty applying mathematics in context.

Secondly, there is some evidence from the results to suggest (as one would expect) that students are better at applying mathematics in context when they are familiar with the context. For example, students who had studied physics in semester 1 scored an average of 4.2 (out of 10) on the physics question, while those who had not studied physics scored an average of 3.1. Similarly, biology students scored an average of 3.4 on the microbiology question, while those not studying biology scored an average of 2.5. The difference was less marked on the computer science question, with computer science students scoring an average of 4.6, and those without computer science an average of 4.1. On the other hand, the top-scoring students performed equally well on all the questions, regardless of subjects studied in first semester, and some students seemed better able to apply their mathematics in the context of a discipline they were *not* studying.

At the time of writing this paper, further analysis of the students' work on the instrument is planned. We have, for example, identified five mathematical skills and seven pieces of mathematical knowledge needed to successfully complete the mathematics questions. Students' responses on all the questions will be analysed in relation to those. We hope to be able to construct an algorithm for calculating a "transferability score" for individual students.

We regarded this trial as a test of the instrument as well as of the students. In this respect some questions, such as whether or not the students could at least attempt all the questions within the allotted time, were easy to answer. (Only half of them were able to do so, despite the fact that the questions had been seriously pruned in response to feedback from the postgraduate students.) Other questions, such as whether or not the instrument reliably tests the ability to transfer, are more difficult to answer. On the assumption that the ability to transfer mathematical skills and knowledge is important for academic success in other scientific disciplines, we compared students' results on the instrument with their university entrance score (UAI), as well as with their WAM (a weighted average of first year university results). The correlation coefficient between "Math mark" and UAI was 0.54, and between "Math mark" and WAM was 0.62. Between "Transfer mark" and UAI the coefficient was 0.47, and between "Transfer mark" and WAM it was 0.57. Despite the non-random nature of the sample, these results are significant enough to lead us to believe that the instrument is a useful test of transferability.

4 Conclusions

There is little argument amongst academics that students do indeed have trouble transferring mathematics. The results of our trial bear this out. With few exceptions, students performed better on the pure mathematics questions than on the "transfer" questions involving the same skills and knowledge. Some reasons for this are obvious. Applying mathematics in context generally involves translating a problem expressed in words into a mathematical statement. Many students have such poor language skills that the problem may well be insurmountable. Nevertheless, it is in the interests of academics in all scientific disciplines to attempt to find ways in which to help students overcome their difficulties in relation to transfer.

Communication between academics is an obvious starting point. The collaboration of academics from four different scientific disciplines on this project has been most instructive. We have discovered that our use of mathematics is often different, in ways which are unlikely to be helpful to students. For example, mathematicians all but ignore exponentials and logarithms to bases other than e, whereas in physics and biology the use of base 10 is more common, and computer scientists generally use base 2. We have also learned that we have different ideas of what is mathematically correct. Physicists at the University of Sydney, for example, believe that it is incorrect to take logarithms of both sides of an equation involving variables which represent quantities with units attached. (So a Sydney University physicist would claim that the equation $q = q_0 e^{-t/RC}$, where q, q_0 are in Farads and t, RC are in seconds, is not equivalent to the equation $\ln q = \ln q_0 - t/RC$.) Further, it would appear that mathematicians expect mathematics to be used much more precisely than other scientists are accustomed to doing. While the use of mathematics in an imprecise way may not hinder scientists in their everyday work, it may be confusing for students. There is much food for thought with respect to the implications for teaching raised by all these differences.

Finally, it should not be forgotten that the acquisition of mathematical skills and knowledge is a pre-requisite for the ability to transfer them. Not surprisingly, the students in our study who performed strongly on the mathematics questions were, in general, much more successful on the transfer questions than were those students whose performance on the mathematics questions was weak. "The power of mathematics as a tool... is that if the working of the tool is understood then it becomes possible to apply it in novel situations" (Gill, 1999a). In teaching mathematics to science and engineering students we should certainly keep in mind the transfer problem. However, our first priority is to ensure that students acquire a good understanding of the tool.

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SOLVING PROCESS OF A NARRATIVE COMBINATORIAL PROBLEM: AN EXPLORATORY STUDY

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Keywords: problem solving - reasoning - combinatorial - narrative

ABSTRACT

Problem solving is one of the main goals of the learning process as it concerns knowledge in action. It is regarded as the search of possibilities, evidences and goals, involving the production of inferences, arguments and strategies to validate or refute a statement. It is related to the way in which the student models the situation and applies or creates solving strategies. Formal and informal reasoning are activated in this process.

Formal reasoning is generally associated to well-defined situations, where all the relevant data are given. It is based on logical inferences where the initial premises imply implicitly a conclusion. Informal reasoning, generally associated to "open" situations, is not restricted by logical operations as it may include inferential processes (developed, sustained and evaluated by a system of beliefs or by common sense).

In the present article, we discuss problem-solving processes that are involved when students solve a combinatorial situation, written in a narrative style, and they have to recognise the relevant data and to decide possible solving strategies. These aspects were analysed in a sample of 70 students, with ages between 15 and 53 years old, that attended different Mathematics courses.

Fifteen aspects were considered as variables, among them: data comprehension, combinatorial focus, representations as support, searching criteria, solving features, formalization level, formalization type, answer type and answer content.

Information was obtained through the application of multivariate statistical techniques. This study leads to the construction of a typology attending to solving process features. The classification analysis of the protocols allowed the identification of four classes, with almost the same number of constituents, which were given the following names: *bewildered*, *rough and ordered*, *heuristic* and *formal and tidy*. They are discussed in terms of the two main bias that emerged through the study: order and formalization.

1. Introduction

Problem solving is one of the main goals of the learning process as it concerns knowledge in action. It can be seen as involving two co-operating sub-processes: the *comprehension* through which the subject organises a mental model of the situation, and the *searching* of possibilities, evidences and goals in order to generate a strategy for its solution (VanLehn, 1998; Johnson-Laird, 1983). These two processes often alternate each other, even if they are incomplete, because comprehension may still be running when the searching process starts.

If the subject deals with the same type of problems many times, she/he may learn how to solve them and she/he may cease to labour through the comprehension and searching processes. As she/he seems to recognise the stimulus of a familiar problem she/he will follow some kind of solution procedure already proved. The collection of knowledge surrounding a familiar problem is called a *problem schema*. It is usually assumed that experts know a large variety of schemas, which may have two parts that match the whole problem unambiguously: one to describe the problem and one to fit the solutions. Teachers, as experts, usually act to provide their students the schemas concerning instructional problems. This fact and the numerous repetitions of the same kind of process make these problems become prototype ones. Though initially these prototype problems could require a hard demand of comprehension and searching of solution to the students, the repetition of a schema finally transforms them in a routine or an exercise.

Frequently the subject who has shown an expert behavior finds obstacles when dealing with non-routine problems. Difficulties arise from different sources: ambiguity in selecting a schema, need of schema combination, the detection of an impossible action during the execution of a solving procedure that produces a halt, a repair of the strategy during the execution due to a failure (VanLehn, op. cit.)

Combinatorial problems provide interesting opportunities to face non-routine problems, partly because of the narrative style in which they are mostly stated. Narrative text has a closer correspondence to everyday experience than expository text does. It involves dynamic events that imply characters, goals and intentions; expositive ones include static contents such as concepts, descriptions and arguments. Many knowledge-based inferences are generated during the comprehension of narrative text, requiring the activation of knowledge structures, schemas and their integration to conform a meaningful representation of the text (Kuhn, 1991). In addition combinatorial problem solving involves the production of inferences to derive progressively typical logical laws, argumentative skills, and strategies to complete the systematic analysis of different possibilities and the exploration of the whole structure of the problem. Formal and informal reasoning are activated in the solving process.

In the present article, we discuss problem-solving processes involved in the resolution of one specific combinatorial situation written in a narrative style. Within an exploratory research, we analyse how a sample of students solved the assigned problem in order to identify indicators of their comprehension, possible bias in the interpretation of premises, the features that oriented their reasoning and the way in which they communicated the solution.

2. Method

Subjects: The participants were 70 students, with ages between 15 and 53 years old, that attended different Mathematics courses. A sample of 20 students (aged 15 to 17) proceeded from a high school, and they performed the test as a requirement to integrate its Mathematics Olympic

Team. The rest attended a first year course at the National University of Rosario, Argentina, 30 of them studying to be a high school Mathematics professor and 20 to be a Mathematics bachelor.

The participants had to read and solve individually a problem, written in a narrative style that referred to a realistic everyday situation including verbal and numerical data. They had 30 minutes to execute the task. The activity was performed before developing the specific contents.

Research Design: The present study is best described as exploratory and interpretative. The aim is to explore patterns of reasoning and strategies that students develop to solve the following narrative non-routine combinatorial problem:

> *Mary tells her friends Bob, Peter and John that she is a "psychic" and to prove it she puts 24 similar chips on a table. Then she covers her eyes and asks one of the boys to take one chip, another to take two and the last one to take three chips. Without having seen who has taken each amount of chips she promises to guess it. But she says that in order to do so, she needs Bob to take as many chips as he has taken before, Peter to take twice the number of chips he has and John to take four times the number of chips that he has taken. Once they have done so, she asks her friends to put away their chips and then she uncovers her eyes. Suppose each boy has done exactly what Mary asked. Will Mary be able to guess how many chips took each boy in the first place? How can she do that?*

In order to perform the study three analysis dimensions were defined: *personal features*, *combinatorial comprehension* and *solving process*. Fifteen variables were selected as indicators of different aspects involved in these dimensions, as follows:

- *Personal features*: this dimension refers to characteristics of the subjects such as gender, age, previous knowledge and current studies.
- *Combinatorial comprehension*: it searches information about the level of assurance and the ways in which the student processes the data, focuses the problem as a combinatorial task and starts to generate an effective strategy. The variables are:
    1. data comprehension: identifies the level of assurance in which the subject understands the information
    2. combinatorial focus: analyses if the subject realises the need of checking all the possible cases
    3. representations as support: searches for evidences of its existence as a guide for the comprehension process
    4. recognised cases: measures the percentage of analysed cases
    5. searching criteria: defines the way in which the subject organises an strategy to look over all the cases
- *Solving process*: this dimension characterizes solving features depicted by the student to arrive to the goal and the way in which she/he arguments to provide an answer. The variables are:
    1. solving features: characterises the process developed to solve the problem
    2. formalization level: takes into account correctness and order of the solving process
    3. formalization type: describes the procedure selected
    4. content of the solving process: refers to completeness, clearness and coherence of the solving process
    5. answer type: describes the tools used to provide an answer
    6. answer content: refers to completeness, clearness and coherence of the argumentation given as an answer.

The specific modalities for each variable, shown in Table I, resulted from a previous analysis of the tasks performed by a subset of 20 students, randomly selected from the sample under study. Based on them, three of the authors performed an individual analysis of the solving activities done by the whole set of subjects. A triangulation process of their different registrations followed this activity.

A data matrix of 70 files (individuals) $\times$ 15 columns (variables), which enclosed a set of 65 modalities, was obtained. A multivariate statystical analysis, applying multiple correspondence analysis and mixed cluster processing (Lebart, Morineau & Fenelon, 1985), was selected and it was used the SPAD software (C.I.S.I.A., 1988) to process the data. The matrix is represented as a cloud of points in the 15-dimensional space of the variables or in the 70-dimensional space of the individuals. The software solves an eigenvalues problem to obtain the principal directions, called factorial axes, of the topological configuration. The first factorial axis is related to the direction of maximal dispersion of the data, and its percentage of inertia measures the contribution of this axis to the interpretation of the initial data matrix. The second principal axis, orthogonal to the first, is oriented in the next greatest dispersion direction. They define the most relevant plane for the interpretation of the projected data, called factorial plane. The meaning of each axis is determined considering the neighboring and oppositions of the modalities within the projection of the cloud on this plane.

The cluster processing provides a classification of the individuals based on their similarities in a reduced number of classes. These are as much homogeneous as possible, and their centers of gravity are related to the individuals, called the *paragons,* that best represent the class because of their characteristics. The classes obtained in the present study, projected on the factorial plane, are shown in Fig. 1.

3. Results

The classification analysis of the protocols allowed the identification of four classes which were interpreted as: 1 - *bewildered*, 2 - *rough and ordered*, 3 - *formal and tidy* and 4 - *heuristic,* constituted by 20 %, 28 %, 26 % and 26 % of the sample, respectively.

Class 1 is constituted by subjects that seem to be bewildered about the task proposed. They do not register any work concerning the problem. In a few cases, after an exclusively mental work - neither supporting representations nor explanations revealing their reasoning were explicitly pointed out - they only provide isolated answers, none of them complete and clear at the same time.

Students of Class 2 partially interpret the data, they do not seem to have realized that it was necessary to check the totality of the cases and their solving process modalities involve mostly features such as incomplete, incorrect and incoherent but ordered. This is the only class in which genre appears as a determining variable: women constitute 84 % of the class.

Students of Class 3 basically check the totality of the involved cases, most of them in a systematic way, using explicit representations as a support of their reasoning. The formalization level is high, their solving processes and answers are correct, clear and complete. The whole class corresponds to students of university level.

Class 4 is completely determined by the age of the students and, consequently, by their scholar level of studies. All the students in this class, with ages between 15 and 17 years old, attend high school. They understand the problem and they basically employ graphical representations to organize their reasoning. The searching is mostly heuristic and disordered but, anyway, they complete the task in a satisfactory way.

Figs. 2a and b show, by symbols and labels projected on the most relevant plane 1-2, the distribution of the modalities that indicate the *combinatorial comprehension* and *solving process* dimensions, respectively. The factorial axis 2 divides the factorial plane in two semi planes: the modalities associated with an adequate interpretation of data lie on the left, and those that indicate misunderstanding, doubts or incapacity to solve the problem lie on the right. The factorial axis 1 (% inertia: 15.43) is conformed, at the negative extreme, basically by the proximity of the modalities: *correct and disordered* as a formalization level, *graphic solving*, *complete content* of solving process, *mixed answer* and a *graphic* solving feature (see Fig. 2a), joining to *yes* for the variable combinatorial focus, a *complete* data comprehension and a *systematic searching* criteria (see Fig. 2b). Opposite to them, on the right, appear the modalities that indicate *omission* to detect the combinatorial focus, *absence* of data comprehension, *no solving* process to recognize the formalization level and *no content* in the solving process. Therefore, the factorial axis 1 is interpreted as that defining the level of combinatorial comprehension achieved which sustains the formalization process.

The factorial axis 2 (% inertia: 8.83) is defined, at the positive extreme, by the modalities: *incorrect and ordered* as a formalization level, *mixed searching* as a criteria to organize a solving strategy, *partial* for the data comprehension and *doubtful* to detect the combinatorial focus, and, on the opposite side, by *correct and disordered* as a formalization level, *omission* of the combinatorial focus and *absence* of data comprehension. Representations as support, recognised cases, solving features and answer type are variables that do not contribute to characterize this axis as their modalities have their projections very close to zero. Therefore, axis 2 reflects the order of the solving process written on the protocols.

4. Discussion

Nature of the combinatorial problem used as instrument in the research

Problems like the "psychic" one have not been typical in the Mathematics classes in Argentina, neither in content nor in style. As regards the content, they refer to an everyday situation that acts as a *challenge to think* (Munby, 1982). Therefore, it is the type of ingenious problem that teachers sometimes offer to their students to solve on their own, assuming that only some of them, the creative and/or clever fellows, will be successful. As problems of this type are not seen as instructional ones, teachers do not work systematically on them and, consequently, students do not construct any associated schema of resolution. It is interesting to point out that this lack of a solving schema, both in students and teachers, conversely provokes that the latter disregard them as instructional, closing a circuit that seems to be hard to interrupt, though the new curricula in primary and secondary school specifically include combinatorial problems.

The resolution of this kind of problems demands a combinatorial reasoning (Piaget & Inhelder, 1951; Halpern, 1996) that may be characterized by two facts:

a) *completeness*, that is, the recognition of all the possibilities relative to a certain event (e.g., in the "psychic" problem all the alternatives in which Bob, Peter and John may pick the chips). This feature may be considered as equivalent to that introduced by Perkins (cited in Garnham and Oakhill, 1996) in the analysis of informal or everyday reasoning, where decisions and conclusions are based on a set of plausible arguments derived from evidences, and from which the subject organizes a situational modeling

b) *organization of an approach to check the possibilities*, as the order of the procedure is relevant for a successful performance of the task (e. g., the "psychic" problem requires also to design a methodical plan in order to explore the possibilities in a systematic manner).

As regards the narrative style of the problem, a previous study (Llonch et al., 2001) has shown that it demands from the student a further transformation of the explicit information into numeric or at least symbolic data in order to initiate the solving process. Relevant implicit inferences may be omitted or the student may fail to "translate" the narrative text into a scientific one (in this case, into a combinatorial language). Therefore, pure narrative problems offer additional difficulties. They involve people performing actions in pursuit of goals and include additional information acting as an obstacle or interference that has to be avoided by the reader. We may also conclude that purely narrative statements activate informal reasoning patterns that lead students:

(a) to the use of systems of beliefs (e.g., a student explicitly stated that *a riddle has nothing to do with Mathematics*) that bias the solution,

(b) to the demand of unnecessary data that adds difficulties to the situation (e.g., some students pointed that *there were insufficient data to solve the problem*).

The demand stated in the "psychic" problem introduces an interesting perspective for future research: the disturbance that may be produced by an unfamiliar question in a mathematical context *(Will Mary be able to guess...? How can she do that?)*, where habitually the student is required for a quantity (*How many*...?), to detect an existence (*Is there*...?) or to find an optimum (*Which is the greatest*...?, ...*the shortest*?,... *the cheapest*?).

About problem schemes produced by the different subject classes

To summarize, the categorization schemes, stated as *bewildered*, *rough and ordered*, *heuristic*[1] and *formal and tidy*, tend to focus on the solving tendencies of the subjects. Transitions among their features are clearly depicted by the trajectories followed by the modalities of the variables *comprehension focus* and *formalization type*, as shown in Figs. 2a and b, respectively.

Basically, the whole set of secondary students understood the task in its combinatorial essence and their spontaneous solving attitude was to organize a graphic array of possibilities – arrows, tables or lines – as an heuristic. The lack of a certain conceptual base oriented them to the representation of the different situations, in coherence with the data and the proposal. The protocols showed a satisfactory searching of possibilities, although the organization to look over the different cases was varied in level of order and systematization.

About the third part of the university students tried to perform elaborate actions, with a marked tendency to introduce analytical or, at least, numerical procedures, with the occasional use of various conceptual labels or symbols to give their answers in a formal academic style. However, the informality of the spontaneous reasoning related to this new type of problem produced some disturbance in their facultylike procedures. This fact was solved by the development of more tidy graphic organizations than those seen in the previous group, which allowed the student to reach successful solutions.

The university group basically characterised by genre (Class 2) failed to detect the demand of completeness required by the task. Although they intended to produce ordered organisation of

[1] Heuristic (from the Greek word *heuriskin* that means *serve to discover*) refers to the procedure that a subject believes as a reasonable possibility to arrive to the solution or, at least, to be close to it. It is an alternative to an algorithmic procedure, that is, a detailed prescription, step by step, to get the goal.

solving strategies, they were incoherent and rough. Basically, they failed to detect the combinatorial nature of the problem.

Finally, it is an interesting fact the presence of a significant group of students that looked like bewildered when faced with a new and unknown situation. We interpret that they lack self-confidence and only act when a known schema guides their work, that is to say, when they feel like "moving on a known land". This negative attitude is still present at university level, to which the majority of the students in this class belong.

High school students of the sample showed a different attitude – they tried to do something –, probably because they were spontaneously interested in the competition and the will to participate gave them the strength needed to persist.

6. Final Remarks

The analysis of the students' performance in solving this type of combinatorial problem provides some evidence about two relevant facts that seem to accompany successful solvers. One of them is related to an attitude of persistency on the searching, when faced with the absence of a known schema. The second deals with the tendency of using a graphic design as an heuristic when neither a numerical nor an algebraic strategy suited properly. The latter provides an interesting framework to introduce in our secondary schools the use of graphs as an alternative and powerful topic, whose methods develop the combinatorial reasoning and constitute the nucleus of Discrete Mathematics. In Argentina teachers still should be trained in these topics, not only in their contents and didactics but also in the knowledge of the strength of their treatment. They should teach combinatorial and graph problems mainly based in the knowledge of the fact that combinatorial ability is one of the basic conditions for logical reasoning (Fischbein, 1994).

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1- Gender ▪ Male ▪ Female	**2- Age: a (in years old)** ▪ $a < 15$ ▪ $15 \leq a < 18$ ▪ $18 \leq a < 21$ ▪ $21 \leq a < 30$ ▪ $30 \leq a$	**3- Previous knowledge** ▪ Incomplete high school ▪ Commercial high school ▪ Technical high school ▪ Bachelor high school ▪ Other university studies
4- Current Studies ▪ High school ▪ Mathematics high school professor ▪ Mathematics Bachelor	**5- Data Comprehension** ▪ Complete ▪ Partial ▪ Absent ▪ Incorrect understanding	**6- Combinatorial focus** ▪ Yes ▪ No ▪ Doubtful ▪ Omission
7- Representations as support ▪ Explicit support ▪ Implicit support ▪ Absent support	**8- Recognised cases** ▪ 100% ▪ Between 70 % and 100 % ▪ Between 40 % and 70 % ▪ Less than 40 %	**9- Searching criteria** ▪ Systematic path ▪ Random path ▪ Mixed path ▪ Uncertain criteria
10- Solving features ▪ Mental ▪ Numerical ▪ Mixed ▪ Graphic	**11- Formalization level** ▪ Incorrect and disordered ▪ Correct and ordered ▪ Correct and disordered ▪ Incorrect and ordered ▪ Non existent	**12- Formalization type** ▪ Literal solving ▪ Symbolic solving ▪ Graphic solving ▪ Numerical solving ▪ Mixed solving ▪ No solving
13- Content of the solving process ▪ Complete, clear content ▪ Incomplete but clear content ▪ Complete content ▪ Incomplete, incoherent content ▪ No content	**14- Answer type** ▪ Literal answer ▪ Symbolic answer ▪ Graphic answer ▪ Numerical answer ▪ Mixed answer ▪ No answer	**15- Answer content** ▪ Complete and clear ▪ Incomplete but clear ▪ Complete ▪ Incomplete and incoherent ▪ No content

Table I: Variables and modalities employed to perform the analysis

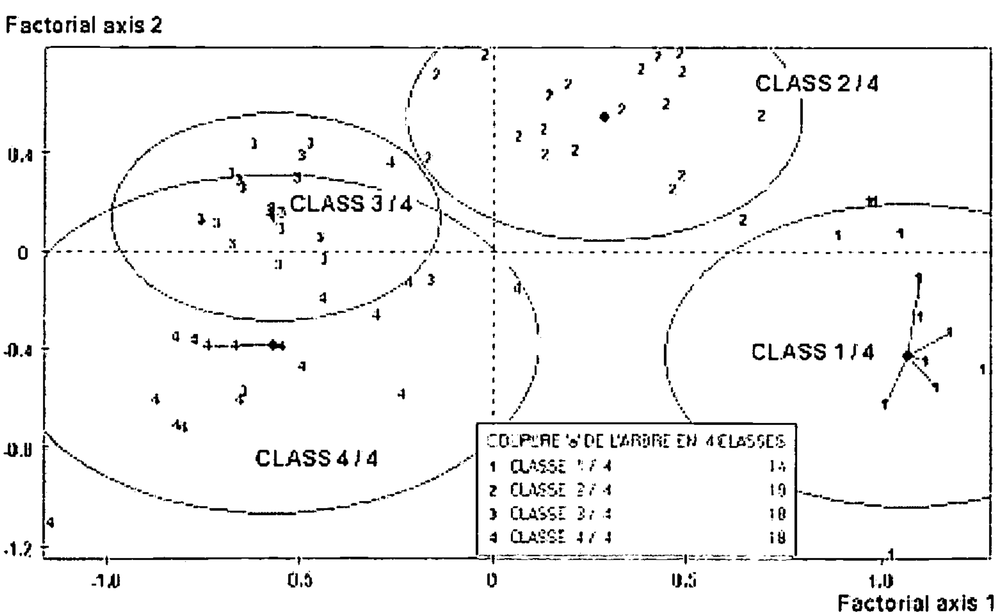
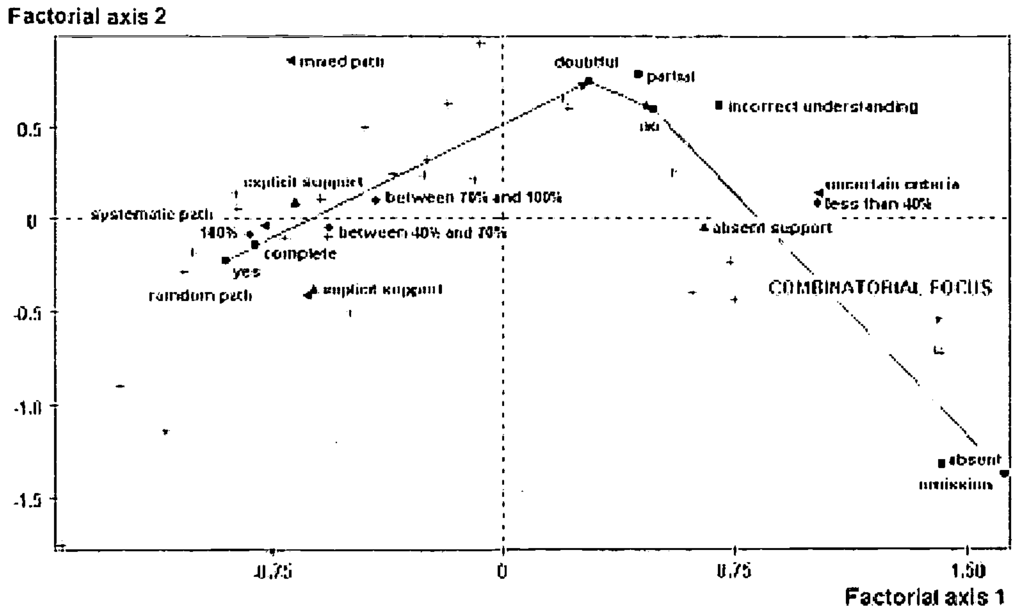
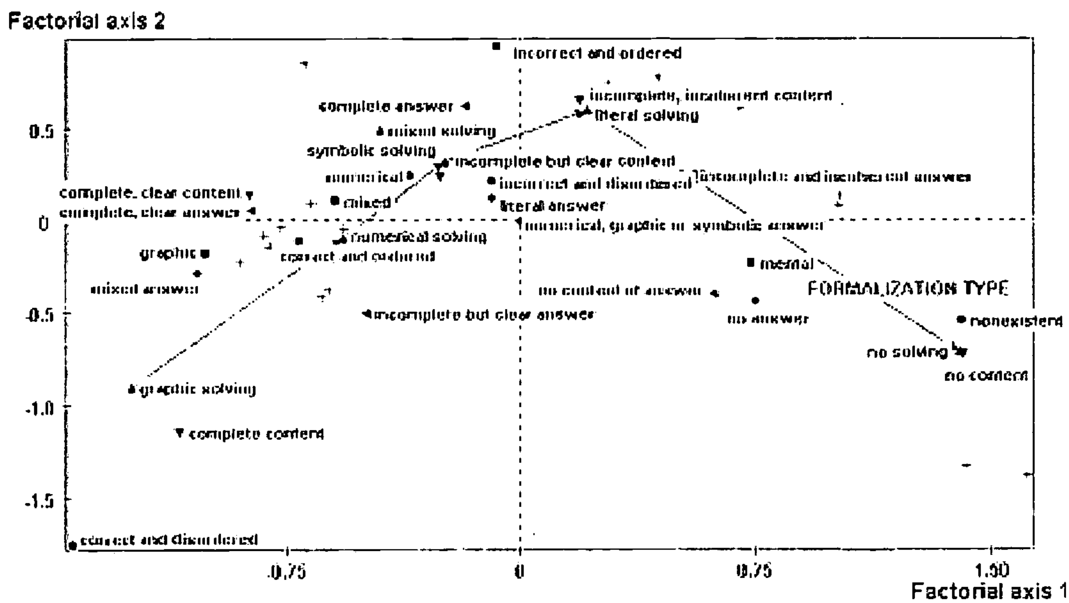


Figure 1: Classification of the students attending their affinities in the problem-solving task



(a)



(b)

Figure 2: Factorial plane defined by the first and second principal axes, with the projections of the modalities of the variables *combinatorial comprehension* (a) and *solving process* (b). The symbols (+) represent the modalities whose labels are indicated in the other figure.

ON THE EFFECT OF USING AUTOMATED REASONING IN TEACHING DISCRETE MATHEMATICS

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ABSTRACT

Programs that reason are playing an increasing role in teaching mathematics at the undergraduate level. This paper is concerned with teaching topics in discrete mathematics for computer science students. It aims to increase the likelihood of using computer programs to understand, represent, and solve problems with the help of automated reasoning. Topics of the course "Discrete Mathematics in Computer Science" include logic, set theory, relations and graphs as well as counting techniques. It is the authors' view that computer scientists must have substantial training in using discrete mathematics if they are to understand these topics and use them well.

1 Introduction

The field of automated reasoning is concerned with the ability of computer programs to reason about a given knowledge and deduce a new one. A number of automated reasoning programs (tools) can provide great assistance in solving a wide variety of problems, answering open questions, designing hardware circuits, and verifying correctness of theorems' proofs. Such tools are rich enough to be used in teaching computer science students various mathematical concepts. These concepts include problem representation (in first-order predicate calculus), quantification, simplification, substitution, splitting hard cases into smaller solvable ones, proof justification, and different ways of deduction like resolution and factoring.

The effectiveness of the automated reasoning programs is amply demonstrated by examining their role in answering open questions, designing and/or validating the design of logic circuits, verifying the correctness of proofs and programs, and constructing bases for domains which students need to understand before working vigorously on those domains.

An analysis to general problem solving leads to the identification of three types of problems: numerical, data-processing, and reasoning. Some problems depend on some combination of the three for a solution to be found. Although, most problem-solving programs currently in use focus on the first two types, there do exist programs that reason. Some of these programs are of commercial value, while others are either shareware or freeware. Examples of such programs are OTTER (McCune 1994), GANDALF (Tammet 1997) , SETHEO (Moser *et al.* 1997), and THEO (Newborn 1997). Except the last one, all of the above examples are freeware and can be obtained from the Internet. Any of these programs can be given some axioms and a statement to be shown correct.

The strength of a computer program that is capable of reasoning depends on how the problem being solved is represented, the completeness of rules employed to draw conclusions, and on the effectiveness of the strategies used to control the reasoning process. These three areas - representation, inference rule, and strategy - are key issues to students learning discrete mathematics.

2 Mathematical System: Axioms, Definitions, and Theorems

With respect to representation, first-order predicate calculus give computer science students the skills needed not only to understand and work on theorems, but also to write computer programs in order to verify the correctness of their proofs. Usually, computer applications deal with finite discrete sets of data items such as arrays and files. Notice that even the set of real numbers is finite in the digital world, because of the limited accuracy of their internal computer representation. Therefore, it is important to show the relationship between the notations used in first-order predicate calculus and their equivalent program codes. For example, let the universe of discourse be the finite discrete set $U = \{x_1, x_2, \ldots, x_n\}$.

- $\forall x[x \in U \Rightarrow p(x)]$

The Scope of x starts with the quantifier, therefore x is local in the function **for_all**. The predicate $p(x)$ is passed as a parameter to the function for_all. T_PF is of type pointer to a function with the profile (x: in U) return BOOLEAN;

```
function for_all(p:in T_PF)return BOOLEAN is
 begin
  for x in U loop
   if not p(x)
    then return FALSE;
   end if;
  end loop;
  return TRUE;
 end for_all;
```

- Similarly, $\exists x[x \in U \wedge p(x)]$

```
function for_some(p:in T_PF) return BOOLEAN is
 begin
  for x in U loop
   if p(x)
    then return TRUE;
   end if;
  end loop;
  return FALSE;
 end for_some;
```

The program code that checks if one and only one element x that belongs to the discrete set of data items U satisfies a specific predicate $p(x)$ is given next.

$\exists ! x[x \in U \wedge p(x)] \Leftrightarrow \exists x[x \in U \wedge p(x) \wedge \forall y[y \in U \wedge y \neq x \Rightarrow \neg p(y)]]$

Programming the above well-formed formula suggests nesting a new predicate as shown below:

$\exists x[x \in U \wedge p(x) \wedge q(x,p)]$, where $q(x,p) \Leftrightarrow \forall y[y \in U \wedge y \neq x \Rightarrow \neg p(y)]]$.

Note: scope of y is local, bound to q, whereas x and p are free in q.

```
function for_one(p:in T_PF) return BOOLEAN is
 begin
  for x in U loop
   if p(x) and q(x,p)
    then return TRUE;
   end if;
  end loop;
  return FALSE;
 end for_one;

function q(x:in U; p:in T_PF) return BOOLEAN is
 begin
  for y in U loop
   if y/=x and p(y)
    then return FALSE;
   end if;
```

```
  end loop;
  return TRUE;
 end q;
```

or by substitution of the body of the function q for the call of $q(x,p)$ in the function **for_one**:

```
function for_one(p:in T_PF) return BOOLEAN is
 begin
  for x in U loop
   if p(x)
    then for y in U loop
          if y/=x and p(y)
           then return FALSE;
          end if;
         end loop;
         return TRUE;
   end if;
  end loop;
  return FALSE;
 end for_one;
```

The above presented examples suggest combining the theoretical course Discrete Mathematics for Computer Science with hand-on programming experience in the form of laboratory work associated with the course.

3 Background

In the course "Discrete Mathematics in Computer Science", the students usually practice modelling combinatorial puzzles as well as mathematical theories by using the first-order predicate calculus, and then by transforming these models into corresponding computer programs (as shown above) that solve these puzzles. Some of the existing programs employ domain-dependent knowledge in their aim to model the reasoning process. While other programs hope to achieve the same objective by using domain-independent systems for modelling the puzzles. As an example, in this paper we model the theory of natural numbers and ask to prove the commutative property of addition.

The difficulty and/or complexity of mathematics should encourage students to use the assistance of automated reasoning programs. These programs can help in proving theorems, checking proofs, developing conjectures, and answering open questions in various domains such as number theory, set theory, group theory, ring theory, field theory, and lattice theory (McCharen *et al.* 1976; Wos *et al.* 1991). Other involved domains include combinatory logic, finite semigroups, Robbin's algebra, and equivalential calculus (Wos 1993; Wos 1988). Subsequently, many problem sets were developed for the purpose of underlying the theory behind those domains; for instance, the Stickel Test Set (Stickel 1988), the Quaife sets (Quaife 1992a; Quaife 1992b; Quaife 1991), the seventy-five theorems for testing automatic theorem provers (Pelletier 1986), and the TPTP problem set (Sutcliffe 1997). In this paper, we present as an example a fundamental system of well-structured theorems in elementary number theory based on the axioms provided by G. Peano in 1889, and which gave raise to the Peano Arithmetic. Our theorems range from quite trivial to moderately difficult. A refutation was obtained for each theorem in the system by THEO.

4 The Peano Axioms

This section illustrates how the program THEO proves the commutative law of addition. But first, we need to formalize the Peano axioms in first-order predicate logic. We will compare the proof obtained with the one usually given to students in classrooms. We use two primitive function symbols: **0**() to denote the constant zero (note that a constant is a function with no parameters), and the successor function **s**. The predicate symbol **N** represents the set of natural numbers and the predicate symbol **EQ** means equal. The logical operation OR is represented by the symbol | and the logical operation NOT is represented by the symbol $\sim$. The fifth Peano axiom is concerned with the induction principle. Anything follows a semicolon ; is considered as a comment.

```
;;;;;;;;;;;;;;;;;;;;;;;;;;;; Peano Axioms ;;;;;;;;;;;;;;;;;;;;;;;;;

A1: N(0())

A2: ~N(x) | N(s(x))          ; N(x) --> N(s(x))

A3: ~N(x) | ~EQ(0(),s(x))    ; N(x) --> ~EQ(0(),s(x))

A4: ~EQ(s(x),s(y)) | EQ(x,y) ; EQ(s(x),s(y)) --> EQ(x,y)

A5: ~EQ(x,y) | EQ(s(x),s(y)) ; EQ(x,y) --> EQ(s(x),s(y))
```

The axioms A4 and A5 indicate that the successor **s** is a one-to-one function. The equality relation **EQ** is an *equivalence* relation. This adds three extra axioms to the system.

```
;;;;;;;;;;;;;;;;;;;;;;;;;;; Equality Relation ;;;;;;;;;;;;;;;;;;;;

A6: EQ(x,x)                      ; (x=x)

A7: ~EQ(x,y) | EQ(y,x)           ; (x=y)-->(y=x)

A8: ~EQ(x,y) | ~EQ(y,z) | EQ(x,z) ; (x=y)&(y=z)-->(x=z)
```

Addition is defined by structural recursion as follows:

$$\{\forall x : x + 0 = x\} \wedge \{\forall x, y : x + s(y) = s(x + y)\}$$

The above well-formed formulae, defining addition over natural numbers, are converted into the first-order predicate clauses in order to introduce the definition to the theorem prover. The conversion algorithm can be found in (Newborn 1997).

```
;;;;;;;;;;;;;;;;;;;;;;;;; Addition Axioms ;;;;;;;;;;;;;;;;;;;;;;;;

A9: +(x,y,A(x,y))                          ; Closure property

A10: +(x,0(),x)                            ; x + 0 = x

A11: ~+(x,y,z) | ~+(x,s(y),u) | EQ(s(z),u) ; x+s(y) = s(x+y)

A12: ~+(x,y,z) | ~+(x,y,u) | EQ(z,u)       ; Uniqueness property 1

A13: ~+(x,y,z) | +(x,y,u) | ~EQ(z,u)       ; Uniqueness property 2
```

Next, we compare the difference between the proof of the commutative law of addition given to students in class (hand-written proof) with that of the theorem prover. Note that the hand-written proof requires the use of associative law of addition: $m + (n + p) = (m + n) + p$.

Hand-written Proof:

m + s(n) = s(m + n) by definition of addition,
n + s(m) = s(n + m) by definition of addition,
since s(n) = 1 + n
We have: m + s(n) = m + (1 + n) = (m + 1) + n by associative law
Thus, m + s(n) = s(m) + n
Which means s(m + n) = s(n + m) as stated above
m + n = n + m. by Peano Axiom A4 QED.

The proof obtained by THEO's is given next:

```
Theorem:CommAddition.thm

Given axioms:

 1# NO
                       } Peano
 2# ~Nx Nsx            }
                       } Axioms
 3: ~EQxy EQsxsy

 4: ~EQsxsy EQxy

 5# ~Nx ~EQ0sx
------------------------------
 6: EQxx
                        } Equality
 7 >~EQxy EQyx          }
                        } Axioms
 8 >~EQxy ~EQyz EQxz
------------------------------
 9 >+xyAxy

10: +x0x
```

```
11 >EQsxy ~+zsuy ~+zux  } Addition
                        } Axioms
12: EQxy ~+zux ~+zuy

13 >~EQxy ~+zux +zuy
------------------------------
14: +0xx                } 0 + x = x

15 >~+axy +xay          } assume a+x = x+a for some constant a

16 >+abc               } let a+b=c, where a, b,& c are constants

17 >+sabd              } let s(a)+b=d

18 >+bsad              } and b+s(a)=d

Negated conclusion:

19S>+sasbk             } assume s(a)+s(b)=k

20S>+sbsal             } and s(b)+s(a)=l

21S>~EQkl              } prove that k = l by contradiction

Inferred clauses: Proof: 25: (21a,8c)  ~EQkx ~EQxl
 26: (25a,7b)  ~EQxl ~EQxk
  27: (26a,11a) ~EQsxk ~+yszl ~+yzx
   28: (27c,15b) ~EQsxk ~+ysal ~+ayx
    29: (28b,20a) ~EQsxk ~+asbx
     30: (29b,9a)  ~EQsAasbk

31: (18a,11b) EQsxd ~+bax
 32: (31b,15b) EQsxd ~+abx
  33: (32a,7a)  EQdsx ~+abx
   34: (33a,13a) ~+abx ~+yzd +yzsx
    35: (34b,17a) ~+abx +sabsx
     36: (35a,16a) +sabsc

37: (16a,11c) EQscx ~+asbx
 38: (37b,9a)  EQscAasb
  39: (38a,13a) ~+xysc +xyAasb
   40: (39b,11c) EQsAasbx ~+yzsc ~+yszx
    41: (40c,19a) EQsAasbk ~+sabsc
     42: (41a,30a) ~+sabsc
      43: (42a,36a) []
```

To begin, the top line is the name of the theorem. The given axioms follow next and then the negated conclusion. Clauses in the proof are printed next. Following each clause number is a :

> to denote the clause is used in the proof.

to denote the clause is eliminated during the simplification phase.

S to denote the clause is derived from the negated conclusion.

Each inferred clause is either a binary resolvent or a binary factor. In the former case, the parents of the clause are printed out and then the clause. For example, the first clause in the above proof:

```
25: (21a,8c)  ~EQkx ~EQxl
```

is clause number 25, it was derived by resolving the first literal "a" of clause number 21 with the third literal "c" of clause number 8. Clause 43 is the NULL clause (denoted by []).

5 Conclusion

In addition to providing evidence that automated reasoning has made rigorous contributions in the theoretical foundation of mathematics, the presented work identifies its significance in teaching computer science students various skills needed in discrete mathematics. As it gives students the ability to express knowledge and test its correctness by writing computer programs and analyze their execution time. This is a good opportunity for combining theory with practice while teaching mathematics that can be expressed in first-order predicate calculus.

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UNDERGRADUATE MATHEMATICS FOR PRIMARY SCHOOL TEACHERS:

The Situation in Portugal

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ABSTRACT

At the beginning of the 1990s a national reform of the mathematics curriculum took place in Portugal. This was not accompanied by a corresponding reform in the training of primary school teachers.

In Portugal teachers are trained in higher education institutions that are officially free to do whatever they believe is appropriate. This leads to wide variation in the training programmes, with some exhibiting a considerable degree of irrelevance (Gomes, Ralha & Hirst, 2001).

This is a worrying scenario, because the curricular reform that took place presents new ways of understanding the teaching of mathematics, imposing new challenges on teachers.

In Portugal it hasn't been until recently that the scientific community has begun to show an interest in the mathematical training of primary school teachers (APM, 1998). There are very few studies in this area, and they mostly deal with the pedagogical knowledge component of teaching, minimizing the importance of teachers' subject knowledge.

In this study we undertake a brief analysis of the pre-service mathematical training for primary teachers currently offered in Portuguese institutions. We shall consider some studies in this area and discuss the possible consequences for the reform of pre-service mathematical education. In particular we pay attention to teachers' subject knowledge of basic mathematics, following the research of Liping Ma (1999)

The process of introducing mass schooling seems to have started, in Portugal, a bit later than in most European countries: it dates from the mid 1970s and it gradually implied considerable changes in both teacher recruitment and training models.

By 1986, an Education Act lists as *specialised functions* within teacher education the following: special needs education, school management, student teaching supervision, curriculum coordination, in-service teacher training, etc. Those specific dimensions came, as expected, to be implemented into the educational system and, in relation to infant and primary teacher education, one also moved from not considering this education a university matter to the creation, in the mid 1980s, in all Portuguese regions of the so called ESE(s) (Higher Education Schools) within the University system. The Government also decided, in 1998, that both infant and primary teachers would have the same academic qualifications as, for example, secondary school teachers; this is called a "licenciatura" degree and it takes from 4 to 5 years to accomplish (Formosinho, 2000). All these important changes within the Portuguese Educational System brought, as can be acknowledged from national assessment reports on the university degrees (CNAES, 2000), considerable reflection on methods of teaching and organisational aspects but did not bring any reflection on the contents of training courses for these "new" teachers supposedly better prepared to deal with modern educational challenges than "old" ones.

In fact, it hasn't been until recently that the scientific community in Portugal has begun to show some interest in the mathematical training of primary school teachers (APM, 1998). Evidence of this neglect can be found, for example, in searches conducted through periodicals such as *Gazeta da Matemática*, which was first issued in 1940 with the specific goals of helping the A-level students and support the A-level teachers (G.M., N. ° 1). There were, then as well as nowadays, no references to the mathematical training of primary teachers or to the problems related to ~~the~~ mathematics teaching at primary schools. On the other hand, a specific search through the magazine *Escola Democrática* reveals some discussion about the mathematics curriculum, particularly at the time of the introduction of the so called "Modern Mathematics". More recently we find several articles concerning primary school mathematics in *Educação e Matemática*, a periodical published by the Portuguese Mathematics Teacher Association. However, no matter whether or not these are specialized mathematical magazines, we have reasons to believe that these articles are not as widely known as one might expect.

The situation appears to be quite different in other countries; in summary

- Using *L'Enseigment Mathématique* as a reference, we can picture the way the so called "elementary mathematics" was treated and the importance given to the mathematical training of primary school teachers, through several articles published for more than a century reporting on the situation worldwide.
- Comparing Portuguese and some British infant and primary teachers' education one identifies

	Students	**Entry requirements**	**Structure**
Portugal	Almost all women. "Regular" students (average age 18 years old).	Upper-secondary; No special requirements for any subject.	4 years degree: 3 years + 1 year in-service training.
England	Majority of women. Three different age groups identified: "regular" students (21 years old average), mature students in their 30s and mature students in their 40s.	Academic requirement for admission to 1st degree studies; To achieve at least grade C in the GCSE examination in both Mathematics and English.	4 years degree: first degree + 1 year PGCE or 4 year Bachelor + QTS

TABLE 1: Comparing Portuguese and some British infant and primary teachers' education.

One can clearly identify a worrying scenario if one adds to the lack of research the fact that a national reform of the mathematics primary curriculum also took place in Portugal at the beginning of the 1990s. This reform, which presents new ways of understanding the teaching of mathematics, imposing new challenges on teachers, was definitely not accompanied by a corresponding reform in the training of primary school teachers. We still have infant and primary school teachers trained in three different kinds of higher education institutions: universities, polytechnics and private ones, that are officially free to do whatever they believe is most appropriate. This leads to a wide range of training programmes with some exhibiting a considerable degree of irrelevance. In an analysis of the mathematics programmes of the different institutions several questions were raised (Gomes, Ralha & Hirst, 2001), namely:

- About the coherence exhibited by the mathematical curriculum.
- About the relevance of some topics such as Topology, Matrices or Algebraic Structures.
- About the number of hours dedicated to the study of mathematics.

Questions	**Mathematical content**	**Coherence**	**Relevance of topics**	**Time dedicated to mathematics**
Analysis	It ranges from a condensed type Mathematics degree (for secondary school teachers) to a condensed type Education degree	The same contents repeatedly appear in different disciplines but the similarities are not explicitly identified. Disconnected topics.	Topics such as Matrices, Topology or Algebraic Structures are often questioned as relevant by most students.	It ranges from less than 6% to 17% of the total training time.

TABLE 2: Analysis of mathematics curriculum in different Portuguese infant and primary teachers' education.

In Portugal, to a large extent, the undergraduate students arrive at the training institutions with a mathematical training equivalent to nine years of mathematics. In an inquiry to the 1st year students of the Initial Teachers Training Course (Gomes & Ralha, 1999), it was verified that 28% of the students had more than 9 years of mathematics. Although almost all students considered mathematics to be interesting and useful, they find it hard to study (66%). Paradoxically, the majority of those asked believe that teaching mathematics to primary school children will be an easy task (72%).

Assuming that elementary mathematics is fundamental mathematics in the sense defended by Ma (1999), that is, even though it is presented in an elementary format it constitutes the foundations of the future mathematical learning and contains the rudiments of many important concepts in more advanced branches of the discipline, then the only sensible path to take seems to be to guarantee solid and efficient mathematical knowledge in the future teachers.

As a starting point to the study of the kind of mathematical knowledge Portuguese primary school teachers should have, we decided to analyse the mathematical primary school curriculum. It was also decided that we should do a pilot study, doing some observations of trainee teachers' classes, in order to gain a clearer picture of the real situation. We focused our attention on the teacher rather than on the children.

Pilot Study

In Portugal there is an official curriculum that results from the reforms mentioned above. Even though the curriculum does not exhibit great changes in respect to mathematical content (in accordance with international trends), it reflects a significant change concerning the main goals and the guiding principles for teaching mathematics.

A total of 6 groups of trainee teachers were observed (18 teachers), one lesson each, over a period of 2 months. The lessons were all video taped. It was our intention to pick up general information to be synthesised and reflected upon in further studies.

Our main focus was on the teacher and his/her approach to mathematical concepts but we also took into account the following hierarchical list of items: language (as used by both teachers and their pupils), approaches to problem solving activities, organisation and planning of mathematical activities, manipulative aids considered (by teachers) to be useful, etc.

A. Mathematics Teaching Goals

The three main goals for the teaching of primary school mathematics are stated as (DGEBS, 1989):

- Development of the ability for reasoning;
- Development of the ability to communicate;
- Development of the ability to solve problems.

This clearly reflects the influence of the NCTM Standards on the Portuguese primary mathematics curriculum even though initial teacher training in Portugal seems to be quite different from that in the U.S.A.

We believe it is crucial to have an explicit understanding of the essence of these purposes in order to avoid the error of only changing some aesthetic aspects of the mathematics lessons. As we observed, even when the classes were organized in groups, the predominant type of work was individual and traditional.

B. Teachers' Role

The main task imposed upon the teachers is to develop children's positive attitude towards mathematics (DGEBS, 1989). The affective component is regarded as a crucial one. There are several studies that relate love/hate for mathematics with success/ failure in the discipline (Renga & Dalla, 1993, McLeod, 1992). Dehaene (1997) also claims that *"children of equal initial abilities may become excellent or hopeless at mathematics depending on their love or hatred of the subject"* (p.8).

In what form does the affective relation between the teacher and Mathematics influence the relation of the pupil with Mathematics? Is it possible for a teacher who does not like mathematics to make students like it? From our observations it appears that when the teacher doesn't like mathematics or feels uncomfortable with the subject he/she tries to spend the least time possible on the subject. However he/she makes a considerable effort not to pass on the negative feelings to the students and also in preparing the mathematical lessons.

According to the curriculum, it is the teacher's responsibility to organize the means and create the proper environment for the fulfilment of the program.

However this responsibility raises some concerns:

- On the quality of the mathematical training of teachers; already it has been said and evidence from international studies proves that nobody can teach what they do not know and it is not enough to have a superficial knowledge of elementary mathematics. In fact, how can one expect that a teacher can create a proper environment for learning if he/she is not confident of his/her knowledge? If he/she repeatedly fails to give satisfactory answers to the questions that the pupils ask him/her? This way, not only will the environment be inadequate but probably it will also generate an atmosphere of unhappiness and frustration among the pupils.

- On the autonomy of the teacher; expecting the teacher to organize the means and create the proper environment for the fulfilment of the program seems to indicate that the teacher is supposed to re-create the curriculum. This attitude seems to be, like many others, imported from the United States, where a good teacher is one who constructs his own curriculum. In accordance with Ball and Cohen, cited in Ma (p.150),
  "this idealization of professional autonomy leads to the view that good teachers do not follow textbooks but instead make their own curriculum"

C. Problem Solving and Manipulatives

The core of the Portuguese curriculum is stated as being problem solving. It appears as if the only goal of mathematical activity is to be able to solve problems. Apparently problems are replacing content, becoming the contents themselves.

This educational approach, while exhibiting some short-term advantages, as for example improving self-confidence and motivation, raises several concerns, namely:

- Concerning the definition of problem. There are several different definitions of problem by different authors. Do teachers know exactly what we mean when we talk about a problem? What kind of problems do teachers use in their classes? The trainee teachers who were observed revealed incapacity to formulate problems. They believe that a problem is something that has a specific context, already exists in textbooks; it is motivating and different from the usual activities. They don't think it is their job to formulate problems and when facing a problematic situation they were unable to explore it.
- Concerning different approaches to mathematics teaching. According to Schroeder & Lester (1989), we can distinguish three different approaches: (1) teaching about problem solving, (2) teaching for problem solving, and (3) teaching via problem solving. What we found was that teachers use only the second approach. Are they aware of the other approaches?
- Concerning the teachers' ability to solve problems. Most of the teachers are not used to solving problems on their own. They look for solutions and just copy them.

The use of manipulatives is strongly recommended in the curriculum. The trainee teachers in the study always took materials for the class. This attitude seems to be justified for two reasons:

- The teachers believe that the use of manipulatives facilitates learning, motivates the students and makes learning more fun.
- Teachers involved in supervision expect trainee teachers to propose different activities, using manipulatives that are not typically used.

However, the use of manipulatives appears sometimes to be unnatural. In fact there were cases in which the teachers imposed the use of manipulatives even when the students didn't seem to need them. Strangely, there were other times when the teacher did not allow students to use the manipulatives.

D. Shape and Space (Introduction to Geometry)

The teaching goals presented in the curriculum for Geometry are:

- Development of the aesthetic sense and creativity;
- Development of the ability to compare, classify and transform;
- Understanding the world of shapes;
- Acquiring vocabulary and elementary geometric notions.

In the observed classes, the contents related to Geometry were less treated than those related to Number and Operations.

We may conjecture some reasons for that:

- Teachers' insufficient geometrical knowledge. Teachers don't feel confident in dealing with geometrical questions so they tend to avoid them.
- Teachers attribute little importance to geometry. It looks as if teachers consider the questions related to number and operations much more important than those related to geometry. Besides, at this level, they think that geometry *"concerns the formation of concepts about space and the mere observation of geometrical entities in space...* [Geometry] *tends at primary level to be all observation and no problems."* (Fielker, p.16).

The main focus of the geometry lessons was on the so-called "arbitrary" contents (Hewitt, 1999) which include names, definitions, notations and things alike, and where pupils can't come to acquire them by themselves and so, they explicitly need to be informed about. It looks as if the only important goal for the teaching of geometry is the recognition and naming of shapes. This attitude seems consistent with the one observed by Clements & Battista (1992) of some American teachers. However, even though the teachers emphasized the knowledge of the "arbitrary contents", we found situations when they were not confident of their own knowledge. For instance they used "right triangle" instead of "right-angled triangle" or the term "diagonal" to mean "oblique". It is significant to report that the teachers consider such incidents unimportant. They assume that students pay no attention to them as if students had some sort of filter that separates the things that are important to understand and memorize from the ones that are not. This is only one example of the lack of importance attributed to rigorous treatment of geometry (and of mathematics in general). This attitude is contrary to the recommendation of the *Geometry Conference* which says that *"the degree of rigor in the teaching of mathematics may vary according to circumstances, but that should never be an excuse to misinform or to mislead the student"* (p.286).

One of the topics in the mathematics program refers to the relative position of two lines in the plane and also in space. Students should be able to recognize parallel and perpendicular lines from the observation of solids. These terms parallel and perpendicular seem to appear in the program as opposites. But two lines that are not parallel don't have to be perpendicular. However the teachers in the pilot-study don't explore the possible relations between two lines. Moreover, they don't even consider the difference between these relations in the plane and in space. The only definition they used for parallel lines was "two lines that never meet". They seem to be unaware of the limitations of this definition.

As for perpendicular lines they defined them as being two lines that meet each other and make a right angle. But at this stage the students lack the notion of angle. So they just memorize the definition without any sort of understanding. This way, although the teachers defend meaningful learning, they are promoting meaningless learning, based on memory, which seems to reveal an insufficient content knowledge on their part (Ma, 1999).

Conclusion

Analysis of the results came to convince us that the so-called "elementary mathematics" is neither easy nor easy to teach. The role played by primary school teachers is crucial in what concerns the introduction of mathematical contents and therefore the mathematical training of these teachers deserves a deep analysis and the achievement of clear evidence.

We are looking for some kind of mathematical training, eventually with some cultural influence, clearly justified that makes the future teachers able to teach elementary mathematics in a more efficient way than the one we have been reporting, in Portugal.

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TACIT KNOWLEDGE IN CURRICULAR GOALS IN MATHEMATICS

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Key words: mathematical teaching – curricular guidelines – tacit knowledge – mathematical knowledge

ABSTRACT

In this paper the authors analyze the current curricular goals in mathematics as proposed for school levels K-7 to K-12 (ages 11 at 16) in different countries. Based on Paul Ernest's view of mathematical knowledge, the authors consider school-acquired mathematical knowledge as multidimensional, in the sense that it involves components from different domains: cognitive and social, beliefs and values. Furthermore, most of those components are of a mainly tacit nature. The authors present evidence to support that the goals identified in those curricula foster the learning of a mathematical knowledge that is mainly tacit in nature. On the other hand, they argue that the curricular guidelines for the teaching of mathematics lack the supports to handle the processes involved in the learning of any knowledge of that nature. Part of the current literature on the subject emphasizes that such knowledge can be learned although it cannot be taught in the traditional sense of the word *teach*, that is, by the teachers' publicly transmitting or stating their knowledge. The same literature, although not dealing specifically with the teaching of mathematics, suggests, for instance, that the act of teaching a knowledge that is mainly tacit is closely linked to the teacher's public actions in face of authentic questions. That is, when he is engaged in a situation which demands the use of his own tacit knowledge. The authors conclude by discussing some curricular implications for the teaching of mathematics, which result from those issues.

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Introduction

In the last thirty years we have witnessed a growing movement of changes in the understanding of what mathematical knowledge is about. In order to understand such changes, one has to remember that traditional mathematical epistemology used to assume that mathematical knowledge could be described, on the whole, through a set of explicitly formulated sentences and thus regarded as *essentially explicit* (Ernest 1998a). Such a conception has influenced the teaching of mathematics for years in what concerns primarily the learning of the formal aspects of the systematization of that knowledge.

In contrast, there is a current tendency in the epistemology to regard mathematical knowledge as a social practice in its wide sense. This tendency is clearly seen in the present-time curricula where we find consensus about the need to fill in the gap between school-acquired mathematical knowledge and some of the practices and processes used by mathematicians to produce mathematics (Romberg 1992, Shoenfeld 1992, Winbourne and Watson 1998, Ernest 1998a, 1988b). Mathematical knowledge is then reshaped: besides the relative components to its justification, it includes other equally relevant components, which are, by nature, mainly tacit. That is, knowledge built on experience or action and which cannot be fully described by rules or words.

An analysis of the current curricular goals set for the teaching of mathematics based on Ernest's model of mathematical knowledge (1988b) shows that such goals foster the learning of a knowledge that is more tacit than explicit in nature. This holds true for students at several school levels and in different countries. However, those curricular guidelines lack the support to handle the process involved in the learning of that kind of mathematical knowledge.

With this in mind, the aim of this work is to promote a critical reflection on the implications of the curricula that result from those issues. To this end, this paper is organized in three sections. *In the first*, we digress on Ernest's model of mathematical knowledge (or, as he says, of mathematical learning). *In the second*, we present evidence to support that the current curricular goals set for the teaching of mathematics in different countries foster the learning of a mathematical knowledge that is mainly tacit in nature. This trend was shown in Frade and Borges (2001) in the discussion about a given level of teaching in the Brazilian case. *The conclusion* discusses some curricular implications arising from that trend.

1. Ernest's Model of Mathematical Knowledge

Following and expanding Philip Kitcher's view of mathematical knowledge, Ernest (1998b) regards mathematical knowledge as a social practice and describes such knowledge through a multidimensional model whose components are classified as either mainly explicit or mainly tacit. As we understand it, for Ernest, mainly explicit mathematical knowledge is the knowledge that can be taught through a propositional language, as for instance, the Pythagoras' theorem. Alternatively, mainly tacit mathematical knowledge is that which is built on experience or action and cannot be fully taught explicitly.

To Ernest, mainly explicit mathematical knowledge includes the knowledge of a set of:

1. Accepted propositions and statements (**PS**)
2. Accepted reasoning and proofs, including less formal ones (**RP**)

3. Problems and questions (**PQ**).

As mainly tacit components he cites:

4. Knowledge of mathematical language and symbolism (**LS**)
5. Meta-mathematical views, that is, views of proof and definition, scope and structure of mathematics as a whole (**MV**)
6. Knowledge of a set of procedures, methods, techniques and strategies (**PMTS**)
7. Mathematical aesthetics and personal values regarding mathematics (**AV**).

For Ernest, the word *knowledge* covers both theoretical and practical knowledge. In the case of mathematics, the latter corresponds to the use of mathematical knowledge. Secondly, the first two components – accepted propositions and statements and accepted reasoning and proofs – are mainly explicit since they are strictly related with warrants in mathematics. As long as they are kept under discussion within the mathematical community, problems and questions relevant to mathematicians are also mainly explicit.

Based on Wittgenstein's (1995) concept that a word is given meaning through its suitable use in a *language game* or in *forms of life*, on Polanyi's (1962) view that any propositional knowledge rests on the tacit knowledge of language, and others, Ernest (1999a) sets his argument according to which the fourth component – language and symbolism – is mainly tacit. To Ernest, meta-mathematics views constitute a tacit element of mathematical knowledge in the sense that the mathematicians acquired and built them up through the enculturation of the mathematics community. And this experience cannot be fully explicitly taught.

With respect to procedures, methods, techniques, strategies, he argues that although they are often applicable to new problems, they are used differently in different situations. Thus, he states that, "(…) while the applications of these procedures and strategies are explicit, the more general knowledge underpinning them normally is not" (1998b, 13). To Ernest, it is not the procedures, strategies and algorithms that are not explicit but that underlying general knowledge of how and when one uses them, for example.

The last component – aesthetics and values – transcends the meta-mathematics views and is mainly tacit as long as the feelings about the aesthetics and the beauty of mathematics are closely linked to personal beliefs and values, which are only partly articulated.

How we interpret Ernest's model and on what it can help us with relation to the aims of that work

First of all, let us interpret Ernest's model as compared to some aspects of Polanyi's (1983) theory on tacit knowledge. Among the various types of knowledge used to support the task of teaching mathematics are Ernest's mainly explicit and mainly tacit components. This understanding gives us a clear example of how mainly explicit mathematical knowledge, as for instance, the Pythagoras' theorem, may become tacit in Polanyi's sense. As we understand it, when we use certain knowledge as subsidiary to another, the former is mobilized as tacit knowledge. In our case, it means that while the Pythagoras' theorem is being used as a tool to solve a problem, that specific knowledge is not explicitly shown (at that moment we may not even be aware of holding such knowledge) as it is not our focus of attention. Thus, what is taken as tacit knowledge depends on the context of situation.

On the other hand, Ernest's use of the expressions *mainly explicit* or *mainly tacit* implies an attempt to stress these two dimensions as complementary to one and only knowledge. Let us think, for example, about this as represented in a scale where the extremes could be one the totally inarticulate and the other the totally articulate knowledge. In such a scale the components of the mathematical knowledge are either close to one extreme or to the other but never reach any of them. Besides, the position of one component in the scale is directly related to its learning: the closer the component is to an extreme, the easier or the more difficult to reach it through a propositional language, depending on what extreme the component is close to.

It is our understanding that Polanyi's *fragmentary clues* - which allow for the identification of the particulars of a given tacit knowledge someone is trying to communicate - can be more or less meaningful, more easily learned or not, depending on how that someone is handling either mainly explicit or mainly tacit mathematical components. In sum, (1) tacit mathematical knowledge is any type of mathematical knowledge (such as, the mainly explicit components and the mainly tacit components of Ernest's model) used as subsidiary to the performance and control of a mathematical task. (2) If a certain type of mainly tacit mathematical knowledge (in Ernest's sense) is used as subsidiary by a first person, the fragmentary clues that allow for a second person to identify them will demand great effort from the second person to apprehend and integrate them.

Secondly, as long as it does not embrace the cognitive/psychological processes involved in mathematical learning, Ernest's model is more closely related to an ontological than to an epistemological model. However, that model helps us understand the kinds of mathematical knowledge, as for instance, concepts, procedures and attitudes or dispositions, which are currently enhanced in mathematics curricula. In fact, according to Ernest, mathematical knowledge is not a single block of knowledge pertaining to a single domain; it aggregates multiple faces or multiple domains: cognitive and social domains, beliefs and values. Furthermore, most of those components are, by nature, mainly tacit. This means that part of mathematical knowledge can be taught through the transmission of propositional knowledge, but most of it cannot. Only in this sense do we understand Ernest's statement that his model is also able to describe the process of learning mathematics. Finally, we cannot forget that Ernest's model describes a knowledge that is mainly tacit. Thus, it must be considered with all the limitations that result from the attempt to explicit any knowledge of that nature.

One should remember that, according to Polanyi (1983), when one tries to describe a tacit knowledge through the closely scrutiny of its particulars or explicit the relation between them, the meanings of that knowledge are effaced and their original meaning cannot be recovered.

2. Tacit Components of Mathematical Knowledge in Current Curricular Goals

In this section we analyze some current curricular goals for the teaching of mathematics in the light of Ernest's model. The aim of the analysis is to present evidence to support the statement that, in different countries, those goals foster the learning of those components of mathematical knowledge that are mainly tacit in nature.

To this end we analyze the Attainment Target 1 – Using and Applying Mathematics – for Key Stages 3 and 4. The material is suggested by The National Curriculum for Math of the United Kingdom (Appendix). Our choice to analyze Target 1 was based on the belief that it expresses the general goals for the teaching of mathematics in what refers to delimiting the context in which the other targets – Number and Algebra; Shape, Space and Measures; Handling Data – are to be developed.

Although we understand that any sub-target of Target 1 can embrace others, if not all components of Ernest's model, from the analysis of each sub-target we identify the *dominant* components of the model. These must then be constructed in order to reach or to accomplish those sub-targets. At the end of the analysis of Target 1 we obtained the identification represented in Table 1.

Table 1 – Dominant components of Ernest's model identified in the curricular goals in United Kingdom

	Target 1 (key stages 3 and 4) - Using and applying mathematics	**Components**	**Nature**
1	**Puplis should be given opportunities to:**	PS	ME
a)	use and apply mathematics in practical tasks, in real-life problems and within mathematics itself;	PMTS	MT
b)	work on problems that pose a challenge;	AV	MT
c)	encounter and consider different lines of mathematical argument.	MV	MT
		AV	MT
2	**Making and monitoring decisions to solve problems**	AV	MT
a)	find ways of overcoming difficulties that arise; develop and use their own strategies;	PMTS	MT
b)	select, trial and evaluate a variety of possible approaches; identify what further information may be required in order to pursue a particular line of enquiry; break complex problems into a series of tasks;	PMTS	MT
		MV	MT
c)	select and organize mathematics and resources; extend their view and reflect on alternative approaches of their own;	PMTS	MT
		MV	MT
		AV	MT
d)	review progress whilst engaging in work, and check and evaluate solutions.	AV	MT
		MV	MT
3	**Communicating mathematically**		
a)	understand and use mathematical language and notation;	LS	MT
b)	use mathematical forms of communication, including diagrams, tables, graphs and computer print-outs;	LS	MT
c)	present work clearly, using diagrams, graphs and symbols appropriately, to convey meaning;	LS	MT
d)	interpret mathematics presented in a variety of forms; evaluate forms of presentation;	MV	MT
e)	examine critically, improve and justify their choice of mathematical presentation.	MV	MT
		RP	ME
4	**Developing skills of mathematical reasoning**	LS	MT
a)	explain and justify how they arrived at a conclusion or solution to a problem;	RP	ME
b)	make conjectures and hypotheses, designing methods to test them, and analyzing results to see whether they are valid;	AV	MT
		PMTS	MT
		MV	MT
c)	understand general statements, leading to making and testing generalizations; recognize particular examples, and appreciate the difference between mathematical explanation and experimental evidence;	LS	MT
		PMTS	MT
		MV	MT
d)	appreciate and use 'if…then…' lines of argument in number, algebra and geometry, and draw inferences from statistics;	AV	MT
		RP	ME
		PMTS	MT
e)	use mathematical reasoning, initially when explaining, and then when following a line of argument, recognizing inconsistencies.	RP	ME
		MV	MT

Key: **Components** – Dominant components of Ernest's model; **Nature** – Nature of the components; ME – Mainly Explicit; MT – Mainly Tacit

In the case of sub-target 4 – Developing skills of mathematical reasoning – the identification corresponding to the letter b, for example, results from our interpretation that the action "make conjectures and hypotheses" is closely connected to a favorable disposition to inquire or pose questions. Such disposition originates from personal experience, beliefs and values about mathematics. On the other hand, "designing methods to test them" involves not only the close observation of specific cases to unveil regularities but also the knowledge of accepted mathematical ways to test hypotheses and results. Finally, "analyzing results to see whether they are valid" demands, among other actions, connecting the old and the new and developing a way of thinking based on evidence or argumentation. Such an action demands a type of knowledge that is constructed through a slow process of enculturation and some understanding of how mathematics works in the context that the results are being analyzed. Such identification exemplifies how the process of analyzing the curricular goals was constructed in this work.

We then find not a single and precise identification of the sub-targets and the components of Ernest's model but a combination of the dominant components involved in each sub-target.

On close inspection it was possible to see the prevalence of the mainly tacit components encountered in Target 1, in particular, the more elusive and slower components to acquire: meta-mathematics views and aesthetics and values. Those are the ones that shape our mathematical way of thinking more deeply as they are, to a great extent, stable in time. This prevalence can be found in the curricular goals proposed in other countries, such as Germany (Table 2), Brazil (Table 3) and Portugal (Table 4). (Refer to the Appendix for documentation)

Table 2 - Dominant components of Ernest's model identified in the curricular goals in Germany

General goals of mathematics teaching (general education)	Components	Nature
• mathematics as a theory and as a tool for solving problems in natural and social sciences, including modelling;	MV	MT
	PS	ME
	RP	ME
	LS	MT
	PMTS	MT
• experiences with fundamental ideas in mathematics like the idea of generalization, the need for proving, structural aspects, algorithms, the idea of infinity, and deterministic versus stochastic thinking;	MV	MT
	PMTS	MT
• methods of getting insights like inductive and deductive reasoning, methods for proving, axiomatic, formalization, generalization/specification, heuristic work;	RP	ME
	LS	MT
	PMTS	MT
• variation of argumentation levels and representation levels in all fields and aspects of mathematics teaching;	MV	MT
	LS	MT
• historical aspects of mathematics.	AV	MT

Table 3 - Dominant components of Ernest's model identified in the curricular goals in Brazil

General goals of mathematics teaching (third and fourth cycles of elementary school)	Components	Nature
• identify mathematical knowledge as a means to understand and transform the learner's surrounding world; understand that mathematics is an intellectual game, and as such, a trigger to promote interest, curiosity, investigative mind, and develop the ability to solve problems;	MV AV	MT MT
• use mathematical knowledge (arithmetic, geometric, metric, algebraic, statistic, arrangement, probabilistic) to make systematic observations about the quantitative and qualitative aspects of the real world aiming at establishing relations between those aspects.;	PS LS MV	ME MT MT
• select, organize and produce relevant information to be interpreted and evaluated critically;	LS MV	MT MT
• solve problems, validate strategies and results, develop various forms of reasoning and processes such as intuition, induction, deduction, analogy, valuation. Use mathematical concepts and procedures and every technological tools available;	MV RP PMTS	MT ME MT
• establish mathematical communication, that is, describe, represent and show results accurately, argue in favor of learner's own conjectures, making use of speech and establishing the relations between speech and various mathematical representations;	LS RP MV	MT ME MT
• establish connections between mathematical subjects from distinct fields and between those subjects and the knowledge of other fields of the curricula;	MV	MT
• feel capable to construct mathematical knowledge, develop self-esteem and persist in the search of solutions;	AV	MT
• interact cooperatively with peers, working collectively in search of solutions for the problems posed. Identify common sense about the subjects discussed and be respectful of peer's viewpoints while learning from them.	AV	MT

Table 4 - Dominant components of Ernest's model identified in the curricular goals in Portugal

Mathematical competence at basic education integrates attitudes, skills and knowledge, and includes:	Components	Nature
• the disposition and capacity to think mathematically, this is, to explore problematic situations, search for patterns, formulate and test conjectures, make generalizations, think logically;	AV MV	MT MT
• the pleasure and self-confidence in developing intellectual activities involving mathematical reasoning and the conception that the validity of a statement is related to the consistence of the logical argumentation rather than to some external authority;	AV MV	MT MT
• the capacity to discuss with others and communicate mathematical thoughts through the use of both written and oral language adequate to the situation;	LS AV	MT MT
• the understanding of notions such as conjecture, theorem and proof, as well as the capacity to examine the consequences of the use of different definitions;	MV RP	MT ME
• the disposition to try to understand the structure of a problem and the capacity to develop problem solving process, analyze errors and try alternative strategies;	LS PMTS MV AV	MT MT MT MT
• the capacity to decide about the plausibility of a result and to use, according to the situation, mental computational process, written algorithms or technological devices;	MV PMTS	MT MT
• the tendency to "see" the abstract structure underlying a situation, from daily life, nature or art, involving either numerical or geometrical elements or both.	MV AV	MT MT

A similar pattern can be found in the curricular goals proposed for the teaching of mathematics in the USA, Spain and Canada where the curricula undergone similar changes in the 1990's. (Refer to the Appendix for documentation).

3. Discussion

In face of the changes in the conception of the epistemology of mathematical learning, many scholars (Schoenfeld 1992, Romberg 1992, Winbourne and Watson 1998, among others) have been putting emphasis on the importance of creating learning environments where teachers and students would be involved in actual mathematical experience. On account of that, we can say that Ernest's model re-signifies mathematical learning when it characterizes it as being mainly tacit. In other words, such an approach tells us that a great deal of mathematical knowledge cannot be either taught or learned by means of explicit transmission.

Although schools have incorporated a discourse in favor of the actions advocated by current mathematics curricula, such a discourse has not been given actual support as the practice keeps treating the teaching and learning of mathematical knowledge as mainly explicit (refer to Romberg, 2001). The reason may be that the curricular guidelines for the teaching of mathematics lack the support to handle the processes involved in the learning of a knowledge that is mainly tacit, as the quotes below suggest:

> My third observation is related to the concept of competence and, in the case of mathematics, the definition of mathematical competence. Doubts and criticism on the presented proposal showed that a broad concept is difficult to be widely accepted. Terms like *disposition* (to think mathematically), *pleasure* (in developing intellectual activities) or *tendency* (to look for the abstract structure) have been especially criticized with the argument that is very difficult to make such things "operational". (Abrantes 2001, 35)

> It is said that thinking mathematically and developing mathematical skills through learning mathematical content is important. However, the meaning of mathematical way of viewing and thinking is interpreted in several ways among university mathematics educators. Among schoolteachers there is some confusion about the meaning. (Kunimune and Nagasaki 2001, 2)

Approaching the subject in the light of the literature on tacit knowledge will only help us understand how to teach and how to learn the various types of school-acquired mathematical knowledge that are being valued at present, for example, those commonly labeled *mathematical competencies* in some countries. We understand that Polanyi (1983) and Schön (1987) imply Ernest's sense of mainly tacit knowledge when both stress that tacit knowledge can be learned. However, they claim that it cannot be taught in the traditional sense of *teaching*, that is, by means of stating the knowledge the teacher holds or by making it explicit.

When evaluating the teaching of architectonic design, Schön suggests that the act of teaching tacit knowledge is closely connected to the teacher's public actions in face of authentic questions, that is, as he is involved in a situation that demands the use of his own tacit knowledge. That means, for example, that the teacher's act of doing standard exercises and solving problems on the board, which does not actually challenge him, does not correspond to that type of practice. As for the learning of a disposition to think mathematically, what we interpret Schön suggests is that the students should be exposed to a number of experiences that would allow them to *see* their teacher *think mathematically.*

More generally, according to Polanyi, a person can learn or know about a second person's tacit knowledge through the apprehension of some of its particulars, which are provided by fragmentary clues, and a great effort to understand the meanings of those few apprehended features. On the other hand, for the latter to be able to communicate the features of his tacit knowledge to the former, it is necessary to provide him with suitable means to express it. Thus, Polanyi says that both the communication and the integration of the particulars of a tacit knowledge occur through their meanings. As we see it, mathematical teaching demands, among other things, a great effort from the teacher to develop a sensibility to apprehend the fragmentary clues provided by the students and observe how they become manifest when students mobilize mainly explicit and mainly tacit components of mathematical knowledge.

From all this it is possible to foresee the consequences of a curricular tendency to value the tacit components of mathematical knowledge in the teaching practice. There is however one aspect yet not stressed but of equal relevance and equal consequences for the teaching of mathematics. We refer to the understanding that most of the assessment practices to evaluate mathematical learning in school are based on the assumption that mathematical knowledge is either of a fully explicit nature or possible to be made explicit in its entirety (refer to Romberg, 2001). As such, it is clear that such practices are potentially inadequate as evaluative of a curriculum that stresses the tacit components of mathematical knowledge.

Teachers can use their previous experience with students' evaluation to understand the difficulty the learner finds in apprehending the tacit components of mathematical knowledge. It bears the same nature - and probably similar or higher intensity - of the difficulty that they have when trying to apprehend the tacit knowledge of their students.

Understanding mathematical knowledge as proposed by Ernest's and finding the most adequate way of evaluating students' development demand the teacher's commitment with both the use of new assessment tools and the awakening and tuning of his own sensibility to the new trend. For the curricular trend here discussed to become effective, it is necessary that the nature, the curricula and the teaching and learning processes that characterize the basic qualification required for teachers be fully reviewed. Those changes must place the teacher's formative process in tune with the curricular goals that value the tacit components of mathematical knowledge. All the same, they must aim at the adequate qualification of the reflective teacher.

Appendix: Documents used in section II.

Canada
Mathematics - The Ontario Curriculum
http://www.edu.gov.on.ca/eng/document/curricul/curr97ma/curr97m.html (12/16/01)

Brazil
1998. *Parâmetros Curriculares Nacionais - Terceiro e Quarto Ciclos do Ensino Fundamental - Matemática.* Brasília: MEC/SEF.

Germany
Kaiser, Gabriele 2001. A Description from Germany. *Proceedings of PME25* 1: 164- 169

Weidig, Ingo. 2 Mathematics teaching in Germany.
http://www.mathematik.uni-wuerzburg.de/History/meg/weidiga2.html (12/16/2001)

Portugal
Abrantes, Paulo 2001. Revisiting the Goals and the Nature of Mathematics For All in the Context of a national Curriculum. *Proceedings of PME25* 1: 25- 40

Spain
1989. *Diseño Curricular Base - Educación Secundaria Obligatoria.* Ministerio de Educacion y Ciencia. Vol. 1: Capitulo 2, 478-549

United Kingdom
The National Curriculum For Maths
http://www.dfee.gov.uk/nc/matks34.html (11/16/1998)

United States
2000. Principles and Standards for School Mathematics. Reston (VA): The National Council of Teachers of Mathematics, Inc.

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Frade, Cristina and Borges, Oto 2001. Componentes Tácitos e Explícitos do Conhecimento Matemático nas Orientações Curriculares para o Ensino de Matemática. *Anais 2001 Anped.*

Kunimune, Susumu and Nagasaki, Eizo. Curriculum Changes on Lower Secondary School Mathematics of Japan – Focused on Geometry.

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Polanyi, Michael 1962. *Personal Knowledge.* London: Routledge & Kegan Paul.

_____ 1983. *The Tacit Dimension.* Gloucester (Mass): Peter Smith.

Romberg, Thomas A. 1992. Problematic Features of the School Mathematics Curriculum. *Handbook for Research on Curriculum.* Edited by Philip W. Jackson. New York: MacMillan. Chapter 27, 749-788.

_____ 2001. Mathematics Goals and Achievement in the United States. *Proceedings of PME25* 1: 180-185

Schoenfeld, Alan H. 1992. Learning to Think Mathematically: Problem Solving, Metacognition, and Sense Making in Mathematics. *Handbook for Research on Mathematics Teaching and Learning.* Edited by D. Grouws. New York: MacMillan. Chapter 15, 334-370.

Schön, Donald A. 1987. *Educating the Reflective Practitioner.* San Francisco: Jossey-Bass.

Winbourne, Peter and Watson, Anne 1998. Participating in Learning Mathematics through Shared Local Practices in Classrooms. *Situated Cognition and the Learning of Mathematics.* Edited by Anne Watson. Oxford: Centre for Mathematics Education Research, University of Oxford. Chapter 7, 93-104.

Wittgenstein, Ludwig 1995. *Tratado Lógico-Filosófico. Investigações Filosóficas.* Tradução de M.S. Loureiro. 2a edição revista. Lisboa: Fundação Calouste Gulbenkian.

SOON UNACCOUNTABLE*

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ABSTRACT

The Mathematics Across the Curriculum Project at Dartmouth College produced a number of new courses integrating mathematics with a humanistic discipline such as literature, art, or philosophy. These courses all were free of any prerequisite and attracted a wide variety of students. The mathematical topics were chosen for their relative modernity and sophistication, e.g. group theory, infinity, or the fourth dimension. How does one come up with math that can be offered in these interdisciplinary courses? How do you present it in a way that isn't trivial? What sort of understanding is it reasonable to expect students to carry away as a result of such a class? Why is it worth the trouble to educate this body of students in this particular way? In this talk we will consider these questions and get a glimpse into some unusual courses.

Keywords: Curriculum development, humanities, liberal arts, interdisciplinary, quantitative literacy, math phobia, math avoider, art, philosophy, literature, group theory, infinity, fourth dimension, history, astronomy

*When I Heard the Learned Astronomer

When I heard the learned astronomer,
When the proofs, the figures, were ranged in columns before me,
When I was shown the charts and diagrams, to add, divide, and measure them,
When I sitting heard the astronomer where he lectured with much applause in the lecture-room,
How soon unaccountable I became tired and sick,
Till rising and gliding out I wandered off by myself,
In the mystical moist night-air, and from time to time,
Looked up in perfect silence at the stars.

Walt Whitman

Proofs and figures

Class begins and the students arrive, homework in hand. The work, consisting of block-printed mandalas and small handwritten tables, is not dropped on the front desk for the benefit of the instructor, but posted on the blackboards for all to see. The first ten minutes of this two-hour class is an art exhibit. The homework problem was to design two mandalas whose symmetry groups were of the same order but not isomorphic. The calculations accompanying each design are the group tables associated to its symmetry group. The designs are beautiful, but most solve the mathematical question the same way. One at a time the students comment on their work, acknowledging that their choice ("Mine are also a D4 and a Z8") was usually the popular one. Once in a while the problem is solved incorrectly and the students themselves point that out. No comments are necessary from the instructor except for an indication when to move on to the next piece. All is going exactly as expected until J.P.'s mandala.

A few weeks earlier J.P. had brought in a sort of paper bracelet with symmetric fishes swimming on the inside and outside of it. He wanted to know if it qualified as a "mandala" for the purposes of an earlier assignment. That discussion (without firm resolution, by the way) together with the instructor's assurance that there were lots more groups out there besides the dihedral and cyclic ones, led J.P. to a clever solution to this week's problem. The paper bracelet he brought in on this day had six horizontally symmetrical motifs and could be flipped inside out to yield a total of twelve symmetries. J.P. believed that the resulting symmetry group was neither Z12 nor D6, but how could he be sure? The answer to his question involved a discussion of clock arithmetic, direct products of groups, and methods for telling two groups apart by counting the number of subgroups of a given order. All finally agreed that J.P. had made an object whose symmetry group is Z2 x Z6.

One purpose of this paper is to argue that courses like the one described above, and experiences such as those of J.P., are essential to educating a population to quantitative literacy. Please notice that I am not saying that every student should take such a course or have such an experience, but students should have access to such experiences if they desire them. I would like to emphasize the point that educating a population to accomplish a certain thing is a completely different proposition from educating a large number of individuals to some dubious Platonic mathematical ideal. J.P. and his fellow classmates had a strong intellectual experience that taught them something about what it is to do mathematics, how mathematics informs the arts, and what modern mathematicians are sometimes thinking about. Such understanding can only come from a freely chosen course of study that speaks to individual interests. It cannot come from a forced march through statistics or calculus or any other course.

The students in J.P.'s course were not generally math or science majors. If anything, most were the so-called "math avoiders" that we sometimes dread to teach. Yet, those twenty or so students now have a deeply embedded knowledge of very elementary group theory. They say that they bore roommates and friends by constantly pointing out examples of D4 in fabric and architecture. They talk about these groups as if they were personal acquaintances, as they are. We can't predict what they will do with this knowledge later. But we can say that the ideas of group theory will be carried into vocations and situations that they would not otherwise have entered. This effect is an extremely pragmatic outcome of educating a population as such, rather than as an aggregate of individuals held to a single standard. Intellectual diversity in the national population would be a blessed thing.

To add, divide, and measure

A class of about fifteen first year students, one of the "first year writing seminars" at Dartmouth, is trying to figure out how to admit the guests who arrive at a rapid rate at Hilbert's Hotel. Hilbert's Hotel has infinitely many rooms, numbered by the natural numbers. Alas, the arriving guests are evacuating a similar motel chain—an infinite sequence of motels each containing an infinite number of guests, all enumerated by the natural numbers. They arrive on a sequence of very large buses, one for each motel. Can Hilbert's Hotel contain them all? The students begin to argue. They are organized into three groups. One group solves the problem by dividing the rooms in half and using only half of them on the first bus. They then divide the remaining rooms in half and use half of those on the second bus, etcetera. The second group assigns a prime number to each bus and an integer to each passenger on a bus. The passenger is put into the room corresponding to the bus's prime raised to the passenger's integer, without overlap. The third group of students tries to make a probabilistic argument. If the rooms are assigned at random, they wonder, then aren't the chances of using up all of them zero? So there should always be some room left, if assignments are made at random. The desk clerk can stop worrying. The instructor, however, looks unconvinced.

The students in the "How many angels?" course are a completely different population from the ones in the "Pattern" course of our first example. Although each of these courses had some math lovers present, in this class all of the students were very interested in math. Indeed, many of them go on to major in the subject and some of that class of sixteen are now in graduate school. We have found that when the required first year writing course centers on mathematics and the humanities, it attracts a room full of prospective majors. When the course touches on advanced subjects it gives the students a foretaste of the courses they will see as juniors and seniors. By its nature, it asks these mathematically inclined students to read and write about mathematics in a different way from standard courses. In this course mathematical literacy is inseparable from the usual sort of literacy, as all mathematical learning is demonstrated through written essays.

A mathematician and a philosopher designed this course jointly. Each helped the other select readings and design exercises that would illuminate both the mathematics of infinity and the philosophical debates that accompanied mathematical developments. The philosopher visited Dartmouth for a quarter to work with the mathematician. Each learned enough of the other's subject to feel at ease with the material. Each of the professors taught a version of this course by themselves in their own department at their own institution. Neither could have done as well had they designed the course alone.

Much applause in the lecture room

One of the most popular interdisciplinary courses at Dartmouth, with over a hundred students in attendance, is a course on "Time", taught by Professor D in the mathematics department and Professor B in comparative literature. The students have studied ancient methods of timekeeping and alternative concepts of time from various cultures. It is late in the term and Professor D is lecturing on relativity and Einstein. The class listens and takes notes. After a while, Professor B interrupts Professor D. She informs him that he is not making any sense. He has made statements, but hasn't explained them to her satisfaction nor argued them convincingly. Then she turns to the students. She points out that if she said completely incomprehensible things about a piece of literature, they would never let her get away with it. They would question, disagree and push hard until they at least understood why she made her claims. Why haven't they questioned the professor of math on his unclear and ill-explained statements? Do they really understand everything he just said? The students admit they do not. Why, then, this reverence for the authority of the scientist? Why this unquestioning acceptance of nonsensical claims? Both professors agreed that the ensuing discussion made that particular class the best day of the term.

This particular course came about because the mathematicians most interested in doing work with the humanities were booked solid. Therefore others were enjoined to "make friends with a humanist" in order to see if there were any areas of commonality that could form the basis for an intellectually strong course. Within a few months of making such a strategy explicit we had two new collaborators in the humanities. In both cases, the mathematicians were flexible, rearranging their interests around the areas of expertise of the humanities faculty members with whom they worked. One goal was to keep the humanist feeling secure in what sometimes seemed to be a risky endeavor. The quite evident security displayed by Professor B in the above example owes its remarkable character to the fact that Professors B and D are a married couple. (You have no excuse for not talking to a humanist, we had told D.)

Up in perfect silence

Rarely is it necessary to consult astronomical charts before offering a class, but the "Renaissance Astronomy and the New Universe" course can only run in terms where a planet is visible in the night sky for most of the term. The laboratory exercise depends on such a planet, preferably Mars. If we are very lucky with scheduling, Mars will show us some retrograde motion before the term is out. Mars offers two advantages over other planets. First, it travels relatively quickly against the backdrop of the stars. Secondly, the "Mars problem" was the source of great dissatisfaction with all predictive models of astronomical behavior prior to Kepler. So, the course is planned around the appearance of Mars in the night sky, which for some reason seems to like to happen during winter quarter. The students are sent out to look for it and track its motion on clear nights when the temperature drops below zero Fahrenheit. They become remarkably quick at locating it.

Our students have a modern education and they "know" that the earth is not the center of the universe, that the stars and planets do not rotate around it, but that our predecessors believed it so due to the appearance of the night sky. They consider the Copernican model to be an obvious truth. Most of them have never really looked at the sky. Many come from cities where you can't see the stars.

Textbooks evidently paint a picture of early astronomers as quaint old gentleman who were at the mercy of a lot of silly assumptions and lack of a telescope. The students learn otherwise as they proceed to make every possible error in measuring the location of Mars. They try to use landmarks to judge where a star is, a mistake Ptolemy would never have made. After some discussion they abandon that method. Of course, they "knew" all along that the night sky moves all night. They learn to use the zodiac the same way our ancestors did. After a few months of observation they learn a few things.

First, there are some kinds of science in which data is absurdly hard to obtain. One of the things Europe imported during the renaissance was the star record kept by Arabic astronomers for a thousand years. As the students go out night after cloudy night, they begin to understand the priceless nature of such data. Sometimes they complain that the course should be held in a less cloudy location, but are reminded by the instructor where Copernicus and Tycho Brahe were living. It was not so different.

Second, the students eventually come to see the Ptolemaic system as the most natural explanation for celestial motion. They make a full turn and begin to wonder why anybody would have believed Copernicus at all. Then the course gets interesting, because this is the right question. Another thing astronomers obtained in Renaissance times was "Euclid's Elements of Geometry", one of the great Greek intellectual mathematical advances. The students read Copernicus (in English) to see how closely he imitated the rhetoric of Euclid. Copernicus had two things at his disposal—a mountain of valuable rare data and an irrefutable form of argument, the mathematical proof. How these factors interacted, which was more important to his argument, and how surrounding beliefs and historical forces contributed to the discussion; all become part of the answer to that question.

Charts and diagrams

All four of the courses de scribed above were a success at some level and we are sure to offer all of them repeatedly at Dartmouth. We have several more successes, including a math and music course and a mathematical science fiction course, not described in this paper. Some courses attract science and math majors, most attract the so-called "math avoiders". Some are very large, some quite small. Some require two instructors, others only one. Some require problem sets, others require papers, artwork, or musical composition. Exte nsive evaluation shows that the courses are a success by delineating for us what kind of things the students learned, whether they felt they learned a lot, and whether faculty felt the students were doing good work. However, for each course the details are different. What a potential math major gets out of the Infinity course cannot be compared directly to what an English major in the Time course takes away. Nonetheless, after many years of tracking these courses we are confident of their varied kinds of worth. The question we must ask is this: what accounts for success? I will offer you some design principles that have guided us in the hope that they will serve you as well as they have served us.

Before creating any courses it is important to achieve some consensus, preferably across departments, as to why you are doing so. What is the goal of all this work? It is fairly easy to get faculty to agree that calculus serves the sciences better than it serves students in other disciplines, but that observation alone does not tell us why we need other courses or what they should look like. It is difficult to get humanists in particular to articulate a mathematical goal for their majors.

It is worth the trouble to do so, however. Here are the sentiments a small group of humanists produced when we locked them in a room with us for several hours:

"A humanist needs to understand that mathematics is a human endeavor too."

"There must be some overwhelmingly important cultural advances in math that everyone ought to know about."

"There should be less of a split between the cultures of science and humanities."

These ideas have been stated more eloquently elsewhere, but no matter. They were ours and so we could use them as a basis for action rather than persuasion.

Plan on offering a variety of courses because one size definitely does not fit all. As our evaluators have shown via survey instruments and interviews, students choose courses based on their current interests, not on their perceived preparation for the course. Unless you have made your entire career out of teaching required courses, you know yourself that this is true. I believe it safe to extend this observation to their actual learning. Mathematics connected with a subject that already interests a student ought to be learned more readily and retained longer than any topic visited during a forced march. Furthermore, every faculty member knows from personal experience the quality of educational experience that results from having a class full of people who have freely chosen to be there.

It is also worth noting that allowing students to sort themselves by their own interests is something mathematics departments have traditionally avoided in structuring mathematics curricula, relying instead on a battery of tests of mathematical preparation and sorting students entirely by level. When courses are arranged in a single linear pattern of prerequisites for each other, you have automatically constructed a "sieve". The probability is high that the student will eventually encounter material that isn't interesting enough to prompt him or her to take the sequel, so the student opts out of the only sequence available. No amount of good teaching or technology can change that: it is forced upon teachers and students alike by the basic structure of the system. When there are multiple points of entry to a variety of kinds of mathematics at wide ranging levels, then at least the potential exists for the system to avoid being a sieve. The potential application of this simple observation to the problem of attracting women and minorities into these classes is obvious, as we say in mathematics all too often.

Work the system because a course that satisfies no curricular need will not be well subscribed. In other words, take an inventory of the types of requirements a student must satisfy. Is there a quantitative requirement? Some of your new math and humanities courses should satisfy it. Is there a writing requirement? Some could be constructed explicitly to satisfy that. At Dartmouth, there is an interdisciplinary requirement, and those courses staffed by two faculty members from different departments can be made to satisfy it. We also have requirements such as technology and applied science and western culture, each of which is satisfied by one of our courses. This part of the strategy is essential to making sure new courses are well attended from the outset and can therefore justify their existence to the administration as well as the department offering the course.

Additionally, the kind of student who enrolls in a course is closely connected with the type of credit offered. A math and humanities course carrying quantitative credit is likely to be taken by "math avoiders" who see it as a palatable way to satisfy that requirement. A similar course satisfying the writing requirement will, instead, be heavily attended by potential math majors.

Plan for variance. Some of the traditional one-track sequencing of math courses can be explained as an attempt to lower variance among students, the better to target content and pedagogy to the correct level. To a large extent it succeeds so that we can have a fairly good idea of the background of someone entering second semester calculus, for example. On the other hand, a prerequisite-free course such as "Pattern", the first example in this paper, has a huge variety of students in it. The variation in math background goes from high school algebra and geometry right through fairly advanced calculus. Furthermore, there is a large art component to the class, and the variance for that subject is far greater than for math. Some are planning to major in art and others have never taken a class in art of any kind, nor art history. And as we all know, when you put those two factors together the variances sum.

The instructor is left with two options. Either one can proceed to deliver lectures and other activities, assuming everyone knows nothing, or else one must rethink pedagogy to capitalize on the variance among students. In other words, the instructor has the option of learning to use the variance in student background to advantage. Carefully thought out assignment of students to groups, coupled with well-chosen activities, can turn students into helpful tutors for one another. The interdisciplinary subject matter lends itself well to this because different students have different areas of strength and can thus be leaders at different moments. Discussions can be very rich.

Sometimes the subject matter lends itself well to the first strategy. In the math and science fiction course (not described here) much of the mathematics has to do with the fourth dimension. It is completely safe for the instructor to assume that nobody in the class knows anything about this topic, and to proceed accordingly. In the best situation, there would be opportunities for both of these approaches in any given course.

Some conclusions. In imitation of the courses it describes, this paper is trying to make several parallel points simultaneously. First of all, interdisciplinary courses in math and humanities are completely viable if built to respond to both faculty and student interest. Students are not allergic to intellectual work in mathematics if they find the topic inherently interesting. Second, a road to quantitative literacy that is broad enough for everyone to travel must have a multiplicity of entry points and natural connections to other subjects. Any one course proposing to solve the Q.L. problem is doomed to failure, because it cannot possibly respond to the needs and interests of an entire student population. Third, diversity of knowledge is a good thing in any population. The population can hold more knowledge that way.

Finally, it is unfortunately necessary to point out that there are some goals of a college education that we do not often acknowledge and that I have rarely heard spoken aloud. We spend a lot of energy discussing preparation for citizenship and the job market, yet life consists of more than our duties. My suspicion is that the courses described above are successful because they also respond to another human need. We crave delight and pleasure, whether in getting a new understanding of how part of the world works or from the joy of fruitful intellectual activity. Surely if we can think broadly enough to put "quantitative" and "literacy" into the same phrase, we might also draw "mathematics", "delight" and "pleasure" in with the same breath of fresh air.

OLD IDEAS: A NEW TOOL FOR TEACHING BASIC MATHEMATICS

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ABSTRACT

Incorporating the History of Mathematics in everyday teaching can be much more than just giving anecdotic events a place in our classroom expositions. In fact, there are ideas and methods used in the past which may be no longer of practical usefulness today, but may nevertheless help our students to grasp the meaning of a theory, an algorithm or a simple formula. Just as, for Elementary School children, manipulating pebbles is a useful way of visualizing some properties of the basic operations, like commutativity, for example, also working at the High School level with Euclide's " Geometric Algebra" or the egipcians' " False position rule" may be of great help for understanding relations and concepts more clearly. In fact, the possibility of realizing that concepts and ideas in different topics of Mathematics are connected is one of the most important benefits for the student exposed to the historic development of some mathematical ideas. On the other hand, the teaching of Mathematics as a field of knowledge which is ever changing, instead of as a rigid set of formulas and algorithms, is, besides a tribute to truth, a way of encouraging our students to interact with the ideas developed in the classroom, since they will necessarily be exposed to several different ways of solving problems, and therefore their creativity will be stimulated. After a 4- year experience in Mérida, Venezuela working with High School teachers who introduced their students to Algebra and Geometry using the approach mentioned above, there have been positive results, especially in their general attitude towards learning Science. Some examples of the use of certain elements of the History of Mathematics in the exposition of basic topics in Algebra and Geometry of the High School level will be shown.

1 Introduction

The teaching of Mathematics at the High School level in Venezuela is, generally speaking, a task which faces great difficulties due to many factors, the two most important of these being : 1)The curricular design, and 2) the academic background of the teachers.

Since the year 1.996, and after obtaining a Doctorate degree in Algebra, the author has participated, as a Professor of Mathematics at the Mathematics Department of the Facultad de Ciencias, Universidad de Los Andes, in a special service program of this University called " Proyecto Palestra". This program was created as a contribution to the improvement in the teaching of Science at the High School Level in our city, and, eventually, in our country. At the beginning, the work was concentrated in the curricular revision of the Mathematics courses taught in Venezuela at 7th, 8th and 9th grades. Also, for the academic year 1996-97, weekly visits to the classrooms were made, watching the Mathematics teachers working with their students in one of the most important High Schools of the city: Liceo Libertador. This institution has made an official agreement with Universidad de Los Andes to incorporate our suggestions on the teaching of Mathematics, including curricular modifications of the official curriculum, into some of their regular Mathematics courses.

In this work we will explain the main ideas that support those curricular modifications, among which the introduction of " Historic Mathematics" in the curriculum, which means more than just adorning the exposition in the classroom with a few anecdotes, is one of the most important aspects of the proposed curriculum.

Liceo Libertador has 4 sections of each course in 7th, 8th and 9th grade, and the Palestra proposal was applied for the first time with one section of 7th grade which began in September of the year 1.997. It was Section C. The students of this section were kept together in the three following years, and the proposal was applied at their Mathematics courses in 8th and 9th grade as well. There were, then, three other sections taking the regular Mathematics courses, and at the arrival of all 4 sections to 10th grade (in the venezuelan system this is called the 1st year of Sciences), observations were made to compare the performance of the Section C students with that of the other sections, not only in Mathematics courses, but in other Sciences, especially Physics.

It may be of interest to say that Liceo Libertador has 10 sections of the 1st and 2nd year of Sciences (10th and 11th grades). This means that our Section C students were also compared to students coming from High Schools other than Liceo Libertador. We should also add that the curricular changes proposed do not affect the total list of topics included in the official curriculum of 7th, 8th and 9th grades , in the sense that , by the end of 9th grade, the students of section C and all the other students of the same level have studied the same topics. The curricular changes in our proposal regard the order in which the topics are taught, the incorporation of some aspects of the History of Mathematics, the special emphasis on the connections between different topics, and the general orientation of the work in class. We will comment on these changes later.

The results that were observed in the academic year 2.000- 2.001 were very much encouraging. The students in Section C had in general a much more positive attitude towards Mathematics and Physics in relation to most students in other sections. The teachers of these subjects observed a clear eagerness of students in Section C to ask questions and participate with their opinions during their classes, in contrast with very passive students in most of the other sections. In other words, the level of motivation for the comprehension of Physics and Mathematics topics was clearly higher in the group of Section C.

As for the grades of the students, the information gathered is still not sufficient for a complete analysis, since only one group of students has been subject to the experimental curriculum, during three years of their studies .

2 Main aspects of the Palestra Experience

In this section, we will briefly comment on the basic curricular changes which were introduced in the teaching of the Mathematics courses of Section C, Liceo Libertador, from 7th through 9th grade.

1) The order in which the topics are taught:

The official curriculum of 7th, 8th and 9th grades courses of Mathematics in Venezuela has several mistakes regarding the order of the topics, especially in Algebra. For example, students in 8th grade are introduced to polynomials, in some cases in several variables, before they have been exposed to quadratic functions or quadratic equations, because these topics belong to the 9th grade curriculum.

In the Palestra proposal, the order of these and other topics was rearranged with the purpose of adjusting to the following premises:

a) The topics should be arranged in such a way that the difficulties due to complexity or degree of abstraction are in a non decreasing order. As obvious as this may seem, the example mentioned above shows that it is not always taken into account.

b) The teaching of Mathematics should emphasize the relations of the discipline with other disciplines and the historical relations between the ideas that originated different topics within Mathematics . The connection between two subsequent topics in the curriculum should be highlighted. The official curriculum does not contribute to this practice. Rather it is a good example of what has been called an " atomic" curricular design: isolated topics to be taught without an explicit connection between them. The student enjoys realizing that there are " hidden" relationships between concepts seemingly belonging to completely different topics of Mathematics.

The introduction of some important events of the History of Mathematics has proved to be very valuable for this purpose, as we will see in Section 3.

c) Mathematics should be taught as we would teach a foreign language: the meaning of each symbol, of each expression, is to become clear, either before or while engaging in learning the basic rules of grammar. Mathematics are taught at the High School level in Venezuela, in most cases, as if the symbols had no meaning, and only the rules of grammar are to be memorized for the next evaluation, and forgotten soon after it. Again, some historic ideas in Mathematics help the students grasp the meaning of symbols and algorithms.

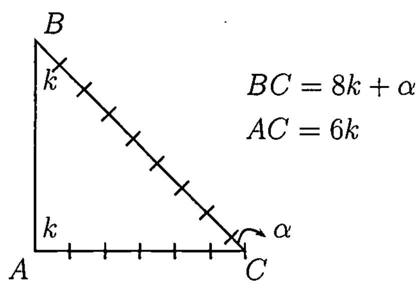
d) The presentation of the cultural context in which some ideas were developed contributes to give the student a different and more realistic perspective over what Mathematics is: a living, ever changing body of knowledge, instead of a rigid set of formulas, algorithms and rules.

3 Some examples of the introduction of Historic Mathematics in the curriculum

I. Crisis in pythagorean Mathematics as the first irrational numbers were considered.

The pythagorean conception of the Universe as being ruled by the natural number, including ratios, collapsed as the isoceles right triangle proved to have a hypotenuse which could not be expressed as $\frac{a}{b}$ with a, b natural numbers. Discussing this episode with 8th grade students who have just looked at right triangles with natural numbers as lengths of the sides, and in that context learned the formula associated with Pythagoras, gives a good opportunity to regard mathematical ideas as connected historically to other cultural aspects of the time: philosophical beliefs in this case created resistance to the evidence of a new kind of number. And this number arose in the simple example of the length of the side of a triangle. After being exposed to this episode, the students engage in working with square roots and irrational numbers in general, having arrived at the topic through geometry .

On the other hand, watching closely the geometrical construction which showed, empirically, that there was no way of expressing the hypotenuse BC of the triangle ABC as a ratio of two natural numbers, gives the student a chance to grasp further the concept of ratio and also to have a geometric intuition of what irrationality means: if we choose a unit of measure such that the sides adjacent to the right angle are exact multiples of this unit, then the hypotenuse is not an exact multiple of that same unit, no matter how small the unit is chosen:



II. Euclid's "Geometric Algebra".

The historical interaction between algebra and geometry has one of its most beautiful exhibitions in Euclid's methods for studying geometric solutions to linear and quadratic equations .

Students, in general, find these methods amusing and the austerity of Algebra is somewhat lightened when Geometry comes into play.

For example, the following method for finding a geometric solution to the linear equation

$$2x + \frac{x}{3} + \frac{x}{2} = 8$$

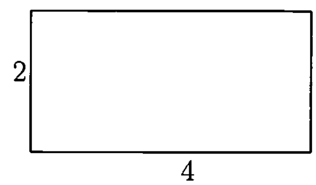
shows the typical creativity of Euclid's reasoning:

If we start by writing the equation in this way:

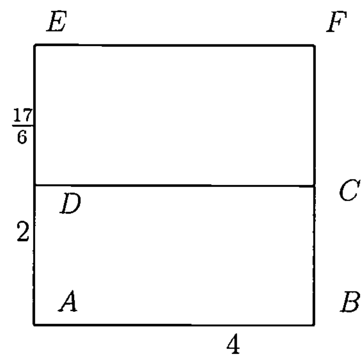
$$x\left(2 + \frac{1}{3} + \frac{1}{2}\right) = (2)(4)$$

then we may interpret it as stating the equality of the areas of two rectangles: one which has sides of lenghts x, $2+\frac{1}{3}+\frac{1}{2}$ and the other having sides of lengths 2, 4.

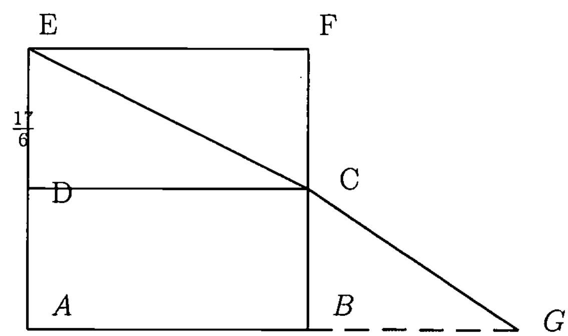
We begin by constructing a rectangle with sides of lengths 2,4:



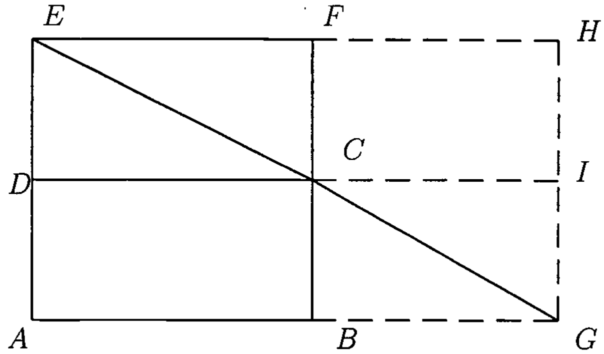
We will now find a rectangle which has the same area as the preceding one, and with one of its sides of length equal to $2+\frac{1}{3}+\frac{1}{2}=\frac{17}{6}$. The conclusion will be that the length of the other side of the new rectangle is x. In order to do this, we will add to the original rectangle, a new one of sides 4 and $\frac{17}{6}$:



We now draw the straight line that contains the diagonal EC of the upper rectangle and call G the point where it meets the extension of the base AB of the lower rectangle:



We draw over this figure the rectangle $AGHE$:



By construction, EC is the diagonal of $EDCF$ and CG the diagonal of $CBGI$. Therefore, the triangles EDC and CFE are congruent, and also the triangles CBG and GIC.

On the other hand, since EG is the diagonal of $EAGH$, the triangles EAG and GHE are congruent. Therefore, the rectangle $CIHF$ has the same area as the rectangle $ABCD$, and the side CI represents the geometric solution to the original equation, because the length of FC equals $\frac{17}{6}$.

In general, students enjoy trying out the same construction, but beginning with a different factorization of 8, and then checking out that the solution segment has the same length as in the first construction.

Usually, we use algebra as a tool for solving geometric problems, but examples such as this one, of the use of geometry for solving algebraic equations are rarely shown in the classroom, and we have found a positive reaction in the students exposed to them, such as an awakened curiosity and a desire to interact with the teacher or the other students in the classroom.

The following construction appears also in Euclid's "Elements" and was learned on the IX century A.D. by al-Khowarizmi, the great arabic mathematician who wrote an important treatise on Algebra, and used it to check out his own algebraic solutions for quadratic equations.

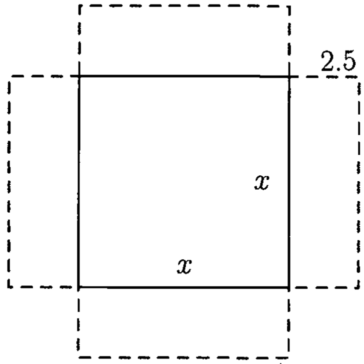
For the equation

$$x^2 + 10x = 39$$

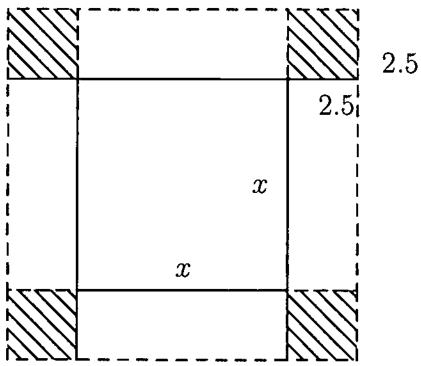
we have the following geometric interpretation:

If x^2 represents the area of a square of side x, and $10x$ the area of a rectangle of sides 10 and x, then the equation states that the sum of the two areas is equal to 39.

We "add up" these two areas geometrically in the following way. We divide the rectangle in 4 rectangles of sides x and 2.5 each, and then place each of these rectangles on the sides of the square of side x, as shown in the figure:



So, the area of this figure equals $x^2 + 4(2,5)x = x^2 + 10x$, which is the same as 39 . Now we complete the square, placing the corners missing, which are, of course, squares of side 2.5 each:



Since the area of this new square is equal to $39 + 4[(2.5)^2] = 64$, then its side equals 8, but this side is equal also to $x + 2(2.5)$, so we get $x = 3$ as a positive solution to the equation.

This construction has proved to be very useful as a tool for introducing quadratic equations and their general solution.

III. The Egyptians"False position Rule" for solving linear equations.

The False Position Rule, used by egyptians circa 1,600 B.C. to solve certain linear equations is a good example of archaic methods that offer the students a chance to find relations between different topics learned in their basic Mathematics courses. In this case, as we explain to the students why the False Position Rule works, it is possible to show connections between linear equations , the idea of proportion, linear functions and similarity of triangles.

Let us use the mentioned egyptian method for finding the solution of the following equation:

$$x + \frac{x}{4} + 3x = 8$$

First of all, we choose an arbitrary value for the x and introduce it to evaluate the expression at the left side of the equation. For example, for $x = 4$, we get

$$4+\frac{1}{4}(4)+3(4)=17$$

Then we state the equality of the ratios:

$$\frac{x}{8}=\frac{4}{17}$$

and obtain

$$x=\frac{32}{17}$$

which is the solution to the equation.

We could explain to our students why this method works as follows:

Let $f(x)=\frac{17}{4}x$. We know that the inverse image of 8 by f is the number that satisfies the equation

$$\frac{17}{4}x=8$$

Since f is a linear function, the expression

$$y=\frac{17}{4}x$$

represents a relation of proportion between the variables x and y. In other words, the line $y=\frac{17}{4}x$ is the set of all points (x,y) in the plane such that

$$\frac{y}{x}=\frac{17}{4}$$

This is why the egyptian method works. As you choose a " false value" , in this case, 4, for the unknown, we discover what is the ratio between all pairs (x,y) that belong to the line associated to the equation

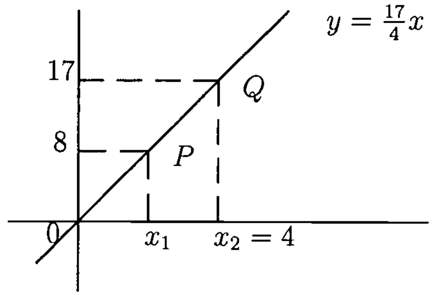
$$x+\left(\frac{1}{4}\right)x+3x=y$$

Once we have determined this ratio, we have

$$\frac{x}{8}=\frac{4}{17}$$

since the pair $(x,8)$ must also belong to the line.

We now use a representation of the situation in the coordinate plane for the purpose of emphasizing its geometrical meaning:



To solve the equation $\frac{17}{4}x = 8$ is to find the x- coordinate of the point P where the line $y = 8$ meets the line $y = \frac{17}{4}x$. On the other hand, we have two triangles to consider in the figure:

$$\triangle OX_1P \text{ and } \triangle OX_2Q.$$

These triangles are similar, because their inner angles are congruent , so the ratios between the corresponding sides are equal:

$$\frac{X_2Q}{X_1P} = \frac{OX_2}{OX_1} = \frac{OQ}{OP}$$

So, we get

$$OX_1 = \frac{(OX_2)(X_1P)}{(X_2Q)} = \frac{(4)(8)}{17} = \frac{32}{17}$$

Showing the students various perspectives for considering a mathematical problem is always a good way of stimulating their own creativity.

IV. The use of the concept of similarity of triangles and its consequences in the calculation of unreachable distances.

The legend of Thales giving an exact measure of the height of one of Egypt's great pyramids using his knowledge of the properties of similar triangles illustrates the power of Thales' theorem when used as a tool for practical calculations.

Also, we can show the students, for the same purpose, the consequences that this idea had in the development of the Alexandrian Greeks astronomy, in particular,in the calculation of the distance from the Earth to the Moon done by Hipparchus, in the II century B.C.

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"ARE MATHEMATICS FOR OTHER DISCIPLINES DIFFERENT MATHEMATICS? "

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ABSTRACT

Teaching Mathematics for different disciplines raises the question of whether the depth, the topics and the intensity of courses are or should be something different according to each area of study. The main argument is that at a basic level there is no difference and, on the contrary, there are enough reasons to avoid the creation of first and second-class Mathematics. Unfortunately, the literature is full of examples of books inviting students of Economics to learn Mathematics for economists or students of Biological Sciences to read only Mathematics for biologists. Nevertheless, at a higher level, there are also good reasons to split the group of students into more specialized, more Applied Mathematics, mainly because of the demand for proper models and the more extensive use of Mathematics in other areas. For the mathematician this implies a challenge because to teach good courses it is necessary to get a broad insight into other disciplines.

Keywords: Mathematics for other disciplines, basic courses, advanced courses, teaching of by mathematicians, curriculum

1 Introduction

Going to the mathematical section of a bookshop one can find titles like: Mathematics for Economists, Economic Mathematics, Mathematics for the Life Sciences, Applied Calculus for Engineers, Linear Algebra with Applications to Economics, and so on. If one looks at the Statistics area, the menu is even broader. The question is whether there exists a different approach, a different Mathematics, to what universities worldwide teach in basic Mathematics or Statistics courses for different disciplines which corresponds to the titles and the contents of these books? What about the teachers – should they be mathematicians, or should each discipline receive what they need in Mathematics from its own people?

This paper will try to answer these questions. Section 2 is devoted to fix a position towards teaching basic Mathematics for other disciplines. Section 3 will answer the question if there exists a moment of switching to a more specialized, i.e. more Applied Mathematics. Finally, section 4 will give some ideas about curricula and give some conclusions.

2 Teaching basic Mathematics for other disciplines

Like a child who wants to learn football, there are steps to be learned like how to kick the ball, how to stop it, how to dribble, i.e. solid foundations that must be learned step by step. Of course, one can apply the method of learning by doing, but if one wants to use your knowledge professionally, with very few exceptions, sooner or later you would run into trouble. A solid foundation of mathematical thinking and techniques is needed to undertake what can be called an application to other disciplines. Mathematics is by definition a rigorous discipline and students of any academic area have to learn not the mechanics of Mathematics or what can be called a mechanistic Mathematics, where for this type of problem this recipe is applied and for that other type another will provide the solution. It is more important to understand concepts than mechanics; nevertheless, a dose of carpentry must also be trained. Therefore, a solid foundation has to be created and this implies that basic Mathematics is the same for all.

Naturally there can be disagreement as to what is understood under basic Mathematics, but I think that at least Calculus - Differential and Integral in one variable, including some Sequences and Series theory – and Linear Algebra are a common denominator. Is this short sequence different for Mathematics students? The answer is not unique. Some more demanding theorems even in Calculus and Linear Algebra should be discussed with this last group because this is the essence of Mathematics itself. Take, for example, the Intermediate Value Theorem: the proof that an odd polynomial expression has at least one real root is not simple. The proof that a convex function is continuous is even more difficult and perhaps the majority of students different from the pure mathematician will not enjoy them and will gain very much trying to follow such developments. Other aspects, even if they are also not easy to fully understand, have a more intuitive application. For example, knowing that as n increases

$$(1+\frac{x}{n})^n \rightarrow e^x$$

has to do with interest rates which are of common use and the proof shows how to pass from a discrete case to a continuous one. So at this level at least there is no good reason to split students of different disciplines into specialized courses. Calculus for Economists or Matrix Algebra for Engineers does not exist. Another reason to maintain students of different disciplines together is to avoid opinions that there are first class Mathematics and other courses, which are not so prominent. And this is exactly what happens if the students are separated "according to their disciplines".

Table 1 in the Appendix shows the marks obtained by a group of students at Universidad de los Andes in Bogotá – Colombia, during five semesters, second of 1997 – second of 1999, whose disciplines are Mathematics, Physics, Economics, Engineering, Business Administration and Biology. They have to take a compulsory Differential Calculus course in the first or second semester of study. It also includes other students, mainly of Law, Psychology and other Social Sciences as well as Architecture who decided freely to take this course instead of others given only for them. The course was given in small sections (at most 30-40 students) and there is no discrimination inside each section according to the field of study chosen. Students can retire their inscription at the middle of the semester and have to repeat in the next semester. The results show that their performances are not high but even, with the exception of the Business Administration and Biological Sciences students who have in general lower marks and also are the most numerous groups with respect to retirements.

Is there an explanation for these results? The most common answer is that these two groups of students are not so dedicated to Mathematics and feel that it is not so important as other courses of their area of study. I disagree strongly with this and am convinced that the main reason for the lower performance has to do with the use and teaching of Mathematics by those teachers in their disciplines who do not apply them in their courses. We arrive so to one of the most important aspects of our discussion: the use of Mathematics by other disciplines. The main argument is that the non-use of mathematical concepts and techniques by some disciplines creates a dichotomy in the students that results in an attitude of indifference if not of total rejection towards Mathematics. So if we accept that even basic Mathematics can and should be taken by other disciplines in mixed courses, the complement of showing in other courses of their own area the applications is a necessary condition. Most courses of Engineering, Economics and other disciplines are using Mathematics each day to model their own theories and these models are studied intensively. Therefore there exists an interaction between Mathematics and these sciences; there is a demand which requires a suitable supply.

Something similar happens with basic Statistics courses. It is possible to give Statistics with little more than elementary Algebra. Nevertheless, Statistics should be preceded by Probability concepts, in particular, the notion of randomness, which allows understanding that in real life few things are deterministic. But Probability ignoring Calculus and Statistics ignoring Matrix Algebra will be poor. Social scientists have expressed in several occasions their need for support research on empirical evidence and it is not enough to hire a statistician who presents some results. Many colleagues, graduated 10 or 15 years ago, feel a vacuum and even are willing to take lessons in Statistics. Resuming, in this area of study there is also a need to establish a solid foundation to be able afterwards to understand more advanced methods.

In basic courses, including Statistics, mathematicians have a lot to say. New techniques like the use of computers and software packages and modern calculators are at order and so these courses can and should be given by mathematicians. But that's not all. Like in the past mathematician had to know some notion of Physics to teach Calculus, today it is necessary to have some notion of Economics and even Biology to teach basic Mathematics for these areas.

3 Teaching advanced Mathematics for other disciplines

Economics, Biology and even journals of Social Sciences disciplines like Political Sciences, Anthropology, Psychology and others are full of research papers, which use Mathematics and/or sophisticated statistical techniques. As soon as one goes into the level of Multivariate Calculus, differences begin to appear. For Mathematics, Physics and Engineering more important than going deeply into Optimization problems are concepts of partial derivates, line integrals, Stoke's and Green's theorems. In contrast, and I will take the case of Economics in more detail, maximum and minimum with and without restriction are extremely relevant. A great part of consumer and production theory is based on Lagrangian and Kuhn-Tucker theorems. More advanced theories like Optimal Control Theory and Dynamic Optimization (see for example Escobar 2001 or Takayama 1996 or Seierstad & Sydsaeter 1987) need strong foundations, both in optimization as in topics like Differential Equations. So here there is a reason, a good reason, to split students according to their area of interest. This doesn't mean that economists cannot take elective courses in areas not directly related to Economics. Nor it is forbidden for engineers, for example industrial engineers, or pure sciences students, to study Convex Analysis exhaustively. But to be consistent with what we mention in Section 2, other disciplines make each day more extensive use of mathematics in their field of study and research and therefore a more specialized mathematics is at order. Should mathematician give this kind of courses? Two reasons, at least, provide a positive answer. First, there are new areas of Applied Mathematics where great contributions have been made in the recent past and new fields appear; for example, the use of Numerical Analysis in economic research or the relatively new developments in Mathematical Finances. Second, following J. Marschak "The fact that an internally coherent and determined theory be or not be formulated in mathematical terms, doesn't change its logical essence; but it is easier to verify its coherence and its determination if it is stated in mathematical terms" (Frechet 1946). Our interpretation is that mathematicians can and are beginning to make incursions into other fields of knowledge.

With respect to Statistics and for those mathematicians who work in this area, their support to other areas is enormous. Sampling is done practically every moment be it in Biology, Psychology, Economics or Business Administration. Social Sciences historically adverse to Mathematics are using them and specially Statistics. It is not surprising to hear students at the end of their first degree where perhaps a thesis or monograph must be written to dispose of a "lot of measurements" but not knowing what to do with them. And such final manuscripts are demanded by their teachers to support empirically underlying theories. Decisions with political, environmental and economic consequences are taken on that basis and it must be said that in many cases a lack of rigor is present. So in this sense statisticians acquire a responsibility with respect to society. The consequences are the same as above:

it is necessary for the statistician to know about the field he is trying to apply to be able to support students in their first research and to give better and more supported courses.

The above arguments have consequences and the most important is that to teach mathematical courses at certain levels demands from our part to delve deep into other areas like the case of Economics mentioned. So all our formation and analytical thinking must be complemented with a broad insight into the parallel discipline. But another consequence that follows is that these advanced courses are also given by mathematicians now for separate groups according to the area of study.

4 Curriculum and conclusions

For first degree if we take a period of about 4 – 5 years, the big problem is how to accommodate the Mathematics courses into this time frame. The three basic courses, Differential Calculus, Integration and Series and Linear Algebra can be absolved during the first year. As argued, there is no need for a split and what methodology should be applied, whether massive courses or small sections or a combination of the two, is not a subject of this paper. It is important because of the use of technology to support learning, but in any case it should not take more time. In this first group of students, Mathematics, Physics, Engineering, Biological Sciences, Economics and Business Administration can be put together. For the Social Sciences during the first year a fundamental sequence of courses in Calculus, Linear Algebra and Probability are sufficient to build upon these a one year course in Statistics which should include Sampling theory. We have therefore identified two big groups; each of them are together at least during the first year. For the second year, and depending of the available space, the first group can be divided into two subgroups. One that goes into Multivariate Calculus with more emphasis in Physics followed necessarily by a Differential Equation course. The other one goes more in the direction of multivariate Optimization. On the other hand, both subgroups can meet once more in a Probability course and a Statistics course. In this meeting the Biologist should not be absent.

Perhaps it sounds easy and simplified but it gathers the ideas expressed in this paper whose main conclusions are that it is not necessary to split students according to their field of study in the basic mathematical courses. The second important conclusion is that for the mathematician to give more advanced courses for other areas it is essential to get a broad insight in the respective discipline. And last but not least, the same applies for the statistician. Some fields of knowledge like Architecture or Medical Sciences were not involved in our analysis. Here is a broad field of study where some concern has been expressed but where proposals are scarcely beginning to be handled. I am also not sure that the perspective adopted in this paper is excessively tailored to conditions imposed in my country and university, but I hope that at least our reflections will serve other colleagues in other countries.

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Table 1 – Marks according to area of study in Differential Calculus 1997 –1999

	Mark frequencies						
	D	C	B	A	Total	Num. Obs.	Weighted mark mean
Engineering	4,4%	29,9%	56,4%	9,3%	100%	2344	3,1245
Economics	6,4%	18,8%	65,7%	9,1%	100%	405	3,2692
Mathematics/ Physics	2,5%	34,6%	50,6%	12,3%	100%	81	3,2026
Biological Sciences	24,6%	36,6%	36,9%	1,9%	100%	415	2,7827
Business Administration	11,1%	41,6%	45,9%	1,4%	100%	370	2,7568
Others	27,9%	24,6%	39,5%	8,0%	100%	276	3,1498
Totals	4,4%	29,9%	56,4%	9,3%	100%	3891	

Note:

(i) Marks go from 1,5 to 5,0

(ii) Maximum mark is 5,0; Minimum to gain the course is 3,0

(iii) C=1,5 - 2,5 ; B=3,0 - 4,0 ; A=4,5 - 5,0

(iv) D=retired before end of semester

INTRODUCING REALISTIC MATHEMATICS EDUCATION TO JUNIOR HIGH SCHOOL MATHEMATICS TEACHERS IN INDONESIA

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ABSTRACT

The study finds its basis on the emergent needs for improving mathematics education in Indonesia, that has for a long period experiencing many challenges. Since the time when the government changed the school subject from arithmetic to mathematics, many efforts to improve instruction have been done. Since 1977 the government has produced over 900 million copies of newly develop textbooks for the students and the teachers, carried out in-service training for most of the teachers, and provided teaching aids to schools. A diagnostic survey conducted by Ministry of Education and Culture in 1996 revealed that yet many mathematics teachers were still using the arithmetic based methods in their teaching.

This present study is called IndoMath (In-service Education for Indonesia Mathematics Teachers) and focuses on the introduction of Realistic Mathematics Education (RME) theory to the Junior High School (JHS) mathematics teachers. The RME theory is developed by some Dutch scholars. Its aim is to enhance the teachers' knowledge of mathematical content and RME pedagogy by means of workshops, instructional practices, and reflections. This introduction has been conducted from 1999 to the present in an effort to improve the teachers' competency. It involved 44 mathematics teachers.

This paper examines the effect of this introduction on the teachers' instructional practices in their respective mathematics class. The result indicated that a carefully planned program of professional development grounded in the principles of effective strategies can significantly impact teachers' understanding of the principles and practice of RME.

Key words: mathematics teaching, Realistic Mathematics Education, in-service training.

Introduction

The teaching of mathematics in the schools in Indonesia has been implemented since 1973 when the government replaced the teaching of arithmetic in the elementary school by the teaching of mathematics. Since then mathematics has become a compulsory subject in the elementary, junior high, and senior high schools. However, the teaching of mathematics has always raised problems, as is indicated by the ever-low achievement of the students on almost every examination, including the final year national examination conducted by the government. Issues of how to increase the students' understanding of mathematics and the students reasoning ability have always dominated the discussions on mathematics education in Indonesia. In response to the criticism of educational professionals and the society at large on the significance of school mathematics, the Indonesia government has lately revised the Curriculum of 1994. However, there is no information yet, about the effect of this revision on the students' performance in mathematics.

Realistic Mathematics Education (RME) seems to be a promising instructional approach that meets the Indonesia need for improving mathematics teaching. In the concept of RME, mathematics is a human activity and should be connected to reality. The concept of RME is characterized by students' activity to reinvent mathematics under the guidance of an adult (Gravemeijer, 1994), and the reinvention should start from exposure to a variety of "real-world" problems and situations (De Lange, 1995). Therefore, it is worthwhile to explore whether RME is an appropriate approach to tackle the problems of mathematics education in Indonesia.

General Research Design of IndoMath Study

The IndoMath study is aimed at designing and evaluating an instructional program to introduce RME to the JHS mathematics teachers. This 'development research' approach has been chosen with the purpose to document the development process, and to learn about the supporting conditions.

Orientation	Development and Evaluation		
	Fieldwork 1 10 teachers	Fieldwork 2 18 teachers	Fieldwork 3 16 teachers
Sep 98- Aug 99	Sep 99-Feb 00	Sep 00-Feb 01	Sep 01-Feb 02
Literature study on professional development program and RME theory	**Program tryouts:** Workshops Classroom practices with peer collaboration Reflections		Program Implementation

Notes: 1. Arrows indicate the cyclical character of development process
2. Increasing gray area means gradual upscaling of project

Figure 1: General design of the IndoMath study

The steps of the study are orientation, development, and evaluation (Fig. 1).

In the orientation phase an in-dept review of literature on professional development and RME was carried out. By doing this, one of the criteria of program quality, namely 'the state of the art knowledge' has been incorporated. For the purpose of understanding the context in program implementation, an analysis of the context was done as part of the first fieldwork in Indonesia during the period from September 1999 until February 2000. Based on the result of this orientation, procedural specifications have been formulated, i.e. specific guidelines on how to design RME based mathematics instruction. The specifications generated a methodological direction for the design and evaluation of the IndoMath program.

In the development and evaluation phases, two tryouts were conducted in Yogyakarta, Indonesia. In the first tryout (as part of the first fieldwork), 10 mathematics teachers of JHS in the Yogyakarta Province participated in the in-service training. This was carried out from December 30, 1999 to January 27, 2000. The second tryout was done as part of the second fieldwork in Indonesia, from September 2000 to February 2001.

The program evaluation was conducted as an integral part of the development process. Based on the results of the first tryout, some revisions were made to program components, RME exemplary curriculum materials, and program organization. The second tryout was focused was focused on the practicality of the program components and usability of the RME curriculum materials for the teachers. The third fieldwork period was conducted to evaluate the effects of the in-service training on the teachers' understanding of RME theory and practice.

Research Method in the Third Fieldwork

Since RME is so new for many people in Indonesia (teachers, teacher educators, curriculum developers, supervisors, and students) research is needed to investigate whether and how it can be translated and realized for the Indonesian context. Using the notion of 'think big start small' in education innovation efforts, it is important that a number of small experiments be carried out as a contribution to the curriculum reform in Indonesia. These experiments are needed to reveal the factors determine a successful implementation on both curriculum and teachers' level. According to Fullan (1991) a complex innovation is characterized by three dimensions, namely changing of teachers' beliefs, introducing new teaching and learning methods, and introducing new curriculum materials. The innovation we are talking about here pertains to all three dimensions. So, for Indonesia we are talking about a complex innovation if we want to introduce RME.

Within this analysis of the problems related to the introduction in Indonesia of the Dutch-based RME, the research problem is: *How can a professional development program be designed to make Indonesian mathematics teachers understand RME and prepare them for effective implementation of RME in junior high school mathematics?*

Within this general research question the focus of the evaluation in the third fieldwork was on the effect of the in-service program. For this purpose, three of the five levels of professional development effects as distinguished by Guskey (2000) were used for formulating the evaluation questions. This has led to the following effect categories:

- *Perception*: Participants' perceptions of the effectiveness and usefulness of program's aspects;
- *Learning effects*: Participants' understanding of RME theory and practice;
- *The use of RME exemplary curriculum materials*: The use of RME exemplary curriculum materials and approach in the participants' mathematics classes.

These in turn has led to the following sub-questions, and success criteria, concerning the effects of the inservice program:

- *Do participants perceive the program as relevant and meeting their expectation?*

  The teachers value the organization and components of in-service program positively, meaning that the program activities (workshops, classroom practices, and reflection meetings) meet their expectation, and are considered as instructive, useful, enjoyable, relevant and informative.

- *Do participants perceive the program activities as helping them to understand RME?*

  This would be indicated by the fact that participants: (a) gain knowledge of the RME theory; and (b) the participants perceive the RME approach, the in-service program activities, and the RME exemplary curriculum materials as positive and useful.

- *Do participants perceive the program activities as supporting them in implementing RME in their classes?*

  This would follow from a perceived change in the participants' confidence on the possible implementation of RME.

- *Do participants understand the RME theory?*

  This would be indicated by the participants' work and their scores on Realistic Contextual Problems Test (RCP-Test) before and after the program.

- *Can participants realize the characteristics of the RME approach in mathematics instruction?*

  This would be indicated by an observed change in participants' knowledge and skills in applying the RME approach in their teaching.

- *Do participants use after the IndoMath program the RME exemplary curriculum materials in their lesson?*

  This would be indicated if the participants use the RME exemplary curriculum materials in their actual lessons or as supplementary material to the governmental compulsory textbook.

- *Do what participants' learn inspire them to use RME method in their teaching for other mathematics topics?*

  An indicator for this would be participants' other mathematics lessons show characteristics of the RME approach (such as using contextual problems and students active learning).

This research (the third fieldwork in Yogyakarta) has been carried out in Yogyakarta with 16 teachers and used six kinds of data collection methods and instruments to evaluate the in-service program:

- *Questionnaires* were distributed to the participants at the end of each workshop session, and at the end of the whole program.

- *Realistic Contextual Problem (RCP) Test* was administered to the participants before and after the program. This test assessed participants' understanding about RME contextual problems and the relevance of the contexts to the current Indonesian Junior High School mathematics curriculum.
- *Classroom observation* was conducted during the program (in RME classes at the junior high schools) to get insight in the ways in which the teachers were implementing the RME exemplary curriculum materials.
- *Reflective reports,* during the reflection meetings, were provided by the teachers about the instructions they carried out in their classrooms using the RME exemplary curriculum materials.
- *Focus group discussion* took place of the researcher and participants after the program, about the program as a whole.
- Two months after the program, the researcher visited the participants' schools for several weeks to conduct *classroom observations* focusing on the effects of the program on the actual daily mathematics classes.

Implementation of IndoMath Program

The IndoMath program has been implemented by using the model of educational change that was based upon the principles of effective professional development (Fig. 2).

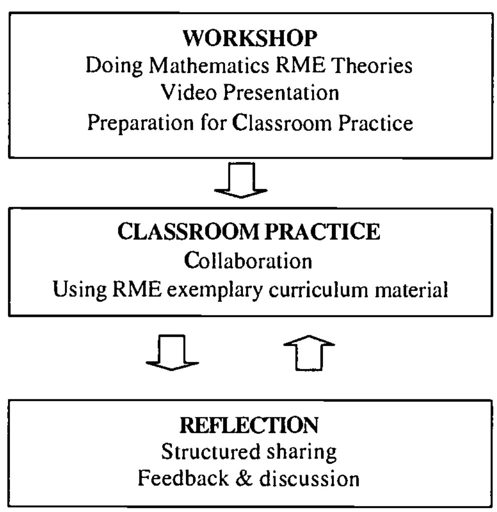


Figure 2: Teachers Development Model of the IndoMath Program

In this model the instructional practice is seen as being influenced by the teacher's subject matter and pedagogical content knowledge, the teacher's opportunity to experience new practices in a real setting, and with collaboration and reflection being the mediating factors between enhanced teacher's knowledge and the implementation of new practice (see e.g. Loucks-Horsley, et al., 1998;

Ball & Cohen, 1996; Borko & Putnam, 1996; Joyce & Shower, 1988, 1995; Van den Akker, 1988, 1998; Swafford, et al., 1999). So, the strategy of intervention in he IndoMath program was a combination of workshops, classroom practices, and reflections.

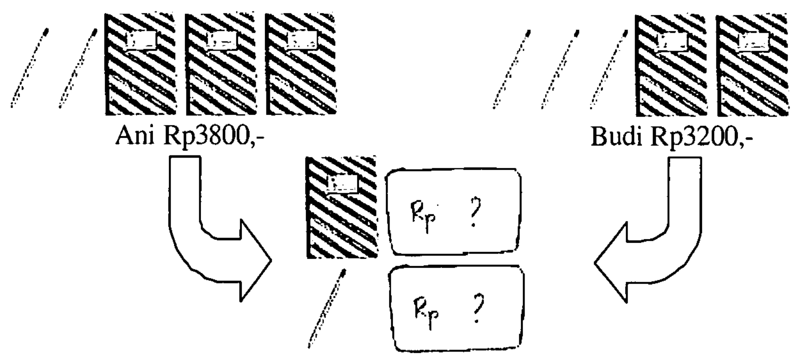
The IndoMath in-service program was held at *PPPG Matematika* (National In-service Training Development Centre for Mathematics Teachers) in Yogyakarta. The in-service course was conducted in period September 20 till October 10, 2001. The time spent for workshops, classroom practices, and reflection meetings was 25 hours (see Tables 1 and 2 for the example of program activities). So, the IndoMath Program can be categorized as an introductory in-service program about RME, as a preliminary effort to support teachers in the implementation of the RME approach to mathematics instruction.

Table 1: IndoMath Program Activities in Workshop I

Program Component	Content and Procedure	Relevance to RME
Session 1: *Doing Mathematics* *(2 hours)*	First, teachers work in a group to solve "the last card problem." Second, they learn how to approach a problem using "4-steps toward problem solving." Third, discussion of their findings.	In this activity teachers learn to find mathematics ideas by themselves, find procedure by themselves in interactive discussion among group member and share the findings with whole class.
Session 2: *RME theories* *(1 hour)*	Instruction on RME theories started froma general review of RME background and history. Trainer facilitates the discussion about *students' reinvention* and *interactivity* based on the results of doing mathematics.	In the previous session teachers learn how to find mathematics concepts by themselves. From this experience they get the idea of *students reinvention*. Since the activity is conducted in a group they experience the idea of *interactivity*.
Session 3: *Video presentation* *(1 ½ hours)*	Teachers watch the video on a lesson using RME material performed by a junior high school teacher.	It gives them visual support how to conduct the lesson, such as starting the lesson by giving students contextual problems that facilitate them to immediately engage in meaningful mathematical activity.
Session 4: *Preparation for classroom practice* *(2 hours)*	Teachers work individually and in a group to solve contextual problems on the topic of *Persamaan Belanjaan* (Shopping Equations).	By solving problems in the RME curriculum material that is being used in the classroom practice teachers will understand the content of the lesson. Teachers also understand *the use of contexts* as one of RME tenets. In this session the trainer acts as a teacher in a way that is typical for the RME approach, thereby participants can mirror from it as they intended to use it in their classroom lessons. In this regard the trainer should be able to be a good role model of RME teacher.

Sixteen mathematics teachers from 8 JHSs in Yogyakarta participated in the program. They were grouped in pairs, two teachers from each school. The day, after the workshop, each teacher wrote a lesson plan for teaching practice in collaboration with his or her partner. The material for the classroom practice was *Persamaan Belanjaan* (Shopping Equations, see Box 1). They performed teaching practice, by emphasizing the mutual observation (the teachers in each pair observed each other in their teaching practice). Teachers experienced important aspects of RME, such as the *lack of authority*, *interactivity*, and *student's free production*.

A student Store at *SLTP Realita* sells school supplies. Students prefer to buy their school supplies in the store because each supply has the same price. Each pencil, of different brands, is of the same price, so is each, etc. Ani bought 2 pencils and 3 books for Rp3.800,- whereas Budi bought 3 pencils and 2 books for Rp3.200,-



By using the above information find the price of a pencil and of a book.

Box 1: Sample of RME Curriculum Materials

After classroom practice teachers came again to the training centre to participate in the Reflection Meeting (Table 2).

Table 2: IndoMath Program Activities in Reflection Meeting I

Program Component	Content and Procedure	Relevance to RME
Session 1: Structured sharing (2 hours)	Each pair presents to other participants the results of their collaboration. They show the works of their students. They explain to the other participants the meaning of their students' free production.	In this session teachers learn that gaining understanding can be achieved by collaborating with their colleagues. This is the way that is also used in RME instruction emphasizing the *interactivity* and *intertwining* in mathematics concept building.
Session 2: Feedback and discussion (2 hours)	The trainer comments on the reports by paying special attention to the issues related to the aspects of RME. The trainer asks participants to share their experiences.	Students' work as the results of classroom practice will be discussed in this session. The discussion is directed to map the learning route of the students from which the teachers learn how to assess the process of students' mathematics learning.

There were two sessions in this meeting, namely *structured sharing* and *feedback and discussion*. This meeting facilitated participants to share their own experience in RME lesson and got information from other teachers as well as received comments and feedback from the trainer.

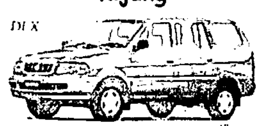
Participants' Understanding of RME

In order to know the participants understanding of RME, the RCP-Test*) was administrated to them before and after the IndoMath in-service course. The RCP-Test consists of four contexts in which some questions were embedded, namely a context of *pencils and books*, a context of *stacking chairs*, a context of *cars, viz. Kijang and Colt L-300* (see: box 2), and a context of *telephones and populations*.

Context 3: Kijang and Colt L-300

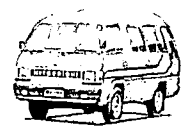
Second grade students from *SLTP Realita* are going to make a camping trip. There will be 96 people going, including the students and teachers. All the luggage, gear, and supplies are already packed into 64 equal-size boxes. The organizers want to rent the right number of vehicles to take everyone to the campsite. They can choose between two different types of vehicles from a car rental agency:

Kijang



Seats: 6 people
Cargo space: 5 boxes

Colt L-300



Seats: 8 people
Cargo space: 4 boxes

1. What combination of vehicles would you recommend to the camping organizers? (Use formal as well as informal mathematics procedure).
2. What mathematics concept, can be explained using the above context? Explain your answer (be more specific).
3. With which topic of the current SLTP mathematics curriculum does that context match? Explain your answer.

Box 2: Sample of Question in Realistic Contextual Problem (RCP) Test

The results of the test were used to find out the change of the teachers understanding of the RME on three aspects:

- teachers understanding of contextual problems (that is, solving the problem using informal as well as formal mathematics procedure);
- teachers understanding of the mathematical concept addressed in the contexts; and

*) The Realistic Contextual Problem Test (RCP-Test) has been tried out with the participants of the IndoMath program in Yogyakarta during the second fieldwork. 18 SLTP mathematics teachers participated in the tryout, and 17 teachers finished the test. Their results were used for the analysis of the content validity and reliability of the items (contexts) in the test. The test appears to be reliable (coefficient alpha .7544) and internally consistent (Pearson correlation is significant at the 0.01 level).

- teachers understanding of the relevance of the contexts to the current Junior High School mathematics curriculum.

All the problems in the test were judged as being on the level of JHS students' knowledge, and appeared to be quite simple for teachers (as concluded from tryout in the second fieldwork). Moreover, all the mathematical concepts in which the problems have their basis are relevant to the current JHS mathematics curriculum. So, for mathematics teachers those problems are solvable. However, the test does not merely assess teachers' ability to solve the problems by a formal procedure, but also their ability to solve the problems using informal procedures. Equally important, the test also explores teachers' knowledge about the concepts behind the contexts, and the relevance of the contexts to the current JHS mathematics curriculum. The results of the test for the participants in the third fieldwork period are presented in Table 4.

Fifteen participants stated that they had never heard about RME until they participated in the in-service course. The result of the pre test also indicates that they had little or no prior knowledge about RME. Particularly, they were not familiar with informal procedure for solving problems. For example, in the context of *Kijang and Colt L-300* most of the participants solved the problem using formal procedure: translating the problem into two linear equations of two variables, then solved the linear equation systems by elimination and substitution methods. Five participants gave no solution to the problem, had no idea about the mathematical concept addressed in the context, and had no idea of the relevance of the context to the current JHS mathematics curriculum.

Table 4: Participants' Scores on RCP-Test

No.	Teacher	Pre test*	Post test*
1	Suw	44	92
2	Sri	35	79
3	Sug	15	33
4	Wij	29	67
5	Kin	35	63
6	Wat	56	38
7	Sen	63	75
8	Wah	67	75
9	Sab	63	67
10	Har	63	79
11	Nug	35	67
12	Sud	27	50
13	Moc	25	54
14	Tut	46	83
15	Ton	25	67
16	Agu	33	75

* The scores are in percentage. Participants' work was also assessed independently by second evaluator. The Spearman correlation between the scores of the two evaluators are 0.789 (pre test) and 0.760 (post test). Correlation is significant at the 0.01 level.

Participants' scores on the post-test indicated that they gained knowledge about the importance of solution variation in solving contextual problems. In the context of *pencils and books*, 10 participants made use of two or more procedures using formal as well as informal procedures. Also, in the context of *stacking chairs* and the context of *Kijang and Colt L-300*, 8 participants made use of two or more procedures. The increase of participants' scores in the post-test are contributed mostly by their ability to solve the problems using different ways.

Teachers' understanding of the variety of possible answers to one contextual problem is important for RME mathematics teaching. Teachers should be aware of the different responses coming from their students in classroom lesson, and should be ready to facilitate discussions.

There were observed changes in teachers' mathematics lesson structure during and after the IndoMath in-service course. The results of the classroom observations during classroom practices indicated participants' ability to translate RME philosophy into classroom lesson. By the support of RME exemplary curriculum materials (student's book and teacher's guide) the teachers could perform instruction that was different from what they usually did (Fig. 3).

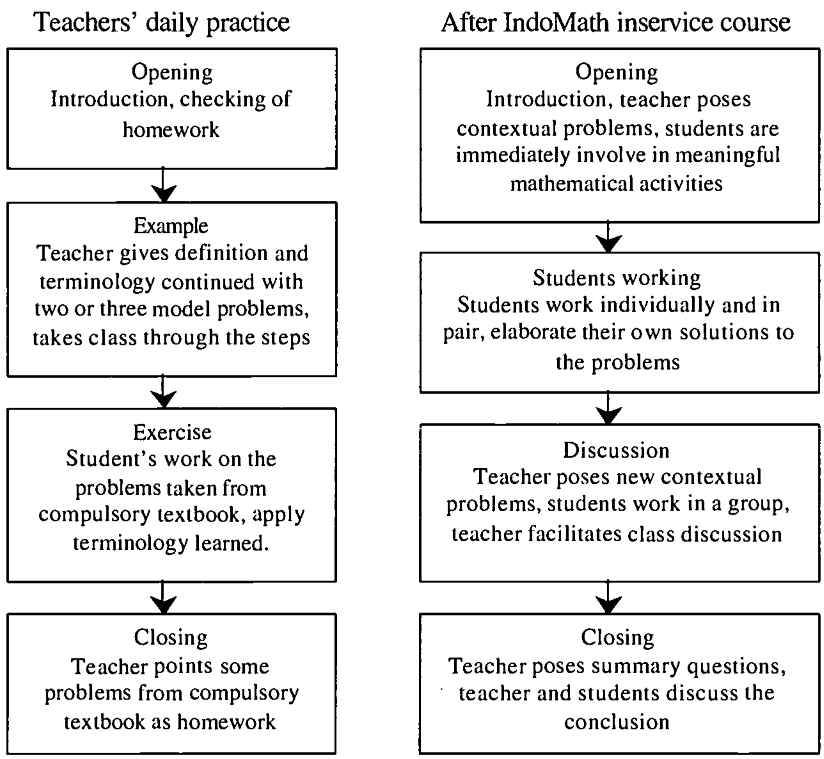


Figure 3: Teachers' Mathematics Lesson Structure

In their daily practice, teachers perform their instruction following the sequence: Opening – Example – Exercise – Closing. Their lesson structure was dominated by traditional "chalk and talk" that put intellectual authority in the hands of the teachers, and students' activities of note taking. Teachers have the tendency to 'spoon-feed' their students. This unfortunate nature of the 'traditional' learning process makes the students to become passive learners and with little responsibility for mathematical thinking and reasoning.

In the classroom practice during the IndoMath in-service course, teachers tried to structure their lessons by emphasizing the student's learning. Although it was rather troublesome because the students were used to being 'spoon-fed', the teachers always ask their students to explain their thought, or to comment on the other student's response, and facilitate discussion.

Conclusion

The IndoMath study used a development research approach which emphasized the design and evaluation of an in-service instruction program. This study has been conducted through a cyclic process of design-evaluation-revision. Now, the researcher (first author) is in the final stage of this development process that is visiting participants' schools. By conducting observation of participants' mathematics class daily, the researcher learns about program effects on the teachers' practical knowledge of the RME approach (developed in the Netherlands) and its feasibility to be implemented in Indonesia Junior High Schools.

The results of the analysis of the data that were collected during three fieldwork periods in Indonesia as well as the preliminary classroom observation indicated that the introduction of this innovation can be done by using a carefully planned program grounded in principles of effective professional development and supported by exemplary curriculum materials. The results also gave evidence that the use of adapted RME exemplary curriculum materials could reduce the difficulty of the introduction of the innovation to the teachers.

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NAME, ARTS, MATHEMATICS, AND TECHNOLOGY

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ABSTRACT

For some years the author has been experimenting with an activity based on designs formed by her students' names. In order to form the design, we transform the student's name into points on the coordinate plane, then we connect points to form a closed polygon, then we subject this basic polygon to three 90-degree rotations. The result is an individualized, frequently interesting and complex polygonal design. Students can then, by coloring in regions, obtain interesting and often quite beautiful designs. One can then ask mathematical questions about these figures, how many pairs of parallel or perpendicular lines there are, and what are the areas of the various regions, and so on. Students seem to enjoy this activity, and the fact that the activity automatically yields individualized projects seems to enhance the students' interest.

When the lines involved happen to pass through points of the graph paper grid, it is comparatively easy to find their slopes and thus determine properties such as being perpendicular. Areas of squares and other quadrilaterals can be computed readily if their vertices lie on grid points. There will be some quadrilaterals whose vertices do not lie on grid points and we must learn how to solve pairs of linear equations to find these vertices. There may be many quadrilateral figures in a given design, and so we encourage the use of calculators to keep the computational labor from being excessive.

We find this project has been helpful to students about to enter calculus, because it affords an amusing and motivated review of the important pre-calculus notions. It also is good for prospective teachers because it gives a way to vertically integrate parts of the curriculum.

The author has been working with this pedagogical device for several years and new ideas still seem to be coming up. In this talk we illustrate in detail how our activity works with the example "ICTM".

Keywords: Art; Calculator; Spiral Curriculum; Problem Solving; Problem Posing

1. Background and Literature Review

The National Council of Teachers of Mathematics (NCTM, 1989, 2000) suggests that the emphasis of the mathematics curriculum should move away from rote memorization of facts and procedures to the development of mathematical concepts, and that students should investigate through problem solving not only to make connections among various representations of those concepts, but also to make these concepts meaningful to themselves.

It is believed that the use of real world representations helps students develop understanding of abstract mathematics (Fennema & Franks, 1992). Real-world problems are commonly used as vehicles to introduce or deepen students' understanding of mathematical concepts and relationships. To be successful problem solvers, however, students must develop inquiring habits of mind. They not only need to seek what are the solutions to problems and to determine why the solutions work, but also to pose questions to answer. Of course, teachers also play an essential role in developing inquiring minds. In particular, teachers must themselves be models of inquiry and must establish classroom context in which questioning and proving are the norm. They should pose questions about the problem situations and challenge students to defend their problem-solving strategies.

House (2001) concludes that a good problem is a problem just keeps giving and giving. What are the characteristics of investigations that can lead to good mathematics problems for students? Good problems (adapted from Russell, Magdalene, & Rubin, 1989; Wheatley, 1991: Clements, 2000):

1) Are meaningful to the students;
2) Stimulate curiosity about a mathematical or non-mathematical domain, not just an answer;
3) Engage knowledge that students already have, about mathematics or about the world but challenge them to think harder or differently about what they know;
4) Encourage students to devise solutions;
5) Invite students to make decisions;
6) Lead to mathematical theories about a) how the real world works or b) how mathematical relationships work;
7) Open discussion to multiple ideas and participants; there is not a single correct response or only one thing to say;
8) Are amenable to continuing investigation, and generation of new problems and questions.

Miller (2001) reports that he uses an interdisciplinary project to teach the mathematics concepts of transformations of periodic functions. Through this project the author has increased the students' understanding of the mathematics concepts and helped them to see mathematics in art. This use of interdisciplinary units in art can satisfy requirements found in NCTM Principles and Standards for School Mathematics (NCTM, 2000).

Mustafa (2001) finds a new method to determine the end point of a segment. In his article the author presents a new method for determining the coordinates of the endpoint of a segment, given its slope, point of origin, and length. For example, given segment AB=12.5, point A (2,2) and slope $m=\frac{4}{3}$, find endpoint B. His method consists of five steps:

Step 1: Z=(2+3, 2+4)=(5,6)

Step 2: $d_1=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=\sqrt{(2-5)^2+(2-6)^2}=5$

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Step 3: $k=\frac{12.5}{5}=2.5$

Step 4: $\frac{\Delta y_2}{\Delta x_2}=\frac{4\times 2.5}{3\times 2.5}=\frac{10}{7.5}$

Step 5: B=(2+7.5, 2+10)=(9.5, 12)

B=(2-7.5, 2-10)=(-5.5, -8)

This seems an easy-to-follow set of steps for finding the endpoint of a given segment AB, but my question is why do we need to find the endpoint? We want students to understand why things such as Mustafa's method are worth knowing. Mathematicians all over the world are trying to find a better way of teaching mathematics and for students to learn mathematics. However, students are always wondering; why do we need to study mathematics, how is it related to us? With the help of technology, plugging into a formula to compute the right answer is not the main issue, what is important is that our students understand mathematics and know the relevance of mathematics to every one of us.

2. The Questions

Question 1: When we ask teachers to "integrate technology into their classrooms" we are asking for the biggest change in educational practice in the last 200 years. This task is so difficult, so painful, so challenging and so directionless. How to help our teachers to try and to try very hard to "integrate technology into their classrooms" in order to improve their teaching and improve their students' learning, will be a main focus of this presentation.

Question 2: Mathematics has been taught based on the chapters of the textbooks. Usually there are few direct connections among chapters of books or among branches of mathematics. In collaborative, open-ended designed problems, how do the teacher and students build and maintain a common understanding of the task? From the teacher's perspective, how can she or he guide and assist them as they invent and design an artefact? From the students' standpoint, how do they know where to start? How do they know what to do and whether they have all the information to complete the task? How are they able to figure out which knowledge is useful or which is not? How do they decide on a goal?

We do not have answers for the above questions. However, we do propose an activity that can be used to tie in some of the mathematics curriculum which can be used as review for calculus students or a spiral curriculum for combining algebra and geometry, and which also motivates student to learn how to use technology.

3. Designed Activity: The Shape of "ICTM"

In the following activity, we combine interactive teaching and collaborative learning. Students become active participants in the learning process. In this activity, this pedagogy, proved by our experience to be helpful, is used to cover topics in pre-calculus, linear equations and inequalities, algebra, geometry and analytical geometry. The goal of this activity is to let students gain competence in mathematics, and be prepared to go on to successful completion of the calculus sequence. We believe that this activity meets most of the characteristics of NCTM standards. It represents a "big idea", uses processes that are appropriate to the discipline, is thought provoking, fosters persistence, develops thinking in a variety of ways, and has multiple avenues of approach, making it accessible to all students.

For general mathematics classes, we use this activity to get the students interested in learning some basic mathematics. For calculus classes, we use this activity as concept review and prior knowledge checking. For prospective teachers, we use this activity in their mathematics methods course to demonstrate how to integrate material into a spiral curriculum.

We now describe our activity in detail, in the following five steps

Step 1:

1) First, we create a rule for relating alphabets to sets of numbers. We use a simple, order-preserving rule for assigning numbers to letters of the alphabet, given by the following Table 1.

Table 1: Chart showing correspondence between alphabets and numbers

1	2	3	4	5	6	7	8	9	10
A	b	c	D	E	f	g	h	i	j
K	l	m	N	O	p	q	r	s	t
U	v	w	X	Y	z				

2) Next, we find a point set corresponding to ICTM (the acronym of "International Conference on Teaching of Mathematics"). The letters I, C, T, M, according to Table 1, correspond to numbers 9, 3, 10, 3. Let these be the x-coordinates of our points. Start from the second number to form the y-coordinates (No specific reasons, just to have the y- coordinates. You can use any method to get y coordinates as long as it is consistent). In order to make the first point's coordinates and the last point's coordinates the same, we will have to add the first number at the last. Therefore the points of (9,3), (3, 10), (10,3), (3,9), (9,3) will form a closed basic shape of our name, ICTM.

Step 2:

Connecting the five ordered pairs (x, y) on the following coordinate plane, we construct the basic figure of our graph (See Figure 1). We can identify some mathematics concepts here. We hope our students will able to pose some questions. For example, what are the lengths of these four lines? What are the slopes of these four lines? What are the equations of these four lines? Do we have parallel lines? Do we have perpendicular lines? What is the intersection point of any two lines? What kinds of geometric shapes do we have? Do we have isosceles triangles? Do we have trapezoids?

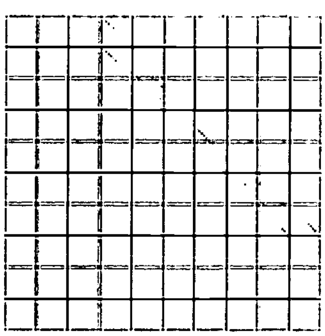

Fig 1: Basic Shape

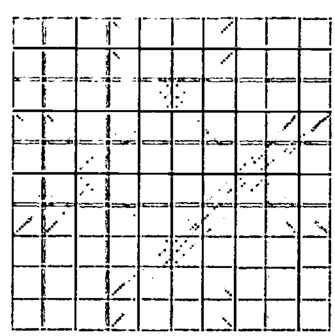

Fig 2: Transformed Shape

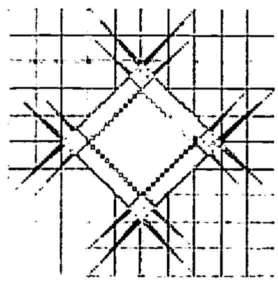

Fig 3: Squared Table

Step 3:

In order to make the shape of ICTM more complicated and interesting, we form the following Table 2, adding on three extra correspondences of ICTM and the numbers. Now we connect all the other transformed ordered pairs of (y, 10-x), (10-x,10-y), (10-y, x) to form Figure 2. What do you see? We can ask the same questions as for the basic shape and ask further questions. For example, are there any squares in this transformed shape? We can color the graph to make some beautiful pictures (See Appendix). What is the name of this picture? Does this picture fit the theme of our "International Conference on the Teaching of Mathematics"?

Table 2: ICTM and the transformed ordered pairs

X	y	(x, y)	(y, 10-x)	(10-x, 10-y)	(10-y, x)
I	c	(9,3)	(3,1)	(1,7)	(7,9)
C	t	(3,10)	(10,7)	(7,0)	(0,3)
T	m	(10,3)	(3,0)	(0,7)	(7,10)
M	i	(3,9)	(9,7)	(7,1)	(1,3)
I	c	(9,3)	(3,1)	(1,7)	(7,9)

In the above Figure 3, the author sees the design of our shape of ICTM as Squared Tables. The squared tables at which mathematicians all over the world are sitting here to share ideas, to talk about teaching strategies, to learn more about our students, to assess new technologies, to make a better environment for teaching and learning mathematics.

Step 4: Exploring the transformed shape and posing mathematics questions

We ask the students to explore their name shapes and find out what kind of mathematical concepts show up. Can the students pose their own mathematical questions, such as are there equal line segments, deciding if there exist parallel or perpendicular line segments, or are there any parallelograms and so on. In the ICTM case, we have the following mathematics concepts show up: 1) plotting points; 2) connecting two points; 3) finding distance between two points; 4) finding slopes of line segments; 5) finding equations of line segments; 6) looking for perpendicular or parallel line segments; 7) solving systems of linear equations; 8) area of geometrical figures; 9) angle of two segments; 10) different geometric figures; and more.

Step 5: Doing mathematics

We pose a problem that uses most of the above mathematic concepts. With the help of a calculator, the students can plug in the formula to get the answers quickly. Rather than giving credits for the answers, we ask students to observe the relationships between the answers. Or we can, for instance, ask them to predict whether two given segments AB and CD have the same length, and use the formula to find out if their prediction is correct and similar questions of making and checking predictions. We can generate more mathematics by, for example, asking students to decide whether the triangle formed by vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is an isosceles triangle or an equilateral triangle. Different students might have different figures to work on.

The question: In our ICTM case, to prove that the triangle formed by vertices A (3,1), B (9,7) and $C(x_1, y_1)$ is an isosceles triangle, where $C(x_1, y_1)$ is the intersection point of line segments formed by points (3,1) and (10,7) and the line formed by points (3,0) and (9,7), respectively.

In order to prove it is an isosceles triangle, we need first to find the third vertex which is the intersection point of line segments formed by points (3,1) and (10,7) and by points (3,0) and (9,7), respectively. In order to do that, we need to solve a system of linear equations. Second, we need to find the equations of these two line segments. In order to find the equations of line segments, we need to find the slopes of line segments. Reverse the operation processes, we find the slopes first and proceed until we find the intersection point.

(1) Find the slopes: $m_1 = \frac{y_1 - y_2}{x_1 - x_2} = \frac{1-7}{3-10} = \frac{6}{7}$ $m_2 = \frac{y_1 - y_2}{x_1 - x_2} = \frac{0-7}{3-9} = \frac{7}{6}$

(2) Find the line equations:

Line 1: $y = mx + b \Rightarrow y - 1 = \frac{6}{7}(x-3) \Rightarrow 6x - 7y = 11$

Line 2: $y = mx + b \Rightarrow y - 0 = \frac{7}{6}(x-3) \Rightarrow 7x - 6y = 21$

(3) Solve a system of linear equations:

$$\begin{cases} 6x - 7y = 11 \\ 7x - 6y = 21 \end{cases}$$

With the help of a calculator, we can find the intersection point is $(81/13, 49/13)$. Our next question is whether the triangle formed by (3,1), (9,7) and $(81/13, 49/13)$ is isosceles? How do we start? Again with the help of a calculator, we can find that the lengths of the segments formed by (3,1) and $(81/13, 49/13)$ and by (9,7) and $(81/13, 49/13)$, respectively, are equal. We can continue to pose questions; for example, what is the area of this isosceles triangle?

The area of this isosceles triangle$= \frac{1}{2}\begin{vmatrix} 3 & 1 & 1 \\ 9 & 7 & 1 \\ \frac{81}{13} & \frac{49}{13} & 1 \end{vmatrix} = \frac{18}{13}$

Our last question for the moment will be, what is the acute angle of this isosceles triangle?

Angle Formula: $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

From the previous calculation we know a=b=4.2552, and c=8.4853. Thus

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{4.2552^2+4.2552^2-8.4853^2}{2\times 4.2552\times 4.2552}$$

Again the calculator helps us to find the angle C that is 2.98934 radians or 171.1968866 degrees. And we can ask the students to explore whether the other two angels are equals and as reinforce checking for the concept of equal sides of triangle have the equal angles. Notice how this question about the isosceles triangle requires students to use much of their prior knowledge.

4. Discussion

In this activity, we ask our students to explore lines that have the same slope, and be able to conclude that they are parallel lines. Or we can ask students to find the slope of two apparently perpendicular lines and explore the relationship between the two numbers and find out what pairs of perpendicular lines have in common algebraically.

We would like our students to find out for themselves why we need to solve systems of linear equations. When the intersection points are on the grids of coordinate system, it is easy to identify the coordinates. However, if points are not on the grid of the coordinates what can we do? These kinds of questions lead to solving systems of linear equations. Students now see a purpose for systems of linear equations, and they will be willing to work on them. If a few problems are not enough to make the students familiar with the mathematics concepts, we can always find further questions or problems. For example, we could ask them to do more, to observe more, to conjecture more and to learn more, there is always more to learn.

In the ICTM case, if we rotate the basic shape 90 degrees, 180 degrees and 270 degrees, then we get the transformed shape. The shape is symmetric horizontally, vertically and diagonally with respect to x=5, y=5 and point (5,5). We can explore different ways to transform names into shapes. For instance, in Step 1, we could choose the y-coordinates to be some other cyclic permutation of the x-coordinates. We could modify Table 1 to make the alphabet letters correspond to numbers 0 through 9, for instance. It would be interesting to see how such changes in the correspondence affect the shape of the names. Also, instead of permuting the x-coordinates to get the y-coordinates, we can let y be a pre-determined function of x. For example, if we take y=x, we have one straight line instead of a closed polygon, and with the transformations we get a figure consisting of two perpendicular lines intersecting at (5,5). If, as another instance, we let y=x/2 and perform the transformations we get four lines enclosing a square. Another interesting function to use is $\left[\frac{x^2}{10}\right]$ (integer part).

5.Conclusion

To be successful problem solvers, students need to develop inquiring habits of mind. Many educators believe students learn better when they have a personal interest in the assigned projects. In our background and literature review we have presented some main features of a good problem according to some educators.

The challenge to teachers is to come up with good problems and activities. For some years the author has been experimenting with an activity based on designs formed by the students' names. We have had enough success to believe this fits Clements' criteria for a good problem. Due to the individualization of this activity, we hope your name produce some interesting mathematics that might be totally different from the example we presented here.

In this paper we have illustrated our name-design activity by focusing on one example; the acronym of our organization. We have seen how a number of interesting questions come up, and we have explored a few of them. Answering all these questions by hand calculations would be too tedious, and students would lose momentum: so we are naturally led to the appropriate use of calculators.

Not mentioned above in our paper, but a facet that makes this activity appeal to students, is that very beautiful designs frequently appear when regions of the name-graphs are colored in various ways. In an appendix, we show several designs based on "ICTM".

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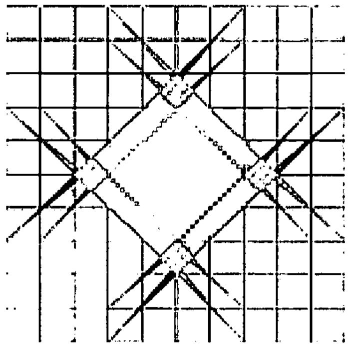
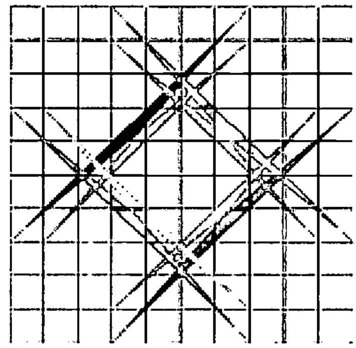
Appendix

The designs of ICTM

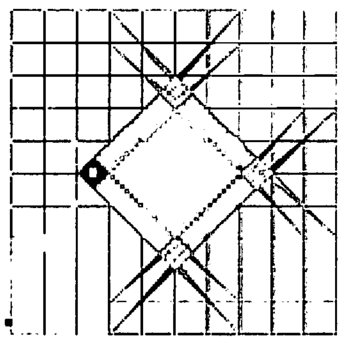
The Squared Tables



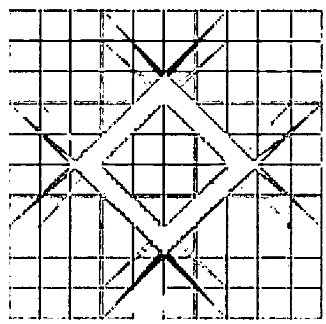
The Isosceles Triangles



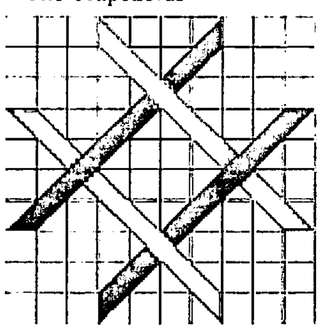
The Tropical Fish (Side View)



The Tropical Fish (Front View)



The Trapezoids



BLENDING TECHNOLOGY AND PURE MATHEMATICS: Is the hard work worthwhile?

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ABSTRACT

Incorporating a computing component into an undergraduate pure mathematics course is well-established practice. Reasons given for introducing technology include freeing students from the grind of hand calculations so that they can tackle more realistic problems, exposing students to the possibility of exploratory work, and allowing graphical as well as numerical representations of the mathematics. Although a small number of courses have abandoned lectures and are taught entirely in the laboratory, most still retain the traditional format and present the computing component as a supplement.

Integrating the computing work with standard lectures and pen and paper exercises requires a clear understanding of the aims of each type of learning activity. Questions to be considered include: what is an appropriate balance between teaching the students about the software and teaching them mathematics, what do students believe they are learning from computer-based sessions, and are students' perceptions of the purpose of this type of activity markedly different from that of the teacher? Designing a new Matlab-based computer laboratory program for an undergraduate linear algebra course with an enrollment of 850 students presented both a technical challenge and an opportunity to investigate these important questions. Student reaction, both critical and favourable, is discussed.

1 Background and purpose

The practice of integrating computing components into undergraduate pure mathematics courses (usually calculus and linear algebra courses) goes back at least twenty years and with ongoing improvements in software is becoming increasingly common. The primary motivation is usually to improve the learning outcomes for students. One hopes that the technology will free students from the grind of hand calculations so that they can tackle more realistic problems, expose them to the possibility of exploratory work, allow graphical as well as numerical representations of the mathematics, and provide more variety in the students' learning experiences.

The commitment in time, energy and resources to run a computing component is substantial, and so it's important to know if the aims are being met. Alexander (1999) reported results of a survey of 104 teaching development projects involving technology (90% of which had the stated aim to improve student learning) in university courses over a broad range of disciplines, which revealed that only a third could report an improvement in quality of learning outcomes because only a third actually tested for this. The remainder restricted their evaluation to a basic student feedback questionnaire of the type that focused on student reaction to the innovation. Alexander suggests a range of fourteen different methods of evaluation of student learning outcomes, including comparative studies, pre- and post-testing, focus groups, expert reviews, observations of student use, and student questionnaires testing experiences and perceptions as well as reaction.

Anecdotal evidence and evidence based on student surveys suggest that a sizeable proportion of students are lukewarm on the use of computers in maths courses. Coupland (2000) reports that asking students for an overall view of their experiences with *Mathematica* in first year courses produced positive, neutral and negative responses in the ratio 25 : 27 : 47. In the study by Galbraith et al (1999), the open-ended question "How do you feel about using computers to learn mathematics?" elicited 15 positive responses, 14 negative responses and 5 containing both positive and negative comments. In a University of Sydney linear algebra course held during 2000 using in-house software, students were asked if the lab sessions had helped them understand the course. There were 110 positive, 79 neutral and 63 negative responses.

The question of appropriate evaluation became relevant when a new Matlab-based computer laboratory program was introduced in 2001 into a large second year linear algebra course at the University of Sydney. Although a computing component had been part of the course for many years, there were several reasons for replacing it with a new program. Firstly, the Engineering departments had moved to Matlab and wanted their students to use the same system in mathematics. Secondly, it was felt that all students would benefit from an introduction to a commercial program widely used in industry. Thirdly, the previous program had no graphics capability and was somewhat dated; the increasing experience and sophistication of students as computer-users meant that attention had to be paid to visual as well as numerical aspects of the program. The new lab program had two aims: to familiarize students with basic Matlab commands and to improve their understanding of the linear algebra concepts.

Prosser (2000) commends the usefulness of open-ended questions in order to accurately reflect student beliefs and perceptions, especially in the evaluation of new technologies. He notes that "level of agreement" questions produce judgements by students on issues determined by academics as important, which may not coincide with issues students consider important. The purpose of this study is to attempt to discover and analyze students' perceptions and experiences of the lab program using both student-focused questions and questions measuring student reaction, as a first step towards evaluating whether the aims of the programs have been met. Other evaluation methods such as those mentioned by Alexander are expected to be used at a later stage.

2 Method

At the end of the course in which the new lab program ran for the first time (semester 1, 2001), 362 students (218 engineering and 144 science students) volunteered to complete a pen and paper questionnaire. Students were asked to indicate if they were enrolled in engineering, but no other personal data were recorded. The questionnaire contained 19 statements, of which 12 related specifically to the computer laboratory sessions. Students indicated their level of agreement with each statement. The responses were scored 0,1,2,3 or 4, a score of 0 corresponding to strong disagreement and a score of 4 to strong agreement. In addition, three open-ended questions invited students to say what they liked most and disliked most about the lab sessions, and to suggest improvements.

In the following running of the course (summer session 2002), a further questionnaire containing open-ended questions on students' experience of the lab program was completed by a much smaller number (n=28). Seven of these were repeat students, while twenty one were new to the course.

2.1 The students and the course

Students in the course are drawn mainly from Engineering (55%) and Science (42%) degrees. This course is compulsory for engineers, roughly 50% of whom had prior Matlab experience. There are two lectures, one pen and paper tutorial and one computer laboratory session per week, for one semester. The lectures cover standard material: elementary vector space theory, linear transformations, diagonalisation and applications of the theory to the solution of recurrence relations, systems of linear differential equations and quadratic forms. Over the years, many students have said that they find this material abstract and somewhat difficult to understand. Labs contribute 10% to the overall course assessment, the balance coming from quizzes, tutorial participation, written assignments and final examination.

2.2 The lab program

The new program uses Matlab with a graphical user interface to provide a step-by-step path through each problem, giving immediate feedback to students on the correctness of their data entry and allowing for automatic registration of completion of questions and recording/marking of their answers to specific assessment tasks. These features were incorporated to manage the large enrollment, and permitted the laboratory sessions to run (after the first month) without tutorial staff. Around 50 problems (numerical, graphical and experimental) and two special animations were devised, tested and incorporated into the program. Students complete four or five problems each week, either at a scheduled time or at any other time when lab space is available. At present, students can access the program only on campus.

3 Results

3.1 Results of the first questionnaire

Statements asking for level of agreement

Statements concerning the computer labs which scored the highest and lowest averages are given below. There were no significant differences in the responses of engineers versus non-engineers to any question except that which asked about previous Matlab experience.

Highest averages, indicating agreement (average score over n=362, standard deviation):
I would prefer to be able to do the lab work from home via the web (2.92, 1.20)
I now feel reasonably familiar with the basic Matlab commands (2.76, 0.89)
The mix of 2 lectures, 1 tutorial and 1 lab session per week was just right (2.69, 0.91)
I appreciated the structured nature of the lab problems (2.62, 0.88)

Lowest averages, indicating disagreement (average score over n=362, standard deviation):
I was an experienced Matlab user before the course started (1.35, 1.43)
The lab questions are too difficult to understand (1.45, 0.88)

Some of the remaining statements, with averages closer to the "neutral" score of 2, were statements that related in important ways to the pedagogical success of the lab program from the students' point of view. The statement
"The lab sessions helped me to understand the course" (2.20, 1.03)
included 169 positive, 100 neutral and 93 negative responses. The statement
"The lab work was interesting" (2.10, 1.00)
included 139 positive, 128 neutral and 95 negative responses. The statement
"The graphics in the lab sessions helped me to understand the maths" (2.20, 1.03)
included 157 positive, 107 neutral and 98 negative responses.

Answers to the open-ended questions

From the open-ended questions, a total of 303 responses to the question "What did you like most about the lab sessions?" and 304 responses to "What did you dislike most about the lab sessions" were recorded. There were 161 suggestions for improvement to the lab sessions. The students' comments were classified under the following general headings. The numbers in brackets indicate the number of times the response was written. The spread of responses from engineers appeared not to differ markedly from the non-engineers and so the numbers recorded are combined. Students usually wrote at most one comment for each question.

Liked most about the lab sessions:
Easy-to-use system, questions were quick and easy to do (94)

Interesting questions that helped understanding of concepts (68)
Ability to work at own pace and at flexible times (44)
Step by step structure of the questions (43)
The graphical questions and animations (22)
Learning Matlab (14)
Ability to use computer and maths together (8)
Ability to experiment and solve realistic problems (6)
Labs contributed to the assessment (4)

Disliked most about the lab sessions:
Old hardware, lab ambience, lab location, occasional bugs (72)
Step by step structure of questions (49)
Lack of tutorial assistance after first month (47)
Questions sometimes boring or too easy (46)
Problems sometimes too hard (24)
Lack of feedback on whether answers were right or wrong (19)
Can't do labs off campus (17)
Method of assessment of labs (11)
Having to use pen and paper as well as computer (11)
Labs not relevant to lectures (8)

Suggestions for improvement to the lab sessions:
Employ tutors for the whole semester (35)
Arrange access to lab program from home (24)
Buy better computers for the labs (20)
Have more challenging questions (18)
Change the way lab work is assessed to provide better feedback (17)
Abolish the lab program (16)
Replace the existing GUI (8)
Provide more graphical questions (7)
Provide a hard copy of the help manual/question bank (7)
More problems on applications (6)
Make lab work more relevant to lectures and tutorials (3)

Some of the replies to the open-ended questions (with both positive and negative views of the program) were very thoughtfully constructed, others were very brief. It is also possible that these responses were influenced somewhat by the content of the previously-answered written statements, which reminded them specifically about particular issues concerning their lab work. For this reason the second questionnaire, conducted during the next running of the course, attempted to gauge students' opinions of the wider issues relating to the lab sessions, free of the influence of a structured survey.

3.2 Results of the second questionnaire

Twenty eight students volunteered to provide responses to the following three open-ended questions. Students' comments were again classified under general headings, to indicate the range of their replies. Students usually wrote one comment at most. The brackets indicate the number of times that answer was mentioned.

What do you think is the purpose of the computer lab component?
(Comments arranged from emphasis on understanding of mathematics to emphasis on learning Matlab)
To help students understand the course (7)
To help students understand the theory by using a computer to eliminate errors (3)
To revise lecture material by doing complicated questions that cannot be done by hand (2)
To solve practical problems related to the theory in the course (2)
To assist visualization (2)
To expose students to advanced software and lift awareness and understanding of how technology can come into mathematics (2)
To be able to answer questions faster and more efficiently (2)
To gain experience with a computational package that will be used in the real world (8)

Your course contains lectures, tutorials and computer laboratories. What relationships should there be between these components?
Temporal/content relationship: lecture first to present a topic, then pen and paper tutorial with exercises to reinforce the same ideas, followed by lab for practical applications (11)
Content relationship: all components should reinforce each other to widen understanding (6)
Balance is wrong: there should be 1 lecture, 2 tutorials and 1 lab (1)
Matlab should be referred to in lectures to show ways in which it can be used (1)
Lab problems are appropriately easy mathematically, because new computing skills are being learned simultaneously with the mathematics (1)

What do you believe you are learning from the computer lab sessions?
Increasing knowledge of Matlab and seeing how it's used to solve real life problems (19)
Increasing understanding of concepts presented in lectures (8)
Observing patterns, seeing how matrices work, predicting (1)
Nothing much (1)

4 Discussion and Conclusions

In the first questionnaire, approximately one quarter of students had a negative response to the statements "The lab sessions helped me to understand the course", "The lab work was interesting", and "The graphics in the lab sessions helped me to understand the maths". It seems that the computer laboratory program has failed to engage a significant number of students, confirming similar data mentioned in the introduction. Galbraith et al (1999), in a study on attitudes to computer use and to mathematics, present correlations which suggest that positive attitudes to computers are more influential than positive attitudes to mathematics in determining active involvement in the use of computers to learn mathematics. It would be interesting to investigate this further to determine if other factors are also involved.

The most appreciated feature of the lab sessions (94 mentions) was that the work could be completed relatively quickly and easily. Linking this with the 44 favourable mentions of the ability to work at their own pace at flexible times, and the quite strong agreement with the statement "I would prefer to be able to do the lab work from home via the web" suggests that students value highly a reasonable workload that can be managed in their own way at their preferred time. The 68 favourable mentions of questions that helped understanding of concepts, together with the 22 favourable mentions of graphical questions and animations (and other related comments with smaller frequency), suggest that about a quarter of students value the "mathematics plus computer" experience. Many of the features that were disliked were echoed in suggestions for improvement. The most disliked feature, the hardware and the lab environment (72 unfavourable comments), was reinforced by 20 suggestions for improved quality of computers. This type of complaint should become less frequent with progressive upgrading of the equipment. There appear to be roughly the same number of students holding opposite extreme views about the lab program (16 abolitionists and 18 who want more challenging questions), and roughly the same number who were pleased by the structure provided by the GUI as were irritated by it.

Though the sample was much smaller for the second questionnaire, some interesting features can be observed. In answer to the question about perceptions of the purpose of the computer lab component, the two most frequent comments identified each of the two aims of the program, but students did not perceive the possibility of a dual purpose. For the question on the relationships between the different components of the course, most responses focused on the temporal/content relationships. This suggests a preference for a course structure in which topics are well defined by lectures and tutorials, and the role of the labs is to demonstrate their applications. Only 6 of the 28 students mentioned the lab program as a source of understanding of the mathematics. In the question asking what they believed they were learning from the lab sessions, responses focus predominantly on the use of Matlab itself and its capabilities in solving practical problems rather than Matlab as an aid to understanding the mathematics.

Do the responses from the two questionnaires help to determine whether the aims of the lab program have been met? A "level of agreement" statement in the first survey shows that even allowing for the subset of engineers who already had prior Matlab experience, the students as a whole claimed familiarity with the basic Matlab commands by the end of their course, and this is reinforced by the 19 responses in the second questionnaire claiming to be learning and using Matlab. This suggests that the first aim of the lab program has been met. However, disappointingly few students perceive the computer lab component as helpful in learning linear algebra. Further testing along the lines proposed by Alexander (1999) will be required to determine whether this indicates an actual lack of learning or simply a lack of perception.

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LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS:
An alternative to the method of undetermined coefficients

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ABSTRACT

Here we describe and illustrate an alternative to the method of undetermined coefficients for obtaining a particular solution of a linear differential equation with constant coefficients. The method requires only polynomial differentiation and some elementary algebra. The procedure has also been expressed as a recursive algorithm. Examples have been included to show the usefulness of the recursive analogue. Both the technique and its recursive equivalent can be suitably reformulated for similar difference equations.

1 Introduction

In this paper we are concerned with nonhomogeneous linear differential equations with constant coefficients. The term which makes the equation nonhomogeneous is a linear combination of the terms of the type $e^{\alpha x}p_n(x)$, where α is a (real or complex) constant and p_n is a polynomial of finite degree n in x.

The method of undetermined coefficients is, perhaps, one of the most widely used procedures for obtaining a particular solution of such a differential equation. The procedure is affected by choosing an appropriate trial solution containing unknown constants. Evaluation of these constants so that the trial solution satisfies the given differential equation leads to the required particular solution. The trial solution can be obtained by the annihilator method, but usually, it is obtained by following a set of rules [1], and unknown constants are determined by solving a system of linear equations. The procedure may be quite involved and often leads to tedious algebra.

For instance, consider the problem of finding a particular solution of

$$\begin{aligned} L(D)y &= (D^6 + D^5 + D^4 - D^2 - D - 1)y \\ &= x^3 + 4(11 - 6x^2)e^{-x} + 24x^2 \sin x + 6e^{-x/2}\cos(x\sqrt{3}/2), \ D \equiv \frac{d}{dx}. \quad (1.1) \end{aligned}$$

Appropriate choice for a particular solution of this differential equation (1.1) will contain 17 constants, to be determined by solving a linear system of 17 equations in 17 unknowns! However, by the superposition principle, its particular solution would be the sum of particular solutions of

$$\begin{aligned} L(D)y &= x^3 && (1.2) \\ L(D)y &= 4(11 - 6x^2)e^{-x} && (1.3) \\ L(D)y &= 24x^2 \sin x && (1.4) \\ L(D)y &= 6e^{-x/2}\cos(x\sqrt{3}/2). && (1.5) \end{aligned}$$

Computation of particular solution for the above equations would respectively lead to solutions of different systems of linear equations in four, three, six and four unknowns. In the following we present a much simpler alternative which uses only differentiation and some very simple algebra.

2 The procedure

Consider the equation

$$P(D)y = \sum_{i=0}^{m} b_i D^{m-i} y = e^{\alpha x}p_n(x), \quad (2.1)$$

where $b_i, 0 \le i \le m$, $b_0 \ne 0$ are constants, D denotes d/dx and other symbols have already been described. In general, the integers m and n are not the same. Set $y = e^{\alpha x}u$ in (2.1) to get

$$P(D)(e^{\alpha x}u) = e^{\alpha x}P(D + \alpha)u = e^{\alpha x}p_n(x).$$

This leads to finding a particular solution of

$$P(D+\alpha)u = \sum_{i=0}^{m} c_i D^{m-i} u = p_n(x), \quad (say) \tag{2.2}$$

Here $c_i, 0 \le i \le m$ are real or complex constants, and we assume $c_m \neq 0$. Now differentiate both sides of (2.2) repeatedly n times, so that the right of the last equation becomes a constant $(= D^n p_n(x))$. Then a particular solution would be given by

$$\begin{aligned} D^n u &= D^n p_n(x)/c_m = constant \\ D^{n+s} u &= 0, s = 1, 2, \cdots. \end{aligned} \tag{2.3}$$

Finally by backsolving we obtain a particular solution of (2.2) and since $y = e^{\alpha x} u$, a particular solution of (2.1) is obtained. It may be remarked here that the procedure leads to a particular solution in terms of lowest order derivative of u appearing in (2.2), which on integration leads to a particular solution of (2.2).

This procedure is based on the description given by Love [2] and its generalisation [3].

3 Examples

To illustrate we obtain particular solutions of equations (1.2)-(1.5).

• Differentiate (1.2) three times to get

$$(D^7 + D^6 + D^5 - D^3 - D^2 - D)y = 3x^2 \tag{3.1}$$
$$(D^8 + D^7 + D^6 - D^4 - D^3 - D^2)y = 6x \tag{3.2}$$
$$(D^9 + D^8 + D^7 - D^5 - D^4 - D^3)y = 6. \tag{3.3}$$

A particular solution of this equations is

$$D^3 y = -6, D^{3+r} y = 0, r = 1, 2, \cdots.$$

Substitute these in (3.2) to get $D^2 y = 6 - 6x$. Combining these with (3.1) we get $Dy = 6x - 3x^2$, and finally (1.2) gives $y = -6 + 3x^2 - x^3$.

•• To obtain a particular solution of (1.3) set $y = e^{-x} u(x)$ to arrive at

$$L(D-1)u = (D^6 - 5D^5 + 11D^4 - 14D^3 + 10D^2 - 4D)u = 4(11 - 6x^2). \tag{3.4}$$

Differentiating this we obtain

$$(D^7 - 5D^6 + 11D^5 - 14D^4 + 10D^3 - 4D^2)u = -48x$$

$$(D^8 - 5D^7 + 11D^6 - 14D^5 + 10D^4 - 4D^3)u = -48$$

Its obvious solution is $D^3 u = 12$, $D^{3+k} u = 0, k = 1, 2, \cdots$. Backsubstituting in the preceding equation we get

$$D^2 u = 30 + 12x,$$

and (3.4) gives

$$Du = 22 + 30x + 6x^2.$$

This is satisfied by $\qquad u = x(22 + 15x + 2x^2).$

Thus Eq. (1.3) has a particular solution $y = xe^{-x}(22 + 15x + 2x^2)$.

• • • By the principle of superposition a particular solution of (1.4) would be the imaginary part of the particular solution of

$$L(D)y = 24x^2e^{ix} \tag{3.5}$$

Here we set $y = e^{ix}u$ so that (3.5) becomes

$$(D^6 + (1 + 6i)D^5 + (-14 + 5i)D^4 - (10 + 16i)D^3 + (8 - 10i)D^2 + 4D)u = 24x^2$$

Proceeding as before we get

$$u = 2x^3 - 3(4 - 5i)x^2 + 3(1 - 24i)x,$$

and

$$Im(ue^{ix}) = 3x(5x - 24)\cos x + x(2x^2 - 12x + 3)\sin x$$

is a particular solution of (1.4).

• • • • Again by superposition principle a particular solution of (1.5) is real part of the particular solution of

$$L(D)y = 6e^{(-1+i\sqrt{3})x/2}. \tag{3.6}$$

The substitution $y = e^{(-1+i\sqrt{3})x/2}u$ in this equation gives

$$(D^6+(-2+3\sqrt{3}i)D^5-(9+5\sqrt{3}i)D^4+(13-3\sqrt{3}i)D^3+\frac{3}{2}(-1+3\sqrt{3}i)D^2-\frac{3}{2}(1+i\sqrt{3})D)u = 6.$$

Obviously this equation has $Du = (-1 + i\sqrt{3})$ as its solution and

$$Re((-1 + i\sqrt{3})xe^{(-1+i\sqrt{3})x/2}) = -xe^{-x/2}(\cos(x\frac{\sqrt{3}}{2}) + \sqrt{3}\sin(x\frac{\sqrt{3}}{2}))$$

is a particular solution of (1.5).

4 The recursive algorithm

We notice [3] that the problems of finding a particular solution of a nonhomogeneous linear differential equation of the form (2.1) is reduced to finding a particular solution of (2.2). In view of (2.3) we rewrite (2.2) as

$$\sum_{i=0}^{n} \beta_i D^{n-i}u = p_n(x). \tag{4.1}$$

This has been obtained from (2.2) by ignoring all the terms containing $D^{(n+s)}u, s = 1, 2, \cdots m-n$ when $m > n$ and adding terms containing $D^{(m+r)}u, r = 1, 2, \cdots n-m$ with zero coefficients when $m < n$. For convenience, we take $\beta_n \neq 0$ in (4.1). The process of n-times differentiation and backsubstitution can be expressed as the recursive relation

$$\beta_n D^{n-j}u(x) = D^{n-j}p_n(x) - \sum_{i=n-j}^{n-1} \beta_i D^{2n-i-j}u(x), j = 0, 1, \cdots n \tag{4.2}$$

For $j = 0$ this gives a particular solution of the equation obtained by differentiating (4.1) n times and its recursive use gives $u^{(0)}(x)(= u(x))$ when $j = n$, which is a particular solution of (4.1).

However, if $\beta_n = 0$ and $\beta_{n-1} \neq 0$, one obtain a solution u' which, after one integration, gives the required u. In fact, this recursive scheme gives a particular solution in terms of lowest order derivative in (4.1).

To illustrate the use of this algorithm (4.2) we obtain particular solutions of some differential equations.

• Following the above remarks, a particular solution of (1.2) is the same as that of the equation

$$(0D^3 - D^2 - D - 1)y = x^3$$

Here we have $n = 3, \beta_0 = 0, \beta_1 = -1 = \beta_2 = \beta_3, p_3(x) = x^3$, and the equation (4.2) takes the form

$$-D^{3-j}y = D^{3-j}p_3 - \sum_{i=3-j}^{2} \beta_i D^{6-i-j}y, j = 0, 1, 2, 3$$

This gives

$$\begin{array}{lrcl}
(j=0) & -D^3y & = & 6 \\
(j=1) & -D^2y & = & 6x - (-1)(D^3y) \\
 & & = & 6x - 6 \\
(j=2) & -Dy & = & 3x^2 - (-1)(-6) - (-1)(6-6x) \\
 & & = & 3x^2 - 6x \\
(j=3) & -y & = & x^3 - (-1)(6-6x) - (-1)(6x - 3x^2) \\
 & & = & x^3 - 3x^2 + 6
\end{array}$$

which gives the same particular solution of (1.2) as obtained earlier.

•• Particular solution of (3.4) is the same as that of

$$(-7D^2 + 5D - 2)v = 2(11 - 6x^2)$$

with $v = Du$. For this equation $n = 2, \beta_0 = -7, \beta_1 = 5, \beta_2 = -2, p_2(x) = 2(11 - 6x^2)$. The scheme (4.2) becomes

$$-2D^{2-j}v = D^{2-j}p_2 - \sum_{i=2-j}^{1} \beta_i D^{4-i-j}u, j = 0, 1, 2.$$

This yields

$$\begin{array}{lrcl}
(j=0) & -2D^2v & = & -24
\end{array}$$

$$
\begin{array}{lrcl}
(j=1) & -2Dv & = & -24x - 5(D^2v) \\
 & & = & -24x - 60 \\
(j=2) & -2v & = & 2(11 - 6x^2) - (-7)(12) - 5(12x + 30) \\
 & & = & -44 - 60x - 12x^2,
\end{array}
$$

as expected, leading to the same particular solution of (3.4) as computed earlier.

• • • The differential equation

$$(D^7 + D^5 - D^4 - D^2)u = 30x^4 \tag{4.3}$$

is the same as

$$(D^5 + D^3 - D^2 - 1)v = 30x^4$$

with $v = D^2u$. Its particular solution is the same as that of the equation

$$(0D^4 + D^3 - D^2 + 0D - 1)v = 30x^4$$

For this equation $n = 4, \beta_0 = 0, \beta_1 = 1, \beta_2 = -1, \beta_3 = 0, \beta_4 = -1, p_4(x) = 30x^4$. The algorithm (7) becomes

$$-D^{4-j}v = D^{4-j}p_4 - \sum_{i=4-j}^{3} \beta_i D^{8-i-j}v, j = 0, 1, 2, 3, 4.$$

This gives

$$
\begin{array}{lrcl}
(j=0) & -D^4v & = & 720 \\
(j=1) & -D^3v & = & 720x \\
(j=2) & -D^2v & = & 360x^2 - (-1)(D^4v) \\
 & & = & 360x^2 - 720 \\
(j=3) & -Dv & = & 120x^3 - (D^4v) - (-1)(D^3v) \\
 & & = & 120x^3 - 720x + 720 \\
(j=4) & -v & = & 30x^4 - (-720x) - (-1)(720 - 360x^2) \\
 & & = & 30x^4 - 360x^2 + 720x + 720.
\end{array}
$$

Since $v = D^2u$, this gives $u = 30x^2(-12 - 4x + x^2) - x^6$ as a particular solution of (4.3).

The procedure used here completely replaces the method of undetermined coefficients for a particular solution of nonhomogeneous linear differential equation with constant coefficients.

5 Epilogue

Here we consider ordinary linear differential equations with constant coefficients in which the nonhomogeneous term is a linear combination of the terms of the type $e^{\alpha x}p_n(x)$, where α is a (real or complex) constant and p_n is a polynomial of finite degree n in x.

The method of undetermined coefficients is commonly used to find a particular solution of such differential equation. Both from teaching and learning point of view, this is usually quite demanding. It would, therefore, be pedagogically interesting to have a simpler alternative to this method. In this paper we have presented such a method which requires only differentiation and some additions. Several examples have been included to manifest its versatality.

It has been observed that the problem is finally reduced to finding a particular solution of a linear differential equation with constant coefficients with nonhomogeneous term being a polynomial of a finite degree. The procedure in such a situation can be expressed as a recursive algorithm [3]. This has also been featured and illustrated by obtaining particular solutions of several differential equations.

Thus the present paper contains a simpler alternative to the method of undetermined coefficients in its totality. The differential equations which are amenable to the method of undetermined coefficients are taught almost everywhere at the undergraduate level, perhaps, due to the fact that their applications to the real world problems can not be over emphasized. In view of this, the procedure presented herein is didactically relevant and should attract the attention of everyone involved in the teaching of ordinary differential equations. Finally, it is remarked that the procedure can suitably be reformulated for similar difference equations [4, 5].

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A CASE STUDY IN THE HISTORY OF CALCULUS REFORM

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ABSTRACT

A case study of calculus reform at the University of Wisconsin-Eau Claire is presented. Instruction of calculus at this institution has passed through four identifiable stages. Assessment of these stages are discussed and reasons for changing modes of instruction are explained. A conclusion is that teaching environments need to be designed to accommodate different teaching styles and learning styles.

THE USE OF THE HISTORY OF MATHEMATICS IN THE LEARNING AND TEACHING OF ALGEBRA

The solution of algebraic equations: a historical approach

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ABSTRACT

The Ministry of Education (MPI) and the Italian Mathematical Union (UMI) have produced a teaching equipment (CD + videotapes) for the teaching of algebra. The present work reports the historical part of that teaching equipment.

From a general point of view it is realised the presence of a "fil rouge" that follow all the history of algebra: the method of analysis and synthesis.

Moreover many historical forms have been arranged to illustrate the main points of algebra development. These forms should help the secondary school student to get over the great difficulty in learning how to construct and solve equations and also the cognitive gap in the transition from arithmetic to algebra.

All this work is in accordance with the recent research on the advantages and possibilities of using and implementing history of mathematics in the classroom that has led to a growing interest in the role of history of mathematics in the learning and teaching of mathematics.

1. Introduction

We want to turn the attention to the subject of *solution of algebraic equations* through a historical approach: an example of the way the introduction of a historical view can change the practice of mathematical education. Such a subject is worked out through the explanation of meaningful problems, in the firm belief that there would never be a construction of mathematical knowledge, if there had been no problems to solve.

Gaston Bachelard (1967, p. 14) has written: "*It is precisely this notion of problem that is the stamp of the true scientific mind, all knowledge is a response to a question.*" That is, the concepts and theories of mathematics exist as tools for solving problems.

Also Evelyne Barbin (1996) has pointed out that. "*There are two ways of thinking about mathematical knowledge: either as product or as process. Thinking about mathematical as product means being concerned with the results and the structure of that knowledge, that is to say, with mathematical discourse. Thinking about mathematical as process means being concerned with mathematical activity. A history of mathematics centred on problems brings to the fore the process of the construction and rectification of knowledge arising out of the activity of problem solving.*"

Algebra (mostly that part relative to the so-called "literal calculus") is the more suitable branch of mathematics for the use of the method of analysis. Such a method is very old and still today one of the best definitions [together with that of synthesis] is that given by Pappus in his *Collection*: "*Now, analysis is the path from what one is seeking, as if it were established, by way of its consequences, to something that is established by synthesis. That is to say, in analysis we assume what is sought as if it has been achieved, and look for the thing from which it follows, and again what comes before that, until by regressing in this way we come upon some one of the things that are already known, or that occupy the rank of a first principle. We call this kind of method "analysis" as if to say* ***anapalin lysis*** *(reduction backward). In synthesis, by reversal we assume what was obtained last in the analysis to have been achieved already, and, setting now in natural order, as precedents, what before were following, and fitting them to each other, we attain the end of the construction of what was sought. This is what we call "synthesis."*." [Pappus, *Book 7 of the Collection* (tr., comm. A. Jones), 2 vols., New York, Springer, 1986]

Simply with reference to an educational point of view, we can say that analysis is a "backward reasoning". In (Rojano & Sutherland, 2001) this method is used for explaining the solutions of word problems.

2. The method of analysis in the construction of an equation

Before considering some algebraic problems drawn from the history, let us consider a typical problem that students deal during the first year of high school as an example of the method of *analysis*.

Problem. *In a rectangle the difference between its sides is 12 m and the perimeter is 224 m; find its area.*

We suppose that such a rectangle exist; to find its area we have to know the base and the height of rectangle; but we know the difference between the base and the height. Therefore

base = height + 12

or

height = base – 12.

However we know the semi-perimeter, that is 112 m. Then, if we take away the base from the semiperimeter, we can get the height. At this point, starting from what the problem asks, we have reached what we are given: the difference between the sides and the semiperimeter.

We suppose that the height is known and call it x and then the base is $x + 12$. From such an analysis we get

$x = 112 - \text{base}$

$x = 112 - (x + 12)$.

From that it is easy to obtain $x = 50$, the height; then we get the base, 62, and consequently we can compute the area, which is 3100 m^2. Geometrically we realise the symmetry of problem (that is, the base can be exchanged with the height).

We can note, from a didactical point of view, that the backward reasoning comes, step by step, from the question: what do I need to compute...? We go on putting these questions until we find, by splitting the problem, something known (given by the text of problem).

3. The Egyptian Rule of False Position

The next problem is one of 85 problems in the Rhind Papyrus, now housed in the British Museum.

Problem. *A quantity whose seventh part is added to it becomes 19.*

In modern notation the problem is equivalent to the solution of the equation $x + (1/7)x = 19$. The Egyptian method of solution, called the *Rule of False Position*, consist of giving to the unknown quantity x at the left side the beginning value 7, so that the resulting value at the right side is $7 + (1/7)\cdot 7 = 8$.

The argument goes on supposing that, if some "multiple" of 8 gives 19, than the same "multiple" will produce the sought number.

Therefore we can solve the problem by the proportion

$8 : 19 = 7 : x$ that is $x = (19/8)\cdot 7$.

The "False Position" in the history of mathematical education

Till the nineteenth century the rule of "false Position" is proposed again to present to the students first-degree equations. In the handbook *Elementi di matematica* [Elements of mathematics] by V. Buonsanto, Società Filomatica, Naples 1843 (pp. 117-119) we find the following passage:

"We shall look for a number, which solve the problem: but you will find it only by means of a *false* number, which does not solve it. This is the *rule of simple false position.* You have been told: A third and a quarter of my money are 24 ducats. How much money has I? Since you don't know the true number of ducats, you suppose that who is speaking gets 12 ducats. *This number, supposed in such an arbitrary way, is called position.* But it is easy to see that such a *supposition is false*, because a third and a quarter of 12 are 4 + 3 = 7 and so your friend should have not 24 ducats for a third and a quarter, but 7. However you can argue in this way. If 7 are the result of the false position 12, what number does 24 come from? You will do 7 : 12 = 24 : 288/7 and 288/7 = 41 and 1/7. Your friend has 41 and 1/7 ducats. To solve such problems you can suppose every number, but it is better to choose it in a way to avoid fractions. It is also better to choose a small number."

The rule of false position was also taught to American students of XIX century and is present in the textbook *Daboll's Schoolmaster's Assistant*, that was, till 1850, the most popular book of arithmetic in that country.

We can find still in recent works some notes on this method. The rule of false position can be used today, besides teaching first-degree equations (Winicki, 2000, and Ofir & Arcavi, 1992), to analyse as the spreadsheet works, e.g. the hidden algorithms (Rojano & Sutherland, 2001).

4. A Babylonian problem considered also by Diophantus

Babylonian algebra consisted of a totally algorithmic method formed by a list of operating rules to solve problems (*rhetorical* algebra). The algorithms were illustrated by numerical examples, however the recurrent use of some terms gives us a first concept of *symbolism*. Instead Diophantus introduces (in his *Arithmetica*) a literal symbolism and a form of language half way between "rhetorical" and "symbolic", that is "syncopated". In particular he introduces the "*arithme*" an indeterminate quantity of units that becomes a real unknown. Diophantus accepts only exact rational positive solutions, while Babylonians accepted also approximations of irrational solutions.

Problem. *Find two numbers whose product is 96 and sum is 20.*

Using modern notation the problem becomes

$$\begin{cases} x+y=20 \\ xy=96 \end{cases} \quad \text{or, in general form} \quad \begin{cases} x+y=b \\ xy=a \end{cases}$$

which is equivalent to quadratic equations $z^2 - bz + a = 0$.

What follows is the rhetorical solution of scribe (*instructions*) and his modern "*translation*".

	instructions	*translation*
1.	Divide by two the sum of numbers 20:2 = 10	$\frac{b}{2}$
2.	square $10^2 = 100$	$\left(\frac{b}{2}\right)^2$
3.	subtract the given area, 96, from 100 $100 - 96 = 4$	$\left(\frac{b}{2}\right)^2 - a$
4.	take the square root 2	$\sqrt{\left(\frac{b}{2}\right)^2 - a}$
5.	the base is $10 + 2 = 12$	$x = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - a}$
	the height is $10 - 2 = 8$	$y = \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - a}$

This method of solution shows that the Babylonians knew some laws of algebraic operations, made substitutions and solved by algebraic methods quadratic equations and systems equivalent to quadratic equations (Bashmakova & Smirnova, 2000).

The following Diophantus' method of solution (also used by Babylonians) is called "*plus or minus*".

1. "The difference between two numbers is two *arithme*" $x - y = 2\varsigma$

2. "If we divide the sum into two equal parts, each part will be half the sum that is 10" $\frac{x+y}{2}=\frac{b}{2}$

3. "If we add to one part and subtract from the other one half the difference of number, that is one *arithme*, we find again that the sum of two numbers is 20 units and the difference is two *arithme*"

$$\begin{cases} x+y=b \\ x-y=2\varsigma \end{cases}$$

4. "Let us suppose the bigger number is 1 *arithme* plus 10 units that are half the sum of numbers; therefore the smaller one is 10 units minus 1 *arithme*"

$$\begin{cases} x=\frac{b}{2}+\varsigma \\ y=\frac{b}{2}-\varsigma \end{cases}$$

5. "It is necessary that the product of two numbers is 96"

$$\left(\frac{b}{2}+\varsigma\right)\left(\frac{b}{2}-\varsigma\right)=a$$

6. "Their product is 100 units minus a square of *arithme*, that is equal to 96 units"

$$\left(\frac{b}{2}\right)^2-\varsigma^2=a$$

7. "And the *arithme* becomes 2 units. Consequently, the bigger number is 12 units and the smaller one is 8 units and these numbers meet the statement "

$\varsigma=\sqrt{\left(\frac{b}{2}\right)^2-a}$ from which $x=\frac{b}{2}+\sqrt{\left(\frac{b}{2}\right)^2-a}$ and $y=\frac{b}{2}-\sqrt{\left(\frac{b}{2}\right)^2-a}$.

The description of Diophantus shows awareness in the use of unknowns that we shall find only in the works of Arabic mathematicians.

5. Algebra and geometry in Euclid and Bombelli

Traditionally Book II of the Euclid's *Elements* (but also part of Book VI) is considered as an example of "geometrical algebra", also if this name can be misleading because the formulation is completely geometrical. We don't want to enter into the merits of debate concerning geometrical algebra (still far from over) that has seen engaged some famous mathematicians as Unguru, Van der Waerden, Freudenthal and Weil. We want instead to stress that the so called *problems of applications of areas*, also if explained and solved in geometrical way, can be considered equivalent to first and second-degree equations.

The first application consists of constructing a rectangle of area S on a given base a and finding its height.

This problem is equivalent to first-degree equation $a\cdot x = S$.

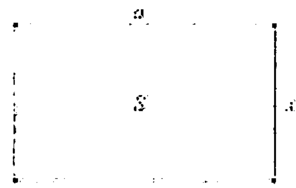

Figure1

Such a problem is solved by Euclid both in Book VI by means of proportions theory (thinking $S = b \cdot c$) and in Book I by means of that we can call a theory of equivalence of polygons.

Bombelli (*Algebra*, Book IV) proposes the same problem again in the following form: "*Find a line that is in proportion to .c. as .b. is to .a.*" Therefore we have to find the fourth proportional after three segments a, b, c. [$a : b = c : x$]

Bombelli in his *Algebra* gives for this problem two different constructions, both taken from Euclid.

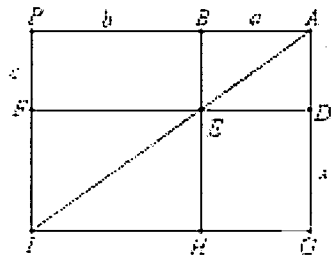

Figure 2

In the first one he considers the rectangle *FPBE* (Fig. 2), whose sides are b and c, then, having set $BA = a$, he joins points A, E, I and constructs the rectangle *PAGI*. The two rectangle *PBEF* and *EDGH* are equivalent for the Proposition I.43 of the Euclid's *Elements* and therefore *DG* is the solution of the equation $ax = bc$, that is

$$x = \frac{bc}{a}.$$

In the second construction (Proposition [73]) Bombelli uses the Thales' theorem and sets $AB = c$, $BC = a$ and $CD = b$ (Figure 3) and using the Proposition VI.12 of the Elements, concludes that $AB : BC = DE : CD$, so that DE is the required solution.

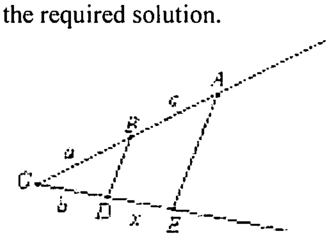

Figure 3

The method of Bombelli can be outlined in the following way:

1. *Enunciation of the problem*
2. *Geometrical construction of the solution*
3. *Solution of a numerical example via algebra.*

Always *the geometrical solution precedes the algebraic one.* Yet it is apparent from the analysis of the single cases, that *it is the equation, or better the form of its algebraic solution, which determines the subsequent steps of the construction* (Giusti, 1992).

We can note that this part of *Algebra* (Books IV and V), devoted to the application of algebra to geometry, marks almost a turning point, and sometimes a bringing forward, of the analytic geometry of Descartes (Bashmakova & Smirnova, 2000 and Giusti, 1992).

6. The Arab algebraists of the Middle Ages, the Italian algebraists of 16th century and the solution of the cubic equation

We owe to Arab algebraists, beside the introduction of word "*algebra*", the more and more aware use of substitutions to simplify the solutions of problems; Diophantus had already proposed such a method.

Moreover we find in the works of *Abu Kamil* (850-930?), more than in those of *Al-Khwarizmi* (800?-847), complicated transformations of expression with irrational numbers as the following problem shows.

Problem. *Divide 10 into two parts x and $10-x$ to get*

$$\frac{x}{10-x}+\frac{10-x}{x}=\sqrt{5}.$$

The relative quadratic equation is

$$\left(2+\sqrt{5}\right)x^2+100=\left(20+\sqrt{500}\right)x$$

that, multiplying by $\sqrt{5}-2$, becomes

$$x^2+\sqrt{50000}-200=10x.$$

But Abu Kamil finds another simpler solution setting $y=\dfrac{10-x}{x}$. He obtains immediately the equation

$$y^2+1=\sqrt{5}y$$

which has the solution

$$y=\sqrt{1+\frac{1}{4}}-\frac{1}{2}.$$

In this way we arrive to the linear equation

$$\frac{10-x}{x}=\sqrt{1+\frac{1}{4}}-\frac{1}{2}$$

that could be solved as

$$\frac{10}{x}-1=\sqrt{1+\frac{1}{4}}-\frac{1}{2}$$

that allows determining the unknown x, but it gives a result with an irrational denominator. Abu Kamil instead finds

$$10-x=\sqrt{1+\frac{1}{4}}x-\frac{1}{2}x \quad \text{that is} \quad 10-\frac{x}{2}=\sqrt{1+\frac{1}{4}}x$$

and squaring both the sides he obtains, after some calculations, the equation

$$x^2+10x=100$$

of which he finds the solution $x=\sqrt{125}-5$.

The method of making a substitution of an unknown to reduce a more difficult equation to a simpler one will become, as we shall see, quite usual.

The Italian algebraists of 16^{th} century have used these substitutions to solve cubic equations. We know that this mathematical "discovery" is the result of the works of *Scipione del Ferro* (1456-1526), *Girolamo Cardano* (1501-1576) e *Niccolò Fontana* (1500-1557) called *Tartaglia* [the "stammerer"]. Del Ferro begins with the equation $ax^3 + bx = c$ that he immediately reduces to the form $x^3 + px = q$ $(p, q > 0)$, dividing by a. Tartaglia, in his famous cryptic poem, assumes that the solution is of the form

$x = u - v$.

Then the equation can be reduced to the form

$u^3 - v^3 + (u - v)(p - 3uv) = q$.

If one imposes on u and v the additional condition $3uv = p$, then u and v can be determined from the system

$$\begin{cases} u^3 - v^3 = q \\ uv = \dfrac{p}{3} \end{cases}$$

Or also

$$\begin{cases} u^3 - v^3 = q \\ u^3v^3 = \left(\dfrac{p}{3}\right)^3 \end{cases}.$$

Putting $z = u^3$ we see that this system is equivalent to the quadratic equation

$z^2 - qz - (p/3)^3 = 0$,

which means that

$$x=\sqrt[3]{\sqrt{\frac{q^2}{4}+\frac{p^2}{27}}+\frac{q}{2}}-\sqrt[3]{\sqrt{\frac{q^2}{4}+\frac{p^2}{27}}-\frac{q}{2}}.$$

Let us consider, as an example of application of this method, the equation $x^3 + 6x = 20$. We set $u^3 - v^3 = 20$ and $u^3v^3 = 8$. We get $u^3 = 6\sqrt{3} + 10$ and $v^3 = 6\sqrt{3} - 10$ or $u^3 = -6\sqrt{3} + 10$ and $v^3 = -6\sqrt{3} - 10$. In both cases we get

$$x=\sqrt[3]{u}-\sqrt[3]{v}=\sqrt[3]{6\sqrt{3}+10}-\sqrt[3]{6\sqrt{3}-10}=2.$$

Remark. It is possible to reduce the standard cubic equation (in modern notation)

$ax^3 + bx^2 + cx + d = 0$

to the form

$y^3 + py = q$,

used by the Italian algebraists by means of the substitution

$$x=y-\frac{b}{3a}.$$

Viète will use a similar method to obtain the quadratic formula.

7. The quadratic equation in Viète and Descartes

The method of Viète

A possible way to obtain the quadratic formula was proposed by Viète in *De aequationum recognitione et emendatione Tractatus duo* (1591). He uses a substitution quite similar to that of Italian algebraists of 16^{th} century. Viète begins with the equation

$$ax^2 + bx + c = 0$$

(of course, he uses different symbols for the unknowns and the parameters). He puts $x = y + z$ and obtains

$$a(y + z)^2 + b(y + z) + c = 0$$

$$ay^2 + (2az + b)y + az^2 + bz + c = 0.$$

To eliminate the first degree term it is necessary that

$$2az + b = 0,$$

from which we get $z = -\dfrac{b}{2a}$. The substitution in the equation gives

$$ay^2 + a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c = 0$$

that is

$$4a^2y^2 = b^2 - 4ac.$$

From this he obtains

$$y = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

and lastly, using again the variable x

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

the well known quadratic formula.

The method of Descartes

In the Book I of the *Geométrie* (1637) Descartes gives detailed rules to solve quadratic equations. He uses, with a different approach, the classic Greek geometry; particularly the problems of applications of areas (Bos, 2001).

Hyperbolic application

a) Equation: $x^2 - ax - b^2 = 0$ $(a, b > 0)$.

Construction:

1. Draw a right-angled triangle AOB with $OA = \dfrac{1}{2}a$, $OB = b$ and $\angle AOB = 90°$.
2. Draw a circle with center A and radius $\dfrac{1}{2}a$.
3. Prolong AB; the prolongation intersects the circle in C.
4. $x = BC$ is the required line segment.

[Proof: BA intersects the circle in D; by *Elements* III.36 $BC \cdot BD = OB^2$, i.e., $x(x - a) = b^2$, so $x^2 - ax - b^2 = 0$.]

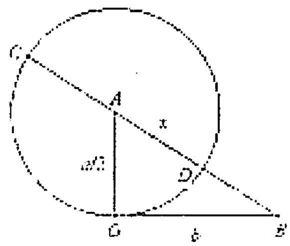

Figure 4

b) Equation: $x^2 + ax - b^2 = 0$ $(a, b > 0)$.

The construction is the same of previous case: it is enough to put $x = BD$.

Elliptic applications

Equation: $x^2 - ax + b^2 = 0$ $(a/2 > b > 0)$.

Construction:

1. Draw a line segment $AB = a$, with midpoint O.
2. Draw a semicircle with center O and radius $\frac{1}{2}a$.
3. Draw the line tangent at B to semicircle and mark on that line $BP = b$ in the half-plane where the semicircle is.
4. Draw a line through P parallel to AB. It intersects the semicircle in Q and S is the projection of Q into AB.
5. $x = SB$ is the required line segment, but also $x = AS$ is a solution.

[Proof: By *Elements* VI.8 $BP^2 = SB \cdot AS$, i.e. $b^2 = x(a - x)$, so $x^2 + ax - b^2 = 0$.]

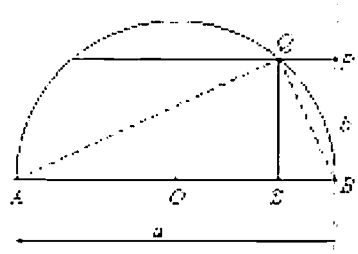

Figure5

Remark. In the cases of hyperbolic application Descartes constructs only the positive solution. Actually, also if he uses negative numbers in calculations, he doesn't give a geometrical meaning of negative numbers and therefore doesn't use negative abscissas.

7. Conclusions

The problems and the selected subject are meant to give relevance to the history and also to motivate and deepen student understanding of subject matter. Student can also see how problems were solved before the use of what to us are familiar equations and realise how a good symbolism make life easier for us in studying mathematics.

On the other hand it is difficult for student to give meaning to the "handling of symbols", when he meets the first time with algebraic equations. In this case the use of geometry, that gives a concrete meaning to symbols, can help student to overcome this epistemological obstacle.

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INTERACTIVE VISUALIZATION IN COMPLEX ANALYSIS

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ABSTRACT

Analytic functions of a complex variable exhibit some of the most striking beauty found anywhere -- but in the ages of black-on-white printed textbooks, this facet has been largely inaccessible to all except a few. Needham's recent text "Visual Complex Analysis" clearly demonstrates the power of a visual language as an organizing principle and as a useful tool to develop promising strategies for analytic arguments.

But modern technology allows one to go much further: We discuss selected implementations in computer algebra systems and especially in JAVA applets whose ultimate interactivity transforms every learner into an experimenter and researcher! Selected examples include zooming into essential singularities, mappings of the complex plane, winding numbers, and convergence of Laurent series.

We report how such tantalizing imagery transformed our own class, where amazing beauty led to inquiry and an urgent sense of "I want to know how/why that works". We contrast CAS-worksheets with model JAVA applets: On one side the user may modify and change everything -- but the algebraic-symbolic language of CAS worksheets usually requires a nontrivial "manual". On the other side, well-designed JAVA applets ideally require no instructions at all. Moreover, by using the "mouse" for input, and a graphic language for output, they take advantage of tactile, kinaesthetic and visual pathways that arguably have been much underutilized in mathematics teaching in recent centuries.

1. Introduction

The study of analytic functions of a complex variable is a center-piece of classical mathematics. Some of its distinguishing characteristics are its beauty and symmetry, which must have led many a researcher to go into complex analysis simply for aesthetic reasons. On the other hand, a large proportion of students in traditional introductory complex analysis classes never reach this level where they truly enjoy this beauty, but instead get stuck in a morass of algebraic-symbolic manipulations.

In the past decade we have seen calculus reform make much mileage from the "Rule of Three": Students achieve a deeper level of understanding when offered (and forced) to address concepts from multiple perspectives, commonly algebraic/symbolic, graphical, and numerical. Complex analysis faces the difficulty that a simple dimension count precludes the naïve implementation of a graphical approach as each of domain and range requires two (real) dimensions. Thus it is no surprise that for well over a century complex analysis was almost exclusively approached from the symbolic/algebraic perspective. However, the last two decades have seen a proliferation of graphical perspectives of complex analysis. The mesmerizing images of fractals and Julia sets, especially the "Mandelbrot set" may well be considered the starting point of this revolution. While Julia sets were known for several decades before that, it was only the dramatic increase of computational power (combined with mathematical ingenuity) that allowed one to "calculate" these beautiful images that result from (complex) function iteration. More recently the spectacular book "Visual Complex Analysis" by T. Needham [13] has much further propelled the move to also include graphical aspects into the complex analysis courses. These days queries of standard search engines yield an abundance of articles, applets and various course materials on the World Wide Web that implement graphical approaches to Complex Analysis. An excellent starting point is the page "Websites related to Visual Complex Analysis" [18].

In this article we give a personal account of teaching experiences using home-made implementations in computer algebra system (CAS) worksheets and JAVA applets, and discuss the merits of key features of such visual aids. One focus is on the balance of minimal start-up-costs (ease of use) versus universality (can do everything). In particular, in some cases a single slide or a movie (animation) is appropriate, whereas in others the kinesthetic aspects of direct interactivity via the mouse appear to be essential. Many of the insights, experiences and course materials shared in this article date back to an introductory complex analysis class taught at ASU in the fall of 1999. The class-size was very small, but the student body was very diverse as this was not a required class for any degree program. While many of our implementations into MAPLE and JAVA may no longer be unique as similar efforts are proliferating, we believe that they still have valuable unique aspects worth to be shared.

2. Visualizing functions of a complex variable

As suggested in the introduction, one of the first challenges encountered with functions of a complex variable is the difficulty to "visualize" their graphs: Upon identifying the complex line with the real plane, the graphs are simply (real) two-dimensional surfaces in (real) four-dimensional space – too hard for almost all human brains. A classical alternative identifies a complex valued function f of a complex variable $z=x+iy$ with the vector field $(x,y)\rightarrow(\mathrm{Re}\ f, -\mathrm{Im}\ f)$, the so-called Polya vector field [15,16], see figure 1.a for the example $f(z)=\cos(z^2)$. This approach helps especially well to connect contour integration from complex analysis with line integrals from vector calculus. But until the emergence of powerful graphing software in the 1990s, plots of vector fields were just as rarely produced by hand and only a few static sample images were found in textbooks. However, modern courses in vector calculus and differential equations very much rely on plots of vector fields, and thus the Polya vector field is expected to become more important in complex analysis.

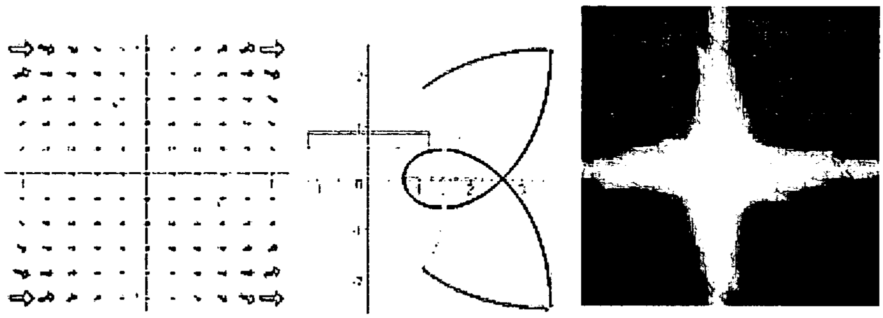

Figure 1 Three different views of the function $f(z)=\cos(z^2)$

Here we shall concentrate on two other ways of visualizing functions of a complex variable. The first and traditional approach investigates images of specific curves (and regions) under the mapping f. For example, the complex exponential maps any line segment of length 2π on the imaginary axis onto the unit circle. Similarly, it maps any rectangular region with vertices r, R, $R+2\pi i$, $r+2\pi i$, (with r,R >0 real) onto an annulus with radii r and R centered at 0.The study of such special cases forms an integral part of the introductory sections of any textbook, and is considered essential for getting an intuitive understanding of the elementary functions.

Today's computing tools make it very easy to automate such tasks. We distinguish those which require algebraic/symbolic definitions of the curves/regions to be mapped (in computer algebra systems, CAS, such as MAPLE, see e.g. [9] for sample implementations) entered from the keyboard, and JAVA applets [10] and similar programs, e.g. [2,11,17] that allow one to specify the input region by "drawing" it with the mouse. The main advantage of CAS implementations is that the learner must face e.g. explicit parameterizations of the curve/region and face the compositions of functions needed to obtain the image. See figure 1.b for the example $f(z)=\cos(z^2)$. The main disadvantage is that it takes quite some effort and time to modify the input. This serves to suggest a more thoughtful, planning approach as opposed to simply playing around. Nonetheless, students in our class spent significant time exploring the mappings using our CAS implementation [9], spending much time trying to understand where intersection points occur, reversal of orientation, "foldovers" that cause the boundary of the image not being a subset of the image of the boundary of the original region etc. A key effort that made this implementation so successful was the attention to detail, like the coloring of opposing edges by red/magenta and green/blue together with the associated internal grids in pastel tones (pink and cyan). This proved to be essential to help track features in more complicated mappings, and to forcefully convey the image of conformality (here preservation of orthogonal angles). On the other hand, JAVA implementations such as [10] and free-standing programs such as [2,11,17] provide much more immediacy and foster a much more playful attitude. They are great tools to get a class excited, but they generally require much more guidance by the instructor to again focus on relevant mathematical questions.

We see their main uses in getting a quick overall feeling, to quickly look up specific cases, and, most importantly, to discover interesting special cases which then warrant further investigation and theoretical follow-up study. A major benefit is that students generally are much more excited to study a question/problem/special case that they asked/posed/discovered themselves rather than a question from the textbook. For a minimalist implementation of the squaring map $f(z)=z^2$, and its (multi-valued inverse) see [10]. For professional large programs see the commercial software [11] and freeware [2, 17].

3. Graphing functions via colormaps

An exciting alternative approach, which yields immediate global images, uses colour maps. The basic idea is to assign to each point of the range a color, and then color each point z of the domain by the color of its image f(z), see [12] for a detailed and more general description. While colormaps have been used for quite some time, e.g. for visualizing curvature on surfaces [9] via the color functions in MAPLE, the first use of color maps for complex functions on the WWW is attributed to Farris [6]. A common color map uses the polar form of complex numbers, mapping the magnitude to the interval [0,1] for brightness (e.g. zero is white and infinity is black) and mapping the argument to some "rainbow" (colorwheel). While the standard graphics on computers uses the RGB color model, we prefer a variation of the LAB model (as found in Photoshop) more suitable as the complex plane has natural 2-symmetries (e.g. conjugation), and thus the primary colors should be assumed by 1, i, -1, and –i. In order to achieve bright colors (but fast calculations) one needs maps from the complex plane (Riemann sphere) into the color cube that "hug" the faces and stay away from the diagonal, compare figure 2.a for our construction.

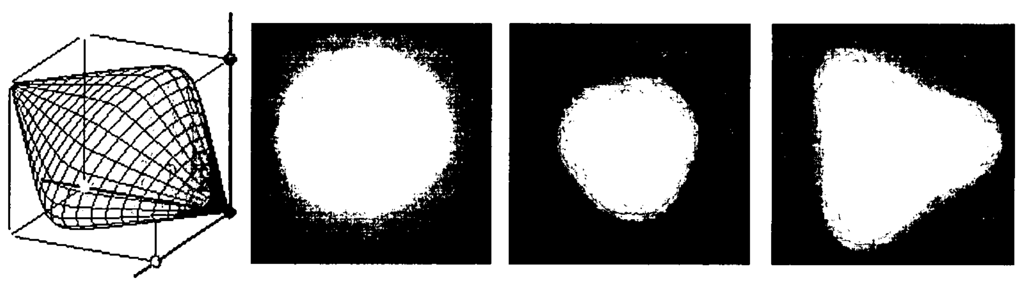

Figure 2 Mapping the complex plane into the colour cube, views of $f(z)=z$, $f(z)=z^3$, and $f(z)=z^3-1$

Figure 2.b. shows the resulting image of the identity (which defines the colormap). Figures 2.c illustrates the image of the map $z\rightarrow z^3$ with beautifully shows that its degree is 3 (traverse the rainbow three times when encircling the origin one). The narrower colored band and the wider whitish region clearly correspond to the growth rate of the real function $|z|\rightarrow |z|^3$. The last image figure 2.d shows $z\rightarrow z^3-1$ with its three simple zeros. Implementations of such colormaps may be found in freestanding packages such as [2,11], applets such as [10, 17], and CAS worksheets such as [9].

4. Zooming in on essential singularities

One of the more exciting uses of such color maps is to zoom in on essential singularities, such as those of $z \to \exp(1/z)$, $z \to \cos(1/z)$, and $z \to \sin(1/z)$. Each of these has an essential singularity at zero. By Picard's theorem, each of these assumes every complex value (with at most one exception) infinitely often in every neighbourhood of zero. Upon first encounter most students, just as the author, react with disbelief: How could that be? What does that look like? We implemented a simple JAVA applet that allows one to successfully zoom in into these singularities, using colormaps as above, compare figure 3 for sample images.

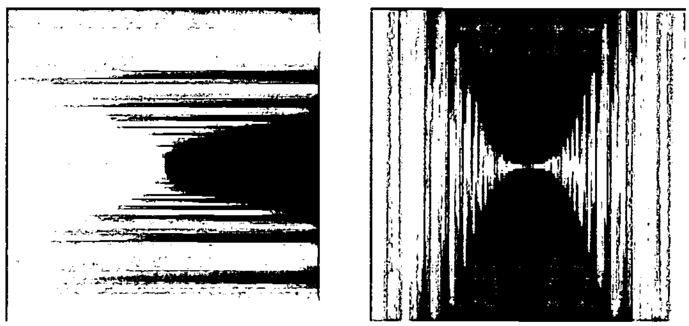

Figure 3 Zooming in on the essential singularities of cos (1/z) and exp(1/z)

The instantaneous reaction in class was one of intrigue – but this very quickly gave way to many mathematical questions being asked. They start with the difference in the colors at infinity of the three maps (different limits), one white and one black blob for the exponential as opposed to two black blobs for the cosine (hint: the Taylor series expansion of the cosine is an even series!). Everybody asked about the apparent parallel lines of constant color – but this was a standard homework exercise (like proving that the level curves of the real and imaginary parts, or of magnitude and argument are circles or lines etc.) However, when asked to prove their conjecture students react much more positively if this is their own observation as opposed to some assignment like "#34 from the exercises in the textbook". A little deeper are questions like the conjectured geometric progression of lines of same color, and the "order of contact" of the two black (or the black and white) blobs in the pictures (basically of the level curves of the magnitude): Here alternating "zooming horizontally only" with "zooming in horizontal and vertical directions at the same rates" quickly suggested that the order of contact is two, i.e. like two circles touching each other.

A technical comment: Our applet [10] uses JAVA 2D-Graphics classes, and runs well in NETSCAPE 6, but not under NETSCAPE 4.7. Incidentally, it was a first try to use these classes and it performs reasonably well, delivering a new 256 x 256 image within a second on a typical PC, recomputing all function values and converting them to our LAB-like colormap. However, this speed is still insufficient to allow for slider-controlled continuous zooming or for more computationally intensive images such as in figure 4 (which were produced in MATLAB).

5. Convergence

One of the most beautiful sequences of images that we obtained in our initial explorations portrays the convergence of a Laurent series of a rational function. Recall, Laurent series expansions are similar to Taylor series expansions, but they also allow for negative powers. Depending on the choice of the expansion, one finds that the series converges on open disks, open annuli, or the complement of a disk. The behaviour on the boundaries generally can be very complicated.

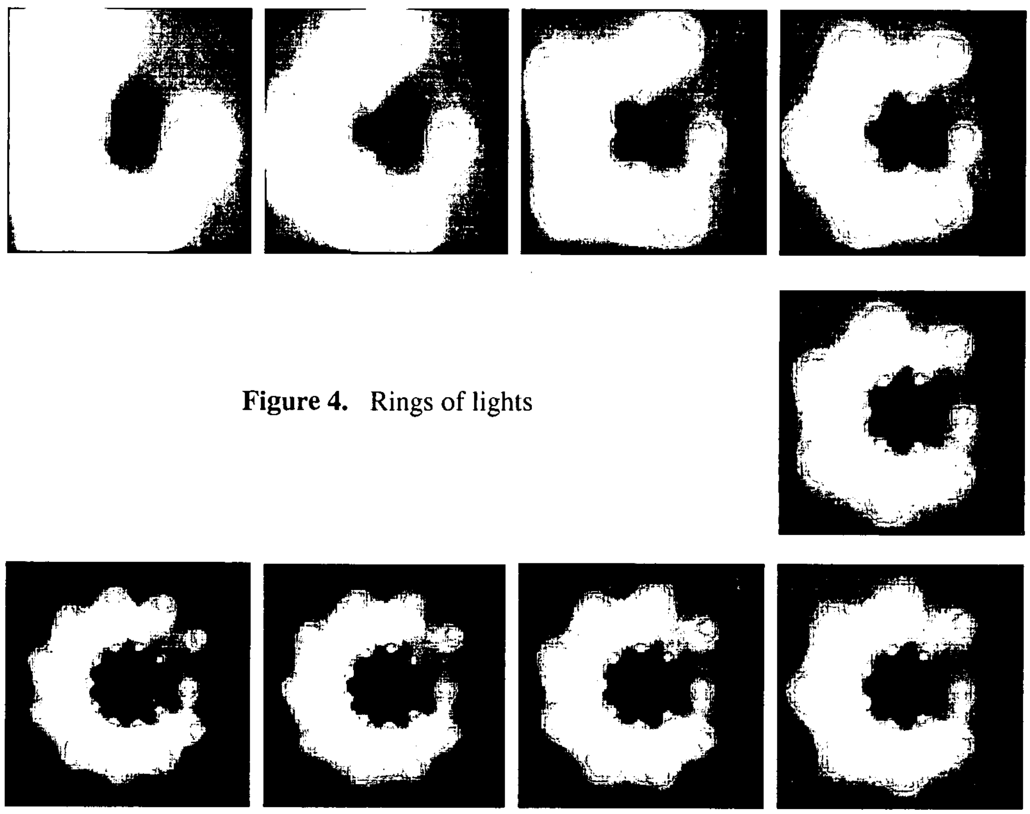

Figure 4. Rings of lights

The series depicted in figure 4 shows truncations (at the same negative and positive orders) of that Laurent series expansion of $1/((z-1)(z-2))$ that converges in the annulus $1 < |z| < 2$. More precisely, the images show the error terms, using different rescalings of the magnitude-to-color-map. (Without such rescalings, the annulus quickly approaches a white color.) What we expected was that as the order of the truncation increases, the regions inside and outside the annulus would become black, with something happening on the annulus itself. But we certainly did not expect the rings of lights. Clearly these are all simple zeros (degree one, single rainbows). Of course such observation, discovery has to be made into a conjecture, which then warrants further analysis as homework/project, trying to prove the statement, and more importantly, finding general hypotheses on the function under which the theorem holds true. Yet in class, we already took pride in the formulation of the conjecture: "As the order of the Laurent approximation increases, the approximating function interpolates the original functions at asymptotically uniformly spaced points on the circles of convergence." (Clearly the zero at z=3/2 and the poles at the singularities z=1 and z=2 of the original function are exceptions.)

Technical comment: These images were produced in MATLAB using symbolically generated code from MAPLE and taking advantage of the superior speed for numerics and graphical rendering. (The new releases of the CAS such as MAPLE 7, should now be competitive, too.)

6. Classroom experience and conclusion

Student reactions in our class to, even the limited use of interactive visualization has been mostly enthusiastic. From the instructor's point of view it appeared that, as a result, the class made more and faster progress on the traditional analytic aspects of the course, too. (But we do not have hard data, only anecdotal evidence comparing our class to that of prior semesters.) Our preference is that students (usually in pairs, or one volunteer in the front) directly, interactively explore – but there are plenty of occasions (such as the ring of lights), where even a single still image (transparency) gives rise to lively mathematical questioning.

While the instructor and author spent much time developing these materials while exploring the subject matter himself, we now expect that with the proliferation of dedicated software, slideshows, CAS worksheets and JAVA applets, one can achieve similar results with only minimal time investment. Moreover, we note that in a typical class only very few minutes, usually at the beginning (and sometimes also near the end) were devoted to graphical explorations. Typically such few minutes already raised so many mathematical questions that could barely be all addressed in the available class-time. The spirit was one of "doing mathematics", in the truest sense of the word, from exploration, observation, formulating conjectures all the way to hard proof.

In the future we expect that such approach becomes ever more common place. At the same time we expect a further increase in the rate at which new dedicated software is developed, and we foresee a healthy competition between different implementations of related topics that pay special attention to various minute details. The color schemes matter as much as the placement of the buttons, and the choice when to specify input data via the keyboard (e.g. formula for f(z), vertices for a rectangular region), sliders (parameter values), or directly with the mouse (drawing regions or "dropping" zeros and poles).

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ELECTROCATALYTIC REACTIONS: AN INTERESTING PROBLEM OF NUMERICAL CALCULUS

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ABSTRACT

Electro – chemistry provides interesting problems for applied mathematicians. An example of this is the adsorption of Carbon Dioxide over Platinum surfaces (Mendez, E., Martins, M.E. & Zinola, CF, 1999). In fact, this problem was studied in other papers, were several methods of linear algebra, ordinary Differential Equations, Statistics and Numerical Calculus were used (Martínez Luaces, V., Zinola, F. & Méndez, E., 2001).

Now, in this paper, we try to show part of the richness of the problem (Martínez Luaces, V., 2001, b), in order to use it in Numerical Calculus courses for Chemical Engineering and other chemical careers.

Important concepts as numerical derivatives, and typical processes as fitting curves and determining coefficients numerically (Mathsoft Incorporated, 1999), can be illustrated in the context of this scientific and technological problem, closely related with other disciplines of these careers.

This kind of problems provides a good opportunity for interdisciplinary work, but not only in their solution. In fact, they can be taught in the same way, by a group of teachers of several disciplines. Also, it is possible to propose project – works to the students, taking parts of the problem or making small changes in order to motivate them with a real – life mathematical and chemical challenge.

We discuss results of these and other situations, experimented in the chemistry Faculty at Montevideo, Uruguay by the Mathematical Education research group ((Martínez Luaces, V., 2001, b) and (Martínez Luaces, V., 1998, a)). Taking into account all these experiences, we propose some conclusions and recommendations for this kind of mathematical service courses for chemical students.

KEYWORDS: Differential Equations, Qualitative behavior, Runge - Kutta, Electro catalytic Reactions.

1. Introduction

Electrochemistry is an interesting branch of Chemistry, which provides motivating mathematical problems of Differential Equations, Linear Algebra, Statistics and Numerical Calculus.

In this paper we will try to show part of the potential richness of one of this problems. More precisely, we will study the adsorption of Carbon Dioxide over surfaces of Platinum.

This is a very important problem to solve, due to chemical and economical reasons. In fact, adsorption, absorption and electro-deposition reduce active surface of Platinum electrodes, and this fact produces a consequently waste of money (Zinola, F., Méndez, E. & Martínez Luaces, V.,1997)

Form the Mathematical Education view point, these problems provide for Mathematics teachers a wide possibility of interaction with other subjects, in order to present real problems to their students. This kind of problems led to a better motivation, for students of Chemical Engineering, Food Technology Engineering, and other chemical careers, as it will be shown later.

2. The chemical problem and the numerical approach.

Electrodes for chemical laboratories and/or chemical industries are made of Platinum Iridium, etc.. All these metals are very expensive, in fact, they are even more expensive than gold. For this reason, it is very important to use these electrodes in the most efficient way.

Electro-chemical and electro-catalytical reactions reduce active surface of these electrodes. For example, the adsorption of Carbon Dioxide over Platinum surfaces is one of these problems studied by researchers (Méndez, E., Martins M.E., Zinola, CF., 1999)

If all reactions are electro-chemical, this problem led us to a system of Ordinary Differential Equations (O.D.E.), as follows:

$$\begin{cases} \dfrac{dA_1}{dt} = -(k+r).A_1 \\ \dfrac{dA_2}{dt} = k.A_1 - s.A_2 + u.A_3 \\ \dfrac{dA_3}{dt} = r.A_1 + s.A_2 - u.A_3 \end{cases}$$

In this system, the variable t is time, A_1, A_2, A_3 are surface-concentrations of carbon dioxide adsorbates and k, r, s, u are kinetic constants. It is important to remark that for physical and chemical reasons, all this variables and constants are always positive numbers (Zinola, F., Méndez, E. & Martínez Luaces, V.,1997).

This kind of ODE System always has a null eigenvalue and this is a consequence of the stechiometry of these reactions. This fact makes impossible to explain two inflection points (Martínez Luaces, V., 2001, a) in the experimental curves of surface-concentration vs. time (Martínez Luaces, V. & Guineo Cobs, G., to appear). So, this is not an electro-chemical process and it is necessary to postulate an electro-catalytical mechanism in order to explain it (Martínez Luaces, V., 2001, a). The O.D.E. system is the same in both cases, but in the electro-catalytical one k, r, s and u are exponential functions depending of variable time.

Then, from this problem we obtain an ODE system with constant coefficients in the electro-chemical case and another one with variable coefficients in the electro-catalytical case.

From the Mathematical Education view point, it is possible to propose a project-work in Numerical Calculus courses, based on this problem.

3. The project - work for Numerical Calculus courses

The O.D.E. system:

$$\begin{cases} \dfrac{dA_1}{dt} = -(k+r).A_1 \\ \dfrac{dA_2}{dt} = k.A_1 - s.A_2 + u.A_3 \\ \dfrac{dA_3}{dt} = r.A_1 + s.A_2 - u.A_3 \end{cases}$$

can be proposed to students of chemical careers (after modeling the chemical problem) and let them try to solve it with different kinetic constants values, using for example Runge-Kutta methods (Dahlquist, G., Bjorck, A. & Anderson, N., 1974). If coefficients remain constant (electro-chemical case), students can solve the ODE system with different non-negative values for k, r, s and u. In this case, they will realize that they cannot obtain the necessary number of inflection points. In fact, experimental curves show at least four inflection points(see graphic 1)

In the other case (the electro-catalytical one), they work with variable coefficients. More precisely, they are exponential functions like: $k(t) = k_1.e^{k_2.t}$, $r(t) = r_1.e^{r_2.t}$, $s(t) = s_1.e^{s_2.t}$ and $u(t) = u_1.e^{u_2.t}$. That means that they have now, eight values to change in the O.D.E. system.

A typical student of Chemistry, would easily realize that k_2, r_2, s_2 and u_2 must be small numbers. If he put not so small values in the exponents, curves will go up (or down) very fast, in contradiction with experimental results. For the same reason, it is not possible to assign big numbers for k_1 and the other coefficients.

Is important to remark that k_1, r_1, s_1, and u_1, are positive numbers (for chemical reasons the kinetic functions $k(t)$, $r(t)$, $s(t)$ and $u(t)$ cannot be negative, for all values of variable t), but the exponent coefficients k_2, r_2, s_2 and u_2, can be positive, negative or zero.

An exponent coefficient zero, reduces strongly the possibility of obtaining inflection points, so it is not recommendable. So, next step will be essay with positive and/or negative small numbers for the exponent coefficients.

If all the exponent coefficients are positive, then, the corresponding curve of A_1 surface concentration shows a negative concavity for small values of variable t. This fact is in contradiction with experimental curves.

Graphic 2 is an example of curves that can be obtained with positive exponent coefficients.

Let's consider again the A_1 surface-concentration curve. This curve depends only on the first differential equation of the O.D.E. system, as can be easily observed. Then, only $k(t)$ and $r(t)$ determine its behavior. In particular, if k_2 and r_2 are both negative numbers, there are not inflection points in this curve (see for example, graphic 3) and if both are positive, there is an unique inflection point, as in graphic 2. Both cases do not correspond with reality, so students must try with one positive exponential coefficient and the other one must be negative.

Graphic 4 shows the numerical solutions with $k_2>0$ and $r_2<0$, and graphic 5 does the same for $k_2<0$ and $r_2>0$ (in both cases s_2 and u_2 are positive). In a first approximation, both cases (graphic 4 and graphic 5) can be acceptable.

Next step will be study the behavior of curves corresponding to A_2 and A_3 surface-concentrations. The most reasonable option would be solving numerically the O.D.E. system with all the sign possibilities, that is, four cases with $k_2>0$ and $r_2<0$ and other four cases with $k_2<0$ and $r_2>0$. These cases are shown in graphics 6 to 11 (remember that graphics 4 and 5 correspond to a pair of these combinations).

With teachers´ help, students observe that from all these graphics, there is only one case, really accurate with experimental data. This combination has $k_2<0$, $r_2>0$, $s_2<0$ and $u_2>0$, so the remaining work consists only in choosing the best numerical values for these constants and also, the best numerical values for the other coefficients, that is: k_1, r_1, s_1 and u_1 (all of them must be positive, as was mentioned before).

Graphic 12 shows the best results obtained trying with different values, taking into account the signs recommended for the exponential constants (k_2, r_2, s_2 and u_2) and for the multiplicative coefficients (k_1, r_1, s_1 and u_1). Then, the best kinetic functions will be:

$$k(t) = 0.091 \cdot e^{-0.27 \cdot t}$$
$$r(t) = 0.0031 \cdot e^{0.08 \cdot t}$$
$$s(t) = 0.31 \cdot e^{-0.07 \cdot t}$$
$$u(t) = 0.7 \cdot e^{0.007 \cdot t}$$

These final values of exponential and multiplicative constants provide a satisfactory numerical solution for the chemical problem, but for this paper, the most important thing is the process, not the results. In fact, this process can be done by students or by groups of students, with help of Numerical Calculus teachers.

This kind of oriented project-work was put into practice in several courses between 1996 and 1999, with very successful results. In fact, it is important to note that this problem (even in the electro-catalytical case) can be solved analytically (Martínez Luaces, V., 2001, a) and in several cases, coefficients and kinetic constants can be obtained using numerical derivatives, combined with statistical methods (Martínez Luaces, V., Zinola, F. & Méndez, E., 2001). For these reasons , both problems (electro-chemical and electro-catalytical) were used to propose small project-works to the students in second year courses, that is: Numerical Calculus, Statistics and Differential Equations. As we will see later, students react positively to this style of teaching based on real scientific and technological problems related with other disciplines of their careers.

4. Results.

In a previous paper (Martínez Luaces, V. & Casella, S., 1996) an expert group of teachers, researchers and university authorities were consulted about Mathematics service courses. Almost all of them mentioned the importance of real-life problems in order to motivate students of non-mathematical careers.

In concordance with experts opinion, students of chemical careers reacted positively to mathematical problems related with other subjects, as it was shown in another paper (Martínez Luaces, V., 1998, a). In fact, their answers to several questions about applications, relations with other disciplines and real-life problems, presented an interesting connection with their answers about mathematical courses motivation.

Recently, some techniques of Multivariate Statistical Analysis were used in order to compare the results obtained by teachers of the Mathematical Department at Chemistry Faculty, in Montevideo (Gómez, A. & Martínez Luaces, V., to appear).

In this case, the instrument was a list of 25 questions about teachers, programs, assessment, etc. This questionnaire was prepared specially by experts in Education of two different faculties (Chemistry Faculty and Engineering Faculty, University of the Republic of Uruguay), that worked together in this project.

The answers of the students remain anonymous and the information was processed automatically by a scanner without any participation of teachers.

The results of this study showed again that students have a good reaction to an applied approach in Mathematics. Obviously, this kind of approach makes an important change in motivation and then in the attitude of students towards Mathematics. Also, it is possible to identify a small group of very important variables: knowledge of the teacher, a good learning environment, order and management of the class and the already mentioned motivation and applied approach. These variables are not independent but their correlation is not easy to understand. For example, there are teachers with good knowledge, who are also able to give an applied approach of what they teach, but unbelievably they are not capable of motivating the students. As a consequence, the final and global result of their evaluation is not good enough. It is possible to show a large group of examples and counterexamples useful to understand the correlation between the variables listed above.

It is important to remark that in the questionnaire, two of the questions proposed asked specifically about applications to other disciplines and connections with real-life problems. A Cluster Analysis of these two questions showed a group of five teachers of the department, separate from the others, as the better ones. Four of this five teachers are professors of second year courses, that is, courses where this kind of problems were presented, discussed and used as a source of project-work for the students (Gómez, A. & Martínez Luaces, V., to appear).

5. Conclusions

Real-life problems and situations related with other subjects, are very useful for Mathematics teachers in order to motivate their students in service courses.

In case of Mathematics courses for Chemical Engineering, Food Technology Engineering, and other chemical careers, these problems can be obtained form Physical-chemistry and Electro-chemistry. These two disciplines are already known for their richness in O.D.E. and P.D.E. problems, but also they provide interesting exercises and problems for Linear Algebra, Statistics and Numerical Calculus courses, as it was showed in this paper.

Searching, developing and solving these problems need an interdisciplinary group, that in the best situation can be integrated by mathematicians and chemists. Implementation of these problems in the classroom also needs the collaboration of teachers of several disciplines.

This interdisciplinary group can work in the design of activities to be carried out in the classroom, but it would be better if this collaborative work extends to teaching and assessment of the project-works proposed to the students.

Several experiences in this direction were developed in Uruguay with excellent results (see for example (Martínez Luaces, V.,1998, b) and (Martínez Luaces, V., 2001, b)).

As it was said in an important paper of ICMI (ICMI, 1986), this collaborative teaching represents "the ideal situation" for mathematical service courses.

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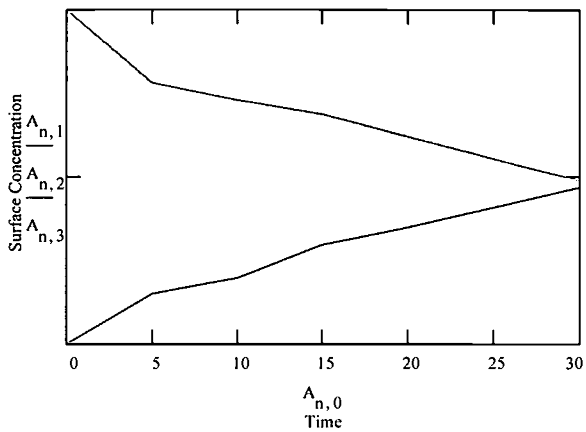
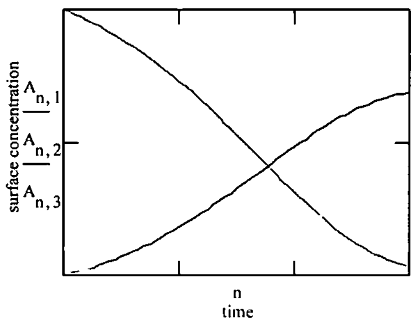
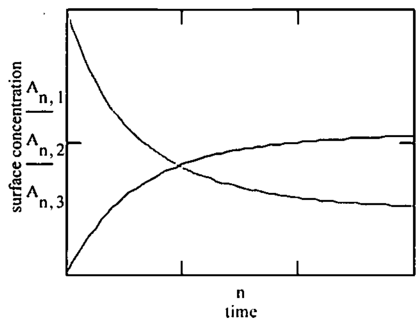
Graphic1:



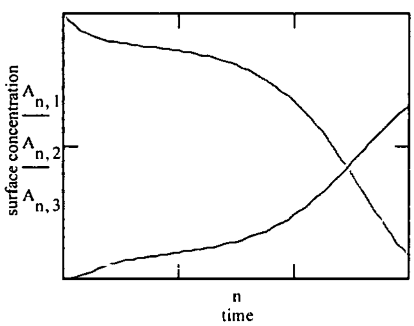
Graphic 2: $k_2>0$, $r_2>0$, $s_2>0$, $u_2>0$



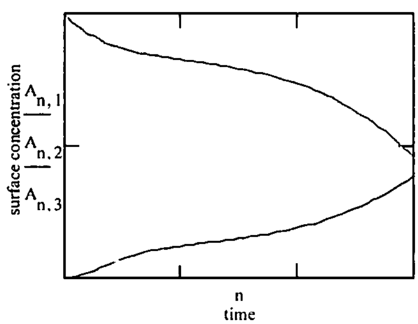
Graphic 3: $k_2<0$, $r_2<0$, $s_2>0$, $u_2>0$



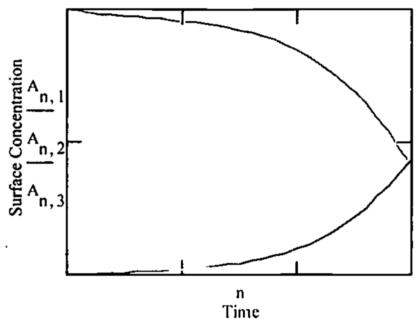
Graphic 4: $k_2>0$, $r_2<0$, $s_2>0$, $u_2>0$



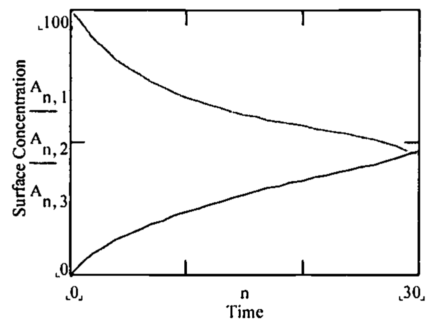
Graphic 5: $k_2<0$, $r_2>0$, $s_2>0$, $u_2>0$



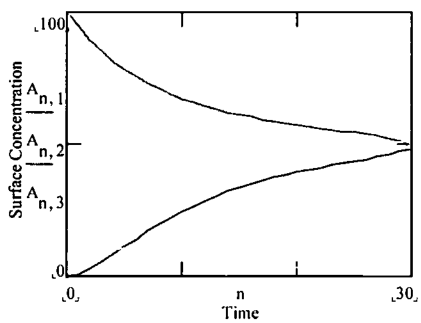
Graphic 6: $k_2>0$, $r_2<0$, $s_2<0$, $u_2<0$



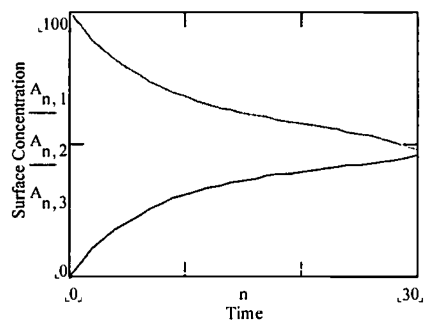
Graphic 7: $k_2<0$, $r_2>0$, $s_2<0$, $u_2<0$



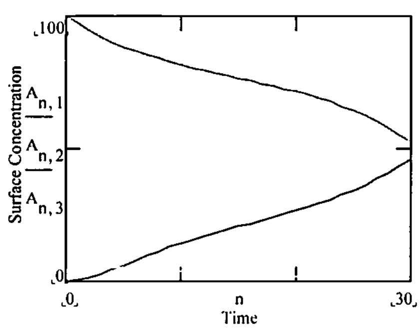
Graphic 8: $k_2>0$, $r_2<0$, $s_2>0$, $u_2<0$



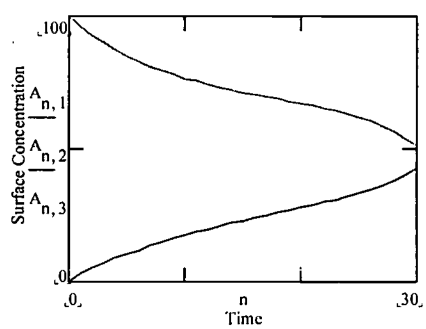
Graphic 9: $k_2<0$, $r_2>0$, $s_2>0$, $u_2<0$



Graphic 10: $k_2>0$, $r_2<0$, $u_2<0$, $s_2>0$

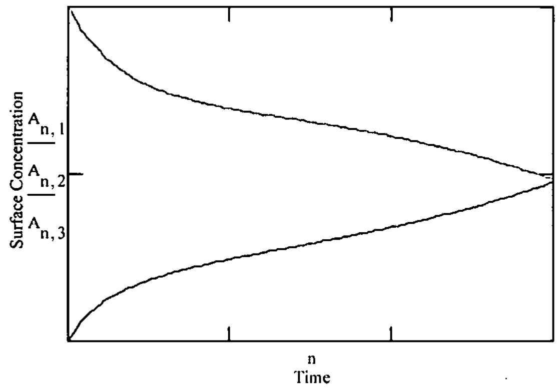


Graphic 11: $k_2<0$, $r_2>0$, $u_2<0$, $s_2>0$



Graphic 12 $k(t) := 0.091 \cdot e^{-0.27 \cdot t}$ $r(t) := 0.0031 \cdot e^{0.08 \cdot t}$

$s(t) := 0.31 \cdot e^{-0.07 \cdot t}$ $u(t) := 0.7 \cdot e^{0.007 \cdot t}$



NUMERICAL CALCULUS AND ANALYTICAL CHEMISTRY:
An example of interdisciplinary teaching

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ABSTRACT

Analytical Chemistry is an almost unexplored source of real – life problems for Numerical Calculus courses in chemical careers (Martínez Luaces, V., 2001).

In this paper, we discuss one of these problems: the pH determination of a weak monoprotic acid aqueous solution (Labandera, F. & Martínez Luaces, V., 1994).

From the mathematical viewpoint, this problem led us to solve very difficult algebraic equations. In several cases is possible to obtain an algebraic exact solution, but in other situations the algebraic approach is not useful. So, if we wont to generalise our methods, we need a numeric approximate solution.

We analyse several algorithms form well known methods as Newton – Raphson, Regula Falsi, Bisection and others (Dahlquist, G., Bjorck, A. & Anderson, N., 1974). We also study a couple of methods, developed specially for this kind of problems.

The variety of situations, and the mathematical and chemical richness of them, suggests proposing an interdisciplinary work in research and teaching. This can be carried out by a group of both Analytical Chemistry and Numerical Calculus teachers. In the same way possible to use these problems for students project – work, with interesting advantages

We comment here, some important results, strongly related with this style of teaching ((Martínez Luaces, V. 1998) and (Gómez, A. & Martínez Luaces, V., 2001)). Finally, we suggest some recommendations for these mathematical service courses in chemical careers.

Keywords: Applied Mathematics, Numerical Calculus, Analytical Chemistry, Interdisciplinary teaching.

1. Introduction

The pH determination of an aqueous solution is a very important problem in Analytical Chemistry.

From the mathematical point of view, this chemical problem is modeled using algebraic equations. As an example we have:

$$\frac{V_b.C_b}{V_a+V_b}+\left[H^+\right]=\frac{K_w}{\left[H^+\right]}+\frac{C_a.V_a}{(V_a+V_b).\left(1+\frac{\left[H^+\right]}{K_a}\right)} \tag{1}$$

In this equation C_a and C_b are the concentrations of acid and base solutions and V_a, V_b are their volumes.The symbols K_w and K_a represent the equilibrium constants for water and acid, respectively and finally $[H^+]$ is the concentration of the hydrogen ion. All these variables are positive numbers and $[H^+]$ is the unique unknown, and then, pH is obtained as - log ($[H^+]$).

Equation **(1)** corresponds to the pH determination of a weak monoprotic acid solution (Martínez Luaces, V., 2001). Obviously, this formula can be easily converted in a polynomial equation of third order.

If we consider now another situation, like the dilution of a phosphoric salt (for example Na_2HPO_4), then, the resulting problem is much more difficult. In fact, in this case (Martínez Luaces, V. & Martínez, F.,submitted), we have:

$$\left[H^+\right]+2.C_s=\frac{K_w}{\left[H^+\right]}+\frac{C_s.\left(3+\frac{2.\left[H^+\right]}{K_3}+\frac{\left[H^+\right]}{K_2.K_3}\right)}{1+\frac{\left[H^+\right]}{K_3}+\frac{\left[H^+\right]^2}{K_2.K_3}+\frac{\left[H^+\right]^3}{K_1.K_2.K_3}} \tag{2}$$

As in the other case, C_s is the concentration of the salt solution, K_w, K_1, K_2, K_3 are equilibrium constants and $[H^+]$ is the concentration of the hydrogen ion. In this equation, as in **(1)**, all these variables are positive numbers and $[H^+]$ is the unknown. Finally, the pH value is obtained using the equation pH = - log ($[H^+]$)

As in the other case, equation **(2)** can be converted in a polynomial one of fifth order.

It is well known, as a result of Galois theory (Grillet, P., 1999), that there are no formulas based on Nth-roots, useful to solve general polynomial equations with a degree greater or equal than five.

Then, we need a numerical approach to solve this chemical and mathematical problem.

In this paper, we will analyze several numerical methods and all of them will be studied from the Mathematical Education view point. It is important to remark that this methods and their applications to the chemical problem already mentioned, provide a source of interdisciplinary work in research and teaching. Moreover, the variety of situations and the mathematical and chemical richness of them, suggest to use these examples to propose project-work for the students, with interesting possibilities.

Finally, we will comment some important results obtained in the last years in our department of Mathematics. Taking into account these results, we will propose several conclusions and recommendations for service courses in chemical careers.

2. The numerical approach.

In this section, we will analyze the applicability of several numerical methods: Functional Iteration, Newton-Raphson, Bisection, Secant and Regula Falsi (Martínez Luaces, V. 1998). All these methods are very common and well known by students.

As a complement, we will mention a couple of methods developed specially for these problems.

- Functional Iteration:

A first easy option is to put equations 1 and/or 2 in the form:

$$[H^+] = g([H^+]) \tag{3}$$

Then, it is possible to find the solution using a fixed point iteration method (Martínez Luaces, V. 1998). In this case, if $V_a = 10$ ml, $V_b = 3$ ml, $C_a = 0.1$ M, $C_b = 0.1$ M and $K_a = 10^{-3}$ (K_w is a constant, and its value is 10^{-14}), then, the correct pH will be 2.69. Unfortunately, if we start the iterative method with a pH of 3 (that is, the best entire approximation), next iterant will be 1.81, and the next one does not exist! (pH is a positive number and in this third iteration we obtain the log of a negative number, and that situation has no chemical sense).

It is possible to show that the same situation takes place for almost all the reasonable values of pH in this case (Martínez Luaces, V. & Martínez, F., 2002), so we cannot recommend this method. It is impossible to use it for this problem.

- Newton-Raphson.

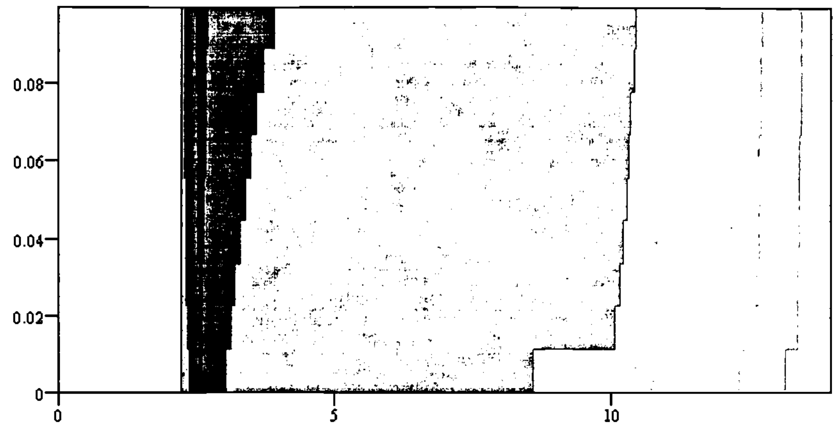
The speed of convergence for this method depends strongly of the initial approximation and the precision required.

For chemical reasons (Kolthoff, I. & Sandell, E., 1943), pH is given with only two significative numbers, so, the maximum precision needed will be 0.01.

The speed of convergence can be measured considering "n", that is the number of iterations to reach the pH value with a given precision "ε".

Taking into account all this facts, we can plot the variable "n" against "pH_0" (the initial approximation) and "ε". We decided to make the plot in $\square^2$ putting "pH_0" in the "x" axis and "ε" in the "y" one, and the value of "n" can be visualized with different tones of blue. In this plot we put a dark blue color (almost black) for a fast convergence point (that is a small "n" value), a blue color for a moderately fast one, and sky blue for slow convergence points (which correspond to big "n" values). Finally, the white color is for very slow convergence, or for points where the iterative method does not converge at all.

All this facts can be visualized in the following figure:



Newton Raphson

A first observation is that if we increase the "ε" value, then zones with a deep blue color predominate. This is really obvious taking into account that if we accept a big error for this method, then we need less iterations.

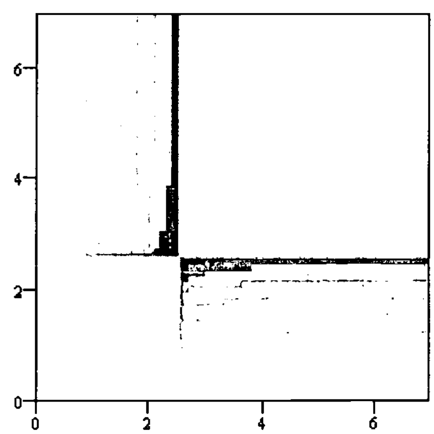
The second, and not so obvious observation, is that for "pH_0" values greater than 2.69 (the correct pH value, in this case), the iterative method converges, but for several initial values (pH_0 less than 2.30) iteration does not converge or is very slow for practical purposes. So, students must be very careful with the initial value, if they decide to use this method.

- Methods with two initial approximations.

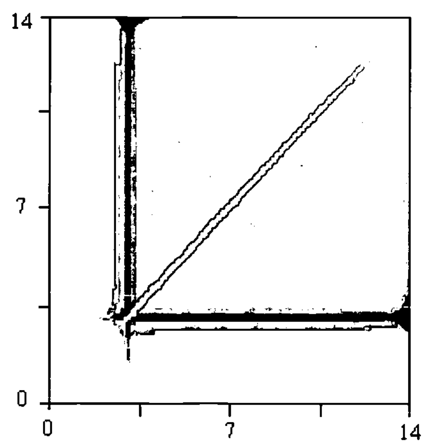
These methods (Bisection, Secant and Regula Falsi) need two initial values "pH_1" and "pH_2" in order to start the iterative process (Martínez Luaces, V. 1998).

As a consequence of this fact, we decided to put "pH_1" in the "x" axis and "pH_2" in the "y" axis. As in the other case, we represent the "n" value with different tones of gray (for Bisection Method), bright blue (for Secant Method) and green (for Regula Falsi), to show the speed of convergence for each point $(pH_1, pH_2) \in (0,14] X (0,14]$

The figures are (for an "ε" of 0.01):



Bisection



Secant

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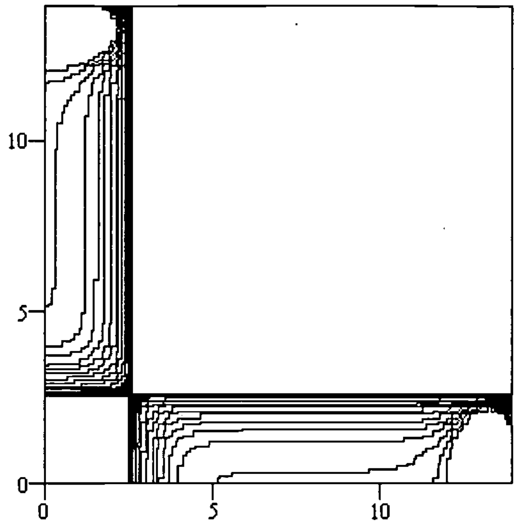
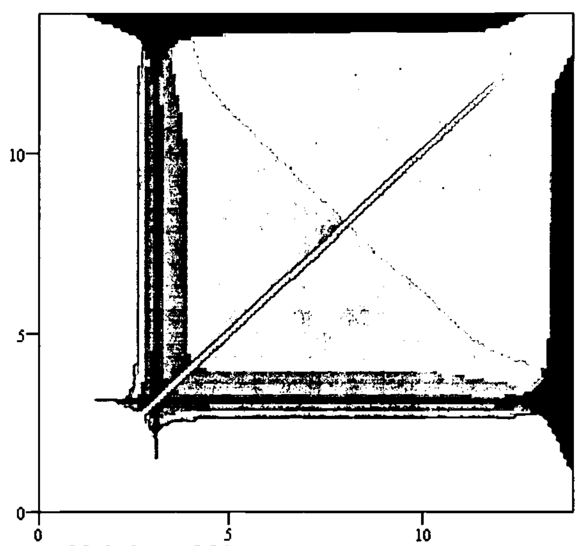


Regula Falsi

The figure of Secant Method is the most interesting. The reason is that Bisection and Regula Falsi methods, need initial values with different signs in their functional values. As a consequence of this fact, important parts of the figures (for these last methods) remain uncolored.

For this reason, we decided to present only the figure corresponding to Secant Method (in black and white), for another "ε" value (this "ε" will be 0.04):



Secant Method: error 0.04

As in the other cases, dark zones become greater when "ε" is increased.

- Two special methods for this problem.

In a previous paper (Martínez Luaces, V., 2001), a couple of methods were presented in order to determine the pH value for certain aqueous solutions.

One of them consists in an approximate equation based only in several chemical considerations (Day Jr., R. & Underwood, A., 1980). With this formula, very poor results are obtained for certain weak acids, so it is not the best option.

For this reason, another method was developed specially for this problem (Martínez Luaces, V., 2001). This last one uses the chemical idea of electro - neutrality (Day Jr., R. & Underwood, A., 1980) in aqueous solutions, and the corresponding algorithm seems like a modification of Bisection Method. It is a very particular iterative method, useful for an analytical chemist, but not very interesting for mathematicians. So, in this paper, this algorithm was used only to confirm the results of the other methods.

3. The Mathematical Education viewpoint:

Typical courses of Numerical Calculus propose to the students pure mathematical exercises, which are not the best way to motivate the group.

In chemical careers, the situation is even more difficult for teachers (in order to motivate students), at least if we compare with Engineering, Informatics, etc. In fact, it is not easy to find real problems, related with other subjects that can be useful for Numerical Calculus courses.

Determination of pH in solutions of weak acids, or aqueous solutions of salts, are exactly what we need for this purpose. As we have seen before, they are real-life problems, with important connections with other subjects (as Analytical Chemistry), and they represent an important source of interesting Numerical Calculus problems.

As can be easily observed, there is no optimal method for this kind of problems. Besides this, results showed a very strong dependence with the initial approximations and with the precision required. Then, students realize that not always real-life problems can be solved in a routinary form. Moreover, in several cases, it is necessary to find a more creative solution.

These problems, among others, were studied and developed by interdisciplinary groups, integrated with both Analytical Chemistry and Mathematics teachers.

In this first stage, three courses (Numerical Calculus, Statistics and Differential Equations), were based on real problems, strongly related with other disciplines. The other three courses offered by the Mathematics Department (Calculus I, Calculus II and Linear Algebra), remained traditional, at least in this first experiment. So, at present time, all second year courses of our department are in connection with other disciplines, while first year ones will be changed probably next year (this will be the second part of this experiment).

There are other important differences between first and second year courses. For example, in second year courses, real problems represent more than fifty percent of final examinations. Moreover, in several cases, these final examinations can be substituted by project-work, where students try to solve this kind of problems with help of computers or electronic calculators (and, of course, with orientation of teachers).

4. Results and conclusions.

In a previous paper, an expert group was consulted, and almost all the experts remarked the importance of teaching significative concepts and procedures in service courses ((Martínez Luaces, V. & Casella, S., 1996) and (Martínez Luaces, V., 1998)).

From a different point of view, Chemistry students showed an important preference for teachers who make the effort of presenting real-life problems, related with their own careers (Day Jr., R. & Underwood, A., 1980).

Finally, Cluster Analysis and other Multivariate Statistical methods showed a very similar situation (Gómez, A. & Martínez Luaces, V., to appear). More precisely, in our group of mathematical teachers (that is, twelve teachers of the Mathematics Department at the Chemistry Faculty in Montevideo), the Cluster Analysis of "Applications", separate a group of them as the better ones (this variable "Applications" consists of an $\square^2$ vector with the average results of two questions: one of them related with real-life problems and the other one about the connection with other disciplines). This group of five teachers was integrated almost exclusively with teachers of second year courses (Numerical Calculus, Statistics and Differential Equations) and almost all of them participated in interdisciplinary work with teachers and researchers of other departments and laboratories. Moreover, two teachers of this group are researchers in Applied Mathematics.

From these comments and results, it is obvious that real applications produce positive reactions in Chemistry students, in concordance with experts' opinion (Martínez Luaces, V., 1998).

In an important paper of ICMI (ICMI, 1986), this style of teaching, where Mathematics is applied to other disciplines, was considered as "the ideal situation" for mathematical service courses.

Other aspect, very important to be considered is assessment. The evaluative process must not be dissociated from the style of teaching. So, if we try to teach through problem-solving of real-life situations, in context with other subjects, assessment must be carried out in the same way. This purpose can be put into practice through project-work, where students (with orientation of an interdisciplinary team of teachers) try to solve real problems of their careers, in order to approve their mathematical courses.

It is important to remark that Analytical Chemistry is an excellent source for this kind of problems. In most cases, they remain almost unexplored in their mathematical richness. Also, this branch of Chemistry provides a good opportunity for interdisciplinary work in research and teaching.

Finally, as it was mentioned before, these problems represent an interesting challenge for applied mathematicians and Mathematical Education researchers.

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LET THE STUDENTS EXPLORE ALGEBRA WITH CAS, TI89

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ABSTRACT

These symbolic calculating tools, CAS, Computer Algebra System, will change the way to teach mathematics more than the start of using graphing calculators did.
With these tools we will get more time over to discuss the concepts of mathematics, more time to let the students explore algebra themselves and more time to increase the understanding of mathematics. The question is not if, but when and how, we should use CAS in our math classes. In my presentation I will show some examples how to work with TI89 and simultaneously reinforce the concepts of mathematics. All students in my class use TI89 and the age of the students are 17-19 years old.

Let the students explore algebra with CAS, TI89.

All my students use TI89 as a tool in my math classes. It is some kind of a project because the school provide it with some financial funding. Further more there is another class that also works with TI89. In all we have 21 classes at a secondary upper school that takes mathematics.

Every test we do is divided in two separate parts. One part is without any helping technical tool and the other parts is with help of graphing calculator or symbolic graphing calculator, CAS. With this system we still demand a minimum bas of knowledge in mathematics with paper and pen.

I made a little investigation among my students and asked them what positive and negative reactions they have concern TI89.

The positive reactions are following:

1. The tool is often used as a control function. "When I work with a task, I often use "Solve" or other tools at TI89 to check if my thoughts are right.", said several students
2. If you get an answer from the TI89 that you really don't recognize, you get a very strong motivation to search and find out how this answer could appear.
3. Just working with CAS, TI89, is very nice and quick
4. You can easier see the general picture of the thoughts.

The negative reactions was:

1. You could sometimes be a little lazy with CAS, TI89
2. Some of the answers don't match the answer back in the book and sometimes you don't understand the answer.
3. TI89 seems little expensive comparing with ordinary graphic calculators.

Although my students now have worked with this tool one and a half year there are still many things to explore in mathematics and how to use this CAS tool.

We have the same tests for students at the same grade in our school. This tests are divided in two parts. One part where no technological tools are allowed, just paper and pen, and one part where CAS tools a are used. Here are some examples from the latest test:

A. If a function f(x) has the derivation f ´ (x) = 3x^2-7x+2. Use this derivation to make a sketch of the function f(x). Your sketch should be mathematical motivated. All calculating where you make conclusions from should be written down.

B. A cars price was from the beginning 265000 SKR. After 7 year has the value decreased to 150000 SKR. Suppose the value has changed exponential during the time.

a) How big is the exponential value decreasing each year?

b) After how long time is the value of the car 95000 SKR?

Motivate all your mathematical steps in the solution.

C. In a village with 974 people the number increase with 3,7% each year. At the same time in another village, with 1090 people, the growth is 2,4 % per year. How many year will it take until the both villages have the same number of inhabitants? Give the answer exactly and approximately. For maximum points you have to write down and motivate all your steps in the solution.

D. Decide exactly the equation(s) of the tangent(s) to the function y = 3x-x^3/4 and which are parallel with the line y = 1-9x.

We also have some national tests, because we have a compulsory curriculum in our schools.

My pupils have succeed very well in this tests.

The new way to teach math with CAS , Computer Algebra System, is now growing all over the world.

We have to face a new approach to make young students understand math better.

One way is perhaps to give them the answer of example from real life and let them make the right questions or the right polynomials. It will be some kind of **"jeopardy"** in math. It could be like this: The answer is that the two zeros are 1 or 6 and the maximum value is 4. *See picture 1.*

What will the question be?

The question will be: What zeros and maximum value will $f(x) = - 1/16\,(x-1)(x-6)$

We can also randomly make a quadratic function equal zero and solve it in one order. You get two solutions (zeros). Look at the *picture 2.* Here from you ask your students if they can suggest one or two equations with these zeros. This is a way to make the students understand that every quadratic function can be at the form $A(x-x1)(x-x2)$

The students can with CAS simplify huge expression and get a very easy answer. But to understand this answer you also can use the CAS tool to explore and find out the steps to make a better understanding for this algebra. *See picture 3 and 4.*

Create and solve differential equations. *See pictures 5,6,7 and 8*

In a little village with 780 peoples a rumor is spreading with a velocity which is proportional to the number of peoples which do not know the rumor and the peoples that know. The constant of proportional is 0,05 %

Make a differential equation over the problem and then solve it exactly and numerical with a slope field. How many people know the rumor after 18 days?

An investigation of cubic functions can be a good test of CAS tool.

If you draw $f(x)= (x-1)(x-3)(x-6)=x^3-10x^2+27x-18$ you discover the three zeros. If you take the mean value of two of the zeros and draw a tangent at this value you will hit the third zero with the tangent. *See picture 9*

After this investigation I ask my students to show that this is true for all cubic functions with three real zeros. Here is the CAS tool TI89 really a good help. The proof of this is at the pictures 10 and 11.

Perhaps this is correct for every line who has three intersections of the cubic function? See picture 12 It seems correct but the proof of this will I turn over to the reader.

Here is another example how to use CAS tool.

If you draw a tangent to the square root function y= □x at any point, you will notice that the tangent will intersect with the x- axe . Investigate how far from origin this intersection will be if you select the tangent point at x = a. See picture 13

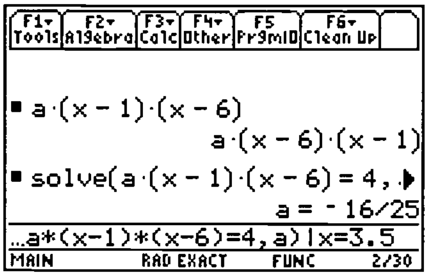
Give a proof that the intersection value is equal to –a . See picture 14

If you have a beam that will hit the square root function at a and reflect at the tangent, the reflection beam will intersect with x-axe at c. Independent what beam you start with the reflection beam always will hit the same value. Try to evaluate the value c. See picture 15.,16,17 and 18. Here you have to notice some conclusions from the above example.

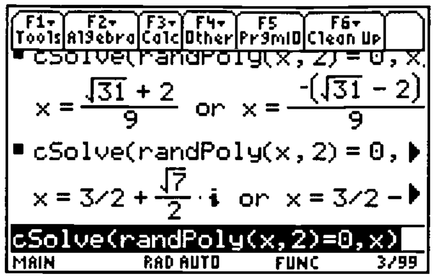
Conclusion: Your students can faster reach the goal and have some time over to reflect about the problems. Even rather weak students will quickly understand the meaning with the mathematical thoughts and will not be disoriented in all the algebraic labyrinth with no exit. They can rise their heads and see the entire meaning of the thoughts.

Bert Waits:
"Some mathematics becomes more important because technology requires it.
Some mathematics becomes less important because technology replaces it.
Some mathematics becomes possible because technology allows it."

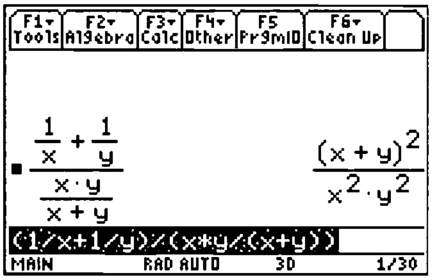
Pictures to "Let the students explore algebra with CAS tool, TI89"



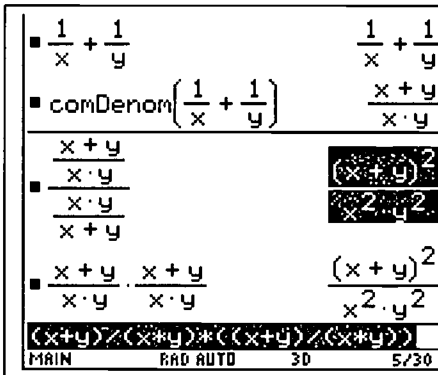
Picture 1



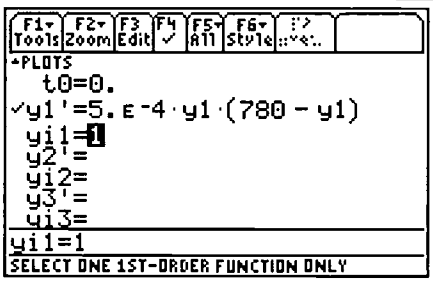
Picture 2



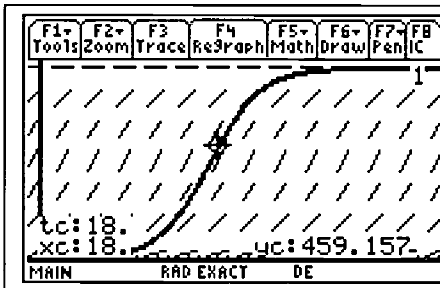
Picture 3



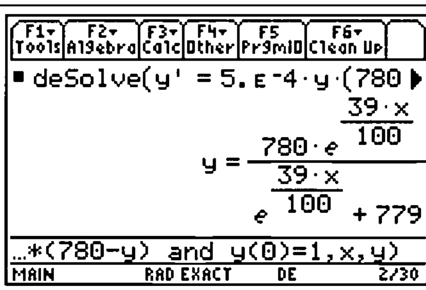
Picture 4



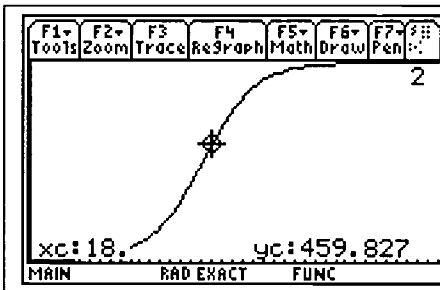
Picture 5



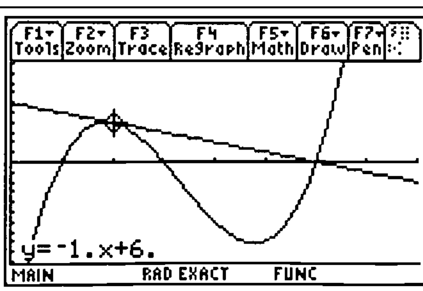
Picture 6



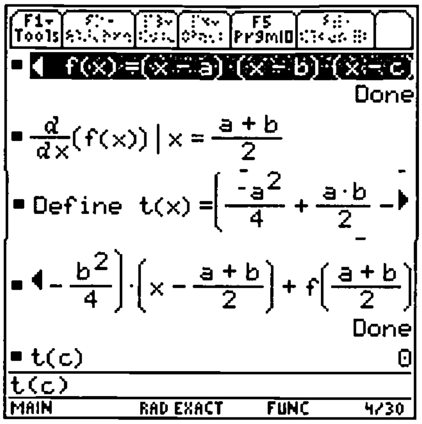
Picture 7



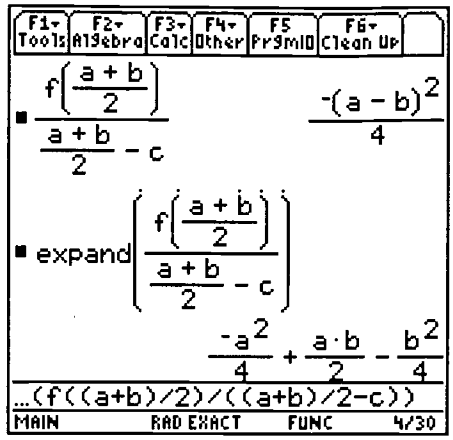
Picture 8



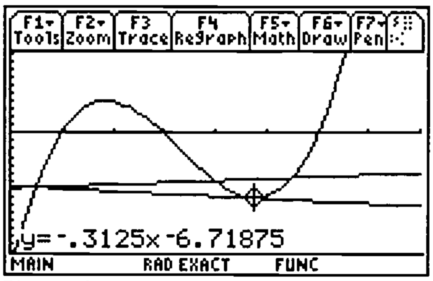
Picture 9



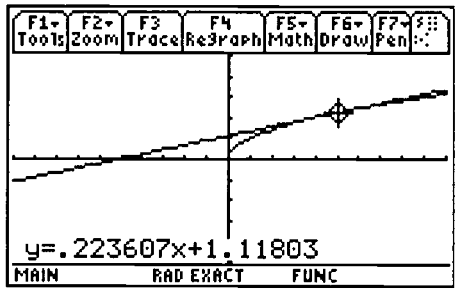
Picture 10



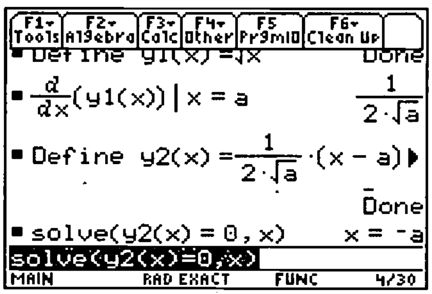
Picture 11



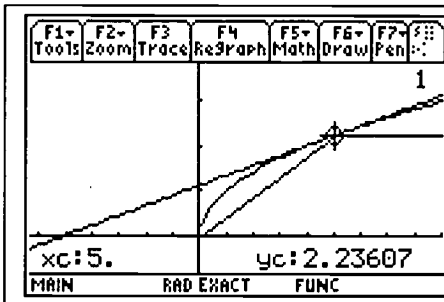
Picture 12



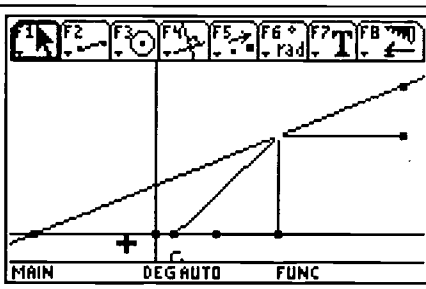
Picture 13



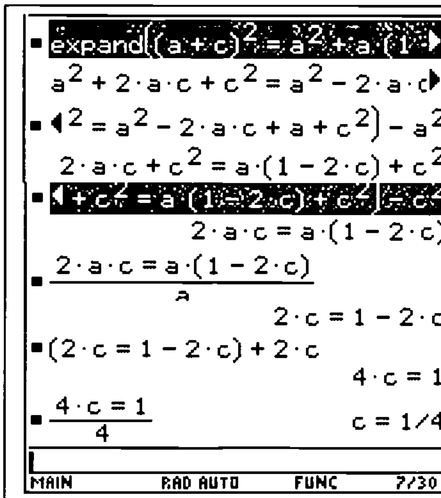
Picture 14



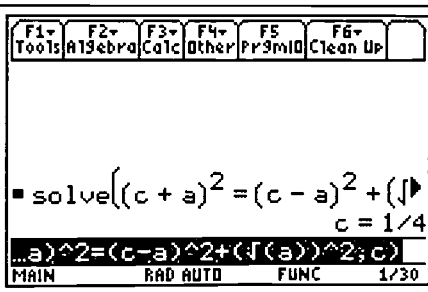
Picture 15



Picture 16



Picture 17



Picture 18

BEST COPY AVAILABLE

"ALL OF A SUDDEN THEY GOT IT": Understanding preservice teachers' perceptions of what it means to know (in) math

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ABSTRACT

In a recent study at the University of Regina, preservice teachers were asked questions about their internship experiences of teaching mathematics. One question in the study focused on asking preservice teachers to recall their most meaningful experiences in the mathematics classroom during their internship, to which many responded with stories of how their students all of a sudden just "got" a concept and how this could even be visually detected. It is interesting to note the comparisons between their responses to this question about meaningful experiences and their responses to other questions concerning their images of math as a subject, their attitudes toward math, and their perceptions of what it means to know (in) math.

Factors other than ability influence students' approaches to challenges, their persistence (or withdrawal) when facing difficulties, and how they use cognitive skills. This paper explores goal theory and achievement motivation as a perspective for examining the issue of what it means to know (in) math. The question of the role of the teacher in how students focus their efforts in mathematics classrooms, or in setting the classroom climate, is also of significance to this discussion.

This paper presents implications for the changing needs of teacher education programs, including the contexts of mathematics education courses as well as critical issues in curriculum development and implementation in general.

Introduction

This presentation emerges out of a study with post-internship preservice teachers in a Canadian university. In this study, we surveyed twenty-seven preservice teachers who had recently completed their fourth-month internship in elementary and secondary schools. Preservice teachers were asked questions about their past experiences as students learning mathematics, and about their internship experiences of teaching mathematics. This presentation will present and discuss some of the implications of the responses to this survey, revolving around the theme of "what it means to know (in) math."

Goal Theory

Factors other than ability influence students' approaches to challenges, their persistence (or withdrawal) when facing difficulties, and how they use cognitive skills (Dweck, 1986). Researchers have demonstrated that nonintellectual dispositions, such as achievement motivation, may improve the prediction of academic success beyond intellectual dispositions. General academic success may rely "more heavily on the ability to adapt to new learning situations and to apply intellectual assets than on the level of academic aptitude alone" (Larose, Robertson, Roy, & Legault, 1998, p. 290).

Achievement motivation involves two classes of goals. Learning goals seem to reflect intrinsic motivation as individuals seek to increase their competence, to understand or master something new (Dweck, 1986). Competence is viewed as developing through effort (Anderman and Maehr, 1994). Students with learning goals tend to seek challenges, pursue task mastery, and persist despite difficulties and obstacles (Dweck, 1986). Performance goals, on the other hand, reflect extrinsic motivation as individuals seek to gain favorable judgments of their competence or avoid negative judgments of their competence (Dweck, 1986). Errors are viewed as evidence of lack of ability or worth (Anderman and Maehr, 1994). Therefore, students with performance goals may avoid challenges and withdraw when faced with difficulty (Dweck, 1986).

Meaningful Experiences in Mathematics

In our study, several survey responses drew our attention to a concern that many mathematics teachers and learners are emphasizing performance, rather than learning, goals. When asked about their most meaningful experiences during internship, several preservice teachers used terminology such as "get it" to describe their students' experiences of learning mathematics. For example, one respondent described her most meaningful experiences as "when my grade two's finally caught on to the concept of 'time'—it was like I turned on a light switch [and] they all of a sudden just 'got it'." This begs the question of how a teacher actually knows when her/his students reach the point of "getting it" in their learning. Perhaps even more importantly, the critical question is what exactly are they getting.

For the purposes of this paper, our interpretation of "getting it" closely relates to what we feel preservice teachers mean by understanding. With this in mind, it becomes critical to look at the connections between the visual detection of "getting it" and what it means to know or understand in mathematics.

> Understanding is not properly attributed to the recitation of steps in a proof, no matter how perfectly the steps unfold from premise to conclusion, but to the "seeing" that occurs when

the products of reason are re-examined—looked at—by intuition for the purpose of discerning or creating meaning. (Noddings & Shore, 1984, p. 53)

The metaphorical use of a light switch being turned on is common for the portrayal of what is perceived as moments when a person reaches sudden understanding. Barnes (2000) describes such times as "magical moments" and adds as a point of clarification that "such occasions may best be described as illumination or insight rather than intuition" (p. 34). One characteristic of a magical moment is described as follows:

> There is a claim to sudden realisation of new knowledge or understanding. Usually this new knowledge is 'seen' with great clarity, or experienced with a high degree of confidence or certainty. (Barnes, 2000, p. 34)

Such an 'aha' or feeling of 'getting it' is believed to be a strong motivating force in a learner's continued participation and persistence in mathematics. These insightful and exciting feelings are important to the learner. As Burton (1999) states: "Far from understanding being something which is *only* driven by knowledge, there is both a *need* to know and an associated *pleasure* in knowing which is its own reward" (p. 29). Our claim, however, is that even though there can be a great deal of satisfaction and excitement in this insightful and pleasurable moment, it may be connected more to the performance of mathematics than to understanding. For example, in a recent secondary mathematics methods class, preservice teachers explored methods for solving quadratic equations. They were quite intrigued by a method that one of their classmates introduced for the process of completing the square; it was virtually a short cut that gave rise to a series of 'aha' responses. When the instructor (i.e., one of the authors of this paper) questioned the preservice teachers about their enthusiasm for understanding this new approach, it was apparent that their motivation merely stemmed from a desire to approach procedural understanding from a different perspective. When prompted for communication about their relational understanding between the algebraic representation and the concrete geometric representation of what it means to *complete a square* and solve quadratic equations in general, the students were not the least bit motivated to explore it further. Their own school experiences of learning and 'doing' mathematics focused primarily on the successful performance of mathematical questions and problem solving tasks. Insight into the how's and why's of such mathematical tasks had never really been a part of their 'illumination' or 'getting it' experiences. It is feared that such levels of understanding never *will* be a part of their teaching unless they experience, at some point in their careers, dissatisfaction with how and why they know mathematics.

In a recent study (Nolan, 2001), an elementary preservice teacher explained that she preferred learning math to learning science because she could see how the math pieces all fit together like a jigsaw puzzle, while her understanding of science did not feel quite so connected.

> Math is easy. It's a game. It's a puzzle. Math is yes or no... I know I'm going to get the right answer. Math to me seems like sort of a closed box (p. 102). I have success with math. When I uncover a piece in math I say, 'oh, that makes sense. That fits in with everything else I know.' (p. 184)

When questioned about the 'fitting in' relationships that she felt she understood, it was apparent that the pieces were predominantly understood as a puzzle would be; that is, procedurally.

We have discussed our concern that the 'getting it' or 'aha' moments might signify nothing more than procedural clarification, which is possibly void of a *deeper* understanding of mathematical meaning and relationships. We also have another critical concern associated with these insightful moments in coming to know. We are concerned that students possess a belief that knowledge and understanding of mathematics travel in waves of these magical and insightful moments. A belief that such an emotional experience is a prerequisite to learning and knowing in mathematics has a profound impact on learners' attitudes and motivation. Barnes (2000) points out that "attitudes are more stable than an emotional experience, however intensely felt" (p. 39). While the 'aha' experience might foster greater motivation and persistence, the absence of such 'aha' experiences could, unfortunately, have an opposite (and detrimental) effect. Students faced with tasks in learning mathematics often embrace the view that you either get it or you don't; that you are either good at it or you are not, as if mathematical ability is innate. In addition to perpetuating an elitist attitude toward the knowing of mathematics, this view directly opposes the belief that a positive attitude and willingness to persist at a task will inevitably lead to greater success in the learning of mathematics. The emotional 'getting it' experience is mistakenly seen as a necessary precursor to learning, and that it will naturally occur if (and only if?) one has mathematical ability. This view, we believe, creates a dichotomous relationship between ability and persistence, as one becomes associated with strength and the other weakness.

In light of these critical concerns for what it means to know (in) mathematics and how learners experience the processes of coming to know, we are advocating the importance of critical epistemological reflection in teacher education programs. There is a need to reflect on learning experiences in order to acknowledge the problematic nature of knowing in mathematics, but not with an intention to focus on the preservice teachers' weaknesses in their knowing. If we acknowledge the problematic nature of knowing then it is more acceptable to critically question the differences between performing and learning mathematics. Davis, Sumara, and Kieren (1996) help illustrate the problematic nature of knowing and learning when they write:

> Learning should not be understood in terms of a sequence of actions, but in terms of an ongoing structural dance—a complex choreography—of events, which, even in retrospect, cannot be fully disentangled and understood, let alone reproduced. (p. 153)

It is critical that these moments of insight not be unquestionably accepted as indicators of a deeper, more relational understanding in learning when they may indicate only a deeper (or more expansive) understanding of procedures. While the feeling of 'getting it' is beneficial as a motivating episode, one must be critical of exactly *how* it is motivating and toward *what* end.

This discussion brings us to the issue of how we can encourage and develop reflective practices in teacher education—reflective practices on the parts of preservice teachers *and* teacher educators.

Reflective Teaching Practices

Many students (particularly women) do not view themselves as participants in the construction of math knowledge but instead see the teacher as an agent who delivers factual information, rules, and formulas which must be memorized (Seaman, Nolan, & Corbin Dwyer, 2001). The teacher plays a major role in creating "situational demands" which influence students' goal development. This may be the primary reason to promote reflexive teaching practices.

Internships may be considered a form of a mentoring program since preservice teachers complete a school practicum under the supervision of a more experienced teacher. When experienced teachers mentor beginning teachers, both report increased reflection on their teaching styles (Flockhart & Woloshyn, 2002). In order to be effective, both parties "must assume *active* [emphasis added] roles in seeking mentoring relationships that will satisfy, sustain, and fulfill them" (p. 51). Some preservice teachers, however, do not make connections between their internship and university courses (Dyson, 2000). As Meyer and Tusin (1999) remind us "teacher educators must help preservice teachers make explicit links among their course work, field experiences, and their pedagogical beliefs" (p. 136).

Beattie (2002) offers a "Holistic and Narrative Pedagogy" in which "practices are focused on enabling prospective teachers to find their voices in relation to the theory and practice of teaching, to use them to articulate their questions and concerns, and take ownership and responsibility for their own learning" (p. 20). Through these practices, preservice teachers would come to new understandings about themselves, their students, their classrooms, their schools, and their communities. The preservice teachers in our study emphasized performance goals in their experience of students' ability to 'get it' despite being taught about learning goals in their education programs. This suggests that links among their course work, field experiences, and their pedagogical beliefs are not being made.

How do teacher educators help preservice teachers make these links? How do students, and teacher and students, relate to one another and what are the 'rules' for these interactions? What are people *really* saying to one another beneath the surface? For example, in spite of advocating a constructivist approach to teaching and learning, are errors (or, false starts) still being viewed as evidence of lack of ability? Are students still seeking, and teachers providing, external reinforcement of competence, particularly those who 'get it'? Teacher educator reflective practice is an essential starting point in making links between theory and practice explicit.

How do teacher educators engage in the process of reflective practice? Beattie (2002) has written about the important role of students' writing, and feedback from teachers, in achieving this. More recently, in a "self-study," she has explored her own writing and inquiry, and uses the feedback she receives from students as a source of her own insights and understandings (2002). Portfolios are another external artifact in which preservice teachers and teacher educators can engage to create new ideas and meanings (Corbin Dwyer & patterson, 2001). They help educators define good practice, stimulate reflection on their own teaching and learning, and acknowledge and refine their own teaching and learning practice (Lyons, 1999). "Portfolios offer particular opportunities for preservice teachers and their instructors to construct meaning about teaching and learning as well as to reflect on learning to teach" (Corbin Dwyer & patterson, 2001, p. 18). While the use of portfolios in teaching is not new, they can be used as simultaneous inter-collegial and self-initiated evaluation (Egbo, 2001).

Schön (1987) described "reflection-in-action" as "the kind of artistry that good teachers" (p. 1) display everyday. Reflection does not have to take only the form of words. Messages are sent from teacher educator to preservice teacher, from supervising teacher to preservice teacher, and from teacher to student "in doing, in performance...The student's performance, for example [says] 'This is what I make of what you have said. This thing that I'm doing now is what I make of what you have said.'" (p. 8). From our study, it appears that the preservice teachers' performance is speaking louder than their words. Many talk the language of learning goals in their description of what it means to know (in) math but their own approach to solving math problems, and to teaching math, indicate an emphasis on performance goals.

Implications for Teacher Educators

Schön (1987) examined what it means to "heal the splits between teaching and doing" (p. 13). We, too, are concerned about healing the splits—between theory and practice, between what preservice teachers say about what it means to know (in) math and what it means to teach math, between learning goals and performance goals. After all is said and done, however, are we, as teacher educators, expecting preservice teachers, our students, to 'get it'—to 'get' that knowing (in) mathematics is not only a matter of magical moments but is also a matter of effort and persistence?

"The act of reflecting on the value of learning activities is another fundamental part of the teacher's work...The question for reflection becomes not, 'How can I get better results?', but 'Improve what, for whom, and how?'" (Tite, 1986, p. 21). As teacher educators, questions for our own reflection include: How can we promote the use of reflexive teaching practices so that teachers do not embed learning goal language within performance goal teaching? If the reality of our students' experience in our current educational system, particularly in university, rewards high grades and uses comparative evaluation, are we being contradictory in our message of the importance of learning goals in the mathematics classroom? How can we, as teacher educators, help them to "walk the talk?"

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ELEMENTS FOR TEACHING GAME THEORY

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ABSTRACT

Game Theory should be included in the undergraduate programs of many majors, specially in those of economics, business administration, industrial engineering and, of course, of mathematics and statistics. It becomes indispensable in a globalized and technified society to become acquainted with theoretic points of view that help make decisions in conflict of interests situations. Game Theory gives a nice opportunity for university lecturers to carry out the essential role of stimulating the attitudes of observing, analyzing and theorizing in our future professionals, as a way to build a better world. Moreover, it is highly formative to know the basic results of a theory developed in the 20th century and to use the elements of probability to examine multiperson decision problems.

In the teaching-learning processes of mathematics, we should be careful about how and when to present the rigorous formalization of concepts and the use of specific techniques since we must always bear in mind the importance of stimulating both an intuitive approach to the concepts that we are introducing and a creative use of the previous knowledge of our students. When we teach Game Theory we have a nice opportunity to apply these criteria through the collaborative learning and solving problems according to the following sequence: understanding the problem (includes organization of the information and representation), intuitive approach to the solution, solution (or attempts of it) using previous knowledge, intuitive introduction of new concepts or theorems related with the problem, solution (or attempts of it) using the new concepts or theorems, formal and rigorous presentation of the new concepts or theorems, formal solution of the problem, search of other ways to solve it, explorations modifying the problem, and a deep study of the theoretical aspects using intuition and formalization. With this didactical propose, I made it easy for my students to understand the concepts of Game Theory, specially Nash equilibrium and mixed strategies for non zero-sum games and their applications.

1. Introduction

A fundamental task of teachers of any subject, but specially of mathematics, is to guide their students in learning to learn, and helping them become self-confident about their learning capabilities. Game Theory is specially favorable for the performance of this task, because it deals with topics related with our daily life, which are becoming more important: situations in which there are conflicts of interests, in which it is necessary to decide looking for the most suitable choice and considering what the other persons, with similar interests, may do. It is very good for the motivation to be aware that these situations happen not only in parlor games, but also in games in a wider sense, which we play or whose play we see day to day: driving a car in a big city, trading the price of a commodity (as buyer or as seller), advertising, defending or accusing a prisoner, proposing a salary, designing economic policies in a country, facing a war, etc. All this favors the motivation and contributes to the presentation and development of the concepts starting from problems and making dynamical and collaborative classes with intuitive approaches prior to the formalizations proper of the theory. The cases of noncooperative games with two players and a finite number of strategies are particularly interesting because the students, appropriately guided in using their intuition and with the aid of relatively elementary mathematics, usually arrive at solutions or criteria that are in fact part of the theory, even though not yet formalized. When the students verify this, they strengthen their self-confidence about their learning capabilities.

Regarding intuition and mathematics, it is appropriate to recall what Efraim Fischbein wrote in his book *Intuition in science and mathematics*. He does not believe intuitive reasoning to be present in certain stages of the development of intelligence only, but instead that typically intuitive forces guide the way we solve problems and carry out interpretations, no matter how old -or young- we are. Furthermore, even when faced with highly abstract concepts, we tend - almost automatically- to represent them in a way that makes them intuitively accessible. However, we must bear in mind that this same author warns that "by exaggerating the role of intuitive prompts, one runs the risk of hiding the genuine mathematical content instead of revealing it. By resorting too early to a 'purified', strictly deductive version of a certain mathematical domain, one runs the risk of stifling the student's personal mathematical reasoning instead of developing it". (Fischbein 1987, p. 214)

The present article is meant to show a way of working with basic aspects of Game Theory, which agrees with the outline of the previous paragraphs.

2. Playing in the classroom

Students are divided into two groups: Alpha and Beta. From each group two students are selected to be the players (P1 and P2) of games whose rules are to be announced. So that in each group there is a P1 and a P2. The idea is to obtain results in the separate groups for later comparison. Each player calls from his group a team of "advisers" that will help him make the best decision. Neither players nor different teams are allowed to communicate, and the decision must be rational.

Game 1

For this game I give each player two cards, named C1 and C2. Each card holds a written demand that I will fulfill:

C1: Give the other player 3 dollars.

C2: Give me 1 dollar.

Each player must choose one card only, and give it back to me. So they must decide which card to choose in order for them to get the greatest possible benefit from their participation in the game.

After a prudential time for discussion with their advisers, players from both groups turn in one card each. After reading them, I fulfill each card's demand.[1]

Understanding the problem is a fundamental stage and generally, after some time for group deliberation, the information is organized in one of the following forms:

- *Lists of payoffs*

Payoffs to P1:

P1's choice	P2's choice	Payoff to P1
C1	C1	3
C1	C2	0
C2	C1	4
C2	C2	1

Payoffs to P2

P1's choice	P2's choice	Payoff to P2
C1	C1	3
C1	C2	4
C2	C1	0
C2	C2	1

- *Matrix tables*

Payoffs to P1:

P2

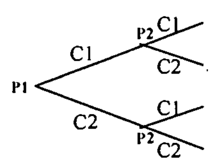
P1	C1	C2
C1	3	0
C2	4	1

Payoffs to P2:

P2

P1	C1	C2
C1	3	4
C2	0	1

- *Trees*



Payoffs to **P1**	Payoffs to **P2**
3	3
0	4
4	0
1	1

It is of great stimulus for the student's learning to learn capacities to realize later that, without consciously knowing it, they had been using concepts and representations that are common use in Game Theory. Thus, their way of organizing the information by means of "payoff lists" corresponds to the *payoff functions* of the proposed game, and the two other ways are just the two major representations for describing games: the *normal form* and the *extensive form*, respectively. It is then a very simple task to resume the two matrix tables in a bimatrix table, just as the ones used for the analysis of normal form games.

P2

P1	C1	C2
C1	(3, 3)	(0, 4)
C2	(4, 0)	(1, 1)

[1] This game is based on Aumann's version of the known game "prisoner's dilemma". (Aumann, R. 1987)

It is generally the case that in both groups, Alpha and Beta, players use the C2 option. When they are asked to explain the rationale behind their choice, they do it by means of the scheme they used to organize information and by certain criteria that are in fact intuitive approximations to the notion of *strict domination of strategies*. It is clear that even when apparently they would be better off using both C1, rationality (and a certain sense of self-assurance) forces them to choose C2. Next they are asked to relate this game to similar real-life situations. In one occasion a group showed that the same situation could be observed in an arms race between two countries: both are conscious of the convenience of decreasing their expenses in weapon systems, but neither will risk to do so without being reasonably sure that the other also would. As a result of distrust, they continue spending enormous amounts of money in weapons.

We continue posing two new problems; both already resumed in their bimatrix form:

Game 2

P2

		Red	Yellow	Green
P1	White	(4, 3)	(3, 4)	(4, 5)
	Black	(0, 6)	(5, 0)	(3, 4)

Game 3

P2

		Red	Yellow	Green
P1	White	(1, 9)	(3, 4)	(3, 8)
	Black	(2, 4)	(0, 4)	(4, 6)
	Brown	(3, 5)	(2, 6)	(3, 4)

Working in groups as before, I give the students enough time to study the problems. By using the notion of strictly dominated strategies, but without any further formalization, they find the solution for *Game 2*: P1 chooses White and P2 chooses Green, and the players receive the payoffs 4 and 5, respectively. Through this problem students learn to work with the rationality of Game Theory; they realize that at first P1 has no strictly dominated strategy, but that on the other hand Yellow is strictly dominated by Green for P2, so this starts their process of finding a solution.

Game 3 brings a particular difficulty: neither player has a strictly dominated strategy. However, students generally come to the solution that corresponds to a Nash equilibrium: the best choice for P1 is Black and the best one for P2 is Green. Difficulties they find to explain how they came to such a solution, added to the lack of formal algorithms, make us think that their solution is purely intuitive. The fact of receiving the teacher -and the whole class's- approval of their solution reinforces their self-confidence; the next task is to find a rational way to arrive at the solution. This is a crucial part of the learning process of Game Theory since the search for a more careful description of the player's rationality is in turn the beginning of an understanding of the rationality behind this theory. At this stage they are not yet informed of formal definitions or techniques, which when given from the beginning lead to a purely deductive learning, and sometimes to a merely mechanical application of techniques, shortening so this important phase of intuitive and creative approach. It takes a little time, but it is generally the case that after a period of discussion within the groups, and between groups, students grasp the idea of thinking what a player would do if he knew the other player's choice in advance. So they start ticking the "most convenient" payoffs in each case, and the solution is then determined by the strategies that correspond to a box having both components of the pair of payoffs ticked. After this experience, it is clear for the students that the absence of strictly dominated strategies does not imply the absence of a solution, and it is interesting to ask them to attempt a definition of the concept of "rational solution", which in the theory corresponds to Nash equilibrium. The students clearly perceive the necessity of formalization, and they are asked to take care of it. Regarding this stage, I had an excellent experience when receiving the following explanation, as an attempt to define a Nash equilibrium for games similar to the given ones:

Two lists are made:

If P2 chose	*then P1 would choose*
Red	Brown
Yellow	White
Green	Black

If P1 chose	*then P2 would choose*
White	Yellow
Black	Green
Brown	Red

Since Green - Black is in the first list and Black - Green is in the second, this pair of strategies is the rational solution for the game. These lists are in fact the *best-response correspondences* for the players; so essentially the definition is that of Nash equilibrium in pure strategies in terms of the best response correspondences that are commonly given for finite two-person games[2].

3. Creating Games

An activity that is frequently given little importance is that of creating problems. This task should parallel that of solving problems, since it stimulates creativity, helps to fix ideas and concepts that are being introduced, and presents new difficulties that require the introduction of new concepts or techniques in order for them to be overcome. It is very attractive and motivating for the students to attempt to get through with the difficulties created by themselves; specially when they are conscious of the criteria they should use, but they find them insufficient. When asked to create games similar to those ones they were faced with, students easily come with games having more than one Nash equilibrium, games in which a player's best response to a certain strategy from his opponent is not unique (this is taken to introduce the concept of correspondence, rather than that of function); and -more interesting- games that have no Nash equilibrium according to the given criterion. After discussing some selected problems, formal definitions of game, payoff function, strictly dominated strategy, best-response correspondence and Nash equilibrium are presented for two-person games. The equivalence of the definitions of Nash equilibrium in terms of the best-response correspondence and of the payoff functions is highlighted. By observing a bimatrix game with a Nash equilibrium, they verify that being (s, t) an equilibrium point, if player 1 changes his strategy while player 2 does not, then the payoff received by the former is never as good as that he would receive in (s, t). A symmetric verification is made for the case of player 2: if he deviated from his equilibrium strategy while player 1 did not, then his payoff would never increase. After that, the statement of Nash theorem is presented: in every

[2] If R_1 and R_2 are correspondences defining the sets of players' best response to each other's strategy, the pair of strategies (s, t) is a Nash equilibrium if and only if $s \in R_1(t)$ and $t \in R_2(s)$

finite game (a game with a finite number of players, each one having a finite number of strategies only) there is a Nash equilibrium.

Here is a selection of games, taken from those presented by the students:

Game (a)

	S	T	U
A	(3, 6)	(7, 1)	(2, 6)
B	(4, 1)	(7, 5)	(5, 8)

Game (b)

S	T
(2, 4)	(3, 9)
(5, 3)	(2, 1)

Game (c)

S	T	U	V
(2, 4)	(3, 9)	(7, 1)	(7, 0)
(5, 3)	(2, 1)	(6, 4)	(4, 1)
(0, 5)	(4, 3)	(3, 3)	(9, 2)

Game (d)

	S	T
A	(2, -2)	(-6, 6)
B	(-3, 3)	(4, -4)

In the four of them students use the technique of underlining the payoffs that correspond to a player's best response, when considering that his opponent uses some fixed strategy.

-In *Game (a)* it is easily seen that player 1 is indifferent to choosing his strategies between A or B if he knew that player 2 will choose T. Similarly, player 2 is indifferent between S and U, as long as he is certain that player 1 will choose A. Using the best-response correspondences, we have:

$R_1(S) = \{B\}$; $R_1(T) = \{A, B\}$; $R_1(U) = \{B\}$

$R_2(A) = \{S, U\}$; $R_2(B) = \{U\}$

It is a simple matter to see that the pair (B, U) is a Nash equilibrium, and we can find this point either by the elimination of dominated strategies, or by observing that $B \in R_1(U)$ and at the same time $U \in R_2(B)$.

-In *Game (b)* two Nash equilibria are obtained. This fact causes controversy on which one should be used, and motivates commentaries on the interchangeability and equivalence of equilibria, as well as on the idea of subgame perfect equilibrium. Furthermore, when the concept of mixed strategy was introduced later, it was very interesting that they found out their proposed game had a third Nash equilibrium.

-In *Game (c)*, formed from *Game (b)* by adding strategies to both players, no Nash equilibria could be obtained. Expectative and doubt arose among students, since it was natural for them to think that a counter-example had been found for the Nash theorem, stated before. Then they were suggested to look for more simple games having this property. That is how *Game (d)* came into scene; the latter has also another interesting particularity: it is a zero-sum game, that is, a game in which the amount obtained by a player is the amount lost by the other.

-In *Game (d)*, in the absence of strictly dominated strategies, and being "unable" to find a clear criterion to guide the players' choice, I suggested them to think that the players have actually more than two ways to carry out their choice. In most cases, students found, as a third way to "choose" an alternative, a random device: tossing a coin. At this point the natural question is: why not to use a dice instead of a coin? Or why not a roulette? Thus, for instance, player 1 could choose between A or B by tossing a coin: if it comes up heads, he chooses A, and if it comes up tails, he chooses

B; and player 2 has the possibility of choosing between S or T by throwing a dice: if the outcome is 1 he chooses S, while if the outcome is 2,3,4,5 or 6, he chooses T. It is clear that when using a dice to "choose" between two alternatives, many different assignments could be done between numbers and strategies. A question is now in order: are these random devices the most convenient? Were the students to accept that random devices are indeed necessary, the formalization suggests the use of probabilities and expectation. With the aid of these tools, the students themselves redefine in a natural way the (expected) payoff for each player, and it is interesting to guide them towards an extension of the definition of Nash equilibrium, by asking them to compute and compare some expected payoffs. For instance, in *Game (d)*, assuming that players carry out their choices by tossing a coin and throwing a dice, respectively, and thinking of the correspondence between outcomes and strategies given above, this means that player 1 chooses A with probability 1/2 and B with probability 1/2 as well; while player 2 chooses S with probability 1/6 and T with probability 5/6. The expected payoff for player 1 corresponding to these probabilities, which we may call $EP_1((1/2,1/2), (1/6,5/6))$, or simply $EP_1(1/2, 1/6)$, can be obtained from the matrix of payoffs for player 1, in which the probabilities are written too:

		1/6	5/6
		S	T
1/2	A	2	-6
1/2	B	-3	4

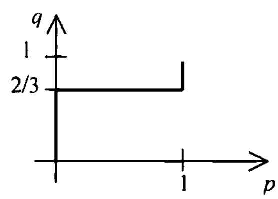
$$EP_1(1/2,\ 1/6) = 2\times\frac{1}{2}\times\frac{1}{6}-6\times\frac{1}{2}\times\frac{5}{6}-3\times\frac{1}{2}\times\frac{1}{6}+4\times\frac{1}{2}\times\frac{5}{6}=-\frac{11}{12}$$

With a similar computation we obtain $EP_2(1/2, 1/6) = \frac{11}{12}$. However, this random device to choose their strategies is not the most convenient for any of them. To see this we can consider, for instance, that player 1 decides to use a dice instead of a coin while player 2 maintains his previous device. In this case, assigning a probability of 1/6 to A and 5/6 to B, we would obtain $EP_1(1/6, 1/6) = 2\times\frac{1}{6}\times\frac{1}{6}-6\times\frac{1}{6}\times\frac{5}{6}-3\times\frac{5}{6}\times\frac{1}{6}+4\times\frac{5}{6}\times\frac{5}{6}=\frac{19}{12}$, which means that player 1 has improved his expected payoff. The moral is that (1/2, 1/2) for player 1, and (1/6, 5/6) for player 2 cannot be a Nash equilibrium. The search for the most convenient device for choosing at random a strategy makes them think of the most convenient probability that should be assigned to each strategy. With a little help they come to realize that the best "practical" device is neither a coin nor a dice, but something like a two-color roulette, with the portion covered by each color being proportional to the assigned probabilities. Thus, for instance, player 1 could use a roulette having 3/5 of its area painted in Green and 2/5 in Blue; if the roulette stops in Green he chooses A, if it stops in Blue he chooses B. After these experiences it is natural to extend the set of strategies for each player, calling *pure strategies* the original strategies they had been working with, and introducing the concept of *mixed strategies* as probability assignments over the pure ones. Restricting our work to two-person games with only two pure strategies for each player, and recalling the best-response criterion used to define the concept of Nash equilibrium in pure strategies, we look at the general expression for the expected payoff for each player and plot the best-response correspondences; next we intuitively conclude that the points where these two curves intersect determine all Nash equilibria, including pure strategy equilibria, if any. Furthermore, looking at the graphics we can figure out that in two-person games with only two strategies for each one, there will always be at least one Nash equilibrium. In the case of *Game (d)*, assigning probabilities p and $(1-p)$ to player 1's pure strategies A and B, respectively; and probabilities q and $(1-q)$ to player 2's strategies S and T, respectively, we obtain:

$EP_1(p, q)=15pq-10p-7q+4 = p(15q-10)-7q+4$.

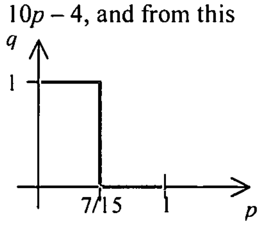
Since p and q can only take values in the interval [0, 1], and since this function is linear in p, it can be seen that player 1's best response to values of q that make the expression $15q-10$ positive (i.e., $q\in]2/3, 1]$) is choosing the greatest possible value for p, that is $p=1$. Analogously, his best response to values of q that turn the expression $15q-10$ negative (i.e., $q\in [0, 2/3[$) is choosing the least possible value for p, that is $p=0$. If $q=2/3$, the expression $15q-10$ vanishes and the expected payoff for player 1 no longer depends on the value he chooses for p; in consequence, p can take any value in the interval [0, 1]. To resume, player 1's best response to the mixed strategy $(q, 1-q)$ of player 2, which we call $R_1(q)$ for short, is

$$R_1(q)=\begin{cases} \{1\} & if \quad q\in]2/3,1] \\ \{0\} & if \quad q\in [0,2/3[\\ [0,1] & if \quad q=2/3 \end{cases}$$; and graphically:

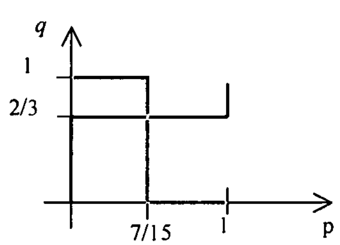


With a similar reasoning, we obtain, $EP_2(p, q) = q(7 - 15p) + 10p - 4$, and from this

$$R_2(p)=\begin{cases} \{1\} & if \quad p\in [0,7/15[\\ \{0\} & if \quad p\in]7/15,1] \\ [0,1] & if \quad p=7/15 \end{cases}$$; and graphically:



When plotted in the same coordinate system, the intersection of these two graphs gives us, for each player, a mixed strategy that is the best response to his opponent's choice. Thus, we see that the only Nash equilibrium is the pair of mixed strategies ((7/15, 8/15), (2/3, 1/3)).



This visualization of Nash equilibria is a very interesting tool for the analysis, creation of problems and the stimulus of research. It is very important to induce the students to make conjectures on the existence of Nash equilibria and on the greatest possible number of these, as well as having them design their own examples and counter-examples to support or discard their conjectures. We can thus obtain a whole rank of cases, from the "intuitive security" of the existence of at least one Nash equilibrium, up to the design of games with infinitely many equilibrium points.

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AN INTERDISCIPLINARY APPROACH TO TEACH ODE - DEVELOPMENT AND IMPLEMENTATION OF THE EV & C UBB PLATFORM

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ABSTRACT

An active learning approach has been developed and implemented to teach Ordinary Differential Equations (ODE) for Food Engineering undergraduate students using an Internet-based package (EV & C UBB).

A variety of learning strategies have been introduced to support and extend the traditional lectures making it easy for instructors to design and deliver online learning. To achieve those goals we have implemented an Internet-based package that includes several sections for learning and teaching, some of them interactive.

The internet-based package works like a distance educational platform, so the student can use it from anyplace. EV&C UBB includes class calendar and bulletin, interactive tests, secure access for students and instructors, homework and projects forum, peer review, and resources area.

The ODE course was originally designed so the student could make his (her) own projects, and be evaluated at the end of each project. Now with the introduction of the Internet-based package the student can receive help whenever he (she) wants, and it is possible to know at every step of his (her) work all advice that has been given by the instructor or peers. We believe that EV & C UBB is an extraordinary teaching aid strategy to learn from research projects in ODE. Students interact with instructors and peers to improve his (her) project. Therefore we have contributed to develop a teaching and self-learning comprehensive system that reinforces active and constructive learning.

Key words: Collaborative learning, Distance Educational Platform, Peer instruction, Mathematical modeling, ODE Lab.Work Book.

1. Introduction

Engineer students at the undergraduate level, very often, have serious problems applying theoretical concepts to real and concrete problems, especially when doing the mathematical modeling. Furthermore, in order to be updated with our fast and growing technology, we have to promote critical thinking development in our university lecture practices (Bates, 2001). Therefore we have developed and implemented a teaching-learning platform system to be used by teachers or instructors with their students. In particular, our university is situated in a city surrounded by rural underdeveloped and economically poor communities. Therefore, our students face for their first time in their life courses with technology that allows them to think and develop own ideas. We believe that the virtual platform developed here would encourage that attitude in any student. The main objective of this device is to guarantee a greater communication with and among students.

Our continuously changing world society triggered the current developmental needs in our global university educational system. The structure of this new educational system is sustained by innovation, which in turn is being supported by didactics, curricula and education itself. This global society has changed because science and technology has played an important role in their cultures. In that sense, our students will have to face a society that works based on high productivity patterns and standardized global requirements. So, the university teaching strategies have to change ahead of time in order to prepare those students for critical thinking behavior, an intellectual tool for a constantly changing world culture (Machamer *et al* 2000). It is well known that traditional lectures have proved to be ineffective promoting critical thinking in students. Such academic training is obtained through other kind of strategies where students are active learners (Brussee, 1999).

We have chosen to apply a distance educational platform strategy to teach ordinary differential equations to undergraduate students. That course requires skills that are difficult to obtain in traditional lectures. Evidence comes from comparing the high failure rate among students that have taken the traditional course in the past, to the greater rate of success among students that learned using the plattform.

We believe that one of the major challenges for teachers is to teach so students can learn, in that way we describe here an active learning approach that has been developed and implemented to teach Ordinary Differential Equations (ODE) for Food Engineering undergraduate students. The device uses an Internet-based package (EV & C UBB) including a variety of learning strategies to support and extend the traditional lectures making it easy for instructors to design and deliver online learning. To achieve those goals, the Internet-based package includes several sections for learning and teaching, some of them interactive.

Distance educational platforms values any kind of learning, favors communicational skills among classmates, strengths interaction between instructor and students, weakens paradigms that are opposite to innovation, and strengths education outside classrooms. Distance educational platforms are a strong complement for on site teaching practices. In this paper we show how this platform works, what is the teacher role in the course, what are the general administrator functions, and how is the course ODE taught using this device.

2. Implementation

The virtual platform "Virtual Education & Science at UBB" (**"Ev & C UBB"** in Spanish), using a remote way, supports academic development for undergraduate students, teachers and other users in general with an easy delivery of contents, knowledge and experience exchange among participants. The platform "Ev & C UBB" is designed so any user (teacher or instructor), from now on called administrator, can have a personal area programmed with a variety of supporting modules according to each user needs.

It also has a working area that contains all different sections or courses saved inside the platform. These courses may be designed by each course administrator with high flexibility, allowing a module development according to each particular course needs.

Technically, the platform has been developed using a professional language for web applications called PHP (Personal Home Page), which interacts with a data base created inside a data base server MySql. Personal Home Page and Mysql are tools used to develop "Ev &C UBB", and altogether can offer a robust distance educational platform.

2.1 THE PLATFORM

The platform EV & C UBB is flexible, may be used for any course by any teacher because it can be designed and modified at any time by the course administrator. Therefore, the administrator has total control of the course and the information students put into the platform; in that way the administrator can view at any time contributions or progress made by the students in the course. On the other hand students may navegate freely through the plattform and can make contributions, talk to the administrator, or with other students, solve a variety of tests and questionnaires on line, get into discussions, review other students contributions, set their own web page, and so on. The internet based package provides the student with general information about the course, the student may have access to all the course members personal information (addresses, phones, picture), and if the administrator wishes, the student may also view his(hers) peers reports or comments.

The administrator can organize the course contents at will. For example, create files titled Laboratory Guides, or Special topics discussion, or Teacher´s lectures notes. The administrator may have access to most frequent questions asked by students and can check when the student have logged in or has made a contribution. The plattform has a private area for the student where documents and links may be saved, and there is a personal calendar to schedule an agenda for the whole year or beyond. This personal calendar may be shared or not by the student with other group members, for example to schedule meettings among them. In the working area the student may read or unload information about lectures, create and be part of forum, read or add documents and links, see the course calendar. The student may also create his(her) own web page using the resource My page.

The cascade forum resource is interesting since the adminitrastor may give a topic for discussion with a date and time limit to make contributions. The student contribution can either answer the administrator ideas or any other student idea, identified by the student's name In that way the package will create a cascade of opinions linked to the original source. The administrator and the students as well, can see all contributions.

The administrator can also have a data base with all the questions for a given test or survey. In that way the questions can be selected for a given group of students or any other purpose. The multiple-choice test can be done once or as many times the administrator wants; the administrator can choose to let the student know about the answer or else give clues to answer properly. The administrator can set a date and time to solve any given test. At the end the administrator can see the statistics for the student´s answer and evaluate performance in the course.

The platform has a variety of modules that can be useful to the administrator and are designed according to the course characteristics and necessities. The modules are Calendar, Files, Cases, Simple Forum, Users groups, Peer review documents, Peer group review, Questionnaires and Surveys, Cascade Forum, Resources, and Webmail.

3. The Ordinary Differential Equations (ODE) course.

The ODE course is taught at the third year of Food Engineering after the students have taken at least two Calculus, Elementary Algebra, and Linear Algebra Course. The ODE course teaching strategy is based on projects design and computer simulated experiments. These methodologies motivate the student to modify their learning styles and therefore develop independent critical thinking. The platform Ev&C UBB with all its modules have helped students to develop analytical thinking, using intuition, and logical arguments. The administrator would give each student or group of students a problem related to real cases from food industry, where it is required to manage ideas linked to areas like heat transfer , and fluids mechanics. The students interact through the platform using symbolic packages like Maple, Modellus and Scilab. This software would help the student to understand and do mathematical modeling about real situations. Softwares like the above-mentioned can be used to simulate experiments (Borreli *et al* 1998), and therefore to infer changes in a discrete field to understand in depth processes in a continuous situation.

The administrator can allow students to peer review their reports on line. This exercise has shown to be effective since students who do peer review with detailed and constructive comments may enhance their own work (Tsai et al 2002). Tsai et al (2002) also suggest that anonymity offered by networked environments may help build up a more objective way of judging peers work.

Transition from discrete to continuous thinking requires adequate problem searching and design of motivational exercises by the instructor. In that way the student can move in a comfortable and increasing way among symbolic contexts, numerical concepts and graphic development. For computer modeling experiments we followed Borrelli *et al* (1992), and for project design problems we used strategies in common use by American universities (Cohen *et al* 1991) plus special and slight modifications (Toledo *et al* 2001). Evaluation could be done at any time using forum modules, questionnaires and survey modules, homework assignments, computer experiment assignments; grading was done based an forum participation performance, weekly tests or quizzes, and 2 or 3 mid-term testing. Furthermore, this Platform can be used during laboratory work where every student is working in his own computer; the instructor would ask questions to be solved by students, the answer can be displayed on the instructor's screen. In that way, the instructor can manage a survey about concepts or skills acquired by students, allowing the instructor to reinforce concepts in case is necessary.

Performance in this course in the past was poor, only 40 to 50% of the students would pass the course. Now, with the current use of the platform students feel that teachers keep them "in their toes", they have a continuous feedback not only from their teachers but also from their classmates.

Good performance in this course has increased and the percentage of students that pass are over 80%. We have done student surveys about the platform and the students say that they feel more comfortable talking to the professor through a machine, another opinion is that they feel they can solve any problem, and is a real pleasure to face one when you think that you are solving applied engineer problems just like in the real world.

4. Conclusions

Platform EV&C UBB has been created and implemented as a complement for student learning. This strategy promotes a greater dynamism and participation among students, teacher assistant, and teacher or instructor. The ordinary differential equations course has been enriched using project design exercises, and computer modeling exercises in the teaching and learning process. These activities has been reinforced continuously with periodic evaluations through all semester by teachers or instructors given the fact that student's work and participation can be followed daily using the platform Ev&C UBB.

Opposite to traditional kind of communication in classroom lecturing or through office hours attention to students, the platform offers an active type of communicational tool. The students feel their case or question can be assisted at all times not only by a teacher but also by peers. The student acquires knowledge, a higher selfsteem, and communication skills through message interaction with peers. The platform offers a collaborative environment, where students can identify themselves and can be influential or accept other ideas towards building a teaching and learning process fitted for everyone individually in a academic community.

We believe that Ev&C UBB is a platform that can be used to collaborate in the teaching and learning process not only in mathematical fields but in any field, so it can be used by all teachers and students. Since the administrator can modify any of the modules adapting the platform to his(her) own uses, we feel that this is a strong communicational and teaching tool for everyone. The language used in the platform is Spanish but we are working to translate it to English so it can be used as a universal learning strategy tool.

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QUALITY CONNECTION: GOING THE DISTANCE

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ABSTRACT

Virginia Beach City Public Schools launched a new initiative in February 1999 – distance learning videoconferencing. The program was initially designed to offer additional curricular choices to students and expanded training opportunities for staff in real time, but the program has grown exponentially. Not only has DL created expanded opportunities, the technology has liberated students and staff from the confines of budget and schedules. In fact, the DL program – Quality Connection: Going the Distance – has revolutionized the way the division does business.

Though modest in its beginnings, with the installation of DL labs at only three of the district's high schools and with only one course offering in discrete mathematics, the DL program expanded rapidly. DL capabilities have continued to expand. Currently, all 11 Beach high schools originate and receive over 20 courses. Most recently, five middle schools have come on line.

In addition to the discrete mathematics offering, other DL courses available to our students include AP Statistics and Pre-IB Algebra II/Trig. Videoconferencing technology supports the various pedagogical strategies promoted by standards-based mathematics educators. The document camera is the heart of most instruction. Technology such as the graphing calculator, algebra tiles, and PC applications are effectively employed. The current emphasis on student learning through communication of mathematics is complemented utilizing site-to-site communications enabled by DL. A demonstration can be arranged provided there is comparable videoconferencing technology at the conference.

The division's motto is "Ahead of the Curve." As far as we are concerned, that is where we are collectively, all 85 schools and 10,200 employees. And, that is where we intend to stay.

Introduction

The use of technology for technology's sake is not a choice that can be made in a field where budgets are tight and the stakes are high. One need only look to recent events in the news regarding technology, whether it be stocks, technology companies, IPOs, or the like, to realize how fluid and pervasive the technology market really is. That being said, it is also true that the effect technology has had in education can be compared to the effect the printing press had on the dissemination of knowledge in the Renaissance. This unprecedented access to knowledge and the potential for providing equity in education as well as enhancement of curricula have never been greater.

For Virginia Beach City Public Schools (VBCPS), technology – specifically, Distance Learning (DL)--has changed the way the division does business. Although the process has not been easy, we have progressed from offering one DL class to 20 students in two high schools to our present offering of 22 content-rich classes to 422 students spread across the division (Apprendix A).

VBCPS launched a new initiative in February 1999 – distance learning videoconferencing. The program was initially designed to offer additional curricular choices to students and expanded training opportunities for staff. The program has grown exponentially from its nascent beginnings and fast outstripped its original purpose. Not only has DL created the above-mentioned expanded opportunities, the technology has also liberated students and staff alike from the confines of budget and schedules. In conjunction with training and courses, DL serves the Human Resources (HR) Department in its teacher recruitment effort. Staff employs the DL lab to interview prospective teachers at college campuses far afield. Students, also, use DL to virtually "visit" college and university campuses and discuss the admissions process and other issues.

Course Selection

In preparation for the creation of our first class, we knew we had several critical components to consider: what class would be taught and which teacher would teach it. Three schools were targeted to pilot the DL labs and the mission was to have one school originate a class and the other two receive it by second semester of the 1998-1999 school year. Discrete Mathematics was selected as our premiere class and had a collective enrollment of 20 students. This videoconferencing medium premiered February 2, 1999, with Princess Anne High School's Discrete Mathematics offering sent to Bayside and Ocean Lakes high schools. By the following school year, another high school shared AP Statistics with two neighboring schools. Currently, Pre-IB Algebra II/Trig is being shared from one of our high schools to our magnet middle school. We have been pleased with the vast majority of the experiences provided for both our teachers and students. The DL experiences tended to provide motivation for those students who were previously unmotivated or unprepared and created additional academic opportunities for gifted and highly motivated students.

Teacher Selection

We made a conscious decision to be inclusive of all who were interested in providing distance learning instruction. This has proven to be a wise choice. The three mathematics classes offered have worked very well through the technology, largely due to the exceptional instruction of the teachers. In our three-year tenure with this program, excellent instruction by excellent teachers has been our greatest gain and the most important feature of a successful program. We have learned that most important to a distance learning program's success is teacher quality, and that exceptional teachers make effective instruction happen, regardless of the facilities. Whether they are excellent communicators and/or performers, unabashed risk-takers, or reticent traditionalists who have built a powerhouse of a program, DL teachers all begin at the same level. Once the teachers are committed to using the distance learning venue, they must be convinced to stay the course. Because most are not technologically savvy, validating the fear that accompanies the lack of experience is important. During training, it is vital to put teachers in front of the cameras and microphones early and have them utilize the document camera and control keypad immediately. The phobias will only persist and grow if the lack of hands-on experience continues.

Equipment

In Virginia Beach all DL rooms are similarly equipped and can be either origination or receiving sites. The equipment is permanently fixed and cannot be moved from room to room. At each site a primary camera (Illustration A) is focused on the teacher who uses a touch pad to manipulate the camera (Illustration B). The instructor also manipulates a document camera (Illustration E) and the cameras at the remote sites, and can select the video sources seen by the students (Illustration C). A monitor allows the teacher to preview each image before it is broadcast. Each classroom has four television monitors, two at the front of the room, two farther back (Illustration D). Each shows images of the teacher's choosing, such as his/her computer screen, document camera image, an instructional video, or the shot from another camera. At the receiving site, a student who wants to ask a question presses the button on a microphone on the table (Illustration F). When the remote site camera zooms in on that student, the microphone allows the question to be heard by the teacher and students at the other sites. If a student wants to show the teacher her work (Illustration G), she uses the document camera in her room.

Special Considerations

It must be noted, however, that instructing through the DL medium is neither for every teacher nor every student. While we determined that student need was a main factor in course and teacher selection, we recognized that there were several other crucial determiners as well. Some might consider these determiners to be self-serving, but the end result was the establishment of a firm foundation for distance learning across the division. We were able to "sell" distance learning to some principals and teachers simply because they had differing agendas. For example, some were avoiding involuntary transfers due to low enrollment, or generating interest in fledgling programs that needed a jumpstart, while some were saving dangerously low enrollment elective courses that were close to being dropped from the master schedule.

DL and Teaching Standards

The question of whether Distance Learning is an effective or an appropriate medium for the teaching and learning of mathematics is a critical one. Best practices espoused in the *Handbook of Research on Improving Student Achievement,* (ERS, 1995), and promoted by the Virginia Beach City Public School System, are addressed by Distance Learning in many ways. For example, the "Opportunity to Learn" is provided to those students who might otherwise not have access to a particular mathematics course due to unavailability of staff or insufficient student enrollment. From another viewpoint, this "Opportunity to Learn" may be more of an equity issue. The National Council of Teachers of Mathematics in its *Principles and Standards for School Mathematics* (NCTM, 2001) paints a vision for school mathematics that demands "high-quality, engaging mathematics instruction" for all students. Its first principle, that of educational equity, "is a core element of this vision." With the growing national shortage of qualified mathematics teachers, the concepts of equity and opportunity to learn will certainly become more critical issues, for which distance learning can provide an answer.

"Openness to Student Solution Methods and Student Interaction" is uniquely enhanced through the use of the document camera. Students can share work directly from their notebooks with their distant peers. The possibility of this occurring can serve as motivation for more consistently organized work. Although a few students exhibit camera shyness, others frequently are eager to experience the new technology and often do so with an elevated air of professionalism. Our teachers report that younger (middle school) students seem particularly willing to "ring in" to ask questions and contribute to class discussion. The opportunity for "Small Group Learning" is not impeded, but does require special consideration in terms of space and accessibility to microphones and camera. "Whole class discussion" takes on a different flavor. It is imperative that participants from each site contribute to the learning process, and herein lies the challenge. The necessity for the camera to focus on the speaker before other sites can hear him/her is for some, a "moment of fame" while others experience an unfortunate rise in anxiety. An impatient few find the moment it takes for the camera to train on the speaker agonizingly slow. They want to voice their input immediately and spontaneously.

The intrinsic motivators of curiosity and ambiguity (Child, 1986) and the brain's innate drive to seek patterns and meaning (Caine & Caine, 1994) can be tapped through the use of concrete materials and calculators. Both tools can be employed in a visually pleasing and effective manner through the use of the document camera. One of the most important themes espoused by the *Principles and Standards for School Mathematics* (NCTM, 2001) is that of connections. Through the use of a graphing calculator and the employment of multiple representations, graphical, algebraic and numeric, connections not previously possible can be discovered. The document camera enables <u>any</u> calculator to be utilized and viewed easily by all students without extra cables or a specialized view screen. The student can display his/her own calculator while justifying individual thinking or posing a particular question. Caine & Caine (1994) speak of a teacher's need to "orchestrate the immersion of the learner in complex, interactive experiences that are both rich and real." The distance learning lab enables an internet linked computer to be experienced by all participants at all sites simultaneously, providing real world, even real time, data and global access with ease.

The mathematics teacher plays an important role in enabling students to construct understanding by providing a variety of rich experiences. Since the lens of the camera magnifies everything from flaw to forte, the distance learning teacher must embody NCTM's Teaching Principle. The teacher must not only possess profound content knowledge, but he/she must be well versed in multiple representations of an idea, able to connect concepts, and possess an expertise in a wide array of pedagogical strategies. Perhaps more importantly, the teacher must be capable of creating an environment that is supportive and conducive to students participating actively in their own learning process. The combination of techno phobia and math anxiety could be a deadly combination. The use of games, simulations, and multimedia presentations has proven to be effective in distance learning. A spirit of camaraderie can be developed through the encouragement of cooperation and competition, which are both valid motivators (Child, 1986). The ability to set an onscreen timer for such activities assists with time management, both from the teacher's perspective and that of the students'.

Feedback

We asked teachers and students how the distance learning technology affected instruction. Teachers were candid in their responses, looking for the positive, and suggesting methods that may improve a continually changing medium. When asked, students provided refreshing, objective candor.

Many of the suggestions offered by the first instructor were excellent building blocks for the program. A stipend for the distance learning teacher, a fax machine located in the classroom, a teacher assistant hired to not only support the receiving classes but the sending classes as well—all are now regular fare in our distance learning program. Two subsequent mathematics teachers had more specific reflections regarding instruction using the distance learning venue. Both agreed that instruction changed dramatically as they utilized videoconferencing equipment. They also found that interactivity decreased and there was a dire need for creative thinking on how to accommodate this feature in a math class. In addition, they discovered that enticing student response using microphones and cameras easily shut down the participation of the most volunteer-phobic student. For those students, the tendency to participate was minimal in a traditional class, but having to use the technology coupled with a the lack of interactivity made it like pulling proverbial teeth in order to get a response.

Teachers found the greatest gains of lesson-planning-for-TV included the following: increased interactivity in instruction; employment of state-of-the-art technology especially the document camera; enhancement of instruction in non-distance learning classes, specifically, better organization, frequent use of power point; and a brisk instructional pace. Overwhelmingly, the mathematics instructors found great value in enriching student academics by providing courses, especially at the upper level, that would not otherwise be available.

Probably the greatest interference to mathematics instruction was the teacher's inability to see the students as they worked and what they could do. In addition, because the interdepartmental mail between 85 schools takes days to deliver, teachers also found that lack of immediate feedback in grading homework and tests/quizzes hampered effectiveness in keeping students current and on task Lastly, all teachers found the effectiveness of the teacher assistants at the receiving sites vital to student success.

Lessons Learned

The one constant of our DL program throughout the three years of its existance has been the continual and successful marrying of mathematics courses with videoconferencing. However, most students will tell you, and we concede the point, that nothing will ever take the place of excellent, live instruction. DL, despite its advantages, will always remain a strong second.

The lessons we have learned are many. This medium is not for every teacher nor is it a venue for every student. The abstract nature of mathematics can create an unbreachable chasm for many. Distance learning can compound the psychological barrier experienced by some learners of mathematics. Therefore, teacher selection will dictate the success of any program since it is the teacher that must bridge the divide. Those professionals who understand that interactivity in instruction connotes success, who demonstrate mastery of their content, and who illustrate effective communication and delivery styles, will thrive using distance learning. Because of the critical nature of these essential skills, found in master teachers, it is not recommended that an inexperienced instructor should ever be directed to teach via DL.

In three years we have learned the many facets of a successful distance learning program. We continue enthusiastically on the journey through a program whose only constant is the fact that it will never remain so.

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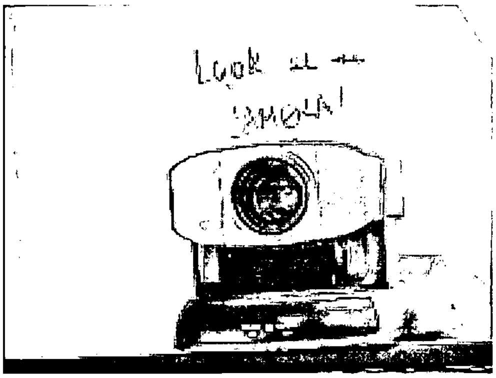
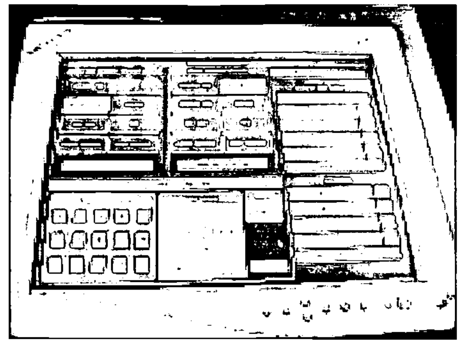
Illustrations of VBCPS DL



(A)



(B)

(C)

(D)

(E)

(F)

(G)

MIDDLE SCHOOL DISTANCE LEARNING BELL SCHEDULE							
	1st Bell	2nd Bell	3rd Bell	4th Bell	5th Bell	6th Bell	7th Bell
Landstown						**Creative Writing (20)**	**Creative Writing (16)**
Princess Anne						*Creative Writing (5)*	*Creative Writing (12)*

HIGH SCHOOL DISTANCE LEARNING BLOCK SCHEDULE 2001-2002

SCHOOL	BLOCK 1A	BLOCK 1B	BLOCK 2A	BLOCK 2B	BLOCK 3A	BLOCK 3B	BLOCK 4A	BLOCK 4B
COX	*Japanese I (7)*		**AP Comp Gov (6)**	*Japanese II (2)*		*Anatomy/ Sports Injury (2)*	*Japanese I (2)*	*Russian II (1)*
BAYSIDE		*Japanese I (7)*	*AP Comp Gov (1)*		*Japanese I (8)*	**Latin IV (2)**	**Latin III (5)**	
FIRST COLONIAL		**Japanese I (15)**	*Japanese II (2)*	*Russian I (3)*	*International Relations (8)*		*Russian II (1)*	**AP Art History (9)**
GREEN RUN				*Russian I (3)*		*Latin IV (2)*	*Japanese I (13)*	*AP Art History (2)*
KELLAM	**Russian I (9)**	*Japanese I (6)*		**Russian I (8)**	*Japanese I (6)*		**Russian II (7)**	**Russian II (3)**
KEMPSVILLE	**French V (4)**			*Japanese II (2)*			*Russian II (1)*	*Japanese I (12)*
LANSTOWN	*Russian I (4)*	*Japanese I (4)*	*Japanese II (4)*		*Japanese I (6)*			*Japanese I (6)*
OCEAN LAKES	*French V (1)*	**Japanese I (12)**		**Japanese II (5)**	**Japanese I (11)**	**Anatomy/ Sports Injury (11)**	**Japanese I (12)**	**Japanese I (9)**
PRINCESS ANNE	*Japanese I (14)*		**Alg. II/ Trig. (PAHS-17)**	*Russian I (9)*	*International Relations (1)*	*Anatomy/ Sports Injury (4)*	**Mus. Theory I & II (18)**	*Russian II (1)*
SALEM	**Japanese I (10)**		**Japanese II (9)**	*Russian I (2)*	*IR (2)*		*Latin III (4)*	*Russian II (1)*
TALLWOOD	*Russian I (10)*		*Japanese II (2)*		**International Relations (7)**		*Mus. Theory I (2)* *Mus. Theory II (1)*	*AP Art History (6)*
KEMPS LANDING MAGNET			*Alg. II/Trig. (2)*					

#'s indicate student enrollments **Bold indicates sending classes** *Italics inidcates receiving classes*

E-STATUS: A WEB TOOL FOR LEARNING BY DOING EXERCISES

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ABSTRACT

The paper introduces the project led by a team of teachers to assist students learn statistics. The goal is to build a tool able to present mathematical problems and to correct the students´ answers. The problems may include random data, so the solution cannot be previously known (if solved before) and the student can reconsider it if necessary. Pedagogical implications are commented, since the method can be effective on the basic and middle domains of learning, as well as on higher levels, specially if careful design of the problems is applied.

1. Introduction

The objective of this work is to present a web-based tool which allows the student to solve probability and statistical inference exercises in the introductory course taught in 2nd year degree in Computer Science Engineering at the Barcelona School of Informatics (FIB) of the Universitat Politècnica de Catalunya (UPC).

Our teachers (eight to ten involved) are teaching a large introductory statistical course to over 250 students. The course usually lasts 15 weeks (one semester) and the student attends six weekly hours of statistics sessions. Students are divided into groups of 80 students or smaller groups for the work in the laboratory.(The student's profile belongs to three different Computer Science degrees by the same university).

We agree with the opinion expressed by Roberts & Simonyi (1997): "A major challenge in teaching introductory courses to a large, diverse audience is the wide variation in background and ability that exists in the undergraduate population, which makes it hard to find the appropriate level of instruction." It is for this reason that we have thought of a product that allows the weaker students to practice many times the more difficult concepts without obstructing the progress of the advanced students.

It is well known that introductory statistics courses require important individual work on the part of each student, who needs exercises and practical cases in order to use the concepts acquired in a very short period. Exercises are undoubtedly a good complement for the "theoretical" lessons: they show how the concepts explained in the classroom are applied in real or simplified cases, and they facilitate the comprehension of the exposed ideas, through a typical process of learning by doing.

Why have we designed a web-based tool? The reason is obvious. Although our students are non-distance learning students, they will have a tool available to practice the statistical problem resolution, very accessible, and it could be used any time, at any place. The only condition is having a computer with Internet access. The difference between our project and most web-based tools (the list of examples would fill many pages) is that our software is dynamic in the sense that every execution presents new sample data randomly generated. The student could do the same exercise again but the data (and the solution) would be different.

The proposed methodology can be useful to many other courses on a mathematical basis, mainly in engineering studies, whose students make extensive use of problem solving in order to lessen the level of abstraction present in the classroom.

2. Motivation

We agree with Garfield (1994): "As goals for statistics education change to broader and more ambitious objectives, such as developing statistical thinkers who can apply their knowledge to solving real problems, a mismatch is revealed between traditional assessment and the desired student outcomes. It is no longer appropriate to assess student knowledge by having students compute answers and apply formulas, because these methods do not reveal the current goals of solving real problems and using statistical reasoning". The development of mathematical thinking in our students is capital for their professional future: their ability to manage a problem and lead it to an efficient solution is highly related to their analytical abilities. However, students will achieve an appropriate level of mathematical reasoning only if they face up to different situations compelling them to apply their higher aptitudes and, hence, reinforcing them. These aptitudes, jointly known under the name of *intellectual habits*, include several levels: 1) comprehension, 2) application, 3) analysis, 4) synthesis and 5) evaluation, and are related to the ability to transfer the knowledge from one field to another, the competence to face up new problems, and in general to what is called critical or reflexive thinking. These levels, based on Bloom's taxonomy, can be revised in Besterfield-Sacre et al. (2000).

The assessment methods used in our statistics course are, Muñoz, P. et al. (2000): 1) one test/quiz after the first 7 weeks including calculations and basic questions, 2) exercises delivered every two weeks and marked in the classroom, 3) one project developed in groups with data simulated from a realistic situation and covering a broad range of course objectives, and 4) the final exam. Obviously, this tool is a complement to all the learning materials used in it.

3. Present framework and further work

Using a computer application able to generate individualised problems to students, getting their response and providing the correct solution is an old ambition for us, teachers of statistics courses in several university centres. A former prototype, *Autoproblem*, was developed by two students as a part of their career final project: it was designed to pose fixed questions about inference with one and two samples, mainly confidence intervals and test hypotheses. The data is randomly generated according to the specifications given by the teacher: the sample size, the probability distribution and parameters representing the population. The student runs a Java applet on a standard internet browser, connected to the remote server installed in our department, where the results are stored in files.

Although it was an interesting experience, it showed the drawbacks of a closed application: more code development would be necessary to include new thematic modules, e.g. ANOVA or linear regression. On the other hand, the management of the information obtained should be improved to a great extent.

e-status is the next proposal, designed to overcome the aforesaid deficits. In the following section we extensively describe the working mode of the tool, designed as its predecessor according to its different applications for the teacher and for the students. At present, the tool is being implemented and we expect it will be available by September 2002. However, we are planning a gradual introduction: first year students will use the tool experimentally, but we will not exploit it as an assessment procedure until we have reached full comprehension of its educational effectiveness through the experience.

Future expansion of the tool will greatly depend on the needs revealed by its use. There are some implications that have not been developed so far, but will as soon as enough resources are available. Some of them are: definition of macros, in order to avoid repetition of code and auxiliary symbols; inclusion of external functions, which will be computed by other application; enriched grammar, now limited to constant strings, logical and numerical expressions, which can be extended to consider graphic objects, very interesting from an educational point of view.

4. Description of the environment

What is a problem?

Usually, in the educational world we think of a problem as a (real or imaginary) situation and a number of unknowns that can be deduced from the explanation. The goal is to find out how students apply their knowledge and reasoning to find the solution.

Many mathematical problems possess a solution that can be deduced analytically: this is the kind of problem considered in this work. From now on, a *problem* is an object consisting of:

— A Situation: a text describing the case

— Formulae: a set of equations according to the structure *symbol* = *expression*

— Data: known parameters of the problem; they are in fact a subset of the symbols appearing in the formulae

— A Quiz: a set of questions addressed to the student, each one composed by a text and the symbol giving the answer

— Metadata: additional items related to the problem, e.g. author, date of creation, lesson, title, difficulty, etc.

Example:

Situation	A farmer wants to know the area of his rectangular field. He measures the length of two sides, X and Y.
Formulae	X = 120
	Y = 80
	A = X*Y
	P = 2*(X+Y)
Data	X Y
Quiz	Which is the area of the field? **(A)**
	What length of fence will the farmer need to enclose the field? **(P)**
Metadata	title: the farmer's field; lesson: the rectangle

How do the students solve a problem?

Let us define an *exercise* as a solution provided by a student to an instance of a given problem. We call *instance* of a problem a particular presentation of the situation, the data and the questions present in the quiz. That is, taking the previous example, the instance would be something like this:

The farmer's field

A farmer wants to know the area of his rectangular field. He measures the length of two sides, X and Y.

X = 120

Y = 80

1. Which is the area of the field? (____)

2. What length of fence will the farmer need to enclose the field? (____)

A good exercise would obtain 9600 as the solution for the first question and 400 for the second. Logically, all the instances of this problem are identical, and this could not be interesting for teachers. But they can change the problem definition:

```
...
Formulae    X = Uniform(1, 100, 140)
            Y = Uniform(1, 70, 90)
...
```

Now, each instance gives random values for the parameters X and Y, and the correct answer can not be known in advance: only values matching the symbols A and P are good.

Architecture

The teachers´ application

A computer application, written in Java, is used for the manipulation of the problems. It allows:

— Management of files: creating new problems, saving them or retrieving them from a database, printing, etc.
— Edition: each component of the problem is edited assisted by the program
— Testing: the author can verify the correctness of the case by creating an instance

The person editing a problem ought to be familiar with the syntax of the expressions appearing in the Formulae section. The grammar defined for the expressions is far from being strange; on the contrary, we have tried to make it as close to standard as possible. Some of its features are:

— Common types considered: integer and real numbers, boolean expressions (true, false), strings of characters; for numbers, we have constants, vectors and matrices.
— Common operations and their precedence order: exponentiation, multiplication, division, addition, subtraction, etc. For the boolean type: negation, logical and, logical or.
— Arithmetic functions: trigonometric, square root, log and exp, etc.
— Special functions: like vector or matrix processing; special attention to the family of probability and statistics functions (random generation, probability distributions, etc.)
— Some functions and operators can be overload, that is, they can manage either constants or vectors, or even matrices, if the operation is allowed and the result is well defined.

Let us see some examples of possible expressions:

Formulae	`N = 10`	
	`X = Normal(N, 100, 10)`	sample from N(μ=100, σ=10)
	`Mean = Sum(X)/N`	`Sum` returns the sum of x_i
	`Var = Sum((X-Mean)^2)/(N-1)`	constant subtracted to vector
	`Stdev = Sqrt(Var)`	
	`Alpha = 0.05`	
	`T = Invcdf(T, N-1, 1-Alpha/2)`	evaluates inverse of a CDF
	`R = T*Stdev/Sqrt(N)`	
	`CI_l = Mean - R`	
	`CI_u = Mean + R`	
	`OK = CI_l < 100 and CI_u > 100`	μ inside the interval?

The previous formulae show how we can obtain the 95% confidence interval for the poblational mean with a sample (that should probably be given to the students as data of the problem).

The students' application

Any student enrolled in a course could access to the application entering a web address supported by the department web server. They have to authenticate themselves with their identifier and password and, once verified that the student is enrolled in thc specified course and the validity of the identification, they can:

— Pick a problem in order to practice freely and enhance their autonomous learning
— Choose a block of problems (they cover different lessons and are like an ordinary exam)

— Take an assignment, maybe as part of the assessment process

— Monitor their own results until that time

The program shows an instance, probably generated with random data, of a problem and waits for the student´s answer. Then it informs the student of the result: correct and wrong responses, time spent and a score to provide an estimate of his/her achievements. Sometimes, a student makes an error, and carries it along to the next questions. Depending on the strictness of the correction (that can be specified as metadata in the problem), dependencies on the responses can be taken into account or not, as these dependencies can be deduced from the set of formulae. We think that allowing this "soft" judgement (that would lessen the penalty of carried errors) is valuable: students will be aware, on the one hand, that they are "fairly" assessed (and not "coldly", as one might expect from a machine) and, on the other hand, that their errors are not innocuous at all.

The Database and its administration

Students, problems and exercises are some of the entities involved in the process. In order to obtain the best possible performance with the operation of the data (mainly searches, input and output), they and their relationships are managed by a Database Management System (like SQL Server). Both the teachers' application and the students' application communicate with the Database remotely, although the teachers will work through a local network and the students' application is designed from the base to work using Internet.

Obviously, some functionalities are included to execute normal operation. They are only allowed to teachers or authorised persons:

— Creating and deleting a course

— Loading a set of students in a course

— Inquiring about a student

— Grouping several problems into a block

— Defining assignments; usually the teacher specifies a timing: when the problem(s) will be available to the students (e.g., the week from May 11 to May 18)

— Collecting statistics or assessments, useful for the evaluation

Moreover, the administrator has to consider profiles other than the students and teachers´: for instance, teachers not involved in a course but interested in its materials, or general guests from anywhere accessing via web.

5. Success factors

The task of the teacher is, broadly speaking, to provide the students with a basis of knowledge, to stimulate their learning and to use suitable tools to measure their performance and achievement of learning goals. Taking into account that some goals can be related to the higher levels of the intellectual domain, requiring thus a direct interpretation of the teacher, the method we present can be clearly useful to the stimulation of the students and their involvement with the subject of the course. Moreover, by returning a score of the exercise promptly we reach an important objective in any educational field: to give the students (immediate) feedback of their progress, making of the evaluation an effective stage in the teaching/learning process.

To conclude, some simple pieces of advice are given. The sought effect is the implication of the instructors in the methodology so they do include it in their collection of educational resources, and preferably as one of the main ones. Their compromise is a key element to achieve a high degree of use on the part of the students. As the reader can see, some of the suggestions appearing below are not different from general rules to compose good "traditional" problems, but we must insist on them in order to bear in mind that technology will not transform a deficient problem into an interesting one.

What teachers should consider

— Compose the wording and the questions accurately
— Write cases for each lesson in the course
— Present a variety of situations (leave aside your old bag of balls)
— Avoid to always use the same type of questions (*"compute the mean of the sample"*)
— Include different degrees of difficulty
— Remark the connections with the course lessons, and organise a suggested sequence
— Try to pose questions referring to each knowledge plane, not just those related to "mechanical" skills
— Elaborate good sets of problems (go over the course contents throughout every lesson)

What students should hear (and take into account)

— Alternate study and problem solving, following directions
— Spend the necessary time, don't answer without thinking carefully
— Doing the same problem several times is advisable and useful, but you should also be aware of when you should change to another problem
— Check your progress regularly, and work to improve the weaker points.
— If you don't understand the questions, or you are repeatedly wrong, look for a way out (e.g., see your teacher about it)

6. Conclusions

This work has to be logically continued with the analysis of the results obtained. Validating the method would be desirable, that is, we would like to say that the academic performance has clearly improved past methods. In fact, we know this kind of conclusions may not be drawn lightly, since one cannot isolate the many factors affecting the learning of the students. In any way, we have considered the organisation of reliable data stored in a database from the beginning of the project, and this point may be capital to achieve a satisfactory validation.

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TEACHERS ESTIMATE THE ARITHMETIC SKILLS OF THEIR STUDENTS WHEN THEY ENTER THE FIRST GRADE OF PRIMARY SCHOOL

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ABSTRACT

Nowadays, in the area of contemporary teaching, the role of the teacher in the class has changed. Teaching is based on students and a great emphasis is given to the communication between teachers and students; the preliminary knowledge of children is taken under serious consideration for the establishment of a new knowledge. Teachers have to know this pre- established knowledge and estimate the abilities of their students, so that they will be able to organize their teaching according to this knowledge.

In the present paper we investigated the predictions and estimations of Greek teachers about the arithmetic skills of their students when they enter the first grade of Primary School. In the first stage of the research, teachers were interviewed and asked to estimate their students' abilities in enumeration, addition and subtraction, writing numbers and solving problems. In the second stage, teachers themselves tested their students, one after the other, in the above - mentioned processes. After completing this test and gathering all the answers, teachers were interviewed again and this time they were asked to evaluate their assumptions about children's knowledge.

The final results of our research show that the teachers' predictions about their students' mathematic abilities, in some cases are away from reality. For example, teachers underestimate their students' abilities in writing numbers, solving simple problems of addition and subtraction, etc. It seems that this perception is enforced by the instructions of the Greek analytical program, which ignores what students already know before they enter school.

Introduction.

During the last years many researches have taken place about teachers' theories and how these theories affect their teaching. One of the most important representations of teachers is relevant to the way by which students elaborate the information they get and learn. However the majority of teaching theories are not specific and they usually refer to educational systems and groups of students. Although it is important for teachers to know these learning theories, it is not obvious to the most of them that they need them in their every day teaching. Late researches on learning theories have focused on the cognitive dimension of learning and have shown new ways of examining students' knowledge and using this knowledge in the teaching process.

Researchers have deliberated on students' mathematical abilities and their knowledge about addition and subtraction (Carpenter, Fennema, Franke, Levi, Empson, 1999). For the purposes of the CGI program (Cognitively Guided Instruction) a number of researches have taken place in order to be examined whether the knowledge of the research findings in addition and subtraction can affect teaching decisions. The results of the research taken place in 1998 (by Carpenter, Fennema, Peterson & Carey) show that teachers are not aware of the strategies that students use when they solve a problem and they do not distinguish the different kind of problems. Their knowledge is organized in a way that does not allow them to understand their students' way of thinking.

In another research taken place in 1989 (by Carpenter, Fennema, Peterson, Chiang & Loef.) the researchers compared two groups of teachers; at first students were asked to solve problems and teachers were asked to predict how the students would solve these problems. In the end it was proved that teachers from the experimental group who knew more about their students' solving strategies had used this knowledge in every day teaching and their students were more capable in solving problems than others. Furthermore, the students from the experimental class had developed their metacognitive abilities the relevant to the understanding of solving strategies.

In our research in 1995 (Lemonidis), we examined teachers' theories about Mathematics and asked their opinions about the best way of teaching Mathematics. According to what they said it seems that during their studies teachers do not take enough special courses in teaching Mathematics. In a later research (Lemonidis 2001) we tested arithmetic skills of students at the first grade of Elementary school. Taking under consideration the findings of this research we decided to proceed a new one.

Firstly, we interviewed teachers asking them to predict their students' arithmetic abilities (for example, in enumeration, addition and subtraction). After that, teachers themselves examined their students' mathematical knowledge. In the end, teachers were interviewed once more and this time were asked to evaluate their predictions comparing them to the answers given by students.

Methodology of research.

Our research, in which 72 students from five classes of the first grade of three schools took part, was taken place in Karditsa. The first school was placed in the center of the city, the second one in the suburbs and the third one in a small town of the prefecture of Karditsa.

1st stage: Prediction interview

As we have mentioned before, first we asked teachers to appraise their students' abilities in mathematical tests. We chose the method of *half-structured interview* that allowed us to use a small general questionnaire adjusting questions according to teachers' answers.

2nd stage: Students examination.

After having completed interviews of the first stage we asked teachers to start examining their students. Throughout the examination we were writing down all the answers concentrating on students' solving strategies.

Students were examined one after the other: 1) in enumeration of objects, 2) in enumeration of dots and in writing the correct numbers, 3) in solving problems, 4) in addition and subtraction. In enumeration and in problems students could use small cubes; in the second test students were asked to find and write down the correct number of dots drawn on separate cards.

3rd stage: Evaluation interview.

Finally teachers were asked to comment on students' performance and evaluate their predictions.

Findings of research.

I. Enumeration.

Every teacher asked her students to enumerate three different collections: one of 5 cubes, a second one of 12 cubes and a third one of 20 cubes. The next table shows the percentages of success:

Table 1. *Success in enumeration.*

	Enumeration of 5 objects	Enumeration of 12 objects	Enumeration of 20 objects
Total Success N= 72	69 (95,8%)	42 (58,3%)	28 (38,9%)

The results of the first test showed that all teachers had predicted correctly; the majority of their students succeeded in the enumeration of the five objects. Although one of the teachers had underestimated her students' abilities, a percentage of 85,7 percent of her students succeeded to enumerate the collection of 12 and a percentage of 78,6 managed to enumerate the collection of 20 objects. On the other hand, one of the teachers had overestimated her students' abilities; she had stated that only two children could not count the collection of 12 objects, while a percentage of 50 percent failed to give correct answers; she had also mentioned in the first interview that the most of the children would succeed in counting the collection of 20 objects, but only two of her students gave correct answers to this test (12%).

II. Counting dots and writing numbers.

In this examination teachers gave children cards with big black dots asking them to count the dots and write down the correct number of each card. Students were given:

a. One card of three dots (in a diagonal order)
b. One card of four dots (in line in a horizontal order)
c. One card of five dots (dots were in line in a horizontal order)
d. One card of six dots (pairs of dots in a horizontal order).

The following table shows the results of this test.

Table 2. Success in counting dots and writing the correct number.

	3 dots	4 dots	5 dots	6 dots	
Success in Counting.	72 (100%)	72 (100%)	71 (98,6%)	53 (73,6%)	
Success in Writing numbers.	72 (100%)	72 (100%)	71 (98,6%)	70 (97,2%)	53 (73,6%)

On this subject teachers had predicted that most of the children would be able to give correct answers. The main procedure that students used in order to find out the correct number of dots in each card was enumeration. It is worth mentioned that for the card of three dots almost 50 percent of students gave their answers subitizing while all the rest followed the procedure of enumeration. In the first interview teachers had reported that students would follow the procedure of enumeration. Only two of them had mentioned the possibility that some students could give correct answers without counting (subitizing).

Writing the correct numbers was the second part of this examination. As we can see in the category of writing number ''6'' there are two columns: the first one refers to children who didn't find the correct number (6), but they wrote correctly the number they found. The second column refers to the percentage of children who not only found, but also wrote the correct number. As we can also see at table 2, all children wrote correctly numbers ''4'' and ''5''. Regarding to the writing of number ''3'' two children identified and tried to write the number, but they made the known *mirror mistake* (they wrote ''ε'' instead of ''3''), something that teachers had mentioned that it might happen. However these answers were listed as correct. It is quotable to cite here an extract of an interview, when a teacher talks about the ability of children to write numbers correctly:

Teacher: *"...Children count, but they can not write before they go to school. There are exceptions of course...for example, in a class of 14 children only two or three children can write".*

Generally teachers agreed that children would face difficulties in writing numbers and most of them stated that students wouldn't write correctly all the numbers. However, table 2 shows that children achieved better results than those that teachers expected.

III. Solving problems

Every teacher was reading aloud the problems to her students, who had objects in front of them (small cubes) in case they wanted to use them to solve the problems. The first problem was of the kind " part - part - all" and the final whole was asked. In the second problem - where subtraction was needed - children had to find the result of the transformation:

- Mary has four balloons. Kiki has two balloons. How many balloons do they have together?
- Helen had five candies. She made her sister a present of three candies. How many candies have left for her?

In problems teachers could use other names familiar to students (for example, names of schoolmates), but they had to maintain the structure of each problem. The percentage of success in solving problems is presented at table 3. The same table shows the procedures that most of the students followed in this test. Percentages of the chosen procedures include both correct and wrong answers of students.

***Table 3**. Percentages of success in solving problems and percentages of procedures.*

	Success	Known fact	sequence counting	Counting all with fingers	Counting all with objects
Problem of Addition.	52 (72,2%)	3 (4,2%)	1 (1,4%)	8 (11,1%)	51 (70,8%)
Problem of Subtraction.	33 (45,8%)	2 (2,8%)	–	9 (12,5%)	32 (44,4%)

In their first interview four of five teachers declared that students could not solve problems. They strongly believed that they had to direct children step by step to the solution of each problem. Only one of the teachers supported that students could easily handle problems of addition and subtraction. The same teacher reported that young children could answer almost immediately in these problems, because it is easier for them to find the solution of a problem (with realistic information) than it is to find the answer to the question: "How much is 4+2?"

Teacher: *"Yes, immediately. The problem is easier. If you ask children to find the solution of the addition 2+3 they will probably use their fingers. I noticed that when they want to solve a problem they calculate more easily".*

IV. Addition and subtraction.

In addition and subtraction every teacher was reading aloud the exercises to children. For example: "I want you to tell me how much is two plus two (2+2), two plus one (2+1)...etc". Students were asked to solve five additions (2+2, 2+1, 3+2, 3+3 and 4+4) and three subtractions (4-2, 5-3 and 6-4). In this particular test children didn't have any objects in front of them. Their teachers encouraged some children to think carefully and use their fingers to find the solution. Table 4 shows the results of this test. Again, the percentages of the chosen procedures refer both to the correct and wrong answers:

***Table 4**. Percentages of success in additions and subtractions and percentages of procedures.*

	2+2	2+1	3+2	3+3	4+4	4-2	5-3	6-4
Success	64 (88,9%)	65 (90,3%)	46 (63,9%)	33 (45,8%)	25 (34,7%)	32 (44,4%)	23 (31,9%)	17 (23,6%)
Known fact	47 (65,3%)	44 (61,1%)	9 (12,5%)	21 (29,2%)	8 (11,1%)	11 (15,3%)	5 (6,9%)	5 (6,9%)
Counting with fingers	2 (2,8%)	2 (2,8%)	5 (6,9%)	5 (6,9%)	3 (4,2%)	3 (4,2%)	9 (12,5%)	5 (6,9%)
Counting all with fingers	17 (23,6%)	20 (27,8%)	41 (56,9%)	25 (34,7%)	36 (50%)	24 (33,3%)	21 (29,2%)	23 (31,9%)

As it was proved teachers couldn't estimate their students' abilities in addition and subtraction:

Researcher: *"What about addition and subtraction? Do you think that they can find how much is 2 plus 2?"*

Teacher: *"No way, they do not even know the numbers 2 and 4."*

Researcher: *"What about 2 plus 1?"*

Teacher: *"No, no...perhaps they know the number (1), if they have seen it somewhere, but they can not calculate with addition or subtraction I do not think so."*

One of the other three teachers said that the majority of children would face difficulties in calculations with addition and subtraction. She mentioned that children would use their fingers to find the solution, because they couldn't calculate without using their fingers. One teacher predicted almost correctly the success in calculations with small numbers and the failure in bigger numbers.

A teacher who said that children would not have a specific difficulty in calculations gave the more optimistic prediction. She reported that 80% of the students would solve correctly the additions and the subtractions, something that was confirmed by the results of the test. The same teacher also reported that students usually remember or learn easily the sums 2+2, 3+3, 4+4, (or else the "double sums").

Second interview: commenting the results

In the second interview, although teachers found out - comparing their predictions to the results - that in main points their predictions were wrong, they did not look surprised; they behaved as if they had predicted correctly. However, in this interview the attitudes of teachers were quite different. For example, one teacher who failed to the most of her predictions persisted in her opinions about her students' abilities, even after the opposite results of the test. When she was interviewed for a second time, among other things, she also said:

Teacher: *"According to the results I predicted correctly. When children come to the first grade, they are able to read something or write their name, but they are not able to use numbers".*

Generally, teachers who had underestimated their students' knowledge discussed the results with us, but they did not look surprised. In a very few cases they admitted that they did not expect these results:

Teacher: *"I did not expect that some children would go so well in the addition..."*
Teacher: *"...Most of them wrote number 6, something that I didn't expect."*

Finally, two teachers who had predicted correctly the performance of their students in some tests simply commented on the performance they did not expect. For example, one of them mentioned:

Teacher: *"From what we can see after the examination, children didn't face any difficulty at all in enumeration; almost 98% of children are able to count from 1 to 20 and some students are able to count over 20."*

This teacher in the first interview had estimated that only a small percentage would actually manage to enumerate 12 and 20 objects. The second teacher commented:

Teacher: *"I was sure that children couldn't find number 6 and most of them would not answer; I did not believe that they could count the dots and find the answer. To my surprise I realized that most of them did very well. I was also sure that that they would count in order to find the number 3; on the contrary most of them did not. They just saw the card of the three dots and answered automatically and correctly."*

From the second interview we can conclude that teachers who had predicted more accurately the performance of their students were more careful during the examination and as a result they evaluated better the abilities of their students.

Conclusions

As we have already seen, teachers' predictions were in many subjects far away from students' real arithmetic skills. Regarding to the enumeration of 5 objects teacher's predictions were verified. One teacher estimated that children were not capable of enumerating collections of 12 and 20 objects and another one overestimated her students' abilities.

Three of the five teachers did not expect that their students would answer immediately without counting the three dots (subitizing). In addition, most of the teachers had underestimated the ability of children to write digits. In the first interview they explained that their students could not be capable to write numbers, because they had never been taught how to do this.

On the subject of the problems four of the five teachers declared that their students could not solve simple problems of addition and subtraction. They had expressed their doubts about how students could ever solve problems since they had not even learned the numbers.

Generally, it seems that most of the teachers underestimate their students' arithmetic abilities and they are not familiar with all the procedures that students use to solve an exercise. They support that students not only need objects in order to calculate correctly, but they also need to be guided to the solution step by step.

From the second interview where teachers were asked to make comments on the performance of their students we concluded the following:

a) The teachers who failed to their predictions did not seem to understand the real abilities of their students and they insisted on their opinions about students' weakness in Mathematics.

b) On the other side, two teachers who predicted more accurately were more perceptive; they were interested in the performance of their students and were willing to compare their first predictions to the results; they commented on their students' performance and admitted that some of their predictions proved by the results wrong.

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TRANSPOSITION OF DIDACTICAL KNOWLEDGE : THE CASE OF MATHEMATICS TEACHERS' EDUCATION

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ABSTRACT

The objects that the didactics of mathematics want to study are not exactly the same as a naïve or only vocational approach could identify as pertinent; didactic tools are an efficient way to analyse teaching situations, and anticipate new ways of learning, but are not always easy to communicate to future teachers, knowing that these students often get a very formal conception of mathematics during their university courses.
The aim of this work is to analyse situations that can be given to novice secondary teachers to help them understand the articulation between advanced mathematical notions and the contents of what they will be teaching themselves. Beyond that, it is to describe some principles of a didactical study of the instruction of trainee-teachers.

It leads to the use of a complex theoretical framework, which:
1) Identifies the didactical contract of the novice teacher;
2) Determines what kind of mathematical and didactical work can ensure the transition from the student's place to the teacher's place, and how this transition will become evident;
3) Uses transposition of the theory of situations due to Guy Brousseau, to build specific situations for young teachers, and to lay out the aims, the criteria and the constraints of such situations;
4) Questions the didactical knowledge on what is useful to drive complex situations in a classroom.
The organization of teachers' education at the IUFM d'Aquitaine is evoked, and examples of situations on the vectors and algebra are expounded.

Keywords: teachers' training, theory of situations, didactical contract of novice teachers, vectors, algebra.

This paper presents an outline of training methods for mathematics teachers, and some examples of the work we offer future teachers. This work has been initialised by the following questions that we are required to meet as educator in a training Institute with young teachers.

What are the conceptions of novice teachers on the mathematics to be taught? On teaching practice? How can we bring these conceptions to light ? What means are at our disposal to make them evolve?

Is theoretical didactical knowledge effective to make future teachers broaden their conceptions of mathematics? Which complex learning situations can be introduced in the preparation of young teachers? Are these situations the same ones as those useful to understand mathematics for oneself?

Which pedagogic knowledge is necessary to help teachers drive complex learning situations ? Does the preparation take care of this knowledge, or is it left to the teachers' own initiative?

This paper consists of four parts:

1) A presentation of the organization of the academic year;
2) The theoretical background we use to analyse the needs and the means of training;
3) Examples of situations (vectors and algebra);
4) Conclusion.

1. Alternate training and conceptions of young teachers

1.1. *Organization of the training*

Once they succeeded at the theoretical examination for teaching, whose content concerns only mathematical knowledge, French students are made responsible for teaching mathematics to a secondary school class, even if they have no experience in teaching, which is frequent. An older and more "expert" teacher has the responsibility for helping the trainee-teacher, and this one must learn from experience their elder's, by attending lessons in the tutor's class, and performing a few lessons under his/her direction. During nine months, novice teachers must also follow some fifteen days of training in one of the twenty-four national training Institutes (IUFMs). During these days of training, trainers of the Institute try to bring young teachers' conceptions of teaching to light and to make them evolve when desirable. The role of the training in the Institute is first to help new teachers to do their job: conceive and perform lessons of mathematics in front of pupils in a secondary school. But beyond that, the role of the trainers is to let the novice teacher build reflexive tools to analyse his/her practice and to improve it, assuming that "improving one's practice" has a clear meaning.

At the beginning of their careers, novice teachers can become aware of dysfunctions only by analysing the pupils' reactions: these can manifest in the form of inappropriate or quite unpleasant behaviour: noise, even the refusal to do the required work. So we see young teachers saying that pupils are lazy, because they do not want to do anything; but the teacher does not wonder whether the given work is interesting or not, or even if the pupils have any real possibility of doing it.

A lever to make the young teachers evolve is the degree of success they meet in their class, but it is difficult because it questions their practice in a very personal way especially if they don't succeed; and because they see no reasons for changing, if they believe they will succeed in a way that the training institution does not consider very pertinent!

A second part of the training has to supply the teachers with mathematical knowledge to help them understand the articulation between advanced mathematical notions and the contents of what they are to teach. This second component leads to revisit some mathematical notions, but differently from the way it was taught at the University: the aim is to make teachers see what a notion means, that is, which problems it allows to solve. A large part of this component of the training is to study teaching organizations, as Thales' Theorem, and try to analyse why it is difficult (for example, that it supposes continuity of real numbers, or that an homothetic path has a length of k the length of the first one, implies to know something about rectification of paths...)

A third component of the training will be the structuring of didactical knowledge, as far as it is possible and useful to future teachers, that in itself is a question of research.

1. 2. *Conceptions of young teachers about mathematics to be taught and ways of teaching*

a) Mathematics

Students often get a very formal conception of mathematics during their university courses, and they are not at all accustomed to solve problems with the mathematics they know. For them, a theorem has to get a proof, but no justification in terms of problem solving; it is seen as a part of a mathematical theory, which is its own justification. They have a very poor culture of problems to be solved with the mathematical tools they have studied at University; and, as many authors have pointed, their own mathematical knowledge is often inefficient (Robert, 2001).

b) Ways of teaching

What are the novice teachers' conceptions of the mathematics to be taught and on teaching practice ? They still keep the illusion that "a good course" of mathematics is done by a teacher in front of the students, and that the teacher "tells the law", that is, the mathematical law. They have no idea that this law could be contested, and no idea that the mathematical law could not be understood, overall, considering that only elementary mathematics are in question at that level. The mathematical formalism seems transparent to them, it is as if it was self-explaining. This is to say that they themselves hardly ever question mathematics, they are accustomed to take what the mathematics teacher said at University for granted and cannot imagine any other behaviour from the students in their own classes.

When these conceptions are brought to light, how can we work with them? The teachers' expertise includes two components (at least) : one of them involves education skills, and is related to vocational habits ; and the other gets an epistemological dimension.

It leads to make the hypothesis that it is necessary to offer novice teachers, both analysis of teaching practice (theirs and the experienced teachers') ***and*** new situations for their students, to make them question the way they are accustomed to be taught themselves. And it is also necessary to provide them with vocational knowledge to drive the situations we propose.

2. Theoretical background

2.1. Didactical constraction of students versus young teachers

At University, the students are not responsible for the mathematics they are taught, either in their dimension of proof or in the global organization of the course; when they become teachers, they have succeeded, so they think that their mathematical studies are achieved, but they know very little about how the mathematics they have learned can be applied at the secondary school. It is therefore difficult for them to get a critical and reflexive point of view on this mathematics. So they receive a real subjection to the didactical transposition of the mathematical knowledge at the secondary level[1].

[1] And young teachers want to become member of the institution "Secondary School", so they tend to be much more conformist than they are expected to, it is a well-known phenomenon of integration.

We saw that young teachers often think that mathematics reduce to the formal point of view; but the didactical transposition in the secondary school emphasizes the pragmatic point of view: mathematics are reduced to manipulation of semiotic tools[2]. The dimension of problem solving is not considered neither by one nor the other institution (the secondary school or the University). It is then difficult for a young teacher to imagine problems relative to a mathematical concept, and all the more problems that can be attainable by students at the secondary level.

Then the didactical contract of young teachers is characterized by :

- an illusion that manipulation of ostensive objects carries sense of mathematical notions ;
- a lack of knowledge about pertinent problems related to concepts taught at secondary level;
- an absence of means to take the responsibility for the mathematical organization of a long course.

How does this contract appear ? We can observe it through the tasks that the students and the teachers consider as theirs. When they are at University, students try to solve roughly the problems at the exams; they do not consider themselves responsible for the exact solution and it is more "profitable" for them to solve more or less numerous questions, than to solve only a little question in detail. It derives from the University's habit: University makes students frequent the mathematics more as large components of theories than exactitude of a closely defined research.

At the opposite, in his/her class the teacher is responsible for the mathematical exactitude, in terms of what is right and what is false. When novice teachers arrive in a training Institute, they very often refuse to write a complete solution of the exercise they give their students: they do not see the use of this work, neither do they see why they should correct an exercise's text.

Similarly, it is difficult for them to anticipate the planning of a week, a month, a term of mathematics with their class. It is also problematic to provide a series of exercises at a given level, because this work does not concern mathematical concept in a usual way (define a new concept, fit it in a well known theory ...), but it is a technical or technological work (Chevallard 1999), a new work : how to express *this* concept for *these* students ?

The didactical contract of the novice teachers must evolve to enable them to :

- organize the didactical time, on a short or long term, and define the objectives they want to reach ; control the schedule of the teaching/learning organization ;
- define the corpus of learning situations and exercises to offer to students, in order to study a mathematical notion, and to reach a given objective ;
- link how to teach and how pupils can learn, and give themselves means for assessment.

2.2. Theory of situations

The theory of situations has proposed some situations for primary school, but not so many for secondary school and college. At this level, the question is not to build ONE good situation (as for multiplication or proportionality - see Brousseau 1997) but to find collections of problems, activities for students that permit to explore the fundamental meanings of a mathematical concept. What we could expect from university knowledge was to enable young teachers to understand these fundamental meanings, but as we already said, young teachers do not know how to converse their formal knowledge into a problematic one.

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[2] On ostensive objects and semiotic tools, see Chevallard 1999.

Another question is the adidactical component of a situation, or, in other terms, the place for the work of pupils or students in the mathematics class (Bloch 1999). One reason for building complex situations is to try and broaden the role of pupils in mathematical research, and their confrontation to mathematical truth through debate.

With young teachers we make the hypothesis that it is possible to play with such situations during the training time, and the objective is :

- to provide them with robust situations to apply in their classes (that is, situations that permit a real mathematical work for their pupils, even if the teacher is novice) ;
- to enable them to make their own mathematical knowledge evolve, thanks to the interactions with situations.

It is also necessary to make the teachers analyse class practice: tools of the theory of situations are also used to analyse the teachers' practice, but this work is not presented here.

So :

a) We have to build situations that can be submitted to the novice teachers to help them understand the articulation between advanced mathematical notions and the contents of what they will be teaching themselves; and research shows that the mathematical knowledge of teachers evolves when they have left University: it broadens in a way, but it is used in restrictive domains, considering the field of university knowledge. What we want to do is to guide this evolution.

b) We try to introduce complex learning situations in the instruction of young teachers in order to enable them to teach in these situations; and we make the hypothesis that these situations are useful to make their mathematical knowledge evolve.

Then we can see that situations for teachers' education are built under a double constraint:

- first, allow the young teachers to question and broaden their mathematical knowledge, by a confrontation with the situation ;
- and, retain components that could be transferable in a class situation, and could be managed by a novice teacher. That is why it is also necessary to anticipate teachers' regulations.

To build situations that are relative to a notion we apply principles from the theory of situations :

- identify a game where the concept is pertinent ;
- make the main didactical variables appear and choose their value ;
- organize the game in two phases : a direct one and an inverse one, the last one being the only one that leads pupils to confront their action to the "milieu".

These conditions will be explained below.

2.3. Didactical knowledge to drive complex situations

What teaching knowledge is necessary to help teachers drive complex learning situations? And if the training takes care of this necessity, how to do it?

It is of course impossible, and would not be efficient, to try and provide future teachers with theoretical didactical knowledge out of a pertinent teaching context: didactical knowledge is always the synthesis of the observation and the analysis of precise situations.

When the teaching device is an adidactical situation, or a partially adidactical one, we analyse difficulties in driving the class, as related to the different phases of the situation: first, devolution of the problem, then, activity of the students, during which the teacher must adjust the work and collect the procedures, and finally, assessment of the best ways of success and institutionalisation of the aimed knowledge.

Organising the phases of a situation: what vocational knowledge is it and how can it be the object of the training work? We make the hypothesis that in this case, a theoretical knowledge is useful, even if not sufficient; and a "theoretical" course is planned, but it is based on realized sessions about well known situations (Bloch 1999, Bloch 2001). It is not presented here.

In the case of "ordinary" practice, the work is based on the analysis of practice, and regulations take place when the trainers of the Institute go in the novice's class to make a visit[3]. The emphasis is laid on the regulations that the teacher can anticipate, and on the place and ways of pupils' work.

3. Examples of situations: vectors and algebra

3.1. Product of vectors by real numbers

A situation to introduce the product of vectors by real numbers has been tested with both novice teachers and pupils. The aim is to build lessons on the vectors that permit to make the functionality of this notion appear in different kinds of problems. It consists of a game, whose support is a grid (see annex 1).

The game that is presented here is the inverse one. The direct game would be to design sums of vectors, and associate them the good points; it must be known before the inverse game, but it is much more common. The inverse game permits a validation, and above all, the pupils cannot succeed if they do not "put the good knowledge" in the game.

The game must be played with trainee-teachers themselves. Afterwards they can build another grid with numbers of points to define with vector equalities; the teacher can identify didactical variables and fix them (supports of the vectors being parallel to the edges of the paper or not; nature of the numbers – natural, rational, irrational ; number of vectors of the system : one – in which case some points cannot be reached, two – in which case all the points can be reached in one way, or three – in which case the points can be reached by different ways).

This situation has got two objectives for teachers:

- doing to understand that the concept of vector is the notion that permits generally to reach every point of the affine plan ;
- show how to build a situation on the link point-vector in such an environment (the grid) ;

and two objectives directed at working with pupils :

- working about the technique : calculate sums of vectors and find the corresponding points ;
- use the grid as a tool for validation in the class, and link this tool with others that will be met afterwards, as coefficient of a straight line.

This situation is particularly interesting to play with young teachers because they generally have a especially formal view of the linear algebra, and they tend to see the situation as quite a gap between what they are accustomed to do on vectors and what they are invited to work on with pupils in this situation. And moreover, they immediately get difficulties to see why the situation is adequate and to find the didactical variables. The idea that it is possible to "put on stage" the concept of basis of a vector space in such a way is very amazing for them.

[3] There are two kinds of visits : formative ones, and assessment's ones.

3.2. Algebra

The second situation is due to Boris Veron: its aim is to work on equations, and more generally literal expressions, considering the difficulties that pupils meet with the notions of variables and unknown quantities. The support is Fibonacci's sequences: the problem is to find one of Fibonacci's sequences, knowing only one of its terms, and its rank (e.g. the tenth term). It allows the introduction of parameters, as it is logical to "do the calculation once and for all". The good didactical variables are available, as to see that elementary algebra is not only a collection of rules, but that it permits to solve problems. Pupils must try to build Fibonacci's sequences, and find one sequence knowing some of its terms (see annex 2). The situation makes the main functionalities of algebra appear as Gascon (1994) describes it.

4. Conclusion

It is difficult to make an assessment of this training: it is not easy to evaluate the situations the trainee-teachers drive in their classes, when we visit them. There are numerous factors which can create conflicts when teachers plan a classroom-situation: even if the situation is interesting, their know-how can be too uncertain to permit the success.

Trainee-teachers get a questionnaire at the end of the training, and they mention that this work allows them to consider mathematics from another viewpoint: they see dimensions they ignored in their mathematical studies.

Anyway, we notice that almost all the trainee-teachers become able to try one of the studied situations in their classes, and this is an important result for us.

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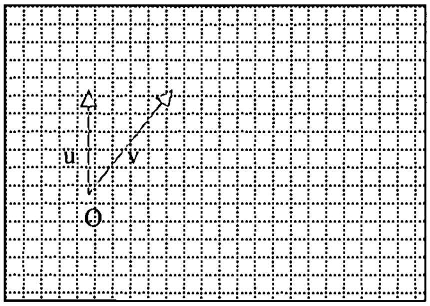
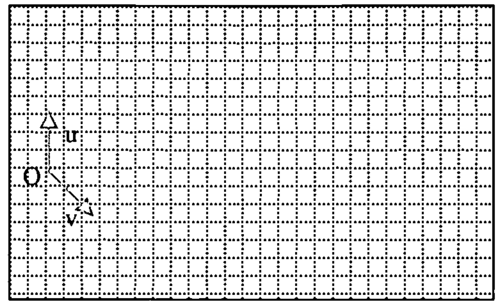
Annex 1 : The grid game

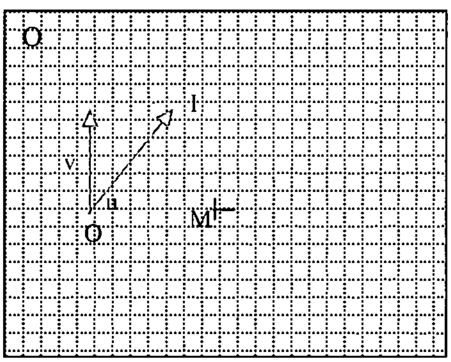
Game n°1
Grid for the receivers



Game n°2
Grid for the receivers

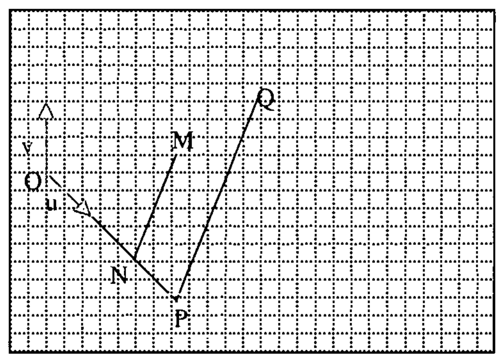


Game n°1
Grid for the transmitters



The other team has got the same grid as you, but with only the point O and the vectors **u** and **v**. Send them a message to place the point M. It is on the circle (O, OI) , and on a straight line orthogonal to v.
But you're not allowed to tell it in your message that must contain only O, **u**, **v** and numbers.

Game n°2
Grid for the transmitters



The other team has got the same grid as you, but with only the point O and the vectors **u** and **v**. Send them a message to place the point M. It is at a place so that (MN) // (PQ) and the points N, P, Q are exactly at crosses of the grid.
But you're not allowed to tell it in your message that must contain only O, **u**, **v** and numbers.

Annex 2 : The Fibonacci's sequences

The first phase (for the pupils) is to explain by examples, what a Fibonacci's sequence is. Then it is to induce them to find a Fibonacci's sequence, knowing its tenth term, or the fifth one.... It appears that there are many solutions. Then the question is : find one Fibonacci's sequence, knowing the first *and* the tenth term. The pupils must put the problem into an equation. This leads them to name x the second term of the sequence and resolve equations, and this work shows the functionality of algebra to solve such problems. Moreover, the calculation of numbers of such Fibonacci's sequences leads to the question: could one do the calculation once and for all? Then the calculation changes its signification, the aim is to show that such a sequence depends only on the first two terms, a and b, seen as parameters (and the underlying structure of vector space is present of course, even if it is not the object of the pupils' work).

But it is possible to go on with this algebraic work, since some questions lead to inequalities: how can one be sure that the terms of the sequence are positive integers ? And this permits further work with systems of equations.

First phase:

Find a Fibonacci's sequence such that:

2	5	7	12						212

Second phase:

Find a Fibonacci's sequence such that:

									178

						51			

									301

Third phase:

Find a Fibonacci's sequence such that:

7									45

9							241

8					77

This leads to name x the second term, and the equations with x have positive or negative coefficients, which is interesting for the algebraic work.

Calculation "once and for all":

a	b	$a + b$	$A + 2b$	$2a + 3b$	$3a + 5b$	$5a + 8b$	...

Possible questions: is there a formula for the nth term? How many Fibonacci's sequences are there, with positive integer terms, and when the tenth term is given?

USING INTERACTIVE DIGITAL VIDEO AND MOTION ANALYSIS TO BRIDGE ABSTRACT MATHEMATICAL NOTIONS WITH CONCRETE EVERYDAY EXPERIENCES

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ABSTRACT

In an attempt to offer a means for better visualization and conceptualization of abstract mathematical notions, we investigated how the analysis of motion contained in a digital video performed in a special computer software environment, can help students increase their understanding on specific topics. Previous research on Digital Interactive Video Technologies (DIVT) was limited to the domain of kinematics and graph interpretation in particular. It was the conviction and in some cases the conclusion of those researchers that students would benefit more from the study of everyday motion as presented in a video, rather than in simulation software. We believe that this is particularly true in the case of mathematics teaching, where students often have difficulty in perceiving the meaning behind an algebraic or graphical representation. Pre-service teachers need to gain a profound understanding on such abstract concepts, as those are usually the ones they have more difficulty teaching. This pilot study is part of a full-scale research that aims to 1) extend the field of investigation using Digital Video Technologies as a connecting link for the Integration of Mathematics and Science, 2) investigate how different dynamic software environments that offer advanced visualization options affect students' learning of mathematics. This paper is the report of the first part of the pilot-study, where the main aspects of teaching with the aid of DIVT were investigated. Five pre-service teachers participated in this study, which consists of two parts, one without and one with DIVT support. The analysis of data gathered indicates that being able to manipulate the reference frame in the environment of the DIVT software and notice how it affects coordinates, graphs and equations of motion, had the greatest impact on the pre-service teachers' understanding on this subject.

KEYWORDS: Interactive Digital Video, Video-Based Laboratories, Motion Analysis, Coordinate Systems, Graphs

1. Introduction

A major problem in the teaching of Mathematics is finding efficient ways of presenting abstract mathematical notions. Sometimes it is hard enough to introduce and explain the definition of an abstract concept alone. The in-depth discussion of such concepts and the revelation of their properties is a task that usually requires advanced visualization and conceptualization strategies that comply to the way students build their network of concepts. What seems natural in developing these strategies is to create a bridge between the world of the abstract (formal mathematics) and the everyday world (the experiences of a person). With the development of technology and its implementation in education we are constantly discovering new ways to teach abstract ideas and support the above statement. As Kaput (1994) notes in a paper discussing the use of technology in connecting mathematics with authentic experience: "The new availability of interactive and representationally plastic media makes possible a wide variety of operative action representation systems, such as coordinate graphs, that can now be manipulated as if they were physical objects. Thus the move of operative symbolism that led to the scientific revolution becomes newly available to enhance the intellectual power of all manner of representation systems".

The term Interactive Digital Video, as used in this paper, refers to computer software tools that allow viewing a movie in digital video format and analyze the motion presented in that movie. Interactive Digital Video and Motion Analysis have mainly been used in the teaching of kinematics in physics, but researchers have also noted the potential use of this technology and related techniques in an interdisciplinary manner to teach Mathematics with the aid of Physics and Technology.

2. VideoPoint and Motion Analysis

Several computer programs have been developed in order to analyze motion presented in videos for the teaching of mathematics and physics (Boyd and Rubin, 1996). The use of this software in the environment of the classroom is most usually referred to as Video-Based Labs or VBL. The program we have chosen to use is VideoPoint by Lenox Softworks. Using VideoPoint, students are able to view videos of motion events and then analyze that motion.

In a motion analysis students begin by marking with the mouse cursor the position of a moving object(s) in successive frames. There is a primary "Reference Frame" present in all video frames and all coordinates are measured with respect to this frame. The collected (object) coordinate data are automatically stored in a table together with the time value attributed to the corresponding frames (Figure 1 – Appendix A). VideoPoint allows dynamic manipulation of data, in the sense that any changes made by the students to object coordinates are automatically updated in the table. Furthermore, they can move or rotate the Reference Frame at any time and view simultaneously how the coordinate data in the table change. This feature offers students a visualization of the abstract concept of Coordinate Systems.

The second step in the motion analysis is the process of setting a physical scale for the movie. So, the position data, which were initially measured in pixels, can be measured in standard units of distance, such as meters, inches or feet. To scale a movie, it is enough that students inform the software about the real length of an object in the desired units.

The next step is usually the construction of graphs corresponding to the time evolution of several physical quantities, such as distance, velocity, acceleration, force, energy, and momentum. Students need only to determine the quantity they wish to display on the horizontal and vertical axis. Subsequently, VideoPoint creates the graph and students are able to change the way it looks in several manners, including its size, the symbols used, and the region being plotted. They can also manipulate the Reference Frame and examine how the graphs are affected. Furthermore, they can try to guess the function that would produce such a graph and compare it to the real graph, or they can directly fit the best curve that matches the graph. There is also the possibility of displaying several quantities in the same graph, e.g. the horizontal component of velocity of two objects versus time (Figure 2 – Appendix A). Another interesting feature is that students can use several Frames of Reference to analyze motion. These Frames of Reference can be either stationary or moving.

Of particular interest is the case where one of the moving objects is selected as the origin of a Reference Frame. Students have the opportunity of investigating how the coordinate data in the table and graphs change when they are measured with respect to a moving Frame of Reference. This technique can offer a better understanding on how coordinates, the shape of a graph, or the equation describing a trajectory, depend on the position of the origin and the orientation of the Coordinate Systems in which they are measured. Teaching cycloid motion with VideoPoint is a good example of how this technique can be utilized. Students can examine the motion of a marked point on the tire of a bicycle as seen by an observer standing on the street or as it would appear to an observer located at the center of the bicycle wheel.

3. Research with VBL – Review of Literature

Research based on VBL has mainly focused on the field of kinematics in introductory physics courses and kinematics laboratories. Beichner (1990 and 1996) has conducted extensive research using software that he designed for this purpose, which is very similar to VideoPoint. His work has mainly focused on student understanding of kinematics graphs. The results of his research indicate that when VBL are integrated in the curricula to an extensive degree then student understanding of kinematics graphs is improved.

In research on VBL, students' misconceptions have been yet another major subject of inquiry. Zollman and Brungardt (1995) focused on students' misconceptions with kinematics graphs and on the way the simultaneous-time presentation of the graphs and the motion event can help them deal with those misconceptions. Their results however revealed that there was no difference in achievement of students using this method, but there was change in terms of student motivation. However, because of the small size of the sample used these results could not be over-generalized and further investigation is necessary.

The innovation of Andrew Boyd and Andee Rubin (1996) compared to previous research on VBL was the use of Interactive Digital Video clearly as means of bridging motion to mathematics. They focused on making connections on how students perceive and/or experience motion in every day life and motion as presented mathematically in graphs and tables. They investigated how students create their own graphs modeling real situations. Digitized video of these situations helps them to revisit and reflect on an object's motion.

4. Our Research – Research Goals

The aim of our research is to extend previous work by adopting a multidisciplinary approach of motion analysis with interactive digital video, for an integrated teaching of Mathematics and Physics. In particular, we wish to investigate students' interpretation of a numerical table of coordinates as a representation of a real motion event, graph understanding and the role of the

Frame of Reference in analysis of motion. These concepts belong both to the domains of Mathematics and Physics. Our hypothesis is that utilizing the methods of both Mathematics and Physics teaching in an integrated activity will lead to increased student understanding of those concepts. Furthermore, we expect that students would be more motivated than in traditional teaching.

5. Research Design

This research was designed as a pilot project of a forthcoming full-scale research on the subject of Integrating Mathematics, Physics and Interactive Digital Video Technologies. The activities and questions used were designed so that the following topics would be mainly investigated: Tables and Numbers, Graphical Representations, Coordinate Systems and Frames of Reference.

Five case studies were conducted with pre-service teachers, students of the Department of Primary Education of the University of Ioannina, three males and two females. Two of them had no computer skills, two had a few and one was an advanced computer user.

The case studies consisted of three parts. In the first part the students were asked to fill in, within an hour, an initial questionnaire that was designed to investigate their skills prior to interaction with VideoPoint. The second part consisted of one to three meetings (depending on their skills and performance during these meetings) where they performed two activities with VideoPoint. Finally, at the third part of this investigation the students answered a modified version of the initial questionnaire this time using VideoPoint. The need for more than one treatment meeting has been documented (Beichner, 1996), as a single treatment meeting cannot produce the desirable change in student understanding.

The questionnaires were based on a movie that showed three moving objects. A screenshot of that movie is displayed in Figure 3 (Appendix A).

6. Results and Discussion

The most important findings and observations based on the pre- and post-questionnaire are summarized in Table 1 (Appendix B).

The analysis of the results is based primarily on students' answers on the pre- and post-questionnaire and in part at the notes kept during the sessions with the students. The analysis is presented in the following nine sections. The first eight sections correspond on the eight findings presented in Table 1. The last section (number nine) presents the analysis of results concerning student motivation and interaction with technology as a teaching medium.

1) Prior to using VideoPoint, students were asked to read a table which consisted of 20 measurements of the x and y coordinate (taken every 0.1 seconds) of three objects moving simultaneously (Figure 4 – Appendix A). In four out of five cases the students tried to estimate the rate of change for x and y, even though that was not requested. Their estimations were either based on mental or written calculations. In the case of written calculations no more than two pairs of numbers were used.

This effort to perform mental or "rough" calculations has been abandoned after they had interacted with VideoPoint in three out of four of the cases. It is our hypothesis that VideoPoint helped them realize that the data contained in the table were not just a collection of numbers but were representing quantities with physical meaning. Using VideoPoint a link was made, between the data in the table and the real motion event. Thus, they realized that it was not possible to make "generalized" or "rough" comments about those quantities, as they concerned the actual motion of the three objects. So, the chance of making the wrong assumptions about their rate of change would now be of significance and not unimportant as in the case of 20 numbers of no meaning.

2) Prior to using VideoPoint, students were asked to identify the points of intersection of three curves displayed in one graph. Only one out of the five was successful in identifying the two intersection points of these three curves. Two more were partially successful as they identified one of the two intersection points. The remaining two were completely unsuccessful. It is very interesting to observe that these two had successfully completed this task after having interacted with VideoPoint!

One possible explanation for this observation is that the "mental" interpolation of the three curves is successfully performed after using VideoPoint, because students had a more concrete and uniform image concept regarding the motion of the three objects.

3) Extending observation (2), we see that only two out of the three students that were able to identify one or two intersection points of the three graphs, named those points, using either their coordinates on the given graphical representation or by associating them with the corresponding video frame. Being able to name a point on a coordinate system is an important task that students should master after being taught coordinate systems. The remaining three students that were not able to name the intersection points prior to using VideoPoint are successful after having used it. This was an expected difference in performance. The obvious justification is that of VideoPoint's dynamic feature of displaying the coordinates (in relation to the given graphical representation) of the user's mouse index when it is within the boundaries of a graphical representation.

4) In the case of three moving objects a graph of the x-coordinate of velocity versus time was given to the students. As two of the objects were moving to the opposite of the positive direction of the x-axis of the Frame of Reference their algebraic value was a negative number.

Students were asked to describe the motion of the three objects by interpreting the meaning of the negative values for velocity. We did not receive satisfactory answers to this question. Though, an interesting reaction was that some of the students noticed that in fact one of the three objects did not have a negative velocity.

In particular, prior to interacting with VideoPoint only two out of the five students noticed that one of the objects did not have negative velocity and marked this on their questionnaire. After interaction with VideoPoint, the ratio has gone up to four out of five. The remaining one student showed no difference prior to and after using VideoPoint.

As the graphs presented to the students prior to using VideoPoint were identical to those provided by VideoPoint in the post-examination, we cannot attribute this change to any of VideoPoint's features. Rather, we can assume that it was the whole activity design that made a difference. Students participating in this activity have a more active role than the one they have when answering a questionnaire on paper. Because of this, we believe, that they were more motivated and more concentrated in their work. The result of placing students in a more active role-control is that they behave as if they are working on a project of their own and not taking some sort of examination. This made students more cautious and suspicious regarding the information given to them.

5) When students were asked to answer questions based on graphs some of them consulted the table in order to provide an answer. This unexpected behavior could indicate that perhaps VideoPoint did not help students improve their understanding of graphs and they resort to the table in order to answer. Another possible explanation, though, is that VideoPoint operates as a link between the different representations of the table and the graphs. Students realize that the table and the graphs both represent a mathematical expression of the motion they observed, so they decide to consult any of the two representations when providing an answer as they are now convinced that they are two versions of the same thing: In both cases they see a mathematical expression of a motion event. In any case, this observation needs to be further investigated.

6, 7, 8) We will explain the change in achievement regarding observations 6, 7 and 8 together, as we believe that it is due to the same reasons.

It is obvious from these three observations that VideoPoint has the potential to make a big difference in students' understanding of the concept of the Frame of Reference. Students were much more successful when answering questions regarding the role of the system of reference after using VideoPoint than before.

Based on the results of the pre- and post-questionnaires, but mostly on the interviews, we could claim that the reason for this change is the dynamic nature of the Frame of Reference in VideoPoint. The Frame of Reference as presented mathematically, is an abstract concept that cannot be conceptualized unless it has been visualized in a drawing representing a motion event. Thus, comprehension derived from this visualization is not enough to provide students with the ability to make predictions of how equations of motion, graphs and coordinates would change if the Frame of Reference were to rotate or/and change position. VideoPoint may serve as a means for an advanced conceptualization.

In VideoPoint the Frame of Reference is a notion of dynamic nature. Students can manipulate it at will and whenever they want and observe how coordinates, graphs, and equations of motion related to it are updated.

It is our hypothesis that being able to "experiment" with the Frame of Reference and observe the change it causes to coordinates, graphs and equations of motion, enhances students' conceptual knowledge on this subject. Students realize that there is a dynamic link between the Frame of Reference's position and orientation and the way that graphs and tables of coordinates look. Furthermore, by bringing the Frame of Reference to particular positions of "special" interest, such as positioning one of the axes to be parallel to an inclined level, or bringing the x-axis vertically and the y-axis horizontally, they can deal with misconceptions and gain a better understanding and insight to the role of a Frame of Reference.

The students that took part in this research seemed to particularly enjoy the part of moving and rotating the system of reference and noticing the change it causes to the graphs and tables. Most of them made remarks on this that indicate some sort of "insight" regarding this topic when it was demonstrated to them for the first time.

9) At the beginning of this research there was some concern regarding students' familiarity with computers. Only one out of five students had advanced computer skills. Two had a few and the remaining two had none. It is very encouraging to see that at the end of the activities all five had almost mastered the skills required to use VideoPoint. They could all run VideoPoint, open a movie, collect data, scale the movie, read tables and create graphs. As two of the students struggled with the use of mouse at the first activity it is amazing that after a maximum of five hours they were able to successfully perform the above tasks on their own. Students were themselves surprised by how well they performed on the computer, which increased their self-esteem. They confessed that they had never thought they could do work on the computer so easily. We consider that this feeling of success was a major factor for the increased motivation that they displayed throughout the activities.

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Appendix A – Figures

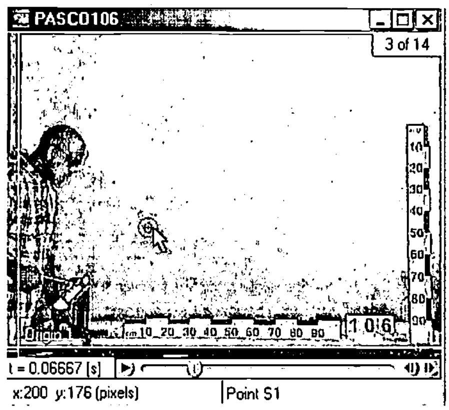

Figure 1

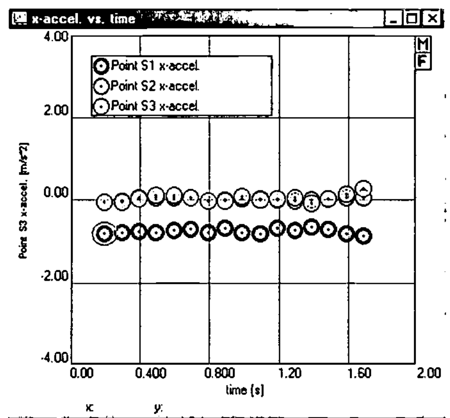

Figure 2

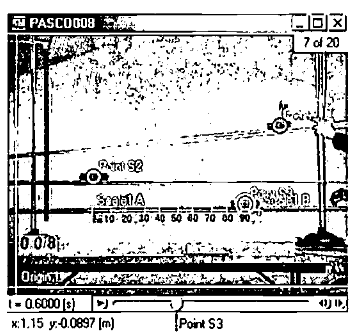

Figure 3

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	Point S3		Point S2		Point S1		
	time [s]	x-pos [m]	y-pos [m]	x-pos [m]	y-pos [m]	x-pos [m]	y-pos [m]
1	0.000	1.678	0.4904	0.1720	0.6433	1.599	0.9490
2	0.2000	1.592	0.4873	0.3312	0.6401	1.564	0.9427
3	0.4000	1.510	0.4873	0.4841	0.6433	1.513	0.9395
4	0.6000	1.420	0.4904	0.6338	0.6433	1.459	0.9331
5	0.8000	1.338	0.4873	0.7803	0.6433	1.398	0.9268
6	1.000	1.255	0.4873	0.9299	0.6465	1.331	0.9236
7	1.200	1.169	0.4873	1.070	0.6433	1.258	0.9140
8	1.400	1.086	0.4841	1.220	0.6465	1.182	0.9076
9	1.600	1.006	0.4841	1.360	0.6433	1.099	0.8981
10	1.800	0.9236	0.4841	1.506	0.6433	1.013	0.8917
11	2.000	0.8408	0.4841	1.646	0.6433	0.9236	0.8854
12	2.200	0.7643	0.4809	1.796	0.6401	0.8312	0.8758
13	2.400	0.6783	0.4841	1.939	0.6433	0.7325	0.8662
14	2.600	0.5987	0.4809	1.971	0.6401	0.6306	0.8567
15	2.800	0.5255	0.4809	1.981	0.6369	0.5255	0.8503
16	3.000	0.4459	0.4809	1.975	0.6401	0.4140	0.8376
17	3.200	0.3662	0.4809	1.978	0.6369	0.3057	0.8280

Figure 4

Appendix B – Tables

Table 1

			Subject 1	Subject 2	Subject 3	Subject 4	Subject 5
1	Using only 0-2 random pairs of numbers to find the rate of change of the data for a list of 20 numbers, even though it was not requested	Pre		●	●	○	○
		Post			●		
2	Ability to identify points of intersection of two or more curves on a graph	Pre	○		●		○
		Post	○	●	●	●	○
3	Ability to name points of intersection of two or more curves on a graph	Pre	●				●
		Post	●	●	●	○	
4	Comparing – crosschecking the truth of information given at the questionnaire with the graphs displayed	Pre	●				●
		Post	●		○	●	●
5	Using the corresponding data table to answer questions about graphs	Pre					○
		Post		●		●	○
6	Prediction of the consequences of a parallel transportation and a rotation of the Frame of Reference regarding position	Pre	●		●		
		Post	●	○	●	○	○
7	Prediction of the consequences of a parallel transportation of the Frame of Reference regarding velocity	Pre					
		Post	●		○	○	
8	Prediction of the consequences of a rotation of the Frame of Reference regarding velocity	Pre				○	
		Post	○	○	○	○	●

● Represents success **O Represents partial success**

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TEXT STRUCTURES IN TEACHING MATHEMATICS

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ABSTRACT

Any good learning material must try to anticipate the learner's problems. The author should take into account that the reader is not with him and understanding his good intentions.

Any given text can be understood as an alphanumeric string that is a rather annoying structure. We can distinguish three dimensions of the text: line, column, and the block one. There are usually many internal relationships between parts of the string.

The transformation between linear and structured text can be explained as two opposite processes: aggregation and decomposition. Natural destruction of the text linearity can be applied to implications, classifications and parallel formulations. The modern word editors offer a large amount of possibilities for structuring texts.

For several years, the Department of Mathematics of the TU in Liberec tries to observe the influence of a mathematical text written in structures on an acceptation of lectures and textbooks. The research started in 1999 and continued in 2000 and 2001 with the goal to verify what type of a mathematical text is better for students – classical linear or structured. The results of the student's polls are presented and discussed which were passed through in exercises of Mathematics. Hundreds of students of five faculties at the TU in Liberec participated in them. The last polls of our research show a shift in the direction of the structured form. According to the student's answers the structured versions of the text are appreciated. We could also read many remarkable, wonderful answers in the student's questionnaires. It could be very interesting for psychologists and pedagogues.

Keywords: Text structures, aggregation, decomposition, students' poll.

1. Introduction

Motto
One structure is better than a thousand of words.
(Paraphrasing Confucius 552-479 B.C.)

In the last ten years of the twenties century, the possibility (sometimes necessity) of the lifelong learning has started to leak in the awareness of the Czech public. The distance learning became the modern form of knowledge acquisition. Hand in hand with it, the progress of the undergraduate mass education or self-learning in mathematics took part in the Czech education system. Transforming the usual full-time studies to the distance learning the requirements to the intelligibility and suitability of learning materials are increasing rapidly. It is clear that many specific features of open learning material could be used for the full-time studies too. From the psychological view, it is evident that a book-like text can attract readers by its design, size (numbers of pages), a graphical form etc. – shortly by its presentation.

Deciding between two textbooks with almost the same contents, a student/reader will choose inadvertently that book which is written in the more readable and understandable form. (Evidently, there is a difference between expert and student decision-maker.) The famous Confucius (552-479 B.C.) saying, "One picture is better than thousand words", stresses the importance of usually neglected attribute of information (esp. textbook), i.e. of its structure and graphical presentation. Everybody knows that the first-quality textbooks and all learning material, the strong basic literature, the brief and effective textbook, a detailed commentary to solved examples, this all can help students to deal with their studies easier and more effectively.

The development of the distance learning at the Technical University in Liberec induced the necessity of writing of mathematical texts several years ago. That is why some teachers of the Department of Mathematics and Didactics of Mathematics of the Faculty of Education started to observe/examine influence of mathematical texts written in structures on an acceptation of lectures and textbooks. It is a well-known fact that the reading of a mathematical text is for non-prepared readers generally and objectively difficult. We have investigated some graphic arrangements emphasising composition of the text and influencing the efficiency of learning. Similar principles could be used for an arbitrary vocational text.

Frequently external observers think that a typical mathematical explanation is of the form "definition – theorem – proof" with prevailing linear writing. But the practice shows that it is more suitable to state a well arranged summary of properties, a summary in tables, mini-graphs etc.

We can look at any given text from three dimensions - line, column, and the block one, and find relationships between them. This text can be structured along the string, across its lines, and on the long distance (between blocks). There exists a transformation between linear and structured text from this point of view. This transformation is based on in principle two opposite processes - aggregation and decomposition (see [Vil]). The structuring of a text means the usage of a natural destruction of the text linearity to emphasise differences, classifications etc. The means may be as standard (tables, Cartesian products of small sets, trees, graphs and mini-graphs etc.), as well as not so usual – different levels of formulations, parallel and/or alternative formulation (the so-called "storey notation") in definitions and theorems, accompanying solutions by intermediary comments, and so on.

The following examples demonstrate the difference between linear version and structured one.

Classical linear version [ThFi-90:189$_{8\text{-}6}$]

Second Derivative Test for Local Maxima and Minima
If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

Commentary to C-version:

This is a typical partly formalised linear text where both of alternatives are in series.

Structured version with miniatures:

Second Derivative Test for Local Maxima and Minima

If $f'(c) = 0$ and $f''(c) \left[\begin{array}{l} < 0, \\ > 0 \end{array}\right.$ then f has a local $\left[\begin{array}{l} \textit{max}\text{imum at } x = c. \\ \textit{min}\text{imum} \end{array}\right.$ 

Commentary to the S-version:

The linearity is survived only at the headline, the following two rows are rewritten almost table-like. The differences and alternatives are column close, they are seen at first sight. The miniature gives the eyes view.

The following (see [ViBi2]) deals with overview of possibilities of a function with respect to real/complex arguments, the number of its variables and range of it shortly and well-arranged.

The structure is Cartesian-like combination of adjectives {real; complex} and dimension $\{1; n\}$, giving 16 possibilities, these can be rewritten as a structure in only two lines:

$$\begin{Bmatrix} (\text{real}) \\ \text{complex} \end{Bmatrix}\begin{Bmatrix} (\text{scalar}) \\ \text{vector} \end{Bmatrix} \text{ function of } \begin{Bmatrix} (\text{real}) \\ \text{complex} \end{Bmatrix}\begin{Bmatrix} (\text{scalar}) \\ \text{vector} \end{Bmatrix} \text{ variable.}$$

The usually left out adjectives/parts are in braces. The symbolic version for a function f with the domain $D(f)$ and range $H(f)$ can be concentrated into the following schema:

$$\begin{Bmatrix} \mathrm{R} \\ \mathrm{C} \end{Bmatrix}\begin{Bmatrix} ^{(1)} \\ ^{n} \end{Bmatrix} \supset \mathrm{H}(f) \xleftarrow{f} \mathrm{D}(f) \subset \begin{Bmatrix} \mathrm{R} \\ \mathrm{C} \end{Bmatrix}\begin{Bmatrix} ^{(1)} \\ ^{n} \end{Bmatrix}, \qquad 1 < n \in \mathbf{N}.$$

Python-like structure can be seen from time to time in textbooks – e.g. [MV-95:$261_{2\text{-}1}$]:

$$\text{Die Summe aller } \begin{Bmatrix} \text{auf einen Knoten} \\ \text{in einem Stab} \end{Bmatrix} \text{ wirkenden Kräfte ist Null.}$$

This can be applied in analysis where the following notion nest is discussed:

{(two); one}-sided (im)proper limit approached
from {(two); one} side(s) at an (im)proper point.

$$\begin{Bmatrix} \text{(two-sided)} \\ \text{one-sided} \end{Bmatrix}\begin{Bmatrix} \text{(proper)} \\ \underline{\text{im}}\text{proper} \end{Bmatrix} \text{ limit approached from } \begin{Bmatrix} \text{(two sides)} \\ \text{one side} \end{Bmatrix} \text{ at an } \begin{Bmatrix} \text{(proper)} \\ \underline{\text{im}}\text{proper} \end{Bmatrix} \text{ point.}$$

[IrRo-98:$51^{11\text{-}13}$] Hidden trichotomy (and three valued range).

Definition. The symbol (a/p) will have the value 1 if a is a quadratic residue mod p, -1 if a is a quadratic nonresidue mod p, and zero if $p \mid a$. (a/p) is called the *Legendre symbol*. ♠

In other books the three valued range is made clear and the meaning of the symbol commented. (Note. The fork with three teeth replaces the former parentheses.)

[Kob-98:$43_{13\text{-}7}$] Clear-cut trichotomy (and three values).

The Legendre symbol. Let a be an integer and $p > 2$ a prime. We define the *Legendre symbol* (a/p) to equal 0, 1 or -1, as follows:

$$(a/p) = \begin{cases} 0 & \text{if } p \mid a; \\ 1, & \text{if } a \text{ is a quadratic residue } mod\, p; \\ -1, & \text{if } a \text{ is a nonresidue } mod\, p. \end{cases}$$

Thus, the Legendre symbol is simply a way of identifying whether or not an integer is a quadratic residue modulo p. ♠

Commentary.

- the almost perfectness is disturbed only by a small *aaa* collision, distinguishing by italics only is often not sufficient.

2. The Realisation and Evaluation of the Research

The research started in 1999 with the goal to verify what type of a mathematical text is better for students – classical linear or a structured one. Writing the textbook for students we would also like to know the students' view. Therefore we prepared four students' polls to verify our hypotheses. These hypotheses were drawn from the long-term experience of significant psychologists and pedagogues, and also from our own practice, passed through in mathematical exercises at several faculties of our university. The first period of this research consisted of four parts and the end of it was in 2001. At the present times we continue in the second period. We show results and opportunities of the first period. The tenets and opinions of the students in four polls are presented. The discussions were organised in exercises of Mathematics. Hundreds of students of five faculties (three of them technical ones) at the TU in Liberec participated in them. The third (last 1999/2000) and the fourth polls of our research show a shift in the direction of the structured form. According to the student's answers the structured versions of the text are appreciated.

We prepared three different topics of Mathematics in classical versions and structured ones to verify type of mathematical text that our students prefer. The first theme was "mapping" (surjection and injection), the second one was "countable and non-countable sets", and the third theme was "Ratio Test and Root Test for number series". We also prepared a questionnaire for students. At the beginning of the lesson the students were divided to 2 groups – Structured and Classical. Then they got the questionnaires with empty upper parts where they wrote their answers to a task written on the blackboard. After 5 minutes they cut off these filled in parts of the questionnaires and gave them back to the teacher. If a subject matter was new and/or the students did not know it, they would give back empty papers. Immediately the teacher gave to each student one of two versions of the research text, structured or classical. After 10 minutes of their studying, the students completed the questionnaires. Then the exercise was running according to normal programme. In the last ten minutes, the students were asked to answer the same task as at the beginning of the lesson on the opposite side of their questionnaires. The teacher gathered them in 5 minutes, thanked the students for their favour and explained their prospective questions.

First of all we tried to present the pre-test to one group of students of the Faculty of Education to verify our questionnaire and the timetable. Then the first part of the own research took place in January 2000 at the Faculty of Mechanical Engineering (174 students) and the Faculty of Textile Engineering (212 students). First of all we wanted to obtain characteristics of students (types of secondary schools, level of their mathematical knowledge etc.), to obtain what forms of mathematical notation they prefer, whether students are able to read mathematical (and/or an arbitrary vocational) text. We wanted also to inform students with the intention of this investigation. It was very important for us to know if students would prefer a graphical emphasising. That is why we chose the theme "mapping" well known from secondary schools and why we did not await considerable improvement. In the second poll (April 2000) students should study the definition of a (un)countable set. This topic was new for almost everybody and so the results were more credible. 241 students of technical faculties took part there. The third theme was investigated in May 2000. Students should acquaint with the Ratio Test and Root Test (105 students).

The experience with filling in questionnaires in the first poll was used to modify the questionnaire in the following polls. Several of them were impossible to evaluate. It was also a sorrow for us to find out that some students were not able to read any mathematical text at all regardless of its style. We did not find essential differences between the structured and classical groups in the first poll. Students must get used to mathematical notations and formulations independently on type of writing. It depends on the type of finished secondary school. Some students are not able to describe a term known from a secondary school and repeated in the first semester. There were cases when students saying, that this term is new for them, tried to formulate an answer before studying the given text. The second task was more interesting for them (we hope so) because it was something new. Many of students tried to explain these terms intuitively according to their names only. However the third poll passed through in accord with our expectations, although seven groups took part in. Several teachers had to finish classification and evaluation of students in the end of the semester.

Analysing the tests only, we see that the second poll does even not show essential differences between the classical and structured variants. However according to the student's answers in their questions, the positive evaluations of the structured text have done. Looking at the "improvement" graph of the third poll of our research, a shift in the direction of the structured variant can be seen. Many students did not answer the first test but then they tried to formulate it. The most of them have got better in the second answers (the improvement about 5 points) but there were students who did not answer again. They said they had been tired. This theme was new for most of them and they did not want to study it after a semester test (students of two groups). It was very interesting that one of participants of our poll, who was in the classical group, used the structured form in her/his answers.

In 2000/01 (the fourth poll), we investigated somewhat-different view. Suppressing the concrete text in the Ratio Test and Root Test, we prepared three versions with them – all in the classical and structured form. One pair of them was without any background, one with an unmarked background, and the last pair had got the marked background. We wanted to obtain any information whether our students would choose the presented text on account of its subject only, and/or they look at the form of it. We expected influence of computers, websites etc.

The questionnaire contained also parts examining frequency of reading and browsing in websites. About 160 students took part in this poll.

The presented graphs express several views in this problem. Looking at the figures (see Appendix) we obtain the first information that our students prefer structures in texts. This evaluation corresponds with the Czech school scale (number 1 is the best). It means they like to study texts with appropriate applications of storey structure, and other underlining means etc.

The Fig. 2 shows the view according to investigated properties and also the ratio of linear or structured variants are seen there. We were interested in five (subjective) aspects – how is the given text (Classical and Structured) intelligible, objective, well arranged, in the ability to remember contents, and its aesthetics. Fig. 3 touches the fourth poll. Let us notice that lots of students prefer a simple background on the given text (an influence of websites?).

Fig. 4 shows the number of hours per week spared to Internet. Several interesting facts can be drawn. Our students think that more complete, lasting and also detailed knowledge can be obtained from books compared with Internet (Fig. 5). More than one third of students had read less than 100 books (fictions) during their whole life (Fig. 6). The minimal number of these fictions is one, maximal number 3 859. Minimum of vocational books during the whole life is also one, maximum 150 (15 in a year). Fourteen students dared to say that they had read more vocational books than fictions.

These graphs present the pilot view in this problem. It is necessary to elaborate assembled data and make the final evaluation more detailed. By that time we hope that the results of our poll will be useful not only for writing texts.

3. Conclusion

The modification of the text to be structured is asked at a practical view. We are going to realise further polls for students of the second years. They are more experienced not only in mathematics but also in other (special) subjects. We are convinced that students are able to understand how to read a structured text. Then they will appreciate its advantages. However, this process is long-termed, it needs teachers' systematic influence and students' practice.

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Appendix – Illustrations of the Pilot Evaluation (Students' Polls)
Technical university in Liberec, Czech Republic

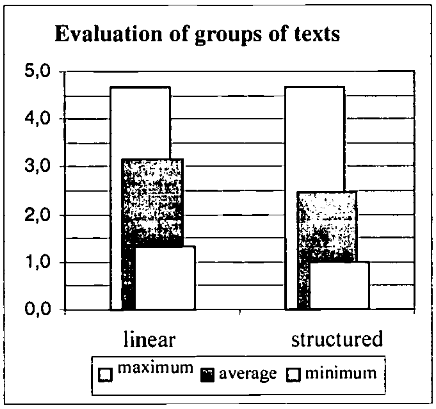


Fig. 1 – Students' evaluation of linear and structured versions of a text

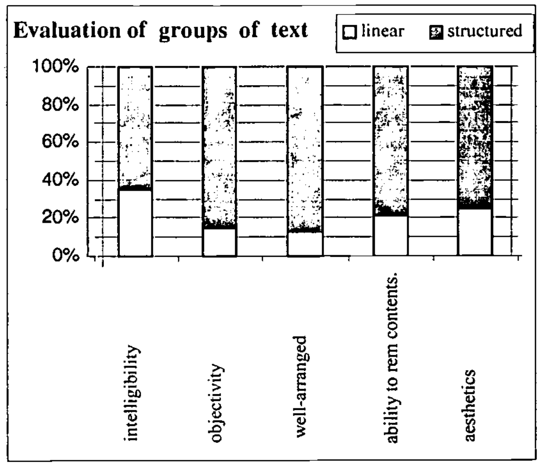


Fig. 2 – 5 aspects of the presented text

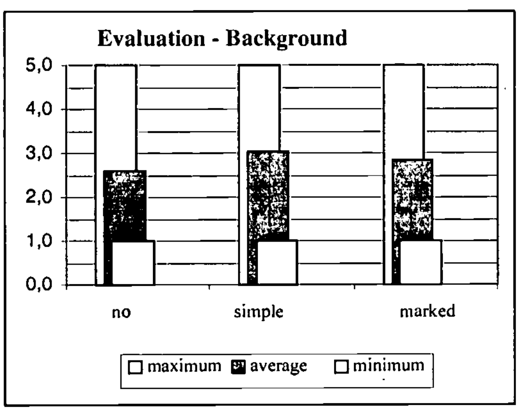


Fig. 3 – Evaluation of linear and structured versions in 3 variants (no background, a simple background, and a marked one)

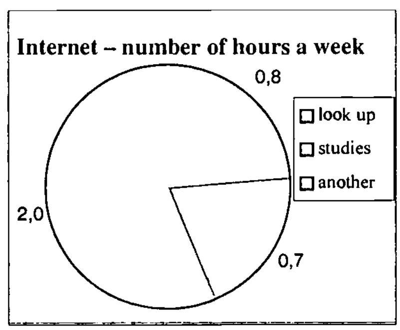


Fig. 4 – The number of hours passing at Internet

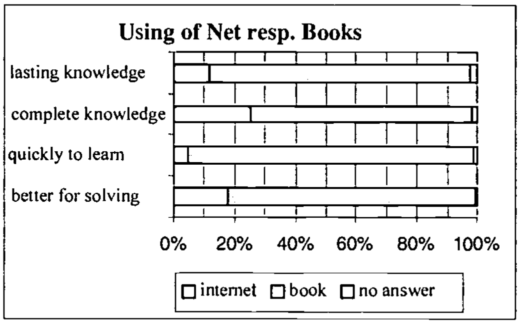


Fig. 5 – Properties appreciated by students using net or books

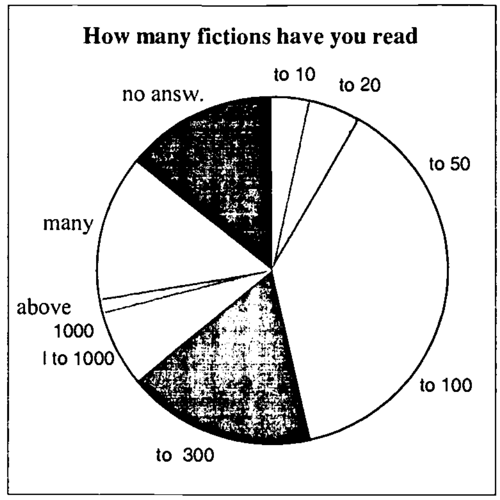
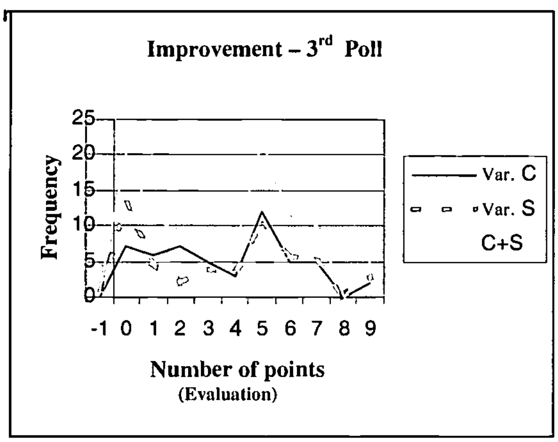


Fig. 6 – Number of books read by Czech students



Improvement	*Var. C*	*Var. S*	*C+S*
-1	0	1	1
0	7	13	20
1	6	5	11
2	7	2	9
3	5	4	9
4	3	4	7
5	12	10	22
6	5	6	11
7	5	5	10
8	0	0	0
9	2	3	5
SUM	52	53	105

Fig. 7 – Improvement of Knowledge in Dependency on the Form of Text

IMAGINATIVE DEPLOYMENT OF COMPUTER ALGEBRA IN THE UNDERGRADUATE MATHEMATICS CURRICULUM

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ABSTRACT

Much of the author's recent experience is attempting to teach Mathematics primarily to undergraduate students following degree programmes in Electronics or Audio Technology. Increasingly, it is found that although such students may be able to perform mechanistic steps such as obtaining a simple derivative, or evaluating a straightforward definite integral, they have little idea as to what these quantities mean. Very few (if any?) would know that these results are connected to a limiting process.

Unless the student's understanding of basic calculus is strengthened, they have little chance of subsequently dealing with the solution of differential equations or the construction of Fourier series. This paper shows how imaginative deployment of computer algebra (*DERIVE*) can substantially assist the understanding of calculus and its applications in the aforementioned areas. In particular, the paper will demonstrate the advantages of using computer algebra as an on-line teaching aid in the classroom compared with using traditional methods of teaching topics such as solving differential equations.

1. Introduction

Mathematics is increasingly perceived as being a difficult subject with the inevitable consequence that many students will try to avoid its study if at all possible. However, it is also well known that knowledge, understanding and competence in certain areas of Mathematics are required for the successful study of many undergraduate courses in Science and Engineering.

Instructors are frequently facing an audience of students, normally with weak mathematical backgrounds [1], who are obliged/forced to study more Mathematics to support their chosen degree programmes. This situation presents considerable challenges to instructors who have the difficult task of motivating reluctant students and of finding ways to facilitate understanding so that such students end up being reasonably competent in the areas taught.

The author believes that imaginative deployment of computer algebra in the undergraduate Mathematics curriculum can greatly assist the understanding of many concepts and applications encountered therein. Using the software package *DERIVE*, this is achieved by the use of built in commands, bespoke user defined commands and visual graphics. In the classroom/lecture theatre, the form of tuition is a combination of traditional methods – white board etc., and interactively generated computer algebra images provided via a notebook PC linked to a data projector.

In this paper, the author gives examples of how computer algebra can be imaginatively deployed to assist with the teaching and learning of differential and integral calculus, solving differential equations and construction of Fourier series. Bespoke user defined commands will be presented for the benefit of instructors. In practice, **the definitions of such commands are normally hidden from students** who simply need to know how to supply the values of the arguments contained in these commands for their own use during workshop sessions.

2. Differential Calculus

When introducing differential calculus, it is customary to begin with the simple function $y = u(x) = x^2$. We obtain a value for the gradient function (rate of change function, derivative etc.) at some fixed point e.g. $x = 3$, by drawing a series of chords with ends anchored at (3, 9) that are decreasing in length and then calculating their gradients. We conclude quite straightforwardly that the gradient function has the value 6 when $x = 3$.

In order to demonstrate this approach for a wide range of different functions, we can employ the User Defined Command (UDC) GRAD_FUNC_POINT(u, x, a) which simplifies to a vector containing two entries namely a and the value of the gradient function evaluated at $x = a$.

$$\text{grad\_func\_point}(u,\ x,\ a) := \left[a,\ \lim_{h\to 0} \frac{\left(\lim_{x\to a+h} u\right) - \lim_{x\to a} u}{h}\right]$$

This UDC was authored as:

GRAD_FUNC_POINT$(u, x, a) := [a, \lim((\lim(u, x, a+h) - \lim(u, x, a))/h, h, 0)]$

Examples of its use are $\text{grad\_func\_point}(x^2,\ x,\ 3) = [3,\ 6]$

(This was obtained by authoring the command followed by an "equals sign", then selecting simplify).

$$\text{grad\_func\_point}(\text{LN}(t),\ t,\ 4) = \left[4,\ \frac{1}{4}\right]$$

$$\text{grad\_func\_point}\left(\text{SIN}(x),\ x,\ \frac{\pi}{3}\right) = \left[\frac{\pi}{3},\ \frac{1}{2}\right]$$

In the case of $\ln(t)$, GRAD_FUNC_POINT can be used for several suitable values of t and, invariably, students are able to conclude that if $t = a$ where $a > 0$, then the gradient function will have value $\frac{1}{a}$. However, the aim is to be able to obtain the gradient function for an arbitrary given function at an arbitrary point. The UDC GRAD_FUNC_POINTS(u, x, b, e, s) simplifies to a matrix of coordinates corresponding to discrete points of the gradient function for $u(x)$, beginning with

```
grad_func_points(u, x, b, e, s) := VECTOR(grad_func_point(u, x, a), a, b, e, s)
```

$x = b$ and ending with $x = e$ in steps of s.

We demonstrate the use of this command on $y = u(x) = \sin x$ by authoring:

```
grad_func_points(SIN(x), x, 0, 2·π, π/6)
```

The matrix of co-ordinates (not shown here but obtained via the ≈ button) can now be plotted to see:

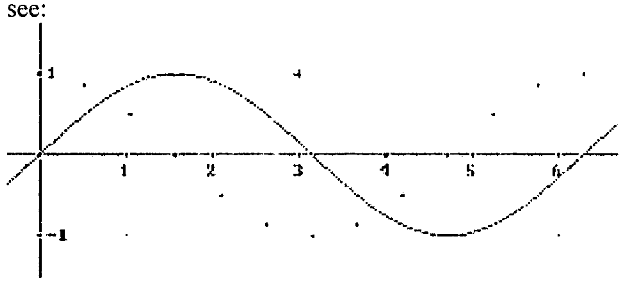


Figure 1 - $\sin x$ *plotted along with discrete points of its gradient function.*

From the plot, it should be apparent that the derivative of $y = u(x) = \sin x$ is $\cos x$. This can now be reinforced by returning to the first UDC and not specifying a numerical value for a.

```
grad_func_point(SIN(x), x, a) = [a, COS(a)]
```

or even

```
grad_func_point(SIN(x), x, x) = [x, COS(x)]
```

!

Hence, CAS has been used to generate the derivative of $\sin x$ using a graphical/visual approach as opposed to solely using an abstract/rigorous approach that students often struggle with. (The reader will recall that students will not be exposed to the definition of the command GRAD_FUNC_POINT).

3. Integral Calculus

It would be unwise for an instructor to launch into definite integration for non specialist Mathematics students (or others?) by starting with the definition:

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0}[f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n],$$

where P is a partition $\{x_0, x_1, \ldots, x_n\}$ of $[a,b]$
yielding the n sub-intervals $[x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n]$ with lengths $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$,

x_i^* is a point taken from $[x_{i-1}, x_i]$ for $i = 1, \ldots, n$, and $\| P \| = \max \Delta x_i, i = 1, \ldots, n$.

A much gentler approach, which will make the above more palatable if the instructor later chooses to expose this to their students, is to associate definite integration with the area under a curve by means of a simple (i.e. equal length subintervals with $x_i^* = x_{i-1}$ or x_i) Riemann sum.

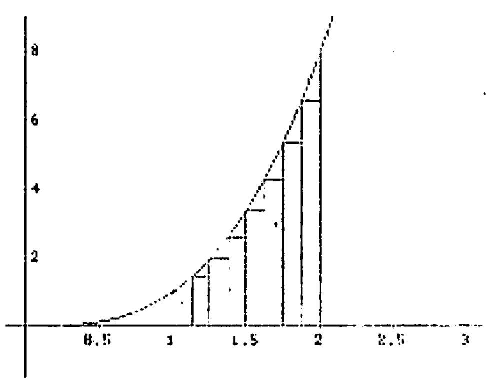
This can be achieved by employing the UDC MAKE_RECTS(u, x, a, b, n) which simplifies to a vector containing (4×2) matrices whose elements are the coordinates of the four corners of the n rectangles, with width $(b-a)/n$, under the curve $u(x)$ between $x = a$ and $x = b$, arranged in such a way as to provide a lower bound for the exact area under the curve for a monotonically increasing function.

The command is authored as :

MAKE_RECTS(u,x,a,b,n) := VECTOR(LIM([[$x + r(b-a)/n$,0],[$x + r(b-a)/n$, LIM($u,x,a + r(b-a)/n$)],[$x + (r+1)(b-a)/n$, LIM($u,x,a + r(b-a)/n$)], [$x + (r+1)(b\text{-}a)/n$,0]],x,a),r,0,n-1)

Plotting this vector of matrices, i.e. the rectangles, gives a visual display which is easy to understand. We demonstrate this by approximating to the area under the curve $u(x) = x^3$, bounded by $x = 1, x = 2$ and the x-axis using 8 rectangles.

MAKE_RECTS(x^3, x, 1, 2, 8) (The matrix of coordinates is not displayed here).



The figure was produced by simplifying the previous command and then plotting the resulting matrix of coordinates.

Display options need to be set to suppress colour changes and to join the vertices of the rectangles in order to construct the rectangles shown.

Figure 2 – *A lower bound approximation to the area under the curve* $u(x) = x^3$, *bounded by* $x = 1, x = 2$ *and the x-axis using 8 rectangles.*

The UDC SUM_RECT_AREAS(u,x,a,b,n) simplifies to a left Riemann sum of the areas of the rectangles produced by MAKE_RECTS.

$$\text{SUM\_RECT\_AREAS}(u,\ x,\ a,\ b,\ n) := \frac{b-a}{n} \sum_{r=0}^{n-1} \lim_{x \to a + (b-a) \cdot r/n} u$$

The command is authored as:

SUM_RECT_AREAS(u,x,a,b,n) := $(b-a)/n$ * SUM(LIM($u,x,a + (b-a)r/n$),r,0,$n-1$) .

Applying this command to the above example gives:

SUM_RECT_AREAS(x^3, x, 1, 2, 8) = 3.32

By increasing the number of rectangles to say, 100, we obtain the following:

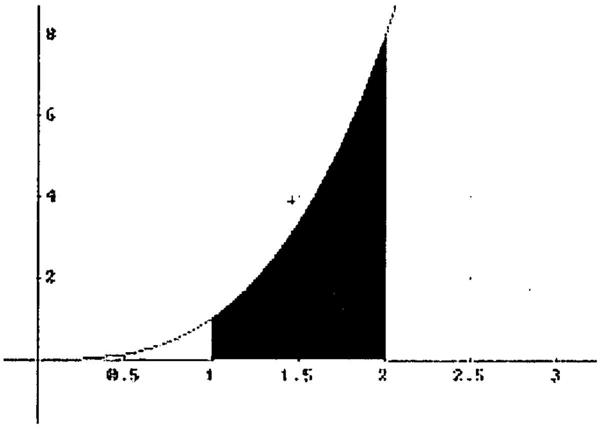


Figure 3 – *A lower bound approximation to the area under the curve* $u(x) = x^3$*, bounded by* $x = 1, x = 2$ *and the x-axis using 200 rectangles.*

This treatment should clearly demonstrate the limiting process inherent in the definition of a definite integral since, visually, we can see that an infinite number of rectangles must correspond to the exact area when summed. The area shown in figure 3 is readily calculated, yielding

$$\text{SUM\_RECT\_AREAS}(x^3, x, 1, 2, 200) = 3.73$$

Leaving the number of rectangles, *n*, arbitrary yields the closed form sum:

$$\text{SUM\_RECT\_AREAS}(x^3, x, 1, 2, n) = \frac{15 \cdot n^2 - 14 \cdot n + 3}{4 \cdot n^2}$$

It is clear that the right hand side can be expanded as $\frac{15}{4} - \frac{7}{2n} + \frac{3}{4n^2}$, with limiting value $\frac{15}{4}$ as $n \to \infty$.

At this stage, students could be shown the relationship $\int_1^2 x^3 dx = \left[\frac{x^4}{4}\right]_1^2 = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$.

Closed form sums are nice to see. Of particular interest is to calculate the left Riemann sum for the area bounded by $\sin x$, $x = 0, x = \pi/2$ and the x-axis using an arbitrary number of rectangles.

$$\text{SUM\_RECT\_AREAS}\left(\text{SIN}(x), x, 0, \frac{\pi}{2}, n\right)$$

Expanding the above command gives: $\dfrac{\pi \cdot \text{COT}\left(\dfrac{\pi}{4 \cdot n}\right)}{4 \cdot n} - \dfrac{\pi}{4 \cdot n}$

We can use *DERIVE*'s limit command (from within the calculus menu) to obtain the exact value

for the area i.e. $\lim_{n \to \infty} \left(\dfrac{\pi \cdot \text{COT}\left(\dfrac{\pi}{4 \cdot n}\right)}{4 \cdot n} - \dfrac{\pi}{4 \cdot n} \right) = 1$

This result also demonstrates that $\lim_{\alpha \to 0}(\alpha \cot \alpha) = 1$!

We note further that
```
lim SUM_RECT_AREAS(SIN(x), x, a, b, n) = COS(a) - COS(b)
n→∞
```
This result can be used to introduce the concept of the anti-derivative. Koepf and Ben-Israel [2] pursue this approach showing that an indefinite integral can be regarded as a definite integral over a variable interval.

It is the author's experience that even students who have encountered integral calculus prior to embarking on their undergraduate course have rarely appreciated that definite integration is connected to a limiting process. CAS enables this important concept to be presented both visually and algebraically by generating, where possible, closed form sums.

4. Differential Equations

Students can often be intimidated by the term "differential equation" and expect these to be difficult at the outset simply because of the presence of one or more derivatives in an equation.

It is useful to begin with a very simple example such as $\frac{dy}{dt} = 2t$. Most students will be able to say that "the" solution is $y = t^2$ and the instructor then normally has to interject to coax out the infinite number of solutions given by $y = t^2 + c$, where c is an arbitrary constant. *DERIVE* can be used here to demonstrate diagrammatically that, in the absence of any boundary conditions, a differential equation will have an infinite number of solutions that can cover the whole real plane.

This is readily accomplished by authoring, simplifying and then plotting the command

```
          2
VECTOR(x   + c, c, -4, 4, 0.5)
```

If we only consider tangent line segments drawn at regular points on these solution curves, then the resulting diagram should give a very good indication as to what the actual solution curves look like.

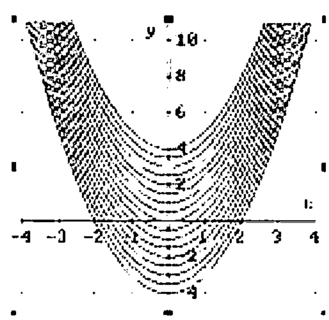


Figure 4 - *Solution curves for* $\frac{dy}{dt} = 2t$

The tangent field can be obtained via the BIC

DIRECTION_FIELD$(f(x,y), x, x_0, x_m, m, y, y_0, y_n, n)$

where $\frac{dy}{dx} = f(x,y)$, x varies from x_0 to x_m in m steps and y varies from y_0 to y_n in n steps.

We now author, approximate, then plot the command:

```
DIRECTION_FIELD(2·x, x, -2, 2, 9, y, -4, 4, 15)
```

Thus, via this very simple example, students can appreciate that much information about the general solution of a differential equation can be obtained from the initial differential equation without the need to solve it. It would be very difficult to convey these ideas to students without the use of a software package.

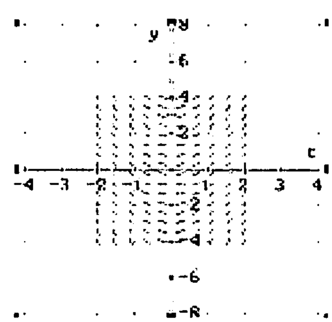
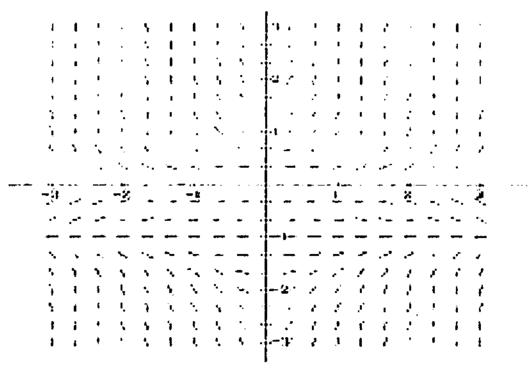


Figure 5 – *Tangent field for* $\frac{dy}{dt} = 2t$

Of course, another obvious advantage of this approach is to emphasise that seeing the tangent field determined by a differential equation is possibly the best we can see with regard to the complete solution curves if analytical techniques cannot be employed to solve the equation. Indeed, we may only be able to generate points for particular solutions using numerical techniques.

Several of the aforementioned concepts can be encapsulated by the following example. We shall consider the solutions of the differential equation $\frac{dy}{dt} = y(1+y)t$, and begin by obtaining a plot of its tangent field.

DIRECTION_FIELD(y·(1 + y)·t, t, -3, 3, 18, y, -3, 3, 18)



Using the approximate command, we obtain a large matrix of coordinates (not shown here) which can now be plotted.

This rather interesting diagram shows the flow of the solution curves and also indicates asymptotic behaviour.

Figure 6 – *Tangent field for* $\frac{dy}{dt} = y(1+y)t$

It is a straightforward matter to analytically obtain the general solution $y = \dfrac{ke^{\frac{t^2}{2}}}{1-ke^{\frac{t^2}{2}}}$. This now presents the instructor and students with a rich mathematical investigation. We may pose the question "for which values of k do we obtain solutions in that part of the plane where $y < -1$, where $y \in [0,-1)$and $y > 0$?" Using *DERIVE*'s SUB command, by experimentation, we can discover that if $k > 1$, we obtain solution curves in the region where $y < -1$. If we choose $k < 0$, we obtain solution curves in the region $y \in [0,-1)$. Both these ranges for k show solution curves that are asymptotic to the line $y = -1$. For $k \in (0,1)$, we obtain solution curves each consisting of three pieces with two vertical asymptotes and the horizontal asymptote $y = -1$. The case $k = 1$ yields a solution curve with different characteristics to the previous cases. A selection of these solution curves is shown below.

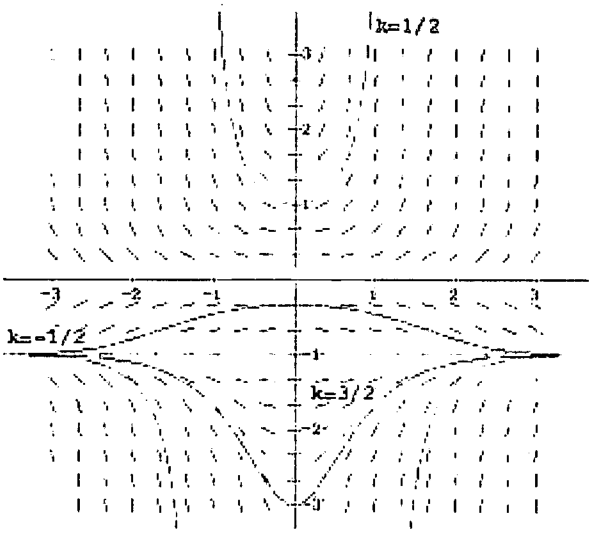


Figure 7 – *Particular solutions of* $\frac{dy}{dt} = y(1+y)t$

For completeness, at this stage students can be informed that sometimes only numerical techniques are available to obtain a numerical solution of a differential equation. *DERIVE* supports a variety of numerical techniques the simplest of which is EULER_ODE($f(x,y), x, y, x_0, y_0, h, n$) and approximates to a vector of $n+1$ solution points of the equation $\frac{dy}{dx} = f(x,y)$ with $y = y_0$ and $x = x_0$ using a step size of h.

We apply the EULER_ODE command to generate solution points on the particular solution passing through the point $(0,-3)$ and contrast these solution points with the exact solution given by $y = \frac{3e^{\frac{t^2}{2}}}{2 - 3e^{\frac{t^2}{2}}}$.

```
EULER_ODE(y·(1 + y)·t, t, y, 0, -3, 0.25, 12)
```

Simplifying EULER_ODE via the approximate command, yields:

0	-3
0.25	-3
0.5	-2.625
0.75	-2.091796875
1	-1.663501132
1.25	-1.387608869
1.5	-1.219527337
1.75	-1.119132491
2	-1.064882785
2.25	-1.028552903
2.5	-1.012833306
2.75	-1.004421989
3	-1.001368420

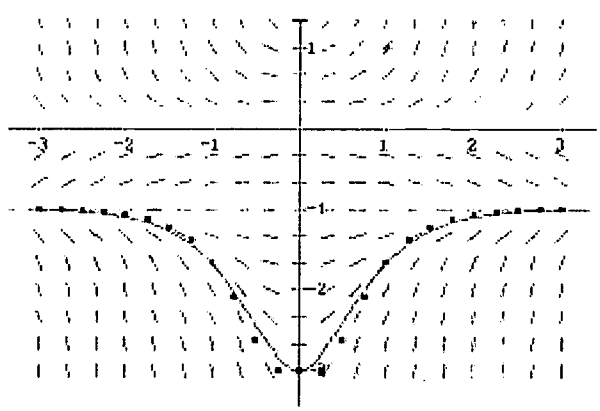


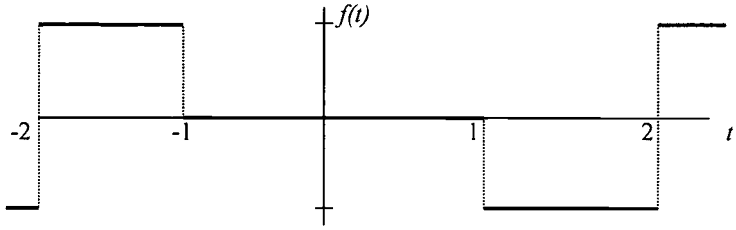
Figure 8 – *Numerical solution of* $\frac{dy}{dt} = y(1+y)t$ *passing through* $(0,-3)$.

Solution points for $t < 0$ are obtained by replacing 0.25 with -0.25 in the EULER_ODE command.

Very few (if any) students have seen tangent fields associated with the solutions of differential equations even though they may already be familiar with solving simple differential equations. It is a revelation for them to see tangent field diagrams "on-line" by a CAS in the classroom and this stimulates them to engage with the topic with greater confidence and understanding.

5. Fourier Series

DERIVE is an indispensable tool for dealing with piecewise defined periodic functions and their associated Fourier Series representations. As an example, consider the function with graph:



Defined as:

$$f(t)=\begin{cases} 1 & -2<t\le -1 \\ 0 & -1<t\le 1 \\ -1 & 1<t\le 2 \end{cases}, \text{ where } f(t+4)=f(t)$$

It is useful to be able to plot the graph of this periodic function using *DERIVE*, so that, later, we can superimpose the graph of its Fourier Series and contrast the two.

Plotting the graphs of piecewise defined periodic functions is achieved by defining the function over the interval (0, T), where T is the period using *DERIVE*'s built-in function $\mathrm{CHI}(a,x,b)$, where $\mathrm{CHI}(a,x,b)=\begin{cases} 1, & a<x<b \\ 0, & x<a \\ 0, & x>b \end{cases}$, and then using the built-in MOD function to take care of the periodicity.

```
f(t) := CHI(1, t, 2)·(-1) + CHI(2, t, 3)·1
F(MOD(t, 4))
```

Plotting the latter expression produces the graph of the piecewise defined periodic function.

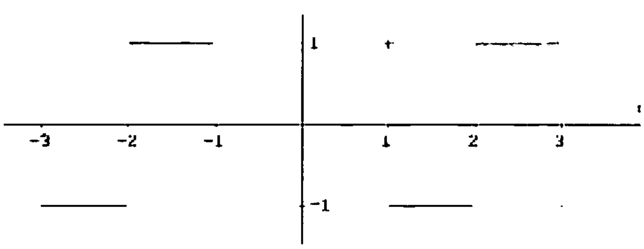


Figure 9 – *Plotting piecewise defined periodic functions using DERIVE's CHI and MOD functions.*

The standard Fourier Series representation for a function with period T is given by:

$$f(t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos\frac{2k\pi t}{T}+b_k\sin\frac{2k\pi t}{T}\right),$$

and the Fourier coefficients are given by:

$$a_0 = \frac{2}{T}\int_{t_1}^{t_2} f(t)\,dt$$

$$a_k = \frac{2}{T}\int_{t_1}^{t_2} f(t)\cos\frac{2k\pi t}{T}\,dt \quad \text{for } k \in N^+$$

$$b_k = \frac{2}{T}\int_{t_1}^{t_2} f(t)\sin\frac{2k\pi t}{T}\,dt \quad \text{for } k \in N^+$$

where $T = t_2 - t_1$.

Since the given example is an odd function, $a_0 = 0$, and $a_k = 0$ for $k \in N^+$. In addition

$$b_k = \frac{4}{T}\int_{0}^{T/2} f(t)\sin\frac{2k\pi t}{T}\,dt \text{ for } k \in N^+$$

It is a straightforward matter to show that $b_k = \frac{2}{k\pi}\left((-1)^k - \cos\frac{k\pi}{2}\right)$

The required Fourier Series is therefore:

$$f(t) = -\frac{2}{\pi}\left[\sin\frac{\pi}{2}t - \sin\pi t + \frac{1}{3}\sin\frac{3\pi}{2}t + \frac{1}{5}\sin\frac{5\pi}{2}t - \frac{1}{3}\sin 3\pi t + \frac{1}{7}\sin\frac{7\pi}{2}t + \cdots\right]$$

As a check, or otherwise, we can use *DERIVE*'s BIC FOURIER$(f(t), t, t_1, t_2, n)$ to generate the first n harmonics of the Fourier Series for $f(t)$ defined over the periodic interval t_1 to t_2.

`FOURIER(f(t), t, 0, 4, 7)` which simplifies to the expression below:

$$-\frac{2\,\mathrm{SIN}\left(\frac{7\cdot\pi\cdot t}{2}\right)}{7\cdot\pi} - \frac{2\cdot\mathrm{SIN}\left(\frac{5\cdot\pi\cdot t}{2}\right)}{5\cdot\pi} - \frac{2\cdot\mathrm{SIN}\left(\frac{3\cdot\pi\cdot t}{2}\right)}{3\cdot\pi} - \frac{2\,\mathrm{SIN}\left(\frac{\pi\cdot t}{2}\right)}{\pi} + \frac{2\,\mathrm{SIN}(3\cdot\pi\cdot t)}{3\cdot\pi} + \frac{2\,\mathrm{SIN}(\pi\cdot t)}{\pi}$$

Superimposing this truncated Fourier Series onto the original piecewise defined periodic function, we obtain:

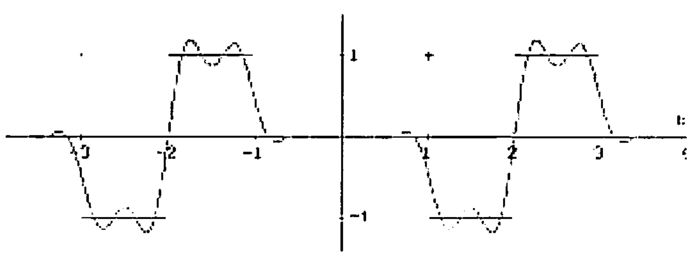
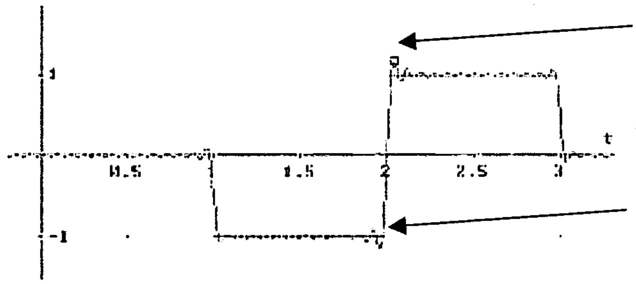


Figure 10 – *Plotting the truncated Fourier Series representation along with the original piecewise defined periodic function.*

At this stage, a discussion can take place over the behaviour of the synthesised function around the points of discontinuity. Classical theory states that in general, the magnitude of the combined

undershoot and overshoot together at a point of discontinuity amount to about 18% of the magnitude of the discontinuity. This is the so called onset of Gibbs' phenomenon.
We can "test" this theory using *DERIVE*'s **trace** facility to measure the lengths of the under and overshoots on a plot of the truncated Fourier Series containing 50 harmonics.



The pronounced "peak" has coordinates (2.04, 1.178) provided by *DERIVE*.

The pronounced "trough" has coordinates (1.96,−1.178)

Figure 11 – *Using DERIVE to explore Gibbs' phenomenon*

In the above example, at $t = 2$, the magnitude of the discontinuity is 2. We can from the coordinates obtained using *DERIVE*'s trace facility, that the distance from the trough to the peak is 2.356. Hence the magnitude of the combined under and overshoot is equal to $2.356 - 2 = 0.356$, and $\frac{0.356}{2} \times 100 = 17.8\%$

The main use of *DERIVE* here is to show, visually, how a Fourier series can generate a given periodic signal function even when it is piecewise defined. Moreover, the ability to measure the onset of Gibbs' phenomenon in such a straightforward manner is particularly appealing.

6. Conclusion

There is no doubt that the ability to perform tedious or repetitive symbolic manipulation using computer algebra focuses the student's mind on the concepts that are very often obscured by the time consuming process of carrying out the manipulation by hand. Furthermore, computer generated plots provide a powerful means of visualising concepts and applications.

Much of the treatment demonstrated in this paper would simply not be viable using traditional teaching methods. Certainly, the interactive use of computer algebra in the classroom both helps to "bring alive" the Mathematics being presented and stimulates interest. The very fact that a computer image is being projected catches the attention of the audience. This type of delivery, coupled with the enthusiasm and pedagogical skills of the instructor can result in a positive, productive and enjoyable experience for the students.

Whenever asked, students invariably welcome the deployment of computer algebra within the curriculum to assist their teaching and learning. This is further demonstrated by the many occasions where this style of exposition has provoked questions from the audience and has inspired dialogue between students and instructor. Common remarks have included statements such as "I never really understood calculus before" and "it is helpful to *see* what solutions to expect before actually finding them" etc.

The author's experience of this type of delivery has been to non-specialist Mathematics undergraduates where the emphasis has been on a less rigorous exposition of the Mathematics needed. However, the software can be used to address important and more rigorous aspects of calculus such as differentiability and continuity where the limiting processes need to be more controlled involving, for example, left and right limits.

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MATHEMATICS EDUCATION FOR SOFTWARE ENGINEERS: IT SHOULD BE RADICALLY DIFFERENT!

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ABSTRACT

Software engineering is a young engineering discipline that is different in many aspects from the classical engineering fields. For me the most distinguishing point is the kind of mathematics that serves the respective fields well. By giving examples I will try to show that classical, calculus based mathematics is of no help for defining central notions in software engineering, like "abstract data type". Thus, mathematics education for software engineering students should be radically different from the traditional curricula for science and engineering students. In particular, the changes to be made go far beyond putting more emphasis on discrete mathematics as done in many math curricula for computer science students.

I will report on our introductory mathematics course that that we have now taught at the Polytechnic University of Upper Austria for several years. The whole first year is dedicated to teach "The Language and Methods of Mathematics". I will also report on experiments with using the THEOREMA language and system in the lab exercises for this course, both about highlights and problems. THEOREMA is being developed by Bruno Buchberger and his team at Risc-Linz and aims at combining general predicate logic proof methods and special proof methods in one coherent system.

An important observation is that students are in no way prepared for this kind of mathematics after high school. Since computers and information technologies continue to gain more and more importance in our lives, the ability to developed software with mathematical rigour will be a crucial asset for the competitiveness of the software industry of any country in the future. This implies that changes in the high school mathematics curricula towards usability for software engineering should also be considered.

1 Introduction

In the early years of computer science and engineering there was a lot of discussion where the field should be positioned in the landscape of university education. It was also not clear how to name the various programs, several of them differing only very little: Computer science, computing science, computational science, information science, systems science, systems engineering, computer engineering, software engineering, information science, informatics, etc.

Since most of the early programs grew out of mathematics and electrical engineering curricula, one main point in the discussion was if computer science should be considered a science or an engineering discipline. The advent of software engineering as its own field has clarified things partially: it is generally agreed that software engineering is an engineering discipline, whereas computer science is (sic!) a science. This fact should be reflected in the respective educational programs. David L. Parnas ([9]) gives an excellent and exhaustive account on this theme in his paper titled "Software Engineering Programs are not Computer Science Programs".

It is generally agreed as well that mathematics plays an important role in the curriculum of the classical engineering disciplines like Civil, Mechanical or Electrical Engineering. Often up to 30% of an engineers education is devoted to mathematics.

If one looks at journals and proceedings in the field of software engineering education, the topic of teaching mathematics does not seem to have much importance. Several well-known software engineering educators such as, again, David L. Parnas ([8]), say that mathematics in a software engineering program should be essentially the same as for the classical engineering disciplines. I strongly oppose this opinion, and on the contrary will try to show the opposite by giving an example that classical engineering mathematics does not really help a software engineer in working in his or her profession. I agree that software engineering clearly is an engineering discipline. But it is different in various aspects, and the most distinguishing point for me is the kind of mathematics that well serves the respective fields well.

I have expressed some of these thoughts already at a conference on Engineering Education ([7]). Here I try to elaborate in more detail on the points that distinguish math for software engineers from math for classical engineers. Additionally, our experience with using the THEOREMA system in lab exercises is more recent.

2 Mathematics for Software Engineers

Civil, mechanical, or electrical engineers usually model aspects of our physical reality where space and time are continuous quantities. Thus it is understandable that classical engineering mathematics is primarily based on calculus. With the advent of the computer discrete mathematics gained more and more importance and thus in a typical computer science curriculum discrete mathematics plays an important role. Often discrete math is even taught before calculus - and then praised as a radical reform in teaching mathematics.

Classical engineering mathematics also differs strongly from pure mathematics. Mathematicians are primarily interested in deep theorems and general properties of classes of functions, expressions, algebras, etc., whereas engineers primarily use well-known mathematical entities to model aspects of reality and to do calculations in these models. Nowadays engineers should know how to use modern tools like program libraries and computer algebra systems, but the mathematical topics they need are settled and hardly change over the years. I therefore believe that it is a mistake to treat math education for computer science and software engineering as basically the same, as done recently in a working group at an established conference on computer science education ([5]).

When we think about a software engineer designing and implementing a software system for controlling robots, he will need a lot of classical engineering mathematics like geometry and differential equations. The point is that he does not need this math because he is a software engineer but because he is working in the area of robotics. If he worked, for example, in the banking area on modelling workflow, he would need a rather different kind of math (if any). Since software engineers can work in any area of the economy it is virtually impossible to teach all the mathematics they possibly could need. We thus propose to place emphasis on teaching the methods of mathematics and thus enable students to learn arbitrary topics by themselves when needed. This approach is described below.

Nevertheless, one should also try to identify the mathematics that every software engineer should learn. Analysing this question, I came to the conclusion that this kind of mathematics is very different from classical engineering mathematics. Let me try to explain this with the following example.

3 An Example: Abstract Data Types

The concept of an abstract data type (ADT) is fundamental in programming. Students learn to base their programs on that concept in the first year. Every introductory textbook on computer science, programming, or algorithms and data structures gives a "definition" of that notion. In [1] it reads like this:

> "We can think of an abstract data type (ADT) as a mathematical model together with a collection of operations defined on that model."

This informal definition usually suffices for daily programming work, but if one wants to *prove* properties of an ADT or relations between ADTs (i.e. does a proposed ADT implement another ADT correctly?) one has to base such proofs on *mathematical* definitions. A math curriculum for software engineers clearly should provide such formal definitions and teach the students how to use them. The best definition of ADT I know is an algebraic one and taken from [3]. Since this book is written in German, we also cite [4] where the definition is similar.

> **Definition:** Let Σ be a Signatur. Then $\Sigma\text{--}ALG$ denotes the category of all Σ–algebras with all Σ-algebra-morphisms between them. An *abstract data type* of Σ is a full subcategory of $\Sigma\text{--}ALG$.

I would like to use that definition early in my math courses for software engineers, but this is virtually impossible. The students are in no way prepared for this kind of mathematics when coming from high school. Traditional, calculus based engineering mathematics does not help here at all.

This should make the dilemma clear: something that even mathematicians sometimes call "abstract nonsense" - which is taught in the late undergraduate or graduate curriculum only - i.e. some concepts of category theory, is necessary to define such basic notions as "Abstract Data Type" in a mathematically precise way. One should see that a mathematics curriculum for software engineers has to be very different from traditional engineering curricula.

4 Teaching "The Methods of Mathematics"

Already in the early eighties Bruno Buchberger (the founder of the RISC Institute) started to develop a new mathematics curriculum for computer science students at Johannes Kepler University in Linz (Austria). I had the privilege to contribute to this project from the beginning and to teach the course for almost two decades. The whole first semester is dedicated to teaching "The Language and Methods of Mathematics". By showing several case studies we try to analyse and teach all those aspects of mathematics that are necessary to treat the whole problem solving process, i.e. starting from the formal specification of a problem, then developing recursive and iterative algorithms for solving the given problem (which often means to conjecture and to prove some mathematical facts), prove the correctness and analyse the complexity of the algorithms, and finally give a structured documentation and presentation of the problem and its solution. The lecture notes of the first semester were published in [2], the whole curriculum is described in [6].

Since the Polytechnic University of Upper Austria at Hagenberg started a four-year software engineering program in 1993 this course is taught as the only mathematics course in the first year. At the beginning Mathematica was used to demonstrate the basic concepts, now we use THEOREMA in the lab exercises. Although we teach basically no mathematical contents that are new to the students, I am convinced that this course serves the students better in developing the skills needed for their professional career than a calculus or discrete math course would do. The more traditional math courses are taught in the second and third year.

One could argue that this kind of methodological training comes too early in the curriculum since there is nothing "to abstract from" at the beginning of a university program. We believe on the contrary that students have been exposed to enough mathematical topics in high school to be able to demonstrate and teach the technique of problem solving with mathematical methods explicitly. We also believe that this methodological training should make it easier for the students to learn and understand arbitrary mathematical contents in the following years.

5 Experiments with THEOREMA

THEOREMA is a language and system which aims at combining general predicate logic proof methods and special proof methods. It is written in Mathematica and thus the user interface and the computational power of Mathematica are available. THEOREMA is being developed by Bruno Buchberger and his team at Risc-Linz. With THEOREMA it is possible to define new notions in "natural" mathematical notation, do computations using them, and prove theorems about these notions in one single environment.

Unfortunately it is not possible to show the nice features of an interactive system like THEOREMA in a paper like this. This will be the main part of the oral presentation at the conference. I suggest to visit the homepage of the project ([10]).

A typical homework example which is given at about mid term of the first semester is the following:

> Define the notion of "a finite sequence is in descending order" according to the following three explanations. A sequence is in descending order if
>
> 1. the left of two neighbouring elements is at least as big as the right one,
> 2. all elements to the right of an arbitrary element are at most as big as this one, and all elements to its left are not smaller than it,
> 3. comparing two arbitrary elements, the left one is at least as big as the right one.

In the second semester, when proving is the main topic, they have to prove that the three definitions they gave are equivalent. THEOREMA can produce these proofs automatically. The presentation of the proofs is essentially the same as if done carefully by hand.

The results of our experiments are promising: students appreciated that they were "forced to rigor". Also, prove construction – a topic that is generally regarded as very difficult – could be "demystified" by showing the computer made but human readable proofs. One drawback still is that the start-up effort is very high. We hope that this problem will vanish with future versions of THEOREMA.

6 Implications for the High School Curriculum

High-school math is based on calculus and thus serves well as a preparation for classical engineering mathematics. Clearly much energy is put into defining the basic notions, like *continuous function* precisely. Comparing this with the ADT example, it is definitely not sufficient to give an informal definition like

> A real function is called *continuous* if it can be drawn with a pencil in one stroke (without gaps etc.)

The formal definition is much more complicated:

$$f \text{ is } \textit{continuous} \text{ at } x :\Longleftrightarrow \bigwedge_{\varepsilon>0} \bigvee_{\delta>0} \bigwedge_{|y-x|<\delta} |f(y)-f(x)| < \varepsilon$$

Nevertheless, every engineer has to learn – and should understand – this (or an equivalent) definition of this important notion, even if he will never in his professional life use it in this way. It is important for him to know that the mathematics he uses is based on such solid grounds.

When heading towards a formal definition of *abstract data type* one first has to introduce the notion of *signature*:

A *signature* $< S, \Sigma >$ consists of a set S, whose elements are called *sorts*, and a family of sets $\Sigma = (\Sigma_{w,s})_{w \in S^*, s \in S}$.

For each $w \in S^*$, $s \in S$, $\Sigma_{w,s}$ is a set whose elements are called operation symbols.

For each $\sigma \in \Sigma_{w,s}$, w gives the *domain* and s the *codomain* of σ. $|w|$ is the *arity* of σ; σ is called *nullary* or a *constant*, if $w = \lambda$.

In this definition S^* denotes the set of finite words over (the alphabet) S and λ the empty word. Instead of $< S, \Sigma >$ we often denote the signature just Σ.

I believe that this definition is easier to understand with the right preparation than the definition of *continuous function* given above. Nevertheless, students at university have big difficulties understanding it since they are not prepared for this different kind of basic mathematics. Today high school math usually culminates in teaching differential and integral calculus and it's Fundamental Theorem, sloppily written:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

In order to prepare students for the mathematics I would like to start with at university I would recommend that, for example, Birkhoff's theorem of 1935 stating the completeness and consistency of the equational calculus be taught already in high school:

$$E \models e \iff E \vdash e$$

(Here e is an equation over a signature Σ and E a set of equations over Σ)

I know that this is just wishful thinking and far from having a chance to become reality. It's even worse: all the nice new tools like computer algebra systems, TI 92, etc. make teaching classical, calculus oriented math easier and sometimes even fun. The more formal kind of mathematics I need plays almost no role any more.

7 Conclusions

I have argued that although Software Engineering clearly is an engineering discipline the mathematics that could serve the field well is quite different from classical engineering mathematics. I would like to explicitly state the following three points to characterize this type of mathematics:

- The main emphasis is on **modelling of non-physical realities**. The mathematical tools needed go beyond traditional discrete mathematics and come primarily from (universal) algebra and logics. Even classical propositional and predicate logics often are not sufficient, so that one has to use others like modal, temporal, or non-monotonous logics.

- **Define the new instead of use the known**: Research on new types of, say, differential equations is not a task of a classical engineer; mathematicians do that job. But if one considers abstract data types to be essentially classes of algebras, as proposed above, defining or specifying new classes of algebras is everyday work for a software engineer. He will not try to detect deep mathematical theorems valid in these algebras but definitely will have to prove some basic properties of the objects and operations he modelled.

- **Prove instead of compute**: In the classical engineering disciplines mathematics is primarily used to compute numerical values. Symbolic computations are almost exclusively done as a preparation for numerical computations, often the results are visualized graphically. A software engineer should use mathematical reasoning to prove properties of the programs or software systems he designed. Training in doing formal proofs should thus be mandatory.

Since high school math in no way prepares for that kind of mathematics, it is very difficult to introduce these topics at the university level. Nevertheless, I believe that it would pay off to take this challenge since it could improve the quality of software in general. Since computers and information technologies continue to gain more and more importance in our lives, the ability to developed software with mathematical rigour will be a crucial asset for the competitiveness of the software industry of any country in the future.

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TEACHING MATHEMATICS IN INDONESIAN PRIMARY SCHOOLS USING REALISTIC MATHEMATICS EDUCATION (RME)-APPROACH

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ABSTRACT

This paper presents a case study about employing Realistic Mathematics Education (RME)-approach to teach mathematics in Indonesian primary schools. Many obstacles, such as the very dependent attitude of the pupils, the pupils who were not used to working in groups, lack of reasoning capability and lack of understanding of basic concepts, were found when the pupils, who were used to the traditional way of teaching, dealt with the new approach (RME). The discussion in this paper is focused on these obstacles and the efforts undertaken to overcome them.

1. Introduction

There is a number of problems in mathematics instruction in Indonesian primary schools. For example, the approach that is used to teach mathematics is very theoretical, and many abstract concepts and formulas are introduced without paying much attention on aspects such as logic, reasoning, and understanding (Karnasih & Soeparno, 1999; Soedjadi, 2000). Besides, the teaching learning-process is always organized in a traditional (teacher centered) way (Somerset, 1997).

The conditions above make mathematics more difficult to learn and understand and pupils become afraid of mathematics. Moreover, the conditions also create unfavorable climate for mathematics instruction in the classrooms. In general, the climate in Indonesian classrooms is similar to those in several African countries as was summarized by de Feiter at all. (1995) and Ottevanger (2001) as follow: pupils are passive through out the lesson; 'chalk and talk' is preferred teaching style; emphasis on factual knowledge; questions require only single words, often provided in chorus; lack of learning questioning; only correct answers are accepted and acted upon; whole-class activities of writing/there is no hands work is carried out.

In our research project (started in 1998 and is partly reported in this paper) we explored the extent to which Realistic Mathematics Education (RME) could address some of the problems in mathematics education in Indonesia, more specifically in the geometry instruction. This aim is realized by developing and implementing the student book and teacher guide based on RME theory through development research (see Akker & Plomp, 1993; Richey & Nelson, 1996).

The paper reports about the very first experiences in Indonesia to teach geometry according to the RME approach, and addresses specifically the research question '*what are the obstacles when introducing the RME approach and how can they be overcome*?' In the next section, the characteristics of RME will be summarized. Then, the RME-based intervention for teaching geometry topics to grade 4 classroom will be described followed by the design of this research. The report of the research findings is followed by some conclusions and reflections relevant for further work in this area.

2. Realistic Mathematics Education (RME)

RME is an approach in which mathematics education is conceived as human activity (see Freudenthal, 1973; Treffers, 1987; Gravemeijer, 1994; De Lange, 1987, 1998). In RME, learning mathematics means doing mathematics, of which solving every day life problems (contextual problems) is an essential part.

There are three key principles of RME for instructional design namely *guided reinvention* and *progressive mathematizing, didactical phenomenology,* and *self developed models* (Gravemeijer, 1994). Even for teaching learning process, RME has five learning and teaching principles: *constructing* and *concretizing*, *level* and *models*, *reflection* and *special assignment*, *social context* and *interaction*, *structuring* and *interweaving* (see De Lange, 1987; Streeflands, 1991; Gravemeijer, 1994). So, in RME-based lessons, pupils should be given the opportunity to reinvent mathematical concepts, and teaching learning process would be highly interactive. The main role of teachers is to determine in which way an optimal result can be obtained, for example by organizing pupils' interaction, individual work, group work, classroom discussion, pupil presentation, teacher presentation, and/or other activities.

Given its characteristics, RME is considered a very promising approach to change the classroom' climate in order to improve mathematics teaching and make it more relevant for pupils in Indonesia.

The Intervention: a series of lessons on topic '*area and perimeter*'

To investigate whether and under what conditions RME can be utilized in Indonesian primary schools, a series of 10 lessons have been designed for pupils at grade 4 (age 9 – 11) on the topic 'area and perimeter'. There are two potentials of RME-based lessons on this topic compare to traditional lessons. Firstly, Indonesian curriculum for topic area and perimeter school contains only the most minimal concept of area that is area as "length times width' or area as counting the squares centimeters in a rectangle or square. Even in the RME-based lessons the concept of area is broaden to other shapes, by relating area to other "magnitudes' (costs, weight, paint, rice, cake, etc.); investigating the relation between area and perimeter; connecting measurement units to reality; integrating some geometry activity (re-shaping, tessellation, etc.). Secondly, the lessons for topic area and perimeter in Indonesian curriculum emphasize only on applying the formulas (after the formulas are introduced deductively using chalk and talk method). In other hand, RME-based lessons would create the situations that due to learning and teaching principles and RME characteristics mentioned above such as pupils centered instruction, pupils active learning (interactivity), pupils free production (reinvention and self-developed models), etc. The principle 'free production' would stimulate pupils' reasoning because the pupils have to share or discuss concepts they reinvent or models they develop in solving contextual problems.

Related to the potentials of RME-based lessons, pupils are expected not only to master the mathematical concepts related but also to pay much attention on the process related. They are expected to know how to work in groups, be active and creative in reinventing the concepts related and developing their model in solving a contextual problem, understand the importance of giving an explanation for a solution. The same case for teachers, they are expected to be able to attract the pupils to solve the contextual problems, stimulate the pupils when they are working in groups, to react upon the pupils' contribution, and to guide the classroom discussions.

As there was no information at all about how Indonesian pupils would react on such a new approach, it was decided to use an '*emergent*' design approach: the series of lessons was only planned in general terms of what content, methods and learner activities should be applied in the lesson series, while the detailed plan for each lesson would be strongly determined by the events and experiences of the preceding lesson(s). This approach implies that only the first lesson a detailed plan was designed.

3. Design of the research

Given the research question and its context, the research reported here has an exploratory character. The research was conducted in a primary school in Surabaya (East Java). As no teacher in Indonesia has experience with teaching RME-based lessons the first author taught the pupils himself, even the teacher and the second author taking the role of observers. The data collection focused on pupils' activities and reactions when they dealt with RME-approach. The instruments used to collect the data were observation scheme, logbook, and interview guidelines. The data analysis in this exploratory research was qualitative and judgmental

4. Research Findings

Below is described what happened in the consecutive lessons to the classroom . The data are presented in narrative form to be able to convey the richness of the interactions and other processes that took place. As the first author acted as the teacher, researcher (formative evaluator) and developer of the lessons, this part of the paper is written in a 'personalistic style'

Finding from lesson 1

The topic for the first lesson is "the sizes of shapes" in which pupils would compare and order the sizes of various shapes. To do these activities, I prepared materials such as: worksheet, tracing papers, drawing papers, and scissors. An important goal of the lesson is to see how pupils would react and act to the change in roles: from passive listening and making exercises towards active working on mathematics tasks. In this meeting pupils worked in groups of 4, in which pupils who sat next to each other were in the same group. The pupils were grouped to make it easier to observe their activities. At the beginning I explained what the lesson is about, what expectations I had from the lesson (the changes of pupils' and teacher's role, compare to traditional method), what activities the pupils would do, and what the nature of the materials was which I provided for. This was what happened when the pupils dealt with the first contextual problem.

Hand Size-fingers

Draw the outlines of your hand size-fingers on a piece of paper then find out who has the smallest hand size-fingers? Explain your answer!

After reading the contextual problem the pupils kept silent. It seemed they did not know what to do and were waiting for instruction. I tried to explain and encouraged them to use any materials in order to solve the problem, but there was none of the pupils started to work. Because of that, I explained how to draw hand size-fingers on a drawing paper/tracing paper. Then, I gave a clue how to use those drawings to find the member of groups who had the smallest hand size-fingers (by putting one drawing on top of the others). Some groups were not interested and just observed their drawings then decided about the answers (without giving any reasoning). When I asked them 'how do you know it is the smallest?', they just looked at each other. Because most pupils were still confuse, I asked them to cut out their drawings in order to make easier to compare the drawings. All groups did this but only two groups (out of ten) succeeded on this task.

Initially I thought the problem was because of poor reading ability. After asking some pupils I discovered that the problem was not in reading but that the pupils never worked on story problems. Besides, they were used to a situation in which the teacher would give first an example, after which the pupils do the tasks that similar to the example.

Working in groups was not running smoothly because only one or two pupils in each group were working seriously, while the others were waiting for the answers. Moreover, the pupils in the mixed groups (boys and girls) did not enjoy working together.

From the first lesson, the following points emerged as lessons learned:

- Most pupils had a very dependent attitude. They lacked initiative very much, and were not self-confident in solving a problem. Every time after they finished a task, they always asked me (the teacher) to come closer and check if what they did were correct or not.
- I had difficulties in organizing the class because the pupils shouting many times asking for helps. The classroom was also too small so that I could not move easily from one group to the others to give guidance.

- In solving a contextual problem, the pupils could not explain about what they did, how they did it, or why they did it, neither orally nor in written.
- The problem in the mixed groups (boys and girls) was because of the pupils' culture. In their everyday life, it is rarely seen that boys and girls are doing activities together. So they were shy to work together in one group.

Finding from lesson 2

The tasks in lesson 2 were similar to those in the lesson 1. Dealing with the problems found before, I made a plan for this lesson as follow:

- using Overhead Projector (OHP) to attract the pupils and to focus their attention to the process of solving the contextual problems;
- minimize the intervention of the teacher in order to reduce dependent attitude of the pupils;
- making agreements on not shouting, putting a hand in the air when wanting to say something.
- in grouping the pupils, they could choose their friends themselves.

However, this planning did not go well. It was the first time the pupils followed an instruction using OHP. Some pupils came closer to see the OHP and played with its light, and the others were laughing when seeing the shadows were moving on the screen. Pupils from other grades (they did not have lessons at that time) were also curious, especially about the use of OHP and presence of the observers in the classroom. They stood in front of the door and made noise.

Most pupils still asked 'what should they do now and next?'. I tried to motivate them to think themselves by giving hints and/or rising stimulating questions. This effort worked for most of the pupils, but still did not work for some pupils who were very weak in basic mathematical concepts. (they could not draw a simple geometry objects; they also still used their fingers to count 3 x 4, and did not know the results of 8 x 7, a half of 6, a half of 9, etc.). These pupils really needed guidance step by step in solving a problem.

The frequency of pupils' shouting in asking for helps and clues was reduced, although sometimes they forgot the rule. The motivation of most pupils to work in groups was increased, and they also started to give the reasons for their solutions orally as well as in writing, although most of those reasons were not relevant to the questions. It was also found pupils' tendency just to get the results and did not pay attention to the process in solving a problem. For example, some groups preferred dividing the tasks among the group members in order to get the answers as soon as possible, rather than having a discussion to find the answers together.

The findings mentioned above can be seen as the effects of the traditional way of teaching as these pupils were almost never work on contextual problems and the teacher never conducted working group. As a consequence, the activity and creativity of the pupils were not developed well because lack of opportunities.

I learned from the two lessons that the pupils needed time to get used to the new approach (RME), therefore some more efforts had to be done to realize it. Below is summarized the efforts were done in the next lessons (3 –10) and the impacts that these had.

Lesson 3-7: the efforts and impacts

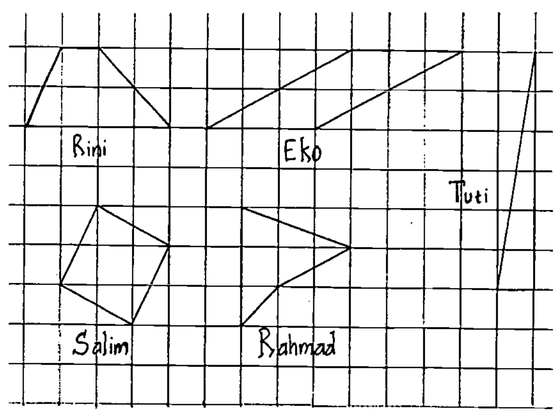
Firstly, the effort related to the condition where the pupils were not used to the contextual problems. In the third meeting I read the contextual problems for the pupils orally, instead of just let them read and solve the contextual problems by themselves. Sometimes I changed the context (became not exactly the same with those in their book) to make the problems more interesting so that the pupils could come inside the problem and then they feel responsible or have motivation to solve them. After reading a contextual problem, I took some times to rise questions; for example: *Who can explain what the problem is about? Who get an idea to solve the problem? Who has* *another idea?* This tactic could work well. The pupils started to give their contribution in solving a problem, though their opinions were frequently not relevant. But by emerging democratic condition (not just saying right or wrong for what the pupils said) in the classroom, the pupils were not afraid anymore to mention their idea.

The positive impact of this effort was found in the fourth meeting. In this meeting the pupils worked in groups of 4 with special assignment in which a member in a group should write down the answers on the blackboard. I observed that most pupils were very enthusiast in doing this task. Each group had a discussion to find the answers instead of dividing the tasks among the group members (as they did before). They were glad when they finished one task then could show the result on the blackboard directly (the groups competed each other).

Secondly, the effort related to pupils' tendency just to get the results and did not pay attention to the process. I succeeded to stimulate them in changing that attitude after applying some rules in the class. I told the pupils that they would not get a maximum mark if they could not show or explain the process and reasons in solving a problem. Moreover, I also wrote the notes in pupils' exercise books, asking them to explain the processes and reasons every time they worked on their homework. This effort had an impact in that the pupils started to give explanations or reasons. Even at the beginning most of their reasoning was very weak, but after few meetings most pupils showed an improvement. The next example shows an improvement of a pupil (Astrid).

In the first two meetings, Astrid was very weak in reasoning. Every time she compared "the size of shapes" she wrote '....... *is bigger than......., because it is looked bigger* or *when I measure it, it is bigger'*. In the third meeting she wrote *'when I compare it, and tried to trace it, I found......'* eight times in solving the problems. However, in the seventh meeting she could come with nice idea when she worked on the problem below.

Rini, Eko, Tuti Salim and Rahmad drew the shapes below. Did they draw shapes with area five square units? Explain your answers.



She used reallotment strategy to explain her answer on this problem:

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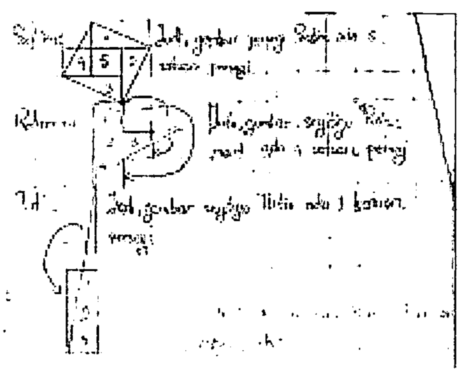
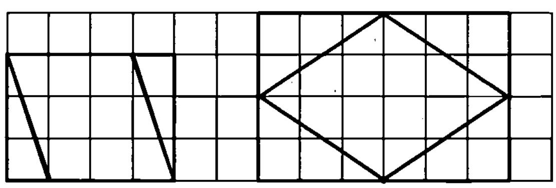


Astrid found that the drawing of Salim was 5 units square, Rahmad was 4 units square units, and Tuti was 3 units square using reallotment strategy.

Attitudes of the pupils and parents

There were also found interesting facts related to pupils' and parents' attitude. Firstly, in checking the solutions of the exercises or homework, the pupils preferred to do it classically so that they could express their happiness (by shouting) if their answers were correct. They also asked me to put the mark on their exercise book every time they finished an exercise or homework. This was not only for the proud of the pupils themselves (especially when they get 10) but also because the parents always ask the marks the children get every time they back home from the school.

Secondly, some parents helped their children doing the homework. But the main reason for this was only to increase the mark of the pupils (the marks for the homework used to be considered in determining the final mark). They did not pay attention on pupils' understanding, because when I asked the pupils about what their parents told them they could not explain. The next is an example of what the parents taught their children.



To determine the areas of shaded figures above, the parents told the children to use the formulas of parallelogram (for the figure on the left) and kite (for the figure on the right). It seemed that the parents only think about topic 'area' as merely playing with the formulas (at this moment the pupils have not learned the formulas yet). In fact, the problems could solve easily using reallotment strategy or by halving (without knowing the formulas).

5. Conclusion

There were many obstacles in applying RME in Indonesian mathematics education. Nevertheless, this first pilot with RME had many positive impacts on the teaching-learning process in the classrooms. The difference in the learning behavior of the pupils found from day to day showed that RME is a potential approach for teaching and learning mathematics. Based on the interviews with a number of pupils it was know that they like the new approach. They realized that there were some positive changes on themselves especially in reasoning, activity and creativity. The teacher himself admitted that there were positive changes on the pupils' behavior after they dealt with RME-based lessons.

In conclusion, RME is an approach to mathematics education developed in the Netherlands, but the exploratory research reported here shows that this approach is not something impossible to utilize in Indonesia. But to realize this, a big effort is needed in the areas of curriculum development, assessment practices, and teacher (in-service) training, all supported by focused development research and formative evaluation to assure that 'local' relevancy will be obtained. The efforts needed should not be underestimated as the change touches on the roots of mathematics education in Indonesia: it is necessary that all stakeholders understand that not only a new curriculum and a new pedagogy is needed, but above all that the notion of what is good mathematics education has to change (see Fullan, 1991). Therefore, a process of changing to the mathematics curriculum and culture towards introducing RME in Indonesia is only possible with the support of the government. The government has to play an important role, in the first place by providing the budget that is needed to facilitate the research and development in all three areas mentioned above. But also in order to develop a policy on mathematics education that provides the formal and 'administrative' support that such a change of the national curriculum and assessment approach needs. Moreover, the teacher training institutes may become the first "targets' for change, as they have to play a central role in preparing the teachers to be capable of teaching and disseminating RME.

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THE STUDY OF MATHEMATICS COMMUNICATION ON INTERNET WITH PALMTOP COMPUTER

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ABSTRACT

Internet technology enables us to develop distance education system with the web site. A number of experimental studies for virtual university on web sites already existed. On the one hand, students need help of tutors or teaching assistants to learn mathematics collaboratively in each course. Instead of graphing calculator, the palmtop computer which enables access to the Internet is expected as strong next generations' mathematical exploration tools for collaboration in classroom (no computer lab) or for tutoring on distance education. For technological innovation of mathematics teaching on this context, the experimental research of mathematical communication with palmtop and Internet environment is necessary.

To design a palmtop environment for mathematics communication over the Internet as the newest mediational means for mathematics and to analyze how it works, this study developed and improved BBS sites. By experimenting with these sites, difficulties are clarified from the perspectives of grounding (Baker et al, 1999) and mediational means (Wertsch 1991). The different BBS designs strongly influenced the quality of communication. In the pilot study, two experiments illustrated that it is not easy for novice users of the environment to get the common ground such as image that is necessary to communicate mathematical ideas but we can communicate and collaborate on mathematics even in a small palmtop environment if we are accustomed to that environment or the environment is good designed for communication task. From this study, two no mathematical content factors were clarified for enabling communication with it. The first involves ways of communication in mathematics such as asking for better mathematical explanations, asking for conditions to be checked, confirming what the other party is saying, and general greetings. The second involves that users have to accustom to use palmtops such as BBS and DGS. Before the experiments, we expected that we easily collaborate as well as the communication on desktops but experiments well illustrated that the different BBS designs strongly influenced the quality of communication. These results implicated the specific environment help us to find how we depending on hidden common ground based on paper-pencil and face to face communication.

1. Introduction

Today, some universities request each student to bring laptop computer. On the other hands, most undergraduate students in Japan have their own mobile telephone which enables access to the Internet and their own electric palm size dictionary. By 2005, each classroom in Japanese schools must have Internet equipment and calculator companies expected the palmtop computer which enables access to the Internet, instead of the graphing calculator, as the next mathematical exploration environment in the mathematics classroom. There are a lot of research studies in education regarding using the Internet on the desktop or laptop environment. For example, we find studies described as 'Computer Supported Collaborative Learning' (Dillenbourg, P., 1999), 'Distance Learning' and 'Distance Education' (Fabos, B. and Young, M. 1999, http://mcs.open.ac.uk/icme/). However, mathematics education research on the palmtop environment has just begun with new palmtop computers for mathematical exploration such as the CASIO Computer Extender (CEx). Indeed, at the undergraduate level, every mathematics course has a lot of teaching assistants who help many students understanding collaboratively. The palmtop computer with mathematics exploration tools must be a strong for their collaboration in distance situation.

With this pilot study, we aimed to develop an experimental environment for mathematical communication on the palmtop computer, to analyse how it works and to recognize what kind of support is necessary. We developed the Bulletin Board Communication System (BBS) on the web site using CGI script for the CEx and researched how it works for mathematical communication. For this purpose, we analysed two experiments from two perspectives Socio-Historical-Cultural perspectives by Wertsch, (1991) of the functions and restrictions of mediational means for describing features of developed environment; and the perspective of collaboration as the grounding process for mutual understanding through communication (Baker, M. et al 1999).

2. Developed Environments and Setting

The Computer Extender (CEx) exists only on a palmtop computer in 2001 that is able to use Mathematics tools such as a Computer Algebra System (Maple), Dynamic Geometry Software (GSP) and Graphing Tool, and can connect with the Internet using Internet Explorer in Microsoft Office for Windows CE 2.0. Based on the experience of our previous study in which the Internet is used for collaborative mathematical problem solving between Japanese and Australian classrooms (Isoda et al 2000), the web pages of BBS for problem posing and communicating solutions were developed with the CEx's window size of 640x240 in mind.

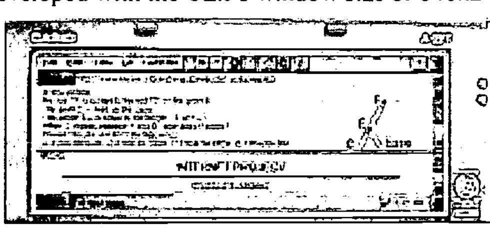


Figure 1. First Design of the Top Page.

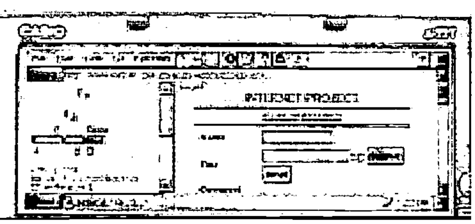


Figure 2. Second Design of the Top Page.

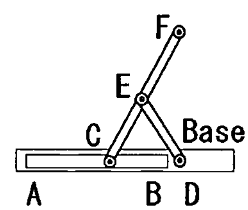
In the first experiment, the page design consisted of two parts divided horizontally, the upper for reading the problem and the lower for communicating solutions (Figure 1). It aimed to show more messages at once because we expected long messages as well as the experiences of the previous study on desktop computers. We found that it was difficult for users to read the problem while writing their solutions. In the second experiment, the page design consisted of two parts divided vertically, the left side for reading the problem and the right side for communicating solutions at the same time (Figure 2).

There are a number of restrictions with Internet Explorer (IE) on Windows CE 2.0. We can download the file through BBS but we have to use Outlook for sending the file. We have to inform recipients to renew BBS content by telephone because IE on the CE 2.0 does not accept automatically renewed settings. CEx with Windows CE 2.0 has a QT keyboard and we can input by Pen on display, but the drawing tool by Pen does not exist.

In the developed environment, BBS worked as the mediational means for communication between both sides. For the experiments, we preferred graduate students who had experienced learning mathematics in English (because CEx is only available in English fonts) and at the same time who were novice users of the desktop computer in mathematics (because experts can be expected to work well in any computer environment). Because we had to teach them how to use mathematics software, we set the communication between the teacher and a subject (student) with the help of a tutor. The subject used the CEx but the teacher used a desktop computer for sending the file through BBS. We recorded the student's working on VTR. We used the following problem which was expected to involve DGS. The easiest way for a novice to use mathematical software on CEx involves using a DGS file for simulation and it is necessary in mathematics communication to share visual images with mathematical language.

Problem

In this picture; the rod CF is joined to the rod ED at the point E.
The point D is fixed on the base.
The length ED is equal to the lengths CE and EF.
When C moves between A and B, how does F move?
F moves along a
a) sinusoidal path. b) curved path. c) circular path.
d) straight line path. e) different path from a-d.



The problem were used for pre-service and in- service teacher program several times and most of teachers could not get right answer but the solutions are very simple. Thus, it is good for collaborative problem solving.

3. The Result of Experiment 1

Experiment 1 using the first BBS design (figure 1) included four episodes (see Episode 1-1 to 1-4). In each episode, the left hand side activity is the subject person's activity based on observations by the researcher, who helps the operation of CEx, and the right hand side is the reported activity of the teacher (another researcher).

At Episode 1-1, the student (she) tackled the problem on paper as in figure 4 and selected c as the answer. From the reaction at S1, the teacher imagined that she had recognized a circle as in figure 5 and asked for reasons. Then, the student understood the conditions at S3 as in figure 6. Until the description of the conditions, the teacher believed they shared their images such as those of figure 5 and figure 6 (but her image is actually like figure 8). The teacher imagined figure 7 from S3's words of 'moving around AB'. Thus, the teacher confirmed the student's response and asked for the center because the teacher wanted to change the student's image of figure 7. But the student's real image was figure 8 and she replied at S5 that C was the center point. The teacher recognized that the student had some mistaken image, and so asked her to read the problem once more at T6. She tried to read the problem but did not work through the whole problem and only read a part of it. At S7, she described her images, and at T8, the teacher lost ground in

Episode 1-1. Teacher began to confuse what student said.

Figure 4 Drawing at S

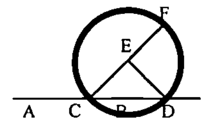


Figure 5
Expected at T2

S1(5/23,13:19) It will be a circle. (Like figure 4)

T2(5/23,13:23) Yes, Nh. Why did you choose the C(circle)? (teacher expected figure 5)

S3(5/23,13:28) Because DE=CE=EF, F is moving around AB. (Student drew the figure 6)

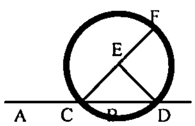


Figure 6 Drawing at S3

T4(5/23,13:39) Hi Nh? You thought point F is moving around AB, did you not? Please let me know which point is the center of the circle? (Teacher imagined figure 7)

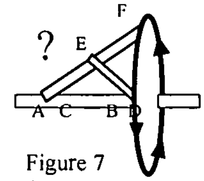


Figure 7
Expected at T4

S5(5/23,13:44) Here C is a fixed point. (Student imagined the figure 8)

T6(5/23,13:50) My question is "Which point is the center of the circle?" You did not read the problem; the C of rod CF moves between AB. Please read the problem.

S7(5/23,14:01) EF is the fixed rod, C, D is the base point. So I think that the center is 03 C while CF is moving around AB. (Drew the figure 8)

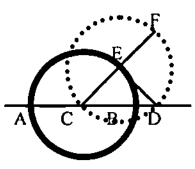


Figure 8 Drawing at S7

?

T8(5/23,14:) Dear Nh. Thank you very much. We want to continue it next session. Please consider the problem for a while and let me know. With Best Regards, Maha.

Episode 1-2 Student used pencils for the model

Figure 9
Modelling at S11

S9(5/25,9:29) Let me know what the problem is.

T10(5/25,9:35) Please let me know if your solution is changed or not.

S11(5/25,9:40) My previous answer was not correct. This will be a circle. (Used pencils to model the mechanics like figure 9 and drew figure 10)

T12(5/25,9:44) Please let me know your answer mathematically. If it is a circle, please let me know the center and radius. Please read the problem once more.

?

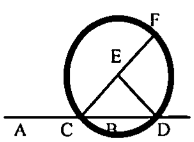


Figure10
Drawing at S11

communication (Baker, M. et al 1999) and thus asked her to reconsider and redo the problem.

At S11 of Episode 1-2, the student reworked the problem with the help of a pencil model and got the locus as a circle. The teacher hoped she would change her invalid image at T10. At S11, the student replied that her answer was not correct, but the answer was still a circle. Thus, at T12, the teacher was unsure what the student imagined, and so asked her for a more mathematical description.

At Episode 1-3, she began to ground as well as the teacher, but they were not successful. Indeed at T13, the teacher began with a greeting as well as reference to previous episodes. Then, at

S14, the student responded with a greeting (it was the first time) and expressed her desire to solve the problem. At T15, the teacher sent her an attached GSP file because he felt it difficult to continue communication without correct grounding of their images. At S16, the student replied that she had found the locus was a line. At T17, the teacher believed that they shared the image of figure 13, but unfortunately the student's image at S16 involved the motion of C. At S16, the student dragged the line AB as well as the point C; she could not focus on the motion of F. Up to Episode 1-3, she had displayed skill in communication via the Internet, but had not displayed skill with GSP.

At Episode 1-4, both student and teacher succeed in synchronising their ideas. The student at S18 began communication with greetings and also displayed skill with GSP.

Episode 1-3 Student simulated with GSP

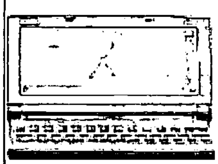


Figure 11 Simulate with GSP

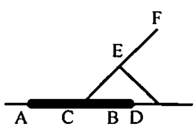


Figure 12 S16 dragged the line AB not only C.

T13(5/28,8:56) Good Morning, Nh. Did you have any new ideas over the last few days?

S14(5/28,9:02) How are you? Yes, I am thinking about the previous matter. Please let me know the problem again.

T15(5/28,9:05) Yes, Nh. Please use the attachment file "GSP" to understand the problem. Move the point C!
(Sent the file *"55.gs4"*, figure 11.)

S16(5/28,9:15) Today in thinking about the previous problem, I reach the conclusion that if the point C moves along the rod AB, then it will be a line and coincide with the rod AB. (Drag as in figure 12 and figure 13.)

T17(5/28,9:18) Yes, Nh. If you move the point C, F moves along a line. Please let me know the reason tomorrow. (Teacher expected figure 13 but student imagined figure 12)

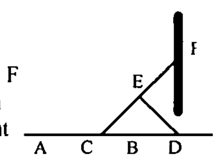


Figure 13 Teacher expected

Episode 1-4 Synchronized communication

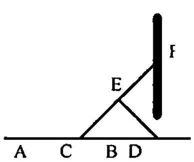


Figure 14

S18(5/30,10:17) How are you? What are you doing now? I am now thinking about the previous matter. The previous matter I misunderstood a little, regarding point F. Let me write: If the point C moves along AB, then it is line that coincides with AB, but F moves along DF; it is a perpendicular. (figure 14)

T19(5/30,10:23) Yes, I am fine and you. Oh, you were thinking about only point C. The problem is the motion of point F, is not it? Thus, as you wrote, the point F moves on the perpendicular line to AB. (figure 15)

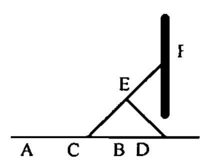


Figure 15

3. Discussion for Experiment 1

Episode 1-1 to 1-4 illustrate the difficulty of mathematical communication in a developed environment, mediational means (figure 1), the selectable strategy for sharing ideas, and where difficulties arise. We will analyze these points from the grounding process for collaboration and Socio-historical-cultural perspectives.

Michael Baker et al (1999) defined grounding as the process for reaching common ground of mutual understanding, knowledge, beliefs, assumptions, presuppositions, and so on that were claimed to be necessary for many aspects of communication and collaboration. A number of research studies report on the difficulty of communication or collaboration over the Internet due to the lack of common ground. Episode 1 also illustrates this difficulty. In episode 1, the most influential grounding factor is the difference between the images of the student and teacher. At episode 1-1 and 1-2, the teacher could not picture the student's images and thus asked her to explain mathematically and read the problem once more. However, the student could not easily begin the problem over the web and explain the motion with appropriate mathematical conditions on the problem. At this stage, the teacher's strategy for grounding is to ask the student to explain the image mathematically and to read the problem to confirm the conditions. The teacher did not succeed, and then, at episode 1-3, preferred using a file for sharing the image as the next strategy. Use of the DGS file was expected to lead to a sharing of the image and the answer. The teacher hoped it would help to construct a pseudoconcept of mechanics before mathematically explaining the mechanical motion of F. Indeed, we had other good experiences to suggest that it helped in explaining the motion without mechanics. But at episode 1-3, the student dragged C and responded regarding the motion of C rather than that of F, because this was her first experience of using DGS. After she became accustomed to using DGS, she found common ground in the images at episode 1-4.

Roschelle and Teasley (1995) defined collaboration as a coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem. Lee (2000) illustrated that collaboration in mathematical problem solving is analyzed from two aspects: object-oriented activity and interaction-oriented activity. Until the collaboration of episode 1-4 with common ground, there are some remarkable changes in the student's responses. At episode 1-1 and 1-2, the teacher gave a lot of interaction-oriented messages such as greetings as well as object-oriented messages such as asking the student to explain mathematically and to read the problem. Interaction-oriented messages are important teacher's strategies to continue communication using restricted mediational means; unfortunately, the student just responded with object-oriented answers. At episode 1-3, she began to reply with interaction-oriented messages as well as object-oriented answers and at episode 1-4, she began the interaction-oriented message by herself. The changes illustrate that the student needed these experiences of Internet communication on the BBS to synchronize with the teacher. At episode 1-4, the student became a user of the mediational means for mathematical communication.

From analysis of the difficulties, the following functions and restrictions of developed mediational means (Wertsch 1991) are clarified. First, the BBS in the design in figure 1 is functional for posing problems and text communication, and enables file download but not file transmission. At the first stage, the miscommunication of images which is not easy to explain by text is unavoidable. Thus, there is a need for a grounding activity to promote the sharing of each other's images. The teacher preferred the strategy of asking for conditions, but until episode 1-4, the student failed to understand the conditions precisely, because she could not read the problem and messages simultaneously. Second, DGS on CEx is designed for sharing images. But if sharing is to occur, the user has to recognise ways of using and observing. If the novice user cannot use the DGS file, she cannot see the same point as clearly as the expert. Third, it is necessary for the student to use traditional tools such as paper and pencil for reasoning. In particular, at episode 1-2, we see that the pencil also functioned as an aid in representing a model of mechanics. The pencil model helped the student's images, but the mediational means (BBS) restricted its use to communication.

4.The Results of Experiment 2

We determined that it was difficult to read both problem and messages on BBS simultaneously with the first BBS design (figure 1). This design was not suited to confirming the conditions of a problem while communicating. Thus we changed the design from that in figure 1 and

Episode 2-1. It looks flexible shape

T1(9:10) Hello Pusan. My name is Mathe. I am looking forward to your reply.

S2(9:21) It may draw a circle.

T3(9:33) Good morning, Pusan. OK, you chose C, a circle. Please let me know the centre of it.

S4(9:37) Oh sorry, I didn't read problem carefully. I would like to change my answer to a or b.

T5(9:40) Hi Pusan. You changed your answer from c to a or b. Can you tell me where F goes when C moves B to A?

S6(9:45) Cause of Point D is fixed on the base and it is a rod. It's not flexible so may be F will may draw a line.

T7(9:49) Oh you changed your answer from a or b to d, right? Why did you image the motion a or b and now you changed the image d. Could you explain me why d must be answer?

S8(9:53) At firs I thought it like a flexible shape so it will be the motion like a wave or curve. But it's a rod it is not flexible so when it move, it will move in a straight way so I prefer chose d.

T9(9:57) Aha, Pusan. You thought the motion of F based on the motion of rod. I attached a very interesting file by "GSP". Please explain your result mathematically.

Episode 2-2. Students looked other part of figure.

Figure 16

S10(10:07) Cause at first I didn't understand what I have to do. Sorry. Well after I view a figure that you sent, I think my answer should be change to b. F may be curve (figure 16)

T11(10:12):Pusan, did you drag the point C? Please use the locus command. Firstly, delight the point c and f. Secondly, chose the locus command from the construction. Then, please drag the point c, again.

S12(10:22):It's very interesting. F may draw a circle and E is a center of it. (figure 17)

Figure 17

T13(10:27):Yes, EF=ED=EC. Thus, there is a circle that the center is point E and radius is EF, ED and EC. Please read the problem once more with comparing the GSP.

S14(10:38):F move with "curve line." (figure 18)

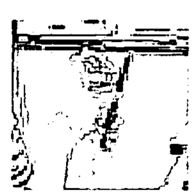


Figure 18

T15(10:48):Pusan, please let me know the locus on GSP, mathematically. I do not think it is a curve.

S16(10:50):F moves on a line

T17(10:53):Yes, F moves on the line, which is perpendicular to the base AD. Can you prove why F moves on the perpendicular line using the conditions you already knew?

experimented with how the second BBS design, shown in figure 2, works. Episodes 2-1 and 2-2 took place within the second design.

At episode 2-1, the effect of the new design is illustrated from the beginning. At S4, the student replied that she read the problem once more without the teacher asking. From S2 to S8, she changed her answers, because she reflected on her solution with the conditions of the problem based on the teacher's questions. At S8, she misunderstood the conditions but then understood that the rod is not flexible. Because the teacher believed they already shared the same image, the teacher sent her the GSP file.

At Episode 2-2, the student changed her answer again at S10. The teacher asked her to use the drawing and locus functions of GSP at T11. At S12, she replied with a different observation of the drawing. Then, at T13, the teacher asked her to read the problem again for reconsideration. At S14, the student changed her answer again and at S16 obtained the correct answer.

5. Discussion for Experiment 2

Comparing experiment 2 with 1, communication was synchronized from the beginning of Episode 2-1, but the GSP file is not helpful for sharing ideas. These results gave us some view of grounding and the function and restriction of mediational means.

First, the different BBS design altered communication significantly. From episode 2-1, the student could review her ideas based on each message from the teacher and the conditions of the problem. We cannot see such synchronized communication from the beginning in experiment 1. The BBS of figure 2 functioned on text as well as the BBS of figure 1, but the design in figure 1 did not enable messages and the problem to be compared simultaneously. The design of figure 2 enabled simultaneous comparison and functioned better for communication because this new BBS supported the student's reasoning. Indeed, even if student and teacher could not share their images, the teacher succeeded in the grounding of images at episode 2-1 without DGS because the teacher's strategy for sharing images functioned well in this case. It was easy to compare the student's images with the teacher's questions and the conditions of problem. In addition, the teacher's strategy in the second experiment changes for the better compared with the first experiment. At episode 2-1, the teacher began his message by confirming what the student said. It enhanced both object-oriented and interaction-oriented collaboration. Second, DGS on CEx also did not work from the beginning in Experiment 2 but did work at the end. Because it was also the first time the student had used DGS, she did not know which part of the figure to observe in the situation. It is difficult for the novice to know what to observe even if we tell them by text. Third, traditional tools are necessary even when the DGS file is made available. Indeed, at episode 2-2, the student used the pencil model as well as the DGS file. For the novice, traditional tools have an important role.

6. Conclusion

In order to design a palmtop environment for mathematics communication over the Internet and analyze how it works, this study developed and improved BBS sites. We successfully experimented with how such sites work and clarified difficulties from the perspectives of grounding and mediational means. Due to the mediational means developed, BBS sites functioned well with respect to text communication but were not easily able to exchange mathematics software files. The endeavor of grounding for sharing images is necessary for communicating mathematical ideas. The different BBS designs strongly influenced the quality of communication. In order to share images, it is necessary to have a simple means to compare the conditions of a problem with questions posed in communications from the teacher. The teacher's strategies of asking the student to provide mathematical explanations and to read the conditions of the problem worked only when the subject could easily compare them. On the other hand, it was difficult for novices to share images with DGS. Thus DGS use could be also seen as a grounding factor in these experiments. The pencil model as a traditional mechanism was a common ground for face to face communication but it is impossible to use over the Internet. These results are in agreement with the idea of affordance from the general theory of cognitive design science (Norman, 1992).

From the pilot study, both experiments illustrate that we can readily communicate and collaborate on mathematics in a palmtop environment if we are accustomed to that environment. Because the Internet provides a new form of communication, users need to accommodate to this environment. This study clarified two factors regarding this. The first involves methods of communication such as asking for better mathematical explanations, asking for conditions to be checked, confirming what the other party is saying, and general greetings. The second involves methods of using the CEx, such as how to use BBS on the Internet and how to use DGS.

It can be expected that the palmtop will evolve into the equivalent of today's desktop. At the same time, we expect that findings relating to the palmtop, such as the design of the BBS, will remain valid into the next generation.

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FOSTERING STUDENT ENGAGEMENT IN UNDERGRADUATE MATHEMATICS LEARNING USING A TEXT-BASED ONLINE TOOL

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ABSTRACT

This paper presents an example of an introduction of text-based online activities in a core unit for non-mathematics science students, which focuses on the development of numeracy skills. The purpose of the online activities was twofold: firstly, they served as an organising device to help students work consistently throughout the semester, and secondly, they provided an opportunity for students to learn from each other. The trial was carried out to address the problem of student disengagement from university life, an emerging trend observed in tertiary institutions which is strongly related to failure and attrition.

The approach of integrating online tasks to on-campus activities is described, and the results of the trial are discussed, including student and staff evaluation. Finally, the paper looks at possible roles that online text-based tasks may take to enrich the educational environment in the context of undergraduate mathematics teaching and learning.

1. Introduction

One of the big challenges academic teachers face today is the decline in student involvement with the university and in their academic performance. An Australian study on trends in the first year undergraduate experience (McInnis, 1995 and McInnis, James and Hartley, 2000) found that students are spending less time on campus and more time in paid employment. The studies suggest that "compared to students who do not work, younger first year students who work part time are more likely to not work with other students on areas of their course, and to study inconsistently throughout the semester" (McInnis, James & Hartley, 2000). Similar trends have also been reported in the US (Astin, 1998 and Kuh, 1998).

Academics are being urged to put forward creative ideas to address this disengagement from university life and apparent lack of commitment, to think of new ways of engaging students that would fit with their lives. Colleges and universities are exploring ways to make students' experience, particularly in first year, more engaging and successful.

Online environments and communication tools offer unparallel opportunities to enrich the learning experience, to provide students with more flexible programs to fit in with their multiple commitments, to foster student-student and staff-student interaction, and to give students a sense of belonging to a learning community regardless of their physical location. However, approaches using online interaction have not been widely used up until now in the area of undergraduate mathematics teaching and learning. The communication technologies available today such as the internet, e-mail and discussion group facilities present serious challenges for the communication of mathematics; these are primarily text and graphics based and are not ready yet for the easy and user-friendly communication of mathematical symbols.

There are, however, ways to support undergraduate mathematics teaching and learning with text-only based online tools. This paper presents one such approach used in the context of a unit that aims at developing numeracy skills for science students, and suggests approaches that could be applied in mainstream undergraduate mathematics teaching and learning.

2. A case study

The students undertaking the Bachelor of Science at Monash University show the same patterns of disengagement and lack of motivation reported in the Australian study on trends in the first year experience. Over the last years, science has also been an area of high student attrition and failure; according to a recent report, science is one of the areas with lowest completion rates, with less than 60% of Australian science students completing their degree (Martin, Maclachlan & Karmel, 2001).

Here is an account of the approach taken in one of the core first year science units with the aim of addressing the problem of disengagement and improving students' first year experience. It was first run as a trial in second semester 2001, with the intention to apply the same approach in two other areas of first year science in the academic year 2002. The approach taken was informed by the growing body of literature which suggests that rich learning environments, active student participation, and a strong sense of community can make a positive difference in fostering student success and engagement (Tinto, 1987).

2.1 A core unit for science students not majoring in mathematics

The unit involved in this case study is the first year core unit *The design of science* taken by all science students enrolled in the Bachelor of Science degree course, and who do not have the intention of majoring in mathematics. Students are accepted to the degree with no prerequisites;

the majority of them have not done mainstream mathematics at school. The aim of the unit is the development of generic skills, with an emphasis on numeracy skills such as experimental design, collection and analysis of data, sample surveys, modelling of data and mathematical modelling. The teaching and learning activities revolve around project work carried out in weekly workshops, in which students conduct investigations following the scientific method and write a report including the methods they followed and their conclusions (Varsavsky 2001).

Given the skills-based nature of the unit, it requires from students a continuous engagement with the unit throughout the semester. It also requires students to work on open-ended projects, where students have to decide how they are going to carry out their projects, rather than follow steps given by the instructor. This appears to be the most difficult hurdle to overcome for first year students which, combined with the students' growing isolation within the university system, leads to frustrating learning experiences.

2.2 The use of online activities

In second semester 2001, online tasks were added to the existing teaching and learning activities as an attempt to help students keep their pace and support collaborative learning. The online tasks formed an integral part of the unit activities together with workshops and projects.

The interface used for the online activities was *InterLearn*, a new collaborative web-based learning tool developed at Monash University. *InterLearn* is an online tool designed to support greater interaction between learner and teacher and between learner and learner by facilitating a shared construction of knowledge and understanding. Its first version, developed by Len Webster and David Murphy (Murphy, 2000), was used with postgraduate students and, given its success, the university is now developing *InterLearn* as part of a suite of flexible learning tools for staff to assist them in developing student-centred flexible learning environments.

InterLearn is built on a database structure that allows students' individual text-based responses to online activities to be stored and viewed on demand. Students log on to an individualized worksite where they complete set activities mostly by entering responses into dialogue boxes. The activities can be shared, meaning that they are available for viewing by all course participants, or individual, meaning only the participant and the teaching team can access them. An important feature is that students' responses can be edited, to allow for the development of their tasks after viewing the submissions made by their fellow students, and so facilitate the construction of knowledge and understanding.

The *InterLearn* worksite for *The design of science* was structured around semester weeks. When students logged on to the site, they saw a week-by-week schedule, and below each week, the unit activities that they were required to complete during that week, both in the face-to-face workshop and in their own time.

Some of these weekly activities were online assessable tasks. The tasks were short and focussed, and although each of them had their own objectives, the common aim was to help students to get ready for the workshop or the new project they had to work on. Before the introduction of these online tasks, tutors always had the difficulty of leading a discussion on the topic of the workshop, mainly because students came unprepared, but in many cases also because students found the open ended projects too difficult to handle. At weekly meetings with tutors, the dominant comment was about the "blank student faces" staring back at the tutor expecting directions from him/her rather than coming up with their own suggestions on how to approach the project under discussion. The weekly online tasks had the aim to facilitate the discussion between students in preparation for the forthcoming workshop, to emulate the discussion at the start of the workshop that in the past was so difficult to lead. There were no tutor contributions online, students had to construct their own suggestions between themselves, through an iterative process of submitting their responses online, reading and assessing other students responses to the tasks, and editing their own responses.

Here are some examples of the kind of online tasks we had in the unit *Design of science* during the trial phase, grouped by categories:

Design of strategies. There was one such online activity for each of the four projects, which had to be completed before the start of the workshop where the relevant project would be discussed. Students were asked to read the project requirements and think about how they were going to carry it out. For example, in the project that involved answering the question "How does the wing length of a gyrocopter affect its flight time", students had to think of a hypothesis and design an experiment to test it and submit their design before the next workshop for other students to view. In the past, this was the topic of a discussion conducted by the tutor at the beginning of the workshop, which proved to be hard to lead because students came unprepared or did not know where to start, and in many cases the tutor fell under the pressure of giving too much guidance. With the online tasks, students were able to write up their own hypothesis and experimental design, supported and re-assured by the responses given by their peers. The tutor, who read students responses before the start of the face-to-face workshop, could tailor the discussion around these, focussing on the main points and clarifying aspects that showed to be poorly understood.

Commenting on and sharing of results found. This was also an activity that appeared very often as all projects involved either collection or modelling of data. This kind of online activities required students to post numerical summaries and interpret their meaning. For example, in the project that asked for the average surname length of people living in Melbourne, after designing the sample survey, collecting the data and calculating the numerical summaries, students were asked to post the mean and standard deviation and explain their meaning. Students then used the summaries posted by their peers to interpret them in the context of the Central Limit Theorem. This exercise might look very simple, but proved to be very useful for students to understand the meaning of the standard deviation and the standard error.

Assessing someone else's work. This approach was used early in the course in the context of scientific writing, with the aim of helping students to become aware of the structure and style used in scientific reports. They had to read two pieces of work from students who undertook the unit the previous year and comment on the good and bad points of each of them. This was another case where the online task, which students carried out by sharing their responses, proved to be much more effective than a face-to-face discussion lead by the tutor.

Reflection. A reflective online activity was included at the end of the semester. Students had to elaborate on what they learned in the unit, what progress they made in the development of the intended generic skills, and where would they apply these skills.

Feedback. In the workspace for each week, students had the option to provide feedback on their personal development, on the unit as a whole or on a particular aspect of it. The feedback could be either anonymous or signed.

2.3 Evaluation

The trial involved an ongoing evaluation including the observation of the development of students' online responses, student online feedback (signed and anonymous), fortnightly interviews to the members of the teaching team, and a student focus group interview at the end of the semester.

Overall, the results of the trial were very positive and encouraging. The tutors already had the experience of running workshops for this unit for at least one semester, so they could compare the student engagement with the unit to their previous experience. Reports from tutors indicate that students kept a more consistent working pattern throughout the semester and that students' understanding of what they were required to do improved. The rate of successful completion of project work also improved significantly. Some workshop groups were however, more successful than others in remaining engaged with the unit and in their performance; it was established that this was due primarily to the tutors failure in conveying to the students the role of online activities in the process of construction of knowledge, and could be prevented in the future with appropriate training.

Observation of the evolution of students' responses and interview with students also indicated that many students were using the sharable and editing facilities of the online tool: very often they modified their responses after reading the responses of their peers. This was particularly more noticeable in the first half of the semester; "feeling confident about what was required to carry out the project" and "too many assignments for other subjects" were the main reasons given for it.

It was also observed that a few shy students, who would normally not participate in class discussion, were very active in the online environment, and often they were the brave ones to publish first the response to an activity for their peers to see.

Through the online feedback, which was unsolicited and had an open format, the most often comment students made about the online tasks was that they helped them to keep the consistent pace required by the unit throughout the semester.

3. Text-based online activities in the context of mathematics learning

Our experience shows that online activities that facilitate the construction of knowledge and understanding between groups of students could have a positive impact on the students' first year experience. It could help students to have a sense of belonging to a learning community and improve their chances of success. It is also an approach that fits better with the current students' lifestyle and commitments.

Our experience also shows that such online activities, even though they are primarily text-based, could also work in the context of mathematics teaching and learning. Text-based online tools cannot be used easily to publish information which includes mathematical symbols, but they can still play an important role in setting a rich collaborative learning environment. All examples of types of activities given in §2.2 still apply in the context of mathematics:

Designing strategies could be used to force students plan ahead how they would tackle a project, what will be the steps to follow and how will they know that the results are correct.

Commenting on and sharing of results found. Very often students solve a problem (either by hand or using a mathematics software) but do not stop to think whether their result makes sense. In many settings, such as statistics, a further activity could involve the use of the results obtained by the whole class group.

Assessing someone else's work could also become a powerful learning experience; with creativity, online activities could be designed which only require text-based assessment. For example, the teacher could publish on a website the solution to a problem given by a former student, carefully labelling the various parts of it, and ask students to explain in words what they think about specific parts of the solution or to provide an overall assessment.

Reflection is also a powerful learning activity, one that is not used very often in the context of mathematics. For example, asking students to elaborate on what they think they learned by doing a particular assignment or after a module was completed, and how did that relate to other things they know, could help them to take deeper approaches to learning.

There are many other possibilities. For example, text-only online activities could be designed to help students understand proofs, with an online task that requires them to comment on a proof (published on a web site with labels for the important parts), focussing on a particular assumption, or explaining why a particular step was necessary.

In summary, the possibilities are numerous, limited only by the imagination of the teachers. The examples given here assume that the teaching and learning of mathematics focuses on problem-solving situations, with an emphasis on explanations, justifications and activities that require students to go beyond blind symbol manipulation.

In the case study presented in this paper the specific online tool *InterLearn* was used which has the distinctive feature of allowing the editing of students responses, but similar although somewhat less powerful activities could be designed with the more widely available tools such as online discussion or conferencing tools. The case presented here involves first year students, but a similar approach could be valuable also in to higher years.

4. Conclusion

Online text-based environments could play a significant role in helping students to feel part of a learning community without requiring them to be physically on campus. As shown in the case study presented in this paper, even though the available communication technologies are not yet ready for the handling of mathematical symbols, they could still be used effectively to foster student engagement and deep approaches to learning in mathematics courses.

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SELF-GUIDED AND CO-OPERATIVE LEARNING – scenarios and materials

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ABSTRACT

What can scenarios of self-guided and co-operative leaming look like? How can knowledge be consolidated by means of intelligent practice? How can new media make the teaching and learning of mathematics more exciting? These are three main questions that the German pilot study 'SelMa - self-guided learning in mathematics in senior high schools' tries to answer.

Teachers of the pilot study have created different scenarios of self-guided learning. They describe the role of new media for mathematical exploration (e.g. CAS) as well as for presenting the topic to be learned (e.g. hypermedia). They focus on co-operative working and point out the teacher's role in each learning arrangement. Up to now suitable classroom material for some scenarios has been developed and systematically tested by other schools (evaluators) to determine if it is suitable for everyday use.

This paper presents the following learning arrangements and materials: a learning caroussel, an electronic learning environment for constructive learning and a jigsaw-puzzle. How learning diaries, mindmaps and communication via email and world wide web can support the individual learning processes will also be demonstrated. Results of first evaluations are included.

The current state of affairs is documented online (http://www.selma-mathe.de). This site (in German) offers a wide range of material that can be tried out and adapted to the teacher's individual needs. It is intended to be a platform for communication and co-operation between teachers working in the field of self-guided learning of mathematics as well.

Keywords: self-guided learning, learner-centred teaching , innovative pedagogical methods, changing role of tutors, learning environment, learning caroussel

1. A pilot study: organization, aims and evaluation

The four-year pilot project 'Self guided learning in mathematics in senior high schools" -called **SelMa** started in early 1999. Its aim is to analyse the interdependencies and interactions between mathematics, learning in general and the use of new media.

Main questions focussing upon learning, mathematics and the use of media are

- Which mathematical topics are suitable for the idea of self-guided learning?
- How should classroom material be arranged and presented? In particular: online or offline?
- How should students and teachers be supported?
- How can the progress of learning be 'assessed'?
- How can new media improve the quality of learning (mathematics) and stimulate life-long learning?
- How can knowledge be consolidated by means of intelligent practice?
- How can telecommunication (platforms for collaborative learning or a "teacher-on-demand" via email) support the learning of mathematics?

A team of 3 to 4 teachers from 5 schools in North Rine-Westphalia, called *'authors'*, began - addressing the issues mentioned above - to create scenarios of self-guided learning and develop suitable classroom material.

A second group of 10 schools, so called *'trial-schools'* were incorporated in the pilot study, when the first projects had been finished and successfully tested by the authors in their own classes. The trial-schools are to evaluate the material and to systematically test whether it works in everyday use. To analyse the materials and concepts from an educational point of view, suitable evaluation tools are created. Monitoring is done by academics and experts of mathematical departments of the universities. Furthermore, authors and evaluators are going to disseminate their practice in in-service-teacher-training, in order to build up networks of schools in the different regions. As the project takes place in an "open workshop" on the internet (www.selma-mathe.de) other schools can participate at an early stage. A wide range of materials that can be tested and adapted to the teacher's individual needs is offered. The aim of the 'SelMa-website' is to be a platform for information, communication and co-operation between teachers. Including *publishers* in our project leads to effect high-quality media (offline and online), for the work in the periods of self-guided and co-operative learning of mathematics.

2. At a glance: different scenarios and concepts

The first projects were based on rather different ideas of self-guided learning. The authors could not base their work on concrete concepts or learning arrangements because in German mathematics education there is still a severe lack of comprehensively documented research material so that ideas could be transferred to other fields within mathematics education.

In one scenario of self-guided learning, **electronic learning environments** are intensively used. Students use these environments in longer periods in mathematics lessons as well as at home. The material consists of a **hypertext** with exercises, contextual aid, a glossary, solutions and general ideas how to optimize individual and collaborative learning in school and at home. Another group of authors established an **independent learning centre** (for all subjects) at their school. Some parts of our mathematics curriculum have been set aside for self-guided learning, that means that these topics are not taught collectively in mathematics lessons but outside the framework of the school timetable. The students have to study them on their own without any support from the teacher. These learning environments consist of a course with a rather linear structure - with graded aid, suggested solutions and a collection of problems, in particular real-life problems of different categories. It provides opportunities for simulation and visualisation of mathematics. It acts as a tool in order to free the students from laborious calculations, which often distract from the actual problem.

Other authors tried to describe a scenario with a systematic change between instruction, self-guided learning by the **group-jigsaw-puzzle-method** (see Figure 3). In the initial phase, several groups work on different problems. They solve the problem, discuss and clarify anything that is still unclear. These 'experts' have to transfer the knowledge gained during a second phase when new groups with experts for different problems are formed. Schools that evaluated the material and this method stated that it worked well on topics that can be seen from different perspectives. Classes which worked on this method for the first time had problems in the beginning of their work because weaker students feared failing as teachers in the second part of the jigsaw-method.

Another means for increasing student activity and self-guided learning is the method of the **'learning carousel'**. Ten to twenty different stations (exercises, real-life problems depending on the subject) are offered to the students. Some stations deal with a special task, a new mathematical context, others invite students to exploration or investigation of mathematical problems using handheld computers. Each station offers special aids on how to approach the task and other hints suitable to the students' needs and a paper with a complete solution. All students receive a 'to-do-list', which informs them about all the stations (number; title; topic; obligatory or additional station; individual, pair or group work, media). Students can choose the order of tasks and might individually (or in groups) choose their learning pace.

These learning arrangements carry certain dangers. During periods of self-guided learning teachers automatically change their roles from acting as instructors to being supporters of individual learning processes. Usually teachers cannot exactly measure how much has been learned by the groups and the individual students. Students must be capable of monitoring their learning progress on their own, but this ability has to be acquired in a similar way as subject matter has to be learned. **Additional tools** like learning diaries, mindmaps and electronic communication tools might support this process of self-assessment (*see 3.3*).

3. Scenarios, material: use in the classroom and evaluation

3.1 Learning environments and evaluation of material

The pilot study 'SelMa' offers two examples of learning environments, *'linear programming/optimization'* and *'matrices'*. Educational research tells us that learning and understanding mathematical concepts and using problem solving strategies work better if there are various approaches with real-life problems of various levels accommodating different types of learners. So we drafted hypermedia-learning environments with some interactive parts concerning visualisation or intelligent practice. As different details, conclusions and relations between single mathematical topics (that are required to understand a mathematical topic) are presented in a linked-up, not a linear structure, the learners are facilitated to create their individual mental network of mathematical knowledge.

'Linear optimization': This learning environment has been created for the revision of concepts around linear functions. The students choose one out of a range of problems (on the basis of brief descriptions of the problems), which make up the 'heart' of the learning environment, and then they are guided through the important steps to solve a mathematical optimization problem At the same

time they revise what they have learned about linear functions at lower secondary level. The learning unit links new contents and mathematical concepts with topics that the students acquired in previous mathematics lessons (and possibly have forgotten in the meantime). One part of this learning environment deals with the learning process and the monitoring by the students themselves. In the learning environment students find, e.g., advice for self-assessment and hints how to optimize group work and their study at home.[9]

'Matrices': In this learning environment students are offered several real-life problems related to the same mathematical topic of the subject 'matrices'. They choose one problem that they are interested in, then they are 'guided' through the problem (which is posed rather openly) not step by step but by more general questions concerning strategies of problem solving, by a glossary or by questions that prepare the formation of the mathematical theory behind the problem. New definitions and theorems etc. will be discussed in whole-class teaching. Students can see that different problems lead to the same mathematical concepts. They can easily built up the theory of matrices with a minimum of help from the teacher. Fundamental operations on matrices are found and correctly defined by using technology for exploration or as a black-box (Derive or handheld TI 89/92 or built-in java-applications).[7]

Both learning environments, intended for the use in the classroom and at home, offer details that support orientation and self-guided learning in hypermedia:

- survey of the subject to be learned
- table of contents
- glossary and review of the topic
- some recommended paths
- different modes of representation and visualization and interactions (as often as possible)
- some interesting historical facts of the subject and real-life applications
- exercises with contextual, graded help
- a chapter concerning learning strategies, problem solving and self-assessment of the learning process.

The material includes practical advice for the teacher, who becomes an *individual adviser* when students work with this learning environment. He acts as *moderator*, when the results of the group work are presented and general methods to solve problems are discussed by the whole-class.

The **results of first evaluations** show that it works very well if the teacher chooses some of the problems leading to the same mathematical topics. If the students are working on different problems, they will often not solve them because they lack parts of the theory that are required for the chosen task. Periods of self-guided learning do not have to be too long. Whole-class-teaching is necessary to deepen theory. As the material is based on HTML, some teachers modified some of the problems, added or reduced hints and solutions or integrated documents, links to websites and interactive visualizations.

3.2 Learning Carousel and its evaluation

The project "Geometry of Circles" consists of two parts. In the first part the students investigate the equation of a circle and then create - using CAS or a graphing calculator - a mathematical description of a logo, a window of a church, a pattern or a model of an existing object containing several circles. Here, students can see the importance of geometry in real life. The students work in groups of two or three and have to present their results on posters or WORD-documents to the rest of the class.

The second part of the project is based on the method 'learning carousel', often practised in elementary schools. It focuses on the development of new aspects of coordinate geometry and it consists of problems that connect the new geometric object 'circle' with other objects like parabola and lines (tangent, points of intersection, ...).

The problems are presented on worksheets and in files, first with the help of concrete exercises, then by generalizing the solution. Each station consists of the worksheet, some helpful questions and a complete solution. Some stations are more graphically oriented, e.g. including investigations of families of curves or a puzzle in which descriptions, graphs and equations of circles have to be matched.

The fact that small groups of students work on different tasks according to their suitable learning pace enhances individual learning. The complete learning carousel consists of 10 stations. The stations are accommodated to different background levels of learning, different speeds of learning, and different modes of working (individually or in groups of two or three)..

Different media are used at different stations, e.g. the CAS DERIVE or the TI-89 calculator. The tasks are usually activity-oriented. The students normally work in groups of two or three and decide together at which station to work next. At each station the materials lie on a table during the whole lesson. There are 3 or 4 copies of each station so that the students really have a choice of what to do next. During the work the teacher answers questions from each group. In our first evaluation we noticed that students only tentatively used the additional aid, which was put on a table further away from the exercises. First they tried to help each other, then they asked the teacher who had much more time to give individual advice than in traditional lessons. Collaborative work is highly supported by this method. The students did not look at the solution provided without trying to solve the problem on their own.

First **evaluations** show that:

- A convincing structure of the different tasks and items of the learning carousel seems to be very important for the organization and the success of learning.
- Students have to be introduced into this way of working and have to learn to get by in the time provided for this task.
- There must be a summary and/or a test after using the learning carousel
- Work with more than 10 stations has to be interrupted by short periods of whole-class-teaching to summarize and to see if any support is necessary or not. Most of the teachers admitted that this method required more flexibility and presentation skills than teaching lessons that are more teacher-oriented.

3.3 Mindmaps, "learning logs" and communication tools and evaluations of their use

Many evaluators of the material state that, especially after long periods of self-guided learning, weaker students sometimes did not know whether they had learned all topics and understood all relations between new and old subjects. **Mindmaps** can support the review of the main steps of the learning process in different ways.

First, a mindmap containing only main topics can be completed individually after a period of self-guided learning. So the individual automatically reflects his or her own learning progress. New facts are linked to details of the 'old' individual network of knowledge. Different mindmaps - that means different points of view - can be presented and discussed in class.

Second, a mindmap of the subject matter can be constructed in whole-class teaching and can be used to summarize the topic with all items of the subjects, with definitions, examples and the relations between them, at the end of a learning unit.

Third, mindmaps can accompany the learning unit to show which aspects of a problem/topic have already been examined, what the next steps are and how many different aspects are still to be analysed. The mindmap is completed when the learning unit is finished. When all topics are numbered the mindmap visualizes the outcome of the course as as well its results.

We and the **evaluators** of the study often used electronic mindmapping tools like '*Mindmanager*'[1] or '*Inspiration*'[2] that offer various features e.g., structuring branches on different levels that can be moved to other positions, annotations and links to different documents and websites. Most of the students were rather acquainted with the method of mindmapping. They liked visualizing the topics and relationships between them in their notes. They added formulas, annotations and other documents by links. When asked why they liked this tool, they answered that they automatically discussed relationships in greater detail and became better aware of the structure and relationships between mathematical topics than before.

Learning logs are a means to encourage students to continuously reflect upon their learning progress. In an introductory session students of Year 11 were informed about the aims of this method and the prospective contents of their learning log. A learning log - only read by the student and the teacher - should contain all important facts of a lesson (steps towards a new topic, definitions, proofs, examples) and may include a personal review (What did I learn? What was difficult for me to understand? How can I memorize it?). The diaries were checked (annotated if there were mistakes) and assessed by the teacher every three months. The SelMa-website[6] presents 25 diaries in the following six fields of reflections: lessons, aha-effects, individual explanations, self-assessment, analyses of mistakes, and further issues. Most of the students who kept a diary with a lot of personal annotations stated that they felt better prepared for the tests in comparison with the beginning of the school year because they had paid more attention to their weaknesses (see: www.learn-line.nrw.de/angebote/selma/foyer/projekte/lerntagebuecher/index.htm)

We have also started to gather experience with **collaborative online tools** like **BSCW**[1] or **Web-CT** that are used in longer periods of self guided learning. Using these we hope to encourage encourage students to share information, to help each other and initiate discussions with experts.

All students had access to the internet at school, most of them at home, too. At the beginning of the first longer period of self-learning we invited students of Year 11 (40 students) to use a workspace in the internet. We prepared a forum with FAQs to post individual questions concerning the topics of the previous lessons. There, students got answers first from the 'teacher on demand', then later from other students as well. Weaker students could find intelligent practice, links to interactive online-tools and visualizations, brighter students sometimes (!) used worksheets with more demanding tasks and experiments e.g. with CAS instead of the given homework.

First evaluations produced different results:

- Groups without any experience with electronic communication (except personal emails) were not convinced of the benefit of this additional teaching aid. They did not like to pose questions in the forum and seldom used the offers for intelligent practice.
- Groups that were acquainted with communication tools used the forum more intensively. They annotated the applets and mathematical online-tools to inform the others whether they were helpful or not.
- The workspace that had been "prepared with different offers" *before* starting was accepted better and more intensively used than a rather waste workspace that had to be filled with FAQ.

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9. project "Linear Optimization": http://www.learn-line.nrw.de/angebote/selma/foyer/02b_hammproj1.htm
10. project "Geometry of Circles":
    http://www.learn-line.nrw.de/angebote/selma/foyer/02b_hammproj2.htm
11. project "Mindmaps": www.mathe-selma.de/
12. Personal website: http://www.mathematikunterricht.de/

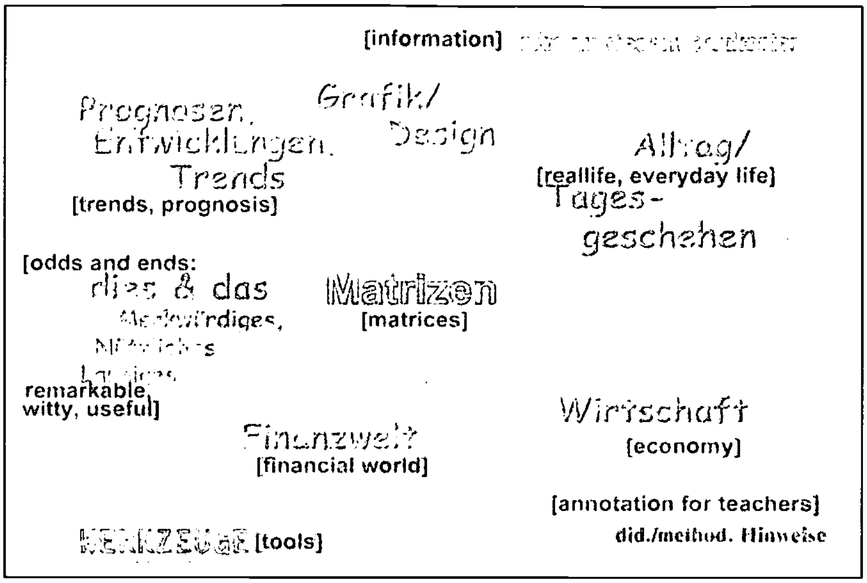

Fig.1: main page of the learning environment 'matrices'.

"To-Do-List" Circles

This is your own "To-Do-List". Please mark the station which you have accomplished, and note questions, which are still open.

Nr.	topic	important to know	how difficult?	o.k.?
1	finding equations of circles	revision	✓	
2	intersection line-circle		✓✓	
3	tangent line		✓✓	
4	points of intersection of 2 circles		✓✓	
5	does every equation fit to a circle?		✓	
6	family of circles	additional	✓✓✓	
7	puzzle of circles, equations and descriptions	revision/self-check	✓	
8	circles and lines	Nr.2	✓✓	
9	more complex exercise	last station?	✓✓-✓✓✓	
10	equation of a tangent line, that reminds of the position of the circle	Nr.2 before	✓✓✓	

Fig. 2: "To-Do-List" for the learning carousel about the subject 'circles'

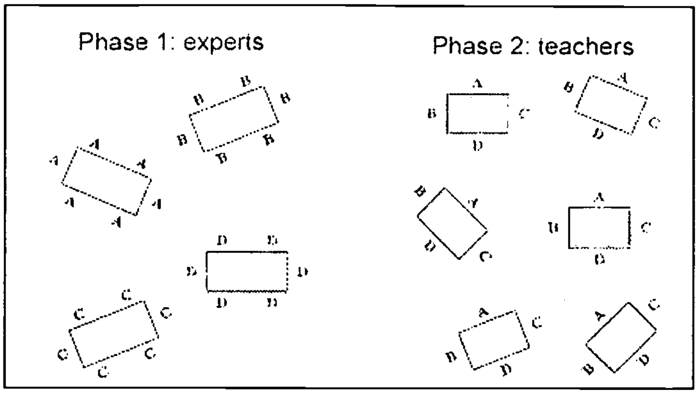

Fig. 3: organisation of a jiggsaw group puzzle

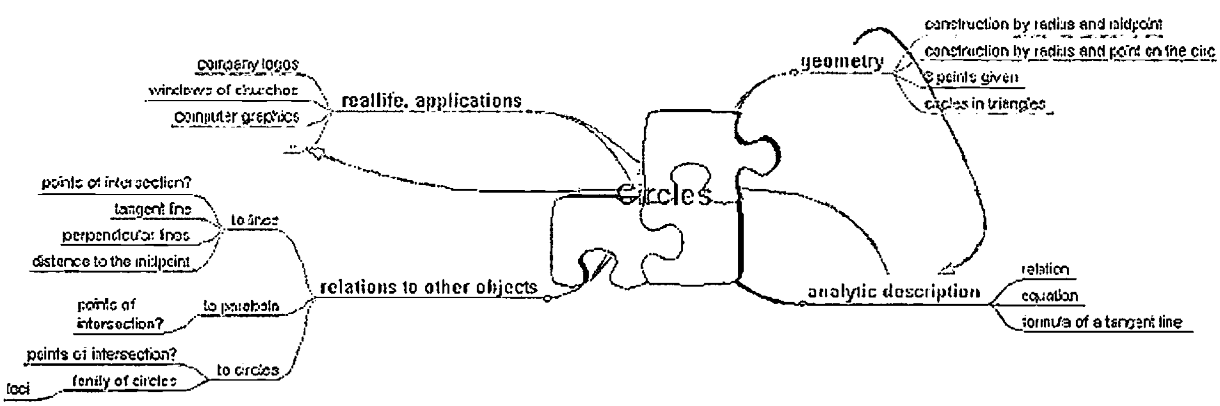

Fig.4: Mindmap of the learning subject "circles"
(translated in English, made with the help of the software 'Mindmanager'[5]).

E-LEARNING IN MATHEMATICS UNDERGRADUATE COURSES
(an Italian experience)

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ABSTRACT

The teaching of mathematics in Italian Universities is going through a period of deep transformations, partly due to general reasons and partly to national ones. The strongest drives are probably:

1) a recent reform of the Italian University system, which allows every single University more autonomy and decisional power than in the past;

2) the deep changes occurred in the past decades all over the world in the perception of the relations between mathematics and its applications;

3) technological innovations, and the major changes they imply both in teaching methods and in the mathematical contents we teach.

In March 2001 a group of nine mathematicians and computer experts working in Università Bocconi in Milan – a well-known business University – started a project focused on integrating heavy e-learning technologies into the traditional structure of Mathematics courses for undergraduates.

We would like to present at Creta ICTM-2 Conference a comprehensive description of our experience: the project (March-July 2001), the courses (September 2001-April 2002) and a first analysis of the results (May-June 2002). We chose to present at the Conference three independent papers (see also the works by M. Impedovo and F. Iozzi); each one takes a different point of view.

The first part of the paper describes the Italian context and our project, following the above framework.

The second part analyses some aspects of the project, referring in particular to the courses in which the author is more deeply involved:

1) a complete e-learning course, specifically dedicated to students with poor performances in mathematics (approximately 100 students);

2) a basic e-learning course, to be further developed next year and to be dedicated, presumably, to all first-year students in Università Bocconi (approximately 2500 students).

Finally, we try to draw some conclusions.

1. The context

In this paper we try to write down a whole year's experience on the use of e-learning technologies as a support to traditional classroom teaching. We describe the context in which it took place, the main features of the project and the reasons for some of our choices, and then we try to draw some conclusions.

We don't claim to point out any new and brilliant way; our main goal is to tell our story, and possibly compare it with others who are or have been in similar situations in other countries.

Since the academic year 2000-2001, a reform of the Italian University system has organised its first steps in two levels:

1) a three-year "short degree";

2) followed by two optional years, ending with a five-year "specialised degree".

Till the year before, the Italian system provided as a first step only a four-year first degree.[1]

Moreover, many reasons led to a considerable stretching of the actual length of the degree, up to an average of more than six years; and also to great many students giving up their University studies, with the percentage of graduates with respect to matriculated students running as low as 30-40%.

It's a widespread opinion that this reform will succeed if and only if:

1) the idea will get through that the "real" degree, that is the one sufficient for the majority of occupations, is the three-year degree;

2) both the lengthening of studies and the abandonment of studies phenomena will be reduced.

It is not difficult to foresee that a fundamental issue about the future three-year degree courses will be that of coping with a reduction in quantity and quality levels of the University studies, while trying at the same time to guarantee an acceptable standard (as least as professional needs are concerned) and to increase the efficiency of the system in a significant way (both in terms of length of studies and percentage of graduates).

With the reform each University has obtained more autonomy than in the past, and can decide for example to reduce or increase the space given to each subject in each degree course, remaining within a broad range established by the central authority.

The distribution of Italian students whose course of studies contains mathematics as an important subject has changed in the last decade: in particular, there has been a reduction in the percentage of students belonging to scientific faculties, and a dramatic cut in strictly mathematical courses; the percentage of students in Engineering and Business Administration courses is either stable or increasing.

Thus, the majority of math students, courses and professors is referable to faculties where mathematics have not a strong academic status; this implies that the spaces for mathematics, in the immediate future, will go through possible reductions or at least through long and difficult negotiations. The relations with other disciplines in the course of studies and with our colleagues will be very important, as well as the existence of reliable and positive evaluations on the effectiveness of math courses for the purpose of the global education of students.

In the last decades, a fundamental change has taken place in the mathematical community with regard to the perception of the relations between mathematics and its applications. The image of the Bourbakists' follower, sitting and writing his neat formulas in an ivory tower, unshakeably sure that in a few centuries' time a prince will understand their great utility, come and bring them

[1] This account of the Italian University system oversimplifies the real situation; many faculties, such as Medicine or Architecture, have a completely different story.

to life with an enchanted kiss is no longer plausible. And this has reflected on mathematical education, of course; though we must say the Italian situation is behind times in this respect.

The observation that a mathematics course in a faculty such as Engineering or Business Administration must be strongly related to the other courses and to the overall and specific preparation required from the students doesn't sound so obvious in Italy; a few years ago many Italian mathematics professors simply did not care about the opinion, prevailing just outside their office doors, that math courses are a separate and almost useless body in the students' curriculum.

But things change. And these changes are mainly due, of course, to the conceptual changes occurred inside our subject; but also to the needs for negotiation of academic spaces mentioned above.

Finally, a very important element is technological innovation.

First of all, due to the changes which have taken place inside mathematics and mathematical education. We all understand that the ways of mathematical research and mathematical education, the greater or smaller importance in this historical period of this or that research field, the choice of the subjects we favour in our teaching are all matters which have been modified by the coming of Computer Era. But also because of external reasons, related for example to the academic world that surrounds us.

The wealthiest and farthest-seeing Universities are today eager to invest energies and resources on the use of technological innovations in education. Presumably some of these investments will turn out to be unproductive, but the idea that in the future the issue of education will not do without a deep technological involvement appears strong and widely shared.

In some Italian Universities, Mathematics Departments and Institutes are curiously unprepared to understand these changes; they even risk to be considered a resistance factor to technological innovations. On the contrary, a correct scientific attitude should naturally lead us to an unbiased judgement towards technological innovations. This would also have the positive and not negligible secondary effect of increasing the esteem of the academic world in our capability of participating to a common project.

2. The project

In the last three or four years, Università Bocconi launched and encouraged many different projects related to the use of technological innovations in undergraduate courses.

In March 2001 I proposed to my colleagues in the Institute for Quantitative Methods, at Università Bocconi, to start a project focused on the integration of heavy e-learning technologies in mathematics courses for first-year undergraduate students.

Two problems showed up immediately:

1. the big increase in the amount of work connected with teaching, implied by this project;
2. the doubts on the compatibility of these technologies with some specific issues of mathematics (for example: the difficulties in manipulation of symbols and formulas, the problems connected with the evaluation process).

The first problem is a very serious one.

The Italian University system has not many ways of encouraging the quality of teaching: a great part of the academic career of a University professor is based on the quantity and quality of his scientific production (I will not consider the strong co-optation mechanisms, more or less effective as far as the quality of the recruited personnel is concerned, which are typical of the Italian academic system). Anyway, Università Bocconi is a private University and has enough autonomy and resources to provide a non-standard academic role, with satisfying contractual conditions and exclusively dedicated to teaching. We decided to count on this kind of academic personnel (*quorum ego*), as it was possible in this case to guarantee correct economical and professional incentives; and to seek only enthusiastic volunteers.

First of all, I involved my friend Michele Impedovo; then we gradually built up a group of nine mathematicians and computer experts. The big push that our University is giving to the issue of technological innovations has done the rest, providing us with favourable working conditions and enough economical and human resources.

The second problem has turned out to be a fake. As it often happens, the statement that "yes, it would be nice, but with the teaching of mathematics things go differently; you can't do that with mathematics" has revealed to be a defence behind which to hide, in order to cover our natural difficulties to come to terms with changes and control them. In this year's work we solved many problems connected with the writing of formulas, with automatic assessments, and other problems; the problems we could not solve, we put them apart or managed to go round them.

In April 2001 we began to build up the web-courses we would carry out in the following academic year.

As far as we know, the use of e-learning technologies in Italian Universities is not very common; when they are used, one of the following two software is employed: *Blackboard*, originally developed by Cornell University (and now bound in a strong partnership with Microsoft), and *Learning Space* by IBM-Lotus.

Blackboard is employed by Università Tor Vergata (Rome), Università Cattolica (Milan) and Università Bocconi (Milan). *Learning Space* is employed by Bergamo, Brescia, Modena, Padova, Pavia and Venice Universities, by Milan and Turin Polytechnics and by Università Bocconi (Milan). They usually organise single pilot-courses, not yet fitted in a comprehensive project; and there does not seem to be a co-ordination of all these experiences, although in the last year CILEA, an Inter-University Consortium, has taken some steps in this direction (see the URL www.teorema.cilea.it).

The only structured projects are, as far as we know, those of Milan Polytechnic and Università Bocconi; both projects utilise *Learning Space*, though in two different versions which are not completely comparable.

Milan Polytechnic has opened, in the academic year 2000/2001, the first On-Line Degree in Italy (in Computer Engineering); here *Learning Space* courses (4.0 version) are meant to be a *substitution* of traditional classroom teaching (see the URL www.laureaonline.it).

Università Bocconi has developed since 1999/2000 a different project, in which *Learning Space* courses (3.5 version) are meant to be an *integration* to traditional classroom teaching; the project foresees that for each traditional course a parallel web-course will be developed, and tries to guarantee a strong co-ordination of these courses by proposing common yet flexible standards (see the URL www.uni-bocconi.it/weblearning).

We got in touch with Roberto Lucchetti, a Milan Polytechnic professor who in 2000-2001 was responsible of an on-line mathematics first-year course for the Degree in Computer Engineering; we understood this direction can be equally fascinating, but decided that we were more interested in e-learning technologies which are not a substitution but an integration to classroom teaching.

In the end, following the suggestions of A.S.I.T., the Department that deals with web-learning technologies in Università Bocconi, our workgroup decided to utilise *Learning Space* (3.5 version) as an integration to traditional courses.

We organised five different web courses:

1) A complete course, dedicated to students with poor performances in mathematics (a better definition is the following: all students registered at Università Bocconi since more than three years, who have not succeeded in giving the first-year mathematics exam); this course concerns approximately 100 students, and we will refer to it with its code number 271.

2) A basic course, dedicated to all first-year students in the Business Administration Degree; approximately 1200 students, code number 5015clea.

3) A basic course, dedicated to all second-year and third-year students in the Business Administration Degree; approximately 300 students, code number 4009clea.

4) A complete course, dedicated to first-year students who are particularly interested in technological innovations (students belonging to a brand new degree called Economics of International Markets and New Technologies); approximately 150 students, code number 5015clemit.

5) A complete course, dedicated to first-year students with a mathematical and quantitative high profile but without particular motivations in technological innovations (students belonging to a degree called Social and Economic Disciplines); approximately 150 students, code number 5015des.

I was in charge of the first three courses, with the help of Giovanni Paolo Crespi and Maria Beatrice Zavelani Rossi; Michele Impedovo was in charge of the fourth one, with the help of Fabrizio Iozzi; Annamaria Squellati was in charge of the last one; Anna Marotta and Marcella Gombos were responsible for the web implementation of the courses; Margherita Cigola contributed to the general framework of the project and to its overall management.

3. The courses

Learning Space (in the 3.5 version) is made up of four main environments: *Schedule*, *Media Center*, *Course Room*, *Assessment Manager*.

The *Schedule* contains the instructions on the available course material, and associates it to the single lessons. The *Media Center* contains the available material, which can be grouped by type or by subject. The *Course Room* is an on-line discussion forum dedicated to all course students and teachers. The *Assessment Manager* allows the construction, distribution, collection and evaluation (either automatic or not) of homework and exams.

As we will see, these four environments have been used in different ways according to each different course.

1. Course 271

The classroom course had the following structure:

a) 80 lesson hours, given by me, on traditional subjects (elementary functions, series, differential calculus, integral calculus, linear algebra, financial mathematics); I rarely used a computer in the classroom, I emphasised on applications to economy and finance.

b) Approximately 80 hours dedicated to *tutoring* and exercises in small groups, partly organised by me and partly by my two colleagues, as a reinforcement to the subjects explained during the lessons.

The on-line course had the following structure:

a) In the period April-September 2001 we built up a large data bank in the *Media Center*, containing all exam papers assigned for that course in the last three years; students have access to the data bank to consult/print complete exam papers or single exercises, recorded under various keywords (for example: all multiple choice exercises assigned on differential calculus, regarding economic applications).

b) During the course, with the help of some students (to whom the University guaranteed the payment of a small sum), we put in the *Media Center* the slides of all classroom lessons; at the end, in April 2002, the entire course will be on line. After many hesitations we chose to scan the hand-written slides (in fact, we scanned a polished rewriting of the slides effectively used in the classroom), and not to create a *Word* or *Latex* version of them. Students considered this material as very helpful, but I must admit its preparation has taken a lot of time.

c) We put in the *Media Center* the exam program, additional exercises, simulations of exams to come, many *Mathcad* files and other material.

d) Our *Course Room* has been rather lively, although mainly centered on teachers' communications and students' questions; it was not the place in which to pose or discuss interesting additional mathematical problems (I considered the peculiarity of the course, which was intended for students with particularly poor performances in mathematics; they were surely much more interested in 'finally passing this exam' than in 'exploring the infinite beauty of mathematics').

2. Courses 5015clea, 4009clea

These were two identical courses, and we kept them separate for formal reasons only.

The classroom course was similar to that of course 271, although it included less *tutoring* hours.

The on-line course was made up of the *Media Center* only; it contained an analogous data bank on previous exams, and also the exam program, simulations of exams to come, many *Mathcad* files and other material. We did not create slides out of the lessons, we did not use the *Course Room*. It was a basic course, really.

We will develop this course next year; one of the most interesting characteristics of this kind of course is, in fact, the possibility of building and modifying it year by year.

3. Course 5015clemit

This was the most interesting and innovative course of all; it is described in detail in the works of my colleagues Michele Impedovo and Fabrizio Iozzi, which will be presented at this same Conference (see references at the end); thus I will give only a short description of it.

The classroom course had the following structure:

a) Approximately 110 lesson hours, covering a larger program than that of course 271 (for example: many-variables differential calculus and optimisation, dynamic systems, a larger number of topics in financial mathematics); computers were largely used during the lessons, and this fact had a considerable influence not only on the presentation of the subjects but also on the choice of the mathematical contents to privilege.

b) Approximately 25 hours dedicated to computer laboratory activities, essentially centered on the use of *Mathcad* software.

The on-line course had the following structure:

a) The *Schedule* contained detailed instructions on the use of materials connected to each lesson.

b) The *Media Center* contained essentially the large number of *Mathcad* and *Excel* files used during the course. As it is a new course, no data bank of the previous exams has been provided.

c) The *Course Room* has been used in a very active and lively way, with a strong interaction between teachers and students; as my colleagues explain in their papers, they have tried to carry out an instance of *computer-assisted collaborative learning*.

d) They used many of the evaluation instruments (both automatic and non automatic evaluation) provided by the *Assessment Manager* environment.

In this course the evaluation process has greatly involved the use of *Learning Space*, *Mathcad* and computer laboratories; in all other courses, neither *'Learning Space* nor any other computer technology has been used in the evaluation process.

*4. **Course 5015des***

The classroom course has followed a program similar, in broad lines, to the program of course 5015clemit. Computers have been rarely used; however, students did some group homework involving the use of *Mathcad*.

The on-line course has been essentially conceived as a notice board; during the course a large number of tests has been proposed as homework in the *Media Center*. They have been evaluated with non-automatic procedures. The *Course Room* has hardly been used.

4. Some conclusions

At the time of writing this paper (end of January 2002) the courses described above are only halfway, therefore we cannot evaluate our results; we will present a first analysis at the Conference. Anyway, let's try to draw some conclusions.

1. We think the integration of on-line technologies in undergraduate traditional courses is a workable, sensible, useful and almost unavoidable way; the quality of our teaching offer has sharply and undoubtedly improved. From the scarce data we have, we got the feeling that those e-learning technologies which try to substitute traditional teaching activities are, at least at the undergraduate level, less interesting; they seem fit to cover a little, important niche sector rather than to expand to a consistent part of undergraduate courses.

2. The choice of how much web technology, of what kind, and how deeply related to the use of computer technology in the classroom is an open question: there are many possibilities, and we do not have a unique recipe at this regard. On the contrary, we think the possibility of different approaches is, at this stage, an essential resource. A lot depends on the kind of students for whom the course is prepared (the courses 5015clemit and 271 are nearly opposite, at this regard!); and a lot depends on the personality and teaching style of each teacher, as it obviously should be.

3. There are more general reasons that lead us to think that the choice of using e-learning technologies is useful and unavoidable, even beyond its effectiveness in strictly educational terms: Universities are investing a lot in these technologies, and our choice contributes to bring us nearer to the center of an important innovative stream.

4. This choice demands a lot of teaching work, there is no doubt; and - frankly speaking – sometimes it is not highly qualified work. A necessary boundary condition seems to be the fact of working in a University where these issues receive appropriate consideration, and where a fair amount of economic and human resources is available; we have been lucky, but this situation may be quite common.

5. Another important condition is the fact that the professors themselves should consider their teaching work as interesting work, strictly connected with their professional and personal growth; in our Italian experience this condition has revealed to be more delicate and difficult to obtain, but the perspectives for the future are encouraging. In fact, we hope that the spirit of the Italian University reform will lead to a reconsideration of the role of University professors, favouring a divarication between the functional profile of the undergraduate professor and that of other

academic categories; and with the acknowledgement of the prominence of teaching activity in the identity of undergraduate professors.

The perspectives for e-learning technologies in mathematics courses in Università Bocconi seem to be very interesting.

Next year we are thinking of extending our e-learning project, at least in a basic version, to all first-year mathematics courses; and we will implement a similar project to some of the second-year mathematics courses (financial mathematics) and to some statistics courses.

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SUSPENSION OF SENSE-MAKING IN MATHEMATICAL WORD PROBLEM SOLVING: A POSSIBLE REMEDY

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ABSTRACT

In common teaching practice the habit of connecting mathematics classroom activities with reality is still substantially delegated to word problems. But besides representing the interplay between mathematics and reality, word problems often are the sole example of realistic mathematical modeling and problem solving. During the past decades, a growing body of empirical research (e.g. Freudenthal, Schoenfeld, Verschaffel, De Corte) has documented that the practice of word problem solving in school mathematics promote in the students an exclusion of realistic considerations and a "suspension" of sense-making and hardly matches the idea of mathematical modeling and mathematization. If we wish situations of realistic mathematical modeling, that is both real-world based and quantitatively constrained sense-making, we have to make changes: i) we have to replace the word problem solving with classroom activities that are more relatable to the experiential worlds of the pupils and consistent with a sense-making disposition; ii) we will ask for a change in the teacher conceptions, beliefs and attitude towards mathematics; iii) a directed effort to change the classroom socio-math norms will be needed. In this paper we discuss how these changes can be realized through classroom activities based on the use of suitable cultural artifacts and interactive teaching methods.

1. Introduction

In normal teaching practice, establishing connections between classroom mathematics activities and everyday-life experiences still regards mainly word problems. But besides representing the interplay between formal mathematics and reality, word problems are often the only means of providing students with a basic sense experience in mathematization, especially mathematical modeling (Reusser & Stebler, 1997). Recent research has documented that the practice of word-problem solving in school mathematics actually promotes in students a "*suspension of sense-making*" (Schoenfeld, 1991), and the exclusion of realistic considerations. Primary - and secondary - school students tend to ignore relevant and plausible familiar aspects of reality and exclude real-world knowledge from their mathematical problem solving.

Several studies point to two reasons for this lack of use of everyday-life knowledge: textual factors relating to the stereotypical nature of the most frequently used textbook problems ("*When problem solving is routinised in stereotypical patterns, it will in many cases be easier for the student to solve the problem than to understand the solution and why it fits the problem*", Wyndhamn and Säljö, 1997, p.364) and presentational or contextual factors associated with practices, environments and expectations related to the classroom culture of mathematical problem solving ("*In general the classroom climate is one that endorses separation between school mathematics and every-day life reality*", Gravemeijer, 1997, p.389). Furthermore, it has been noted that the use of stereotyped problems and the accompanying classroom climate relate to teachers' beliefs about the goals of mathematics education (Verschaffel, De Corte, and Borghart, 1997).

This indicates a difference in views on the function of word problems in mathematics education. The researchers, and probably the drafters of new curricula such as the Italian one, relate word problems to problem solving and applications. The student-teachers (and probably teachers in general) see another role for word problems. That is as nothing more, and nothing less, than exercises in the four basic operations which also have a justification and suitable place within the teaching of mathematics, though certainly not that of favoring "realistic mathematical modeling", which is "*both real-world based and quantitatively constrained sense-making*", Reusser (1995).

If we wish to establish situations of realistic mathematical modeling in problem-solving activities, changes must be made.

1. The type of activity aimed at creating interplay between reality and mathematics must be replaced with more realistic and less stereotyped problem situations, founded on the use of concrete materials.

2. We must endeavor to change students' conceptions of, beliefs about and attitudes towards mathematics; this means changing teachers' conceptions, beliefs and attitudes as well.

3. A sustained effort to change classroom culture is needed. This change cannot be achieved without paying particular attention to classroom socio-mathematical norms, in the sense of Yackel and Cobb (1996).

In this paper we discuss how these changes can be realized through suitable classroom activities. These activities are related more easily to the experiential world of the student and which are consistent with a sense-making disposition must be designed. They make extensive use of cultural artifacts that could prove to be useful instruments in creating a new link between school mathematics and everyday-life, which incorporates mathematics. We will show how suitable cultural artifacts and interactive teaching methods can play a fundamental role in this process.

2. Connections between classroom activities and everyday-life experience

The connection between students' everyday and classroom mathematics is not easy because the two contexts differ significantly. Just as mathematics practice in and out of school differs (Lave, 1988; Nunes, 1993) so does mathematics learning (Resnick, 1987). Masingila, Davidenko, and Prus-Wisniowska (1996) outlined three key differences between in- and out-of-school practices (goals of the activity, conceptual understanding, and flexibility in dealing with constraints). In out-of-school mathematics practice in particular, people may generalize procedures within one context but may not be able to generalize to another since problems tend to be context specific. Generalization, which is an important goal in school mathematics, is not usually a goal in out-of-school mathematics. On the other hand, many studies have pointed out that local strategies developed in practice are more effective than algorithms which are usually taught in school to give students powerful general procedures, but which are, in fact, often useless in out-of-school contexts (Schliemann, 1995).

Although the specificity of both contexts is recognized, we think that the conditions that often make out of school learning more effective can and must be re-created, at least partially, in classroom activities. Indeed, while there may be some inherent differences between the two contexts, these can be reduced by creating classroom situations that promote learning processes closer to those arising from out-of-school mathematics practices.

Through our studies, and the paradigmatic example that we will present, we wish to make a contribution towards resolving the problem of 'permeability' between school and life experiences (Freudenthal, 1991). As in the Realistic Mathematics Education (RME) perspective of the Dutch school of thought, we think that progressive mathematization should lead to algorithms, concepts and notations that are rooted in a learning history which starts with students' informal experientially real knowledge. In our approach everyday-life experience and formal mathematics, despite their specific differences, are not seen as two disjunctive and independent entities. Instead, a process of gradual growth is aimed for, in which formal mathematics comes to the fore as a natural extension of the student's experiential reality. The idea is not only to motivate students with everyday-life contexts but also "*to look for contexts that are experientially real for the students and can be used as starting points for progressive mathematization*", Gravemeijer (1999, p.158).

Furthermore we stress that the process of bringing "reality into mathematics" by starting from student's everyday-life experience, is fundamental in school practice for the development of new mathematical knowledge. However it turns out to be necessary, but not sufficient, to foster for example "*a positive attitude towards mathematics, intended both as an effective device to know and critically interpret reality, and as a fascinating thinking activity*", as is stressed for example in the Italian primary school program. We contend that these educational objectives can only be completely fulfilled if students and teachers can bring mathematics into reality. In other words, besides *mathematizing everyday experience* it is necessary to *"everyday" mathematics*. This can be implemented in a classroom by encouraging students to analyze '*mathematical facts*' embedded in appropriate '*cultural artifacts*', and which for brevity we might call "*cultural mathfacts*" or "*social mathfacts*". There is indeed a great deal of mathematics embedded in everyday life.

Cultural artifacts embody theories that users accept, even when they are unaware of them (Saxe, Dawson, Fall, & Howard, 1996). Their use mediates intellectual activities and, at the same time, enables and constrains human thinking. Through these subtle processes social history is brought into any individual act of cognition (Cole, 1985).

The cultural artifacts we introduced into classroom activities (e.g. supermarket bills, bottle and can labels, railway schedules, a cover of a ring binder), or those to be constructed by students, e.g. calendars, are concrete materials which children typically meet in real-life situations. We have therefore offered the opportunity of making connections between the mathematics incorporated in real-life situations and school mathematics, which although closely related, are governed by different laws and principles. These artifacts are relevant to children; they are meaningful because they are part of their real life experience, offering significant references to concrete situations. This enables children to keep their reasoning processes meaningful and to monitor their inferences. As a consequence, they can off-load their cognitive space and free cognitive resources to develop more knowledge.

We believe that immersing students in situations which can be related to their own direct experience and are more consistent with a sense-making disposition, allows them to deepen and broaden their understanding of the scope and usefulness of mathematics as well as learning ways of thinking mathematically that are supported by mathematizing situations. This allows students to become involved in mathematics and to break down their conceptions of a remote body of knowledge. Only in this way can we encourage a positive attitude towards school mathematics.

Obviously, usefulness and its pervasive character are just two of the many facets of mathematics that do not entirely capture its special character, relevance and cultural value; nonetheless these two elements could be usefully exploited from the teaching point of view.

3. Cultural artifacts in classroom activities

The use of cultural artifacts in our classroom activities has been articulated in various stages, with different educational and content objectives.

First, the dual nature of the artifacts, that is belonging to the world of everyday life and to the world of symbols, to use Freudenthal's expression, allows movement from situations of normal use to the underlying mathematical structure and vice versa, from mathematical concepts to real world situations, in agreement with '*horizontal mathematization*' (Treffers, 1987). Using a receipt, which is poor in words but rich in implicit meanings, overturns the usual buying and selling problem situation, which is often rich in words but poor in meaningful references (Basso & Bonotto, 1996).

As we will see, these artifacts may also become real "*mathematizing tools*" with some modification, e.g. removing some data. On the one hand they create new mathematical goals, on the other they provide students with a basic experience in mathematization. In this new role, the cultural artifact can be used to introduce new mathematical knowledge through the particular learning processes that Freudenthal (1991) defines '*prospective learning*' or '*anticipatory learning*'. We think that this type of learning is better enhanced by a 'rich context' as outlined by Freudenthal, that is a context, which is not only the application area but also a source for learning mathematics. The cultural artifacts and classroom activities we introduced are part of this type of context. These experiences have also favored the type of learning "*retrospective*" that occurs when old notions are recalled in order to be considered at a higher level and within a broader context, a process typical of adult mathematicians.[1] This different use of the artifacts also made it possible to carry out '*vertical mathematization*', from concept to concept, compatible with grade level. Vertical mathematization may be described as the process of reorganization within the mathematical system itself, for instance discovering connections between concepts and strategies and then applying these discoveries.

The use of some artifacts, receipts, bottles, labels, the weather forecast from a newspaper, a cover of a ring binder (see for example Bonotto, 2001, Bonotto & Basso, 2001, and Bonotto 2003), allow the teacher to propose many questions, remarks, and culturally and scientifically interesting inquiries. The activities and connections that can be made depend, of course, on the students' scholastic level. These artifacts may contain different codes, percentages, numerical expressions, and different quantities with their related units of measure, and hence are connected with other mathematical concepts and also other disciplines (chemistry, biology, geography, astronomy, etc.). It could be said that the artifacts are related to mathematics (and other disciplines) as far as one is able to make these relationships.

To summarize, the artifacts can be used

- as tools to apply 'old' knowledge to 'new' contexts, thus becoming good material for 'meaningful exercises';
- to reinforce mathematical knowledge already possessed, or to review it at a higher level;
- as motivating stepping-stones to launch new mathematical knowledge.

Furthermore we ask children

- to select other cultural artifacts from their everyday life,
- to identify the embedded mathematical facts,
- to look for analogies and differences (e.g. different number representations),
- to generate problems (e.g. discover relationships between quantities).

In other words children should be encouraged to recognize a great variety of situations as mathematical situations, or more precisely "mathematizable" situations. In this way children are offered numerous opportunities to become acquainted with mathematics and to change their attitude towards mathematics, in contrast with the traditional classroom curriculum.

From our experience, children confronted with this kind of activity also show flexibility in their reasoning processes by exploring, comparing and selecting among different strategies. These strategies are sensitive to the context and number quantities involved, and are better mastered and controlled from the meta-cognitive point of view. They are therefore closer to the procedures that emerge from out-of-school mathematics practice.

4. The basic characteristics of the teaching/learning environment

Besides the use of suitable cultural artifacts discussed above the teaching/learning environment designed and implemented in our classroom activities is characterized by:

[1] Freudenthal (1991, p.118) states that "*prospective learning should not only be allowed but also stimulated, just as the retrospective learning should not only be organized by teaching but also activated as a learning habit*".

- the application of a variety of complementary, integrated and interactive instructional techniques (involving children's own written descriptions of the methods they use, individual and class discussions, and the drafting of a text by the whole class);

- an attempt to establish a new classroom culture also through new socio-mathematical norms.

Regarding the first point, most of the lessons follow an instructional model consisting in the following sequence of classroom activities: a) a short introduction to the class as a whole; b) an individual written assignment where students explain the reasoning followed and strategy applied; c) a final whole-class discussion. We consider that the interactivity of these instructional techniques is essential because of the opportunities to induce reflection as well as cognitive and metacognitive changes in students. This process may be very important for teachers also, since it enables them to recognize and analyze individual reasoning processes that are not always explicit (corresponding to the individual written report). In the collective discussion, comparing the different answers and strategies, children's first attempts at generalizing, and further remarks made during the discussion, lead to collectively drawing up a text aimed at socialization of the knowledge acquired, which completes the activity.

As far as the second point is concerned, we expect students to approach an unfamiliar problem as a situation to be mathematized, not primarily to apply ready-made solution procedures. This does not mean that knowledge of solution procedures plays no part, but the primary objective is to make sense of the problem. In practice, it is often a matter of shuttling back and forth between interpreting the problem and reviewing possible procedures or results. At the same time, the teacher is expected to encourage students to use their own methods, exploring their usefulness and soundness with regard to the problem. The teacher should stimulate students to articulate and reflect on their personal beliefs, misconceptions and problem-solving strategies. Other possible strategies for solving the same problem when it appears next are emphasized and students are encouraged to make comparisons between strategies.

According to the socio-constructivist perspective, these norms are not predetermined criteria introduced into the classroom from outside. Instead, the understandings are constructed and continually modified by students and teacher through their ongoing activities and interactions. The development of mathematical reasoning and sense-making processes is seen as inseparably interwoven with their participation in the interactive constitution of taken-as-shared mathematical meanings and norms (Yackel and Cobb, 1996).

5. Conclusions and open problems

In this paper we discuss some classroom activities based on the use of suitable cultural artifacts, interactive teaching methods and on the introduction of new socio-mathematical norms was combined in an attempt to create a substantially modified teaching/learning environment. This environment focused on fostering a mindful approach toward realistic mathematical modeling, that is both real-world based and quantitatively constrained sense making (Reusser & Stebler, 1997).

We do not suggest that the activities described here are a prototype for all classroom activities related to mathematics, although in agreement with Verschaffel, L., De Corte, E, et al. (1999, p.226), we think that "*the development of mathematical problem-solving, skills, beliefs, and attitudes should not emanate from a specific part of the curriculum but should permeate the entire curriculum*".

We do believe however that by enacting some of these experiences, children are offered an opportunity to change their beliefs about, and attitudes towards school mathematics. Immersing students in situations more relatable to their direct experience and more consistent with sense-making, provides a means to deepen and broaden their understanding of the scope and usefulness of mathematics as well as learning ways of thinking mathematically that are supported by mathematizing situations. Using appropriate cultural artifacts, which students can understand, analyze and interpret, we can present mathematics as a means of interpreting and understanding reality and increasing the opportunities of observing mathematics outside the school context. Teaching students to interpret critically the reality they live in, to understand its codes and messages so as not to be excluded or misled should be an important goal for compulsory education. The computer, as well as other more recent multimedia instruments, has a remarkable social and cultural impact and huge educational potential that perhaps has not yet been fully explored.

For a real possibility to implement this kind of activity, there also needs to be a radical change on the part of teachers. They have to try i) to modify their attitude to mathematics; ii) to revise their beliefs about the role of everyday knowledge in mathematical problem solving; iii) to see mathematics incorporated into the real world as a starting point for mathematical activities in the classroom, thus revising their current classroom practice. Only in this way can a different classroom culture be attained. On the basis of the experience of this and our other studies, we entirely agree with Freudenthal (1991), that the main problem regarding rich contexts is implementation requiring a fundamental change in teaching attitudes. As in other studies (Verschaffel, De Corte et al., 1999), the effective establishment of a learning environment like the one described here makes very high demands on the teacher, and therefore requires revision and change in teacher training, both initially and through in-service programs.

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INTEGRATING REAL MEDICAL STUDIES INTO TEACHING BIOSTATISTICS
A Hands-On Experience

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ABSTRACT

This paper describes an innovative way of teaching Biostatistics (or Biostat) at the undergraduate level. Statistics is a fundamental subject in all courses. In particular, senior students taking up pre-med courses enrol in the subject Biostat. However, there is not much difference between the methods of teaching Biostat and the fundamental statistics. The course content (or curricula) is the same for both except for the case studies. To make this difference strikingly clear to the students, they were asked to do Biostat with medical practitioners. Notably, students experienced the applications of statistics software package SPSS® and learned diagnostic tests and other statistical analysis tools which are not found in their Biostat curriculum We summarize their studies and the proposed changes to the curriculum of Biostat that their collaborations with medical doctors brought about.

1. Introduction

The ongoing challenge of learner-centered curriculum is to help students learn by active inquiry rather than by memorizing facts. This is opposed to the traditional design or subject-centered curriculum [9]. The emphasis of the later is on making the learner absorb as much knowledge as possible on the subject matter.

In the former, learning is built upon the activities students engage in. Under this design, learning activities may be based on the actual (or presupposed) needs and interests of the students. They choose what they want to learn and the teacher serves as guide, pointing where to get the necessary information. After the learner has completed his investigation of the problem that he has chosen, he makes a presentation to the teacher or takes a test on the subject matter.

The purpose of this paper is to report such experience for a group (n=8) of senior biology students, 6 girls and 2 boys. The students were grouped into 3 teams. Team I consisted of 3 girls and worked with a female Obstetrician. The other 3 girls were grouped as Team II and collaborated with a female Gynaecologist. The two boys formed team III and cooperated with a male Pediatrician. The doctors were in their second year of residency in the same hospital. They have already collected their data and would just need assistance in applying statistical tools. The doctors' studies were all due in two weeks. This circumstance provided the cap to the extent of time and work that the students would have to spend with the doctors.

In Sections 2, we tell the experiences of Team I in their journey to learning clinical diagnostic tests such as sensitivity, specificity, etc. Team II explored the flexible statistical analysis and data management system of SPSS® in Section 3. In Section 4, Team III made clear the importance of graphical representations. Finally, we give the conclusions of these learning activities in Section 5.

2. Team I: Diagnostic Tests

Often, medical doctors want to know whether the tests that they perform match the actual findings. Team I worked with an Obstetrician who wanted to know whether ultrasound (USG) test on expectant mothers can detemine anomalies (harelip, sunset eyes, hydrocephalus, etc.) in their babies. Commonly used diagnostic tests that measure the accuracy of such procedure are the sensitivity and specificity analysis. The data are shown in Table 1a.

Table 1a. Distribution of cases according to USG test against the outcome

		Anomaly Outcome		
		Present	Absent	Total
USG Test	Present	16	5	21
	Absent	4	73	77
	Total	20	78	98

Among the 98 cases, 16 anomalies detected by the USG test were observed in the babies delivered. Five anomalies detected by the USG test were not found in the babies delivered. Four anomalies were found in the babies but not detected by the USG test. Seventy-three cases were detected by USG test as anomaly-free and not found in the babies.

The team was not familiar with the diagnostic tests required by the doctor. They included a glossary of the terms in their report, which is found in Appendix A. The summary of the results of diagnostic tests is found in Table 1b.

Table 1b. Diagnostic Tests

Sensitivity	80.00%
Specificity	93.59%
False Positive	6.41%
False Negative	20.00%
Positive Predictive Value	76.19%
Negative Predictive Value	94.81%
Overall Accuracy	90.82%
Prevalence	20.41%
*p value	0.00000<0.05 S

*Fisher's Exact Test, 2-Tail, 95% confidence interval

Based on the formula in Appendix B, the computations were as follows:

Sensitivity = 16 / 20 = 0.8 or 80%
Specificity = 73 / 78 = 0.9359 =93.59%
False Positive = 5 / 78 = 0.0641 = 6.41%
False Negative =4 / 20 = 0.2 or 20%
Positive Predictive Value = 16 / 21 = 0.7619 or 76.19%
Negative Predictive Value = 73 / 77 = 0.9481 or 94.81%
Overall Accuracy =(16+73) / 98 = 0.9082 or 90.82%
Prevalence = 20 / 98 = 0.2041 or 20.41%

Here, they got a very high sensitivity, specificity and overall accuracy rates. This led them to the conclusion that USG test can detect anomalies in babies before they are born.

The team also learned from another doctor, a Gynaecologist, about a study that required diagnostic tests for a 3x3 distribution table. The doctor wanted to know the accuracy of the frozen section test in determining the actual stage of cancer in 339 patients. Table 2a gives the distribution and Table 2b summarizes the results of the desired diagnostics tests.

Table 2a. Distribution of benign, borderline, malignant cases according to the Frozen Section test against the final diagnosis

		Final Diagnosis			
		Benign	Borderline	Malignant	Total
Frozen Section Test	Benign	267	4	0	271
	Bordeline	2	13	3	18
	Malignant	0	0	50	50
	Total	269	17	53	339

Table 2b. Diagnostic tests

	Sensitivity	Specificity	Positive Predictive Value	Negative Predictive Value
Benign	99.3%	94.3%	98.5%	97.1%
Bordeline	76.5%	98.4%	72.2%	98.8%
Malignant	94.3%	100.0%	100.0%	99.0%

The formulas used were just derived from the results of a previous study (source unknown). The computations were as follows:

(i) Sensitivity:

Benign = 267 / 269 =0.993 or 99.3%

Borderline = 13 / 17 = 0.765 or 76.5%

Malignant = 50 / 53 = 0.943 or 94.3%

(ii) Specificity:

Benign = (13+0+3+50) / (17+53) = 0.943 or 94.3%

Borderline = (267+0+0+50) / (269+53) = 0.984 or 98.4%

Malignant = (267+2+4+13) / (269+17) = 1 or 100%

(iii) Positive Predictive Value:

Benign = 267 / 271 = 0.985 or 98.5%

Borderline = 13 / 18 = 0.722 or 72.2%

Malignant = 50 / 50 = 1 or100%

(iv) Negative Predictive Value:

Benign = (13+0+3+50) / (13+0+3+50+2+0) = 0.971 or 97.1%

Borderline = (267+0+0+50) / (267+0+0+50+4+0) = 0.988 or 98.8%

Malignant = (267+2+4+13) / (267+2+4+13+0+3) = 0.99 or 99.0%

Here, the students learned that the benign stage has the highest sentivity rate and the malignant stage has the highest specificity rate when using the frozen section test.

3. Team II: Estimating Risk in a Case-Control Study

Team II worked with a Gynaecologist. It is reported that premature rupture of fetal membrane (PROM) occurs in 4.5 –7-6% of pregnancies. The doctor wanted to evaluate the clinical usefulness of a new bedside test, called PROM test, for the detection of ruptured fetal membrane (ROM).

Among the 28 patients evaluated for suspected ROM, the PROM was positive in 8 cases and negative in 20 cases. Among the PROM test- positive group, 6 patients had preterm delivery while among the PROM test- negative group, 2 had preterm delivery. Table 3a summarizes the number of cases.

Table 3a. The number of patients who had preterm delivery in the PROM Test groups

	PROM Test		
	+ Group	- Group	Total
Cases (Preterm delivery)	6	2	8
Control	2	18	20
Total	8	20	28

Using the statisctial software SPSS®, the Student's unpaired t-test was used for continuous variables (age, weeks of gestation) and differences in the distribution of discrete variables were computed using Fisher's exact test. The result was compared with that of the Likelihood ratio shown in able 3b. They also estimated the relative risk using the odds ratio (OR) shown in Table 3c.

Table 3b. Chi-Square results

	Value	DF	Significance
Likelihood Ratio	8.85838	1	0.00070
Fisher's Exact Test (2-Tail)			0.00176

Table 3c. Relative Risk Estimate

	Value	95% Confidence Bounds	
Case Control (odds ratio)	27.0000	3.09261	235.7233
+ group risk	7.5000	1.89761	29.64250
- group risk	0.27778	0.08291	0.93067

Women with suspected ROM and a positive test result had a 7.5 relative risk, odds ratio 27, 95% confidence interval (CI) 1.89-29.64, p value < .05, of preterm delivery. The 95% CI does not include 1 so we can conclude that the two incidence rates are significantly different.

Here, the students learned about the difference between p value and 95% CI. They contain the same kernel of information, but the 95% CI contains more information. The following was the discussion that occurred:

Suppose there were other two studies that showed the same odds ratio (OR). They showed different CIs, however, at p value < .05.

Study #2 showed an OR of 27 with a 95% CI from 1.45-36.21

Study #3 showed an OR of 27 with a 95% CI from 0.4-25.6

What can be said from these results?

Studies 1 and 2 were "statistically significant," with a p value < .05 because the CI does not include 1.

Study 3 included an OR of 1 in the 95% CI, and therefore the p value was not < .05.

Study 1 had a more precise estimate of the true OR, with a very small 95% CI.

4. Team III: Testing Hypotheses about Mean Differences

Team III worked with a Pediatrician. The subjects were 243 full term infants born from the periods of January 1999 to January 2000. The records were reviewed by the researcher-pediatrician and the following data were recorded: sex, type of milk feeding, i.e. purely breast-fed versus purely formula-fed, weight, height, head circumference and illnesses encountered from birth, 6 months of age and at 1 year. These ages were chosen since not all the records contained complete data for ages between birth to 6 months and from 6 months to 1 year.

The mean and standard deviations of their birth weight, height and head circumference were shown in Table 4. However, due to constraints, the presentation here is limited to the results on weights.

Of the 243 infants reviewed, 140 were male and 103 were female. Furthermore, out of the total population only 37 subjects (15.2%) were purely breastfed from birth to 1 year of age while the remaining 206 subjects (84.8%) were given milk formulas.

Table 4. Population Characteristics

	Breastfed	Formula-fed	p Values
Birth weight	3.18 +/- 0.63	3.10 +/- 0.54	p=.000 (S)
Birth length	49.52 +/- 3.48	49.14 +/- 2.79	p=.000 (S)
Head circumference	34.12 +/- 1.96	34.03 +/- 1.71	p=.000 (S)
Sex ratios (M:F)	3:2	4:3	p=.000 (S)

*mean +/- standard deviation

A t-test between breast-fed and formula-fed infants with 95% confidence interval for difference was made. At value $p<0.05$, the null hypothesis that there is no significant difference between the growth parameters of babies given breastmilk and milk formula from birth to 12 months was accepted. As such, there is evidence that milk formulas are comparable to breastmilk in terms of affecting weight measurements in infants below 1 year. This study also compared the results of the subjects' growth curves with existing growth tables such as the Food and Nutrition Research Institute & Philippine Pediatric Society Anthropometric Tables and Charts for Filipino Children (FNRI-PPS) [2] and the National Center for Health Statistics Percentiles Tables and Charts (NCHS) [1]. The graphs are shown in Figures 1a & 1b.

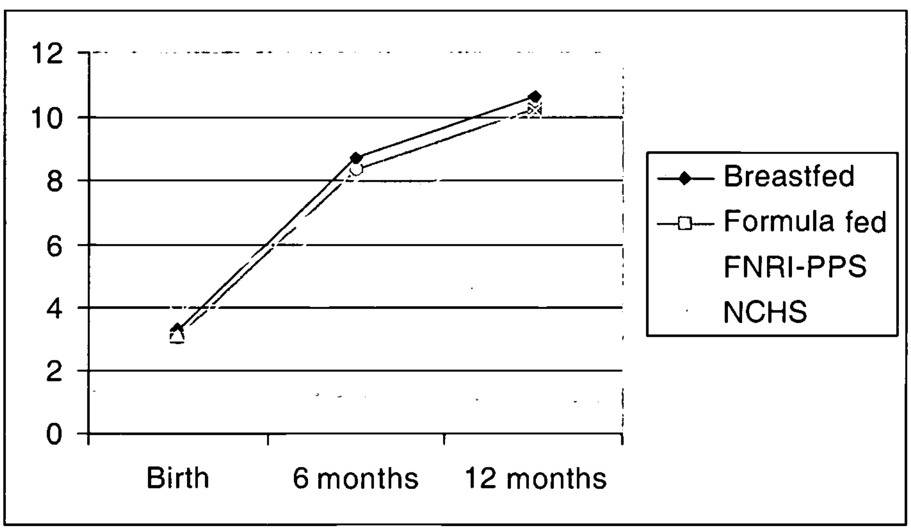

Figure 1a. Comparison of weights between types of milk feeding vs established growth curves in male infants. The values were plotted with those of FNRI-PPS and NCHS tables of boys 0-12 months.

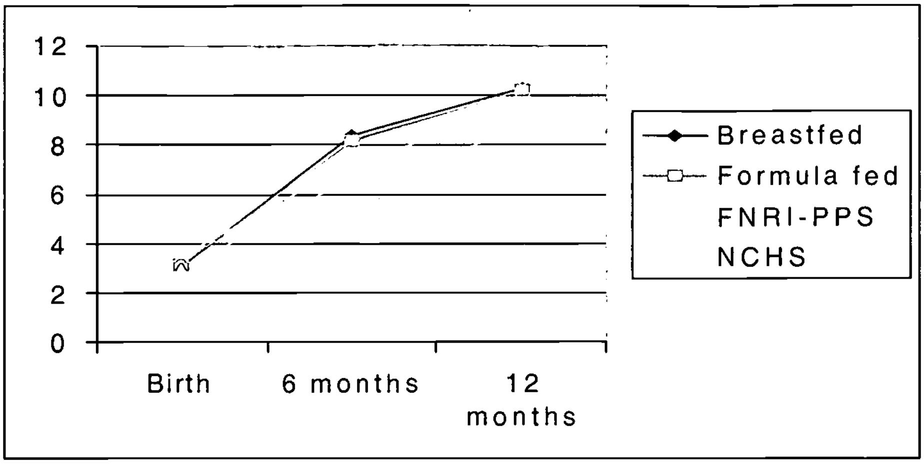

Figure 1b. Comparison of weights between types of milk feeding vs established growth curves in female infants. The values were plotted with those of FNRI-PPS and NCHS tables of girls 0-12months

Finally, a chi-test to determine the relationship between the type of feeding and occurrence of common illnesses was formulated. All chi-test showed the value $p=0.000<0.05$, thus rejecting the null hypothesis. Therefore, there is evidence to show that the occurrence of common illness is dependent on the type of milk feeding (Figure 2).

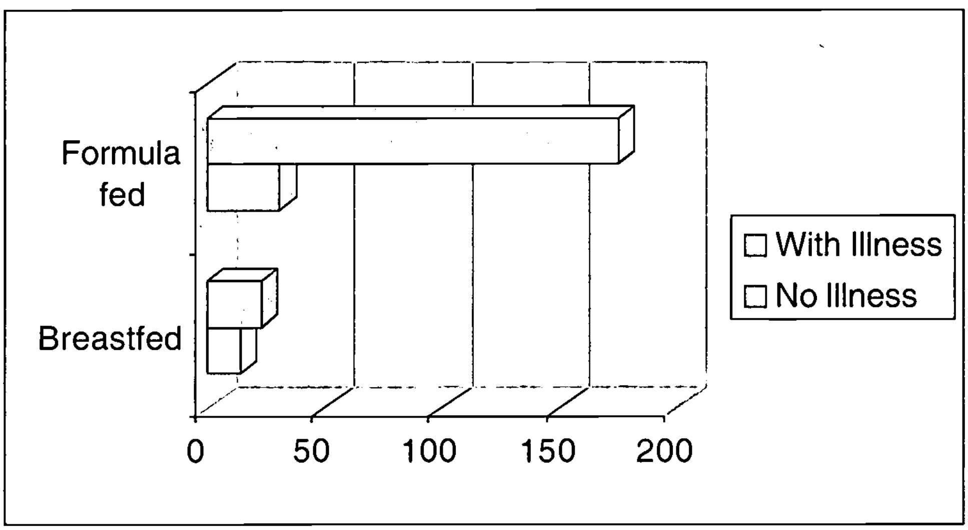

Figure 2. Comparison of type of feeding versus existence of illnesses during the 1st year of life

This study showed that present day milk formulas are comparable to that of breastmilk as to weight gain at least for the first year of life. The growth curves of breast-fed infants versus formula-fed infants did not differ significantly as opposed to previous studies that state that breast-fed infants are leaner. Formula-fed infants are, however, more prone to develop illnesses compared to their breastfed counterparts. The FNRI-PPS growth tables may need further examination in terms of applicability to the Filipino population since even growth curves of breast-fed babies, specifically the weights were significantly different from existing weight values.

5. Conclusion

For most statistics classes, the projects have been simply to conduct surveys which concepts most closely matched the lessons. In contrast, the above learning activities had made a concerted effort to create lessons directly aligned with biostatistics concepts. The lessons were unique because learners worked on real-medical data from a respected medical center that promotes research, and were classroom-ready.

The 8 students involved in this learning activites signified their intention to continue their study in medicine. Two factors, intentionally designed into this particular course, may have contributed to this disposition.

- The On-The-Job Experience. Collaborating with medical practitioners and working with real data, helped them become socially responsible, proactive individuals. It enabled them to plan and realize social improvement at the local and global levels.
- The Application of Technology. The work that they did using MS Excell® and SPSS® showed that statistics can be learned and applied with 'less' mathematics. Grievous math computations were removed, enabling them to focus on the understanding of statistical concepts and the interpretation of the results.

The doctors themselves expressed their trust and gratitude to the students for helping them in the statistical section of their studies. Without such partnership, they expressed concern whether they could have finished their studies on time due to their hospital load as resident doctors. They would recommend to other doctors this collaboration with senior students enrolled in Biostat classes.

Finally, the proceedings of all the three studies were documented for inclusion in the next prints of learning materials in Biostatistics.

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APPENDIX A
Evidence Based Medicine Glossary

Incidence Rate: Number of new cases of a disease in a specified period / average population during that period. Rate is usually expressed as per 100,000. [5]

Likelihood Ratio: The likelihood that a given test result would be expected in a patient with a disease compared to the likelihood that the same result would be expected in a patient without that disease. [4]

Negative Predictive Value (NPV): The percentage of people with a negative test who do NOT have the disease. [4]

False Negative a test result that wrongly excludes an individual from a diagnostic or other category. [3]

False Positive, also known as a false detection or false alarm, a test result that wrongly detects a disease in an uninfected individual. [8]

Positive Predictive Value (PPV): The percentage of people with a positive test result who actually have the disease. [4]

Prevalence Rate: Number of people with a disease at a given point (period)/ population at risk at a particular point (period). Rate is usually expressed as per 100,000. Prevalence = Incidence X duration [5]

Odds ratio is used in case control trials: Odds of a case patient being exposed divided by odds of a control patient being exposed. [6]

Relative Risk: Event rate in treatment group divided by the event rate in the control group. Also known as risk ratio. RR is used in randomized trials and cohort studies. [6]

Sensitivity: The probability of the test finding disease among those who have the disease or the proportion of people with disease who have a positive test result. [4]

Specificity: The probability of the test finding NO disease among those who do NOT have the disease or the proportion of people free of a disease who have a negative test. [4]

Statistical vs. Clinical Significance: Statistical significance means the likelihood that the difference found between groups could have occurred by chance alone. In most clinical trials, a esult is statistically significant if the difference between groups could have occurred by chance alone in less than 1 time in 20. This is expressed as a p value < 0.05. Remember that a trivial difference can have a very low p value if the number of subjects is large enough. Clinical significance has little to do with statistics and is a matter of judgment. It answers the question: "Is the difference between groups large enough to be worth achieving?" Studies can be statistically significant yet clinically insignificant. [7]

APPENDIX B
Diagnostic Tests Formula

Table B1. 2x2 Distribution Table of Test Outcome against Actual Outcome [4]

		Disease		
		Positive	Negative	
Test	Positive	True Positive (TP)	False Positive (FP)	TP + FP
	Negative	False Negative (FN)	True Negative (TN)	FN + TN
		TP + FN	FP + TN	

SENSITIVITY = TP / TP+FN

SPECIFICITY = TN / TN+FP

POSITIVE PREDICTIVE VALUE (PPV) = TP / TP+FP

NEGATIVE PREDICTIVE VALUE (NPV) = TN / FN+TN

FALSE POSITIVE (F+) = FP / TP+FP

FALSE NEGATIVE (F-) = FN / TP+FN

NEGATIVE PREDICTIVE VALUE (NPV) = TN / FN+TN

LR(-) = [FN / (TP + FN)] / [TN / (FP + TN)]

LR(+) = [TP / (TP + FN)] / [FP / (FP + TN)]

Table B2. 2x2 Distribution Table of Outcome of Case-Control Study [6]

		Outcome	
		Event	No Event
Exposure	Case	a	B
	Control	c	D

RELATIVE RISK = a/(a+b) / c/(c+d) = a(c+d) / c(a+b)

ODDS RATIO = a/c / b/d = ad /bc

INTEGRATING WEB-BASED MAPLE WITH A FIRST YEAR CALCULUS AND LINEAR ALGEBRA

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ABSTRACT

This paper reports on extensive work carried out over the last eight years at the University of Queensland to adapt Maple to each of the topics of a first year Calculus and Linear Algebra and the results of this implementation.

The course has about a thousand students mainly engineering and science students with a few from biological science or arts. Most students start with little if any CAS skills, though some have used Derive or graphics calculators at school.

Each topic in the course is introduced by discussion until the analytical background is established. Once this has been covered and digested Maple applications are illustrated on the computer in the lectures and then students work through similar ideas and extensions in their next lab tutorial.

Each student has a one hour computer lab every week. From week two students are introduced to Maple and they can work through the twelve tutorials at their own rate, though one a week is recommended. The tutorials are on the web and students can download them. Week one provides an introductio n to Maple followed by introductions to arithmetic, algebra and calculus so that, by week five, students have some understanding of Maple commands and syntax.

The next tutorials take students through Taylor and Maclaurin series and their uses in approximating π and e and sine, cosine and log functions. Tutorial 7 is a tutor marked test which allows students to judge their progress. The last tutorials cover numerical integration and then Linear Algebra, including vectors, matrices, linear independence, Leslie matrices and the start of programming and finally eigenvalues and eigenvectors.

Projects include practical applications to numerical approximations and using Leslie matrices for predicting changes in populations and dominating eigenvalues to estimate asymptotic distributions.

This paper reports on the evaluations undertaken over the last eight years of the advantages and disadvantages of such an approach.

Keywords CAS, Web-based material, Algebra and Calculus

1. Introduction

At the University of Queensland we have large first year Calculus and Linear Algebra classes of 600 – 700 students comprising students from Science, Engineering and Information Technology. Each student in these courses has three hour lectures, one hour tutorial and one hour computer laboratory every week for thirteen or fourteen weeks of semester.

Peter Galbraith of Graduate Education and I have monitored the introduction of CAS to students for over ten years now and this has changed my attitudes. Initially I was so pleased that CAS had become available (first in the form of our own package and later Maple) that I rushed in and tried to introduce students immediately to its uses and applications with poor results – students could not cope with the syntax and complexity and soon became frustrated with the whole experience. We instigated student surveys and discovered that students felt that computing, as presented to them, was not helpful to their understanding of the course material. I then reacted and probably introduced Maple too late in the course to be effective. For the last eight years I think I have probably got it right (according to the student surveys) by having Maple on the web and starting computer labs in week two of semester with ample introductory explanations.

For their first lab session students have a printed sheet, which tells them how to get into the web site. Once in, there is an explanation of what Maple is and then four introductory sections on arithmetic, algebra, calculus and graph plotting. Following this are nine tutorials illustrating where Maple can be used in conjunction with the lecture material. Students have printed notes for each session but can download any section and print them in the lab. They can work at their own pace – many work at home from their own computers or come in to other lab sessions when there is space available - but it is suggested they do one section per week. There is also a Maple quiz after week six, which we mark and return to students together with solutions, so that they can judge how they are progressing.

Maple is also presented in every lecture to illustrate its applications where appropriate and we discuss in the lecture any problems associated with its implementation and the advantages of using Maple for each application.

2. Content of the Maple tutorial sheets

Tutorial 1 introduces students to the basic syntax of Maple, the system constants for π and e and the five operators for +, -, x , ÷ and exponentiation, with several exercises. Then the difference between exact integer arithmetic and floating point and how to convert to floating point and how to choose the number of digits required. More practice problems.

Tutorial 2 introduces students to algebra and how to assign and unassign variables and the setting up of Maple expressions and functions and the difference between them and composite functions. Examples. Then the six useful algebraic operators *expand, factor, simplify, normal* (combining expressions over a common denominator), *sort* and *collect*, with examples of how Maple can use these commands to perform many useful algebraic manipulations and simplifications. Finally, the use of *solve* for exact solutions (where they exist) and *fsolve* for floating point solutions (where they exist).

Tutorial 3 is the first tutorial, which mirrors what is being presented concurrently in lectures – limits (finite and infinite), differentiation (the *diff* and *D* operators) and integration (definite and indefinite, finite and infinite ranges).

Tutorial 4 supplements work in lectures on sequences (several ways that Maple can create a sequence), infinites sums (especially $\sum \frac{1}{n}$) and products. Then there is a section on various ways of plotting functions (the course does not specifically contain this, but it is so useful) – simple and multiple plots, parametric and implicit plots, with many examples and practice examples.

Tutorial 5 is really the start of serious uses of Maple and again mirrors lecture work on Taylor series. It first introduces students to finding Taylor series of several common functions and how to convert them to polynomials and the use of the *op* command to extract terms from these polynomials. We then see how to approximate e and π as accurately as required, using the above.

Tutorial 6, as with lectures, is about Maclaurin series and error estimates for series of positive terms and Alternating series and using Maple to approximate various trig and log functions to prescribed accuracy and then examples for students to try for themselves.

Sheet 7 is a quiz which gives students twenty questions with no hints (they can, of course, use the web material) to be done in 50 minutes. These are marked and returned with full solutions.

Tutorial 8 mirroring lectures uses Taylor series to approximate definite integrals, which have no closed form to, prescribed error. This is the end of the calculus section.

The second half of the tutorial starts with the Linear Algebra part of the course by introducing students to Maple's *linalg* package and how to set up vectors and matrices and perform addition and scalar multiplication and matrix multiplication. Several examples and exercises on this and whether certain given matrix products are defined.

Tutorial 9, following lectures, takes students through linear systems, augmented matrices and the start of Gaussian Elimination and the use of Maple's *linsolve* to solve systems of linear equations. Then several ways to look at linear independence and bases. Again many examples and exercises.

Tutorial 10 has a nice illustration of the use of Leslie matrices applied to a marsupial population (of course!). The students are introduced to programming in Maple and this is applied to setting up a generational profile for the population.

Tutorial 11 mirroring lectures is about setting up Gaussian elimination step by step and all the commands needed to do this (augmentation and various row manipulations) and then examples for students to do for themselves.

Tutorial 12 as in lectures takes students through five different ways of establishing whether a set of vectors is linearly independent or not, or whether the matrix formed from these vectors is singular or not.

Tutorial 13 is the last topic of the course and shows students how Maple can find eigenvalues and eigenvectors and the characteristic polynomial and factorize this polynomial.

Assessment I have used two variations – three assignments worth 3%, 3%, and 4% and two worth 5% each. This is the same weighting as given to the course tutorials – 10% each. Among topics included are: solving systems of linear equations; finding conditions for sets of equations involving a parameter to have a unique solution; no solution; an infinite number of solutions; simplifying complicated algebraic expressions; zeros of functions; areas under graphs; problems with $\sum_{n=1}^{N}\frac{1}{n}$; using Taylor series to estimate functions and integrals; problems with Leslie matrices using the dominating eigenvalue result to find the final relative age distributions.

3. Discussion

With Peter Galbraith we set up various instruments to monitor problems encountered by students and applications which helped students. The results were as follows.

General Remarks We discovered that the timing of the introduction of a CAS is important – if students encountered a CAS too early they will only treat the whole experience as irrelevant and confusing and if too long after the material is discussed in lectures they will miss the relevance of the applications. Again, if a CAS is merely kept to the computer laboratory, students will believe that the lecturer feels it is not an integral part of the course. It is essential for the lecturer to bring the CAS into lectures and demonstrate it so that students can see its applications and how to implement it successfully and **discuss** any problems they might have as it is presented. I always have **three** forms of information in the lecture theatre while demonstrating a CAS – the actually computer display, one overhead for the CAS commands and one overhead where we can make remarks and answer student queries.

Our surveys also showed there are essentially three types of students – the purists (usually female and good students) who refuse initially to do any computing as they believe all mathematics can be done (and **should** be done) **analytically**. To convert these I always show them the depot problem, which demonstrates clearly that you cannot solve even some of the simplest problems analytically. To the other extreme there is always a group of students (usually male) who love to spend vast amounts of time fiddling around with computers (left over from their computer games) who will lose the main analytical thrust of the course in their quest for arcane computer methods – it is important not to let a CAS take over a course. In the middle is the largest group who are willing to try it out but whose confidence in the outcome is always in the balance of how well you implement the presentation and timing and relevance of your CAS programme.

Specific remarks.

Syntax is the main source of problems and frustrations for students. Those students who always had troubles with brackets soon find they cannot even enter simple expressions like $\frac{2^9+3^8}{4^7+5^6}$ without getting a syntax error. Worse cases were e.g. $(a + b)^3$,where they say to themselves: "a + b cubed" and enter a+b^3 , which, of course, returns a **wrong** expression with **no** syntax error! We actually interviewed students and discovered that those who have always had poor algebraic skills merely carried this incapability through to their computing – the problem goes back to Grades 8 and 9 (13 and 14 year olds). Counting the number of opening and closing brackets sometimes helped but their main problems always were actually getting their first expression " to work".

Choice of algebraic commands was another source of problems e.g. to find the zeros of a polynomial they would use the *simplify* or *expand* command and, of course, Maple merely returns the original polynomial. Knowing that zeros are associated with **factors** is a vital piece of information – if you do not possess this you will get nowhere. However, those with adequate algebraic skills who could enter expressions correctly found the capability of simplifying complicated expressions very useful and soon realized that Maple could save them literally hours of grinding away through lengthy algebra.

Problems and advantages with the calculus section Students found it helpful to be able to find derivatives of complicated functions (especially quotients) and where possible simplify them, provided, of course, they could enter the function in the first place. Typical here was the exponential being entered as e^x instead of exp(x), **despite** the introductory warning about the representation of e.

Confusion also occurred when trying to evaluate the derivative at a given point, as they were not sure if the result of using the *diff* or *D* operators were **expressions** or **functions**. What they found extremely useful was the capability in maximum and minimum problems of finding the derivative (however complicated), plotting it (so they could see how it behaved) and using *fsolve* to approximate where the derivative was zero.

Probably **most useful** was evaluating integrals that students always find difficult like $\int x^n e^{ax}\,dx$, $\int e^{ax}\cos bx\,dx$, $\int x^n \sin bx\,dx$ where n is sufficiently large to make it tedious. Also for integrals involving square roots (what to substitute?) and those involving partial fractions, which students find fraught with numerical mistakes – Maple's capability of finding partial fractions helps simplify things greatly.

This section brings up an important point for discussion – after doing it students will ask **why do we have to learn how to do these integrals when Maple can do them for us? – aren't we wasting our time? – this, of course, is one of the big debates surrounding the teaching of any CAS – how much material should we remove from our traditional syllabus and replace by CAS?**

Plotting Students are very poor at sketching graphs of even relatively simple functions and all report this is **one of the greatest aids**. It also means that you can use more complicated and realistic functions for your examples.

Taylor series and numerical approximation Every year students find understanding Taylor series the most difficult part of the course – until they actually use them to approximate various expressions with desired accuracy they cannot envisage how they work. **All reported this was the most useful area for them especially as Maple can easily produce Taylor series to any order and evaluate them to any given accuracy.**

The Maple quiz (after six weeks) acts as a good guide for their progress – we get them to print out all their attempts so we can see where their approach goes wrong. The most common errors were still **syntax errors,** using the **wrong algebraic commands, confusing solve and fsolve** and **expressions and functions**. In general, though, most have mastered the content so far quite adequately.

Vectors and matrices Many students report being alarmed when they load the *linalg* package and forget to end their command with a colon resulting in a vast description of the package in front of their eyes – they felt as though they have done something terribly wrong. Once this was rectified they found vectors easy to enter and perform addition and scalar multiplication with them.

Many found it difficult to enter matrices, usually leaving out one of the final] brackets and if they achieved this successfully they forgot to include *evalm* to view the results of their matrix calculations. However, once this was overcome, they found the **capability of finding inverses, determinants, powers etc. very useful**, especially when large size matrices were involved. To be able to find the determinant of a large matrix, factorize it and hence discover for which values of the parameter(s) it is singular (or not) was very helpful.

Gaussian elimination They found setting up the process step by step pretty difficult, but liked the *gausselim* command, which did it all for them, very helpful as they could easily see the three possibilities for solutions – unique, none and infinite.

Bases Again, students found the whole concept of a basis very difficult and they reported that the sheet which took them through several ways of attacking these problems were helpful.

Eigenvalues and eigenvectors I always find I have to return to eigenvalues and eigenvectors in second year before students fully understand the concepts. By hand students can only really manage to find them for 2X2 and 3X3 matrices and problems are always encountered if there is multiplicity. Maple provides the characteristic polynomial which can be factored easily to give the eigenvalues and thus the eigenvectors via *linsolve*. Maple can also return eigenvalues and eigenvectors immediately together with the multiplicity – this is an easy way out for students.

Assignments We mark all assignments and monitor students' problems. After one semester students still exhibit problems with syntax – I expect they will retain this problem for all their undergraduate days. Most other problems stem from their misunderstanding of the course material which flows through to their incapability of using Maple successfully. Whichever CAS one must use there will always be these problems.

4. Conclusion

Whatever CAS you might choose I suspect there will always be advantages and disadvantages associated with its implementation. The timing is vital – it must be introduced after the relevant material is covered in lectures but sufficiently soon enough afterwards that students still have it fresh in their memories. Further it is essential to reinforce its impact by having CAS in lectures where we can all discuss problems associated with its application.

Our studies showed that if students either have CAS introduced too quickly or in not sufficient explanatory detail they will react negatively and find the whole experience worse than having no CAS at all. Again, if CAS is not presented as an integral part of the course it appears to students that it cannot be relevant and they will not treat it seriously.

There will always be some students who will encounter problems with syntax and this leads to frustration and withdrawal from the programme. For those who overcome the various pitfalls there can be many advantages and in many cases CAS can illustrate and extend ideas which students otherwise find hard to grasp. We found this especially true with these topics: simplifying complicated algebraic expressions; finding derivatives and integrals of complicated functions; graphing difficult functions; setting up Taylor series and using them for numerical approximations; finding inverses, determinants and powers of large matrices; finding eigenvalues, eigenvectors and characteristic polynomials.

In conclusion, I think the whole exercise is probably worth the considerable effort of setting up the required materials.

Acknowledgement I should like to thank Peter Galbraith for all his help, good advice and wisdom, and without whom none of my research in this field would have started or continued.

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Introduction www.maths.uq.edu.au/computing/local/maple/m.html

Tutorials www.maths.uq.edu.au/~mrp/mt151/index.html#maple

THE "PLUS" PROVIDED BY GRAPHICS CALCULATORS IN TEACHING UNDERGRADUATE STATISTICS

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ABSTRACT

It has long been accepted that the use of technology, in the form of computer packages, is beneficial in teaching undergraduate statistics. However, having recognised the potential of graphics calculators with inferential statistics capabilities, the relative roles of the different forms of technology were investigated. Initially, the focus was on calculators versus computers, evaluating the students' preferences. It soon became clear that it is technology as a whole that is important in a statistics course, rather than one particular form. Consequently, during 2001 the emphasis has been on providing access to learning with a whole range of technologies. Through surveys and interviews, the students have indicated that, whilst they recognise the need for computer packages in future work situations, their learning has been greatly enhanced by the use of graphics calculators. This seems to be due, in part, to their existing familiarity and confidence with the calculators as much as to the calculator's capabilities. Graphics calculators are required in the school leaving examinations in Western Australia and the majority of science students arrive at Murdoch University owning one that has statistical inference facilities. (Typically, about three-quarters of the students have a graphics calculator capable of statistical inference in their final examination.) The benefits to effective learning gained by incorporating, as an extra learning tool, facilities that the students already have at their fingertips have definitely outweighed any extra time required in developing appropriate learning activities.

KEY WORDS: Inferential statistics; technology; graphics calculators; computer packages

1. Introduction

The teaching of statistics has changed enormously over the last few decades with the development of calculators and computers. Their use has not only made the computations easier it has changed the way that people think and teach. The statistician David Moore puts this transition very clearly.

> While the impact of fast, easily accessible computing has had an impact on mathematics as a whole, it has revolutionised the practice of statistics. An obvious effect of the revolution is that more complex analyses on larger data sets are now easy. But the computing revolution has also brought about changes in the nature of statistical practice. In the past, statisticians conducted straightforward but computationally tedious analyses based on a specific mathematical model in order to draw conclusions from data. Instruction in statistics showed a corresponding emphasis on learning to carry out lengthy calculations. Now the paradigm statistical analysis is a dialogue between model and data. ... All [methods] are computationally intensive, and the most widely adopted make heavy use of graphic display. ... Statisticians ... have welcomed calculators and computers as a liberating force. Calculating sums of squares by hand does not increase understanding; it merely numbs the mind. In these circumstances, it is natural for a statistician to urge the use of calculators and computers in instruction at all levels. (Moore 1990, p99-100)

This comment seems to reflect the move towards using technology in the learning and teaching of statistics. Current work is looking at ways of developing appropriate technology tools to enhance student understanding (Chance, Garfield & delMas 2000) and it's importance is being recognised by joint initiatives such as the *Maths, Stats & OR Network* in the UK (Bishop & Davies 2000). The integration of technology into the teaching and learning of statistics brings with it the need to determine the appropriate roles of calculators and computers in various programs of study. This paper looks at the role of both calculators and computer packages in the teaching of introductory statistics courses at the undergraduate level and the merits of using a range of technologies for student learning.

2. Background

Since the mid 1980s it has been common practice in the teaching of statistics courses at the undergraduate level to include the use of a statistics computer package. By taking away the drudgery of tedious calculations more emphasis can be placed on the understanding of data types, methods of data collection, choice and interpretation of analyses and determination of conclusions. Initially, many packages were run on mainframe computers with complexities in access and coding similar to the complexities in the calculations that were being replaced. More recently the advent of desktop & laptop computers, combined with the continual upgrading of computer software has led to very simple access to menu driven packages that rapidly provide sophisticated output for even complex analyses of large amounts of data.

However, the latest "computers" that can perform inferential statistical processes are even smaller than laptops, being the hand held graphics calculators. Whilst the authors admit that these calculators would not be the first choice for performing major analyses, we feel that they can play an important role in a student's first learning experiences with inferential analyses. These calculators are familiar to an increasing number of students, they have the ability to perform simple analyses with summary data as well as raw data (a facility that is often not available on larger packages) and they can provide students with an easily accessible experience of inferential statistics (Kemp, Kissane and Bradley 1998).

At Murdoch University there are three introductory level statistics courses (equivalent to $1/8^{th}$ of a students first year of study). One is intended primarily for commerce and business students and the other two are for students who take the course as part of their studies for degrees mainly in the biological, biomedical, environmental, marine and veterinary sciences, in biotechnology, ecology and molecular biology. Each course consists of lectures and tutorials (of up to 20 students) with marked homework assignments, mid-semester test and final examination. Most of the students would have recently completed at least one mathematics course in the final year of secondary school leading to participation in the Western Australian school leaving Tertiary Entrance Examinations (TEE). Students have been required to use a graphics calculator in these examinations since 1998. Therefore, many students arrive at Murdoch University owning and, more importantly, being very familiar with a graphics calculator. The possible exceptions are interstate, overseas or mature age students.

This paper will focus on the use of technology in the two courses intended for science students. These courses are coordinated by the first author, with the second author providing support classes for the students experiencing difficulties in the courses. The use of computer packages has been fully incorporated into the courses for fifteen years (Bradley 1996). However the authors decided in 1999, after extensive experience with graphics calculators in non-statistics courses (Bradley, Kemp & Kissane 1994), to look more carefully into the roles of different technologies from a teaching and learning perspective. As an ongoing part of this analysis of the role of the graphics calculator as well as other technologies in the courses, the various facilities available on the technologies and relevant to the courses were collated and are given in **Table 1** (at end of paper). The three computer packages are those that have been used in the introductory courses at Murdoch University. The Casio, Texas Instruments, Sharp and Hewlett Packard calculators referred to here are those with descriptive and inferential statistics capabilities, the HP incorporating a specially written aplet. The table also indicates the ability of the technology to use raw data and summary data (such as means and standard deviations).

Until 1999 the computer package *MINITAB* was used in these courses but subsequently, due to financial cuts, a switch was made to using *Excel* and *SPSS* from 2000 onwards. The site licence for *MINITAB* became too expensive and it was assumed that many students would have access to *Excel* at home. The decision to change was rued by the authors as the *MINITAB* statistics package seems to be the most appropriate for this level of student.

As can be seen from the table, the graphics calculators provide more of the facilities used by students in the statistics course. Whilst the displays may not be as large as those produced on computer packages, the graphical displays themselves are often easier to produce and modify to see the effect of data changes. There is also the advantage that students will have been using some of these facilities on their calculators as part of their high school studies. The calculators have the advantage that students can use either summary or raw data for the simple analyses in the courses.

3. Early student responses

Since 1999 the coordinator of the courses has consistently incorporated both graphics calculators and computers into the teaching and learning of statistics. Lectures include demonstrations with graphics calculators and interpretation of computer output and students are required to attend computer laboratory sessions to learn how to use the computer packages. In their initial work during 1999 and 2000 the authors research focus was on students' perceptions of the benefits of computers versus calculators. At that time not as many students owned graphics calculators with the inbuilt statistical inference facilities as they do now. All students had to learn how to use the computer packages, as their use was required for some marked assignment tasks and interpretation of outputs was expected in the final examination. Those students who had graphics calculators were encouraged to use them and, in addition, the authors made calculators available for use in tutorials and in the library; indeed some students decided to buy them during the semester.

In 1999 students were asked for their comments on the graphics calculator versus computer use. There were positive comments for both *MINITAB* and the graphics calculators, with some students being in favour of both technologies in various ways. Their responses included the following comments:

Firstly for ***MINITAB***:

- *MINITAB* output is good.
- Good diagrammatically, has everything written on it.
- Better for more complex tests.

Comments supporting the **calculators** included:

- You can see more.
- It is better value.
- You can take it wherever you are.
- You get a greater understanding of what the calculator tells you than *MINITAB*.
- Had to understand first by doing it by hand. The graphics calculator enhanced understanding.
- The relevance of the P-value was brought home by the graphics calculator.

On whether they would **recommend the calculators**

- Yes, even before books.
- Yes, because they are used in other subjects and there is a future for them.

In 2000 the students were using *Excel* and *SPSS*. The internal students undertaking the two courses for science students were surveyed about a quarter of the way through the course with about 63% responding. As can be seen from **Table 2** (end of paper) and **Figure 1** (below) although the majority of students had access to *Excel* at home, they still preferred to use their own calculators or tables books for normal probability calculations. This preference for using calculators over computers can also be seen in **Table 3** (end of paper) giving survey results for week 3 tutorial exercises in one of the courses. Students were given detailed instructions for calculating means and standard deviations using scientific and graphics calculators as well as computers and asked to indicate, with reasons, their preferred method.

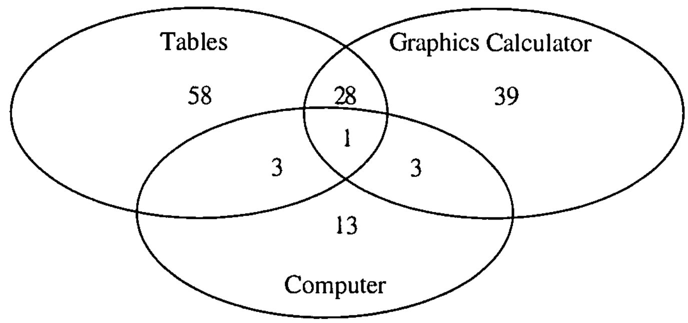

Figure 1: Methods used to calculate normal probabilities on assignment during week 4 - 2000

Throughout the semester the number of students using graphics calculators steadily increased (**Table 2**). This was in part due to the aplet provided about half way through the semester by Hewlett Packard for the HP 38G calculators providing automatic confidence interval and hypothesis test calculations. A number of students in both courses indicated that they were using the calculators in the Reserve section of the library (available for 2 hour or overnight loan). All students who wished to use a graphics calculator for the tests and final examination but did not own one were able to borrow them.

Apart from the survey results, it became evident during lectures and tutorials for both courses that the enthusiasm for graphics calculators with statistical inference increased over the semester. When students did not have access to computers, or when they were not required to use computers by the nature of assignments or tests, they were comfortable with using graphics calculators and became more so over the duration of the course. The students appreciated the extra support classes, provided by the Teaching and Learning Centre, which focused both on the content of the courses and the practicalities of using the calculators. (It is especially helpful for this kind of support to be provided when students come to class with a number of different makes and models of graphics calculators.)

4. Teaching strategies

After reflection on the students' comments and discussion with colleagues in Australia the course coordinator decided to take a slightly different approach with a view to improving student awareness of the power of the different packages and calculators. This new approach in 2001 incorporated all the previous uses of technology. However, this was combined with a strong emphasis on how each kind of technology could contribute to each stage in the development of statistical ideas. This included discussion of the advantages and disadvantages of the different aspects of the various technologies for both learning and performing statistical analyses. Bearing in mind that the major aim of the course is to teach students introductory statistics as well as the use of technology this has to be done quite carefully.

As part of the teaching process, students are given comprehensive technology guidelines and examples in three different ways. Firstly, course notes are prepared giving instructions for using *Excel* and examples of *Excel* output as well as details of the required statistical techniques. Secondly, during the lectures examples of *SPSS* and graphics calculator output are used. Finally, in tutorials students receive detailed written step by step instructions for both *Excel* and *SPSS* together with instructions for all different makes of calculators. Calculators are taken along to the tutorials for students who wish to borrow them and are available for use in the university library.

For assignments students are often given a choice of using *Excel*, *SPSS* or calculators but they need to be able to read *SPSS* output for the final examination.

Students are introduced to the value of using computers to easily handle a large data set. During the first lecture of both courses students fill out a questionnaire giving details such as their gender, height, eye colour, dominant hand and eye, number of brothers and sisters, method and time to travel to the campus each day. Other details relate to their studies, recycling habits and number of pets. The information is anonymous and, whilst the students can identify themselves by birth date (not including year) plus other details such as degree program etc, other members of the courses (including the coordinator) would find it very difficult. Altogether 17 fields are recorded for each student giving data sets each with well over 100 records. These are then used in lectures, in tutorials, for assignments and even examination questions. Over the eight years that the first author has been collecting and using the information, the students have indicated that they really find it motivating to be using a large data set that actually relates to themselves. Performing analyses on assignments and discovering whether, for instance, environmental science students are more likely to recycle than vet students seems more relevant than some standard text book questions. It also helps in talking about the difference between populations and samples and whether the samples are random. As well as being used for analyses the files (created in *Excel* but readily readable by *SPSS*) are useful for practice in manipulations such as sorting, converting text to number and vice versa, combining information (such as brothers + sisters = siblings), splitting files, combining data from two files and so on.

Students come to appreciate that a graphics calculator is a very powerful, portable tool for learning statistics. The more recent calculators can perform all the operations required in an introductory course. The visual representation of the P-value in hypothesis testing produced by graphics calculators helps the students better understand concepts (see **Figures 2 & 3** below). As students become more proficient they become more impressed by the facilities of the calculator and more likely to use them confidently.

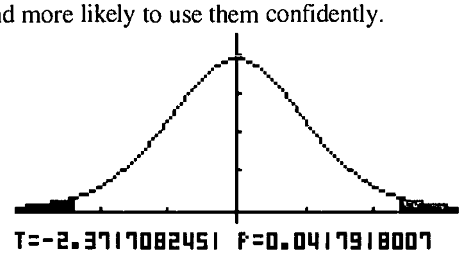

Figure 2

P-value for two tailed t-test - Reject H_0

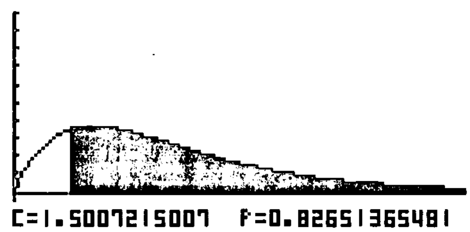

Figure 3

P-value χ^2 test for association - Do not reject H_0

In 2001 students could complete the courses without using a graphics calculator but not without using either *Excel* or *SPSS*. More students than in previous years entered the two courses owning a graphics calculator with statistical inference capabilities and 75% of them chose to have one in the final examination. On marked assignments students were asked to indicate which technology they had used for those questions where there was an option. More often than not they had used the graphics calculators. When they had to use a package they seemed to prefer *SPSS* to *Excel* even though they all had to learn how to use *Excel* for some of the file manipulations.

The authors strongly believe in the total integration of technology, including graphics calculators, into all aspects of a course, including assessment (Kissane, Bradley & Kemp 1994;

Kemp, Kissane & Bradley 1996; Kissane, Kemp & Bradley 2000). Whilst this has been achieved for graphics calculators in some pre-calculus and calculus courses that are taught at Murdoch University, it has been more difficult to do so with the statistics courses. If the use of graphics calculators were required then every student would need access to a calculator and, unfortunately, the large numbers of students means that the resources are not available to ensure this. The calculators are still considered too expensive to require students to own one for a statistics course. Consequently there are no questions on marked assignments, tests or final examinations that require students to use a graphics calculator. Although access to computer laboratories is available to all students at allocated times this by no means implies that they have access whenever they wish. In recent years the move has been away from complex computations towards interpreting given output, especially in examinations. The only calculations that students are expected to complete in timed tests are those for basic hypothesis testing and confidence intervals. However if students have access to graphics calculators they may choose to use them for the actual calculations. A few, even those with calculators, prefer to do it all by hand, others prefer the calculator and many say that to start with they do it by hand then check using the calculator but, once they feel proficient and have limited time, will rely on the calculator.

5. Student Responses

From interviews with students in both 1999 and 2000 it was clear that all forms of technology had important roles to play in the courses. Students indicated that they found different aspects of the different technologies useful and could not place one clearly above another. In the early stages of the courses the calculators had definite advantages in that the simple tests and confidence interval calculations were not readily available on the computer packages. Familiarity with the technology, ease of access and 24 hour access were often cited as **pluses** for the calculators. Towards the latter end of the courses the computer packages had definite advantages for the more complex analyses - not so much in the performing of them - but in the printing of results. Not only is the print out easier to obtain, but it is also more comprehensive.

During 2001, interviews with the students indicated student views consistent with previous years. One change was the fact that more students were coming to university accustomed to using graphics calculators. The familiarity the students had with the technology of their graphics calculators helped overcome their fears and dislikes of studying statistics. (About 99% of these students are doing the course because it is required not because they want to.) Learning about extra features was just seen as a natural extension of their previous use. Many students expressed delight that the calculators continued to have a role in their studies. Some who had passed their calculators on to younger siblings soon got them back or purchased new ones. Being able to check hand calculations and use of tables on the calculators increased their confidence and they particularly liked the visual representation of the P-value on the calculators, which enhanced their understanding of interpreting hypothesis tests.

6. Conclusion

There are advantages to using and making explicit the appropriateness of the different technologies at different times in the courses. Students see the importance of having access to computer packages that are likely to be used in their own research or future careers. Using data that directly relates to the students gives extra relevance to the course as well as producing some interesting information and discussions along the way. On the other hand students also value the graphics calculators as portables aids to learning; the 24-hour access to the calculators through ownership or borrowing through the library is seen as a definite **plus**. Students who have mastered their use in one of these courses continue to use them in later courses.

Lastly, but not least, and possibly as a reflection of the technological age we live in, more students are deciding to do further statistics courses in their second and third years. The increasing relevance of the introductory courses helps students to enjoy and value these courses. These students can then add a statistics minor to their life sciences major and in so doing make themselves far more employable.

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Table 1

Descriptive and inferential statistics facilities

Test/CI	MINITAB	Excel	SPSS	HP	Casio/TI/Sharp
Histogram	R	R	R	R	R
Box and whisker	R	R	R	R	R
pie chart	R	R	R		
Scatter diagram	R	R	R	R	R
CI mean (z)	R			R S	R S
z-test mean	R			R S	R S
CI mean (t)	R		R	R S	R S
t-test mean	R	(R)	R	R S	R S
CI proportion	R S			(R) S	(R) S
z-test proportion	R S			(R) S	(R) S
CI 2 dep means (t)				(R) (S)	(R) (S)
paired t-test		R		(R) (S)	(R) (S)
CI 2 means (z)	R			R S	R S
z-test 2 means	R	R	(R)	R S	R S
CI 2 means (t)	R		R	R S	R S
t-test 2 means	R	R	R	R S	R S
CI 2 proportions (z)	R S			S	S
z-test 2 proportions	R S			S	S
χ^2 association	T		T		T
ANOVA 1 way	R	R	R		R
Regression	R	R	R		R

Key: CI - confidence interval;

R - raw data; S - summary data; T - data in tabular or matrix format

(R) indicates that raw data can be used but only after some manipulation eg sorting for proportions and calculating differences for paired t-test;

(S) indicates that summary data for the calculated differences can be used.

Table 2

Survey results - 2000

Week 4 of Semester	
Access to Excel at home	112 (77%)
No	34 (23%)
Own graphics calculator with inference	41 (28%)
Own graphics calculator but without inference	73 (50%)
No	32 (22%)
Used only GC for normal probabilities - own	29
Used only GC for normal probabilities - borrowed	10
Final Examination	
Used graphics calculator with inference	113 (50%)
Used graphics calculator without inference	36 (16%)
Did not use a graphics calculator	75 (33%)

Table 3

Calculation of means and standard deviations during week 3 tutorial - 2000
(instructions for all forms of technology given) (one course only)

Preferred method		Typical comments
scientific calculator {also owned GC}	8 (9.9%) {0}	"not good with computers" "computer packages awkward to use"
graphics calculator {owned GC}	48 (59.2%) {44}	"faster, more familiar with" "more used to it and it will be available in exams" "don't have to log on" "less steps to take & simpler (even though I am unfamiliar with it)" {student did not own GC}
computer {also owned GC}	25 (30.9%) {13}	"data entry was easier" "the screen is bigger and easier to read"

MATHEMATICS OR COMPUTERS? CONFIDENCE OR MOTIVATION? How Do These Relate To Achievement?

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ABSTRACT

The use of computers in the teaching and learning of undergraduate level mathematics raises many still unanswered questions about the relationships between students' perceived abilities and attitudes towards mathematics and computers (both separately and interactively), and their performance on assessment tasks.

This paper reports on an investigation of the correlations between first-year mathematics students' performances on a range of assessment items, and the following affective factors:

- students' levels of confidence in their ability to do and learn mathematics
- their motivation when doing mathematical tasks
- their levels of confidence in the use of computers
- their motivation to use a computer
- their attitudes to technology in the learning of mathematics.

The study targeted a class of students in a typical first-year Australian Linear Algebra and Calculus course. Support for the use of MATLAB was integrated into their learning, and students did both hand exercises, and tasks requiring the use of technology, in tutor-supported weekly computer laboratory sessions. The USQ MTech scales and Galbraith-Haines scales, instruments already well tested for internal consistency and reliability, were used to assess students' confidence levels with mathematics and with computers, their mathematics motivation and computer motivation, and their attitudes to technology in the learning of mathematics.

Scatter plots and correlation coefficients are offered where appropriate, to illustrate the relationships between the students' mean scores on each of these scales, and their achievement levels on a range of assessment items: three assignments and two examinations. The trends and significant findings are discussed in relation to the overall nature of the assessment items. The data collected are also used to further establish the reliability and validity of the scales used.

KEYWORDS: Mathematics, technology, attitudes, undergraduate, achievement.

1. Introduction

1.1 Outcomes and effects: Increasing student access to the use of technology is providing impetus for the development of a wide range of innovative programs that invite or compel undergraduate mathematics students to interact with computers for learning and for problem solving. This raises many as yet unanswered questions about the effects, both cognitive and affective, of technology-rich learning experiences. While many developments in this area seem to offer exciting and stimulating new approaches to learning, there are relatively few careful attempts to assess the effects of the increasing role of technology on learning preferences and on attitudes.

One reason for this neglect is that outcomes are often difficult to measure and compare. Resources and timetabling often make controlled studies difficult, if not impossible. Equity issues also arise when different levels of access to technology are granted to different groups of students. Many commonly used methods of assessing learning outcomes are unreliable when extended to comparisons between different learning environments. Crucial questions about our objectives and instruments must be answered before we can fairly compare the performance of students who have been exposed to different tasks, approaches and emphases.

Clearly it is necessary to establish what common outcomes we seek, both cognitive and affective, and to investigate ways to assess these.

1.2 The critical balance of Affect and Cognition: Reported studies have continued to pose the direction of the relationship between *attitude* and *performance* as an open question. Thus while Tall and Razali (1993) argued that the best way to foster positive attitudes is to provide success, Hensel and Stephens (1997) concluded that "it is still not totally clear whether achievement influences attitude, or attitude influences achievement". Shaw and Shaw (1997) noted that among a certain group of engineering undergraduates (labelled downhillers) performance and motivation both deteriorated during tertiary studies - leaving the direction of any causal mechanisms open. Certainly if a learning experience is unpleasant for the student, any gains in cognitive achievement and performance may be offset or diminished by attitudinal losses. Raised levels of dislike or feelings of inadequacy may deter the student from studying further in the area. When evaluating learning programs, therefore, our goal of cognitive gain must be tempered by attention to affective outcomes. We might refer to this critical balance as ACE: that combination of Affective and Cognitive outcomes that yields an Effective learning program.

Cognitive issues have long been a primary focus of attention in assessment. While there is much debate about the value of different types of assessment, most educators feel that at least some of the cognitive outcomes of a mathematics learning program can be assessed by evaluating students' performance on a carefully balanced range of assessment tasks, usually a combination of tests, assignments and projects. Affective issues, outcomes and their measurement, on the other hand, have been seriously neglected (McLeod 1992) and have produced far less consensus. Yet their importance is undeniable in an era when a growing number of attractive alternatives are enticing students away from the study of mathematics.

It seems unlikely that affective issues are under-valued, for teachers report frequently and quite strongly on students' attitudes and reactions - but usually relatively informally. Many published reports on innovative programs address affective outcomes in a relatively ad hoc way, if at all. Most common are summaries of student responses to a course evaluation questionnaire, specially designed or generic to the institution. While they may be informative about that particular program, such evaluations do not enable comparison with programs elsewhere.

How can and should we assess the cognitive and affective outcomes of our mathematics programs? How should we balance them? And in particular, on the attitudinal side:

- What common affective goals do we have for mathematics programs?
- Are the goals different for technology-enriched mathematics programs?
- How can we measure the affective outcomes of such programs?

1.3 Significant attitudes, and scales for their measurement: Recent work done independently by two sets of researchers in this area has aimed at designing and testing instruments for measuring attitudes to mathematics and to computers in technology-enriched undergraduate mathematics programs. Most existing instruments, including the well-known Fennema-Sherman scales (Tartre & Fennema 1995), designed for school level students, are inappropriate for assessing attitudes in this particular environment.

The University of Southern Queensland (USQ) project team (Cretchley, Fogarty, Harman & Ellerton 2000, 2001) identified 3 fundamental affective factors, *Mathematics Confidence, Computer Confidence,* and *Attitudes to Technology in the Learning of Mathematics*, and developed three Likert-style attitude scales for their measurement.

Galbraith and Haines (University of Queensland, and City University, London) identified six relevant factors; *Mathematics Confidence, Computer Confidence, Mathematics Motivation, Computer Motivation, Mathematics Engagement,* and *Computer-Mathematics Interaction. Mathematics Engagement* correlated very strongly with *Mathematics Motivation* so five Likert-style scales were retained (Galbraith & Haines 1998, 2000).

A comparison of the above sets of scales reveals that the respective *Confidence* scales seek remarkably similar attributes. The notable difference is that whereas the G-H scales deliberately separate *confidence* and *motivation* into four 8-item scales, the two slightly broader USQ *confidence* scales (11 and 12 items, respectively) include some measure of *motivation*.

The two interactive mathematics/technology scales measure quite different attributes, however. The USQ *MathTech* scale assesses attitudes to the notion of using technology for learning mathematics, and is worded so that it is appropriate for a wide range of students (from those who have little or no experience of using technology for the learning of mathematics to those who are very experienced). The term technology is used to include graphics calculators as well as computer-based resources. A sample item:

"I like the idea of exploring mathematical methods and ideas using technology".

The G-H *Computer-Mathematics Interaction* scale is more computer-specific, and refers to specific types of reaction. Sample items:

"I rarely review the material soon after a computer session is finished"

"I find it helpful to make notes, in addition to copying material from the computer screen or obtaining a printout".

Both sets of scales have been tested in a number of universities over several years and demonstrate strong reliability and internal consistency, yielding Cronbach alphas of around 0.8 and higher, well above frequently cited benchmark values for internal consistency reliability.

Used quite independently in different technology programs, the scales have produced some remarkably robust findings (Galbraith, Pemberton & Cretchley 2001). For example, both sets have yielded consistently low correlations between attitudes to mathematics and attitudes to computers. Furthermore, both sets have indicated that attitudes to technology in the learning of mathematics are much more strongly associated with computer attitudes than with mathematics attitudes.

1.4 The Research Questions: With the background and objectives outlined above, this study targeted both the affective and cognitive domains in the first semester of a technology-enriched undergraduate mathematics program in Australia. Students' perceived abilities and attitudes towards mathematics and computers were investigated both separately and interactively. Based on the literature and observation, mathematics confidence and motivation, and computer confidence and motivation, were selected as factors likely to impact on progress in that kind of learning environment, as were attitudes to technology in the learning of mathematics. The specific questions posed were:

A: What relationships exist between the five affective factors listed below, as defined by student responses in a technology-enriched mathematics program?

- students' *confidence* in their ability to do and learn *mathematics;*
- students' *motivation* when doing *mathematical* tasks;
- students' levels of *confidence* in the use of *computers;*
- students' *motivation* to use a *computer* generally;
- students' attitudes to using *technology* in the *learning of mathematics*?

B: How does each of these attitude scales correlate with performance on a range of assessment items?

C: What is the significance of these findings for course design?

2. The Study

The investigation targeted a class of first year undergraduate students in the Linear Algebra and Calculus course at the University of Southern Queensland, Australia, in the first-semester of 2001. Support for the use of MATLAB was integrated into their learning, and students did both hand exercises and tasks requiring the use of technology, in tutor-supported weekly computer laboratory sessions. A literature survey revealed no more appropriate or carefully developed scales than the University of Southern Queensland (USQ) and Galbraith-Haines scales, to measure students' attitudes to the factors listed above. Hence pre- and post- administrations of the following scales took place in the first and last lectures of the semester. An initial letter G indicates a Galbraith-Haines scale-otherwise scales are USQ.

- *mathematics confidence*: *MathConf* and *GMathConf* scales
- *computer confidence*: *CompConf* and *GCompConf* scales
- mathematics motivation: *GMathMotv* scale
- computer motivation: *GCompMotv* scale
- attitudes to technology in the learning of mathematics: *MathTech* scale

The Galbraith-Haines *Computer-Mathematics Interaction* scale was not appropriate for the pre-test because at that stage many students had not yet used a computer for learning mathematics.

The Likert-style attitude questionnaire containing the items invited students to place a cross on a *continuous* scale from 1 to 5 with 1 indicating strong disagreement, 3 a neutral view, and 5 strong agreement. Intermediate responses were recorded to the nearest decimal place. Almost all students present in the first lecture completed the pre-test (N=196), and performance scores on 3 assignments and 2 end-of-semester examinations were obtained for most of those students. Because of the pressures of the course, post-test attitudinal data could only be obtained from 92 students who attended the final class late in the week before the examinations. A full set of pre- and post-data, as well as assignment and examination data, was therefore available for 82 of the original 196 students. It could reasonably be assumed that this subgroup contained conscientious students.

Students' mean scores were calculated for each of the 7 attitude scales, and relationships between these were investigated graphically and analytically. Students' performances on each of the assessment items were explored for relationships with the affective factors, and correlations calculated where appropriate. Relevant Pearson correlation coefficients are provided below.

It is recognised that correlations do not enable directional inferences to be made about relationships within the data. However it has been noted that the direction of causality between attitude and performance appears to be left open in the literature, and the approach here is consistent with that conservative stance.

3. Analysis and Findings

3.1 Attitude scale data and correlations: Students' mean scores on each of the six attitudinal scales were roughly normally distributed, with pre-test data yielding the group means and standard deviations shown in Table 1. Group means were all above 3 indicating positive attitudes, on average.

Table 1: Group mean scores on the attitude scales (1 = min, 3 = neutral, 5 = max)

	N	*Mean*	*Std. Dev.*		*N*	*Mean*	*Std. Dev.*
MathConf	176	3.66	.60	*CompConf*	176	3.87	.72
GMathConf	176	3.51	.60	*GCompConf*	171	3.69	.67
GMathMotv	174	3.46	.57	*GCompMotv*	171	3.58	.68
				MathTech	172	3.67	.54

Table 2: Pearson Correlations between *Confidences*, *Motivations* and *MathTech* Attitudes

	MathConf	*GMathConf*	*GMathMotv*	*CompConf*	*GCompConf*	*GCompMotv*
MathConf	1.00					
GMathConf	.83** (.83)	1.00				
GMathMotv	.76** (.84)	.62** (.80)	1.00			
CompConf	.12 (.02)	.16* (.01)	.17* (-.01)	1.00		
GCompConf	.14 (.07)	.21 (.04)	.12 (.02)	.87** (.85)	1.00	
GCompMotv	.14 (.09)	.13 (.10)	.22 (.14)	.79** (.73)	.75** (.65)	1.00
MathTech	.28** (.18)	.31** (.20)	.28** (.14)	.49** (.50)	.51** (.47)	.58** (.66)

** Corr. is significant at the 0.01 level (2-tailed). * Corr. is significant at the 0.05 level (2-tailed).

Table 2 gives the Pearson correlation coefficients for the pre-test data (N=196) with the post-test data coefficients (N=92) shown in brackets. These indicate the following:

- Mathematics and computer attitudes (both confidence and motivation) correlate surprisingly weakly (up to a maximum of 0.22 for this data).
- Attitudes towards technology in the learning of mathematics correlate far more strongly with computer confidence and motivation than they do with mathematics confidence and motivation (0.47 and above, compared with 0.31 and below).
- Confidence and motivation data correlate strongly within the mathematics scales and within the computer scales, as expected. In particular, *post-test mathematics* motivation data yielded very high correlations of 0.84 and 0.80 with the 2 mathematics confidence scales.

The correlations confirm earlier findings (Cretchley et al 2000, 2001, Galbraith & Haines 1998, 2000), and establish the stability of these findings over a period of some years in which there has been further steady growth in the use of computers generally.

Administering the USQ and G-H confidence scales in parallel revealed the following:

- There are consistently very strong correlations between the two mathematics confidence scales (0.83) and the two computer confidence scales (0.87) (0.85 on post-data).
- The pre-test *GMathMotv* motivation data correlate more strongly (0.76) with the *MathConf* confidence data than they do with the *GMathConf* data (0.62). This may be the effect of a few items in the *MathConf* scales that target some aspects of *motivation*: for example, "I don't understand how some people seem to enjoy spending so much time on mathematics problems".

3.2 Mathematics attitudes and performance: Examinations A and B covered a range of tasks which, for equity reasons, were designed so that manipulation of data could be done quite easily and quickly by hand. However, graphics calculators were permitted in both A and B, and laptops were permitted in B. Exam A tested the basic concepts and techniques of the course far more directly than Exam B, which placed greater emphasis on applications and required more problem-solving skills. Appendix A elaborates this distinction.

Appendix A outlines typical tasks on the assignments. Tasks in Assignments 1 and 3 required direct use of technology to the value of 10% and 18% of the respective totals. Assignment 2 did not include any computer-based tasks. Hence while use of technology could be readily avoided in the examinations and Assignment 2, its non-use presented an impediment to the efficient completion of Assignments 1 and 3, and its use could enhance performance on Exam B. Tables 3 and 4 offer correlations of performances on these assignments and examinations, with the *pre*-test mathematics attitudes data measured at the start of the semester, and with *post*-test attitudes measured only a week before the examinations.

Table 3: Pearson correlation coefficients for *pre*-test *mathematics* attitudes & performance on assignments/exams (N = 130)

	MathConf	*GMathConf*	*GMathMotv*	Asn1	Asn2	Asn3	ExamA	ExamB
Asn1	.28**	.21**	.27**	1.00				
Asn2	.33**	.31**	.19**	.63**	1.00			
Asn3	.17	.16	.14	.66**	.51**	1.00		
Asn Ave	.29	. 23	.20					
ExamA	.47**	.37**	.34**	.65**	.67**	.59**	1.00	
ExamB	.45**	.34**	.29**	.57**	.55**	.50**	.85**	1.00
Exam Ave	.46	.36	.32					

Table 4: Pearson correlation coefficients for *post*-test *mathematics* attitudes & performance on assignments/exams (N ≈ 81)

	MathConf	*GMathConf*	*GmathMotv*
Asn1	.41**	.48**	.39**
Asn2	.45**	.34**	.42**
Asn3	.44**	.45**	.37**
Asn Ave	.43	.42	.39
ExamA	.65**	.63**	.59**
ExamB	.60**	.55**	.50**
Exam Ave	.63	.59	.55

** Correlation is significant at the 0.01 level (2-tailed).

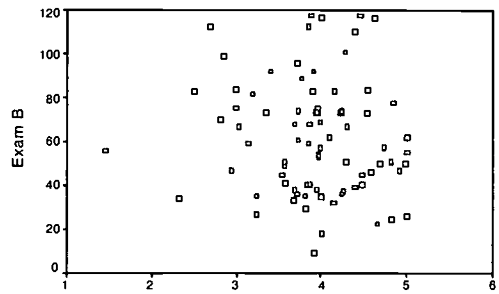
Corresponding coefficients in Tables 3 and 4 indicate that *post*-test attitudes correlate better with performance on all the assignments and examinations than do *pre*-test data. Since post-test data collection took place closer to the timing of Assignment 3 and Exams A and B, this finding is not surprising for those three items. However, post-test attitudes also correlate better with performance on Assignments 1 and 2. This may be due to the nature of the post-test sample – conscientious students who attended the optional final class and completed the course.

The following trends are worthy of note:

- Columns 1, 2 and 3 of Tables 3 and 4 indicate moderate correlations between mathematics confidence and motivation levels, and performance on the assignments and examinations. The *post*-test data reveal much stronger correlations than the *pre*-test data: in particular, Assignment 3 correlations with post-test data were significant – not so the pre-test data.
- Mathematics *motivation* yielded slightly weaker correlations with performance on average than did mathematics *confidence*.
- Despite considerable differences in the type of questions in Examinations A and B, correlations with the three mathematics attitudes scales were quite consistent.
- Correlations of mathematics attitudes with performance on the 3 assignments were similarly consistent, despite differences in the range of concepts and the nature of the tasks.
- Correlations with mathematics confidence and motivation were consistently lower with the 3 assignments than they were with the 2 examinations.

Computer attitudes and performance: Graphical investigation of the relationships between *computer attitudes* and performance on the mathematics-based assignments and examinations revealed very scattered data. Figure 1, for example, is a plot of students' levels of performance on Examination B against their post-test (N=82) computer confidence levels. Statistical analysis confirmed the lack of correlation generally, and hence no tables corresponding to Tables 3 and 4 are presented for computer confidence and motivation. This lack of correlation with performance is perhaps not surprising when we consider that the assessment tasks were strongly mathematical, and that computer attitudes and mathematics attitudes correlate weakly.

Figure 1: Scatterplot of Exam B results against Computer Confidence levels



What is of interest here is the lack of strong correlations between computer confidence and motivation levels and performances generally on mathematics tasks in a technology-rich mathematics learning environment. That lack of correlation is evident with performances on *both* examinations, even on Exam B in which students were encouraged to use a computer (see Appendix A). It is further suggested by the fact that computer attitudes did not yield significantly different correlations with performance across the assignments, despite the different composition and relative weighting of computer-based tasks: 18% of Assignment 3, 10% of Assignment 1, and 0% of Assignment 2. Tasks requiring the use of technology or inviting its use (see Appendix A), were generally well done by the majority of students, not only by those who were confident with and enjoyed using computers.

4. Summary and Conclusions

This study confirmed the weak relationship between mathematics and computer attitudes (both confidence and motivation), and that students' attitudes to using technology in the learning of mathematics correlate far more strongly with their computer attitudes than with their mathematics attitudes.

Mathematics attitudes (both confidence and motivation) correlated quite strongly (up to P=0.65) with levels of achievement on a wide range of mathematical tasks, some of which invited the use of technology. Mathematics attitudes measured late in the learning program correlated much more strongly with performance on assessment items, even the earliest ones, than did attitudes measured early in the course.

Computer attitudes demonstrated little or no correlation with performance on mathematical tasks, even on items of assessment that invited or required the use of technology. This raises questions about how we can best harness the enthusiasm for computers that some students have, and what types of computer-based mathematical tasks might capitalise on strong positive computer attitudes. This area clearly needs much more investigation, but it is possible that computer confidence is a poor predictor of the likelihood of a mathematics student being empowered by the use of technology in learning mathematics. To those who seek to use technology to enliven and empower the learning of mathematics, such a finding remains a continuing challenge. Of particular interest, because of the potential for technology to advance or hinder learning, are those students with mixed confidences: high computer confidence but low mathematics confidence, or vice versa. Future research has been planned that aims at identifying more particularly the learning characteristics of such students, as part of the wider search for methods that will empower the learning of student groups within which a wide range of attitudes prevails.

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Appendix A

	Hand tasks: Of the standard typical of those in first-year texts.	Examples of the set computer-based tasks	Technology skills: MATLAB or similar
Assignment 1	Sketch vectors with given properties, use vectors to investigate properties of a parallelogram, applications of dot and cross product, applications of projections, finding equations of lines, planes & applications thereof. Investigate properties of given functions, find & use the inverse of a function.	Plot and explore the graph defined by $f(x) = (\ln x)(2 - \sin x)$. Establish the domain, range & explore the properties of f & f^{-1}. Find or confirm function values like $f^{-1}(1)$ & $f(f^{-1}(1))$. Plot a given exponential growth function and use it to predict populations. Approximate rates of change from a graph using the difference quotient with decreasingly small intervals.	Generating appropriate input values, typing in functions correctly, plotting, zooming, reading the scale correctly, axis control, overlaying graphs, labelling plots, printing.
Assignment 2	Find intersections of planes, set up systems of linear equations to fit a polynomial to 5 given points, model supply & demand systems. Interpret the meaning of derivatives and definite integrals, & find them algebraically & numerically. Applications to rate of change, average value & distance.	Use of a computer was not permitted.	
Assignment 3	Find determinants, find the inverse of a matrix by row-reduction and via the adjoint, apply matrix algebra to elementary networks and cryptography. Use derivatives to investigate slope & acceleration, curvature & concavity. Use calculus for optimisation. Find the area under a curve. Approximate a definite integral with Riemann sums.	Find the inverse of a given 3x3 matrix by row reduction. Use technology to calculate values and confirm properties of matrix inverses & determinants. Solve systems of linear equations & matrix equations using technology in different ways: unknowns typically 3x1 or 3x3 matrices. Plot a graph of the amplitude of a spring and use it to confirm rates of change & accelerations found analytically. Calculate Riemann sums to approximate a definite integral with increasing accuracy.	Defining & using pre-defined matrices, det & inv commands. Solving linear equations using rref or rrefmovie, the \ command, and matrix inverses, where possible. Defining & refining intervals for left & right Riemann sums, calculating function values, summing products.
Exam A	Emphasis on demonstrating understanding of fundamental concepts and mastery of basic techniques. A broad range of typical first-year mathematics major exercises, on topics like those listed above.	No access to computers was allowed. Graphics calculators were permitted but not required.	
Exam B	Quite different to Exam A: An open book exam, with emphasis on modelling and problem solving. Typical introductory applications of basic linear algebra and calculus, including a few tasks quite different to those attempted over the semester.	Laptops & graphics calculators were permitted but not required. Though all tasks were designed to facilitate reasonably quick hand calculation, there was ample opportunity to use a computer: for matrix multiplications (2x3, 3x3), row reduction (2x4), to plot graphs and find range, signs, average value and optimum values, and to calculate Riemann sums.	Most of the above skills would have been useful.

ON THE RELATION BETWEEN MATHEMATICS AND PHYSICS IN UNDERGRADUATE TEACHING

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ABSTRACT

The historical development of Mathematics and Physics suggests that:

(a) Mathematics and Physics have always been closely interwoven, in the sense of a "two-ways process":

• Mathematical methods *are used* in Physics. That is, Mathematics is not only the "language"of Physics (i.e. the tool for expressing, handling and developing logically physical concepts and theories), but also, it often *determines* to a large extent the *content* and *meaning* of physical concepts and theories themselves.

• Physical concepts, arguments and modes of thinking *are used* in Mathematics. That is, Physics is, not only a *domain of application* of Mathematics, providing it with problems "ready-to-be-solved" mathematically by already *existing* mathematical *tools*. It also provides, ideas, methods and concepts that are crucial for the creation and development of *new* mathematical concepts, methods, theories, or even whole mathematical domains.

(b) Any distinction between Mathematics and Physics, seen as general attitudes towards the description and understanding of an (empirical, or mental) object, is related *more* to the point of view adopted while studying particular aspects of this object, than to the object itself.

Points (a) and (b) imply that:

(c) Any treatment of the history of Mathematics independent of the history of Physics is necessarily incomplete (and vice versa).

(d) By accepting the importance of the historical dimension in education, the relation between Mathematics and Physics should not be ignored in teaching these disciplines.

It is possible to illustrate the above points with the aid of many important examples, which can also be didactically relevant by following a historical-genetic teaching approach. In this paper, we illustrate this qualitatively by means of three examples (at the same time surveying author's work in this area in the last few years):

- The possibility to introduce and/or illustrate important geometrical and algebraic concepts on the basis of Relativity Theory.
- The complex, deep interconnections between Differential Equations, (Functional) Analysis and Quantum Mechanics.
- The physical origin of many basic concepts and theorems of the theory of Dynamical System and of Ergodic Theory.

1. Introduction

The present paper rests on the following two points:

(a) The appreciation by many mathematicians, mathematics educators and historians of the significance of the introduction of a historical dimension in Mathematics Education (ME).

(b) The well-known fact that there has been a close interrelation between Mathematics and Physics throughout their historical development.

Both points can have a lasting effect on the way Mathematics is taught and learned. In what follows, I will elaborate on (a) and (b), connecting them and illustrating them by means of 3 examples at the university level. Details can be found in the literature; hence the present paper is also a survey of author's work in this area in the last few years.

1.1. Comments on (a): At least implicitly, the way Mathematics is presented and/ or taught reflects a philosophical and epistemological point of view about the *nature* of Mathematics. In particular, that Mathematics is conventionally presented deductively reflects a point of view, according to which Mathematics is simply a collection of axioms, definitions, theorems, and proofs, that is, *only* the *results* of the mathematical activity. As a consequence, Mathematics is supposed to evolve more or less by a linear accumulation of new results (cf. Lakatos 1976, pp.1-2). Hence, what is essential is to learn these results in their final "polished" form. Such a point of view has a lasting effect on what parts of Mathematics should be taught and how this should be done (Schoenfeld 1992, p.341). This is particularly evident at the university level, given that there, it is often tacitly taken for granted that once the student has made his/her choice to study (either "pure", or "applied") Mathematics, he/she has to learn it independently of the way it is presented.

However, in this way it is not appreciated that Mathematics is a human enterprise, hence that "doing Mathematics" is an equally important aspect of Mathematics itself that should not be left out (cf. Grugnetti, Rogers et al. 2000, §§ 2.2.2, 2.3.3). On the contrary, there is an ever-increasing agreement that helping students to become aware of the evolutionary nature of Mathematics may lead them to a deeper and more solid understanding of Mathematics.

Therefore, if Mathematics is conceived, not only as a collection of logically complete finished products, but also as the *process* by which these products are conceived, formulated, developed and justified, it becomes clear that a historical dimension in teaching and learning Mathematics is helpful, or even necessary. Actually, history makes clear that the deductive organization of any mathematical domain is a posteriori (i.e. once this domain is sufficiently mature). At the same time, history provides a natural framework for helping students to become aware of Mathematics in the making. Introducing a historical dimension into ME has important advantages that cannot be analyzed here (see Tzanakis, Arcavi et al.2000, §7.2 for a comprehensive analysis), and can be done in a variety of ways depending on several factors, like the emphasis one wants to put on the subject taught, the level of education etc (see Tzanakis, Arcavi et al. 2000, §§7.3, 7.4). I will focus on two advantages only.

- History constitutes an important resource of *relevant* questions, problems and expositions, valuable both in terms of their content and their potential to *motivate*, *interest* and *engage* thc student. Thus, historically inspired exercises, problems, or small research projects, may stimulate the student's interest and contribute to enhance curriculum alongside those exercises and problems, which may seem 'artificially' designed. In this way, aspects of the historical development of a subject include "real" Mathematics, so that they become part of the student's "working knowledge". Consequently, history *in* ME no longer appears as something alien to "Mathematics proper", but forms an integral part of it.

- History reveals interrelations among different mathematical domains, or, of Mathematics with other disciplines and suggests that mathematical activities and results may be interdependent. Thus, integration of history in teaching may help to interrelate domains, which at first glance appear unrelated. It also provides the opportunity to appreciate that fruitful research in a scientific domain does not stand in isolation from similar activities in other domains. On the contrary, it is often motivated by questions and problems coming from apparently unrelated disciplines and often, having an empirical basis. This is especially true for Physics and leads us to point (b) mentioned above.

1.2. Comments on (b): What has been said above about the role of history in teaching and learning Mathematics is equally valid for Physics as well (Tzanakis & Thomaidis 2000). On the other hand, as it has already been mentioned, history shows clearly the close, interconnected development of Mathematics and Physics, which cannot be ignored in teaching and learning these disciplines, in view of what has been said in §1.1. This close interrelation can be seen in two different, but *complementary* perspectives:

(1) *From a historical point of view*, there are 3 different ways by which Mathematics and Physics are interrelated, influencing each other (Tzanakis 2000):

(a) Physical theories and the appropriate mathematical framework evolve *in parallel*, often as the result of the work of the same persons. This is the case of the foundations of infinitesimal calculus and of classical mechanics in the 17th century, mainly through the work of Newton and Leibniz; or, the parallel development of vector analysis and of electromagnetic theory in the second half of the 19th century, mainly by Maxwell, Gibbs and Heaviside (Crowe 1967).

(b) *New mathematical* theories, concepts or methods are formulated in order to solve *already existing physical problems*, or to provide a solid foundation to methods and concepts of Physics. The emergence of the basic ideas of classical Fourier analysis, through the study of heat conduction constitutes a typical example. Dirac's introduction of his delta function in quantum mechanics, and its later clarification in the context of the theory of generalized functions is a more recent example (Lützen 1982, ch.4, part 2)[1]. Finally, the introduction in the second half of the 19th century of Boltzmann's ergodic hypothesis in classical statistical mechanics led to the foundations of ergodic theory in the 1920's and 1930's through the work of G. Birkhoff, J. von Neumann and E. Hopf (Sklar 1993, ch.5; see also section 3.3. here).

(c) The formulation of a *mathematical theory precedes its physical applications*. Its use is often made *after* the corresponding physical problems naturally indicate the necessity of an appropriate mathematical framework. A famous example is Einstein's work on the foundations of the general theory of relativity in the period 1907-1916, on the basis of riemannian geometry and tensor analysis developed in the second half of the 19th and early 20th centuries, mainly by Riemann, Christoffel, Ricci and Levi-Civita (Pais 1982, ch.12). Another example is provided by the fact that on the basis of spectroscopic data, Heisenberg realized in 1925 that atomic magnitudes have the algebraic structure of (infinite dimensional) complex matrices and he was thus led to the formulation of matrix mechanics (Mehra & Rechenberg 1982 ch.3; see also section 3.2 here).

[1] Of course, it is well known that the delta function appeared much earlier, in the 19th century in the work of many mathematicians and physicists, in a number of equivalent forms (Lützen 1982, ch.4, §34).

These examples are indicative of the intimate relation between Mathematics and Physics and lead us to look at this relation from another perspective (Tzanakis 2001)

(2) *From an epistemological point of view* Mathematics and Physics are much closer to each other than it is usually thought:

(a) Mathematics and Physics have always been closely interwoven, in the sense of a "two-ways process":

•Mathematical methods are used in Physics. By this I mean that not only Mathematics is the "language" of Physics (i.e. the tool for expressing, handling and logically developing physical concepts and theories), but also it often *determines* to a large extent the *content* and *meaning* of physical concepts and theories themselves.

•Physical concepts, arguments and modes of thinking are used in Mathematics. Thus, Physics not only constitutes a reservoir of problems "ready-to-be-solved" mathematically (i.e. a *domain of application* of already *existing* mathematical *tools*), but it also provides *ideas*, *methods* and *concepts* that are crucial for the *creation* and *development* of *new* mathematical concepts, methods, theories, or even whole mathematical domains.

(b) Any distinction between Mathematics and Physics, seen as general attitudes towards the description and understanding of an object[2], is related *more* to the point of view adopted while studying particular aspects of this object, than to the object itself.

The general characteristics of the relation between Mathematics and Physics described in (1) and (2) above can be integrated into teaching in several different ways. In the next sections I will illustrate the above points in terms of 3 different examples and with the aid of what may be called a historical-genetic approach

2. A Historical-Genetic Approach

As already mentioned in the previous section, a historical dimension can be introduced into teaching in several ways that have been discussed elsewhere, depending on several factors (Fauvel & van Maanen 2000). The discussion here is confined to what may be called a historical-genetic approach, presented in more detail in the literature (Tzanakis, Arcavi et al 2000, Tzanakis 2000, Tzanakis & Thomaidis 2000).

It is an approach adopting the point of view that a subject should be taught, only after the learner has been *motivated* enough to do so by means of questions and problems, which the teaching of the subject may answer (cf. Toeplitz 1963, Edwards 1977). In other words, the subject to be taught should acquire a necessary character for the learner, so that he/she can appreciate its significance in clarifying particular issues and in answering specific problems. This character of necessity of the subject constitutes the central core of the meaning to be attributed to it by the learner. Therefore, such an approach emphasizes less the *way of using* theories, methods and concepts, and more the *reasons* for which these theories, methods and concepts *provide answers* to specific problems and questions, without however disregarding the "technical" role of mathematical knowledge.

It is clear that such a point of view is not restricted to Mathematics only. In particular it is equally applicable to Physics (Tzanakis & Coutsomitros 1988, Tzanakis & Thomaidis 2000). For both disciplines, a historical perspective offers interesting possibilities for a deep, global understanding of the subject, according to the following general scheme:

[2] By this term I mean not only concrete, empirically conceived objects, but also mental objects like concepts, questions, problems etc.

(1) The teacher has a *basic* knowledge of the historical evolution of the subject, so that he/she is able to identify the *crucial steps* of this historical evolution and appreciate their significance. These steps consist of key ideas, questions and problems, which opened new research perspectives and enhanced the development of the subject.
(2) (Some of) these crucial steps, are *reconstructed*, by explicitly, or implicitly integrating historical elements, so that these crucial steps become didactically appropriate.
(4) Many *details* of these reconstructions are incorporated into exercises, problems, small research projects and more generally, *didactical activities* that give the opportunity to the learner to acquire technical skills and a better sense of the concepts and methods used. For instance, one may use sequences of historically motivated problems of an increasing level of difficulty, such that each one presupposes (some) of its predecessors. Their form may vary from simple exercises of a more or less "technical" character, to open questions which presumably should be tackled as parts of a particular study project to be performed by groups of students.

This general scheme forms the basis of what can be called a *historical-genetic approach* and seems to have distinct advantages that have been analysed in the references given above. Here we add only a few comments:

One may argue, that an obvious possibility to use history in the presentation of a mathematical and/or physical subject is to retrace its historical evolution. However, the formulation of the problems which led to its birth, and are presented today as part of modern Mathematics and/or Physics, would be too advanced for the learners, or may look completely foreign to them. Usually, its *strictly* historical presentation, in which all the fine details of the historical development are given, is not didactically appropriate, even at the university level. This is due to the fact that the historical evolution of a scientific domain, contrary to what is sometimes naively assumed, is almost never straightforward and cumulative. On the contrary, it is rather complicated, involving periods of stagnation and confusion, in which prejudices and misconceptions exist and it is greatly influenced by the more general cultural milieu, in which this evolution takes place. Moreover, the conceptual framework and the mathematical terminology and notation vary from one period to another. Finally, the didactical, social and cultural conditions of the students today are very different from the corresponding conditions in which mathematicians, who created and developed the subject under consideration, were living. Hence, strictly respecting the historical order makes the understanding of the subject more difficult (Thomaidis & Tzanakis preprint).

Therefore, integrating history in teaching Mathematics and/or Physics, should mean that a historically motivated thinking framework for the learner has been created, in which various aspects of the mathematical subject under consideration can be illustrated. In this respect, the *crucial* steps of the historical evolution of the subject are didactically important because whether or not a step in the historical evolution is crucial, is judged *a posteriori*. In other words, such *a step is crucial* exactly *because* it opened new research paths, it clarified the meaning of new knowledge, it suggested the most convenient and clear formulation of this knowledge and in general *it enhanced the development of the subject. Therefore*, such a step in the historical evolution is *in principle* didactically relevant.

It is in the above perspective that I will comment in the next section on three specific examples, which at the same time illustrate the deep, continuous and multifarious interrelation between Mathematics and Physics.

3. Examples

3.1 Algebra, Geometry and Relativity Theory

Einstein laid the foundations of the Theory of Relativity in two seminal papers. In 1905 he presented the Special Relativity Theory (SR) and after many years of intensive work and unsuccessful attempts, in 1916, he arrived at a new theory of gravitation, the General Relativity Theory (GR), in a long paper where he presented both its physical foundations and the mathematical methods to be used (both papers are reprinted in Sommerfeld 1952).[3] Although his papers were fairly complete, and full of fundamental consequences, both theories were developed further by many others in the next years.

Today, SR is a standard subject in undergraduate curricula for Physics students, whereas an introductory course in GR is usually addressed to postgraduate, or advanced graduate Physics (and occasionally, Mathematics) students. However, basic aspects of both GR and SR that played an important role in the development of new Mathematics, and enhanced the development of our understanding of physical phenomena, can be presented at a much earlier stage as an illustration of this new Mathematics and their place in the scientific edifice (both inside and outside Mathematics). This can be done by following an approach inspired by history, along the lines suggested in the previous section.

Some crucial historical elements[4]:

(a) SR is based on the so-called Lorentz transformations (LT) that gives the transformations between inertial coordinate systems. Einstein gave the derivation of these transformations in 1905 using the basic principles of SR, namely, the *Principle of Special Relativity* (the laws of Physics are invariant under a coordinate transformation between inertial systems, i.e. systems moving with constant velocity with respect to each other) and the *Principle of the constancy of the light speed* (light has the same speed in all inertial systems, whether or not the source is moving). His derivation is elementary and appeals very much on physical intuition and some tacit assumptions (about the homogeneity of space).

(b) Others (Voigt in 1887, Larmor in 1900, Lorentz in 1899 and 1904) have derived the LT earlier as a consequence of the search for the coordinate transformations that leave unaltered Maxwell's equations in electrodynamics. By the end of the 19^{th} century, it had been realized that these were the transformations between inertial systems, as a consequence of the famous Michelson-Morley experiment (and other similar ones).

(c) Poincaré in 1904 derived the LT by following a more mathematically oriented approach. He explicitly used the group structure of the sought transformations and determined their general form, as well as, fundamental consequences of SR, like the relativistic law of velocity addition.

(d) In 1908, in a seminal lecture (reprinted in Sommerfeld 1952), Minkowski introduced the concept of spacetime and revealed the rich geometrical content implicit to Einstein's 1905 paper on SR. This was the crucial step, without which GR could not have been developed.

(e) In his conceptual analysis of the physical and mathematical foundations of GR, Weyl (in 1918) argued that its basic physical principles imply that spacetime has the structure of a conformal rather than a (pseudo)riemannian manifold (i.e. only ratios of infinitesimal

[3] Perhaps it is less known that Hilbert arrived almost simultaneously to the field equations of the theory by following a different route. I will not touch upon Hilbert's contribution here (for a detailed study see Mehra 1973, Pais 1982 §14(d)). I simply mention that Hilbert knew Einstein's struggle for a new theory of gravitation and approached the subject from a different point of view.

[4] More historical details and references to the original literature can be found in Tzanakis 1999.

spacetime distances have a meaning, not the infinitesimal distances themselves; Weyl 1918/1952, p.204). As a consequence, he considered that the physically relevant basic geometrical structure of spacetime is not its (pseudo)metric. To proceed further, he argued that the basic structure is parallelism (i.e. the existence of a connection), an important concept introduced in 1917 by Levi-Civita and Hessenberg (Weyl 1918/1952, p.202). In this way, Weyl was led to introduce and study the first example of what later became known as gauge theory and gauge transformations (Weyl 1918/1952, Weyl 1921/1952, section 16).

It is beyond the scope of this paper to give a detailed epistemological analysis of points (a)-(e), which supports the claims made in §1.2. This will be done implicitly, by commenting on the didactical relevance of (a)-(e) along the lines of section 2.

(1) It is possible to derive the LT in two dimensions (one spatial and one temporal) by following Minkowski's key ideas: (i) the introduction of the concept of spacetime as a natural idea implied by Einstein's 1905 analysis of the relative character of simultaneity of events and (ii) its immediate consequence that the constancy of the light speed trivially implies that the sought transformation leaves invariant the so-called light cone (i.e. the surface on which light signals lie).

This derivation uses elementary matrix algebra and proceeds in close analogy with the determination of the form of plane rotations in analytic geometry: Rotations conserve the Euclidean distance x^2+y^2 in the xy plane, whereas LT conserve the Minkowski (pseudo)distance $x^2- y^2$ (which is zero on the light cone, y being the time coordinate). For details see Tzanakis 1999, section 3.

(2) Strictly speaking, conservation of the light cone implies only that the transformations are conformal, a fact whose significance seems to have been appreciated first by Weyl (see (e) above). It is more advanced, especially in 4 dimensions, to show that in the context of SR these conformal transformations are indeed isometries of the Minkowski (pseudo)distance if we assume that they map straight lines to straight lines, a consequence of the validity of Newton's law of inertia in SR. This derivation may constitute a small project, which can proceed along the lines of Poincaré, using explicitly the group structure of the transformations sought. A number of fundamental consequences of SR can be obtained in this way (velocity addition, length contraction etc), at the same time illustrating important abstract concepts, like group, commutativity, pseudo-Euclidean structure, conformal transformations etc. (for details and references to the original literature see Tzanakis 1999).

(3) Conformal transformations in the special case of similarities can be also introduced naturally by looking for the symmetry group of Maxwell's equations. It is a nice example to consider this problem for both the Laplace and the wave equation and to arrive at the orthogonal and the Lorentz group of transformations respectively. This is in fact the idea behind the pre-relativistic derivations of the LT by Larmor and Lorentz (papers reprinted in Schaffner 1972, part II, sections 9 and 11 and in Sommerfeld 1952, paper II; see also Whittaker 1951, pp.31-33). This point can be used in connection with (1) above, in the sense that they are dual to each other.

(4) Conventionally, parallelism and the concept of a connection on a differentiable manifold are introduced in a rather abstract and unnatural way. Weyl's geometrical interpretation of the basic physical principle of GR, namely the *Equivalence Principle* (all bodies in free fall in an infinitesimal region of spacetime have the same acceleration), leads to a natural definition of parallelism that is equivalent to its modern abstract definition (see (e)

above and Siu et al. 2000, §8.4.8). The proof can be structured as a sequence of exercises in tensor algebra and differential geometry.

(5) On the other hand, Weyl's analysis, mentioned in (e), led him, to consider that spacetime has a conformal rather than a metric structure, to identify the conformal factor with the electromagnetic potential (a physically wrong but mathematically fruitful idea!-see just below) and to introduce the concept of gauge transformation. It was the first, very early attempt to develop what much later came to be known as a gauge field theory, especially in connection with gauge invariance of electrodynamics. The similarities stressed by Weyl between the geometrical concepts of GR and the dynamical concepts of electromagnetism were elaborated later and led to important developments in differential geometry and its relation to Physics, namely, the theory of connections and of fibre bundles and the formulation of gauge field theory (Pais 1982 pp.339-340, Cao 1997 §9.1). Although this is a rather advanced subject, Weyl's procedure can provide a natural introduction to concepts and methods, which are equally used by mathematicians and physicists and which play an equally important role in pure Mathematics and in Theoretical Physics.

3.2 Differential Equations, (Functional) Analysis and Quantum Mechanics

It is well known that since Newton's time, differential equations have always been one of the main links between Mathematics and Physics, leading to important developments both in analysis and in the concise and fruitful formulation of physical theories. It is perhaps less known that many important concepts of functional analysis originated in the study of quantum theory (QT) and conversely, that it was only through its concepts and methods that a deep understanding of atomic phenomena became possible.

Below, I outline only a few, but fundamental points of this really complex, continuous and deep interrelation that has been so fruitful both mathematically and physically.

Some crucial historical elements[5]:

(a) Already in the 18th century it was realized that there is a close formal similarity between Fermat's *principle of least time* in geometrical optics, and Maupertuis' *principle of least action* in classical mechanics. In the 1830's, on the basis of this similarity, Hamilton formulated the two disciplines in a unified way and developed a general method for solving 1st order partial differential equations (PDE) that became central in the formulation and solution of mechanical problems as well (Hamilton-Jacobi method).

(b) About 90 years later, Hamilton's ideas stimulated de Broglie to take the aforementioned formal similarity as an indication of a deeper relation between mechanical and optical phenomena and to predict the wave nature of atomic particles. Schrödinger, in turn, further elaborated this idea, and arrived in 1926 to the formulation of wave mechanics.

(c) In the 1920's, atomic physics was a complicated mixture of classical mechanics and electrodynamics with additional semi-empirical rules and heuristic arguments. People were trying hard to develop models of atomic phenomena and to understand their mathematical structure. Heisenberg in 1925 developed a kind of algebraic manipulation of atomic quantities, in analogy with Fourier series operations, the novel idea being that in this manipulations, the Fourier frequencies and coefficients were *doubly* indexed as a consequence of the so-called *Ritz combination principle* in atomic spectroscopy. It was immediately realized by Born that Heisenberg's calculus was just the algebra of (infinite, in

[5] References to the original literature and to secondary sources can be found in Tzanakis, 1998, 2000, 2001.

general) matrices. This led to matrix mechanics, the first formulation of modern quantum mechanics (QM).

(d) After Schrödinger's formulation of wave mechanics in 1926, physicists were puzzled by the existence of two conceptually and mathematically very different theories of atomic phenomena (matrix mechanics and wave mechanics), which nevertheless gave identical, empirically correct results. Schrödinger himself and von Neumann tackled the problem. Both showed that the two theories were mathematically equivalent. Von Neumann's approach was more rigorous and led him to introduce for the first time the concept of an abstract (separable) Hilbert space, to show that all such spaces are isomorphic and to resolve the puzzle by making clear that the two physical theories were based on two different, but isomorphic such spaces (l^2 and $L^2(\mathbf{R})$ respectively).

Points (a)-(d) can be integrated into teaching in a number of ways, depending on the course given, its emphasis, the time available etc. I will give some possibilities below, in which the emphasis is on Mathematics, rather than Physics:

(1) The least action principle and the principle of least time, constitute natural examples of variational principles, leading to mathematically interesting and physically relevant equations, the Hamilton-Jacobi equation, which is central in classical mechanics, and the so-called eikonal equation of optics. On the other hand, they are generic examples, in the sense that it is possible through them to establish a general result in the theory of PDE's: the solution of a 1st order PDE is *equivalent* to the solution of a system of first order ODE's, the so-called Hamilton's system of the associated canonical equations (Courant & Hilbert 1962 section II.9, Gel'fand & Fomin 1963 section 23). As it is well known, this result is of central importance both in the theory of differential equations and in mechanics. In fact, one can proceed very close to Hamilton's and Jacobi's approaches to illuminate the subject from two different, but important view points (Dugas 1988, part IV, ch.VI, Klein 1928/1979, pp.182-196).

(2) Schrödinger's elaboration of Hamilton's mathematically unified treatment of classical mechanics and geometrical optics mentioned above, was based on *arguing by analogy*: If classical mechanics is mathematically similar to geometrical optics, and since geometrical

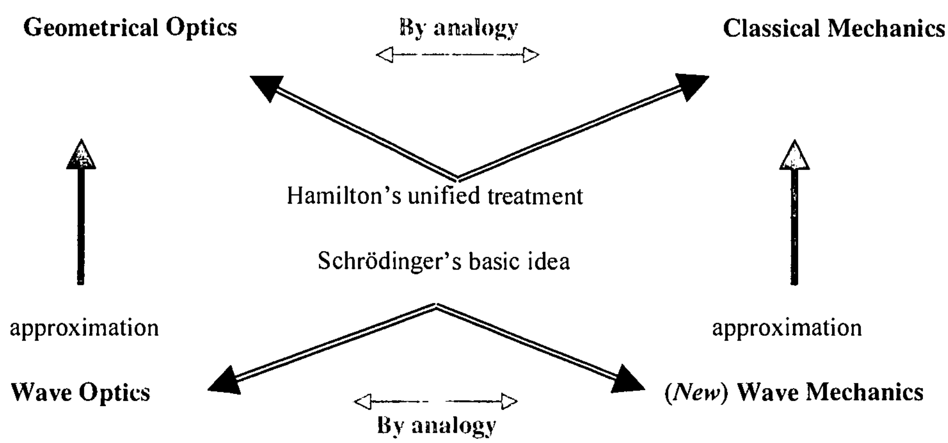

Figure 1: A schematic representation of Schrödinger's reasoning by analogy

optics is an approximation to wave optics, perhaps classical mechanics as well is an approximation to a wave mechanics, which is similar to wave optics in the same sense that classical mechanics is similar to geometrical optics. In this way, Schrödinger's equation results as the mechanical equivalent of the wave equation, schematically given in figure 1.

This derivation can be presented in relation with (1) above (for a detailed treatment, see Tzanakis 1998). This is a characteristic example that makes clear the important role of *analogy as a mode of reasoning* of great heuristic value (for a detailed discussion both in Mathematics and in Physics, see Tzanakis & Kourkoulos 2000, §2). Another such example is provided by Heisenberg's approach described in (c) above and schematically represented in figure 2 (see also Heisenberg's own account in Heisenberg 1949/1930, appendix, which can be used for didactical purposes in a slightly restructured form, as well as, his original paper reprinted in van der Waerden 1967, paper 12).

Classical quantities q, p		**Quantum** quantities q, p
		Looking for a
Fourier representation		*New* representation because of
Single index frequencies		***Double*** index frequencies
$\nu_k = k\omega$		ν_{nm}
		obeying ***Ritz principle***
$\nu_k + \nu_l = \nu_{k+l}$		$\nu_{nl} + \nu_{lm} = \nu_{nm}$
$q(t) = \Sigma\, q_k \exp(\mathrm{i}\, \nu_k t)$	$\xleftrightarrow{\text{By analogy}}$	$q(t) \sim (q_{nm} \exp(\mathrm{i}\, \nu_{nm} t)$

Operations

$p+q \longrightarrow (p_k+q_k) \exp(\mathrm{i}\, \nu_k t)$	$\xleftrightarrow{\text{By analogy}}$	$p+q \longrightarrow (p_{nm}+q_{nm}) \exp(\mathrm{i}\, \nu_{nm} t)$
$pq \longrightarrow (\Sigma_l\, p_l\, q_{k-l}) \exp(\mathrm{i}\, \nu_k t)$	$\xleftrightarrow[\text{By analogy + Ritz Principle}]{}$	$pq \longrightarrow (\Sigma_l\, p_{nl}\, q_{lm}) \exp(\mathrm{i}\, \nu_{nm} t)$

Hence, $pq \neq qp$ which leads to Heisenberg's uncertainty relations!

Figure 2: A schematic representation of Heisenberg's reasoning by analogy

Although analogy seems to play a central role as a discovery pattern both in Mathematics and in Physics, no enough attention is usually given to it in teaching. These examples are important in this respect as well.

(3) Many of the basic concepts of functional analysis can be motivated in a natural way (i..e. avoiding logical gaps), in the context of questions and problems of atomic Physics. The concept of an abstract Hilbert space mentioned in (d) above is a characteristic one. Schrödinger's formal proof of the equivalence of matrix and wave mechanics may serve as a

general motivation to look for a more rigorous proof. This in turn leads to appreciate the significance of the existence of a (complete) orthonormal basis and the various equivalent conditions (e.g. Parseval's relation). Von Neumann's approach is very clear and can be followed closely (von Neumann 1932). Other examples can also be given, like the concepts of a hermitian and of aunitary operator and their generalization, a normal operator, the concept of spectrum and important theorems associated with these concepts that followed as a result of von Neumann's work on the foundations of QM (for more details, see Tzanakis 2000 §3.4).

3.3. Statistical Mechanics, Dynamical Systems and Ergodic Theory

A somewhat more advanced example, which shows the deep and fruitful influence that Physics can exert on the development of new mathematical concepts, methods and theories is provided by the historical development of statistical mechanics and ergodic theory. Only some aspects of this subject are briefly discussed below.

Some crucial historical elements:

(a) Implicit to the work of Boltzmann (in 1871, 1884, 1887) and Maxwell (in 1878), is what later became known as the *ergodic hypothesis*, a desired basic property of the systems with a large number of degrees of freedom studied in statistical mechanics: The phase space trajectory of a mechanical system passes through *every* point of its energy surface.[6] If this conjecture were true, the phase space average of any quantity would coincide with its time average along the trajectory of the system. This was an utterly important conclusion in the foundations of statistical mechanics. It was gradually realized that this hypothesis leads to contradictions on the basis of important mathematical theorems according to which space-filling curves cannot be smooth as required in (statistical) mechanics (Sagan 1994). In an attempt to overcome this obstacle, the Ehrenfests in 1912 distinguished the ergodic hypothesis from the *quasi ergodic hypothesis*, according to which the phase space trajectory of the system is a *dense subset* of the energy surface, hoping that the latter could offer a better foundation of statistical mechanics (Ehrenfest & Ehrenfest 1912/1990 §10 and note 98).

(b) The formulation of the quasi ergodic hypothesis and the significance in the context of statistical mechanics of the coincidence of phase space and time averages (what in fact later was taken as the definition of an ergodic system), were the basic motivations for the important investigations of Birkhoff, von Neumann and Hopf that led to the proof of the first ergodic theorems. It was a crucial starting point for the development of what later became known as ergodic theory (Sklar 1993, §5.II.1, Farquhar 1964 ch.3).

(c) The stability of the solar system was an old problem investigated by many great mathematicians since the 18th century. It was a main motivation for the study of periodic motions in N-body systems and more generally in dynamical systems and it strongly influenced the development of the qualitative theory of differential equations, especially through Poincaré's work on celestial mechanics and Birkhoff's investigations on general dynamical systems that paved the way to the development of the modern theory of dynamical systems (Poincaré 1957, Birkhoff 1927; for a short review, see Moser 1973, §§ I.1, I.2).

[6] It is not clear to what extent Boltzmann and Maxwell thought of this hypothesis as a fundamental element of statistical mechanics (Boltzmann 1954, pp.11-12 and footnote on p.297). Apparently, it was the Erhenfests' review of 1912 that stressed the importance of the ergodic and quasi-ergodic hypotheses and the difficulties inherent to them (Ehrenfest & Ehrenfest 1912/1990).

(d) The existence of (quasi) periodic motions of dynamical systems was an important problem systematically investigated by Poincaré in connection with the stability of the solar system and more generally with the N-body problem. Kolmogorov made significant progress in 1954, by proving the existence of such motions under quite general conditions and contrary to what one would expect if dynamical systems were ergodic (see (b) above). Kolmogorov's ideas were elaborated by others, especially Siegel, Arnold and Moser[7] and led to a revitalization of classical mechanics in the last 40 years, by fruitfully combining concepts and methods of such diverse fields like measure theory, differential equations, topology and differential geometry. Conversely, new, essentially physical, concepts, like ergodicity, mixing property and entropy of a dynamical system etc, were introduced that further enhanced the development of ergodic theory and dynamical system theory, into an interdisciplinary domain that touches upon many diverse areas of pure and applied mathematics and theoretical physics; e.g. probability and measure theory, differential topology, number theory, statistical mechanics, fractal geometry etc.

Ergodic theory and the theory of dynamical systems are certainly advanced topics and at most an introduction to their basics can be incorporated in an undergraduate curriculum. Even the definition of its most basic concepts, like an abstract dynamical system or ergodicity, are rather technical and require some knowledge of various areas of Mathematics (e.g. measure theory, differential geometry, topology etc). My main point is that even these basics cannot be grasped properly if their introduction is *decontextualized* as it is usually done. On the contrary, their introduction in the proper context in which they have naturally appeared historically, namely, in connection with specific, difficult and physically important problems, can greatly enhance their understanding.

(1) Most of the basic concepts of ergodic theory have a deep physical meaning and were introduced in an effort to understand specific physical problems. Thus, ergodicity of a dynamical system (in the sense of the coincidence of phase space and time averages), its entropy, the mixing property etc can be motivated by the ergodic problem in statistical mechanics (see (a) and (b) above) and Boltzmann's probabilistic definition of (macroscopic) entropy of a physical system in terms of microscopic quantities.

(2) The importance of the ergodic hypothesis in statistical mechanics constitutes a natural (but of course, not the only) framework for discussing the interesting and independent subject of space filling curves (e.g. Peano's curve) and the associated deep problems of the definition of the concept of dimension, especially in connection with the fascinating concept of a fractal and its relation to ergodic characteristics of specific dynamical systems (see e.g. Falconer 1990, ch.13).

(3) The significance of the stability of the solar system is self-evident, even in a general cultural context. A historical introduction to this subject, in which the nature of problems and achievements are explained without proof, is helpful (cf. (c) above). If students appreciate the difficulty of these problems *and* their physical importance, they can also appreciate better why one has to work out and understand in detail the dynamical behaviour of many, somewhat artificial, low dimension systems.

(4) For a long time classical mechanics was considered as a dead research domain. It served only as a subject for introducing basic mathematical methods of Physics, or as a first

[7] Kolmogorov 1954/1978, Arnold & Avez 1967, Siegel & Moser 1971, Moser 1973 are the standard references containing the basic theoretical concepts, mathematical tools and results. See Sklar 1993, §5.II.3.

step of a theory that one had to overcome in order to understand phenomena beyond the everyday world (in the atomic or astronomical scale). This picture has changed in the last 40 years, at least as far as research is concerned (cf. (d) above). This came as a result of two different, but dual in character and interconnected lines of research: (i) On the one hand, there was a struggle for understanding the ergodic properties of specific physical systems. Often, this was done in the hope to show that in some sense ergodic systems are the majority of physically relevant multidimensional systems, hence that dynamical motions are mainly non-periodic for systems with many degrees of freedom. (ii) On the other hand, there has always been a continuous interest in and research on the stability of motion of specific dynamical systems and the determination of (quasi) periodic trajectories. For a long time it was believed that this could be true for systems with a few degrees of freedom. We now know that none of the beliefs underlying research along (i), or (ii) is strictly true (there are low dimension ergodic systems and "a lot of" periodic motions in systems with many degrees of freedom). This is a fascinating development in Mathematics, some elements of which can be given to our students, illustrating (i) and (ii) above by means of elementary examples taken from such diverse fields like classical mechanics, riemannian geometry, or number theory.

4. Final Remarks

All three examples support points (a), (b) in §1.2.(2), although lack of space does not allow a detailed epistemological analysis in support of these points. At the same time, the basic historical facts presented for each example, constitute a natural framework for introducing new mathematical concepts and methods and by linking them to mathematically relevant and physically important questions and problems, which may serve as a meaningful motivation for students. They can be adapted so that they become didactically appropriate and they can form the basis for the development of teaching sequences in which many technical details are incorporated in the form of exercises, problems and small research projects. Details can be found in the given references.

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TARTINVILLE AND CABRI II

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ABSTRACT

We have examined the role that 'Tartinville's method" can have in the qualitative analysis of parametric second degree equation, and in the teaching of geometry using Cabri II software.

Modern students do not know this method of analysis while they know Cabri II software.

First we examined the methods used to solve second degree problems since Euclid up to now, through the contributions of Diofanto, Pappo, Brahmagupta, Descartes, Newton.

Then we demonstrated how the Tartinville method could be geometrically interpreted via Cabri II, and how the different geometrical situations could be dawn up as a graph.

The most important aspect of the situation was reproduced through the drawing of Macros which made the solution of the problem easier.

At the end, we showed two relevant examples.

Keywords: parametric second order's equations, Tartinville's method, Cabri II^

1. Introduction

This article is the result of a debate evolved from a complicated subject like "Tartinville's method" [3], and how it can become pleasant at school exploiting the main property of Cabri II, that is, the movement associated to a geometrical figure.

Tartinville (1847-1896) was a professor at Lycée Saint Louis in Paris.

He was well known because of his method of displacing the elements that permit to analyse and to solve a mixed system with one unknown, made up of a second degree equation $f(x) = 0$, whose coefficients depend on a real parameter k and one or two linear inequality of the type x â, or x á, or á x â.

2. Historical outline

The solutions of a second-degree problem by the intersection of straight lines and circle are a subject already present in Euclid's *Elements* (3^{rd} century b.c.).

Exact or approximate numerical solutions of particular equations are in Heron of Alexandria (who lived a century before or a century after the vulgar era) and also in Diophantus (250AD) who never accepts either the negative solutions or the irrational ones.

The algebraic solution appears in Brahmagupta's works (850); Descartes (1596-1650), who was the first to introduce the method of coordinates, quotes him in his own works.

In his work *Geometrie*, in three books, the rule of the signs, called Cartesius', and the problem of Pappus is treated: draw through a point a straight line so that the part determined on it by two other straight lines is equal to a given segment.

At the beginning of the 18^{th} century, Newton's *Arithmetica Universalis* (he was born in 1642 and died in 1727) was published, in which the most famous arithmetical problems are examined, the methods of separation of the roots and their approximation; for example Pappus' problem is used for the drawing of the roots of a cubic or biquadrate equation by the intersections of a straight line with a conchoid.

The qualitative analysis of the problems, in the modern sense of the word, has been the result of the conquest of Algebra, from the second half of the 19^{th} century to the beginning of last century, and above all, from the discovery of Fourier and Sturm's theorem, which allows to assign the number of real roots of an algebraic equation falling in a certain interval, solving implicitly, the problem of the qualitative analysis of the equation of degree 2,3, 4,... without having to look for the algebraic solution.

Secondary teaching and Mathematics, in particular, was flowering since the second half of the 19^{th} century. In the admission exams to several types of schools, problems requiring the qualitative analysis and/or the solution of second-degree problems or leading to them, were assigned.

From the publications of Academies and purely scientific memoirs, Mathematics started taking part in the debates of a more and more numerous audience. The so-called democratisation of elementary Mathematics started and the necessity of disciplining the methods of the qualitative analysis of the elementary problems had its origin.

3. Tartinville and Cabri II

The first scholar who dealt with such subject in a very direct and clear way was Tartinville, once "sadly" famous among the students of Liceo Scientifico.

The name of the French mathematician, Tartinville, is exclusively linked to the problem of the qualitative analysis of the second degree equations, and this method and he was widely studied at Liceo Scientifico, actually, that was the only method used to solve the problems assigned at the final exam at Liceo Scientifico up to 1969.

Thanks to the protest carried out by B. de Finetti, this method in particular and the qualitative analysis of the problems, in general, disappeared both from the syllabi and from the final exam at Liceo Scientifico.

Teachers who have taught this method usually believe that Tartinville's qualitative analysis is boring and cumbersome and it does not provoke any curiosity in the students but only a passive study of the subject; but we think that, not only the qualitative analysis of the problems, but also the method can be newly presented at school, using Cabri II software as useful tool for the teaching of geometry.

This idea should be placed in a wider cultural environment recognizing the importance of "external" events in the development of the mathematical thought.

So we decided to use the computer to make the teaching of geometry lively, to animate the geometrical object, a characteristic diffused in the geometry treatises of the 16^{th} and 17^{th} centuries.

Cabri II is an excellent support, in this sense.

4. General question

Once the figure representing a given problem has been drawn, we associate the unknown length x of a straight line segment with a point of abscissa x on the real line. After drawing the parabola which graphs the equation solving the problem, the pupil can verify that:

1- as k varies, the parabola of the sheaf varies with it and consequently the intersections with the segment vary;

2- as x varies, k varies with it and consequently it is possible to deduce the k values connected with one or two or no solution;

3- to each particular x value corresponds a geometrical interpretation of the problem of immediate representation;

4- in particular cases the figure degenerates and we can see why that happens.

At this stage, the student realizes that the link between mathematics and reality is very strong and he is urged to "materialize" an animated drawing, which reproduces the mental scheme of the mathematical tool.

However, being involved by the power of the images in movement on the screen, we have to avoid underestimating the necessity of proofs.

We conclude with a few reflections:

1- before examining a problem, it is necessary to make sure that the students own the necessary requisites;

2- under this circumstance, group work should be encouraged as it allows more constructive comparisons and debates, while the teacher should keep a discrete role;

3- during the correction of the mistakes, the teacher's presence should be more active and the completed works should be commented.

5. Operations whit Cabri II

In order to proceed to the drawing by means of successive **Macros**, it is necessary to define a few fundamental operations; i.e.:

Given two segments x and y, draw the segment of length x^2, the segment x+y, the segment xy.

MACRO: X^2

Draw a straight line r, on which pick the points 0,1,X

Draw the straight lines orthogonal to r through the point 0

Draw the circumference with centre 0 and radius 1

Draw the intersections between the straight-line s and the previous **circle**

Draw a straight line t through 0 different from s (*e.g.*.: the bisector of the angle r0s)

Draw the straight line k through 1 perpendicular to r
Draw $B = t \cap k$
Draw the segment BX
Draw the straight line p through X perpendicular to r
Draw $A = p \cap t$
Draw the straight line m through A parallel to BX
Draw $r \cap m = C = X^2$
Initial objects: (r,0,1,X)
Final objects X^2
Name: SQUARE OF X.

MACRO X+Y

Given the straight line r, the point 0, the point 1, the point X, the point Y, the segment sum x+y is obtained drawing the symmetric of 0 with respect to (X+Y)/2.

Draw the straight line r, and on it, the points 0,1,X,Y
Draw M = midpoint of X and Y
Draw the symmetric of 0 with respect to M
Initial objects: (r,0,1,X,Y)
Final objects: X+Y
Name: SUM OF X+Y.

MACRO XY

Draw r,0,1,X,Y
Draw the straight line s through 0 orthogonal to r
Draw the circumference (0,1)
Draw a straight line b through 0 different from r (e.g.: the bisector of sOr)
Draw the straight line p through 1 perpendicular to r
Draw $A = b \cap p$
Draw the segment AX
Draw the straight line m perpendicular to r through Y
Draw $C = m \cap b$
Draw the straight line k through C parallel to AX
Draw $r \cap k = XY$
Initial objects: (r,0,1,X,Y)
Final objects: XY
Name: PRODUCT XY.

By means of these Macros, varying the order of the initial objects we can obtain the following results:
the segment x-y (r,Y,1,0,X) on X+Y
the segment x/y (r,0,Y,1,X,) on XY
the segment x^3 (0,X,X^2) on X^2
the segment 1/x (0,X,1) on X^2.

6. Applications

We show a couple of examples; in the first example the student is asked to analyse qualitatively a parametric second degree equation applying successive Macros and to draw the parabola whose graph represents the solving equation in such a way he can see that, as the parameter k varies, the intersections of the parabola with the x axis vary with it and he can proceed to calculate the x value associated with the value of k. The values are pointed out by means of colours in the figure.

It is a useful exercise in the application of geometry to do algebra.

We suppose we have the parametric second-degree equation

$$x^2 + x(4-k) + k-5 = 0$$

with the conditions $-1 \leq x \leq 2$.

Given a system of Cartesian orthogonal axis, as Cabri II does not recognize the two axis separately, we draw the straight line r coinciding with the x axis and choose the points 0,1,x,k, on it (preferably 0 and 1 coinciding with points of the grid).

Next, we draw the segments x^2, $x(4-k)$, $k-5$ and, at the end, the segment sum of $x^2+x(k-4)+k-5$ with M as one of its extreme points.

Such an end point will belong to the x-axis. We draw its symmetric with respect to the bisector of the first and third quadrant in order to move it on the y-axis. We draw the straight line through x perpendicular to the x-axis, the straight line through M perpendicular to the y-axis; their intersection point P generates the appropriate locus when x varies.

We can see how the parabola varies as k varies.

At this stage, if we want to work on the parabola as a conic section we should draw it picking 5 points belonging to the locus. Selecting the segment with end points -1 and 2 on the x-axis, clicking on k, the parabola moves and its intersections with the x-axis vary, accordingly.

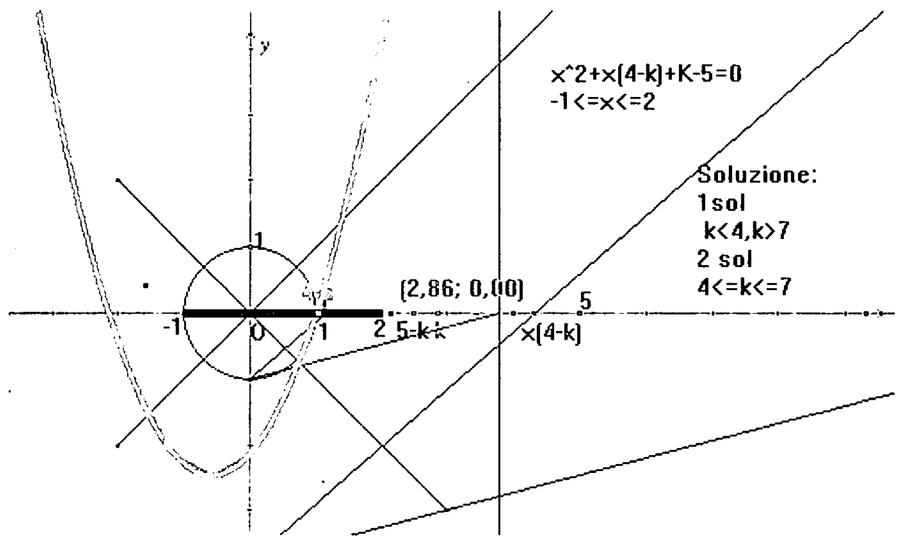

Fig. 1

The second example stems from the following problem:

Given the equilateral triangle ABC with length 1 sides determine on the side BC a point P so that $AP^2+PB^2=k*AB^2$. Qualitative analysis.

The student has to remember the drawing of the equilateral triangle with the compasses and the rule or else to open the Macro already defined in Cabri.

Taken X on the x-axis, we consider the point associated with it by means of the quadratic equation.

At this stage, we associate the point X on the x axis with the point P on the side BC so that PC=x. Animating the figure we can see how the initial figure is transformed as x varies.

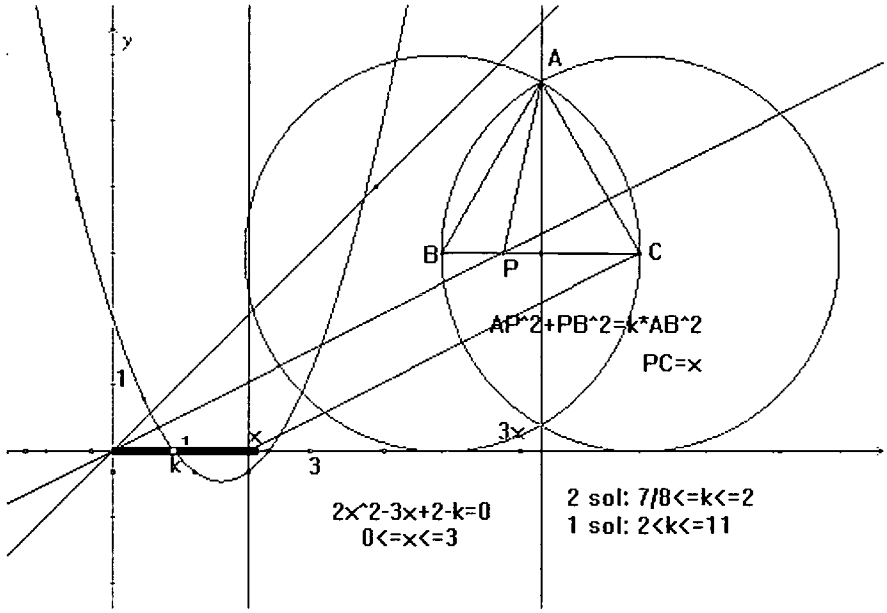

Fig. 2

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THE UNIVERSITY GOES TO HIGH SCHOOL

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KEYWORDS: Educational, Research, Project, Mathematics, University, High School.

ABSTRACT

In Brazil the number of students in high school interested in mathematics has been decreasing in the past ten years. In this paper, we address the way we tried to deal with this problem at the Institute of Mathematics of Federal Fluminense University, Niterói, Rio de Janeiro. In 1998 we had 3.8 applicants for each vacancy. In 1999, we formed a team of three teachers in order to develop an educational research called "The University Goes to High School". The objective is to attract better students in mathematics.

The main activity is to give explanatory lectures in high schools, to students who had not yet made a decision about their career. We address issues as varied as the presentation of problems in Topology, the possibilities of obtaining support of financial agencies during the course, the job opportunities, the University's Distance Learning of Mathematics etc.

An unexpected favorable by product of this action is the return of the high schools teachers to the University. Indeed they begin to pay more attention to continuing education, in order to update or broaden up their knowledge in our University's Specialization Course on the Teaching of Mathematics. The number of regular students has steadily increased since our project was set up.

Another goal we pursue is to detect gifted students with an outstanding talent for Mathematics, and to put them in contact with teachers of the University in order to develop a study program as earlier as possible.

In 2001, we had the greatest number of graduate students with major in mathematics, 6.78 applicants for each vacancy, and we strongly expect to achieve, with this research, a better selection in the coming years.

In the world, there are, nowadays, around 50 million undergraduates, out of which, 4% belonging to Brazil. This number is quite irrelevant if we regard our potentialities as an emergent nation as well as a leader in our continent. Some experts estimate that we should have, at least, triple the present number of undergraduates in order to catch up in percentage terms, with nations such as Argentina and Chile, which are, at least quantitatively, in a superior and better position than ours.

At Federal Fluminense University - UFF, a university which is located in Rio de Janeiro and counts on around 20000 students nowadays, some courses, including the one in mathematics, received new students in 1998, with averages around 3.5 - out of 10 - or even lower, and the number of candidates interested in the mathematics faculty had reached a level - 3.8 candidates per vacancy -, which is considered very low. Believing that there students who present a better educational achievement, and that they would need some encouragement in order to accept their vocation to a technological field, especially in mathematics, the project "The University goes to High School" has been more effectively planned. In the beginning, we thought of the procedures of how to support the high school, stimulating the students, with mathematical vocations, to enroll in courses related to the technological area at UFF, especially in the course of mathematics. This procedure, in our opinion, would permit an efficient growth of the level of demands by the professors throughout the courses, implying, this way, a better formation of professionals heading for the job market.

Considering that professors, researchers and citizens are constantly worried about the continuity of the technological development of our society and, therefore, concerned about the constant improvement of our new undergraduate students, three professors of our Institute of Mathematics (IM-UFF) have begun, thus, to develop the project "on screen", since 1999, beginning with the course of graduation in mathematics, at plenty of establishments of both public and private high schools in the cities of Rio de Janeiro, Niterói and S. Gonçalo.

We have been looking foulard at providing, through this educational research, the access to information as a fair way to promote the university education for all citizens, based on exclusive reasons, not being admitted, therefore, any sort of racial, sex, language economical, social or physical prejudices.

The project has been developed by its basic activity, simply stated, through explanatory lectures (normally one in every two months) of about one hour and thirty minutes in the educational institutions cited above. We intend, among other things, establish a bigger contact with professors and students at that level of knowledge, presenting, despite of the brief way, a scope of this University, by professors inserted in the academic life, as an institution that develops its critical and social roles, which is fundamental to any country that aspires to development. Seventeen institutions where visited until the end of 2001, and four other schools have already been set up for visitation during March of the present year.

It is possible to say that the aim of the project "The University goes to High School" promotes the integration education/service/society, due to the fact that the conferences are, principally, dedicated to students of both introductory grades (the first and the second years of high school) who, most of times, have not decided about a university career so far, and would not have access to this kind of information of a strictly academic concern.

This conference/lecture also emphasizes the quality of the course of mathematics at

UFF, the meaning as well as the goals of the courses of licentiate and bachelors degrees, highlight the methods of work of the graduation field together with the post-graduation one on IM-UFF - which refers to the high level of those courses of mathematics at UFF together with Brazilian Ministery of Education; presents the wide scope of professional opportunities and the current and future job market yet to be reached by the mathematicians; and divulges the criteria for financial support offered by fomentation institutes of the government for the best students of the course (training, monitory and research scholarship).

In each conference, the lectures present mathematical problems emphasizing topology and geometry in order to encourage students to get into the challenge and make the exhibition more interesting. The meaning of what has been done in mathematics is also taught to the young students. Picturesque aspects of mathematics are shown in a very accessible, easy and convincing language. Questions are always welcome.

Another important aspect of our research, and perhaps at a medium period of time that can produce excellent results, is related to the recognition of precocious young students with extraordinary skills for mathematics, which, as a matter of a fact, already counts on two students (aged 15 or 16 years old) who are effectively participating in the tutorial program, another part of the project. Concerning this activity, the cooperation of teachers who work at the visited institutions has been extremely worthy and relevant when it comes to pointing out the skillful students. These students receive the necessary orientation of tutors at the Mathematics Institute, chosen by Professor Celso José da Costa (PhD - IMPA), ex-coordinator of post-degree in mathematics at UFF, current coordinator of the licentiate course of distance learning in mathematics, and member of this project.

Another participant in this research, Professor Marisa Ortegoza da Cunha (PhD - PUC-Rio), is works on the assessment of Extension's projects at IM-UFF and works on an extension program of the Institute, headed for licentiate undergraduates, because during the lectures, we also intend to encourage students to take up the opportunity of working at schools, which is a totally needy area lacking competent professionals, in our State, mainly concerning high schools.

At each conference, we begin establishing a closer communication, at times, intense, between IM-UFF and the visited schools.

At first, we attract teachers and students of that field in order to make part of academic events (such as conferences, lectures, etc.) in our Institute.

It is also revived, in teachers who work in high schools, the will to get down to study again. In our Institute, we have, for years, besides the Masters course, another post-degree course (Lato Sensu) in specialization in Mathematics in activate process. This course had, in 2000, the greatest number of registration of all its history, some of which related to the teachers who have attended to the conferences of the project "The University Goes to High School". It is also important to highlight that, among other accomplished goals, during the last college entrance exams of UFF, we have observed a considerable raise of candidates interested in the course of mathematics (6.78 applicants), who have registered to this area of knowledge.

During these meetings, we have divulged other activities organized and coordinated by IM-UFF, and also with a great deal of emphasis, since august 2001 in the course of licenciate in Mathematics (Distance Learning), formed by five public universities of the state of Rio de Janeiro, which has already organized its first contest of enrollment in September of 2001, and it is already activate in four municipal districts of the state, initially with about 160 students. The Institute of Mathematics, is the institution which is responsible for the coordination, for the enrollment of professors, for the didactic material and for tutors of this kind or category of education, pioneer in the area of mathematics in our state and in our country.

The next challenges of the university, for the coming years, regard the application of new technology of information; however, it is essential establish new criteria not only concerning the triangle education/research/extension to form responsible citizens, but also the improvement of the quality of human material which will make part of the university at the moments of globalization and Internet, and which will cooperate in the process of development of the country. The scholars of this subject one engaged in order to make these new students be the main actors of this process of learning. The project "The University Goes to High School" is, thus, head for this direction and it seems to be in perfect harmony with those ideas.

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ON SOME IMPORTANT ASPECTS IN PREPARING TEACHERS TO TEACH MATHEMATICS WITH TECHNOLOGY

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ABSTRACT

The introduction of technology in the classrooms at all levels of education has brought forth a need to change some teaching practices. Together with the modernization projects of undergraduate instruction for Engineering and Science courses, it is especially important to focus the attention on Courses for prospective (and in-service) teachers in Mathematics.

The use of technology in the teaching-learning activities can be regarded as a new communication language in developing the construction of knowledge. The recognition of this role of technology in education would contribute to a better preparation of the future teachers in selecting right teaching strategies, not only technology. This paper aims first to discuss this aspect of technology in the undergraduate instruction, through a systematized classification of the use of technology in the classrooms based on the forms of activities, illustrated with examples.

Furthermore, one of the advantages of the technology as teaching aid is the possibility of more realistic modeling in problem solving and interdisciplinary activities, so new and reformulated disciplines in the curriculum of teacher preparing courses come up. Regarding this aspect, we point out that the critical interpretation of the computer/calculator outputs demands an awareness of the kind of mathematics needed when using technology. That means that solving a problem with the use of technology requires from the user a deeper understanding of the importance of concepts like units, scaling of units, significant figures, approximation/numerical methods, parametric representation and implicit representation, interpolation methods, structure of algorithms, etc., along with the proper theoretic concepts underlining the problem. The careful use of technology as a teaching strategy would enrich in this way the lectures and the preparations of activities by teachers. The second aim of this paper is to illustrate these considerations exhibiting an example for teachers.

Keywords: Teacher Education Courses, Technology in Education, Mathematics of technology-based activities.

1. Introduction

The introduction of technology in teaching and learning environments brings up a general concern of educators of all levels of instruction, particularly of those involved with the formation of future teachers. The general issue is how to prepare the teachers for a generation for whom the technology is already familiar, and also to update the in-service teachers with new methodologies.

In this paper we will be focusing on the education of prospective and in-service mathematics teachers in the presence of technology.

On the importance of technology in professional development of teachers, Oldknow (Oldknow, 2000) says that "...the effective use of (Personal Computer Technology) in supporting the mathematics curriculum is in the hands of teachers. They need to know more about the use of technology than can just be found from manuals, teaching materials and other information sources."

Also, in the same Reference we find a quotation of Cornu:

"Mathematics is evolving and changing under the influence of computers and informatics. Therefore, teachers need to maintain their mathematics knowledge and to practice mathematics from an informatics viewpoint. Mathematics is becoming more experimental, more algorithmic, more numerical; teachers must be able to follow the evolution of mathematics, and to acquire new competencies and new attitudes and to be able to carry out new activities in mathematics."

The statements contained in the citations above are examples of recommendations alike that can be seen in many documents and papers requiring the change of attitudes of teachers regarding the use of the technology. Training the use of equipment and the particularities of some educational software are obviously not enough to achieve educational results in modern classrooms. Besides the necessary mathematical background, a question is what the prospective teachers should know about teaching with technology before trying the many existing materials or creating their own activities.

In (Lingefjärd & Holmquist, 2001) the authors say, "teachers of today need an understanding of mathematics that allows them to produce and interpret technology-generated results, to develop and evaluate alternative solution paths, and to recognize and understand the mathematical limitations of particular technological tools". Also they say "teachers must be well informed about its (of technology) place and role in a didactical process".

Therefore, some natural questions are posed:

What is teaching with technology? What may change if one uses technology to teach? What are the different ways of the use of technology in education, and which one is the most effective to reach educational objectives?

One great challenge that a mathematics teacher faces when he/she plans to introduce the technology in his/her classes is that, in general, he/she does not know when, what and how to use (sometimes why), even when he/she has previous knowledge about equipments and several educational software. This challenge is faced also by the faculty of Teacher Preparing Courses at university level, who has the responsibility to prepare adequately the future teachers with the mathematical as well as the pedagogical background required in a modern classroom.

This paper is based on the reflections of the author in introducing different teaching methodologies with technology to wide ranging classes, from engineering and sciences students to prospective and in-service teachers of basic level schools, and studying their responses.

The first aim of this paper is to discuss the different ways of communicating mathematical content with the aid of technology, proposing a systematic classification of the use of technology in the teaching context, in order to help the teachers to understand the role of technology in education and consequently facilitating the effective use in their classes of much information already available.

The second aim is to discuss the importance in the teacher preparation courses of mathematical concepts underlining some elementary activities suited for technology-based classes, indispensable to the formation of teachers as a user of technology. This is a key issue when teachers realize the didactical potential of mathematical modeling with the support of technology.

2. Different ways of communicating Mathematics with Technology

The activities done in the teaching/learning context are traditionally the following: a) expository classes in which the teacher introduces concepts and develops problem-solving, exercises, etc; b) working out problems, either individually or in group, repetition classes with drills, etc; c) homework, projects, etc; d) evaluation tests.

In each of these activities one can easily recognize who plays the active role, and also it is clear what are the objectives of each activity. The main difficulty of prospective and secondary level teachers is the perception about the possible changes of these activities into technology-based activities.

An effective teaching/learning process is a communication process between teacher and student that involves the principle of action and reaction, that is, each action taken by either a teacher or a student provokes a response from the counterpart that stimulates a new action. The completion and the repetition of this cycle as many times as necessary are actually required to the results been assessed properly. The technology may take part in this process as an asset to improve the communication between the teacher and the students.

We propose the following classification of the role of technology as teaching aid, based on the forms of communication and the recognition of the active-passive role of each part:

I- In a traditional expository class, in which *the teacher* is the active user of technology;

II- In a laboratory-type class and activity, where *the student* is the active user of technology;

III- In a different type of activity, where *the teacher and the students* **are** active users and together participate in the construction of knowledge.

When a teacher uses an **expository approach** to introduce and develop mathematical concepts, the student is a *passive* recipient of the lecture, and his/her understanding of the topic depends on teacher's communication skills and the interest of the class. The presence of technology in this type of class includes the slide-projection, the overhead projectors with transparencies, videotapes, computer software (CAS, DGS, GC (graphic calculators), etc.) combined with projectors, etc.

In particular, the possibility of using powerful educational software with computer or calculator to develop better examples and more realistic illustrations turns the ordinary exposition into a more exciting and meaningful class, where the main actor is the teacher. The computer-algebra systems and graphing capabilities, combined to fast calculation capabilities turns it possible to the teacher present examples where the use of technology is actually necessary, explore situations to confirm the theory being presented. The visualization effects and the animation feature found in much software are definitive allies to enhance the communication between the lecturer and the class, especially in basic level schools.

This is a nice role of technology that improves greatly the didactical transposition of mathematical concepts. Many experiences reported early on the use of technology in teaching environment started this way, in general. For example, the classical illustration of the slope of tangent lines to the graphic of functions related to the concept of the derivative, the classical illustration of the proof of Pythagoras' Theorem, the graphical study of the concept of limits and convergence, the integral curves and field-plot of differential equations, recurrent use of calculators to study progressive sequences and limits, and many others. Today there are many very good works using CAS, DGS, spreadsheets, calculators, etc.

Now, we point out that in this type of class the learning environment does not change much, the student is a passive observer of the technology and the evaluation of the achievement of the knowledge relies usually on traditional tests.

Other important observation in this type of class is that the teacher takes the most benefit of the technology in the sense that he/she uses the facility provided by the technology to deliver better lectures, and also he/she can feel the pleasure of creating his/her own activity. This last part show to the teacher the necessity of a good knowledge of mathematics, often more advanced than the topics he/she teaches, and of mathematical language of computing tools in order to create a good teaching material, therefore the importance of mathematics curriculum of Teacher Preparing Courses becomes clear.

Some ready-to-use programs, worksheets and files made available in the educational and personal websites are examples of the technology that are offered to those teachers who want to take the advantage of technology to enrich their classes but do not feel comfortable enough to make their own, or do not have time to develop them. Still, the knowledge about the mathematical limitation of the technology and properness of the activities to be used in the classes is required.

Soon it became clear from the experiences that the effectiveness of the technology in the educational context is to put the technology in the hands of the students. In (Waits & Demana, 2000) the authors say "change can occur if you put the potential for change in the hands of everyone".

Thus we discuss the **second category** of the classification. The technology makes possible new methodologies for teaching mathematics. The most important is its participation as a facilitator in those **student-centered** activities. In the laboratory-type classes the students actually manipulate computer software or graphing calculators, therefore playing the active role as user of technology and in learning process. The most representative activities of traditional teaching/learning strategies that can be compared in this category are exercises at the classrooms such as the activities of algebraic manipulations, homework with drill-exercises, etc.

The teaching material for this category, for either individual or group use, may be: a) a hypertext type programmed instruction, in which the student advances his/her understanding about some subject through step-by-step activities; b) an interactive worksheet allowing the student to manipulate the data, speculate and formulate conjectures; c) a worksheet to test the achievement of knowledge; d) laboratory-type activities for problem-solving and activities requiring the use of technology; e) homework and projects requiring the use of technology; f) games of mathematical content to stimulate the interest and to evaluate the mathematical abilities; and many others.

A look at this (incomplete) list shows the didactical potential that the technology can offer to do different teaching activities in or out of the classrooms.

Also, we classify in this category the student's use of technology to document and write reports on their activities, the Internet sites to be accessed by the student, the distance education material, and ready-to-use type didactical material.

The most important property of this use of technology is that the student changes the behavior from passive to active; he/she becomes a responsible participant of his/her instruction. The output produced by the student is his/her own effort and he/she can feel the satisfaction of accomplishment and being a real participant of his/her education. Also the teacher can accompany the rate of the learning progress continuously and in personal basis. The teacher plays an important role of advisor and supervisor of the activities.

The **third category** of the classification represents the most innovating feature of the technology as teaching aid and it is the most promising in making changes for the future strategies. Many researches in mathematics education related to technology point to the transition of traditional expository classes to those based on problem solving activities with mathematical modeling, and also technology aided development of mathematical reasoning and proofs of theorems.

The technology allows to teacher and student to communicate each other in a renewed process of understanding and constructing mathematical concepts. Through activities of modeling, visualizing, conjecturing, testing, confirming, etc, the teacher has the opportunity to show to the students the mathematical language and reasoning, building **together** the paths of the construction of results and connections to real life. This category actually reunites the properties of first and second categories of the classification, strengthened by the capabilities offered by the technology.

Each one of the categories described above has its importance and own place in teaching/learning process, and a teacher must be able to plan his/her class, choosing the right strategy. Yet, in every situation, either making his/her own activity or using the existing material, he/she must be prepared to use a specified software or equipment, knowing its mathematical capabilities and limitations, as well as be prepared with the mathematical language and concepts required to make (or use) the activities, sometimes beyond the content of the topic related to the activity. This is the case if one makes his/her own material, for example programming scripts with algorithms, numerical methods, or designing figures requiring notions of parametric and implicit representation, etc.

This subject is one of most important issues in mathematics education of teachers, and just problem solving or modeling strategy deserves a proper discussion, which is not the scope of the discussion brought in this paper. In (Baldin, 2002) we present examples of topics in teacher preparation courses regarding the limitation of technology.

3. Example

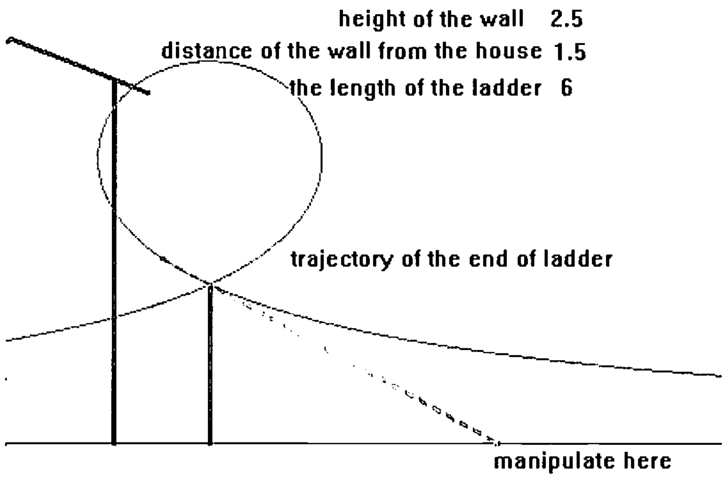
Due to the limitation of the pages, we exhibit one example of teaching situation in basic level in which the three roles of technology as teaching aid can be seen, and also the mathematical concepts and language in this activity that should be expected from a basic level teacher. Other examples from author's experiences can be found in the references.

Consider the classical "ladder problem":

"There is a two story house surrounded by a wall, height 2.5m. The wall is 1.5m distant from the house. Some firemen want to reach the house from outside, using a ladder of length 6m touching on the outside wall. How far from the wall the ladder must be put?"

Although very simple, the students (prospective and in-service teachers) are often surprised with the different use of technology that can turn the problem solving an amusing experience.

First, a DGS activity, previously prepared, illustrating the problem through the profiles of the house, wall and ladder is shown, in order to situate the problem and to facilitate the modeling.



A simple example like this, connected to real life situation, when illustrated with DGS, allowing a manipulation to experiment the possibilities can clarify some mistakes that one may make only by guessing, such as "the solution is unique" or "once the ladder touches the house, you may slide it along its wall to get other solutions in a continuous manner", etc.

The visualization of the problem suggests naturally a geometric modeling of the problem, recalling the concepts of "similitude of triangles" and "Pythagoras's Theorem". This produces a non-linear system of 2 equations, with appropriate variables.

$X*Y = (1.5)*(2.5)$ and $(X+1.5)^2 + (Y+2.5)^2 = 6^2$

Solving this system is clearly a task that needs the use of technology. The **algebraic approach** is the natural try of everybody. If CAS like Maple is used one can get the solution immediately, but this means that the user is not doing mathematics, is transferring the job to the software. A teacher can do better to explore the mathematics behind a simple problem with a pedagogical use of technology.

A student with the Graphic Calculator, (e.g. TI-92), is able to train his/her algebraic reasoning, by substituting $Y= (1.5)*(2.5)/X$ into the second equation, which can be transformed and conducted to a polynomial function of 4^{th} order in X to get the final solution. The commands on the calculator follow the natural syntax, more friendly than those of Maple, and actually are very didactical to realize the mathematical language. The algebra of polynomial functions can be connected to this problem at this moment, and the teacher must be aware of The Fundamental Theorem of Algebra. The graphic plot of this polynomial function can show also the behavior of such a function and the meaning of the zeroes.

The problem can be treated through a **geometric approach**. From the first equation one get an explicit function Y (X) = (1.5)*(2.5)/X, defined on the open interval (0,∞). The second equation can be recogn ized as the equation of a circle with center (-1.5; -2.5) and radius 6 in a system of rectangular coordinates. Therefore the solution is given by the intersection of the graphics in such a system.

With the graphic calculator (TI-92), one can get one of explicit forms of Y (X) from the second equation, choosing the adequate formula. The simultaneous plot of two graphics give the solution displayed in an interactive screen, in which the cursor on the intersection points "reads" the solution. It is quite exciting to the students to solve an algebraic system without doing algebraic calculation!

We observe now that we have in hands an opportunity to explore the capabilities of DGS in exploring the concept of function arisen in the explicit formula of Y (X) from the first equation.

In Maple or the plot editor of TI-92, the graphics of functions are produced from the **expressions** Y(X), and this is the general understanding of students that leads them to make frequent confusions between the **concept of a function** and the **expression** that defines it. Using a DGS, like Cabri-Géomètre II, we can reconstruct didactically the concept of a function, following the order of mathematical elements of the definition as well as to study dynamically the dependence between the varia bles.

The strategy is to explore the "Locus" tool, summarized briefly as: 1) construct the *domain* of a given function as an object on the X-axis; 2) construct a point X **on** the domain; 3) calculate the abscissa of X; 4) calculate the expression of Y (X), **inserting** the value of X into the **interactive calculator** of the program; 5) construct the point Y on Y-axis with the result of the calculator; 6) construct the (X, Y) point in the plane; 7) construct the Locus of points (X, Y) in the plane, depending on X.

A teacher can follow together with the student the conception of a function and its graphic step by step in the procedure above, and study each part of a function (domain, correspondence law, image) in right order. With this construction, the function of the example can be explored as an **inversely proportional** function defined for $x > 0$. This property has a real meaning in the problem! The "Compass" tool provides the graphic of second equation as a circle, and its equation confirms the result.

The intersection points of two plots would give the solution. Yet, Cabri does not confirm on the screen the first equation, because Locus is not a constructed object. Can a teacher solve this trouble? Give up Cabri? A teacher should know that the first equation is a rectangular hyperbola, so the "Conic" tool can be used to get a conic constructed on the previous locus. The equation confirms the first equation. A teacher must know from Linear Algebra why this fact is true, and also that 5 points are sufficient to determine a non-degenerate conic to understand the "Conic" tool.

Connecting algebra and geometry is a very important aspect of basic education and the problem above illustrates how the technology can help in this task, using all the communication features.

4. Conclusion

The understanding of different ways the technology can be used in basic education would help the curriculum of teacher preparation courses to include analysis of the software in the light of mathematical foundation, as well as to in-service teachers to feel more confident in choosing the activities for their classrooms and to take profits from the literature on mathematical education.

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SEMANTIC UNDER-LOADING: THE LESSON OF LOGS

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ABSTRACT

The semantics of some of the most fundamental elements of arithmetic and algebra, rather than either modes of leaming or teaching, or the conceptual complexity of the elements themselves, may be barriers to understanding. In the treatment of logarithms there is an over-supply and under-use of terms that describe the same concept. *Power, index and exponent* supposedly are synonyms and are invoked when *logarithms* are defined. There are supposedly distinct theories, "The Rules of Exponents" and "The Laws of Logs".

Questions on logarithms in algebra are unpopular, although understanding the nature of logarithms does not seem to be a prerequisite for applying logarithms to numerical problems. Nonetheless the mystification that arises from poorly specified and "under-loaded" symbols in the theoretical treatment of logarithms must result in a disheartening loss of understanding.

Students' difficulties with the semantics of elementary mathematics need to be acknowledged where possible and remedies sought, at undergraduate level if necessary. To this end a case is made for abandoning the term *logarithm*, despite its longevity, and for rationalising the terminology in the area of powers and exponents.

Keywords: mathematical education, logarithm, under-loading

1. Introduction

"You cannot teach logarithms to illiterates"
- *"Stand and Deliver", Warner Brothers, 1988*

In June 2002 logarithm tables will have been used for the last time in a public examination in the Republic of Ireland when grade 10 students take the Junior Certificate Mathematics Examintion. This will have brought to an end more than a century of association between log tables and school arithmetic.

Thirty years ago *every* student who took the Republic's Leaving Certificate Examination, (the grade 12 school-graduation exam at "Honours" and "Pass" levels), was examined not only in the use of log tables but in his/her knowledge of the properties of logs. The following is part of a specimen question from 1970:

"$\log_{10} 2 = x \quad \log_{10} 3 = y$.
Express in terms of x and y, (i) $\log_{10} 6$, (ii) $\log_{10} 24$ (iii) $\log_2 3$ (iv) $\log_{10} \sqrt{8}$ "

In the years since then the number of students staying on to take the Leaving Certificate exam has grown from 30% to 90% of the population cohort, and their range of ability has widened. This has necessitated the introduction of syllabi at three ability-levels, (Higher, taken today by 17%, Ordinary by 73% and Foundation by 10% of the cohort) and only at the highest level of these do questions such as the one quoted above appear. Significantly, not only were the grade 8 class of September 2000 (the grade 10 class of 2002) the first to be allowed to use calculators, but the "theory" of logs was at the same time removed from all three Junior Certificate syllabi.

Questions on logs are unpopular. The Irish Chief Examiner's Reports for the Junior Certificate exam of 1996 and 1998, and the Leaving Certificate exam of 2000 show that logarithms are the least or second least popular question on the exam papers and that this is an annual trend.

Some teachers find logs difficult. A team of experienced U.S. textbook authors, Hornsby and Lial, recently conceded in background material to their College Algebra book:

"Without a doubt, the concept of the logarithm is one of the most difficult for algebra students to grasp. The authors of your text admit that even they did not fully understand the concept until taking follow-up courses and teaching logarithms in their classes! So if you find this topic difficult, don't feel as if you're alone".

This paper examines possible reasons for the unpopularity and difficulty of logarithms.

2. The Tradition

It is helpful to look at how logs are introduced in textbooks.

There are two distinct modes for doing so, the algebraic and the analytical. Typically in the latter, the *base a exponential function* $y = a^x$ is firstly defined, and then $y = log_a x$ ($a>0$, $a\neq 1$) is defined to be the inverse of $y = a^x$ (Finney *et al.*, 2001).

This paper concentrates on the alternative, algebraic, treatment because of its potential for being more intuitive than the analytical, and because it is more likely to have been the mode of introduction to the topic experienced by students before they enter third-level education.

In the tradition of Irish and British school-mathematics, Hall and Knight's *Elementary Algebra for Schools*, first published in 1885, is a seminal book. The eighth edition published in 1907 was, along with its companion text *A School Arithmetic* by Hall and Stevens, reprinted more than 30 times, and remained with schools all the way to the New Maths of the late 1960s.

Before going further note that to denote, for example, the "2" in the symbol x^2, the term "index" is more common in Irish and British schools than the internationally-used "exponent".

The Eighth Edition of Hall and Knight defines *log* as follows:

"The **logarithm** of any number to a given base is the index of the power to which the base must be raised in order to equal the given number. Thus if $a^x = N$, x is called the logarithm of N to the base a".

Hall & Stevens (1908) had a similar definition and followed it with: "Since every logarithm is an *index*, it follows that the rules which govern the use of logarithms are deducible from the laws of indices".

But within a few years we had:

"The logarithm of a number to a given base is the power to which the base must be raised to equal the number" (Jones, 1913)

Note the omission of "index".

From the New Maths era onwards came:

"Logarithms are another form of indices...The logarithm of a number to a base is the power to which the base must be raised to give the number" (Holland and Madden, 1976)

"The logarithm of a positive number N to the base a is defined as the power of a which is equal to N" (Bunday and Mulholland, 1983).

"The *logarithm* function is the inverse of the index function " (Solomon, 1997)

"**Logarithm** is another word for **power, exponent** or **index**" (Sherran and Crawshaw, 1998)

"Logarithms are mirror images of exponentials...the logarithms are the exponents" (Strang, 1992)

" $\log_b x$ is defined to be that exponent to which b must be raised to produce x" (Anton, 1999).

It need hardly be said that the intended meanings of the above quotations are all the same. The language however is inconsistent. A logarithm is variously defined as an index, exponent, or power. While the first two of these are synonymous, they are not synonymous with the third.

Also, while a logarithm is indeed an exponent the adjectives "logarithmic" and "exponential" are given opposing (inverse) meanings.

3. Power

The greatest confusion surrounds the word "power". *Power* and the phraseology associated with it are so embedded in the language of algebra since the sixteenth century that they are difficult to deconstruct.

8 is a power. It is, among other things, the 3rd power of 2.

The use of the ordinal, 3rd, appears safe until we start "raising" things. When we read $8 = 2^3$ as "8 is 2 raised to the 3rd power", "*raised to the*" points to the superscript 3, as if *it* were the power as well as being the "exponent".

Repeated use of the cardinal, "8 is 2 to the power (of) 3" or "raised to the power 3" leads to the situation, common at present, in which *power* is used as another term for the exponent, and the original meaning of power as a product of copies of the base is ignored. Yet this meaning is recalled in usages such as "$x + x^2 + \ldots$ higher powers of x".

Similarly, while $\log_2 M = x$ is correctly called the "log form" of the relationship between M and x, its inverse, $M = 2^x$, is often referred to as the "index or exponential form" (which should mean log form!) - when its correct name is "power form".

Because of the identification, in textbooks and teaching, of the concept of an exponent/index with the term "power" it is not hard to deduce from terminology in current use that

logarithm = exponent = **power** = antilogarithm

and that

index form = inverse of log form = inverse of index form

The fact that students do not protest at this lack of consistency and clarity in the language is not proof that they are comfortable with it. The evidence points to the contrary. It is tempting, if flippant, to say that it is possible to teach logarithms *only* to illiterates. A likely result of the inconsistency is that the students lose confidence in their ability to understand the logarithm concept, and settle for engaging with logs at the procedural level only.

4. Teaching Logarithms

In an effort to counter-act the inconsistency of the language a group of 40 first-year college computing students were given a basic course in powers and indices (exponents), with substantial drilling in the "rules of indices", $a^m a^n = a^{m+n}$ and so on.

The students were then introduced to the terms used to describe the *power equation* $8 = 2^3$, comprising the power (8, or 2^3), the base (2) and index (3). "Logarithm" ("log" for short) was given as another, synonymous name for the index when the base was > 0 and not 1.

Next a natural-language description of $8 = 2^3$ was progressively transformed to mathematical "shorthand":

The index of 8, when 8 is written as a power of the base 2, is 3.

The index of 8 when 8 is written to the base 2 = 3.

The log of 8 when 8 is written to the base 2 = 3.

log of 8 to the base 2 = 3.

$\mathbf{\log_2 8 = 3}$ ("log form" or "index form")

Drill was then given in transforming between *power form* and *index form* equations. Subsequent work emphasised the identification of log with index by reinterpreting a subset of the rules of indices as rules of logs and giving drill-exercises in these.

Practice was also given in the application of logs to solving equations containing an unknown exponent, in particular to finding *n* in the compound growth/decay formula $A = P(1 + {}^r/_{100})^n$.

Six weeks later as part of a wider test the students were asked the following:

Q1. What is a log?

Q2. What are logs useful for?

The answers to Question 1, summarised in Table 1 show that 15 of the students returned at least the answer "an index" or "index or power", one student (who had started the Higher Leaving Certificate course but switched to the Ordinary Level) gave a more complete definition ("a log is the inverse of a power"), and four others defined a log only as a "power" but may have meant "index".

In Table 2 the answers to Question 2 are tabulated against the answers to Question 1, and show that in their knowledge of a *usage* for logs those who could <u>not</u> say what a log is, performed no differently from those who could.

Of the four students who defined a log as a power, two could describe a use for logs and two could not. When these students are shared evenly between the two groups who got Question 1 right, the outcome is Table 3, which merely confirms the evidence of Table 2.

Four of the students in the class had passed Higher Leaving Certificate Mathematics before entry to the computing course. None of these four was able to describe what a log is, suggesting both a lack of understanding of the concept by them at school, and an indifference to learning a new treatment of the topic at college.

Of course being able to answer that "a log is an exponent/index" is only a beginning, but it will now be argued that it is a most important realisation.

5. Can logs become popular?

The experience of the author from teaching at both second-level and third-level is that while logs will probably never move to the top of the popularity list in examination questions, they can certainly move off the bottom.

The language problem must first be solved. Napier's term *logarithmus* (logos+arithmos, a "reckoning number"), in the form *logarithm* or *log* does not offer an intuitive notion of the role that a log performs, in the way that, say, *index* (which 'points' to how often a base is multiplied by itself), *base*, and to an extent *power* do for their roles. A suggestion is made later for abandoning the term "logarithm" altogether. Until that happens it needs to be introduced carefully. The key to this lies in admitting to our students that "logarithm" is an superfluous word, hallowed by tradition, for a concept with which they are familiar - the 3 in x^3. They already know this object as the "exponent" or "index". The freedom to interchange the words "logarithm" and "index (exponent)" is to be their lifeline.

Students should be shown the semantic transformation from $8 = 2^3$, with which they are familiar, to $\log_2 8 = 3$. Next should come drill in changing between $A = x^n$ and $\log_x A = n$. The "laws of logs", such as logAB = logA + logB, can be presented as mere rewordings of the rules of exponents.

Students should come to feel that in using a redundant word like "log" they are only humouring their teacher/lecturer. They will not be intimidated by drill-exercises such as "Expand $\log_a \frac{P^3Q^4}{R}$ as a sum or difference of logs" or its opposite "Write $2\log_4 D + 3\log_4 F - {}^1/_2\log_4 G$ as a single log". At all times they know that there is an unbroken thread from the land of *logs* to the familiar ground of *indices/exponents*. It is a "thread" of two strands: the translation at any stage of *log* to *index*, and the skill to switch fluently between a *log form* equation and its *power form*. By this means students can be led into a "labyrinth" of questions such as the one quoted from 1970, and equations requiring logs for their solution. Hopefully they will reach a stage where the thread is no more than an underlying confidence, a feeling that <u>if required</u> they could get back safely to indices/exponents! And this is enough. It should by then be as easy to go forward as to go back, but at least they can advance without the insecurity that ill-defined terminology brings and without the suspicion that their leader, the teacher, isn't sure of the ground.

6. Changing the Terminology (i): *Index* for *log*

In an ideal world none of this remedial work would be necessary. Mathematics would not have redundant terminology, technical terms whose work could be done by terms already in the field.

The job of *logarithm* could be done by the under-worked (hence "under-loaded") term *index/exponent*.

The logic of this is that the word *logarithm* should be dropped from Mathematics. But how easy is it to replace?

Despite its restricted use internationally *index* would be a better candidate to take the place of *log*, than would *exponent*.

$81 = 3^4$ could be semantically transformed in the style shown earlier to $index_3\, 81 = 4$, or for short, ind_3 81= 4, or Ind_3 81 = 4. The form $index_a$ would require that the base a be > 0 and not 1, and a reason would be given for this (This is not so strange, the fraction $^a/_b$ of primary school becomes restricted by $b \neq 0$ in Junior school).

When the base is 10, i.e. *d*ecimal, the form *Ind* 10000 = 4 could be used.

When the base is the *n*atural number e, the form *In* 5 = 1.609... could be used.

In reverse, Ind_3 81 = 4, the *index* form, transforms to $81 = 3^4$, the *power* form.

Rules of Indices having been justified, $x^n.x^m = x^{n+m}$ would imply that the index of a product equals the sum of the indices of the factors, or

Ind AB = Ind A + Ind B

(Strang (1992) uses a direct approach in this sense to justify the properties of logs).

Of course if *logarithm* were dropped from algebra there would be consequences for the language of functions:

In an ideal world one might attempt a new terminolgy:

x^4 is a *variable-base power* function. 4^x is a *fixed-base power* function, as is e^x.

Let e^x be distinguished by the label *ebasepower* function, or *ebp* for short, (both labels have no more syllables, or letters, than *exponential*):

$$ebp(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

"Exponential decay" would become "fixedbasepower(ful) decay", and "growing exponentially" would be "growing fixedbasepower(ful)ly".

fixedbasepower functions and *indicial* functions would be inverses, replacing *exponential* and *logarithic* functions.

7. Changing the Terminology (ii): *exponent* for *log*

The difficulty in choosing *exponent* to replace *logarithm* would lie in the fact that variations of *exponent* are already in use and have well-entrenched meanings.

$exp(x)$ is the so-called exponential function e^x: it has the property that

$$exp(A + B) = expA.expB.$$

If *exponent,* or *exp* for short were to replace *log,* the first law of logs would become

$$exp(AB) = expA + expB,$$

which would be hard for teachers to adjust to.

Furthermore, while the exponential function e^x might become the *ebasepower function* described above, the current *logarithmic function* would become the *exponential function,* which would cause headaches for existing mathematicians and the literature.

8. Acknowledging the Difficulties

But something needs to be done if logs are to become more digestible than they have been for the past 100 years. With the abandonment of logs in arithmetic there is time and space to improve their accessibility in algebra. The notation in current use gives rise to the mystification of logs, and their popularity is as low as ever.

As educators we should not stand idly by. Either we purge the notation of its redundancy at an early stage or we engage in remedial work at third level. Part of the remedial work must be to acknowledge the difficulties of the material in the manner of Hornsby and Lial above. Such openness in mathematical education can both have a reassuring effect on the student ("I'm not so stupid after all") and act as a challenge to him/her to rise to the task required.

In the case of logs, an acknowledgement of (and an apology for) a non-intuitive and careless terminology should be part of every introduction to the topic.

9. Conclusion

The theory of logarithms as currently presented in algebra creates difficulties for many students, although the application of logarithms to numerical problems is more easily pursued. If the mathematical community were prepared to make a beginning on untangling the language surrounding logs it would be interesting to assess the effect of this on the understanding and application of logarithms at undergraduate level. While a start could be made by reducing to one the number of names for an *exponent* and using this name (with the usual restriction) to replace *logarithm*, rationalisation of terms on the scale suggested above may be too radical to hope for.

Yet not to attempt *some* change is to be complacent about a terminology that is non-intuitive, archaic and inconsistent, and to accept with resignation that logarithms are for comprehension by the more able students but not by the majority.

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Student answer:	"Index", "Index or power",…	"Power"	"Inverse of a power"	Other (term, base, number, etc)	No answer
Number of students	15	4	1	10	10

Table 1: Replies to Question 1: What is a log?

	Q1. "Index" or "Inverse of a power"	Q1. Other
Q2: Gave full or partial example	12	18
Q2. No example	4	6

Table 2: Comparison of answers to Question 1, and Question 2 ("Give an example of what logs are useful for").

	Q1. "Index", "Power" or "Inverse of a power"	Q1. Other
Q2: Gave full or partial example	14	14
Q2. No example	6	6

Table 3: Adjusted comparison of answers to Question 1 and Question 2.

A PROJECT-BASED APPROACH TO NUMERACY PRACTICES AT UNIVERSITY FOCUSSING ON HIV/AIDS

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ABSTRACT

In this 'Information Age' it is increasingly important that students at university are numerate at an appropriate level for their discipline. This paper reports on an attempt to achieve this through a project-based curriculum component in an 'Effective Numeracy' course. The practice of numeracy within relevant contexts is emphasised, rather than the decontextualized acquisition of skills. We explore the way in which students engage with curriculum-embedded projects, how they draw on their representational resources in the production of 'texts', and the manner in which the projects contribute to changes of attitude towards numeracy competencies.

The choice of the HIV/AIDS epidemic as the project topic is motivated by the need to raise awareness of the magnitude of the threat and its social implications. Its obvious social relevance is also essential to motivate the students to engage fully with the project. The project develops an appreciation of the relevance of numeracy, by requiring the students to practice numeracy in a context where there is close linkage with other vital competencies, such as writing and information and computer literacies.

The project design provides opportunities for co-operative learning, and includes the provision of scaffolding, especially for and through writing. Students were required to present their research in a range of genres, which enabled different kinds of engagement with the material, and different affective reactions to the tasks. In all cases, the learning was not only through reception, but through synthesis and transformation of knowledge in the processes of production.

1. Introduction

Many first-year students at the University of Cape Town (UCT) arrive without the appropriate quantitative literacy, language competence or computer literacy to enable them to succeed in their chosen course of study. South Africa is still suffering the consequences of the Apartheid policies on education, which results in a large proportion of the population being 'educationally disadvantaged' in terms of basic numeracy, visual, linguistic and conceptual practices. The traditional approach to various literacies at schools and universities in South Africa is to teach 'skills' in a very compartmentalised way. It is also generally assumed that a student who has studied mathematics to a sufficiently high level in school will automatically be able to apply mathematical knowledge to real-life situations. The Numeracy Centre at UCT administers an 'Effective Numeracy' course that aims to provide for the needs of some of these students, by increasing their quantitative and computer literacy, and their ability to exercise these competencies appropriately in the variety of contexts they will encounter in their studies. This paper thus argues for an approach to literacy and numeracy which sees them as practices embedded in particular social contexts.

2. Numeracy as practice in context.

There is an ongoing debate about the meaning of the term 'numeracy' or quantitative literacy and its relationship to 'mathematics'. In talking about numeracy, we adopt the proposed working definition of numerate behaviour from the Adult Literacy and Lifeskills Survey:

> Numerate behaviour is observed when people manage a situation or solve a problem in a real context, and involves responding to mathematical information that may be represented in multiple ways; it requires the activation of a range of enabling knowledge, behaviours and processes (2002: 9).

This emphasis on real context, responding to information, and multiple processes has led to the adoption of the following guiding principles for curriculum design:

- Numerate behaviour is always embedded within a context
- Numerate behaviour can be thought of as a practice involving the exercise of several related competencies, not just arithmetic skills.
- A numerate University student should be able to exercise these competencies to express their understanding of numerical information in the form of a 'text', which we define in the largest sense as communication, in written, oral or visual mode.

Numerate behaviour, as opposed to mathematics, is embedded within a context.

An important component of numeracy, often mentioned in the literature, is the ability to operate in a context. Yet, the dominant pedagogical practice, particularly in South Africa, of teaching numeracy in the restricted context of the formal mathematics classroom is at odds with this idea. Hughes-Hallett (2001) summarises the difference between quantitative literacy or numeracy and mathematics as follows:

> ...mathematics focuses on climbing the ladder of abstraction, while quantitative literacy clings to context. Mathematics asks students to rise above context, while quantitative literacy asks students to stay in context. Mathematics is about general principles that can be applied in a range of contexts; quantitative literacy is about seeing every context through a quantitative lens (94).

In thinking about 'context', Usiskin (2001) warns against the use of contrived 'real-life' examples masquerading as 'reality' in the mathematics classroom, such as treating word problems as if they are applications. Teaching quantitative literacy requires the use of real contexts, which need to be understood as clearly as the mathematics that is being applied. This is why students often experience a numeracy course as rather challenging, even if the mathematics required is quite elementary. Students often avoid or skim over the quantitative aspects they encounter in their disciplines. For a university student to be numerate, they would have to be able to see the contexts they encounter in all the courses in their programme of study 'through the quantitative lens'. For instance, understanding graphs in a discipline like Psychology as opposed to learning these graphs in a rote fashion.

Numerate behaviour can be thought of as a practice involving the exercise of several related competencies, not just arithmetic skills.

Viewing numeracy as a set of identifiable arithmetic skills, construes 'it' as a set of techniques that can be taught and learnt without reference to social contexts and are therefore seen as universal across time and space. Baker, Clay and Fox use the term numeracy to draw attention to the parallels and links between numeracy practices and literacy practices, to refer to "the collection of *numeracy practices* that people engage in – that is the contexts, power relations and activities – when they are doing mathematics" (1996: 3). Regarding numeracy as a social practice alerts us to the fact that power relationships and possible contests over meaning and values might arise. To consider numeracy as a social practice is to question whether numeracy is value-free or to what extent it is positioned in cultural contexts and value-laden. Chapman and Lee (1990) also argue that it is not possible to draw an artificial separation between the notions of numeracy and literacy, but rather that numeracy should be situated within a larger notion of literacy that involves many competencies: "reading, writing and mathematics *are* inextricably interrelated in the ways in which they are used in communication and hence in learning." (279). Focusing on numeracy and literacy practices is an excellent way of integrating the curriculum by combining subjects, genres, conventions, and creating new forms and new ways of knowing. This is in line with a multiliteracies approach to pedagogy and curriculum design. A multiliteracies approach emphasizes competencies in different semiotic systems: numbers, written language, visual design or graphical representation (Cope and Kalantzis, 2000).

A numerate university student should be able to exercise these competencies to express their understanding of numerical information in the form of a 'text'.

Being numerate does not only encompass an ability to interpret information, but also the ability to *express* information of a numerical nature coherently in a verbal and visual form. Contextualized writing reinforces understanding of concepts in context because it requires the student to retrieve, synthesize and organize information in meaningful ways. In dealing with quantitative or mathematical ideas in context, students should be able to *interpret* ideas presented verbally, graphically, in tabular or symbolic form, and be able to make *transformations* between any of these forms. This is consistent with a multiliteracies approach which emphasizes the importance of being able to transcode between semiotic systems as evidence of learning. Kress (2000) defines learning as the movement between modes and the transformation of meaning.We would like to argue that 'numerate behaviour' furthermore requires the ability to choose the appropriate form for the expression of a quantitative idea, and to produce a 'text' that expresses that idea. This synthesis and transformation of knowledge in the process of production is vital in the learning process.Thus, the 'practice' of numeracy at tertiary level must include the ability to put together a particular document for a particular purpose in a particular social, political or other context.

3. The Effective Numeracy course.

The 'Effective Numeracy' course is based in the 'Gateway Programme' which is a four-year extended curriculum programme in the Humanities faculty at UCT. Students enter this programme in their first year and proceed to a variety of economics related programmes of study. Effective Numeracy is one of the core courses in first year, the others being Microeconomics and Philosophy (Quantitative reasoning). The philosophy and development of this course over the last five years is described by Brink (2001) and Frith and Prince (2001). The classroom sessions are run as 'workshops' with limited presentation of course content. Students sit in groups and engage with the course materials provided as printed worksheets, while lecturers and tutors act as facilitators. Interactive computer-based tutorials are used to support the learning of numeracy/mathematics concepts, where appropriate.

Projects

The course design includes a project-based approach to learning numerate behaviour, which creates the opportunity for contextualized teaching practice and encourages student participation. Students were given a choice of four projects on the topic of HIV/AIDS. This is a particularly relevant topic in South Africa since predictions of the HIV/AIDS pandemic are very alarming, especially for teenagers. The project tasks, criteria for assessment, reference list and reading materials were made available on the web. Students had a choice of genre between pamphlets, poster and reports:

Genre	Objectives	Audience
Poster	Awareness of risk of infection and prevention	Primary Health Care Clinic
Pamphlet	Awareness of projected impact of HIV/AIDS on industry	Human Resource Managers
Report	Motivating the need for educating teenagers about HIV/AIDS	School governing body
Report (independent research)	Development of HIV/AIDS epidemic over the last decade	Non-governmental organization

Students were encouraged to work in pairs on the projects, but not compelled to do so. Some of the students mentioned how co-operation is vital, rather than working in isolation, and astutely mentioned that it seemed to be a very feature of the subject of maths itself: "It was easy to work as a group - I couldn't have done it alone".

Scaffolding

Significant scaffolding for the projects was built into the curriculum, including a range of tasks, that functioned as both formative and summative assessment. The classroom materials included comprehension exercises on newspaper articles dealing with the HIV/AIDS epidemic, with titles such as: "R120m for poor in AIDS battle"; "AIDS The facts behind the smokescreen". These exercises required the understanding of numerical information embedded in a text, and the ability to produce brief written expressions of this understanding. This prepared the students for the kind of writing that would be required for the production of the project.

In the tasks for assessment, the criteria were always made explicit, and a mark for writing was included. One student's comments on these tasks:

> They were useful – introduced a new aspect to (the classroom work), something you hadn't gone through in thorough detail in the exercises – incorporating classroom and lab work together, so you had to put what you learned in lab and in class together. They encouraged exploration.

Scaffolding also took the form of guiding the students through the writing process. In the UCT context, acquiring academic discourse is complicated by the fact that English is a second language to most students. The Writing Centre analysed the strengths and weaknesses in student writing on the homework assignments and provided feedback to both the students and the lecturers on the course. The Writing Centre also offered workshops for all students and one-to-one consultations for those students who needed more assistance. The students were given clear guidelines regarding the specifics of the different genres and different registers required in writing for different audiences. Each student (or pair) was required to produce a first draft and discuss it with their 'supervisor'. This student highlights the usefulness of the scaffolding activities:

> I enjoyed the project. It is at the heart of what is happening in the country. I enjoyed it and worked hard. I never expected to get 78%. But we got help in class with the writeup and we also took our work to the Writing Centre, where the lady spoke to us and guided us in a help session. I was aware of the criteria for marks and I had to think about whether the writing was really relevant to the subject - was my interpretation of the graphs applicable to the specific piece of work?

4. Analysis of the Projects and Students responses

If one views all sign-making or production of texts as based on 'interested action', the emphasis focuses on students' motivations for the uses of particular forms; rather than on incompetence and error. In looking at student representations of an important issue such as HIV / AIDS, it may be interesting to look at the degree of personal involvement and how this influences the representation of 'data'. As opposed to the more depersonalised and 'objective' language of the written report, many students battled to operate within an appropriate register in creating the posters. Perhaps this reveals a degree of personal involvement which was not enabled to the same extent through the written report, which tended to be more linear, objective, factual and formal. The posters concentrated less on 'argument' and more on persuasion through 'display'. For instance, the use of abbreviated visual symbols pointing to a host of meanings: ! for caution, a stop sign, predominant use of red indicating both 'blood' and 'danger'; and red ribbons to indicate solidarity with AIDS sufferers. Students also managed to insert a sense of identity with signifiers irrelevant to the overall message of the poster, such as red, yellow and green colours to indicate kinship with the Rastafarian movement.

As noted in an introductory course in engineering, there is often an element of parody in students' use of dominant genres, and the production of multimodal texts in the academic context tends to enable this play with form to a greater degree (Archer 2000). The HIV/AIDS issue was represented in the poster genre as a gothic skull and crossbones experience; a detective adventure story ("The killer is on the loose!"); a photostory; a 'comic-style' narrator that leads the reader through an argument. All of these choices may also be ways of dealing with the sensitive subject matter. Some students took the opportunity to exploit the innuendos generated by the topic. " Use your shaft wisely" was the title for a pamphlet in the mining industry.

The interpretation of the poster genre was broad, ranging from dense written text to almost purely visual texts. The visual, verbal and graphical elements were integrated with varying degrees of success. One poster chose another mode, a three dimensional model of a female figure (See figure 1). The emphasis is on the body as 'text', where the inner and outer workings blur into one organism in a kind of depersonalised medical way. The body is represented as permeable, vulnerable and relatively distasteful (red and blue wire for veins, splashes of blood, grimacing teeth). This inside/outside dichotomy is echoed in the depiction of Africa within the belly; the 'body' represents the larger body politic where the individual is responsible for the collective well-being of society. The integration of the different modes is highly problematic in this poster. The students have done extra research but have represented the information in an inappropriate form. They have chosen dense text rather than graphical representation. The graphical representations they have included do not relate to the written text and are not explained in any way. They are also copied from the reading provided. In copying these charts, the students demonstrate an ability to read and decode the charts, but an inability to *produce* graphical representation of data.

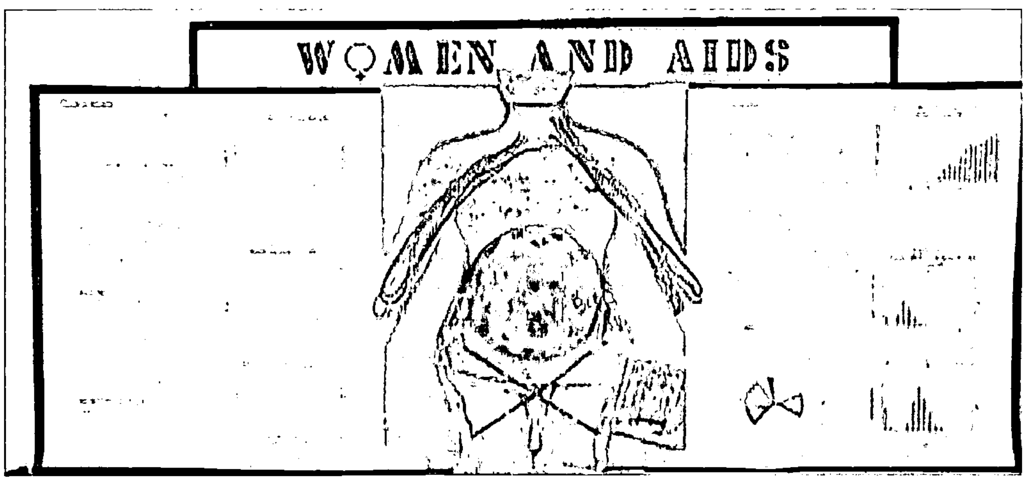

Figure 1

In figure 2, the students have understood the data, internalised the message and represented it in a visually appropriate way, which is not necessarily mathematically correct. In our definition of 'being literate' as being able to choose from a range of semiotic resources to produce a message deemed appropriate for a particular audience, this poster certainly succeeds. Although, it may not be completely accurate mathematically, the visual representation of the data has a specific impact. It manages to put a human face to the talk about statistics; as opposed to the depersonalised 'medical model' looked at above. This human face is important in a discussion about HIV/AIDS where numbers can easily become a distancing mechanism from the issue.

The audience is situated in relation to the numbers and through the informality of language used: "Stats show that 52% of newly infected females are between 20 and 24. This number is said to rise. How old are u?" Here mathematics is used to persuade rather than to inform. The key point is that there is no right answer for this kind of project as it is dependent on context, and is not abstract mathematics.

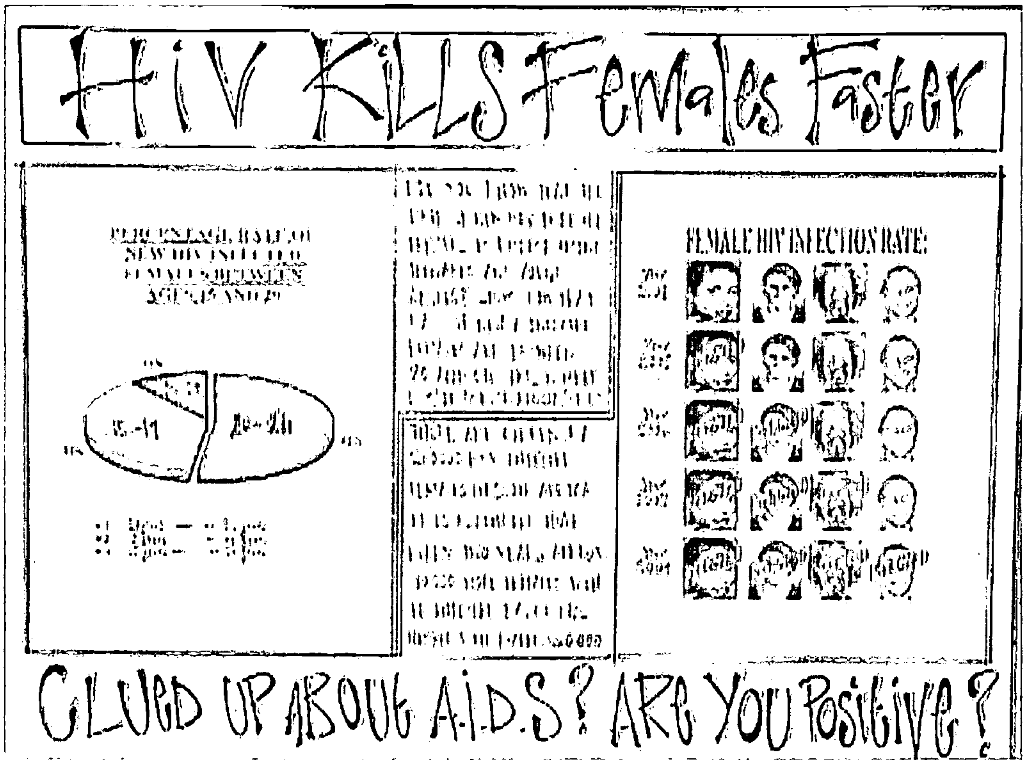

Figure 2

5. Final comments

In talking through past experiences of mathematics, students typically provided highly personal accounts using strong words such as 'hate' and 'fear'. Our project-based multimodal approach hoped to combat anxiety and build student confidence by drawing on their representational resources and different knowledges, and engaging them on an affective level. The general principles for curriculum design to emerge are largely focused around these issues.

- Frame numeracy as a 'behaviour' or 'practice' in context rather than a collection of separate and definable 'skills'.
- Chose the contexts for study carefully, as relevant and interesting to the people for whom the intervention is being designed.
- Encourage the 'production'of multimodal texts as an outcome of numerate practice, not only the 'reception' (understanding and interpretation) of existing texts.
- Incorporate a multiliteracies approach, where different knowledge and competencies must be displayed and exercised by a student in order to achieve the required outcome.
- Build in scaffolding throughout the curriculum. This includes pre-tasks to develop the context, writing and computer competence. Scaffolding also includes making assessment criteria explicit and giving guidelines around specific generic conventions.
- Provide students with an unthreatening and supportive environment, and opportunities to succeed.

In the interviews with a random sample of students, working together on the projects was mostly perceived positively, and even as 'fun'. The project was variously described as challenging, creative, eye-opening, interesting and relevant. Students appeared to appreciate the opportunity to research and write, which they did not usually associate with a mathematics course. The aim was also to design a curriculum which would accommodate and validate the diverse social, and cultural backgrounds of our students, as well as address the inequitable educational opportunities afforded them. This kind of cross-genre, cross-disciplinary, multimodal approach to teaching numeracy and literacy practices has important implications for democracy, equal opportunities and social justice, which is of crucial importance to South Africa at this time.

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PROBLEM SOLVING FOR FUTURE TEACHERS - AN INDIVIDUAL LEARNING COURSE

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ABSTRACT

At the Faculty of Education, Charles University, Prague, problem solving represents one of the key subjects in the preparation of future teachers. For four years, the first of a series of problem solving courses has been organised as an individual learning course as the only course during the study. By that we mean that students have no scheduled classes, they work individually and meet their teacher for consultations. The main aims of this form of study (besides the obvious goal to teach different strategies of problem solving) are to acquaint students with the range of mathematical books and textbooks and to develop their ability to (a) work independently, (b) take responsibility for their learning, (c) critically evaluate mathematical texts, (d) write mathematically.

The course comprises three topics: Equations and their Systems, Number Theory and Plane Geometry. Students have to submit one seminar work for each topic which includes solutions to (a) problems given by the teacher (different for each student), (b) problems chosen by students from the assigned literature, (c) an 'extra' problem chosen by students from any book but with a short justification of their choice. The fourth and last seminar work has a different character – it is an essay in which at least two books or textbooks used during the course are evaluated according to a student's criteria. Finally, students sit for a short test.

After the term, students are asked to write a short anonymous evaluation of the course (they mention advantages and disadvantages of an individual form of work and give suggestions for improvement). These written evaluations and their analysis contribute to the running modifications and improvements of the course over time.

Keywords: individual learning course, problem solving, evaluation, design of a course, student teachers

1. Introduction

Students at the Faculty of Education, Charles University in Prague, start their studies with the aim of being mathematics teachers from the outset. This means that they do not study with future mathematicians as in some other countries. They will qualify to teach students in the age range 11-19 years.

When students leave secondary school and enter the university, they have to learn a different way of working. They are supposed to take responsibility for their learning and rely less on their teacher, to organise their study themselves, to study literature independently and choose relevant information, to be able to communicate their mathematical ideas in writing and as future mathematics teachers, to be able to explain their solving procedures. At the university, more stress is put on home study rather than class learning. This change is not always easy for students. Taking their difficulties in the above areas into account, it was decided in the Department of Mathematics and Mathematical Education that a new type of course should be designed – an individual learning course. When looking for suitable subject matter for this type of the course, we concentrated on less formal courses which could include the study of a variety of books (so, for instance, abstract algebra or calculus were ruled out). Finally, the course 'Problem Solving I' was chosen. The course is offered in the fourth term of study, i.e. students are in the second year of their five-year study.

2. Framework

At the university level, many types of teaching-learning situations can be determined, some similar to those in elementary or secondary schools, others specific to university teaching: tutorials, lectures, seminars, individual instruction, demonstration, class discussion, home study, etc., among non-standard teaching-learning situations we have, for instance, scientific debate (Alibert, Thomas, 1991), and using constructive, interactive methods involving computers and co-operative learning (Leron, Dubinsky, 1995).

In recent years, problem solving has become one of the most important activities of school mathematics, the main reason probably being that it "places the student in the role of actor in the construction of his/her own knowledge" (Grugnetti, Jaquet, 1996). There has been a considerable body of research concerning its use in teaching mathematics (see e.g. Frank, Lester, 1994, Schoenfield, 1992). Problem solving at the university level is explored e. g. in Yusof, Tall (1998).

In this article we will present a non-standard way of teaching problem solving to future mathematics teachers which we call an individual learning course. By that we mean a course which does not include any scheduled classes and consists mostly of individual home study (even though students can co-operate) and consultations with the teacher.

3. Design of the individual learning course

3.1 Aims of the individual learning course

The course has two types of aims and goals. First, there are the goals specific to the *content* of the course: that students are aware of various techniques of problem solving and learn to solve problems out of context[1]. Second, there are aims specific for the *form of work*, i.e. the individual learning course:

- to widen the range of different forms of work with students
- to develop a student's ability to take responsibility for his/her own learning
- to acquaint students with the relevant literature which can be used both for their problem solving at the university, but also for teaching problem solving at school
- to develop a student's ability to write mathematically and formulate a mathematical text
- to develop a student's ability to work independently
- to develop a student's ability to read and understand mathematical texts written for different purposes and audiences, and critically evaluate them in terms of their suitability for a certain purpose
- to enable students work both individually and in teams

3.2 Content of the course

The course 'Problem Solving I' is the first of a series of problem solving courses which focus on basic methods of problem solving. It is the only one which is organised as an individual learning course, the others are organised in the classical way via seminars. It comprises three topics – Equations and their Systems, Number Theory, Plane Geometry. Its content will be briefly illustrated by several problems from individual topics which are taken from a <u>Booklet</u> for students (see below).

Topic and subtopics		**Illustration**
Equations	systems of equations solvable by a 'trick'	Solve a system of equations using a method other than the Gaussian elimination method: $x_1 + x_2 + x_3 = 6$, $x_2 + x_3 + x_4 = 9$, $x_3 + x_4 + x_5 = 3$, $x_4 + x_5 + x_6 = -3$, $x_5 + x_6 + x_7 = -9$, $x_6 + x_7 + x_8 = -6$, $x_7 + x_8 + x_1 = -2$, $x_8 + x_1 + x_2 = 2$
	equations which include the integer part of a number	Solve the equation in **R** ($[x]$ is the integer part of x): $[(5 + 6x) / 8]=(15x - 7)/5$
	graphical solution to a system of equations	Solve the system of equations graphically: $x^2 + y^2 < 11 - 2(x - 2y)$, $x^2 + 4x \geq 2y - y^2 + 4$
	system of equations with a parameter	Discuss the number of solutions of the following system of equations in terms of a real parameter m, x is the unknown: $x^2 + y^2 = 4$, $(x + m)^2 + (y - m)^2 = 1$
	equations solvable by a suitable substitution	Solve in **R**: $2x^2 + 6 - \sqrt{2x^2 - 3x + 2} = 3(x + 4)$
	more difficult systems of equations with parameters	Solve in **R** the system of equations with the unknown x, y, z in terms of real parameters $a, b, c > -1$: $y + z + yz = a$, $z + x + zx = b$, $x + y + xy = c$
	more difficult systems of equations of a higher degree	Solve the system of equations in **R**: $x^4 + y^4 + 3x^2y^2 = 109$, $x^2 + y^2 + xy = 13$, $z(x + y) = z + x + y$
Number Theory	systems of Diophantine equations	For which x will the numbers $(x - 3)/7$, $(x - 2)/5$ and $(x - 4)/3$ all be whole numbers?
	Proofs of theorems on the divisibility of numbers	Prove that for all natural numbers n $3^n \mid 111...11$ (there are 3^n of ones).

[1] In view with Arcavi (1998) we believe that this goal is very important as in "traditional courses problems and exercises are often sequenced in such a way that students can easily find solution techniques".

	'algebrograms' – looking for numbers with certain properties	Find all four-digit numbers which are the squares of a natural number so that their thousand's digit is the same as their ten's digit and their hundred's digit is bigger by one than their unit digit.
	least common multiple, greatest common divisor	For which natural numbers n is (a) $\mathrm{NSD}(n + 6, n + 2) = 4$? (b) $\mathrm{NSD}(6, n + 3) = 3$?
Plane Geometry	construction of a triangle – non-trivial problems	Construct a triangle ABC, if we know: $b - a$, v_c, r (r is the radius of the inscribed circle, and v_c the height from C)
	construction of a quadrilateral and other polygons	Given points S, O and a line p, construct a triangle ABC so that the centre of the circumscribed circle is S, the centre of the inscribed circle is O and its side lies on the straight line p.
	problems on proofs of relations between elements of polygons	Consider a triangle ABC in which the angle ABC is not right. On side AB construct a square $ABKL$, which does not lie in the half-plane ABC. Similarly, on side BC construct the square $CBMN$, which does not lie in the half-plane CBA. Prove that the triangles ABM and KBC are congruent.

3.3 Set literature

The set literature consists of about sixteen books which range from secondary school collections of mathematical problems to books for university students organised in the way 'definition – theorem – proof – problems'. Many of them are organised as 'exposition – examples – exercises'. Students can also use various collections of Mathematical Olympiad problems and some journals on mathematics education. The books are also from different times so that students get to know the style of writing from different periods of the development of mathematics. Books do not only include problems from the three topics above, but other topics too, so that students have to choose parts relevant to their course.

It is important to stress that these books are not specifically designed for individual learning. Some of them are meant for the classroom use, while others are for tutorial/seminar use.

3.4 Organisation of the course

The core of the course work lies in the student's independent work and his/her solving of mathematical problems and then summarising their solutions for a seminar assignment. The interaction with a teacher is limited to his/her office hours when a student may but does not have to come to see him/her. The organisation of course work will become clear in the next section on the course implementation.

4. Implementation of the individual learning course

The course was first implemented in the school year 1996/97 in the spring term and has been offered in the spring terms in subsequent years. Its content and organisation differed a little from one year to another according to the students' and teachers' evaluation (see below). The description below fits the current state of affairs.

4.1 Students' work

A student is given a Booklet (prepared by a teacher) which includes: (a) worksheets with problems from each of the three topics, (b) details of literature he/she should study for each topic: which parts, which problems, to what extent, (c) details of the seminar work for each of the three topics,

(d) the assignment of the essay (fourth seminar work), (e) details of the written test, (f) deadlines for submitting all seminar assignment.

Each of the three seminar assignment has the following content:

- compulsory problems from the worksheets (a); these are assigned in such a way that if possible, no two students have the same problems
- problems chosen from a certain part of the recommended literature (b); for instance, for the topic Equations students have to study a certain booklet and choose two problems which are not solved there and solve them
- an 'extra' problem which can be chosen according to the student's liking from any literature, but which must be relevant to the topic and at an appropriate level. The student must justify his/her choice

The fourth seminar assignment, the essay, differs from the previous ones. It includes a student's evaluation of the literature (at least two publications) from the point of view of their use in the subject 'Problem Solving I'. There is no limit set for the extent of this project.

During the term, students work on the four seminar assignments individually or they can co-operate. They study at home, go to the library or can ask the teacher to be allowed to study in the department library where all the literature is available for them. While doing so, they can go to see the teacher and consult their work.

During the examination period, a test is written which consists of six problems (two from each topic) which are chosen from a given set of problems (e) (some from the worksheets, some from the literature). An example of such a test is given below.

4.2 Teacher's work

The teacher prepares the Booklet and gives it to students at the beginning of the term. This booklet is constantly revised after each term. He/she sets four deadlines during the term by which students have to submit their seminar assignments[2] and the date of the written test. He/she sets office hours and is available to students during these times to discuss their problems. He/she gives a test to students during the examination period.

The focus in the subject lies in seminar assignments. It is a teacher's task to assess them as students hand them in[3]. Each seminar assignment is evaluated by means of points and is accepted if a minimum number of points is gained. If this is not so, the teacher discusses the work with the student and he/she can correct it and submit it again.

This course represents a student's first opportunity to 'write mathematically'. They gradually learn how to do it. Among the most frequent problems is their inability to explain in a logical way their solution strategy. They sometimes write in a too succinct a way, omitting important parts of the explanation because they do not realise that writing mathematically has different rules than when they directly explain their solution to the teacher.

The assessment of the essay is a subjective one. The teacher takes into account:

- if the choice of books is appropriate
- if the text is structured clearly
- if the mathematical language used is accurate
- if the student chose appropriate criteria for the evaluation of the books

[2] The setting of deadlines is essential, otherwise students tend to hand in their work towards the end of the term and the teacher is not able to correct them all at once.

[3] It is our desire that students get feedback on their work as soon as possible and that the teacher speaks with each student about at least one seminar assignment during the term (unless the work must be redone, of course). However, it is not always possible and it depends on the number of students who enroll in the course.

- to what extent the student evaluates the books critically (if he/she expresses his/her opinion and not only lists the content of the books, if it is clear from the text that he/she knows the books sufficiently well, etc.)

The test comprises six problems from three topics and each topic must gain at least 60 percentage points in order for the test to be accepted. Students can write the test three times[4].

Example of the test:

1. Solve in **R**: $x_1(x_1+x_2)=9, x_2(x_1+x_2)=16$
2. Solve in **R**: $\sqrt{x^2+x+7}+\sqrt{x^2+x+2}=\sqrt{3x^2+3x+19}$
3. Prove that for each n natural is $57 \mid 7^{n+2}+8^{2n+1}$.
4. Find all primes which are at once a sum and a difference of two suitable primes.
5. Construct a triangle ABC, if we know v_c, γ, $\omega = |\angle BCD| - |\angle ACD|$, where D is the foot of the altitude v_c.
6. What is the sum of the inner angles of a polygon, which has 52 diagonals more than sides?

Students get a credit provided their three seminar assignments and the essay have been accepted and that they successfully wrote the test.

5. Evaluation of the individual learning course

After its first implementation, the course was evaluated both by the students and by the teachers. The results of this evaluation led to the redesign of the course both in its content but mainly in its organisation. This evaluation has been repeated several times since and each time led to additional changes in the organisation. Below we present both the students' and the teachers' evaluations.

5.1 Students' evaluation

Several years ago, when the individual learning course took place for the first time, we were not sure about its positive and negative aspects. Therefore we started with the students' evaluation of the course. They were asked to submit a written anonymous evaluation of the course. Their comments were taken into account and the course was redesigned. Moreover, in the essay students sometimes spontaneously express their opinions and suggestions. Some of them, which we consider to be typical and important for the course, will be given below. Some are positive, some are negative, some include suggestions.

- *For the first time I was forced to look up literature and solve independently problems which I had not met before.*
- *We are at least forced to search through literature which we would not normally see.*
- *The advantage is that one has to study independently, learn how to work with literature, to be active and patient.*
- *One can solve problems when it is most convenient and this is not dependent on the timetable.*
- *One can study at home 'in peace' and can consult a teacher if necessary.*
- *I believe that the course's main goal was not to solve the problems accurately, but to learn where the problems can be found and to look at them from two perspective: as a problem solver and as a teacher who will be using similar problems him/herself during teaching.*
- *It is very subjective to evaluate books.*
- *It was difficult to get some books from the set literature.*
- *It was too much work to get a credit, in other courses it is easier.*

[4] In the Czech universities, students usually have three attempts to pass an examination.

- *It is a disadvantage that we could not feel the presence of a 'mediator' of knowledge, someone who could react immediately to our questions.*
- *The test should be abolished, seminar assignments themselves are enough.*
- *I would like to know if the problems I solved could have been solved differently.*
- *It is not assured that the students solve the problems independently. The teacher should speak to each student and ask him/her how they solved the problems. Then he/she could be sure that the student understands the problems.*
- *It might be good to include several teaching lessons during the term.*

5.2 Teacher's evaluation

Here, we will summarise the main features of the course.

From the first time it was used, the individual learning course has been constantly evaluated by both students and teachers and has been redesigned several times to meet students' need and the course aims. Therefore, we can claim that it is flexible.

Even though students do not meet in scheduled classes, they know each other (unlike in traditional distance learning), they meet during other courses and often co-operate when working on their seminar works. Such co-operation is desirable provided that it is meaningful for all participating students and that one does not merely copy the other's work.

There is a limited amount of interaction between the teacher and the student which means that students cannot benefit from the immediate exchange of ideas with the teacher. They do not get an immediate response to their queries. However, on the other hand, they are made to try to find the answers themselves or look them up before approaching the teacher. This contributes to the students' ability to study independently and organise their own learning.

It is important to strengthen the feedback – to speak with the students mainly about the mistakes and imperfections in their work. Sometimes to let them explain their thinking, to show them that there was a more 'elegant' solving strategy, etc. However, the individual learning course is time consuming not only for the students, but for the teacher as well. Therefore, it is necessary to find a balance between the number of students and the number of teachers.

6. Conclusions

When we take into account both the students' and teachers' evaluation, the students' results in the written test and seminar assignments, we believe that the individual learning course has its place among other more traditional courses in the preparation of future mathematics teachers and that it serves our aims well. However, we are well aware of its drawbacks and try either to remove them or to compensate for them in other courses. For instance, in the course 'Problem Solving II' more stress is put on the teacher-student interaction and group work. At present we are considering some changes in the written test. Instead of using problems from a set of problems, which students know in advance, we would like to use problems similar to those in the books. Students could then use any literature they want during the test.

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GEOMETRICAL TRANSFORMATIONS – CONSTRUCTIVIST ANALYTIC APPROACH

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ABSTRACT

The contribution illustrates a constructivist approach to the teaching of geometrical transformations to future mathematics teachers at the Faculty of Education, Charles University in Prague. Traditionally, this subject was presented as a series of logically connected definitions and theorems and students were asked to apply them in problems. A lot of material was covered like this, however, students' understanding was often formal and superficial. Several years ago, the course was completely re-designed in such a way as to let students deduce most knowledge themselves through a series of carefully prepared problems. A textbook adopting the Klein approach to geometry was written for the course (in Czech). Only isometries and affine transformations in the line and plane were covered, however, our experiences show that the investigative approach leads to a better understanding of the subject matter and improves students' ability to study transformations independently of the teacher.

A year ago, the author taught geometrical transformations in English to a group of practising teachers and the course was refined. Where it was possible, no mathematical result was presented as a ready made product, students had to discover it for themselves. As the analytic approach to transformations lends itself to using software (e.g. *Maple*), the emphasis was placed on its use to help with tedious calculations. The article concentrates on the basic characteristics of the course: emphasis on the connection between synthetic and analytic approaches, connections between geometry and algebra, investigative learning, use of computer and non-traditional assessment. An illustration is given of a student's investigation of the general matrix for a glide reflection. Examples of problems for the final test are discussed.

Keywords: constructivist approach, investigation, analytic and synthetic geometry, *Maple*, geometrical transformations, isometry, affine transformation

1. Introduction

In the traditional (and prevailing) teaching of university mathematics, we often try to pass as much knowledge as possible to students and present "the finished and polished product into which that well known, unassailable, fully accepted segment of mathematics has grown" (Dreyfus, 1991). However, this does not necessarily mean that students understand the mathematics they are being taught. Their knowledge is often formal.

In the nineties, research in mathematics education (not only) in the Czech Republic has taken into account constructivist approaches, which are gradually finding their way to the teaching of mathematics at the primary and secondary school (e.g. Hejny, Kurina, 2001, Jaworski, 1994). However, as far as we know the instances of using the constructivist way of teaching at the university level have been rare. Moreover, we realised that when student teachers are prevented from experiencing constructivist approaches during their university study, they can hardly be expected to use them in their own teaching. Therefore, we attempted to remedy the situation and redesigned the course of analytic geometry. Here we will concentrate on the part of the course which focuses on geometrical transformations.

2. The course of analytic geometry - history

A course on analytic geometry has always had its place in the preparation of future mathematics teachers at the Faculty of Education, Charles University in Prague[1]. It used to be given in a traditional form: 'definitions - theorems - proofs - exercises'. In 1995, Prof. Hejny redesigned the course so that it better reflected constructivist teaching. It meant, among other matters, markedly cutting down on the content of the course and presenting the content at a less advanced level and in greater detail than was customary. A university textbook (Hejny, Jirotkova & Stehlikova, 1997) was prepared in which more stress was put on student investigations. Most theorems emerge only as a result of a series of carefully selected problems; some of them must be formulated and proved by students themselves. It must be stressed that the textbook is unsuitable for the use as a reference book (it is far too 'chaotic'), it cannot be read, it only can be studied. It also requires a teacher who is prepared to teach in a constructivist way.

The author of this paper has used the textbook for four years at the Faculty of Education and later in a course for practising teachers at a foreign university. This enabled her to further reflect on the course and the way it is delivered, and to modify it. Here we will concentrate on this modification.

3. The goal of the course and its outline

The course main goal is **not** to teach students as many different concepts, definitions and theorems as possible and to show them a finished 'building' of Euclidean and affine geometry, but rather to open the world of geometrical transformations to them and to make them aware of methods they can use for their own study of transformations. It is hoped that the course will make the subject more engaging and meaningful for them.

The course assumes a basic knowledge of isometries and similarities (taught earlier in the course of synthetic geometry) and of group theory and linear algebra (matrices). It starts with the

[1] In the Czech schools, geometry is given relatively more attention than abroad.

geometry of the Euclidean line and plane, which is well known to students, and progresses to affine geometry by extending the group of isometries into the affine group (in plane).

Course outline

1. Isometries in E^1 (Euclidean line): translation, symmetry. Synthetic and analytic views (equations). Products of isometries in E^1.
2. Revision of isometries E^2 (Euclidean plane) from the point of view of synthetic geometry: basic properties, algebra of isometries, and decomposition into the product of reflections.
3. Isometries in E^2 preserving the origin of the co-ordinate system. Their analytic description via matrices. Parallel between the multiplication of matrices and product of isometries.
4. Group of isometries, synthetic and analytic view. Its subgroups. Group generators.
5. All isometries in E^2, their matrices. Product of isometries. Inverse isometries.
6. The group of affine transformations in A^1 (affine line). Matrices of affinities. Products of affinities.
7. Affinities in A^2 (affine plane). Geometric interpretation of a matrix of an affine transformation.
8. Classification of affinities in A^2. Invariant points and invariant lines. Lines of self-corresponding points.
9. Affinities with a line of self-corresponding points. Perspective affinities. Shear, oblique reflection. Euclidean and affine plane. Metric properties and affine properties.
10. Decomposition of affinities into the product of affinities with a line of self-corresponding points.
11. Similarity – a synthetic and analytic view.

4. Main characteristics of the course

4.1 Emphasis on the synthetic and analytic approaches to transformations

Transformations are treated both from the synthetic and analytic way and when possible, problems are solved in these two ways. Students are encouraged to compare the suitability of the first or second approach for certain types of problems.

When investigating isometries, students start from their geometric characterisation and proceed to their analytic (matrix) description. With affine transformations, the process is reversed. Students start with a matrix of affine transformation (see below) and look for the geometric characterisation of the transformation which it represents. By a geometric (synthetic) characterisation, we mean determining some properties of the transformation, such as the properties it preserves, what its fixed points and fixed lines are, etc.

4.2 Emphasis on the connection between geometry and group and matrix algebra

We adopt the Klein approach to geometry, i.e. that geometry can be thought of in terms of a space and of a group acting on it. Moreover, in agreement with Schattschneider (1997), we consider it important to use the study of transformations for the visualisation of the abstract concept of a group and also for "de-emphasising number systems as examples of groups, allowing students to see that not every group has all the nice properties of number systems".[2]

[2] Schattschneider (1997) suggests using the program. *Geometer's Sketchped* for the visualisation of isometries and similarities. In the course of synthetic geometry which precedes the course in question on analytic geometry, *Cabri geometrie* is used for the same purpose at our faculty.

When we consider geometries in this way, it is often convenient to have an algebraic representation for the transformations involved. This not only enables us to solve problems in geometry **algebraically**, but also provides us with formulas that can be used to compare different geometries.

In the course, we use matrices for isometries and affine transformations. However, at the beginning of the course when isometries in E^1 and isometries in E^2 preserving the origin are studied, only equations are used because there is actually no need for matrices. They come to the fore only when isometries not preserving the origin begin to be studied. Unlike most textbooks, which use only equations for transformations or 2 x 2 matrices, we use matrices 3 x 3:

Isometries: $\begin{pmatrix} a & b & c \\ \pm(-b) & \pm a & d \\ 0 & 0 & 1 \end{pmatrix}$, where $a^2 + b^2 = 1$. Affinities: $\begin{pmatrix} a & b & i \\ c & d & j \\ 0 & 0 & 1 \end{pmatrix}$, where $ad - bc \neq 0$.

4.3 Investigative learning

While investigative learning in primary and secondary schools is quite common, it is, in our opinion, undervalued at university level. If it is used at all, then this is usually in problem solving courses. Some tutors believe that most concepts of abstract mathematics are inaccessible to students in this way and even if students could discover them, it would take too much time. However, we believe that this time is not wasted and that the insight students get from their own investigative work is more valuable than acquiring the knowledge of many concepts introduced to them as ready-made products. The understanding and skills the students acquire by investigative learning makes up for the reduction in the content covered in the course.

In the course of analytic geometry, students are asked to derive knowledge for themselves. For instance, instead of being told what the general matrix for rotation (of α about the point (p,q)) is and then asked to try some examples, they have to deduce it themselves on the basis of their knowledge of the properties of rotation. Similarly instead of being told the basic theorems of affine geometry, they are asked to explore several concrete matrices of affinities and their properties and then to formulate theorems and prove them (such proofs are usually easier for them as they can use their experience from the previous experiments). Thanks to the use of 3 x 3 matrices, students cannot easily find the answers in the textbooks.

4.4 Use of a computer (*Maple*)

Nowadays, mathematical computer programs like *Mathematica* or *Maple* play an important role in the teaching of mathematics at university level. Many courses make use of them, especially calculus courses (e.g. Brown, Porta & Uhl, 1991, Devitt, 1993, many contributions in the Proceedings of ICTM, 1998). For geometry, *Geometer's Sketchpad* (e.g. Schattschneider, 1997, Parks, 1997) or *Cabri geometrie* (e.g. Dreyfus, Hillel & Sierpinska, 1999) are mostly used. Some research on the use of technology in advanced mathematics has been summarised in Dubinsky & Tall (1991).

As taught originally, some parts of the course of analytic geometry caused problems. The calculations, which were required to enable a student to deduce a matrix for a certain transformation, or to find the product of several transformations, were long and tedious. Therefore, in the modified course, the stress was put on the use of *Maple* as a means of helping a student to concentrate more on the overall strategy rather than on the calculation itself. The tutor started to use *Maple* herself for this purpose and produced *Maple* worksheets for the students which (projected by a data projector) formed the basis of the class work. The tutor's notes were sent to the students each week both to revise what had been done in class and to work on new problems.

Here we would like to illustrate our strategies using the example of the general matrix for a glide reflection. In the original course, it was virtually impossible to ask students to deduce this matrix and later to interpret the matrix geometrically. Thus, the teacher usually asked them to find one particular example and interpret it and supplied them with the geometric interpretation herself. The use of *Maple* enabled us to ask students to carry out the whole procedure themselves.

Illustration – How to find the general[3] matrix for a glide reflection and conversely, how to interpret a matrix for a glide reflection geometrically

Students know from synthetic geometry that glide reflection is the product of a reflection in a line and translation. We assume that earlier in the course they found the matrix for reflection in a line with an inclination α and the matrix for translation. Later, they are asked to find the matrix for a glide reflection and interpret it. The process has three parts (the following headings represent the tasks given to students, the text underneath is a student's solution). The figures can be found in the appendix.

1. *Find the matrix for a glide reflection*

It can be done by multiplying (in any order) a matrix of reflection in a line with inclination α and a matrix of translation through vector **u**[k cosα, k sinα] (vector **u** must be parallel to the line of reflection) and simplifying the calculations (*Maple* result is given in fig. 1, α is the inclination of the line of reflection, u, v are co-ordinates of any point on the line of reflection, k is any real number).

2. *How do we distinguish a matrix for reflection and a matrix for glide reflection?*

The matrix in fig. 1 is the same as the matrix for reflection in line (fig. 2) in that when we get a matrix of isometry of the form $\begin{pmatrix} a & b & m \\ b & -a & n \\ 0 & 0 & 1 \end{pmatrix}$, we cannot decide immediately which matrix it is. We must use the properties of both isometries to be able to make a decision. Unlike glide reflection, reflection in a line has a line of fixed points. So using the general matrix G in fig. 2, we compute the fixed points (it is a standard procedure for students by this stage of the course). We get a system of two equations and using knowledge from algebra[4], conclude that the system is solvable (i.e. there exist fixed points and the matrix must be the matrix of reflection in a line) iff $d = n\sin\alpha + m\cos\alpha = 0$. Otherwise, i.e. if $d \neq 0$, it is a matrix of glide reflection.

3. *Given the matrix in fig. 2 (i.e. we know a, m, n), interpret it geometrically.*[5]

The task is to find out the line of reflection and the vector of translation. We will write down two equations (which we get by comparing the matrix in fig. 2, which can be both a matrix for a line reflection and glide reflection, and the matrix in fig. 1, which is a matrix for a glide reflection) and solve them in terms of v and k. In fig. 3, the process of determining the equation of the line of reflection and the co-ordinates of the vector is illustrated.

4.5 Non-traditional assessment

From the very beginning, we felt that a new type of course also required a new type of assessment. The traditional way of assessment used to be a written test comprising problems, definitions and possibly theorems and students could only use a calculator. Students very often learnt the content of the course by heart and were only able to solve standard types of problems.

[3] In the following text, we will omit the word 'general'.
[4] All the calculations are done in *Maple*, however, due to the limited space we cannot illustrate everything.
[5] Prior to this general problem, students are asked to interpret one particular matrix geometrically, which makes the general considerations easier.

Prof. Hejny proposed a change of the form of the test such that now the students can use any aids they wish, including their notes from the course, textbooks, computers, etc. (but they must work independently). This form, however, puts greater demands on the tutor and the types of problems he/she has to prepare. They cannot be mere variations of problems solved during the course but on the other hand, students must be able to solve them using the knowledge and skills they acquired in the course. The first sets of problems were prepared by Prof. Hejny, later the author contributed problems too. Below there are three illustrations of problems from the test.

1. Let ABC be an isosceles triangle with the orthocentre O and the basis $|AB| = 4$. Let us denote $u = AC, v = BC, w = AB$. Let p be a line. We know that the following properties hold: $(s_u s_v)^3 = h_C, s_u s_p = s_p s_v, s_p(s_w(O)) = Q$. Find the distance $|OQ|$. Find all solutions.
2. Given a triangle *KLM* and points *N* (a midpoint of *L* and *M*), *O* (a midpoint of *K* and *M*), and *P* (a midpoint of *L* and *K*). An affine transformation *f* is given by *f*(*LPN*) = *OKP*. Express *f* as a composition of $f = tg$ where t is a translation and g is an oblique reflection (it is sufficient to find one solution). Find fixed lines of *f*.
3. Describe via matrices a group G generated by three reflections in lines $x - y = 1, x - y = -1, x + y = 2$.

In the first problem, the students have to use knowledge from synthetic geometry of the basic properties of isometries. They must know how to compose them and how to work with transformational equations. It is necessary to draw a picture. The analytic approach is counter-productive here, the calculations are far too complicated.

The second problem combines synthetic and analytic approaches. This requires a lot of experimenting. Students must know how to find the object point for oblique reflection and translation. For the second part, they must introduce a co-ordinate system to be able to determine the matrix of *f* and find its fixed lines.

The third problem is best solved in an analytic way from the very beginning. Note that this task is not one, which asks students merely to verify that a certain structure is a group, but rather to generate a group, which includes certain objects.

We must stress that allowing the students to use any material during the test was at first[6] an inhibiting factor for them. Some of them thought that no studying was needed prior to the exam because they would be able to find the answers in their notes or in textbooks! This meant that they were very surprised by the problems, which they were asked to solve. They claimed "it is unfair because we did not do such problems in class". Only later, when they did study for the exam, solved problems given in class, etc. could they see that in order to solve the problems in the test, they just had put all the pieces of knowledge gained from the course together.

4.6 Connection with other approaches to geometrical transformations

The approach we have chosen for the course and the use of 3 x 3 matrices for transformations means that students cannot find answers to problems easily in other textbooks. Later in the course, they are encouraged to use other books as well and to see how other authors' approaches differ or are similar to the approach in the course. For instance, while in our course affine transformations are divided according to the number of fixed points and all considerations evolve from the idea of perspective affinities[7], in Gans (1969) the central concept is primitive transformations. The

[6] Later, they shared their experience with other students who subsequently did not underestimate the exam quite as much!

[7] It is because perspective affinities can be studied in a synthetic way relatively easily and every affine transformation can be decomposed into two perspective affinities.

comparison makes students aware that there is not a single 'ideal' approach to teaching mathematics and that different approaches have their advantages and drawbacks.

5. Conclusions

We have shown that it is possible even at the university level to teach some parts of curriculum in a constructivist way provided that we cut down on the content and stress a student's independent work. We are aware that it would be too time consuming and in some cases impossible to use this type of teaching in all subjects. However, we believe that it is worth doing at least in some courses and especially so in the preparation of future mathematics (and elementary) teachers.

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Appendix

$$\begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) & k\cos(\alpha)+u-u\cos(2\alpha)-v\sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) & k\sin(\alpha)+v+v\cos(2\alpha)-u\sin(2\alpha) \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 1

$$G := \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) & m \\ \sin(2\alpha) & -\cos(2\alpha) & n \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 2

- By comparing the two matrices we get the system of equations:

$m = u(1-\cos(2\alpha)) - v\sin(2\alpha) + k\cos(\alpha),\ n = v(1+\cos(2\alpha)) - u\sin(2\alpha) + k\sin(\alpha)$

- We will solve it in terms of v and k.

```
> solve({m=u*(1-cos(2*alpha))-v*sin(2*alpha)+k*cos(alpha),n=v*(1+cos(2*alpha))-
  u*sin(2*alpha)+k*sin(alpha)},{v,k});
```

$$\left\{k = n\sin(\alpha)+\cos(\alpha)m,\ v = \frac{1}{2}\,\frac{n\sin(\alpha)^2 - n - 2u\sin(\alpha)\cos(\alpha) + \sin(\alpha)\cos(\alpha)m}{\sin(\alpha)^2-1}\right\}$$

```
>map(combine,(solve({m=u*(1-cos(2*alpha))-*sin(2*alpha)+k*cos(alpha),n=v*(1+cos(2*alpha))-
  u*sin(2*alpha)+k*sin(alpha)},{v,k})));
```

$$\left\{k = n\sin(\alpha)+\cos(\alpha)m,\ v = \frac{n + n\cos(2\alpha) + 2u\sin(2\alpha) - m\sin(2\alpha)}{2+2\cos(2\alpha)}\right\}$$

- We can see immediately that number k equals the number d which is a determining factor for a matrix to be a matrix of a line reflection or glide reflection. It remains to be seen how the equation of an axis can be found.
- Remember that u, v are co-ordinates of any points on the line of reflection. Therefore if we write x instead of u and y instead of v in the above expression for v, we must get an equation of

$$\left\{y = \frac{1}{2}\,\frac{n\cos(\alpha)+2x\sin(\alpha)-\sin(\alpha)m}{\cos(\alpha)},\ k = n\sin(\alpha)+\cos(\alpha)m\right\}$$

the axis.

The equation of the line of reflection is: $x\sin\alpha - y\cos\alpha + \dfrac{n\cos\alpha - m\sin\alpha}{2} = 0.$

- The co-ordinates of the vector are: $\mathbf{u}[k\cos\alpha, k\sin\alpha]$, where $k = n\sin\alpha + m\cos\alpha$.

Figure 3

USING COMPUTERISED TESTS AS SUPPORT FOR TUTORIAL-TYPE LEARNING IN LOGIC

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ABSTRACT

The course "Introduction to Mathematical Logic" (32 h lectures + 32 h exercises + 56 h independent work) is compulsory at the Faculty of Mathematics and Informatics of the University of Tartu. In 1987-1991, we designed four programs for the exercises: Truth-Table Checker, Formula Manipulation Assistant, Proof Editor and Turing Interpreter. As result, we transferred 70% of the exercises to the computer class. The blackboard-based practical training was preserved for predicate logic. In the subsequent years we added one more program for truth-values of predicate formulas. The main benefit of the first round of computerization was the acquisition of real skills in two areas: Turing machine programming and proof construction. At the same time, rejection of blackboard exercises reduced the possibility of assigning to students small but important questions concerning new topics.

In 1998, we decided to use a test administration system APSTest to introduce tutorial-type support for lectures. After each 1-2 lectures, a test comprising 10-15 questions is available in the faculty network. The test has randomly selected questions and can be solved a number of times; before the beginning of the next lecture, however, the students shall have achieved at least a 75% success rate. The weekly duty to keep oneself up to date with theoretical material has reduced the dropout rate and improved the average grade results.

The paper describes:

- Test system APSTest and our test structure,
- Intent of questions for different types of lecture topics,
- The use of different types of questions,
- Eligibility of test-type questions for the different parts of the course,
- Some conclusions about the students' learning strategies drawn on the basis of the data saved in the database,
- Topics and questions proving to be more difficult,
- Changes made over time in test management.

KEYWORDS: Mathematical Logic, Computer Assisted Assessment

1. Introduction

For any course of lectures to be efficient, it is necessary that the students familiarize themselves with the material already covered by doing independent work. Written exercises used in teaching mathematical subjects do not satisfy the need fully, for they usually deal with technical problems requiring a longer solution. The computerization of exercises may even aggravate the problem, for the problems and questions that the existing software is incapable of addressing may be disregarded altogether. An introductory course of mathematical logic has some qualities that increase the need for tutorial-type work. The course introduces a large number of new but relatively simple concepts. Concepts, formulas, canonical forms, rules, etc. are created by interrelated groups. The elements of a group develop fairly similar relationships with one other as well as with other objects. Consequently, only a small part of them are analysed in lectures and textbooks while the remaining part is left to the students themselves to prove by analogy, or sometimes even to discover and invent on their own.

This article describes the use of computerised tests to support the weekly independent work required for the learning of the theoretical material contained in the introductory course of Mathematical Logic taught at the Faculty of Mathematics and Informatics of the University of Tartu. During a semester, the students independently take 10-12 tests in the computer class. The tests are generated from a databank currently containing approximately 500 questions. The system has been used for three years, which allows us to evaluate both the questions and their effect on the learning of the discipline.

Part 2 of the article gives an overview of the essence of the course as well as the problem-solving software created in previous years. Part 3 describes some features of the test administration package APSTest used in the work, and our organization of tests. Part 4 describes the questions used in the tests and examines the topics of the course they cover. Part 5 investigates, on the basis of the data stored in the database, students' working strategies in doing tests, including inappropriate behavioural patterns. Part 6 evaluates the current situation in the implementation of the system. In comparison with (Croft, Danson, Dawson & Ward , 2001), our experiment is more directed to knowledge and less to skills.

2. The course and computerised exercises

The course Introduction to Mathematical Logic has been on the curriculum of the Faculty of Mathematics and Informatics of the University of Tartu for some time already. The course is compulsory, and most of the students take it in the spring term of their second year. 70-90 students usually attend the course. The discipline has been planned to consist of 32 hours of lectures, 32 hours of workshops and 56 hours of independent work. The lecture themes of the course are presented in Table 1.

In 1987-1991, we designed four computer programs for doing exercises: Truth-Table Checker, Formula Manipulation Assistant, Proof Editor and Turing Interpreter (Prank 1991). The main purpose of the work was to create problem-solving environments for two difficult domains – Formal Proofs and Turing Machines. In addition, the two first mentioned programs were designed for computerising the main problem types contained in the first chapter of the course. As a result, we transferred 70% of the exercises to the computer class. The blackboard-based practical training was restricted to predicate logic.

Introduction to Mathematical Logic Lectures (16 × 2h)
I. Propositional Logic (Model theory) 1. Introduction. Sentences, truth-values, propositional connections, formulas, truth-tables. 2. Tautologies, satisfiability, logical equivalence. Main equivalences. 3. Expressibility by $\{\&, \neg\}$, $\{\vee, \neg\}$, $\{\supset, \neg\}$. Normal forms and their properties.
II. Predicate Logic (Model Theory) 4. Predicates, quantifiers. Validity of formulas for finite models, N, Z, R. 5. Signature, first-order language, interpretation, expressibility. 6. Tautologies, logical equivalence. Main equivalences. Prenex form. 7. Proofs of main equivalences.
III. Axiomatic Theories. 8. Introduction. Axioms and rules of propositional calculus. Examples of the proofs. 9. Consistence. Implications for proof-building. 10. Completeness of propositional calculus. 11. Predicate calculus. Examples of the proofs. Consistence. Completeness (without proof). 12. First-order axiomatic theories. Group theory. Formal arithmetic.
IV. Algorithm Theory 13. Introduction: Concrete algorithms and algorithm theory. Turing Machine. Computing numerical functions on TM. 14. Operations with TM (composition, branching). 15. Enumeration of TM. Halting problem. 16. Overview of decidability problems in mathematics and computer science.

Table 1. Themes of Lectures

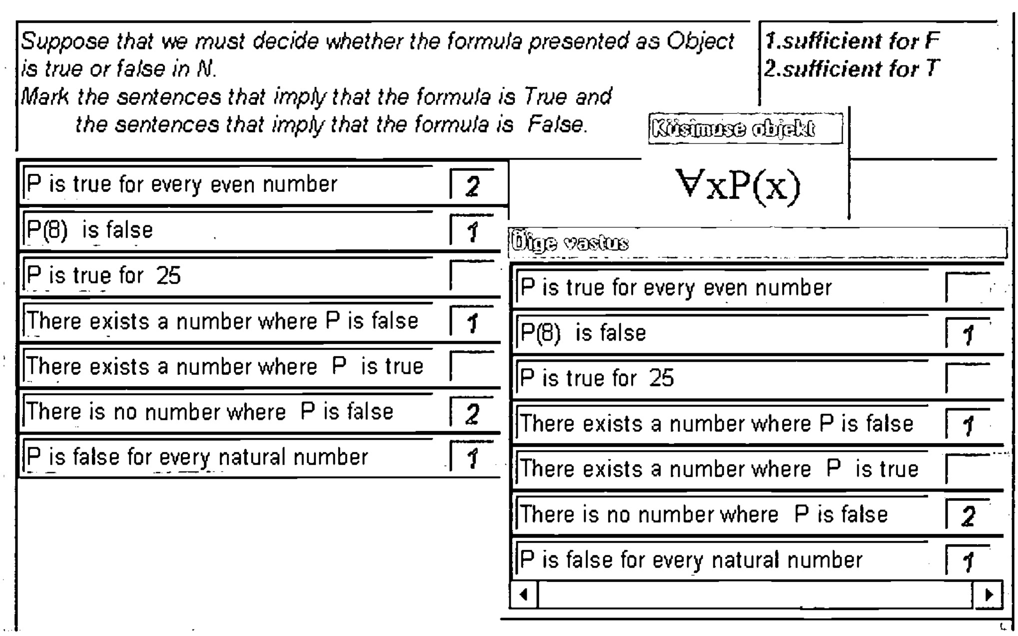

Figure 1. A class-assigning question with student's and correct answer.

In the subsequent years, we have made small improvements to our programs, and renewed the user interface. In addition, we added a new program for exercises on the interpretations of predicate formulas. The computerisation of exercises indeed contributed to the learning of the central concepts of the last two chapters, rendering the concepts less abstract through their practical use. At the same time, however, the students' interest in computer exercises tended to eclipse the deeper meaning of the discipline, for the learning of which the exercises were created in the first place. There arose a need to find a better balance between the lecture material and the exercises in students' work.

3. The test system APSTest and our test structure

In 1998, the test administration package APSTest was created as one of the software projects within the state-funded programme aimed at the computerization of Estonian schools. A characteristic feature of APSTest is the availability of a large number of question types:

1) Yes/no questions,
2) Multiple-choice questions (in a list or between the words of a text),
3) Marking questions (in a list or between the words of a text),
4) Matching questions,
5) Class assigning questions (Figure 1),
6) Grouping questions,
7) Sequencing questions,
8) Filling the blanks (with free input or using multiple-choice),
9) Short-answer questions,
10) Numerical questions.

The program enables to vary many characteristics of tests and to compile tests for different purposes. APSTest saves the following data for each try: the time (beginning and duration), the points scored and the success rate, the number of correct, nearly correct and incorrect answers, the questions asked, the time spent on each question and the answers given. Using queries, the teacher can then build tables concerning the data of interest to him. It allows the teacher to relatively simply draw conclusions about both the work done by a particular student and the level of skills mastered in different domains of the discipline, as well as to pass judgements on the level of questions and the general characteristics of the tests. APSTest runs under Windows. The data can be stored in different SQL-based database systems.

Following the launch of the APSTest package, we decided to test its applicability to providing tutorial-type support to lectures. Within the space of a few years, the following test structure has developed. After each 1-2 lectures, a test of 10-15 questions is put out. It can be solved in the computer classes of the Faculty. A test contains randomly selected questions, and it can be solved several times; before the beginning of the next lecture, however, the students shall have achieved at least a 75% success rate. The time for doing a weekly test is not limited. For finding answers, the students are advised to examine the lecture notes and the respective literature. Cooperation between students is also allowed. Thus, two or three students doing tests on adjacent computers while consulting with each other and studying their lecture notes is a regular sight in computer classes. The average grade for the tests accounts for 10% of the total grade for the course. At the end of a semester, a summary test is conducted on the entire material (approximately 30 questions), which also accounts for 10% of the total grade.

4. The questions

This part of the article describes the questions: for what purpose and for what topics of the course they were composed and in what form they were presented. In the two first chapters of the course, each lecture introduces no less than whole series of new concepts. Thereafter, a lecture usually deals with just a few characteristic cases for each issue; the rest of the material needs to be learned by the students themselves using analogy. The students need to learn and memorize tens of equivalences binding different logical operations and quantifiers as well as a number of derivation rules and proof strategies. The best method of mastering this knowledge is exercises where concepts, formulas and other things need to be compared with each other and applied. First steps in this work can be presented as test questions applicable to achieving different educational aims.

1. The definitions of new concepts formalize certain ideas about sentences, truth-values, proofs, algorithms, etc. Test questions can be used to make the students think about what we have actually postulated and what choices we have made. Let us give an example of a question about the concept of sentence:

Many hypotheses of unknown validity have been formulated in mathematics.
What is the mathematicians' attitude towards such sentences?

1) They do not have any truth- value; therefore they are not sentences.

2) They actually have a logical value, and the fact that we do not know it is not important.

3) They can be considered sentences in terms of Propositional Calculus if their logical value is established.

2. Questions concerning the exact wording of definitions, rules, etc. can be asked using multiple-choice blank filling. This is particularly appropriate when several similar concepts, equivalences, etc. are being studied.

3. After a concept (formula, rule) has been introduced, the students can use test questions for training its execution in direct (1-2-step) applications.

Which of the following figures can be the exponent of 3 in the Gödel number of command of the Turing Machine? 0/1/2/3/4/5

Figure 1 shows a quite difficult class-assigning question concerning the universal quantifier.

4. Some test questions are also applicable to comparing similar and interrelated concepts (formulas, equivalences) and finding relationships between them. In addition, they facilitate the distinguishing of valid principles from their invalid analogues:

Mark the pairs of equivalent formulas:

$\forall x(A(x)\&B(x)) \equiv \forall xA(x)\&\forall xB(x)$? $\quad \forall x(A(x)\vee B(x)) \equiv \forall xA(x)\vee\forall xB(x)$?

$\forall x(A(x)\supset B(x)) \equiv \forall xA(x)\supset\forall xB(x)$? $\quad \forall x(A(x)\sim B(x)) \equiv \forall xA(x)\sim\forall xB(x)$?

5. Some questions are also applicable to giving concrete examples of the relationships between mathematical logic and other branches of mathematics:

Mark the operations on the set of rational numbers that are applicable to interpreting some binary functional symbol f(x,y):

$x+y$, $x-y$, $x\cdot y$, $x:y$, x^y

6. Students with only a superficial acquaintance with a certain problem contained in the course (such as manipulation to normal form) know in general what operations need to be performed for solving the problem. Sequencing questions are applicable to inquiring them about the sequence of steps in an algorithm as well.

7. Matching questions are applicable to building formulas from the "prescribed material":

Express the formula $\forall xR(x,y)\supset P(y)$ *in prenex form, matching the necessary strings and symbols with numbers 1, 2, 3, ... and leaving the rest unpaired:*

$\forall x$, $\exists x$, $\forall y$, $\exists y$, $R(x,y)$, $P(y)$, $($ $)$, $\supset$

1, 2, 3, 4, 5, 6, 7

8. Watching the students solve proof problems led us to the idea of supporting proof building by test questions on the "local" problem. While building the proof tree from root to leaves, it is possible to apply some rules to the sequence under construction in such a way as to generate a sequence above the line that is not valid and therefore cannot be proved. Insofar as predicate logic is concerned, the program is unable to diagnose the errors, and if the student does not notice his error himself, he will try to solve an insoluble problem from that point on. To direct attention to the possible effects of the steps, we gathered material concerning the mistakes actually made in the solutions, and, after examining other possible proof tasks in predicate logic, added a large number of questions concerning analogous situations. One example is given in Figure 2.

Let us now examine the use of weekly tests on different parts of the course. The first two chapters of the course deal with the introduction of the languages of Propositional Logic and Predicate Logic, their interpretations, main equivalences, inferences, different canonical forms, etc., their simple applications and their relationships to different domains of mathematics. Accordingly, test-type questions are very suitable for achieving many educational aims on these themes. Therefore, abundant use of questions is made in teaching the material of the first six lectures. The bulk of the more voluminous exercises (problems based on truth-tables, formula manipulation and the logical value of predicate formulas in concrete interpretations) are solved in computer class during workshops or as independent work. Blackboard-based workshops are used for expressing propositions through formulas and doing exercises on inference and equivalence proof/disproof.

In the chapter on axiomatic theories, the number of questions to be tackled is much smaller. The rules of Grentzen-type Propositional Calculus and Predicate Calculus are introduced and argued. The bulk of the lecture time is spent on the proof of the properties of the systems. The building of formal proofs is virtually the only type of tasks solved in classrooms. We have special software for that. The first test is conducted on the material taught at the first lecture of the chapter and it covers the general concepts and rule properties. At the end of the chapter, two tests are solved, where problems of step selection described under Item 8 are supplemented with those dealing with the properties of quantifier rules and concrete first-order theories.

In a similar manner, the chapter on algorithm theory has been built around one central concept. In workshops, Turing machines are constructed for calculating various numerical functions using unary and binary codes. Two tests are solved, of which the first one is built primarily on the material presented in the introduction while the second deals with the specifics of the enumeration of Turing machines.

As concerns the types of the questions used, the current database of approximately 500 questions contains marking questions (42%) and class assigning questions (41%), along with multiple-choice and matching questions (both 5%), numerical and short-answer questions (both more than 2%), and all the remaining types (each less than 1%). Such a distribution of types is apparently attributable to the authors' intention to support theoretical learning and provide questions on comparison, evaluation and classification promoting the making of generalizations rather than just ask for the facts. In the opinion of the author of this paper, such a distribution also testifies to the functionality of the implementation of class assignment questions as a separate type in the test system.

We have to prove that $\exists x(A \supset B(x))$ and A imply $\exists x B(x)$. Evaluate the following rules/decisions as possible ways to make first step in this proof		*1.Reasonable first step* *2.Not applicable (syntactically* *3.Results in unprovable seque* *4.Useless (but may be correct)* *5.This is uncorrect rule*
To prove $\exists x P(x)$, we choose an appropriate object m and try to prove $P(m)$.	3	
We know that a formula $\exists x P(x)$ is true. Let m denotes such element that $P(m)$ is true.	1	
For to use that $P \supset Q$ holds, we prove that P is true. Then we can use Q.	2	
To prove $P \supset Q$, we add P to assumptions and try to prove Q	2	

Figure 2. A question on possible steps in the proof

Classify the pairs: the formulas are equivalent, one implies other or they are independent.			*1.right formula implies left* *2.equivalent* *3.left formula implies right* *4.neither implies other*
$\forall x P(x) \vee \forall x Q(x)$	$\forall x [P(x) \vee Q(x)]$	3	
$\exists x P(x) \vee \exists x Q(x)$	$\exists x [P(x) \& Q(x)]$	1	
$\forall x P(x) \& \forall x Q(x)$	$\forall x [P(x) \& Q(x)]$	2	
$\exists x P(x) \& \exists x Q(x)$	$\exists x [P(x) \& Q(x)]$	1	

Figure 3. Questions on the comparison of formulas proved to be difficult

5. Students' working strategies and results. Changes in the organization of tests

First of all, we must speak about a few problems that arose upon the launch of the test system. Technically, these concern exactly the free use of the test system where tests can be solved for an unlimited number of times.

The author composed the first weekly tests of exercises in such a manner that after giving an incorrect answer the student was able to press a button and see the correct answer. However, the students started to look up and write down the correct answers before solving the tests (at that time, the number of parallel questions was still fairly small). Thus, we had to disable the correct answer feature, and the students then need to work on their lecture notes in order to understand their errors.

Next, the students discovered that it is possible to run several copies of APSTest on the same computer at the same time. Running several "trial copies" alongside the "main copy", they were able to try several variants for answering a question, until the program acknowledged one of them as true. The programmers then had to improve the programme to deny them this opportunity.

Based on the analysis of the database, we have discovered different strategies used by students for taking tests allowing an unlimited number of tries.

1. The most noticeable were the students who set the goal of achieving a 100% or nearly a 100% success rate. Stronger students usually needed 1-3 tries for that. They carefully considered each question of each try, spending an average of 0.5-2 minutes per question. Quick reply was only given to relatively simple questions as well as those whose answer was known from a previous try. Occasional bulkier questions took them 3-4 minutes to complete. Some students solved a test 3-4 times even after they had already achieved the maximum score. Their intention was to obtain a better knowledge of the entire material by answering the different variants of the questions. Such an approach, which requires approximately one hour per week, might be considered optimal. For instance, at the spring semester of 2001, 31 students out of 89 had scored a 95% or a better success rate in at least 8 tests out of 10. More than one half of these students have a behavioural pattern similar to that described under this item. At all semesters, the number of students scoring a maximum result is smaller at the beginning and increases with subsequent tests; however, this is mainly due to the students taking more tries.

2. Some students looked through a test once or twice without answering any questions before actually doing it. By doing this, they found out which themes of the lecture were represented in the questions and which were not. After that, they read the lecture notes and then scored a try.

3. A small number of students (less than 10) only tried to solve each test once and made no attempts to score the maximum points. They only took another try if they did not succeed in hitting the required 75% mark. Of that number, the students who were stronger often scored a result that was quite close to the maximum.

4. For students who were weaker than average, it took 5-8 tries to achieve the desired result. Very few of them took a break to examine the lecture notes after an unsuccessful try. Usually, they started a new attempt immediately after they finished the previous one. Due to the large number of tries, they had already memorized some questions by the time they reached the last tries. The tables in the database show that they worked through the last tries mechanically, with the time spent on answering being less than half a minute per questions; in multiple-choice questions it was often just a couple of seconds.

5. There were also students who regularly took 15-20 tries per test. It took them at least three hours. Apparently, they tried to do the test without scrutinizing the theory, giving an incorrect answer to even rather simple questions. Often, such students broke off the test after a weak score from the first questions and started from the beginning again. After a while, they had memorized the correct answers to all the variants of the first questions yet a subsequent set of questions led them to new break offs. As a result, they more or less memorized the answers to the first themes, until it sufficed for achieving the 75% success rate. The last themes of the material, however, were practiced less than the previous ones. In most cases, these students displayed no change of strategy over the course of the semester.

6. Analysis of the current situation

The experience gleaned from the three academic years has shown that the tests provide support for students in learning the course. The students have rated the use of computer tests as positive in both informal conversations and formal questionnaires. The need to do tests requires periodical work on lecture notes, which, in turn, improves understanding at subsequent lectures. There has been a decrease in the number of students who drop out from the course towards the end of a semester for having done too little work during the first months. Furthermore, even the leading students of the course admit that they have plenty of food for brain racking in the tests.

From the teacher's perspective, the tests provide us with a means allowing us to secure, without much extra work, that the students are familiar with the material of the previous lecture before they start to learn a new one. On the other hand, it is a means that allows us to obtain feedback on the learning of both individual questions and comprehensive themes. The investigation of the answers of current tests has allowed us to add an item on the spot or explain a question that has remained obscure. A fairly efficient means of discovering the most difficult themes of the entire course is an analysis of incorrect answers found in the summary tests of about twenty most successful students. The most difficult theme of the tests was the set of questions added in the last year for evaluating the suitability of possible steps in concrete proof-building situations (see 4.8). Even better students made fairly many mistakes here, which did not disappear even in the summary test. Questions on the comparison of formulas in predicate logic also proved difficult (Figure 3). Besides the themes posing difficulties, the summary tests also reveal occasional weaker answers that point to certain gaps in lectures/study aids or to differences in the approach of different workshop groups.

The themes and the organization of tests have steadied; we have a bank of questions for generating a reasonable number of parallel variants. To discourage the mechanical solving of tests, the author is currently considering the imposition of a limit to the number of tries or the linking of the effective score to the number of the try. However, we do not consider taking very complex measures. Even in their current form, the tests provide students with enough opportunities and motivation for reasonable work.

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CORRELATION BETWEEN STUDENT PERFORMANCE IN LINEAR ALGEBRA AND CATEGORIES OF A TAXONOMY

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ABSTRACT

This paper concerns a study of the performance of students in a recent linear algebra examination. We investigated differences in performance in tasks requiring understanding of the concepts with those that required only the use of routine procedures and factual recall. Central to this study was the use of a taxonomy, based on Bloom's Taxonomy, for characterising assessment tasks, which we have described in previous publications. The full taxonomy has 8 categories, which fall into 3 broad groups. The first group (A) encompasses tasks which could be successfully done using a surface learning approach, while the other two (B and C) require a deeper learning approach for their successful completion. Tasks on the examination paper were put into one of the three groups and comparisons were made concerning the performance of individual students in each of these areas.

There are several interesting areas to investigate. The first is to identify those students whose performance in group A was markedly different to their performance on groups B and C. There is considerable disquiet amongst mathematics lecturers at tertiary level as to the routine algebraic skills of incoming students and of students studying mathematics at university (see for example the *ICMI Study into the Teaching and Learning of Mathematics at University Level*, 2001). There is a conjecture that students who have poor technical skills are not able to succeed in university mathematics. The contrapositive conjecture that good technical skills (such as algebraic dexterity) are necessary for success in university mathematics is often taken for granted. The taxonomy allows us to test this hypothesis as we can compare performance in group A tasks (routine) with performance in higher level B and C tasks.

We have also investigated whether or not the data supports any systematic effect of differences in sex or language background in the performance on the three groups.

The sample contained a large cohort of students with who had a home language other than English. We tested the hypothesis that such students would have difficulty with the conceptual aspects of the course, since these normally require greater language facility. This proved not to be the case.

1. Introduction

This paper investigates students' performance on an examination—and by extension their learning in the subject—from the point of view of a taxonomy of mathematical tasks. It examines various hypotheses about factors that may affect the nature and success of students learning.

Assessment is a central feature of teaching in formal institutions and can take a multitude of forms, fulfilling many functions, both intended and unintended. Ideally assessment should be linked closely with student learning. We look at a taxonomy for learning in mathematics (Smith *et al* 1996) that is related to that of Bloom (1956). It transforms the notion that learning is related to what we as educators do to students, to how students understand a specific learning domain, how they perceive their learning situation and how they respond to this perception within exam conditions.

We will particularly look at examinations because we believe that a major component of the final grade will continue to be contributed by examination of individual students. As Krantz (1999:57) says '*The principle device for determining grades is the examination*'. There are many reasons for this. Firstly, it is a practical, cost-effective way to assess large numbers of students. Secondly, examinations are seen by many as objective with no favouritism and providing equity, as all students are treated under the same conditions. Thirdly, examinations provide quality assurance and accountability, especially for administrators. Fourthly, examinations have a long historical precedent in mathematics and in educational areas where certification is involved. All of these reasons for maintaining exams focus on their format and administration.

Whether we focus on examinations or other forms of assessment, we can use a range of techniques to assess the nature and extent of student learning. Our decisions about just which forms of assessment we choose are likely to be affected by the particular learning context and by the type of learning outcome we wish to achieve. Essentially, good assessment processes:

- **Encourage meaningful learning** when tasks encourage understanding, integration and application.
- **Are valid** when tasks and criteria are clearly related to the learning objectives and when marks or grades genuinely reflect students' levels of achievement.
- **Are reliable** when markers have a shared understanding of what the criteria are and what they mean.
- **Are fair** if students know when and how they are going to be assessed, what is important and what standards are expected.
- **Are equitable** when they ensure that students are assessed on their learning in relation to the objectives.
- **Inform teachers about their students' learning** (see Brown *et al*, 1997, Brockbank & McGill, 1998 or Biggs, 1999 for greater discussion on the relations between assessment and learning).

With regard to the importance of assessment, Ramsden (1992) says that '*From our students' point of view, assessment always defines the actual curriculum. In the last analysis, that is where the curriculum resides for them, not in the lists of topics or objectives. Assessment sends messages about the standard and amount of work required, and what aspects of the syllabus are most important. Too much assessed work leads to superficial approaches; clear indications of priorities in what has to be learned, and why it has to be learned, provide fertile ground for deep approaches*' (p187).

It follows that students will look carefully at the range of assessment tasks—including examinations—that are involved in any course of study. In mathematics courses, students usually have access to previous examination papers and these very papers give a clear indication of the nature and extent of their course, and the sorts of things that they need to concentrate on in order to achieve high marks or grades in their courses.

2. Examinations

The nature of examinations themselves will change in both content and format. Online delivery with individualised questions will supplement paper-based and oral examinations providing a range of flexibility. The content will change with access to technology, which makes many routine skills less important. Employers are looking for cognitive and communication skills in graduates and this will be reflected in the questions asked in examinations. Students will be able to use a variety of tools in the examination, open-book and/or computer if appropriate. These changes take place in the context of changing classroom environments where higher order conceptions of learning are encouraged through the use of supporting student focused activities (Reid & Petocz, 2001). Assessment is a tool that can be used by students to develop the depth of their understanding of a topic, and also to demonstrate this depth to their teachers. Examinations have the same potential but often send a contrary message. This contrary message is generated by the weighting given to certain questions and thus to the relative importance given to them by students. Hence academics setting examinations need to consider the examination as part of the students' overall learning experience and accordingly need to focus the exam on issues and contexts that encourage a continuation of higher order conceptual thinking. It is important to remember that one quality of higher order conceptions of learning is that they are inclusive and integrated. This means that by encouraging higher order conceptions through class activities and assessments, we are also encouraging the use of routine activities within that context. Crawford *et al* (1994) show this clearly in their categories that describe student learning of mathematics. In their work on innovative examination questions, Smith *et al* (1996) and Ball *et al* (1998) show how the nature of the examination questions directs students toward demonstrating either their understanding of ideas or simply their ability to perform routine functions.

Our categories of mathematics learning, developed from Blooms' taxonomy, provide a schema through which we can evaluate the nature of examination questions in mathematics to ensure that there is a mix of questions that will enable students to show the quality of their learning at several levels.

3. Use of a taxonomy

We have been using a taxonomy (Table 1) to ensure that examinations contain a mix of questions to test skills and concepts. The taxonomy was developed due to our desire to encourage a deep approach to learning. Previous studies have shown that many students arrive at university with a surface approach to learning mathematics (Crawford *et al*, 1996) and that this affects their results at university. There are many ways to encourage a shift to deep learning, including assessment, learning experiences, teaching methods, and attitudinal changes. The taxonomy addresses the issue of assessment. It can be applied to all assessment tasks but in this paper it is specifically applied to examinations. The taxonomy has eight categories, falling into three main groups (Smith *et al*, 1996). Group A consists of tasks which students will have been given in lectures or will have practised extensively in tutorials. In group B tasks, students are required to apply their learning to new situations, or to present information in a new or different way. Group C encompasses the skills of justification, interpretation and evaluation.

Group A	Group B	Group C
Factual knowledge	Information transfer	Justifying and interpreting
Comprehension	Applications in new situations	Implication, conjectures and comparisons
Routine use of procedures		Evaluation

Table 1. MATH Taxonomy (after Bloom). Smith *et al*, 1996

In a previous study (Smith & Wood, 1998), when we looked at the contribution of group A to the total mark gained by the student, we found a significant difference between the performance of males and females. The contribution of group A to the total mark was greater for females, even though there was no significant difference between males and females on the total score. This finding was also investigated with the present data.

The categories of the taxonomy are context specific—proving a theorem when the proof has been emphasized in class is a group A task, while proving the same theorem *ab initio* is a group C task. The taxonomy encourages us to think more about our first attempts at constructing exercises. Whether we act consciously on this influence or simply make changes instinctively, it provides a useful check on whether we have "tested" all the skills, knowledge and abilities that we wish our students to demonstrate.

4. Construction of the examination

We have taken a typical examination of the subject Linear Algebra. This subject was neither taught nor assessed by any of the authors of this paper. The examination was a formal 3-hour university examination in June 2001 with students being able to use scientific calculators and no other aids. Eighty-five students completed the paper and we have data on their marks in all subsections. We also have data on their sex, language background and the number of years in Australia.

The examination consisted of 88 marks of group A tasks, 15 group B and 27 marks in group C, for a total of 130 marks. It is obvious from the weighting of the group A tasks that the lecturer considered that routine tasks were the most important aspects of the subject, or perhaps was setting the exam in a "traditional" way, without using a broad range of question types.

- An example of a group A task (routine procedure) on the paper is
  - (i) Find the eigenvalues and corresponding eigenvectors of the matrix
$$A=\begin{pmatrix}1 & 5\\ 5 & 1\end{pmatrix}.$$
  - (ii) Hence or otherwise, find the diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$
  - (iii) Calculate the spectral decomposition of A
  - (iv) Use the spectral decomposition to calculate the inverse of A

  In this task, the main requirement was for the student to reproduce work done in class.
- An example of a group B task on the paper is

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Explain how the LU decomposition of a matrix A is used to solve the system of linear equations $A\,\boldsymbol{x} = B$.

In this task, the student is required to transform their knowledge of a routine skill to the meta-knowledge of explaining the skill.

- An example of a group C task (justifying) on the paper is

  Let $T = \{v_1, v_2, \ldots, v_r\}$ be a linearly independent set of vectors and let A be an $n \times n$ matrix. Show that the set $T = \{Av_1, Av_2, \ldots, Av_r\}$ is also linearly independent.

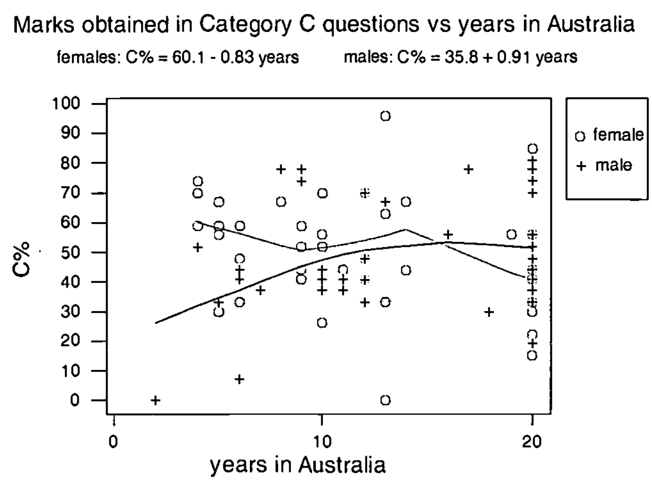
The examination was long, so none of the students completed the whole paper. So although students could have answered all sections, the length of the paper meant that in fact they could choose which sections to attempt. The majority of students started from the beginning and did not make full attempts at the later questions. This did not influence their results on the A, B and C tasks because they were distributed throughout the paper. It did influence the average mark for the examination.

5. Results

The correlations between the scores on group A, B and C tasks were significant and high (the correlations were 0.83, 0.67, 0.65) indicating that all components were measuring the same skill or that students were able to work equally across all groups. On average, students obtained 46% of the available marks for group A, 40% for group B and 49% for group C: the differences probably reflect the marking scheme rather than the difficulty of the questions.

We used a general linear model to investigate the differences between various groups of students in the marks they obtained for questions in group A, B and C. The models used sex, non-English speaking background, length of time in Australia and the particular course of enrolment as explanatory variables.

Figure 1. Marks obtained in group C questions vs years in Australia



The only statistically significant differences were due to interaction between sex and recent arrival in Australia on the marks achieved in "group C" questions. We investigated these differences first by categorising students into those who had arrived in Australia since 1989 (ie those who were likely not to have done the *whole* of their schooling in Australia). We then categorised students into those who had arrived in Australia since 1994 (ie those who were not likely to have done their *secondary* schooling in Australia). Finally, we used years in Australia as a covariate (making the assumption that the students born in Australia were 20 years old).

Looking at the students who had arrived since 1994, the males obtained significantly lower marks (mean C% = 27, p = 0.001) in group C questions than all the other groups—female recent arrivals and males and females who had been done their secondary schooling in Australia (mean C% = 55, 53 and 47 respectively). Looking at students who arrived since 1989, the pattern of results was similar although the differences did not quite reach statistical significance (p = 0.067). Using years in Australia as a (continuous) covariate, the sex-by-years interaction was significant (p = 0.014) and showed the same general picture: males who had not been long in Australia performed lower than other groups on group C questions.

With the exception of this one finding, no other variables or (two-way) interactions showed any significant effects on performance.

6. Conclusion

People who did well overall scored evenly on all groups. This need not have been the case, since the high proportion of group A tasks made it possible to reach high scores without doing particularly well on groups B and C. On the other hand, students who did badly had a mixed performance on the various groups. Two students performed very well in group B and C tasks but not in A. One of these students had a sick wife and, whilst he understood the work well, did not have time to practice the routine procedures. This unusual case shows that it is possible for students who do not perform well at routine procedures to demonstrate deep learning. In general, though, we find that the correlation between A% and the average of B and C% is a very high 0.83. Investigation of outliers may give interesting insights to learning.

There is considerable disquiet amongst mathematics lecturers at tertiary level as to the routine algebraic skills of incoming students and of students studying mathematics at university (see for example the *ICMI Study into the Teaching and Learning of Mathematics at University Level*, 2001). There is a conjecture that students who have poor technical skills are not able to succeed in university mathematics. The contrapositive conjecture that good technical skills (such as algebraic dexterity) are necessary for success in university mathematics is often taken for granted. The taxonomy allows us to test this hypothesis as we can compare performance in group A tasks (routine) with performance in higher level B and C tasks. We have shown in isolated cases that it is possible for students to do well in groups B and C and not in group A. It would be interesting to investigate this further. Clearly a base level of algebraic dexterity is necessary but what is that base?

In retrospect, the examination that was analysed was not ideal in that the questions contained a strong emphasis on routine skills. We suspect that the length of the examination benefited those students who had memorised material and who had practiced techniques. The finding in our previous study (Smith and Wood, 1998) that females scored a higher percentage of their total mark on group A tasks was not replicated. In the present study the same pattern was evident, but was not statistically significant. More work along these lines would be interesting.

Without setting out to test this particular idea, we found that male students who had recently arrived in Australia (but not female recent arrivals) scored significantly lower on group C questions. We are not sure what this suggests. It can't be simply be due to language, or the females would show the same pattern. We need to investigate this further by interviews with these students and consider teaching interventions to improve their performance.

The hypothesis that non-English speaking background students had "difficulty with the conceptual aspects of the course" was investigated. The variable showing language background was not significant in any model, singly or in interaction with any other variable. In fact, both groups scored an average of 49% on the C questions.

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"INTERDISCIPLINARY PROJECTS IN THE ARTS AND SCIENCE PROGRAMME AT MCMASTER UNIVERSITY"

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ABSTRACT

Interdisciplinary component of the Mathematics course taught in the Arts and Science Programme at McMaster University is implemented in several ways, varying in depth, width and level of involvement of other courses. Of a number of issues related to the course, this paper uses instructors' experience and examples of students' writing to discuss the features of the narrative in mathematics. Used as a vehicle to enhance understanding of mathematics and to build and improve research and communication skills, good writing is a key to a successful and productive interdisciplinary mathematics course.

1. Introduction

Arts and Science Programme at McMaster University[1] is an interdisciplinary program that offers students an opportunity to use their university years to further their intellectual growth through a study of significant achievements in both arts and sciences. The main goal of the program is to give students an understanding of sciences, arts, and technology, to help them develop skills in communication, in qualitative and quantitative reasoning, and to help them become critical and independent thinkers.

Acquiring valuable skills as undergraduate students, Arts and Science graduates have always been in demand; even today, in what is perceived as a 'digital and high-technology economy.' The following is an excerpt from the statement, signed by leaders of Canadian high-technology corporations, underscoring the importance of liberal arts education[2]:

> "A liberal arts and science education nurtures skills and talents increasingly valued by modern corporations. Our companies function in a state of constant flux. To prosper we need creative thinkers at all levels of the enterprise who are comfortable dealing with decisions in the bigger context. They must be able to communicate - to reason, create, write and speak - for shared purposes: for hiring, training, managing, marketing, and policy-making. In short, they provide leadership."

Mathematics has always played and important role in the Arts and Science Programme curriculum. The mission of the Programme (as outlined in the opening paragraph) creates an ideal environment for learning mathematics the way it should be learnt. The Arts & Science Mathematics course (Mathematics course, for short) exposes students to all aspects of mathematics, from its 'rigid' and 'abstract' sides (axioms, theorems and definitions) to its applied (modeling) and 'non-mathematical' sides (history, ethnomathematics). This two-semester course, taught in the first year, reveals mathematics at its foundations, presents its theoretic aspects and investigates its meaning and purpose in social and cultural contexts. Although its 'backbone' is differential and integral calculus, the scope of topics discussed in the course is much broader. The skills that the students develop in the course (formation of precise mathematical and logical arguments, written and oral communication, research, problem-solving and critical thinking skills) are the skills that are not needed just in the 'digital and high-technology economy,' but rather in any area of human endeavour.

In this paper, I plan to describe aspects of the Mathematics course that are related to the interdisciplinary mission of the Programme. The second part is devoted to a discussion of the use of writing in the course, in the context of knowledge construction and acquisition, criticality and depth in approach, and originality and creativity in thought and presentation.

2. Arts and Science Mathematics as an Interdisciplinary Course

Appropriate examples, problems, and ideas selected from other disciplines motivate students and stimulate their interest in mathematics content. The main purposes of an interdisciplinary approach are

[1] McMaster University is a medium-sized, full service university located in Hamilton, Ontario, Canada

[2] from: 'Hi-tech CEOs Say Value of Liberal Arts is Increasing,' http://www.trentu.ca/news/ceo.html

to deepen students' knowledge and understanding of major ideas and concepts in mathematics, and to develop their research, communication and critical thinking skills. Well designed interdisciplinary projects will enable students to place mathematics into historic, cultural and societal contexts.

Interdisciplinary approach needs to elevate learning to a new level, by providing something new in all disciplines involved - something that otherwise would not be present.

In the Arts and Science Mathematics course interdisciplinary approach is implemented in several ways - varying in depth, width and level of involvement of other courses:

- First and foremost, the Mathematics course itself is interdisciplinary in nature.
- "Cultural Meaning of Mathematics or Science" is a project that links Mathematics with the course on formal logic and writing (via team-teaching).
- "Standard" links with the statistics and physics courses in the Programme have been established.
- "Science Inquiry" course - under construction at the moment - will use the 'powers' of mathematics, physics and chemistry to investigate questions in biology.
- Interdisciplinary themes (such as "Symmetry," "Knowledge and Popular Culture," "Infinity," or "Construction of Reality") link courses across several disciplines and across all levels (years one to four) in the Programme.

I will describe the first two models in some detail, and then say a few words about the remaining ones.

The Mathematics course itself is interdisciplinary in nature.

Besides introducing new material and establishing connections with the previously taught mathematics material, lectures in the course are used to broaden students' viewpoint and understanding - by presenting historic and cultural aspects of the development of mathematics and by discussing related topics. An example: construction and definition of the definite integral is motivated by a real-life, 'applied' problem - how to compute the area of a plot of land (Ancient Egyptians paid taxes based on the amount of land they owned). The amount of material needed to construct a temple, or a pyramid, was based on calculations of volume. In lectures, students are shown how ancient (and very often intuitive) methods of computation of areas and volumes got formalized in the framework of 19th century calculus.

Number theory is probably one of the most fascinating fields of mathematics - and, quite possibly, one that is among the easiest to discuss on an elementary level (to a certain degree, of course). Yet unproven conjecture stating that every even number greater than two can be expressed as a sum of two prime numbers - so-called Goldbach conjecture - uses mathematics concepts that are understood by an average high school student. In my Mathematics class, Goldbach conjecture is used in a two-fold way: on the one hand, it illustrates the difference between a theorem and a conjecture. Students are asked to articulate what would be needed to prove Goldbach conjecture, and also what would be needed to disprove it. Creation of the conjecture itself mimics a process of creating mathematics. By 'playing' with numbers, we are actually performing an investigation - conducting an equivalent of an experiment in chemistry or physics. Sooner or later - hopefully - we start noticing a pattern (e.g., even and odd numbers behave differently when we try to express them as a sum of two prime numbers). Based on the pattern, we try to formulate a conjecture (that is not a theorem unless we prove it).

Writing about mathematics is one of the best ways of learning mathematics. Only when we are able to clearly and unambiguously communicate an idea, or a result of a computation to somebody else (and answer their questions about it), we can claim that we have learnt and understood. An example: it is a matter of technical expertise to compute a horizontal asymptote. But how does one explain the idea to somebody who has not heard of limits? In their written answer to "what is a horizontal asymptote," my students are not allowed to say "as x approaches infinity;" instead, they are expected to explain in words how "values of a given function f(x) can be made arbitrarily close to some number by taking x large enough." Then, they must further elaborate on statements "can be made arbitrarily close" and "large enough." Finally (now thinking of talking to a mathematically sophisticated audience) they are asked to return to mathematics, and to translate their English statements into mathematics symbols and formulas. Early in their narrative students are encouraged to identify examples of horizontal asymptotes in 'real life' (or argue why they cannot find any) but that is by no means the only goal of the exercise.

"Cultural Meaning of Mathematics or Science"

In order to investigate and discuss mathematics in contexts of society, history and culture, Arts and Science Mathematics course requires that students complete a project, tentatively called "Cultural Meaning of Mathematics or Science." The aim of the project is to investigate one mathematical (or scientific) issue and to explore the cultural significance of it. To start, students are asked to formulate a question within the given categories. The categories are quite broad: assess popular myths about mathematics (science) or competing histories of the origins and/or models of the development of mathematics (science); assess mathematics (science) as an authoritative and powerful institution controlling knowledge production; is mathematics (science) value-free; consider gender, class, race, non-Western approaches and contributions, etc. This project is done jointly (team-teaching) with the course on writing and formal logic, and the final essay and oral presentation are the parts of the requirements for both courses.

After receiving a feedback from instructors and teaching assistants, students revisit their question, reformulate it or narrow it down if necessary. They must identify a reference that they will use (could be several pages, or a chapter from a book, or a newspaper article)[3], and then write a critique of it. Their work should not be merely a summary, or an apology, or celebration of science or mathematics. Rather, it should interrogate and assess the role of mathematics or science in relationship to society.

[3] Several references are listed here, to show the variety of students' interests and the topics they investigated:
* Ascher, Marcia, Code of the quipu: a study in media, mathematics, and culture. Ann Arbor: University of Michigan Press, 1981 (cultural history and sociological aspects of scientific discovery)
* Golinski, Jan. Making Natural Knowledge: Constructivism and the History of Science. 1998 (study of the recent histories of science and their connections to culture)
* J.A Paulos, A Mathematician Reads the Newspaper. New York: Anchor Books, 1995 (use and abuse of mathematics and mathematical reasoning in media)
* LaTour, B. and Woolgar, S., Laboratory Life: The Social Construction of Scientific Facts. 1979 (classic in sociology of science)
* G.H. Hardy, A Mathematician's Apology. Cambridge: University Press, 1940 (why mathematics - by one of the most famous 20th century mathematicians)
* Menninger, K. Number Worlds and Number Systems. New York: Dover, 1969 (cultural history of numbers)
* E. Rothstein, Emblems of Mind. New York: Evon Books, 1995 (among other topics, explores the relation between music and mathematics)
* Henrion, Claudia. Women in Mathematics: the Addition of Difference. 1997 (profiles of professional mathematicians)

The question that a student formulated helps her/him focus on one issue. The final part of the project consists of oral presentations, followed by a question-and-answer period and a discussion.

Other Interdisciplinary Models in the Arts and Science Programme

It is very easy to identify topics that are common to mathematics, probability and statistics, and physics (these three courses form a major part of the core of the science curriculum in the Programme). For example, concept of the area introduced in calculus is revisited in the sections on continuous probability distributions in the statistics course; data from physics experiments is analyzed using statistical methods, etc.

Instructors for the three courses hold regular meetings. Organized initially to adjust and synchronize the syllabi of the courses, the meetings provide a forum for discussions on a variety of topics related to teaching science.

The "Science Inquiry" course - presently under construction - will use the 'powers' of mathematics, physics and chemistry to investigate questions in biology. Students will be assigned to work (in small groups) on a project in biology that will use at least one of mathematics, physics or chemistry in a significant way. The final product - depending on the level of involvement and depth of investigation - will be an essay, a final course report or an undergraduate thesis. In any case, it will be a narrative piece.

Interdisciplinary themes (such as "Symmetry," "Knowledge and Popular Culture," "Infinity, " or "Construction of Reality") link courses across several disciplines and across all levels (years one to four). Last year's theme, called "Bodies of Knowledge," involved students, faculty and guest speakers from several departments within the University. Unlike other interdisciplinary projects in the Programme, this one is not a part of a specific course, and students do not get a credit for participating in it.

3. Writing in Mathematics and Writing About Mathematics

The fact that writing in mathematics - and writing about mathematics - are good for learning mathematics can be taken as an axiom; or, in the least, it is an easily provable theorem (as shown by a significant body of literature in mathematics education). Writing helps students learn mathematics better and teaches them how to communicate effectively their ideas to others. Students' writing assignments represent a valuable resource for the teachers: among the many benefits, they could reveal a nature of students' conceptual misunderstandings and problems.

If we expect our students to write about mathematics, we need to teach them how to do it first. Moreover, if the project they are involved with is interdisciplinary, we should clearly state the expectations in terms of each discipline involved.

Stephen King said that " ... the only way to learn how to write is to read a lot and to write a lot."[4] The same is true if we replace 'write' by 'write mathematics' and 'read' by 'read mathematics.' My experience tells me that one of the most efficient ways of teaching how to write mathematics is to analyze samples of good and bad mathematics writing - both of which are easy to find, especially the latter. I usually use books on popular mathematics and my students' old essays.

[4] 'On Writing: a Memoir of the Craft.' Scribner, 2000.

The most important aspects of mathematics writing include knowledge construction and acquisition, criticality and depth in approach, and originality and creativity in thought and presentation.

What does one write about in mathematics? Most common approach for an elementary interdisciplinary topic is to try to address a 'real-world' situation (such as building the most optimal box, or using exponential growth to model a population, etc. - mathematics textbooks are full of those). However, one must be a bit critical about it. Whose 'real-world' is it that is being investigated? Almost every calculus textbook has a story problem about a person on a ladder. The bottom end of the ladder is sliding away, so the unlucky person on the ladder is falling down. The problem usually asks to compute how fast is the top of the ladder falling. Is that a 'real-world' problem? To whom is it really relevant? Does it present a good opportunity for a (short) narrative in math?

Linked to political goals of 'accountability' in universities, investigation of 'real-world' problems is promoted as a tool that will motivate students, and provide them with better understanding of mathematics. This is true, but only in some cases. Working on 'real-life' problems requires an appropriate level of mathematical sophistication - it cannot be done too early, when students are still struggling with technical intricacies and basics of mathematics concepts (can one appreciate reading a poem without being able to recognize all letters?).

The calculus textbook that we have been using in Arts & Science Programme[5] contains a primitive model of a blood flow, but does not provide sufficient clues as to how the formula has been arrived at. About all a student can do is to answer the questions from the book - which are mostly of technical nature. Actual models of (blood) flow are far too sophisticated for a first-year calculus student.

Without understanding background mathematics, investigation of a 'real-life' problem - unfortunately, in many cases - reduces to repetition of material presented in the text and memorization without much understanding or sophistication.

Let us consider an example, taken from a student's essay:

> "... take for example Edward Lorenz's discovery of the butterfly effect ... an assumption was widely held [in science] that the rounding of numbers would have little effect on the final answer of a calculation, because the rounded values would cancel each other out. Lorenz proved that, by rounding, a discrepancy in value would compound itself until the final value was completely incorrect ... this discovery went against a basic scientific concept, but still proved to be valuable, as it underlies the unpredictability and consistency of weather."

On top of obvious problems - such as not explaining why is the phenomenon under investigation is called the butterfly effect, and what is meant by 'unpredictability and consistency of weather' - the student missed to mention a crucial fact: the described type of behaviour characterizes non-linear systems, and does not occur in linear systems.

A common misconception among students (and not only among students) is that any use of mathematics objects is mathematics; e.g., an essay on the appearance of number seven in the Bible is mathematics; or, the existence of half-tones and quarter-tones in jazz shows that jazz is somehow linked to mathematics. Likewise, talking about three-dimensional objects (say, in architecture) is not geometry – it will become geometry, if one proceeds by asking good and 'provocative' questions ... how do the shapes of the buildings fit together, how do they relate to each other? What mathematics

[5] James Stewart, 'Calculus: Concepts and Contexts.' Brooks/Cole, 2001.

functions would best describe their shape? Finally, when the investigation is finished, is it possible to use the experience and knowledge in a different context - say, to visualize regions used in double or triple integration?

How do I explain to my students what it means to be critical? There are no convenient definitions or recipes - I start by discussing examples. A student once wrote that

> " ... it has recently been proved that the number of prime numbers is infinite."

What is the meaning of the word 'recently' in this sentence? How long ago was it - hundred, thousand, two thousand years ago? Another student wrote:

> "... parabola is another form that appears repeatedly in nature; the curve is created by gravity, and can be observed in the pathway of a flying stone, spear or arrow, water drops of a fountain or cascade [...] a rainbow takes a similar shape."

What does it mean that 'a rainbow takes a similar shape?' Actually this is an excellent opportunity for an investigation - exactly what is the shape of the rainbow? Flying stone is affected by gravity - does the same principle work for the rainbow? Consider one more example:

> "... Greeks saw the structure of math as beautiful ... Greek math avoided the irrational number because they did not believe that such a thing existed. The concepts of numbers and theories were described as being good. Evil presided with the unknown, the irrational numbers and theories. The irrational number is ugly and frightening ..."

Good start, but a math essay should not end here. Investigating why was an irrational number 'ugly and frightening' should lead a student towards a concept of a rational number - that, in turn, can lead to more sophisticated math topics, such as Diophantine equations.

Mathematics writing requires a good degree of originality and creativity. How does one explain a difficult mathematics concept (or a formula, or a computation) to a layperson? One has to simplify the content, without placing the integrity of mathematics in jeopardy. A creative approach will look at experiences of the expected audience, and try to incorporate it into the presentation. A good story is one of the elements I am looking for in my students' essays.

For example, an interdisciplinary project could start by exploring symmetry, say, in the work of M.C. Escher. Then, it proceeds towards investigating mathematical foundations of symmetry, such as concepts of rigid motions and groups. The works of Escher can now be revisited and described in the newly acquired mathematical framework. A final step could aim at identifying symmetric objects in calculus, algebra, or differential equations.

4. Conclusion

Interdisciplinary component of the Mathematics course taught in the Arts and Science Programme at McMaster University is implemented in several ways, varying in depth, width and level of involvement of other courses. Of a number of issues related to the course, this paper uses instructors' experience and examples of students' writing to discuss the features of the narrative in mathematics. Used as a vehicle to enhance understanding of mathematics and to build and improve research and communication skills, good writing is a key to a successful and productive interdisciplinary mathematics course.

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UNDERSTANDING EPISTEMOLOGICAL DIVERSITIES IN MATHEMATICS CLASS

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ABSTRACT

Recently, the notion of community has been increasingly popular in theoretical discourse of mathematics education and become a basic unit for analysis of classroom interaction. In this context, as part of ethnography in university level mathematics classes in the US investigating social transformation in mathematics education, this paper intends:

(1) to examine the notion of "mathematics classroom as community" as a place of learning and

(2) to identify some educational implications for teaching mathematics.

The data were collected through classroom observation and interviews. The analysis focused on comparing notions of mathematics shared among different groups of mathematicians, i.e., novices and old-timers. Through the comparison, I found that there are not only differences but also similarities in their understanding of what mathematics is and that they are intricately related to one another to constitute a practice of mathematics as a whole. Such complexity leads to a conclusion that mathematics class as a community is neither closed nor self-contained. It is interacting with outside communities. Each participant in mathematics class is representative of a community that s/he is committed to and his/her way of thinking mathematically reflects the epistemological standpoint of the community. This suggests that mathematics class is a community where diverse communal epistemological standpoints are renegotiated and that it creates its own unique mathematical culture through the negotiation among its participants. From this point of view, the understanding of the epistemological diversity is important for mathematics teachers to support successful learning.

KEY WORDS: Mathematics Class, Community of Practice, Ethnography, Indigenous Epistemology, Epistemological Diversity

1. Introduction

Recently, the notion of community has been increasingly popular in theoretical discourse of mathematics education and become a basic unit for analysis of classroom interaction to provide deep insight into teaching and learning mathematics in school (Cobb & Bauersfeld, 1995; Lampert & Blunk, 1998; Voigt, Seeger, & Waschescio, 1998). Compared to traditional educational research regarding mathematics class as a semi-laboratory where value-neutral skills are transmitted from a teacher to students in a vacuum of meaning, sociocultural approaches to mathematics education have revealed ways of speaking, seeing, thinking mathematics particular to school mathematics class. In the perspective, mathematics class is a community where participants negotiate their mathematical meanings and ways of doing mathematics to create its won mathematical culture through daily practice of mathematics (Cobb & Bauersfeld, 1995; Cobb, Wood, & Yackel, 1996; Ju, 2000; 2001; Lampert & Blunk, 1998; Voigt, 1985; Voigt, Seeger, & Waschescio, 1998).

In this context, this paper is to present the result of a comparative analysis of understanding of "what is mathematics" between different groups of mathematicians[1], that is, old-timers and novice mathematicians in a university mathematics department in order to reveal the intricateness of the notion "a community of practice" as a place for teaching and learning. Some educational implications for teaching mathematics will be presented based on the findings from the analysis.

2. Research Setting: Mathematics Classes in a University

This is part of the ethnographic research in university level mathematics classes in the US during 1998-1999 academic year. In the research, the author had collaborated with an experience mathematics teacher who had taught mathematics at the university for nearly thirty years. The data were collected through participatory observation of the professor's mathematics classes at three different levels: an introductory calculus class, an advanced undergraduate mathematics class, and a graduate mathematics class. Some sessions were video-recorded for further discourse analysis. In addition to classroom observation, forty people were interviewed. The purpose of interview was to learn about notions of mathematics shared in the mathematics department. The interviewees were selected to reflect the diversities in cultural backgrounds and in the level of mathematical expertise in the mathematics department. The interviews were audiotaped for later detailed analysis.

As mentioned, the analysis of this paper will focus on comparing notions of mathematics shared by two different groups of mathematicians, i.e., novices and old-timers.[2] In the

[1] In interview, some interviewees brought up the point that mathematics is "what they do daily". Based on that, the term "a mathematician" will be used rather inclusively, that is, people "doing" mathematics instead of referring to a professional with a degree. However, it is important to note that the term always carried cultural connotation of legitimacy, as a graduate student pointed out, "I have something new to offer, so a new insight, [to the community of mathematics]. Then I would consider myself a mathematician, a research mathematician."

[2] Although the mathematicians in the department did not unanimously agree upon the idea of that time or the advance in status does exactly predict the development of some essential qualities of a mathematician such as creativity, it was often observed that "time" was taken as one of the most prevalent dimensions in defining a position of a mathematician in the department. Also, it was possible to distinguish kinds of practice of mathematics in terms of the length of engagement with mathematics.

analysis, language is taken as the unit of analysis based on the assumption that use of language reflects the cultural organization of lived experience in a society (Gumperz & Levinson, 1996; Hill & Mannheim, 1992; Hymes, 1974; Whorf, 1956):

> "Facets of cultural values and beliefs, social institutions and forms, roles and personalities, history and ecology of a community may have to be examined in their bearing on communicative events and patterns" (Hymes, 1974, p.4).

3. Different Kinds of Practice of Mathematics

It is well know that novices are more likely to regard mathematics as a product, that is, fixed body of skills independent of human beings, while old-timers think of mathematics as a process of problem solving. This kind of tendency was confirmed by the data collected in the mathematics department. The beginning mathematics students regarded mathematics as a logically structured fixed network of mathematical products such as mathematical laws and rules. On the contrary, the old-timers considered mathematics largely as a process, in other words, "what people do" and refused the notion of mathematics as a fixed structure of impersonal knowledge. For instance, in the advanced mathematics classes taught by the professor, there were several students who retook the classes. They already had taken the courses but with another lecturer, and revisited the course then. If mathematics were fixed, immutable, and impersonal knowledge, for what did they come back to learn "the same things" over again?"

> Interviewer: Why do you take the class?
>
> Interviewee: I already took the class last semester with another professor. But I know..I knew that the professor is teaching it again this spring. And I thought that because very professor has a different point of view..it is just like humanity. Everybody has a different point of view and different experience that they bring to any class. And I knew the professor has a great expertise in the subject. So I knew that he would invaluably have many insights and he would have wealth of experience to share with us. So although I was taking the class before, I want to sit in just to hear his point of view.

As mentioned, the old-timers of the mathematics department regard mathematics as their daily practice as a whole, that is, "what we do" rather a definite structure of mathematical propositions. In their practice, mathematicians are personally engaged with mathematics and the structure of mathematical knowledge is continually evolving through the practitioners' creative imagination. The evolution is deeply related to their personal mathematical experience which provides unique meaning to logical connections in a mathematical structure. In this regard, as the interviewee suggested in the above transcript, mathematics that the professor actually taught in his classes was significantly different from mathematics given in "a book" or in the official descriptions of the courses given in the General Catalog of the university. Basically, he covered the contents of the courses. But he did not simply regurgitate definitions and theorems as given in a book. The majority of class was allotted for interpretations in order to give a student a perspective on a mathematical product under discussion: for instance, what does a concept or a theorem tell, how is it connected to a

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Thus, the categories of "novices" and "old-timers" will be used for the purpose of the comparative analysis in this paper.

broader structure of mathematics" what is its implication for the future development of the subject, and so forth.

This kind of knowledge, that is, "a mathematical point of view", is rarely found in "a book". Rather, it manifests itself through daily practice of mathematics including mathematical communication among colleagues. In this regard, for old-timers, mathematics includes not only a set of final statements but also evolving intersubjective meaning. And it is this latter kind of mathematics which old-timers emphasize in their teaching and learning.

So far, the comparative analysis has highlighted differing understanding of mathematics by different groups of mathematicians. However, this does not suggest that their practice of mathematics follow an either-or scheme. Indeed, it is important to note that the dichotomy "a product vs. a process" provides only a reductive model to understand mathematical practice of each group. In general, it is considered that a deeper scrutiny into daily practice of mathematics will disclose a complicated picture behind the dichotomy and provide a more meaningful understanding of mathematics classroom as a place of learning, which following further analysis purports.

Although the old-timers considered mathematics as a process of developing a perspective on the world, they never underestimated the importance of the aspect of mathematics as a product in their practice of mathematics. For instance, in mathematics class, the old-timers taught specific definitions, theorems, algorithms, computing procedures, and so on. Advanced mathematics students tried to memorize definitions, theorems, algorithms, and mathematical proofs as beginning mathematics students did. The old-timers may be doing these kind of technical things for practical purposes such as preparing for exams. Thus, despite the differing understanding of what mathematics is, it turns out that practice of old-timers is also concerned with mathematics as a product in a certain way. Based on this similarity, the further analysis is to reveal difference in practice of mathematics at a more fundamental level by arguing that mathematicians apply different meanings, or more generally speaking, cultural epistemological standpoint, to their practice of mathematics. I will elaborate this idea by showing different meaning imposed on the shared interest in mathematics as a product.

As noted, although the old-timers values creativity over technical perfection in their practice of mathematics, technical development cannot be separated from developing mathematical creativity:

> "Somebody comes up with some ideas and the idea itself somehow brings some form already and another whole set of questions."

As the interviewee describes in the above, a mathematician begins his/her creative inquiry with a question based on a rather vague idea. Through creative mathematical investigation, the question evolves into "a form", that is, a mathematical product such as a theorem. In turn, the form initiates another mathematical inquiry to lead to another mathematical discovery, and so forth. This recursive process suggests that mathematical creativity is the origin of the formal objective mathematical proposition. In addition, the process implicitly assumes that formal mathematical knowledge is a language to encode mathematical creativity of a mathematician. In other words, mathematical creativity is firmly grounded on factual knowledgeability in mathematics.

Furthermore, the process does not happen in a vacuum of meaning but is shaped by the invisible hand, "the culture of the mathematics community":

"There are a number of possible combinations of axioms for example. It's infinite. And if we make some random deduction and publish a paper, that's silly...There is so to speak a sixth sense that tells you whether something is significant or not."

A mathematical discovery is usually the object of examination in the community of mathematics. The members of the mathematics community scrutinize its logical perfectness and meaningfulness of a mathematical discovery with respect to the current mathematical structure, and more importantly, its creativity and productivity for future development of mathematics, in other words, judging whether the research has something new to offer to the mathematics community. In this regard, mathematical products such as computational skills and theorems are the culmination of the cultural norm of doing mathematics in the sense that they have acquired its social status as a consequence of on-going social review based on the social norms. Thus, not only a step by step guideline to solve a particular problem, a mathematician learns communal voice behind technicality such as what it is concerned with, what is the idea behind it, what and how it suggests doing to further a mathematical idea, and how to put a creative idea in a culturally meaningful way.

Therefore, while techniques are usually a terminal point for a novice mathematician, they become departing point for a future practice of mathematics for old-timers. Furthermore, techniques are cultural language to materialize creative vision for future practice of mathematics as communicating cultural norms of mathematics community to a practitioner. Through the process of communication, a mathematician becomes socially transformed according to the cultural norms of mathematics community and the communication becomes intense as a mathematician grasps the spirit of communal practice through participation. Put differently, when old-timers deal with mathematics as a product, their practice follows the cultural norms of doing mathematics shared in the mathematics community. Therefore, this tells that it is the notion of cultural norms, more generally communal epistemological standpoint, which will be defined in the next, what produces the differences between mathematical practice of old-timers and of novices at the surface level.

4. Communal Epistemological Standpoints in Practice of Mathematics

Indigenous epistemology is concerned with both the theory of knowledge and theorizing knowledge, including the nature, sources, frameworks, and limits of knowledge sociohistorically developed in a cultural group. Specifically, indigenous epistemology refers to a cultural group's ways of thinking and of creating, reformulating, and theorizing about knowledge such as who can be a knower, what can be known, what constitutes knowledge, sources of evidence for constructing knowledge, what constitutes truth, how truth is to be verified, how evidence becomes truth, how valid inferences are to be drawn, the role of belief in evidence, and related issues via traditional discourses and media of communication, anchoring the truth of the discourse in culture (Gegeo & Watson-Gegeo, 2001). Since an epistemological system is socially constructed and informed through sociopolitical, economic, and historical context and processes, it is a community that is a primary epistemological agent and that provide a basis for theorizing knowledge (Alcoff & Potter, 1993; Gegeo & Watson-Gegeo, 2001).

In the above analysis, different ways of doing, specifically different understanding of what is mathematics has been compared. In particular, the analysis focused on the old-timers' practice of mathematics -- not only how to construct logical mathematical reasoning but their understanding of legitimate conduct of mathematics, in general --, and showed that their practice is constituted by sociocultural values and norms of doing mathematics, that is defined as an indigenous epistemology.

For last several decades, in diverse disciplines, research has shown that knowledge is socioculturally constructed and mathematics is not an exception (See Berger & Luckmann, 1966; Bloor, 1991; Joseph, 1994; Restivo, 1994). Based on that, the notion of "ethnomathematics" has been developed to contribute to the awareness of sociocultural aspect in mathematical reasoning, especially "culture" in mathematics learning (Ascher & Ascher, 1997; D'Ambrosio, 1985; Powell & Frankenstein, 1997). However, it is necessary to point out that early sociocultural studies were based on a superficial interpretation of "culture" as a definite repertoire of behavioral patterns and as a result, theory of multicultural education tended to reductively treat culture as "colorful customs of other people" (Watson-Gegeo, 2001). In this regard, the notion of indigenous epistemology make it possible to investigate culture of mathematics classroom at a deeper level to provide stronger theoretical perspectives to improve teaching and learning mathematics.

For the purpose, I consider that it is important to extend the notion of indigenous epistemology argument in order to explain the mathematical practice of novice mathematicians? Put differently, novices as well as old-timers should be considered as communal being instead of as isolated atomic individuals but possessing communal epistemological point of view different from that presented by old-timers. In their practice of mathematics, novices apply the indigenous epistemological standpoints of the community that they have been committed to rather than that of the mathematics community in which they just begin to participate.

For example, in daily conversation with students in introductory mathematics classes, it was often observed that novices' discourse about mathematics was organized around the notion of "economy" such as "time management" -- e.g., spend "less" time to get "more" grades --, "exchange" -- e.g., need A to apply to medical school --, "possession" -- "I know everything about derivative" --, and so forth. These ways of speaking about mathematics reflect the epistemological position shared in communities outside of the mathematics community. Due to the lack of engagement, outsiders rarely have opportunity to develop a sophisticated understanding about what mathematics is than people who practice mathematics daily. In daily practice, mathematics keeps emerging every moment of engagement by a mathematician. However, mathematics has a smooth and perfect outlook only at a distance. Moreover, most often, mathematics is presented as completed knowledge in school and at home by adults through their expectation. And in modern society, mathematics is regarded as knowledge with most potential for future production (Stehr, 1994; Popkewitz, 1991)

Therefore, it can be said that novice mathematicians' practice of mathematics is deeply situated within their understanding of mathematics shaped through their lived experience in the communities they have been socialized into such as home, high school, a capitalistic society. In this respect, it is interesting to point that this kind of pattern could be found among some students in the advanced mathematics class. In the advanced undergraduate mathematics class, there were several students who came from outside departments such as school of engineering. Compared to the students in the introductory mathematics class, they had participated in more mthematics classes and had a strong mathematical background. Despite the difference in mathematical expertise, they were similar to the beginning mathematics students in the sense that they evaluated the practice of mathematics in the class from the viewpoint of the community that they have been committed to. For instance, a Ph.D. student from the civil engineering department compared mathematics to civil engineering to criticize its technicality in the interview:

> "So for mathematicians, he is probably surprised if you cross the street to engineer department and try to learn something about continuum mechanics. They are very different. Because we are not so rigorous, we can do things faster for example. I think that it is very interesting. Like you say your mathematics is less rigorous and fast. But for example, the Bay Bridge, that's constructed by an engineer."

The interviewee came to the mathematics class because he thought that mathematics would provide a valuable insight into physical phenomena. However, he did not agree to the way of doing mathematics in the mathematics department because that does not match his communal epistemological norms concerning what is a valuable kind of knowledge. On the contrary, such kind of mathematical practice gives joy and meaning to people in the mathematics department, as a graduate student of the mathematics department says in the following transcript:

> "You're absolutely right. It is difficult. But at the same time, that's exactly what makes fun that when you finally do understand something. It is really wonderful. And very frequently it turns out to be quite beautiful, the answer. And then those are quite of motivation, I think."

As a mathematics major student describes in the above, the technicality and the abstractness of mathematics causes difficulty in their practice, but ultimately brings meanings. Old-timers have developed "enlightened eyes" to see the beauty of the mathematical structure they have created historically. But the meaning and the beauty cannot be grasped by people who do not share the epistemological norm of the mathematics community.

5. Over the Wall: Is a Difference a Sign of Deficit?

Sociocultural approaches to mathematical problem solving have revealed sociocultural nature of mathematical reasoning and the research findings have been related to development of new theoretical perspectives on how to improve teaching and learning in school mathematics classroom (See Lave,1988; Nunes & et als., 1993). However, cumulating sociocultural research findings imply that cultural influence must be much more fundamental. From the perspective, this study intended to investigate "deep culture" of mathematics class.[3]

For the purpose, this paper presented the result of a comparative analysis of practice of mathematics in a mathematics department, in particularly, focusing on shared notion of what mathematics is. Based on the comparison, this paper introduces the notion of indigenous epistemology and argues that mathematics class consists of diverse kinds of participants in the sense that they bring diverse kinds of communal epistemological standpoints to the class. Moreover, each participant practices mathematics according to an indigenous epistemological standpoint of a community that s/he has been committed to.

[3] By "deep culture", Watson-Gegeo refers to a deeper level of thinking and understanding, in other words, "below the surface level of behavior and the linguistic level of morphology and syntax, a deep set of propositions and images that shapes perception, information processing, and the assignment of values" (Watson-Gegeo, 2001, p. 10).

This suggests that a mathematics classroom as a practice community is neither closed nor self-contained. It is deeply related to outside communities in the sense that each participant represents the indigenous epistemological standpoint of the community that s/he is committed to. Through interaction, a student begins to grasp different ways of doing mathematics, different epistemological style and begins to change. When considering that epistemology is not restricted to cognition in a narrow sense, such change is fundamental. That is, it is negotiation of worldview of a learner. And through the change, the epistemological standpoint of the community also becomes transformed and in fact, it is the product of such historical contingency created by the interacting participants instead of an immutable transcendental given.

As refuting Eurocentrism in mathematics education, sociocultural studies of mathematics have provided theoretical basis for understanding difference to improve teaching and learning mathematics in school. In this aspect, it is important to point out that it is epistemological difference and confliction due to such difference that initiate learning and make the impact of learning more fundamental. And this provides a new perspective on difference, which has been seen as deficit in traditional mathematics classrooms (Voigt, 1998). When considering that a mathematics classroom is a practice community with a particular epistemological standpoint and that indigenous epistemology is much broader than a set of mathematical concepts and skills given in a curriculum, it can be said that misunderstanding of epistemological difference affects teaching and learning in an important way.

For instance, students in the introductory mathematics class evaluated mathematics as boring, repetitive, focusing on minor things, not creative, and so on. This kind of perception often resulted from their negative learning experience of mathematics and more interestingly, lack of resource that they can rely on. Most students wanted something creative in mathematics class. However, one can be creative in mathematical practice only when his/her practice is firmly grounded on the culture and history of the mathematics community. Beginning mathematics students had harder time to understand the significance of a mathematical theorem because of the lack of their knowledge about history of the mathematics community: For them, the theorem was singled out from the historical context and as a result became less meaningful.

It is difference in epistemological style that makes one feel "others" strange. However, it is confliction due to epistemological difference that initiates learning and makes the impact of learning more fundamental, that is, negotiation of worldview of a learner, when considering the broad meaning of indigenous epistemology. A student comes to mathematics class with a limited and often a differing kind of epistemological standpoint from that shared in the mathematics community. As s/he interacts with different kinds of mathematicians in class, particularly mathematics teachers who have already been socialized into the communal epistemology, s/he becomes to see the practice of mathematics from a different perspective, especially, the indigenous epistemology of the mathematics community and begins to change as a whole person. In this regard, mathematics education can be seen as a process in which a mathematics teacher, as an old-timer, invites a student into the world of a vast inheritance historically accumulated to experience a specific mode of thought and awareness and helps him/her get transformed according to the indigenous epistemology of the mathematics community. However, it is important to note that such process of transformation is neither unilateral nor passive. Specifically, when considering that a communal epistemology is a product of historical contingency, it can be said that it is only one of standpoints providing a vision for future. Thus, a mathematics teacher's support based on the awareness of such differences will be essential for successful learning in mathematics class. In this perspective, investigation of deep culture in mathematics class will provide a theoretical ground for the improvement.

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PERCEPTIONS OF DIFFICULTY

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ABSTRACT

Seventy students from a first semester calculus course ranked 8 mathematics tasks as to perceived difficulty before attempting these tasks and actual difficulty after completing the tasks. Students also completed two examinations, one based on facts and procedures and the other based on applications and concepts. The tasks were designed to fit into a taxonomy of mathematical skills.

We have found that students perceive questions to be difficult for a number of reasons. In general, questions requiring conceptual understanding are regarded as more difficult that those which require factual recall or the use of routine procedures. There was not a strong link between familiarity with the question type and ranking of difficulty. Students were sufficiently familiar with the some types of question to be able to perceive inherent difficulties, such as a complex differentiation.

We found that in five out of eight cases, students' perceptions of the difficulty did not change after they had done the task. In one case they found the question to be more difficult than expected and in two cases to be easier. It is not clear to us why students found one question to be more difficult than expected. It may be that some of the complexities (such as the use of the intermediate value theorem) were not immediately apparent. It is also significant that NESB students rated this question as easier than ESB students. This was the case both before and after attempting the question. Student comments are also presented.

1. Introduction

It is commonplace for students to speak of an assessment task as being "easy" or "hard". Frequently, these judgements are at odds with the perception of the person setting the task. Academics will often express astonishment at students' inability to answer "easy" questions. In this paper, we investigate students' perception of the difficulty of a carefully chosen set of questions, with the aim of identifying the type of questions that students perceive as easy or difficult. This may enable us to modify our teaching practices and empower students to attack "difficult" tasks with more confidence.

In this paper we will consider the following: What types of questions do students perceive as difficult and what do they perceive as easy? Do their perceptions change after they have done the tasks? Are their perceptions based on familiarity with the type of question? Are their perceptions based on the conceptual difficulty of the question? Is the language of the question important? Do students with a non-English speaking background (NESB) or male/female students perceive questions differently? Is there a difference in performance in examinations on different types of questions?

In a previous paper (Smith *et al*, 1996), we developed a taxonomy to classify assessment tasks ordered by the nature of the activity required to successfully complete each task, rather than in terms of difficulty. The taxonomy, listed in Appendix C, has eight separate categories and we investigated the links between students' perceptions of difficulty and the categories of the taxonomy.

In relation to their perceptions of difficulty, we examined students' performance on two separate examination papers. One of these was designed to test factual knowledge, comprehension and use of routine procedures (Group A in the *MATH* taxonomy, Appendix C). This examination was of two hours duration and students had no aids. The second examination was designed to test higher-level skills (Groups B and C in the taxonomy). Students had three hours to complete the examination and could use one handwritten A4 sheet of notes and calculator. We believe that it is important to analyse the link between perception and success in assessment. Are their perceptions of difficulty born out by examination results? Do students avoid questions that they perceive as difficult?

Previous studies have considered students performance on statistics examinations in relation to the complexity of language in the question (Smith *et al*, 1994). This study of 186 students showed that there was no correlation between performance on examinations and the linguistic complexity of the question as measured by lexical density. This was also demonstrated with a study of 660 first year calculus students (Craig, T, 2001).

Craig's (2001) thesis considers calculus word problems and students perceptions of the difficulty of the problem. She looks at the variables of concrete versus abstract and the types of representation in the problem. The important variables for the perception of difficulty were familiarity of the problem, the context and whether there was a visual representation. Smith *et al* (1994) concluded that the conceptual difficulty of the mathematics was the important variable in the students' performance.

In the present study we consider a series of tasks requiring differing conceptual skills the students' perceptions of their difficulty. We examine the students' perceptions before and after completing the task to identify any changes that may have occurred. We look at students' performance on the end-of-semester examinations to see whether learning has occurred and to find which variables may cause significant differences in performance.

2. Method

Sample. Seventy students from a cohort of 90 in a first semester university calculus course were included in the study. The survey was voluntary and students were asked to sign an ethics approval form to use their data. Those that did not give approval were not included. There were 31 female students, 37 male students and 2 for whom this information was missing. Twenty-nine students spoke English at home, 39 spoke a language other than English and there were 2 missing data points. There were 13 students who had been in Australia less than 5 years.

Survey. The survey consisted of two parts. Students were asked to read a set of 8 questions (see Appendix A) and rank them in order of difficulty. The questions were representative of the eight categories of our taxonomy, but were presented in no particular order. The students also rated each question for skills required, level of difficulty, clarity and previous experience in answering those types of questions. Students were then asked to attempt the questions and re-rank them in order of difficulty. There was opportunity for open-ended comments. The survey items are listed in appendix B.

Questions. The questions were sample examination questions that the students had not seen but were related to the material they were studying in class. They were chosen as examples that would fit into the categories of the *MATH* taxonomy (Smith *at al*, 1996, Ball *et al*, 1998).

Examinations. As described in the introduction, we studied student performance on two different types of examination paper. The results were analysed for significant differences in student performance due to sex, language background and length of time in Australia.

3. Results

Examinations. Firstly the results of the examinations were analysed to determine if there were any differences between groups of students and to analyse whether learning had occurred. Data on sex, home language background and years in Australia were available. There was high correlation between the two examination results (0.67) and most students achieved satisfactory results. We can conclude that the majority of students reached the objectives of the subject. The only significant difference between groups was for the students who had recently arrived in Australia. Their results on examination 1 (routine skills) were significantly higher than for students, who had been in Australia longer (mean 64 for recent arrivals, mean 44 for others, $p = 0.012$). On the second examination paper (conceptual skills) there was no significant difference between the groups (mean 56 for recent arrivals, mean 59 for others). The students whose home language was not English also did better on the routine skills but this was not significant ($p = 0.062$). There was no significant difference in the sex and language interaction, as had been noted in earlier studies (Smith & Wood, 1998).

Rankings. The rankings before and after were analyzed and compared with the *MATH* taxonomy order (Table 1). There was no *a priori* reason for the *MATH* taxonomy rankings to reflect difficulty, since this was not the rationale for its development. Rather, it was designed to reflect conceptual complexity. There is considerable agreement between the taxonomy categories and the ranking given by the students. The 3 Group C categories were in the 4 questions perceived to be most difficult, the 3

Group A categories were in the 4 questions perceived to be easiest, while the 2 Group B categories were in the 4 questions perceived to be in the middle range of difficulty.

The pre- and post-rankings were compared using paired t-tests. The significant changes in rankings were:

Question B: harder ($p = 0.001$)

Question C: easier ($p = 0.012$)

Question G: easier ($p = 0.008$)

Taxonomy ranking	Pre ranking	Mean	Post ranking	Mean
C	F	2.59	C	2.64 **
F	C	3.37	F	2.87
A	H	4.26	G	3.77 **
D	A	4.54	H	4.67
H	G	4.56	A	4.71
G	D	4.96	D	4.71
B	B	5.06	B	6.03 **
E	E	6.67	E	6.57

Table 1: Rankings of difficulty of questions before doing the question and after. Significant change indicated by**.

Questions F and C were considered easy before and after doing the questions. Questions B and E were considered the most difficult before and after. The other questions were of a similar ranking before doing the questions. Of the middle group, only G changed significantly in the post ranking.

To investigate the reasons why students chose the rankings, we asked whether the questions were clearly worded, whether they understood the questions and whether they had see that style of question before. In each of these areas, there were significant differences in the responses over the 8 questions.

There were significant differences between the questions as to students' familiarity with the type of question. For example, question B (mean 2.3 on 5-point scale) was considered a familiar question but was ranked as difficult. Question A (mean 1.5 on 5-point scale) was considered very familiar but was not ranked as very easy. Students were familiar enough with the type of question to perceive that the presence of square root would increase the algorithmic complexity. Question D was ranked in the middle for perceived degree of difficulty but students had not seen this type of question before (mean 3.3). Question E was the most unfamiliar question (mean 3.7) and ranked the most difficult. B, C, F, G, H were assessed as having similar familiarity but were ranked very differently. The mean scores are presented below (Table 2).

The language is very clear (5 -point scale, 1= very clear, 5= too hard to understand)

A	B	C	D	E	F	G	H
1.36	1.66	1.74	1.67	2.37	1.36	1.58	1.73

I understand the question (5-point scale, 1= I understand the question, 5= I do not understand the question)

A	B	C	D	E	F	G	H
1.32	1.94	1.78	1.54	2.41	1.30	1.58	1.69

Similar questions (5-point scale, 1= I have done similar questions before, 5= I have never done this type of question before)

A	B	C	D	E	F	G	H
1.46	2.35	2.66	3.30	3.74	2.25	2.70	2.35

Table 2: Mean scores for questionnaire analysis

Differences between students. The data were checked for differences between groups of students, in particular with regard to the variables sex, language background and years in Australia. There were no significant differences between male and female student for any rankings. There were significant differences between students who spoke languages other than English as their home language for question B (mean (NESB) = 5.19, mean (ESB) = 7.21, $p = 0.000$) and question D (mean (NESB) = 5.34, mean (ESB) = 3.83, $p = 0.001$). These differences persisted in the post rankings. When one looks at the question, it is not surprising that NESB students perceive question D to be difficult. It requires competence with English. It is not clear why there was such a significant difference between NESB and ESB students on question B but the simplicity of the language, that is very few words, may be the reason.

Open-ended comments. Students were invited to comment on their perceptions. They obviously enjoyed the task of ranking the questions and made some interesting comments. The comments are coded by the student number assigned as part of the anonymity provision of the ethics approval.

There were several students who generally underestimated the difficulty of the questions:

Reading questions may sometime seem easy but when you actually start to do them is when you start to see the difficulty. (12)

A question may look easy enough to do but applying all the information that you know to the question may be quite difficult. In all I underestimated what was asked of the question. (63)

We totally underestimated the hardness of the questions at first glance. Closer inspection of the question revealed the exact nature of the question. (72)

Yes my perceptions about the questions changed as a result of doing it because when I started to read or many other people started to read they got a misunderstanding of the question. Some questions are hard but first look very easy and vice versa. So when I actually sat down to do the question I found out it is harder than I expected it to be. (32)

Other students overestimated the difficulty of the questions:

When I first looked at the questions briefly they appeared quite hard, but on closer look and actually attempting them, they were actually relatively easy. (65) (68 very similar)

When I first looked at some of the questions they seemed really hard, but when I read over them and understood what they were asking, I found them less difficult than I originally thought. (36)

Some students found certain types of questions easier than they expected:

My perceptions have totally changed because the questions that dealt with definitions and explanations have tended to be easier than the questions where practice is necessary. (30)

My difficulty ranking has changed as a result of doing it. I thought that the questions that involved memorising facts or rules like E would be more difficult than other questions since it requires memory of facts/rules rather than logically proving. (42)

Some students realised that they needed to do more revision:

My perceptions about the questions have changed. They are not difficult to do if I had studied a bit more, or a whole lot more. (35)

Yes I read them and I understand what can I do but when I perform them I impact from problems like the rules, memories or calculation etc. (33)

My perceptions have changed because of lack of revision in the subject; I was unfamiliar with the types of questions asked. (46)

Comment from a NESB student who is articulating the difficulty with English that was demonstrated in the previous section.

The order of difficulty does not change much. The hardest question for me is still the theorem (E), i.e. language problem. All calculation is all right for me, except some question need to know more English. (66)

A couple of the students commented on question B. We think that they enjoyed solving it.

My perception didn't change much except that A was easier than I thought. B required a lot of thought – more than I expected. (73)

A was harder than I originally thought, B was impossible but I thought and worked it out, H got easier, D got harder. Basically first impressions don't really count. Only after close consideration can one judge the difficulty of a question. (60)

One student articulated the idea that a familiar question was easier. This was not demonstrated in the numerical data.

We may find some questions hard at the beginning because we think that we have never done that type of question before. (76)

4. Conclusions

We have found that students perceive questions to be difficult for a number of reasons. In general, questions requiring conceptual understanding are regarded as more difficult that those which require factual recall or the use of routine procedures. There was not a strong link between familiarity with the question type and ranking of difficulty. Students were sufficiently familiar with the some types of question to be able to perceive inherent difficulties, such as a complex differentiation.

We found that in five out of eight cases, students' perceptions of the difficulty did not change after they had done the task. In one case they found the question to be more difficult than expected and in two cases to be easier. It is not clear to us why students found Question B to be more difficult than expected. It may be that some of the complexities (such as the use of the intermediate value theorem) were not immediately apparent. It is also significant that NESB students rated this question as easier than ESB students. This was the case both before and after attempting the question.

Not surprisingly, Question D, which required sophisticated language skills for its answer, was rated significantly more difficult by NESB students. Although the students understood what was being asked, they realised the need for language skills to answer it.

The close agreement between the *MATH* taxonomy and the ranking of difficulty given by the students is some evidence that the perceived difficulty is related to conceptual difficulty of the question.

The open-ended comments after completing the ranking showed that students found the exercise interesting and were surprised at the differences between their perceptions and the reality. Many commented that their ranking had not changed but that they had either underestimated or overestimated the difficulty of all the questions. Other students found that questions that dealt with definitions and theorems were easier than they expected. NESB students articulated their difficulties with answering questions, which required English skills.

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Appendix A. Questions for ranking of difficulty.

A. Sketch the graph of $f(x)=\sqrt{\frac{x}{x+1}}$ showing the main features.

B. Show that $x^3+cx+d=0$ has only one root if $c \geq 0$.

C. What is the formula for the linear approximation to the function $f(x)$ at the point $x=a$?

D. Describe, in about 10 lines, the ideas of the mean value theorem. Imagine that you are describing the theorem to a student about to start university.

E. The mean value theorem is a powerful tool in calculus. List three consequences of the mean value theorem and show how the theorem is used in the proofs of these consequences.

F. Explain the differences between instantaneous velocity and average velocity.

G. Explain why the mean value theorem does not apply to the function $f(x)=|x+1|$ on the interval $[-3, 1]$.

H. Sketch a function $f(x)$ where $f(x)>0$, $f'(x)>0$ and $f'(x)>0$.

Appendix B. Questionnaire (data collected before students attempted the question)

1. I will need the following skills to answer this question. Feel free to circle more than one letter.
   (a) memorised facts and rules
   (b) the ability to justify what I am doing
   (c) practice in answering this type of question
   (d) the ability to describe what I am doing
   (e) the ability to apply my knowledge in a new situation
2. I would rate this question as:

Very easy	Easy	moderately hard	quite hard	impossible
1	2	3	4	5

3. The language is:

Very clear	clear	moderately hard	quite hard	Too hard to understand
1	2	3	4	5

4.

I understand the question				I do not understand the question
1	2	3	4	5

5.

I have done similar questions before				I have never done this type of question before
1	2	3	4	5

Appendix C. MATH Taxonomy (Smith *et al.* ,1996)

Group A	*Group B*	*Group C*
Factual knowledge (Question C)	Information transfer (Question D)	Justifying and interpreting (Question G)
Comprehension (Question F)	Applications in new situations (Question H)	Implication, conjectures and comparisons(Question B)
Routine use of procedures (Question A)		Evaluation (Question E)

STUDYING THE EVOLUTION OF STUDENTS´ THINKING ABOUT VARIATION THROUGH USE OF THE TRANSFORMATIVE AND CONJECTURE-DRIVEN RESEARCH DESIGN

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ABSTRACT

The paper describes how the transformative and conjecture-driven research design, a research model that utilizes both theory and common core classroom conditions, was employed in a study examining introductory statistics students' understanding of the concept of variation. It describes how the approach was linked to classroom practice and was employed in terms of research design, data collection, and data analysis. The many possibilities that the design offered for systematically researching students' conceptual change are contrasted to the limitations of the prevailing methodology employed by researchers examining conceptions of data and chance.

Introduction

The prevailing methodology employed by researchers examining conceptions of data and chance is to take snapshots of how students might make sense of stochastic phenomena at a specific point in time. Rarely does one do any follow up of students' initial thinking to watch for future transitions (Shaughnessy, 1997). Researchers such as Pratt (1998) are casting doubt on the validity of this research tradition, which ignores the influence of the setting on the shaping of intuitions, and stress the need for investigation of students' conceptions and beliefs in natural school contexts, for a prolonged period of time.

A recent trend witnessed in educational research is an increase in the study of exemplary instructional practices based on the argument that new classroom practices need to evolve from these "best practices" (Confrey and Lachance, 1999). However, this type of research might not be ideal for wide-scale implementation. A pressing need exists for designs which allow a more speculative classroom research by relaxing some the constraints of typical classrooms while keeping others in force. The paper describes the experiences from adopting such a design, called the *transformative* and *conjecture-driven research design* (Confrey and Lachance, 1999), in a study examining introductory statistics students' understanding of variation. It provides an overview of how the conjecture guiding the study was developed and was linked to classroom practice and outlines how the transformative and conjecture-driven approach was employed in the study in terms of research design, data collection, and data analysis. The rich insights into the evolution of students' thinking that have originated from this research are then briefly discussed.

Developing The Conjecture

Definition of Conjecture

The conjecture is a very important aspect of the kind of research described in this paper. It has two dimensions, a content dimension and a pedagogical dimension. It is also situated within a theoretical framework, which helps to structure the activities and methodologies used in the teaching experiment and link together the content and pedagogical dimension of the conjecture. A conjecture is "*not* an assertion waiting to be proved or disproved", but "an inference based on inconclusive or incomplete evidence"(Confrey and Lachance, 1999, p. 235). In research following the positivistic paradigm, hypotheses or theses are set beforehand and the study's sole purpose is to confirm or disprove their truth. In contrast, a conjecture-driven research design perceives theory development as an inductive process. The purpose of the conjecture is to serve as a guide and not to constrict the collection of data. During the course of data collection and analysis, as experience with the setting increases, the conjecture is subjected to several alterations and modifications.

Variation as the Central Tenet of Statistics Instruction Conjecture

The conjecture driving this study was that if statistics curricula were to put more emphasis on helping students improve their intuitions about variation and its relevance to statistics, we would be able to witness improved comprehension of statistical concepts. The motivation for the study gave the results of a previous study of students that had just completed an introductory statistics course. The results of that study (Meletiou, Lee, and Myers, 1999), agreed with the main findings of research in the area of stochastics education. We had found that the students we interviewed, regardless of whether they came from a lecture-based classroom or from a course that made heavy use of technology and interesting activities, tended to have poor intuitions about the stochastic and to think deterministically. This led me to conclude that student difficulties might stem from inadequate emphasis paid by instruction to the role of variation in statistical reasoning, as well as to students' intuitions. I decided to conduct a teaching experiment that adopted a nontraditional approach to statistics instruction with variation as its central tenet. Pfannkuch's (1997) epistemological triangle, as shown Figure 1, seemed well suited for meeting my research aspirations.

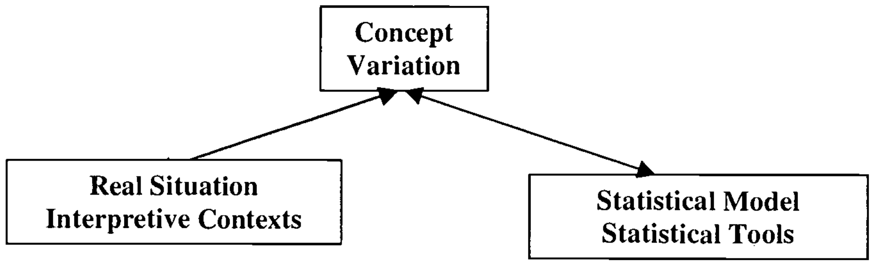

Figure 1: Pfannkuch's epistemological triangle

Pfannkuch's epistemological triangle views variation as the broader construct underlying statistical reasoning. In encouraging students to develop their understanding of the concept of variation it, at the same time, aims at promoting richer understanding of all the other main statistical ideas. The epistemological triangle indicates that for conceptualization of variation, a combination of subject and context knowledge is essential (Pfannkuch, 1997). The inter-linked arrows indicate the strong linkage that has to be created between the statistical tools and the context of the problem. The assumption underlying the epistemological triangle is that the concept of variation would be subject to development over a long period of time, through a variety of tools and contexts (Pfannkuch, 1997).

Pfannkuch's model, which bases instruction on contexts directly connected to students' experience, was a good alternative to typical approaches to statistics, which attempt to develop probabilistic reasoning through standard probability tasks. The model recognizes that adequate statistical reasoning requires more than understanding of the different ideas in isolation. It demands "*integration* between students' skills, knowledge and dispositions and ability to manage meaningful, realistic questions, problems, or situations" (Gal and Garfield, 1997, p. 7). Content is no longer a sequenced list of curricular topics taught in isolation, but "an interrelated repertoire of conceptual tools that can assist one in making sense of, and gaining insight and prediction over interesting phenomena" (Confrey, 1996).

Developing The Teaching Experiment

A transformative and conjecture-driven experiment is a planned intervention, taking place in a regular classroom over a significant period of time. What makes this research model unique and leads to a re-definition of the research-practice relationship is the dialectical relationship between the conjecture and the different components of instruction. Its research questions and methods of data collection are informed both by the conjecture and the components of instruction. Classroom research is speculative and while some of the constraints of typical classrooms are relaxed, others do remain in force (Confrey and Lachance, 1999).

Due to the need to continuously discuss and refine plans and interpretations, a transformative, conjecture-driven teaching experiment requires a team of researchers working together. In this study, I worked jointly with Dr. Lee, the course instructor towards designing the different aspects of the curriculum, towards refining and elaborating the conjecture and the components of instruction. Dr. Lee is a statistics education researcher with whom I had been collaborating for three years. He was therefore very familiar with the conjecture and acted as a research collaborator, providing invaluable insights that led to much better understanding and elaboration of the conjecture.

Context and Participants

The site for the study was an introductory statistics course in a mid-size Midwestern university in the United States. There were thirty-three students in the class (nineteen males and fourteen females). Most of these students were majoring in a business-related field of study. Only few had studied mathematics at the pre-calculus level or higher.

Curriculum and Classroom Setting

The design of the intervention was guided by the conjecture, while at the same time taking the time constraints and confines of the curriculum into account. Instruction included the set curriculum typically covered in an introductory statistics course, but was expanded in such a way as to include throughout the course activities that aimed at raising students' awareness of variation. The different topics were approached through the lens of the conjecture. The instructional approach employed in the course was based on the following two principles (adapted from Wild and Pfannkuch, 1999):

1. Complementarity of theory and experience: Statistical thinking necessitates a synthesis of statistical knowledge, context knowledge, and the information in the data in order to produce implications, insights and conjectures. If the statistics classroom is to be an authentic model of the statistical culture, it should model realistic statistical investigations rather than teaching methods and procedures in a sequential manner and in isolation. The teaching of the different statistical tools should be achieved through putting students in authentic contexts where they need those tools to make sense of the situation. Students should come to view to value statistical tools as a means to describe and quantify the variation inherent in almost any real-world process.

2. Balance between stochastic and deterministic reasoning: Instruction should view as an important precursor of statistical reasoning students' intuitive tendency to come up with causal explanations for any situation they have contextual knowledge about. It should present statistical thinking as a balance between stochastic and deterministic reasoning and should stress that statistical strategies, based on probabilistic modeling, are the best way to counteract our natural tendency to view patterns even when none exists, to distinguish between real causes and ephemeral patterns that are part of our imagination.

Instruction in a conjecture-driven teaching experiment changes over the course of the intervention in response to students' needs and inputs. In this study, curricular activities were designed to be flexible and open-ended. The instructor adapted them in response to feedback received from students. He would always situate instruction within contexts familiar to the learners. He would use analogies from students' everyday experience and would simplify mathematical relations in order to help build links to students' intuitions. He emphasized the complexity of real-life situations rather than making simplistic assumptions that would conflict with students' common sense. When, for example, discussing independent events, and after students had given typical examples of independent events such as coin tossing and die rolling, the instructor asked the class whether the success of a *"free throw"* of a basketball player is independent from the success of his previous *"free throw"*. Students argued that it depends on how the player responds to pressure, on how well he did on the previous throw etc. The instructor agreed remarking: *"In real life it's hard to say with a straight yes or no"*. He did not reject students' causal explanations although "hot hand" is an example often used by statistics educators and researchers to point out that people's tendency to detect patterns (hot hands) is often unwarranted. Tversky and Gilovich (1989) showed, using empirical data, that a binomial model well explains runs (streaks) in basketball player failures. According to this model, the chance of success in a shot is independent from the previous shot. One need not look for specific causes like nervousness since there is no other "pattern" than chance pattern explaining the data. However, the instructor understood what Biehler (1994) has pointed out – that even when the binomial model well explains the variation in a dataset, one should not exclude the possibility of alternative models, which give better prediction and which suggest causal dependence of individual throws. Similarly, when talking about slot machines in a casino, he noted: *"Although in theory when you put a coin and you pull it down and then you put another coin and you pull it down, although those two events should be independent, mechanically they might not be."*

The idea of making conjectures ran throughout the course. Students would state what they believed might or might not be true, and then looked critically at the data to evaluate their statements. Evaluation of conjectures would typically begin informally by using one or more graphical displays. The instructor would encourage students to describe the main features of the distribution displayed by the graph(s), always emphasizing the need to take into account not only the center, but also the spread. Students would look at the displays and try to give explanations for the patterns observed and for the departures from those patterns. Sometimes these explanations would be proposals for a possible model to summarize the dataset. The evaluation of conjectures would then become more quantitative. An analysis using appropriate numerical summaries would be made to support or refute the conjectures originally made by students. At the start of the course, the analysis was made using simple numerical summaries. Eventually, more tools were added to the students' repertoire and the mathematization of the data gradually became more formal.

Assessment

In order to enhance the understanding of the research setting and be able to provide answers to the research questions, a transformative and conjecture-driven experiment needs to use multiple forms of data generation. In examining students' learning progress and outcomes, a variety of both qualitative and quantitative data gathering techniques were employed. By assessing students' understanding prior to instruction, and then monitoring changes in their thinking throughout the course, the study attempted to develop a detailed description of the processes students go through in order to become able to intelligently deal with variability and uncertainty. The data gathering techniques employed included: (1) direct and participant observations, (2) interviews with the students and the instructor, (3) video-taping of group activities, (4) pre- and post-activity assessments, (5) fieldnotes, (6) samples of student work and (7) other relevant documents. Drawing data from several different sources permitted cross-checking of data and interpretations. The assessment strategies used to support and evaluate students' conceptual development helped students further clarify their reasoning strategies. The continuous monitoring of the effect of instruction on student learning was constantly supplying valuable information on their levels of concept attainment. This informed instruction, which was adjusted to promote deeper understandings, while also guiding the evolution of the conjecture.

Data Analysis

In a transformative and conjecture-driven experiment, there are two types of data analysis. The first type is the ongoing preliminary analysis, taking place throughout the course, guiding instruction and pointing towards necessary curricular revisions. This preliminary analysis, which begins simultaneously with the data generation process, is necessitated by the design's anticipation of emerging issues. Throughout the course, I would meet with the instructor on an almost daily basis. Each time we met, I would present him with some preliminary analysis of the data I had collected since our previous meeting. The implications of the feedback gained from students guided our decisions as to how instruction should proceed and what modifications of our plans were necessary. In addition to substantial revisions of the curricular interventions, this initial analytical work of cycling back and forth the existing data also led to revisions and elaborations of the conjecture, which however were of a smaller magnitude than curricular changes. Fledging hypotheses continuously got tested and evidence began to build. This analysis generated ideas for collecting new and often better quality data.

After the data collection stage was completed and all data had been generated and transcribed, the process of analysis continued in a more formal and explicit way. At this final stage I attempted, using a variety of both qualitative and quantitative analysis techniques, "to construct a coherent story of the development of the students' ideas and their connection to the conjecture" (Confrey and Lachance, 1999, p. 255).

Findings

The conjecture driving this study was that the reform movement would be more successful in achieving its objectives if it were to put more emphasis on helping students build sound intuitions about variation and its relevance to statistics (Ballman, 1997). Findings from the study suggest that the emphasis of instruction on the omnipresence of variation and the complementarity of theory and experience was indeed helpful in building bridges between students' intuitions and statistical reasoning. Students' understanding of graphical tools and numerical measures of center and spread was much more sophisticated than that of students in the previous study we had conducted. Instruction proved quite effective in achieving one of its main goals – helping students move away from "uni-dimensional" thinking and integrate center and variation into their analyses and predictions. Although not totally letting go of their deterministic mindset, students were much more willing to interpret situations using a combination of stochastic and deterministic reasoning. The course increased significantly their awareness of variation and its effects.

The investigation of students' conceptions and beliefs in a real school setting has also allowed me to gain wealth of information about the source of student difficulties and to enrich my initial conceptualization of the conjecture. I found, for example, the different meanings that students attached to variation as being one of the main sources of difficulties they had with comprehending sampling distributions. Several students viewed variation as sample representativeness and thus argued that the variation of a sample increases with increase in sample size. Similarly, others who viewed variation as range also argued that variation goes up with increase in sample size. These beliefs regarding variation of individual samples affected how students perceived the relation between sample size and variation of sampling distribution. Both of these groups of students shared the belief that the bigger the sample size, the higher the variation of a sampling distribution. Other critical junctures and obstacles to the conceptual evolution of the role of variation that emerged included the following: (1) Understanding of histograms and other graphs; (2) Familiarity with abstract notation and with statistics language; (3) Appreciation of the need to be critical of data and always examine the method by which it was collected; (4) Distinguishing between population distribution, distribution of a single sample, and sampling distribution; and (4) Understanding the reasons behind finding confidence intervals when producing an estimate of some parameter based on a sample. A detailed description of the rich insights gained from the study can be found in my doctoral thesis (Meletiou, 2000).

Conclusion

Hawkins (1997) stresses the need for more systematic research to guide developments in statistics education. The transformative and conjecture-driven design proved to be a promising alternative to the prevailing methodology employed by researchers examining conceptions of data and chance. It allowed thorough investigation of introductory statistics students' intuitive understanding of variation and use of the knowledge acquired to design, implement, evaluate, and refine meaningful interventions that helped students develop and expand upon their understandings. By examining how students' intuitions evolved during the course, I was able to identify structures that facilitated, as well as structures that inhibited, the articulation of intuitions about the stochastic. The wealth of information that emerged from the study is an indication of the potential of this research model for expanding our understanding of the components that promote development and growth of students' understanding.

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THE NT (NEW TECHNOLOGY) HYPOTHESIS

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ABSTRACT

I teach a first-year undergraduate mathematics course at a business university. The course, which is part of a three-years "Degree in Economics of International Markets and New Technology", deals with those subject one would expect, such as: pre-calculus, calculus, linear algebra.

I believe that today there is a great opportunity to improve the teaching and the learning efficiency, as well as student interest, by using and letting students use a Computer Algebra System (CAS) or, more generally, mathematics software like DERIVE, MAPLE, MATHCAD, or graphic and symbolic calculator as TI-89, TI-92 Plus.

My hypothesis is that students have at their disposal all the time (*during classes, while studying at home or in the University and for any assignment and examination*) mathematics software with the following features:

- symbolic and floating-point manipulation
- plotting and exploring function graph
- capability of defining a function (with as many arguments as necessary)
- capability of running simple programs

Given this hypothesis (that I will call the **NT Hypothesis**), how would a mathematics course have to change? And in which way should contents, teaching of mathematical objects, problems, exercises and finally evaluation instruments be modified?

At the Creta ICTM-2 Conference, we would like to present a comprehensive description of our work, including: the project (March-July 2001), the course (September 2001-April 2002), and a first analysis of the results (May-June 2002). We decided to present at this Conference three separate papers (see also papers by G. Osimo and by F. Iozzi); each of them takes a different point of view.

1. Introduction

I teach a course in "General Mathematics" at Bocconi University, a school for the study of economics based in Milan, Italy. It is one of the first-year courses of a three-years program called: "Degree in Economics of International Markets and New Technology".

The "General Mathematics" course, attended this year by 140 students, comprises 120 hours of lectures per year and deals with the following subjects:

- One-variable Calculus
- Linear algebra
- Calculus of several variables
- Unconstrained and constrained optimization
- Dynamical systems
- Financial mathematics

Every subjects covered by the course relates to various applications in the economic and financial fields.

If we want to teach a mathematics course that takes advantage of the use of technology, we can do it at two different but complementary levels:

1. By using an e-learning environment along with classical mathematics teaching: for this purpose, in my "General Mathematics" course, I used the Lotus Learning Space software, produced by Lotus.

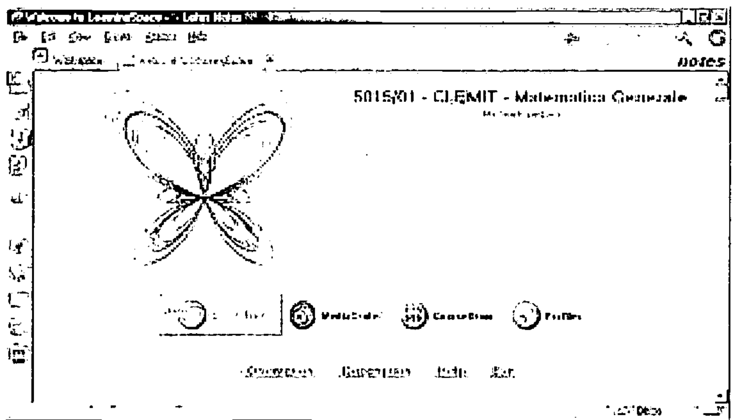

Figure1

2. By using a CAS (Computer Algebra System) software. For this purpose, I used the Mathcad software, produced by Mathsoft Inc.

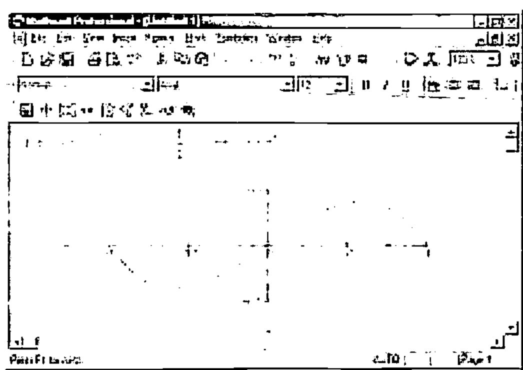

Figure2

All of Learning Space functions were used.

All information about the course was enclosed in the "Schedule" area. This included the timetable and modalities of the course; the rules for the final examination; the subjects covered by each lesson; and a list of reference papers and books for class use.

After each lesson – one lesson lasts approximately two hours – I placed two *.doc* files at my students' disposal: the first one was a 3-4 page summary of the lesson itself; the second one included a list of problems and exercises related to the subjects discussed during the lesson.

In the "Media Center" environment, I prepared for my students' use, a set of about 50 Mathcad worksheets (and a few Excel worksheets) related to the lesson's subject.

I wanted these worksheets to play an integral role in the lessons, because on one hand they show the powerful and syntax of Mathcad, and on the other, they allow us to approach the mathematical problems from a symbolic, numeric and graphic point of view.

The "Course Room" environment is the forum in which the realization of our "computer-supported collaborative learning" project is discussed. In this space, students can post their suggestions and questions with regard to issues dealt with during the lesson. I was surprised to observe that the discussion mainly developed among the students themselves, who tried to explain to each other their own solutions to specific problems. I seldom had to participate in the forum in order to drive the discussion to the correct solution of any problem.

The "Assessment" environment was used to create exercises and simulations of examinations. In particular, in the Computer Science Laboratory, the students took two mid-term tests: a multiple-choice questionnaire and a problem that had to be solved by creating a Mathcad file. For the second part, students had to read the text of the question in the Learning Space environment, use Mathcad to build functions, calculate, plot graphics; and finally go back to Learning Space to post their solution.

Mathcad is a powerful software for numeric calculation and, only partially, for symbolic calculation. It has been used with differing functions: first of all as a "super-blackboard" - the instrument used by the teacher to show mathematical objects to the class in order to improve understanding of the concepts. For example, the following pictures represent different examples of convergence.

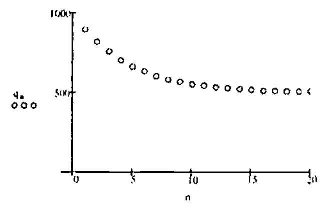

Figure 3

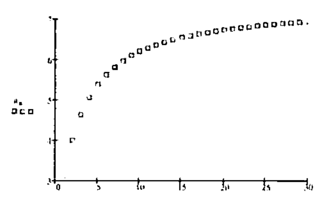

Figure 4

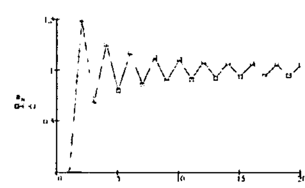

Figure 5

The students spent twelve course hours in the Computer Science Laboratory. This time was entirely dedicated to learning Mathcad syntax and analyzing its power with regard to calculation and graph-plotting.

The purpose was to provide students with an instrument for automatic calculation that they can use in every step of their learning process (during classes, training and examinations). Using this instrument, students could:

- develop both numeric and symbolic calculations using numbers, expression, functions, vectors and matrices;
- define own functions from $\mathbf{R}^n$ to $\mathbf{R}$;

- draw and analyze graphics;
- work out simple programs.

What has been described until now is what I call the *NT hypothesis* (where NT stands for New Technology): in this new context, how does a mathematics course have to change? And in which way do the contents, the teaching of mathematical objects, the problems, the exercises and finally the evaluation instruments have to be changed?

2. The contents

It's obvious that some traditional skills are going to become obsolete. For example, one of the most important issues of traditional mathematics is *function analysis*: starting from the algebraic expression of a given function, we analyze its behavior and finally plot its graphic. Function analysis is the search for a graphic representation of the qualitative shape by studying its limits and derivatives. This skill becomes superfluous when students are able to obtain the graph of the inquired function using Mathcad; even more if we consider that generally it takes longer to explain how to calculate derivatives than to understand what the derivative of a function really represents. If we can save precious time during classes simply by not asking students to do a lot of calculations of derivatives and integrals, but using automatic calculation (both for numeric and symbolic calculations), we will have a lot of time left to explain their applications, instead of the calculation techniques.

Therefore, we asked students to be able to calculate in the traditional way only two patterns of derivatives and anti-derivatives: the power function $x \to x^a$ and the exponential function $x \to b^x$ (the trigonometric functions are not very interesting for economics-related subjects). For all other functions, we can use CAS.

In Linear Algebra, students have to prove their skill in using Mathcad to work with vectors and matrices (product and power of matrices, inverse matrix, rank and determinant). The idea is that students have to calculate the easiest examples in the traditional way (for instance the product, the rank and the determinant of a 2x2 matrix), while they have to apply their Mathcad knowledge to deal with more complicated problems.

For example, if a student is in front of a stochastic matrix $\mathbf{M}$ and a status vector $\mathbf{v}_0$, the student must be able to explore Markov chains ($\mathbf{v}_{n+1} := \mathbf{M}\mathbf{v}_n$) without regard to their length; or must be able to solve the equilibrium equation that defines a Leontief input-output model ($\mathbf{x} = \mathbf{A}\mathbf{x}+\mathbf{d}$, where $\mathbf{A}$ and $\mathbf{d}$ are respectively the *consumption matrix* and the final demand of a productive system).

In this way, not only do we achieve our purpose of spending more time on the semantic of mathematical objects instead of on their syntax, but we provide a kind of mathematics that is:

- much more advanced, because we can investigate problems that are too complicated for traditional mathematical techniques;
- much more interesting for the students, because they are able to create their own functions and objects;
- free from rigorous scheme.

With the *NT Hypothesis,* all classical ways of studying calculus have to be reconsidered. First of all, because in a school of economics, where mathematics is mainly a tool to create patterns that help us understand specific problems related to economics, the theoretical instruments employed might be too "expensive" for the purpose. Is it really necessary to build the whole theory of limits and derivatives to find out a minimum or a maximum, when we can obtain the same result with desired precision just by exploring the graph of the function (or a rather dense table)?

I will bring an example: an exercise asked to calculate the maximum point of a profit function, given an income function $R(x)$ and a cost function $C(x)$, with respect to the sold amount x, defined on an interval $[a,b]$. The traditional method foresees for this case the calculation of the derivative of $R(x) - C(x)$ and the search for the zeroes of this function. A student came to a different solution just by considering a table of 21 rows and two columns containing x values from a to b with step $(b-a)/20$; the student asserted that the maximum point was the value x^* corresponding to the maximum value among the values of $R(x) - C(x)$ in the table. How can we evaluate this solution?

There is another important consideration: we usually consider that a function is differentiable in every point of its domain (without discussing whether this is true or not), but in real world we come across functions defined only in some range, not differentiable or even not continuous in some points.

Let me use another reality-based example: in Italy the main tax on citizens income (called IRPEF) is a function that is continuous but not differentiable in the range $[0, +\infty)$; in fact, the tax y depends on the income x in the following way:

18.5% from 0 to 10,000 €

25.5% from 10,000 € to 15,000 €

33.5% from 15,000 € to 30,000 €

39.5% from 30,000 € to 70,000 €

45.5% over 70,000 €

Using Mathcad, it is easy to define and plot the tax function IRPEF(x) and the mean tax rate IRPEF(x)/(x).

$$\mathrm{IRPEF}(x) := \begin{array}{|l} 0.2\cdot x \text{ if } 0 \le x \le 10 \\ 0.2\cdot 10 + 0.25\cdot(x-10) \text{ if } 10 < x \le 15 \\ 0.2\cdot 10 + 0.25\cdot 5 + 0.3\cdot(x-15) \text{ if } 15 < x \le 30 \\ 0.2\cdot 10 + 0.25\cdot 5 + 0.3\cdot 15 + 0.4\cdot(x-30) \text{ if } 30 < x \le 70 \\ 0.2\cdot 10 + 0.25\cdot 5 + 0.3\cdot 15 + 0.4\cdot 40 + 0.45\cdot(x-70) \text{ otherwise} \end{array}$$

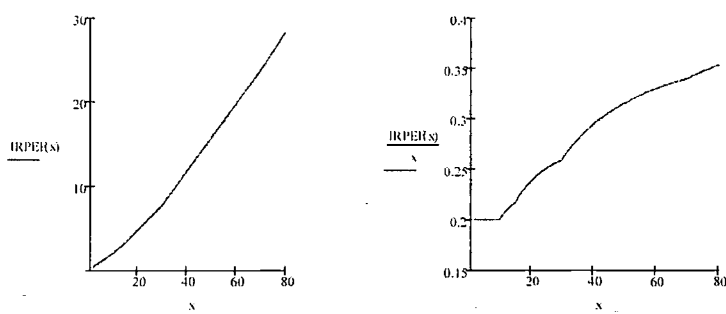

Figure 6

Recently, the new Italian government has proposed to simplify the pattern by reducing it to only two income classes, according to the following function:

23% from 0 to 100,000 €

33% over 100,000 €

Who is going to benefit from this changes? A student solved the problem simply by comparing this new function to the old one

IRPEF_2002(x) := 0.23 if $0 \le x \le 100{,}000$

0.23*100,000 + 0.33*(x-100,000) otherwise

For these kind of functions, solving the problem in the traditional way is very "expensive", while the use of a graphic comparison is much more practical because it provides all features of the function with required precision. Other similar patterns, sometimes even not continuous, are those concerning the costs of transportation of goods, energy costs, and so on.

A very important feature that confers an aspect of innovation to this new way of teaching and learning mathematics, is the contraposition symbolic-numeric and continuous-discreet. In Italy, the symbolic and the continuous are more common. Real functions and symbolic solutions are favoured (the sequences do not appear in the secondary curriculum). For example, for the question:

$$\int_2^3 \frac{1}{x^3+1}\,dx =$$

the usual answer is

$$\frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{5\sqrt{3}}{3}\right)+\frac{1}{6}\ln\left(\frac{16}{21}\right)-\frac{\sqrt{3}}{9}\pi,$$

and it is very difficult to understand what the student has learned from this exercise: the symbolic solution is even more obscure than the question itself, and from a semantic point of view, it does not represent an improvement.

In this case, mathematics is only a close loop: it doesn't have to uncover its mechanism.

3. Visualization of math objects and concepts

The use of a mathematical software leads to a considerable improvement in the efficiency of teaching: the visualization of calculations and graphs, the possibility to use animations and to present a huge number of examples, as well as the fact that any modifications of given parameters take effect immediately on the worksheet, make the learning process easier, faster and more efficient.

Teachers can use a "super-blackboard" that helps them and the students create something new. For example, when I introduced the concept of derivative, a worksheet like the following one was of great help:

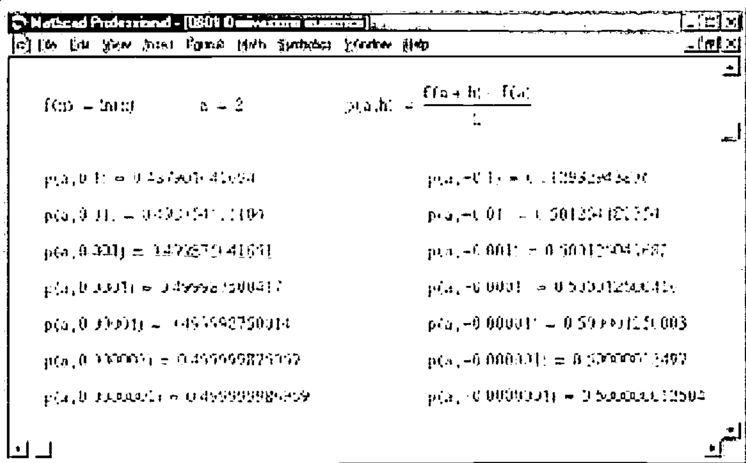

Figure 7

If we modify the function $f(x)$ and the point a, the whole worksheet refreshes and shows that $p(a, h)$ converges to $f'(a)$. In this way, simply by modifying the a value instead of proving the symbolic expression of $f'(x)$, we promote the students' attitude to make conjectures. It takes only a few attempts to understand the trick: the derivative of $\ln(x)$ seems to be $1/x$. In the early learning phase, making conjectures is one of the most important activities: it is a signal that students have understood which problem we are trying to solve; and very often this is more important than giving a "politically correct" answer.

The opportunity to use directly in the classroom a software like Mathcad is also helpful, not only to develop mathematics knowledge, but also to simplify it. For example, let us consider the definite integral $\int_a^b f(x)\,dx$; it could be introduced with this easier definition:

$$\int_a^b f(x)\,dx := \left| \begin{array}{l} \Delta x := \dfrac{b-a}{n} \\ c_k \in \left[a+(k-1)\Delta x,\ a+k\Delta x\right] \\ \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \Delta x \cdot f(c_k) \end{array} \right.$$

In this case, by using an experimental analysis based on approximation, we can make the Fundamental Theorem of Calculus our goal, that is much more important than proving it.

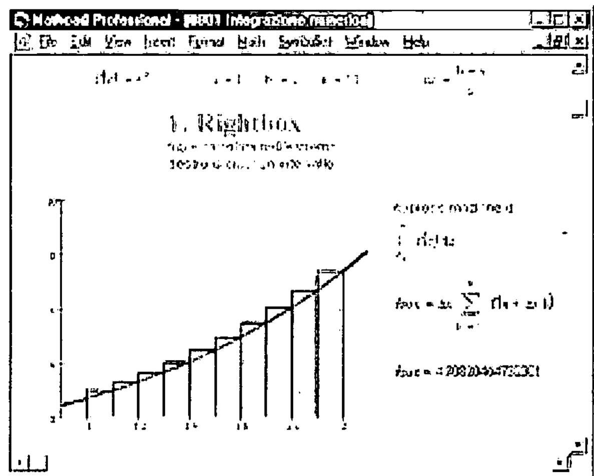

Figure 8

4. Problems and exercises

The *NT hypothesis* also affects the kind of exercises and problems that students have to solve. It is obvious that the students will not be asked anymore to "study" a function or to calculate on paper a derivative or an integral, but just to *project* the calculation so that CAS can do it for them. For example, during the final examination, students must prove that they know how to use Excel to prepare a loan amortization schedule with constant payment: the Excel worksheet must be parameterized, meaning that if only one input data (for example the rate of interest) is modified, the whole scheme changes.

Students who use Mathcad do not distinguish between a function that "can be integrated" and another one that "cannot be" (i.e. the ones that do not have an elementary function as anti-derivative), as

$$f(x) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$

because the result provided is numerical. The complexity of the calculation is one of Mathcad's tasks. Students have to prepare some electronic worksheets in which they have set the Gauss function. To solve a problem like the following:

The average height of a population of adult males is 174 cm with a standard deviation of 12cm. Calculate the portion of the population that is taller than 180 cm.

Students have to change only the values of the two parameters "mean" and "standard deviation".

In the same way, students must be able to calculate a least-square line using Mathcad. If we only know some points of a function, for example a demand function, and we believe that the demand decreases linearly with the price, it is very important to know what a linear regression is. (It is not really important that students are able to calculate it, but rather that they know *what it represents*: why do we choose as parameters the "mean" and the "standard deviation"?).

The suggested problems can be easily solved with some spirit of curiosity. During classes, when we introduced the rates at which functions grow, we presented the following problem:

How many solutions does the equation $x^{100} = 2^x$ *have?*

Most students plotted the graph of the function and they found what they were looking for: there are two solutions.

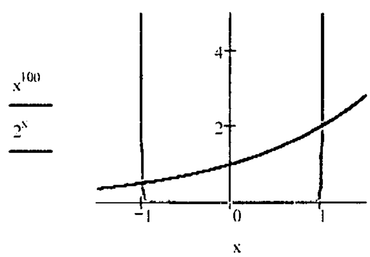

Figure 9

Some of them laboriously looked for a third solution corresponding to very high values of x.

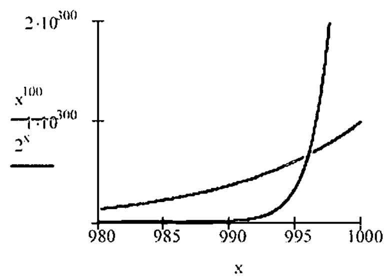

Figure 10

Only one student remembered what he had learned about logarithms, and simplified the problem.

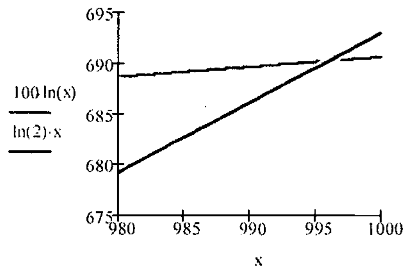

Figure 11

The one presented above is a good example of how much mathematics is mainly a matter of thinking on a particular problem: the experience is useful and enriches our ability of abstraction, but it cannot replace our capability of thinking.

5. The assessment

The course is divided in two semesters; each semester is divided in two further parts and we scheduled an examination at the end of each part. After these four examinations, follows a final oral test.

Students took their second and third examinations in the Computer Science Laboratory, where each student could work on an individual computer, using both the Learning Space environment and the Mathcad software. The examination lasted 1½ hours and included an eight-question multiple-choice test and one problem.

The questions were built in the Learning Space environment and the students just had to click directly on the right answer. When they finished, students had to return the questionnaire, which was automatically and immediately corrected. At the end of the examination, students could see their score in the Learning Space "Portfolio".

The problem had to be solved by creating a Mathcad worksheet and to be reported as "assignment" in the Learning Space environment. This enabled me to correct all 140 examination worksheets in a very short time, giving a judgment to complete the global evaluation, together with the questionnaire score.

To fully understand what kind of examination we proposed, here are both the questionnaire and the problem.

General Mathematics (code 5015) CLEMIT

10 January 2002, 14.30 hours

Second mid-term Test - Type A

1. The average value of the function $f(x) := 20(1-x)e^x$ on the interval [0, 1] is

A: $20e$ B: $20(e-1)$ C: $20(e+1)$ D: $20(e-1)$ E: $20(e-2)$

2. Loan amortization of 14,000 € in 5 equal payments at the annual interest rate i=7%. The first payment is

A: 3570 € B: 3780 € C: 3390 € D: 3420 € E: 3650 €

3. A financial operation consists of investing 2500 € with the result of cashing in 1000 € after 1 year and 2000 € after 4 years. The IRR (Internal Rate of Return) is:

A: 5.8% B: 6.1% C: 6.4% D: 6.7% E: 7%

4. The weight of a population of adult males is described by a Gauss function with average value m=78 kg and standard deviation s=12 kg. The percentage of the population weighting more than 90 kg is about

A: 4% B: 8% C: 12% D: 16% E: 20%

5. The maximum value of the function $f(x) := 50\,x\,\exp(-50x)$ on **R** is

A: 1 B: 1.23 C: 0.37 D: 0.04 E: 0.02

6. Consider the continuous function $f(x)$, positive and decreasing on the interval $[a, b]$. The function $G(x) := \int_a^x f(t)\,dt$ on the interval $[a, b]$ is

A: positive and decreasing B: increasing and convex

C: increasing and concave D: decreasing and convex

E: decreasing and concave

7. Let's consider the linear and decreasing demand function $q(p) := 480 - 10p$. The demand elasticity corresponding to the price p_0= 28 is

A: −1.4 B: −0.4 C: − 1 D: −0.7 E: −2.4

8. Consider a function $f(x)$ with second order derivatives in **R**. If $f(2)$=3 and $f'(2)$=4 and $f''(2)$=10 then $f(2.1)$ is approximately

A: 3.35 B: 3.4 C: 3.45 D: 3.5 E: 3.55

PROBLEM

A transportation company has two different truck models, called A and B. The demand function of the offered service is $q(p)$:=400 − 20p, where q is the amount of transported goods in tons and p is the price in Euro per ton. The cost function of the company depends on which truck model is used. Using the A model the cost function is $C_A(q)$:= 4q + 400, while for B it is $C_B(q)$:= 2q + 800. The company always chooses the truck model that is more convenient according to the amount of goods it has to transport.

Calculate which amount is necessary to transport, in order to realize the maximum profit; which truck model should be used; the obtained profit; and the required price.

The solution must be worked out in a Mathcad worksheet, named

<surname> <name>.mcd,

for example **impedovo michele.mcd**

6. Conclusions

From an educational point of view, the innovation of the *NT hypothesis* is that students can use automatic calculation throughout all the phases of the learning process, and especially during examinations.

In front of this hypothesis we are obliged to review our knowledge and our competences as well as focus on the new skills that we want to teach our students. What kind of skills do we have to transfer to the automatic instruments? The answer is not easy (also because mathematics is only a side subject in a degree program in economics). We will have to choose one path and take our responsibility for it. A possible answer is: students must prove not only their knowledge of mathematics but also their skill in using a CAS. The automatic instrument for calculation is not just an additional one, but it is required as fundamental.

As one can see, the *definition* of a mathematical object (a typical sketch of an oral examination: "What does it mean that a sequence converges to the number 5?", " That it approaches 5!")

becomes much more important than in traditional teaching. There is a shift: students are not requested anymore to do complicated and tedious calculations, but, therefore, they must know what a mathematical object is.

Our task is not anymore to teach students how to solve difficult algorithms; a lot of light and suitable bits can be used for this purpose.

What we have to focus on is the semantics of the concepts that we want to communicate, and we have a lot of time left for this.

The new technologies could become the paradigm of "doing mathematics". If I know what I am looking for, and I observe a syntax, I can obtain a result in a very short time. This enriches my experience, leads me to more complicated problems and, therefore, makes me more independent.

If we adopt this perspective, we can review our programs and our teaching items, and at each step, wonder what is appropriate that students to calculate using the electronic instruments. It is an extremely exciting adventure.

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THE DIFFICULTIES AND REASONING OF UNDERGRADUATE MATHEMATICS STUDENTS IN THE IDENTIFICATION OF FUNCTIONS

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ABSTRACT

In this paper we investigated the difficulty levels of the identification of functions in different representations of mathematical relations. The relative difficulties associated with functions and developmental levels were examined through a written test administered to 38 first year undergraduate students. The results appear to support the assumption that there is a developmental pattern in students' thinking in identifying functions from their symbolic and graphical forms.

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1. Introduction

Representational systems are the keys for conceptual learning and determine, to a significant extent, what is learnt. The ability to identify and represent the same concept in different representations allows students to see rich relationships, and develop deeper understanding (Even, 1998). The difficulty of representing different topics in mathematics has been studied extensively. Some researchers interpret students' errors as either a product of a deficient handling of representations or a lack of coordination between representations (Greeno & Hall, 1997). A common conclusion in most of these studies is that students have deficient understandings in relation to the models and language needed to represent or illustrate and manipulate mathematical concepts (Tall, 1991).

Several researchers in the last two decades address the importance of representations in understanding mathematical concepts (Aspinwall, Shaw & Presmeg, 1997)). However, not enough attention has been given to the reasoning and difficulties of students in representing mathematical concepts at the university level. The primary goal of the present study is to explore students' understanding and reasoning of the concept of function through its multiple representations.

2. Theoretical Background and Literature Review

The concept of function is of fundamental importance in the learning of mathematics and has been a major focus of attention for the mathematics education research community over the past decade (Dubinski & Harel, 1992). The understanding of functions does not appear to be easy, given the diversity of representations associated with this concept (Hitt, 1998). Aspinwall, Shaw and Presmeg (1997) asserted that in many cases the graphical (visual) representations can cause cognitive difficulties, because the perceptual analysis and synthesis of mathematical information presented implicitly in a diagram often makes greater demands on a student that any other aspect of a problem.

The standard representational forms of some mathematical concepts, such as the concept of function, are not enough for students to construct the whole meaning and grasp the whole range of relevant applications. Mathematics instructors, at the secondary level, have traditionally focused their instruction on the use of algebraic representations of functions. Most instructional practices limit the representations of functions to the translation of the algebraic form of a function to its graphic form. Vinner (1992) stated that a function, as taught at schools, is often identified with just one of its representations, either the symbolic or the graphical - the former can result in interpreting function as "formula". Sfard (1992), on the other hand, found that students are unable to bridge the algebraic and graphical representations of functions. Similarly, Norman (1992) found that even secondary school teachers pursuing their masters' degrees in mathematics tended to call up one particular representation of a function, often a graph. In general, they did not take into account verbal and intuitive representations. Furthermore, most teaching approaches do not take into consideration the movement from one type of representation to another, which is a complex process and relates to the generalization of the concept at hand (Yerushalmy, 1997).

Although there are a lot of studies dealing with students' conceptions of functions and their difficulties in coming up with the function concept (Tall, 1991), there remain issues to be examined in relation to the representations of functions and the connections between these representations (algebraic, graphical, verbal, tabular, etc.). This study purports to contribute to the ongoing research on representations in functions by identifying the levels of difficulty of fundamental modes of function representations. The literature does not provide the kind of coherent picture of undergraduate students' representational thinking in mathematical functions that is desirable for the improvement of current approaches to instruction. In this paper, we seek to define the difficulty level and the developmental trend of translations in the representations of a mathematical relationship. To this end, we used the SOLO taxonomy (Biggs & Collis, 1991). The SOLO taxonomy provides a systematic way of describing a hierarchy of complexity, which learners exhibit in the mastery of academic work.

SOLO describes five levels of sophistication, which can be found in learners' responses to academic tasks: Prestructural – the task is not addressed appropriately, the student hasn't understood the point; Unistructural – one or a few aspects of the task are picked up and used (understanding as nominal); Multi-structural – several aspects of the task are learned but are treated separately (understanding as knowing about); Relational – the components are integrated into a coherent whole, with each part contributing to the overall meaning (understanding as appreciating relationships); Extended abstract – the integrated whole at the relational level is reconceptualized at a higher level of abstraction, which enables generalization to a new topic or area, or is turned reflexively on oneself (understanding as transfer and as involving metacognition) (Biggs & Collis, 1991).

3. The Goals of the Present Study

One of the main objectives of this study is to define the reasoning and the difficulties experienced by students in identifying the concept of function through its symbolic and graphical representations. This study is motivated by practical concerns and theoretical needs. The practical concerns focus on the difficulties experienced by students in grasping the concept of functions. By taking into account different systems of representations, we can identify specific variables related to cognitive contents, and, in this way, organize didactical approaches to promote the students' articulation of different representations in a meaningful manner. The theoretical needs come from the lack of a systematic theoretical framework of representations capable of supporting the kinds of understandings, which are necessary for university students to identify and use the concept of functions. Both practical and theoretical concerns are interwoven in understanding the relations between the multiple representations of functions.

Specifically, the purpose of the study was twofold:

(a) To define the level of difficulty in identifying the concept of function through its graphical and symbolic representations, and

(b) To trace the developmental trend (if any) in the student's ability to identify mathematical functions in different modes of representation.

4. Method

Participants

The participants in this study were all first-year students in the department of mathematics at the University of Cyprus (N=38). These students were attending a freshman calculus course. There were 13 male and 25 female students, who graduated from lyceums where the emphasis was on mathematics and physics and succeeded in the university entrance examinations. They attended a one-year calculus course during their final year at the lyceum and graduated with very high marks in mathematics.

Instrumentation

The instrument used in this study to collect information of students' understanding of function representations was a questionnaire, which consisted of two parts involving 20 tasks in total. The first part included 9 relations and the students were asked to indicate whether or not the relations could describe one or more functions (see Table 1). The second part involved 11 graphs and students were asked to decide which of these graphs resulted from functions of the form y=f(x) (see Table 2). In both parts students were asked to justify their answers by writing their explanations.

5. Results

The Difficulty Level and the Developmental Trend

In order to search for a possible developmental trend and difficulty levels in the identification of functions among freshmen, we analyzed the data using latent class analysis. Tables 1 and 2 summarize the "difficulty level" of each of the tasks of symbolic and graphical representations of functions, respectively.

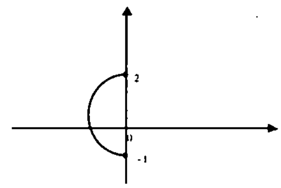
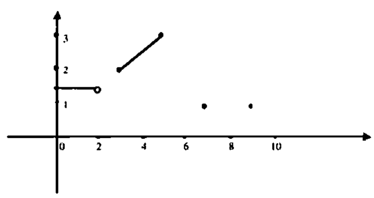
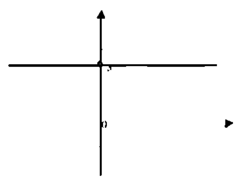
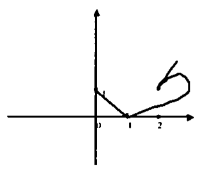
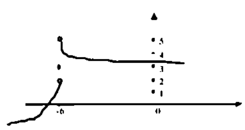
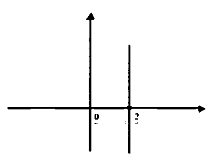
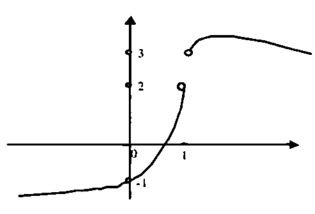
Table 1: The Difficulty Level of the Functions Represented by Symbolic Forms

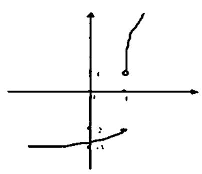
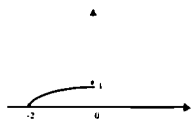
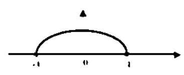
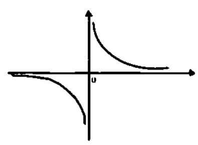
Situation	Relations	Mean	Std. Deviation
a*	$x^2+y^2=3$	0.2105	0.4132
b	$y=\int_0^1 \sqrt{x^3+x+1}dx$	0.2895	0.4596
c	$a^2-b=0$	0.5000	0.5067
d	$f(y)=e^y$	0.6053	0.4954
e	$x^4=3y$	0.6842	0.4711
f	$a=\sqrt{2}$	0.8158	0.3929
g	$f(x)=3$	0.9211	0.2733
h	$y=x^2$	0.9211	0.2733
i	$s=3t$	0.9474	0.2263

* For each situation, the subjects were asked to indicate whether the symbolic representation corresponded or not to a function.

Table 1 shows that situations s=3t, $y=x^2$, and f(x)=3 were the easiest symbolic functions identified by freshmen ($\overline{X}_{(s=3t)}=0.95$, SD=0.23; $\overline{X}_{(y=x^2)}=0.92$, SD=0.27; $\overline{X}_{(f(x)=3)}=0.92$, SD =0.27), while situations a and b were the hardest for students to determine whether the relation was a function or not ($\overline{X}_a=0.21$, SD=0.41; $\overline{X}_b=0.29$, SD=0.46). Situation c was correctly answered by half of the students, while the situation h, which is equivalent to c, was correctly identified as a function by more than 92% of the students.

Table2: The Difficulty Level of the Functions Represented by Graphical Forms

Situation	Graphs presented to students	Mean	Std. Deviation
A*		.2368	,4309
B		0.4474	0.5039
C		0.5000	0.5067
D		0.6842	0.4711
E		0.7105	0.4596
F		0.7632	0.4309
G		0.7895	0.4132

H		0.8421	0.3695
I		0.8684	0.3426
K		0.8947	0.3110
L		0.9211	0.2733

* For each situation, the subjects were asked to indicate whether the graphical representation corresponded or not to a function.

Table 2 shows the difficulty level of the tasks given in graphical forms. The graph depicted in situation L was correctly identified as a function by 92% of the students ($\overline{X}$ =0.92, SD=0.27). Situation A was the hardest task for students since only 24% of them answered it correctly. Situations B and C were also difficult for students, while situations I, and K were answered correctly by the great majority of the students (87%, and 89%, respectively).

Multivariate analysis of data showed that there were statistically significant differences among the situations in symbolic and graphical forms. Students identified functions from symbolic representations more easily than functions from graphical representations, confirming, to an extent, Vinner's (1992) results. The presence of a consistent trend in the difficulty level across translations seems to support the assumption for the existence of a specific developmental pattern. Thus, on the basis of the respective frequency quartiles, the students were ranked to success; four classes were defined: low achievers--Class 1 (n=9), below average achievers --Class 2 (n=9), above average achievers --Class 3 (n=11), and high achievers --Class 4 (n=9).

Table 3 shows the tasks successfully performed by more than 50% of the students in each class. The data included in Table 3 indicate that there is a developmental trend in students' abilities to complete the assigned tasks because success on any translation by more than 50% of the students in each class was associated with such success by more than 50% of the students in subsequent classes.

Table 3: The Developmental Trend of Students' Abilities to Identify Functions

	Class 1	Class 2	Class 3	Class 4
Level 1	*i(89%), h(89%), g(89%), f(78%), L(78%), K(89%), I(89%), H(66%), G(89%), F(66%), E(78%)	i(100%), h(100%), g(89%), f(78%), L(89%), K(89%), I(100%), H(78%), G(78%), F(78%), E(78%)	i(100%), h(100%), g(90%), f(90%), L(100%), K(89%), I(89%), H(90%), G(78%), F(78%), E(78%)	i(100%), h(100%), g(100%), f(90%), L(100%), K(100%), I(89%), H(100%), G(89%), F(78%), E(89%)
Level 2			d(72%), e(72%) D(82%)	d(77%), e(90%) D(78%)
Level 3				c(77%), b(66%) C(56%), B(56%), A(52%)

* The small and capital letters refer to situations shown in Table 1 and 2 respectively. The numbers in parentheses indicate the percentages of students' successful answers in each situation.

Cognitive Developmental Levels

The findings seem to support the hypothesis that there are at least three cognitive developmental levels, which characterize students' thinking in the identification and discrimination among symbolic and graphical representations of functions. Class 1 and Class 2 students seem to successfully perform the same tasks; however, students in Class 2 responded with greater facility as shown by the percentages of successful answers shown in Table 3. The fact that students were unable to successfully perform a higher level task unless they could perform tasks of the preceding level seems to provide compelling evidence that the levels, as identified, may generate a hierarchy of thinking. We claim that the three levels of thinking used in identifying functions from their symbolic or graphical representations correspond to the three of the five levels of cognitive thinking identified by Biggs and Collis (1991), ie., the unistructural, multistructural, and relational levels. In what follows, the hypothetical levels and the major characteristics of each developmental level are described in relation to Biggs and Collis' thinking levels. To this end, we used students` written explanations, which were provided during the completion of the questionnaire.

Level 1: At this level, students identify some kinds of function representations but are then distracted or misled by an irrelevant aspect. Thus, students attempted to identify mathematical functions from a given symbolic or graphical form but their approaches were not always systematic. Students recognized functions from symbolic relations only if the relations were expressed in terms of the dependent variable as in situations a, f and h. Students also identified the symbolic representations of functions when the relations included symbols that are commonly used in their textbooks or during instruction. For instance, students at this level identified functions when x and t were used to denote the independent variables, and y and s are used for the dependent variables. However, level 1 students did not always provide correct answers when the above symbols had a different role in the relations as shown in case e (x^4=3y), where the relation was solved in terms of the independent variable.

Students at level 1 identified functions from graphs when the graphs depicted functions with interval domain or the union of successive intervals as in situations H and L. Situation F is the only graph where it was correctly recognized by students that it did not represent a function, because it depicted an extreme situation where x =2 corresponds to real numbers. In most cases, students were not able to reach a final decision or to provide a consistent answer. For example, although the tasks in situations h and c (see Table 1) were equivalent, very few of the students at this level performed successfully both of these tasks, probably because they were distracted by the context of the relationship or the symbols involved. In the same way, students' responses in identifying functions from graphs were inconsistent (see Table 2).

Level 1 appeared to be a period of transition that is characterized by the students' naï ve and often inflexible attempts to identify functions from their symbolic and graphical forms. Their thinking was more indicative of what Biggs and Collis (1991) termed as the unistructural level in the sense that one aspect of the function concept is usually pursued. For example, many of the students incorrectly identified situation D (see Table 2) as a function, focusing their attention on the left side of the graph and ignoring the right part, which probably confused them. The unistructural nature of students' thinking at this level was also exemplified by their responses to situation b. Most of them identified it as a function but they could not recognize that it was a constant function and thus students proceeded to define the domain as $(-\infty, +\infty)$.

Level 2: In contrast to level 1, students exhibiting level 2 thinking, when faced with representational situations of functions, demonstrated a readiness to recognize and discriminate symbolic and graphical functions in a consistent way. The characteristic of this level is that students improved their ability to identify the functions involving the symbolic and graphical modes with the exception of the functions in situations c, b, C, B, and A (see Table 3). Students at this level recognized more than one relevant feature of function representations and attempted to explain their reasoning in a way that integrates their knowledge about the concept. Students at this level identified functions even in the cases where the symbols played a different role in the relations as in situation d or the relations were solved in terms of the dependent or the independent variable as in situation e. Level 2 students identified not only the graphs that level 1 students did but they also identified that "strange" graphs such as situation D did not represent a function.

Students assessed at Level 2 appeared to exhibit characteristics of the multistructural level within the symbolic and graphical forms (Biggs & Collis, 1991). The following extracts from students' written answers indicate how students' reasoning at levels 1 and 2 differed with respect to the justifications they provided for their responses. Level 1 students (unistructural level) who thought the equation $x^2+y^2=3$ could be described by one or more functions gave the following reasons for their responses, suggesting that they had focused on one aspect of the problem: "This is a circle with radius 3", "It's a function since you can express the equation as $y=\sqrt{3-x^2}$ ". On the other hand, students at level 2 provided answers that suggested that they had concentrated on more than one aspect of the concept of function (multistructural level): "It can describe a function if you restrict domain", "You can solve for y and look at only the + or the – square root. Thus, you will have two different functions", "The circle can be broken into two half circles".

Level 3: Students exhibiting Level 3 thinking made precise connections between the graphical and symbolic representations of mathematical functions. This was evidenced by the consistency of students' answers in the identification of functions in the symbolic and graphical forms. The fact that students at this level successfully performed most tasks indicates that their thinking is consistent with the characteristics of the relational level. That is, they integrate the concept of functions with its multiple representations into a meaningful structure and are able to generate abstractions in mathematical relationships (Biggs & Collis, 1991). However, situation a was not correctly answered even by the students at this level, implying that there is another level, the extended abstract level, which was not considered in the present study.

6. Conclusions

Representations enable students to interpret situations and to comprehend the relations embedded in problems. Thus, we consider representations to be extremely important with respect to cognitive processes in developing mathematical concepts. The main contribution of the present study was the identification of hierarchical levels among the graphical and symbolic representations of mathematical functions. An association was verified between the students' ability to identify various representations of the mathematical functions. Specifically, it was found that representations that could be identified as functions by low achievers were identified with greater ease by students in higher achievement classes, whereas the mathematical functions in some situations could only be performed by top students.

The present study is a first attempt to develop a framework for describing and probably predicting first year university students' thinking in the identification of mathematical functions from their symbolic and graphical forms. This framework recognizes developmental levels and is in agreement with neo-Piagetian theories that postulate the existence of sub stages or levels that reflect the structural complexity of students' thinking (Biggs & Collis, 1991). The analysis revealed that students exhibit three developmental levels. Students exhibiting level 1 tend to adopt a narrow perspective in identifying mathematical relationships as functions. They do not provide complete and consistent answers. There is a tendency to overlook the data in the given representations, that is, to focus on one aspect, rather than on the elements of the concept of function in combination. Students who demonstrate level 2 thinking recognize functions by combining more than one aspects of the concept and tend to provide systematic justifications for their reasoning. However, they lack the ability to consistently relate the symbolic and graphical forms of functions, which is the characteristic feature of Level 3.

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THE ROLE OF A PEER SUPPORT PROGRAM IN LEARNING

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ABSTRACT

A newly established institution Sabancı University offers highly challenging and interdisciplinary programs. The institutional structure and academic programs are based on the utilization of interdisciplinary approaches, and the traditional departmental structure does not exist providing the students with the opportunity of choosing their academic programs, to realize their goals. The students during their first two years of university education are required to take the same courses independent of their future aims, to be engineers, natural scientists, political scientists, economists, historians, art historians or artists. The university has a guidance system that includes various units to promote student success and to support the realization of the academic programs. This paper aims at presenting an academic support program that is structured as a subdivision of the guidance system and runs totally by undergraduate students. The task of motivating the students with different interests and diverse backgrounds as well as giving equal opportunity to each student in the assessment of their class work, calls for extra effort. Not surprisingly one of the main aims of the program is to motivate and encourage students to understand mathematical concepts, mathematical modeling and to use mathematical tools in various contexts. To reach its aim, program offers extra curricular activities in line with the university's academic programs and is subject to systematic evaluation. Program activities, office hours, tutorials and workshops are held by freshmen and sophomores in a friendly atmosphere encouraging peer discussions and sharing academic knowledge and experience. The evaluations and statistical results have revealed the significance of peer support as well as the role of the program in building a learning environment and a healthy academic campus climate (see www.sabanciuniv.edu).

1 Introduction

In any field of arts and social sciences or engineering and natural sciences, it is impossible for professionals to attempt to work under any global standards without the knowledge of other fields; thus, Sabancı University has an interdisciplinary organizational model allowing different faculties to interact and collaborate in contrast to traditional organizational structures of discrete institutional units. As a result, students have the opportunity to be exposed to different subjects and have degrees in the field of their choice. All students go through a common program in the first year of their education that will equip them with an interdisciplinary training so as to assist them to conceive the disciplines as a whole. While this may seem so exceptional in a global platform, it is so in Turkey. To promote the student academic performance as well as to support their individual and academic development the university has several units. One of the support programs is the "Peer Tutorials", which is different to the traditional top-down educational system in Turkey as the "Peer Tutorials" program encourages active involvement of students and peer support.

Social sciences and natural sciences are the basis of the first year undergraduate program. The courses are structured around a lecture addressing to all students and are supported by discussions or problem solving sessions for smaller groups. Freshmen, with diverse backgrounds and interests are treated and their performances are assessed uniformly in all these classes. A number of quite competitive students with a wide variety of knowledge and ability and some lacking motivation in certain subjects, need additional assistance in the first year program. The evaluations and statistical results have revealed that the "Peer Tutorials" program had a significant role in promoting student academic performance. It is worth to state here that Sabancı University will have its first graduates in year 2003, and the institution is very young as well as the peer support program.

2 The Peer Tutorials Program

The main principles of the Peer Tutorials program are "interactive learning" and "peer support". The program is modeled on two intertwining components, which are professional supervision and the tutorial sessions. The system has a dynamic structure with a feedback mechanism.

There are four different stages of tutorial sessions: 'peer tutorials', 'individual tutorials', 'advanced tutorials' and 'workshops'. Peer tutorials are peer study or peer discussion groups moderated by a student. In peer tutorial sessions students are encouraged to share their academic knowledge and experiences, and study in a friendly atmosphere. Individual tutorials offer individual guidance, in accordance with specific student needs, the types of guidance may range from teaching, to practicing study skills. Advanced tutorials are study groups in which the group has the chance to study a specific subject intensively. The workshops are for moderately large groups of students to meet their further needs and requests and focus on supplementary subjects that are determined in line with the incoming feedback from the tutorials, instructors or students. A freshman or a sophomore holds each component of the program and acts as a moderator or a mentor. All the tutorial sessions aim to improve the student academic performance as well as to assist the students in acquiring and using various academic skills in the course-related subjects and in general learning. To participate to the program students can drop in during the work hours or can schedule an individual appointment. A student, coordinates the program activities for each course, and helps the formation of the study groups in accordance with the students' needs and requests. In addition, a group of student coordinators does the event scheduling. Needless to say the training of the students who work for the program and supervisioning them, are the integrated components of the support program.

There are 14 freshmen and 5 sophomores working for the program, and they organize services primarily for the students in the 'Calculus', 'Science of Nature' and 'Society and Politics' courses.

2.1 Some Cases:

The peer tutors often use analogies to explain mathematical concepts and they relate the new concepts to some others that are well known. Unlike the experts, students do not care to choose their examples from a "real case" or make their explanations "mathematically correct" instead they tend to give an idea or produce a mental picture to explain a mathematical concept. On the other hand, being a tool and a way of thinking mathematics lies at the common denominator of many subjects and facilitates the peer interaction.

The following cases are presented for illustration.

The contents of the Science of Nature and Calculus courses are not synchronized. At the time when kinematics (motion), Newton's laws, force, work, kinetic energy, potential energy, conservation of energy and gravitation have been summarized, the formal introduction of the derivative and integral are yet to be done. Hence, a number of tutorials are organized for the students that feel less confident about their backgrounds either in physics or in mathematics.

The list below is used at a peer study group.

continuity e.g.: trajectory	instantaneous velocity at t_0 is the limit of average velocities at t_0	instantaneous velocity is the derivative $\frac{dx}{dt}$	acceleration is the rate of change of velocity and $\frac{d^2x}{dt^2}$

Where, dt is explained as the time interval that is even smaller than any attainable time and dx is explained to be the distance that is smaller than any possible distance travalled.

Furthermore the distance travelled is explained to be roughly the summation of the distances travelled in a time interval Δt. Thus the formula $x_2 = x_1 - \int_{t_2}^{t_1} v(t)dt$ where x_i are the position vectors at times t_i , $i = 1, 2$ and $v(t)$ is the velocity of the particle, has been sufficient to give an idea about the definite integral. The product rule, chain rule and integration are practiced through the work-kinetic energy theorem as: $F_{net}dx = d(\frac{1}{2}mv^2)$, where a constant net force of magnitude F_{net} acts on an object of mass m. The trigonometric functions and uniform circular motion are simultaneously covered. Projectile motion is used to explain several concepts that includes the geometric meaning of the derivative. It is observed that the peer tutorials of this sort not only have made the students feel confident in the Science of Nature course but also helped them later to understand the mathematical concepts in calculus.

The book (Calculus, Hughes-Hallett, Gleason, McCallum, et al.) that is used for the Calculus course focuses on conceptual understanding, and presents the topics geometrically, numerically, analytically and verbally. This approach, not only helps the students to improve their problem solving skills and master mathematical concepts but also has helped them to carry out discussions. For instance, a discussion at a peer study session on chemistry about the covalent bondings and organic compounds, have revealed that for most of the students it has been difficult to visualize the molecules in 3-dimension. To solve this problem a workshop is designed and 3-dimensioanal illustrations are used to explain the subject. This workshop has helped the students to grasp the matter thoroughly and develop themselves furthermore. During the workshop students have discussed the bond angles of the molecules among themselves and not only have discovered that the bond angles in a tetrahedral molecule are the same and are equal to 109.5^0 but also ended up providing a geometric proof of this fact, although this has not been the aim of the workshop.

2.2 Assessment of the Program

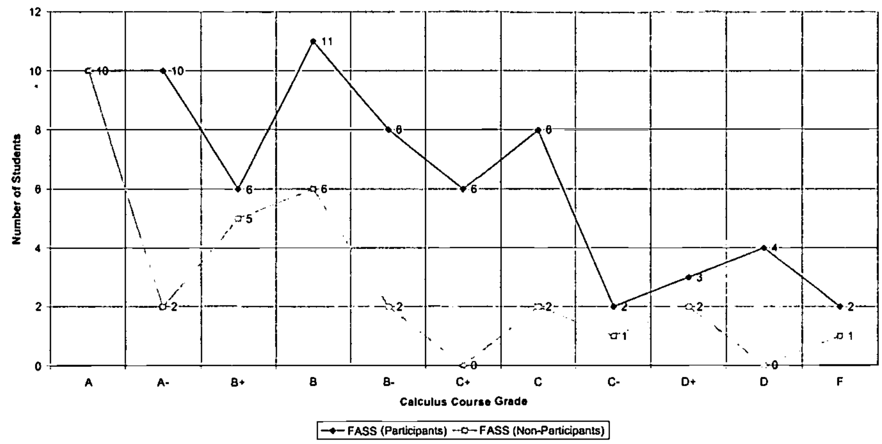
The Peer Tutorials program is evaluated through reports that consist of program participant performances. The mentors as well as the moderators provide a written report after each tutorial session. The reports include the duration of the sessions, the names of the participants and the subjects studied, as well as remarks about the effectiveness of the sessions and the progress of each participant. The results of these reports are taken into consideration for the development of the program and are used as future references. The reports have given us reasons to believe that the peer tutorial sessions, are natural platforms for the inquisitive young minds where they can question each others' interests and learn from different perspectives.

At Sabancı University there are two main groups of freshmen with respect to their educational backgrounds and future interests, namely the students in the Faculty of Arts and Social Sciences and in the Faculty of Engineering and Natural Sciences. Freshmen from both faculties with diverse educational backgrounds and motivations, are treated uniformly in all the first year courses. While the Engineering and Natural Sciences Faculty students are quite competitive in science and may have strong backgrounds in physics, biology or in chemistry, the Arts and Social Sciences Faculty students are competitive in social sciences. The mathematics backgrounds of all the students are good but their levels of mastering the mathematical concepts may vary. To obtain an even distribution of the grades among the faculties is the most desirable out come for each course, since the students' educational backgrounds and interests show a great difference. Therefore, the program aims at supporting the students of the Faculty of Arts and Social Sciences, to promote their academic performances in the Calculus and as well as in the Science of Nature course.

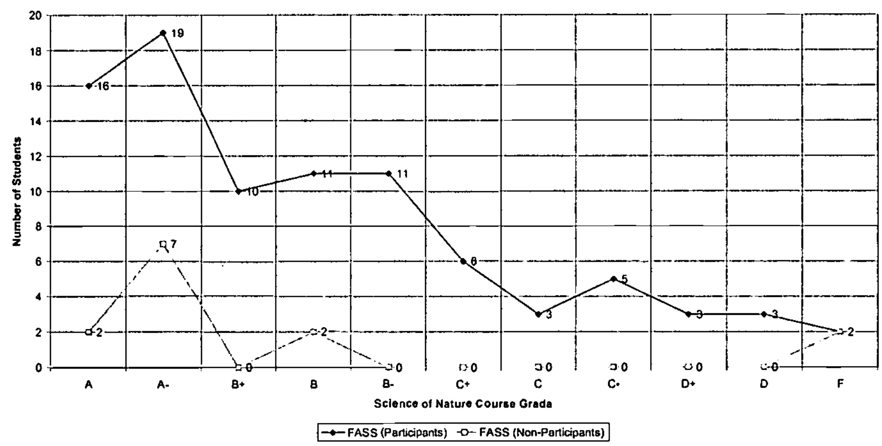
During this academic term, 300 peer tutorial sessions are organized and 261 of the 320 freshmen, have volunteered to attend these sessions and %90, %70 and %67 of the attendants had peer support to strengthen their backgrounds in subjects that are related to the Science of Nature, Society and Politics and Calculus courses respectively.

For the Faculty of Arts and Social Sciences, the grades of the students that have participated and not participated in the peer tutorials are compared. See the charts 1.C and 1.S. for the Calculus and Science of Nature courses.

1.C. Calculus Course: The Grade Distributions among the Peer Tutorials Program Participants and Non-Participants for the Faculty of Arts and Social Sciences (FASS)

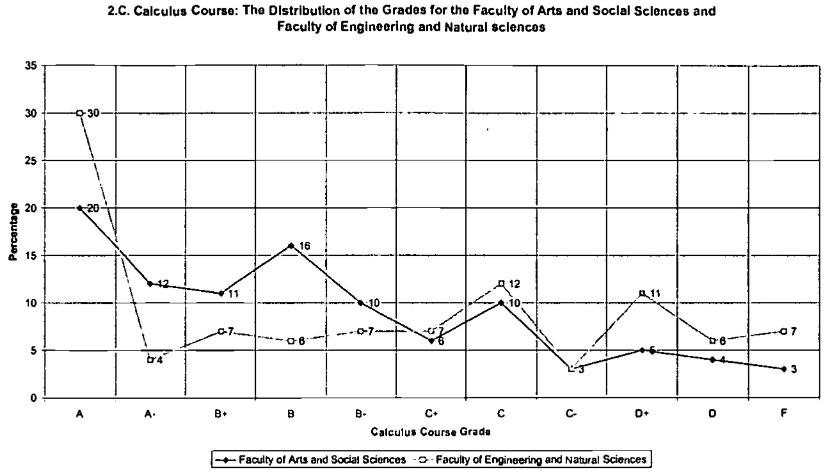


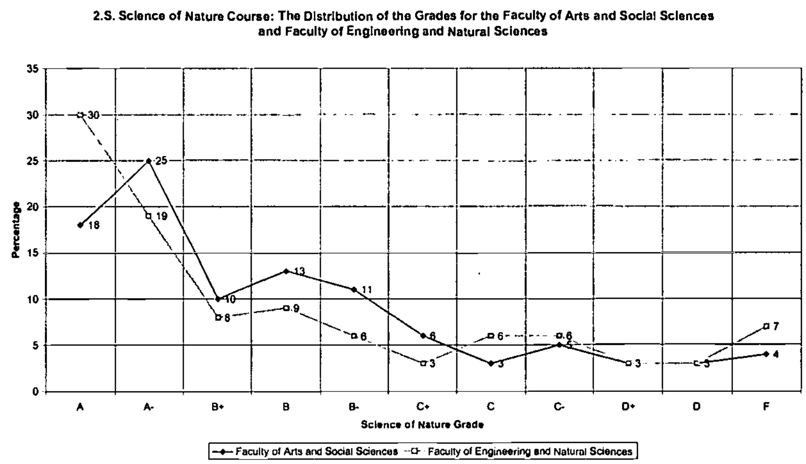
1. S. Science of Nature Course: The Grade Distributions among the Peer Tutorials Program Participants and Non-Participants for the Faculty of Arts and Social Sciences (FASS)



The Peer Tutorials program provides services to all students regardless of their faculties and %85 of the freshmen in the Faculty of Arts and Social Sciences and %83 of the freshmen in the Faculty of Engineering and Natural Sciences have applied to the Peer Tutorials program to have peer support. For a general view, the grade distributions among the faculties are compared and it is observed that the grades are distributed quite evenly between the two faculties and the students of the Faculty of Arts and Social Sciences have performed better than expected (see charts 2.C. and chart 2.S.).

The program is also evaluated through questionnaires. According to the questionnaire results, %86 of the 148 program participants are highly satisfied, %13 are satisfied





with the program activities and %1 of the participants remained to be indifferent about the activities of the Peer Tutorials.

The peer tutorials not only promote the academic and individual student development but also encourage teamwork, collaboration, cooperation and interaction among peers. Since the students are strong in different subject matters, interaction among the peers have had a role in building up mutual respect and understanding among the students and also had a positive effect upon maintaining a learning environment.

On the other hand, along with its advantages the program has a number of drawbacks. The tutorials and workshops, may discourage students from attending classes and students may become reluctant to share what they know in addition to having a tendency to plagiarize homework To this end, utmost effort must be put in rising awareness about plagiarism and the tutorials must not be supplementary for the lectures or classes.

Although the program has been very popular among the students and seemed quite successful, the underlying reasons for its achievements are attributed to the design and content of the academic programs as well as the supportive faculty members of the Sabancı University. Furthermore, the Peer Tutorials program does not aim to organize tutorials for the sophomores, juniors or seniors since, university is a culture where creativity blooms and students need to grow up on their own to research, discover and create.

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PREPARATION OF TEACHING MATERIALS ON SELECTED MATHEMATICAL TOPICS FOR DISTANCE COURSES

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ABSTRACT

The purpose of the presentation is generalization of teacher's work with students at Kharkiv G. S. Skovoroda Pedagogical University, faculty of physics and mathematics, specialty «mathematics and informatics (information science)». The aims of studies are formalization and representation of educational material on the topics "Symmetry" and "Polyhedra". Further use of this material in the system of distance education in Ukraine is supposed.

Keywords: distance courses, teaching materials, mathematical topic «Symmetry».

Introduction

One of the main directions of distance education development in Ukraine is teacher training. Many universities and specialists in this field should contribute to the creation of whole-Ukrainian teacher training system in distance education. Some experience in this field has Kharkiv G. S. Skovoroda Pedagogical University. The work in the area of distance education is carried out together with National Technical University (Kharkiv Polytechnical Institute) [1].

The program of training teachers in distance education includes such issues as computer literacy, basic knowledge of Internet and distance education, psychological and pedagogical issues of distance education, hypermedia in distance education, technology of distance course design and distance course management, managing the quality of distance education, tutor training for distance education [2].

Since 2000 the research laboratories of distance education of both universities started inviting teachers for professional training. As a result a trainee must work out a small distance course.

The presentation is devoted to one of such works - the representation of an educational material on the selected mathematical topics. The students – future teachers of mathematics - train during the course of informatics. The database, created by the students, consists of small units of teaching material.

Topic «Symmetry». The Example of Its Formalization and Representation

One of the most important directions in the teaching course of informatics in our university is the study of the ways of constructing training programs for high and higher schools. Undergraduate students train during this course and writing the diploma works.

In the process they consider the following topics:

• Psychological and pedagogical principles of constructing programs;

• Functional peculiarities and structure of hypertext;

• Use of the program package PowerPoint, HTML language, visual programming languages, other program environments for building of teaching programs on the hypertext base;

• Computer graphics as a tool of preparing illustrations to the teaching programs.

As a result of work in this direction was the creation of educational course with the appropriate educational materials: a program of the course, a hypertext manual, a course of lectures and practical training, materials for discussion.

A program of the course includes subjects:

•symmetry as the special kind of geometrical law;

•symmetry and geometry of natural forms;

•movements of the first and second kind;

•composition of movements.

A large attention in the course was given to study of the topic "Ornaments". The students participated in developing the hypertext manual "Ornaments". The hypertext manual gives the concept of an ornamental motive. The electronic textbook offers exercises and individual tasks. It contains samples of simple tasks for solving in small student groups.

The manual consists of two parts: the first part titled «Drawing ornaments» describes various types of ornaments; in the second part titled «The graphical editor Paint» instructions are given for drawing illustrations.

The manual tells that the concept of symmetry is one of fundamental concepts of mathematics. The symmetry was studied by artists, mathematicians, naturalists and philosophers [3]. The manual describes the peculiarity and specifics of an ornament, various types of symmetric patterns, kinds of motives for constructing ornaments (geometric, non-geometric, plant-like, animal-like, human-like and resembling various objects); also mentions borders, net ornaments.

An example of a sample page from the manual is shown in Fig. 1.

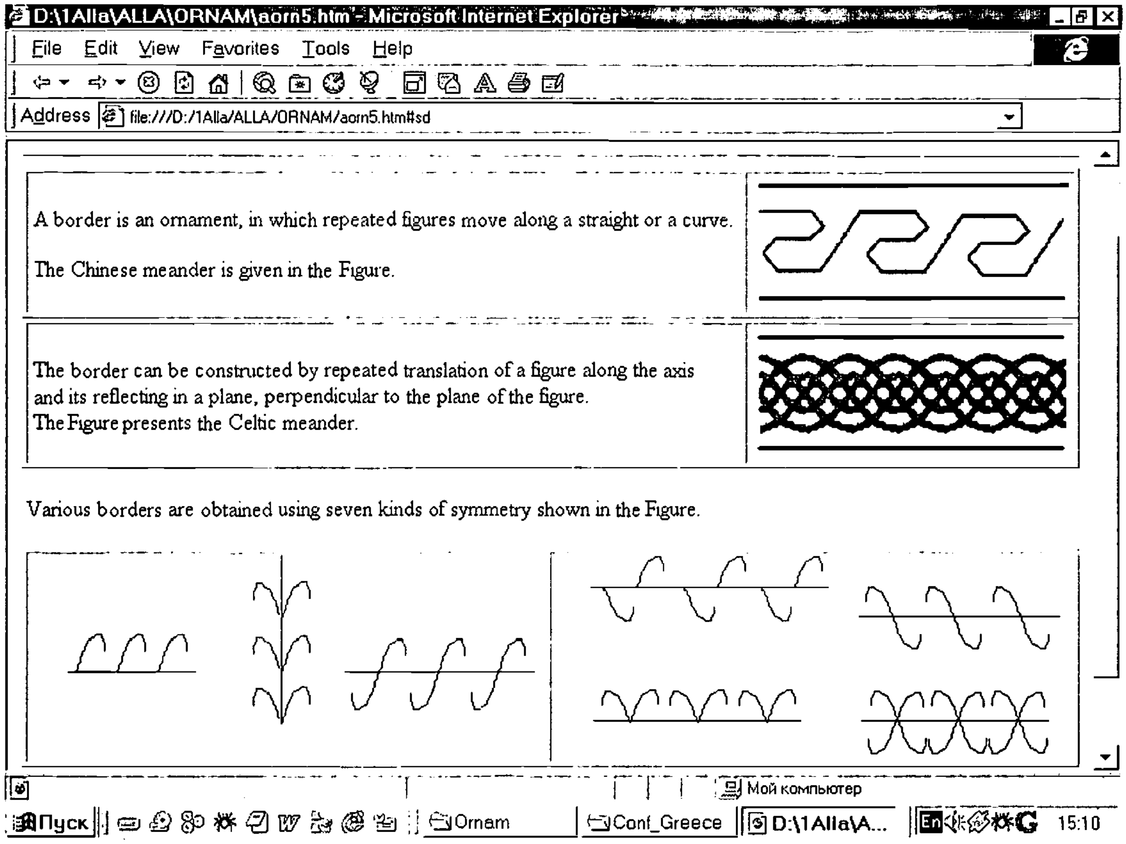

Fig. 1. A page from the manual

Before beginning the performance of the individual tasks it is recommended to tell the students about the remarkable feature of mathematics: in solving different problems quite unexpectedly appear similar concepts and methods. This miraculously similarity may be demonstrated on the example of the elementary transformation group. The transformation group appears everywhere, where symmetry is present. So, samples of architectural and art ornaments are connected with geometry: ornamental figures are symmetric.

Students used geometric and symmetric transformations: symmetry with respect to a point, symmetry with respect to a straight line or plane, rotation, parallel translation, homothety and similarity in their individual work.

Let's consider a simple example borrowed from the ancient Greek art. We regard the rotation of a plane, which maps the plane onto itself. In Fig. 2 the meander element is shown.

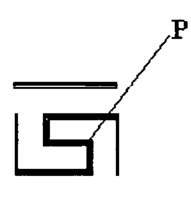

Fig. 2 A meander

The border in Fig. 3 is obtained from the meander by means of the rotation. The rotation axis passes through a point P perpendicular to the planes of the image. The rotation is done by the angle of 180° (we have the second order rotation axis). The border unlimited from the left and from the right has rotation axes of two types. The rotation axis of one type passes through any point equivalent to the point P. The rotation axis of another type passes through a point of other kind. Around of each of axes it is possible to carry out turn on 180°. Thus, in presenting this topic the question may be raised on the second rotation point of the border.

Fig. 3. A border

The patterns connected with this topic are presented in Fig. 4.

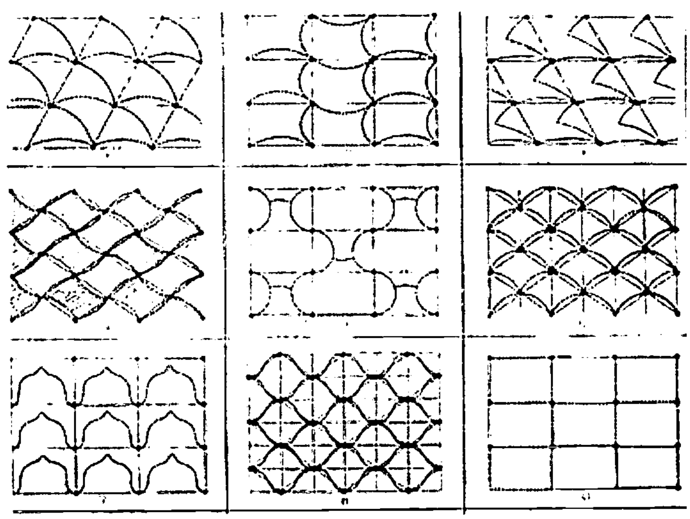

Fig. 4. Some patterns for the tasks

The illustrative material obtained as a result of solving such tasks, enables students to draw conclusions:

1) Any symmetry operation is a movement, and under its action the most part of pattern changes the position.

2) If any object possesses a symmetry element corresponding to the symmetry operation, which it undergoes, then the operation does not change the external sight of the object.

It permits one to classify objects according to those symmetry operations, which leave the objects invariant, and owing to this to reveal their internal essence.

The students' samples are a part of an educational material on the topic "Symmetry". The samples collect in the database. The databases prepared by students contain figures, solved tasks, and text problems. The students' database contains materials for their use at different levels of displaying an educational activity.

In parallels the instructor or teacher forms the corresponding knowledge base. It contains the needed theory, practical skills, tasks and links between them. This set of items and connections among them is called the logical structure of a teaching material. Depending on the teaching purpose elements and connections between them are determined in different way. Two examples of different splitting of teaching material are the consideration of subtopics of a teaching course on the one hand and the analyses of concepts entering this course on the other hand. In our case we have version 2.

Elements of the considered material also include the fragments of the preceding materials of the course. In Fig. 5 the classification of movements is shown used in constructing the semantic network. It reflects the structure of the teaching material of the topic «Symmetry».

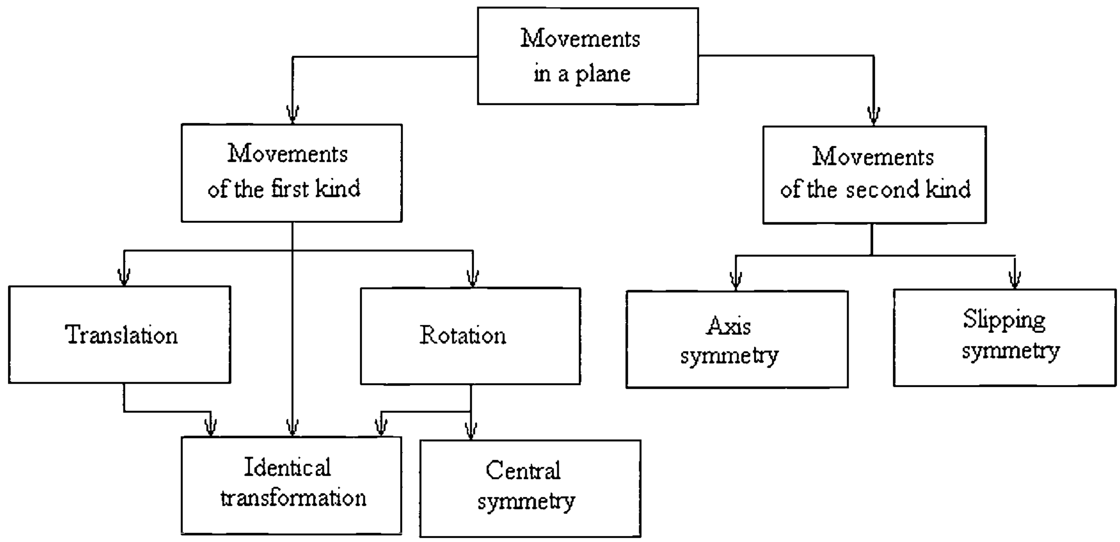

Fig. 5. A classification of movements

The educational tasks look like exercises, control questions, which are usually resulted at the end of the units of the textbook.

The structure of a concrete course is a subset of the object domain model.

The structure of an educational material includes theoretical elements of an educational material and connections between them. They form its theoretical substructure.

The theoretical knowledge is a basis for formation of the appropriate knowledge at trainees of distance courses.

A practical substructure describes the samples of activity and connections between them.

Practical skills are a set of the needed algorithms and activity samples.

A common substructure of educational tasks includes the educational tasks in the terms of elements of these two substructures.

In the practical part we distinguish between two classes of elements: basic skills and formed skills.

The basic skills are the activity samples present in the current teaching material; they are also used in the preceding teaching material.

Formed skills - samples of activity contained in the given educational material and which was not met in previous educational materials.

Therefore large attention in the manual is given to examples.

Let us consider an example of the structure of teaching material in the part «Central symmetry ».

We shall start with theoretical elements. Consider definition. Objects, that have a point O, such that if for any point Õ of each object there exists the point $\tilde{O}_1$, such that lies on the straight line ÎÕ at the distance $\hat{I}\tilde{O}_1$, equal to ÎÕ, are called center-symmetric relative to the point Î.

Exercises.

1) Prove that if a figure has two symmetry centers O and $\hat{I}_1$, then it has an infinite number of such centers and the figure is unbounded. The examples of such figures are a straight line, a strip and a circular cylinder.

2) Prove that if a figure has three centers of symmetry O, $\hat{I}_1$, $\hat{I}_2$ that do not lie on one straight line it has an infinite number of centers on a plane $\hat{I}\hat{I}_1\hat{I}_2$. They form a parallelogram lattice.

3) Prove that if a figure has four centers of symmetry that do not lie on the same plane, it has infinite set of centers. All these centers form a parallelepiped lattice.

Questions.

1) How many centers of symmetry have a straight line, a plane, 3D-space?

2) Give examples of 1D objects which have: à) not more than one center of symmetry, b) infinite number of symmetry, c) do not have symmetry centers at all.

3) Is it possible to fill 3D-space with regular hexahedral prisms?

Practical skills.

1) Construct center-symmetric cube vertices images with respect to the points: a) the intersection of the cube diagonals, b) one of the cube vertex.

2) Which regular polygons can cover the 2D-plane? Construct an example.

The result of work of the students during one semester consists of 50 pages of the electronic tutorial in HTML language and from more than 100 illustrations. Database includes exercises, control questions on topics, samples of the tasks.

It is clear that general (analysis, synthesis, comparison, abstraction, concretization, generalization) and specific (determining the concept and the opposite operation) intellectual operations enter to the structure of the cognitive activity while mastering new mathematical concepts.

The object domain «Polyhedra» is processed in the similar way as «Symmetry». The work is based on Ref. [4].

The illustrations of the regular, semi-regular, star-shaped polyhedrons and some models of polyhedrons, exercises, and questions on topic have come in the database on a topic "Polyhedra". This work was realized in Visual Basic language.

Conclusion

This work describes an object database: its contents and how the students created it.

The peculiarity of the databases is the fact that they include both theoretical elements and samples of practical activity, and a way of combining these two levels.

We further shall expand and modify a database, especially in the direction connected with the construction of educational material of the topic «17 crystallographic groups».

As a whole, the idea of the presentation is forming visual and intellectual vision, visual perception and thinking, external visible and internal figurative form. Thus, the formalization of educational material is shaping intellectual and visual vision of future teachers [5].

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CHECKING THE EQUIVALENCE OF EXPRESSIONS IN COMPUTER ALGEBRA SYSTEMS – APPLICATION POSSIBILITIES IN MATHEMATICS EDUCATION

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ABSTRACT

In connection with the spread of computer algebra systems (and algebraic calculators), the natural question arises: how to change the requirements and emphases of mathematics syllabuses? One possible domain that might be given more consideration in the future is checking the equivalence of expressions. It plays an important role in solving equations, manipulating expressions, and other domains (be it performed on a computer or by hand). In this paper, we examine some possibilities of integrating checking the equivalence expression in computer algebra systems more fully into the educational process. We present different schemes that describe the teacher's and student's activities in different situations, considering the particular goal, problem setup, student's preparedness, the specifics of the computer algebra system, access to computer algebra systems, etc. The schemes are based on problems of manipulating expressions selected from various areas of college algebra. They include step-by-step (line-by-line) solutions as well as solutions in which computers are used for solving larger blocks in one step. The described variants are titled:

-The student writes on paper,
-Computer algebra system as a text editor,
-Computer algebra system as an assistant in discovering errors,
-Computer algebra system as the maker of the next step,
-Computer algebra system as a solver,
-Computer algebra system as a component of an intelligent tutoring system,
-The student as the evaluator of computer algebra system.

Such a description of schemes may be beneficial not only to teachers, syllabus developers, textbook authors and the like but also to developers of computer algebra systems. The features expected from computer algebra systems for the realization of these schemes are described. The schemes are primarily designed for the current computer algebra systems (Derive, Maple, Mathematica and MuPAD); however, apart from the available features, mention is made of those that do not (yet?!) exist directly. For certain schemes, user interface is important.

Finally, some potential trends of research for the future are pointed out.

Keywords: Mathematics Education, Computer Algebra Systems

1. Introduction

In connection with the spread of computer algebra systems (and algebraic calculators), the natural question arises: how to change the requirements and emphases of mathematics syllabuses? How should we respond to the fact that the computer is capable of solving many problems of school or college mathematics? Is it possible or necessary to exclude some topics from the syllabus? What are the topics that can and should be emphasized? (See Herget, Heugl, Kutzler & Lehmann 2000.) To what extent should future students learn the step-by-step solving of algebraic problems, for instance? This article proceeds from the premise that step-by-step solving methods will remain on the syllabuses, at least in the nearest future; however, there may and should be changes in the approaches. One possible domain that might be given more consideration in the future is **checking the equivalence of expressions**. It plays an important role in solving equations, manipulating expressions, and other domains (be it performed on a computer or by hand). In this article, we examine some possibilities of integrating expression equivalence checking more fully into the educational process. With regard to problems, the article focuses on the ones dealing with expression manipulation (they are described in more detail in Part 2).

Putting it simply, it is possible to use or not to use computer algebra systems for solving problems. Between these two extremes, there can be plenty of other variants. The subject matter of this article (which is described in Part 3) is schemes that describe the teacher's and student's activities in different situations, considering the particular goal, problem setup, student's preparedness, the specifics of the computer algebra system, access to computer algebra systems, etc. All schemes are usable in practice. To what extent they will actually be used, however, depends on a number of factors. Such a description of schemes may be beneficial not only to teachers, syllabus developers, textbook authors and the like but also to developers of computer algebra systems, if they want to keep abreast with the educational market.

The features expected from computer algebra systems for the realization of these schemes are described in Part 4. There, a brief analysis is also provided of the capabilities of the currently widespread systems (Derive, Maple, Mathematica and MuPAD). In many respects, the computer algebra systems can cope well with checking the equivalence. Nevertheless, they may encounter some challenges as well. For certain schemes, user interface is important.

Promising is the harnessing of a computer algebra system, with all its mathematical capabilities, to an intelligent tutoring system as an expert module. In this case, the user cannot see the computer algebra system; instead, he communicates with the "shell" created specifically for teaching and learning, which exchanges mathematical information with the computer algebra system. The described schemes are useful for such tutoring system.

Finally, Part 5 points out some potential trends of research for the future.

2. Expression Manipulation Problems

The schemes discussed in this article are applicable to expression manipulation problems. In many school and college algebra problems, the texts are: *Remove parentheses and simplify, Combine into a single fraction and simplify, Combine like terms, simplify, Factor out factors common to all terms, factor by grouping terms, Simplify, and write answers using positive exponents, simplify, write in simplest radical form, etc.* (The topics are Polynomials, Exponents, Radicals, Logarithms, for instance). As a rule, the student needs to solve these problems step by step (line by line). In expression manipulation, all lines need to be equivalent to one another (as it is, the equality sign is put between them). Consequently, the checking of expression equivalence is very important. However, it is usual that textbooks and teachers do not pay much attention to it, confining themselves to the performance of certain steps. For instance, the formula $\frac{x^2-2x}{2x-4}=\frac{x(x-2)}{2(x-2)}=\frac{x}{2}$ is varied, overlooking the fact that x = 2 would render the initial expression undefined.

Analogously, the schemes discussed in this article are more or less applicable to checking the equivalence of equations and inequalities as well. However, the solution of equations and inequalities is a slightly different matter in that in some types of problems (for instance, equations involving radicals), steps are deliberately made that may change the set of solutions.

Apart from equivalence, another important issue regarding a new line in expression manipulation is rationality, of course. We may find a large number of lines that are equivalent to the previous line; however, they may not take us any closer to the solution. In this article, the issue of rationality is not tackled.

3. Schemes for Role Distribution

The use of computer algebra systems (and checking the expression equivalence) in solving mathematical problems may be sectioned into different schemes based on the roles of the student, the teacher and the computer algebra system. The boundaries of the schemes presented herein are fairly subjective, and different role distribution schemes are undoubtedly possible. The use of the *black* and the *white* (also called *glass*) *box* methods for a computer algebra system has been discussed for years already (for instance, by Buchberger 1990). In simplified terms, it means that when using the black box method, one is only able to "see" the problem and the solution whereas the white box method allows one to follow the entire solution procedure (regardless of whether it is presented by a human or a computer algebra system). The distribution given in this article represents an attempt at creating a "grayscale" specifically in terms of equivalence checking. Not all schemes are rational to be applied to all types of problems. Their rationality is contingent on various factors.

The student writes on paper

For a full scheme system, let us start from the conventional and common variant (marked with "A" in tables in this article). Under this variant, the student writes the solution procedure on paper without being aided by a computer algebra system, and the teacher checks it, also without a computer algebra system. However, if the teacher is able to use a computer algebra system in correcting the papers (B), he or she can simplify his work by checking the equivalence of the lines of manipulation using a certain strategy (for instance, binary search) for locating the error(s). This variant requires no access to a computer algebra system on the part of the student, and this is important in view of the fact that the teacher's access to a computer algebra system is easier to organize.

Computer algebra system as a text editor

Although the use of a computer algebra system as just a text editor is clearly an underutilization of its capabilities, this variant should still be given some consideration. As it is, attempts have been made to create reasonable capabilities for entering mathematical text on computer algebra systems (using buttons, palettes and key combinations, etc). Computer algebra systems are appropriate for entering the solution procedure. Of course, this makes checking the tests much easier for the teacher (D), since they have no need to type in the solution (in full or in part) themselves. In addition, it allows the student and the teacher to communicate over the Internet, for instance, which enables distance training. If the student has access to a computer algebra system and the teacher has not (C), it is possible, in principle, to check the printout of the solution. In this case, the only practical benefit for the teacher is the better readability of the script.

Computer algebra system as an assistant in discovering errors

Next, it is natural to consider the variant (E) where the students, after entering the line, uses computer algebra system for checking its equivalence to the previous line himself, and corrects his own work, if necessary. This is a variant that is perhaps the most promising (see Kutzler 1996). Of course, the correct establishment of equivalence does not necessarily guarantee the rationality of a particular step. We may also consider a variant where the student checks not his own answer but a prescribed solution (F). This creates various possibilities, from simply checking a work made by a classmate to checking a solution procedure with a "subtle" error hidden in it.

Unlike the previous schemes, this variant expects a lesser role from the teacher and a greater activity from the student.

Computer algebra system as the maker of the next step

Delving deeper into the capabilities of a computer algebra system, it is natural to desire that the computer algebra system do the next step itself. Computer algebra systems have various commands which could be applied to a part of or to the entire expression (equation), and which could take the student to the next line (G, H). (However, the length of a step made by these commands would often differ from that made by a human.) Several computer algebra systems are equipped with commands like *Factor, Expand, Simplify*, which herein may be called *step operations*. (In principle, the systems may provide the teacher with the possibility of programming such commands of different levels himself). The student selects the operation and the computer algebra system performs it. In this case, it may seem that the student can trust the computer's work and skip the equivalence check. However, let us present a somewhat surprising example here. This is an issue that pertains more to the user interface to enable the selection of a sub-expression. Let us assume that in Mathematica it is necessary to perform the following factorization. There is the possibility of applying a command (for instance, "Factor", "Expand", etc.) to only a part of an expression. It is a good possibility. Unfortunately, erroneous results are possible even there, if, for instance, the user chooses a wrong sub-expression and applies factorisation in response to the expression $x^2 - 4x$

$$x^2 - 4\,x$$

$$((-2+x)\,(2+x))\,x$$

If we now check it, we find no equivalence.

Computer algebra system as a solver

If we do not want to follow the steps, we may have the system solve the problem as a black box. This is perhaps the most widespread application of computer algebra systems in educational setting today. Here we can distinguish between a variant where the option of having the computer algebra system solve the problem as a black box is selected right from the beginning (I) and one where the student has already solved part of the problem (J). Indeed, there are special commands available in computer algebra systems, such as *Solve*, *Simplify*, *Factor*, *Expand*, etc. (which herein may be called *final operations*). Depending on the system and the situation, the same commands may also be executed for a single step.

As a rule, computer algebra systems are able to solve the problems of school and college algebra. In this case, expression equivalence check is not important either, considering the reliability of computer algebra systems.

Mention may also be made of the variant where the computer algebra system provides a textbook-like solution (K). In this case, the intermediate steps are also observable. Currently, this option is not directly available in computer algebra systems; in principle, however, it is programmable. Such a variant would provide the student with the opportunity to familiarize himself with the steps of the solution. The teacher would be able to obtain sample solutions and examine the suitability of problems for students. The full-solution approach would take us to the next variant.

Computer algebra system as a component of an intelligent tutoring system

If we had a feature that would fully present the steps of a solution for certain types of problems, the next variant conceivable would be checking the student's steps of solution against those presented by a computer algebra system. Of course, a problem can be solved in several different ways that are all correct, and to require the student to strictly adhere to a prescribed set of steps would be too one-sided. However, expression equivalence check would still play a major role in checking both the "prescribed" and the "innovative" steps of solution performed by the student.

Considering the fact that there are other capabilities suitable for an intelligent tutoring system (student module, tutor model, etc.), we can speak about intelligent tutoring systems. Since computer algebra systems already possess a number of the required features, they may have a future as expert modules of intelligent tutoring systems (Prank & Tonisson 2001). This means that they will be accessed for performing expression equivalence check, for instance.

The student as the evaluator of computer algebra system

The last variant (M) is one where advantage is taken of the fact that a computer algebra system is never perfect. Thus, the student can be assigned the task of checking whether there really is equivalence between expressions as shown by the computer algebra system. Emotionally, it is a fairly interesting variant. It seems to be more suitable for stronger students. However, this variant tends to be short-lived as the computer algebra systems are being steadily improved.

	Student	**Computer Algebra System (CAS)**	**Expression Equivalence Check**	**Teacher**
A	writes on paper			checks
B	writes on paper	assists teacher	teacher searches for errors	enters and checks expressions using CAS
C	writes in CAS	is a text editor		checks (without CAS)
D	writes in CAS	is a text editor and teacher's assistant	teacher searches for errors	checks, doesn't need to enter expressions himself
E	writes in CAS and checks the equivalence between a line and the previous line	searches for errors	student searches for errors	performs different operations depending on the approach
F	searches for errors in another person's solution	searches for errors	student searches for errors	performs different operations depending on the approach
G	writes and chooses the next step (e.g. Factor, Expand, etc.) for a part of expression	performs the operation	checks the correctness of the operation	performs different operations depending on the approach

H	writes and chooses the next step (e.g. Factor, Expand, etc.) for the entire expression	performs the operation	checks the correctness of the operation	performs different operations depending on the approach
I	writes and selects the final operation (e.g. Simplify, Solve, sometimes also Factor) at the beginning of the solution process	solves (from beginning to end), shows only the final result	checks the correctness of the operation	performs different operations depending on the approach
J	writes and selects the final operation (Simplify, Solve, sometimes also Factor) in the middle of the solution process	solves a certain part starting from the middle, shows only the final result	checks the correctness of the operation	performs different operations depending on the approach
K	examines the solution procedure	solves the problem, shows individual steps	(dependent on the structure of the solution procedure)	obtains problems and solutions
L	uses an intelligent tutoring system	is the expert module of an intelligent tutoring system	is important in checking expressions entered by student	obtains data on each student's errors, progress, etc.
M	checks CAS		may contain errors	is passive

4. What should a computer algebra system offer?

The following table presents evaluations of the necessity of one or another feature of a computer algebra system for the use of a particular scheme. Some evaluations are unambiguous (if a particular feature is unavailable then a particular scheme is inapplicable) while others are ambiguous. The necessity of the availability of a computer algebra system is expressed by the columns *CAS To Student* and *CAS To Teacher*. Availability here means the presence of both the possibility of and the skills for using a computer algebra system.

A computer algebra system's features may be listed with different degrees of detail. The list presented here represents only one possible way of doing it. It enables the user (teacher) to determine what schemes can be implemented using the computer algebra system at his disposal. Likewise, the computer algebra system developers can obtain ideas for improving their computer algebra systems.

The capabilities of expression equivalence check are described in the following columns. The column *Expression Equivalence Check* evaluates the necessity of a particular checking means in general, without imposing particular requirements on it (except that correctness should perhaps be assumed). Variant M is the only one to assume that we do not trust the expression equivalence check performed by a computer algebra system. For the implementation of the other schemes mentioned above, it is necessary that computer algebra systems correctly cope with checking the equivalence of expressions in practice. All the computer algebra systems under study (DERIVE, Maple, Mathematica and MuPAD) allow for the possibility of checking equivalence *Simplify(expression1 – expression2)=0* (or *Simplify(expression1/expression 2)=1* in checking the equivalence of expression1 to expression2. It appears that many school or college algebra problems are readily surmountable by computer algebra systems; however, there are also those that pose difficulties for them (Tonisson 2002).

Some computer algebra systems have special commands for checking the equivalence of expression equivalence (for instance, *testeq* in Maple) that perform a probabilistic check. (Tonisson 2002). Such commands can also be programmed using the programming tools available

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in the computer algebra systems. The column *Equivalence Check Command* presents an evaluation of the necessity of such a separate command.

Considering the needs for the above-mentioned schemes, it is important, particularly for the student, that equivalence check be easily performable in terms of the user interface as well. If much effort is needed for entering expressions, there is not much hope for efficient use. Palettes offer some advantages in this respect (Fuchs & Dominik 1999). The column *Comfortable Equivalence Check* is dedicated to the very availability of a button, a palette or some other handy option (for example tool of selecting two expressions).

Also provided are three more columns that are directly related not so much to expression equivalence check as to the schemes: *The Possibility Of Stepwise Operations*, *The Possibility Of Final Operations* and *The Presentation Of Full Solution*.

The evaluation was performed on a five-point scale, and the meanings of the grades are as follows:

2 – availability inevitable, urgently needed

1 – availability recommended

0 – no difference, 0? – depends on the approach

-1– availability not recommended

-2 – availability unacceptable (or the feature must be inaccessible for the moment)

(The possibility of making one or another feature of a computer algebra system inaccessible for educational purposes would be important in several instances.)

(* – for teacher, ** – for intelligent tutoring system)

	CAS To Student	CAS To Teacher	Expression Equivalence Check	Equivalence Check Command	Comfortable Equivalence Check	The Possibility Of Step Operations	The Possibility Of Final Operations	The Presentation Of Full Solution
A	-2	0	0	0	0	0	0	-2
B	-2	2	2	1	1	0	0	1
C	2	0	0	0	0	0	0	-2
D	2	2	2	1	1	0	0	-2 (1*)
E	2	1	2	1	1	-1 (0*)	-1 (0*)	-2 (0*)
F	2	1	2	1	1	0	0	0
G	2	0?	2	1	1	2	0	0
H	2	0?	2	1	1	2	0	0
I	2	0?	2	1	1	0	2	0
J	2	0?	2	1	1	0	2	0
K	1	2	0	0	0	0	0	2
L			2**	1**	0	1**	1**	1**
M	2	0?	2	1	1	0	0	0
	dependent on the country, school etc.		exists somehow in every CAS	exists in some CASs	insufficient	partially implemented	partially implemented	not implemented

The very brief comments about the existence of the features in present computer algebra systems are placed in the last row.

5. Future trends

The scheme system presented in this article is just a sketch. Naturally, all the schemes presented can be described in more detail, and their efficiency can be investigated by conducting further experiments. The schemes can be expanded and adapted to be applicable to other mathematical topics. Much can be made for the improvement of computer algebra systems, both in terms of their user interfaces and their mathematical capabilities. Quite a few items are already programmable in the existing computer algebra systems. However, a promising trend seems to be the creation of a separate interface where the mathematical capabilities of computer algebra systems (including equivalence check) could be realized more efficiently.

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MODELLING AND INTERPRETING EXPERIMENTAL DATA

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ABSTRACT

A lot of time is spent in traditional math courses for analysing graphs, computing minimal and maximal values, intersections and asymptotes of given functions. In some other examples the students have to find functions fitting to some given data, having special properties. Data are always given by the teachers and the situations treated seem rather artificial to the students.

Instead of this the students can do some experiments from physics and chemistry by themselves and try to model the data obtained by these experiments using mathematical concepts. However, it is rather time consuming to gather large lists of experimental data.

The collection of data during real experiments is supported by the CBL (Calculator Based Laboratory) and CBR (Calculator Based Ranger) from Texas Instruments. It is quite easy to transfer these data to graphic calculators for visualisation and further mathematical manipulations.

Various practical as well as mathematical skills of the students are trained by carrying out experiments, analysing the results and finally using functions for fitting data points obtained by the experiments.

We report about experiments being carried out in the years 1999 until 2001. In eight different classes consisting of students at the age of 16 to 18 experimenting with CBL, CBR and TI-92 was integrated within regular classes. About 50% of the students were girls. A special course for high ability students at the age of 14 was installed during the school year 2000/01 also carrying out experiments with CBL. In 2000 a group of students were testing the water quality in regular classes using CBL and ion selective probes from Vernier. The main goal of these projects was to train cross curriculum reasoning by the students.

The main basic skill in mathematics was to recognise the functional interdependency of experimental data and to find suitable fitting functions. For this reason the students needed some knowledge about different types of functions. They should know the typical shapes of the graphs and how the graphs change if the occurring parameters are varied.

Writing summaries of the experiments they understood the background of the respective experiments and some of the students wished to repeat the experiments to obtain better results. Interpreting results was difficult for the pupils especially in the course of testing water quality.

The students were really motivated. According to questionnaires and feedback forms they enjoyed practical work and felt free of the "pressure of learning".

It was also new for the students to work in groups. They had to distribute work to different group members in accordance to their abilities. Finally, it was quite difficult to find a fair grading for the students according to their individual achievements.

1. Introduction

Cross curriculum reasoning is a principal objective of natural sciences in Austrian high schools [Aspetsberger 2001]. In science courses students have to understand laws from physics and chemistry written in mathematical terms and to apply them to real situations [Aspetsberger 1999]. On the other hand they should be able to model processes in physics and chemistry by mathematical functions, to see the interdependencies of experimental data and to interpret the results of calculations and the occurring parameters in fitting functions [Aspetsberger 2000 b].

In math courses students often do not see the necessity for introducing new mathematical concepts. Examples demonstrating the use of these new concepts are often quite artificial and the students have even problems to understand these examples. [Laughbaum 2000] suggests to use data collected during experiments carried out by the students themselves to promote mathematical understanding of (new) concepts.

It is very motivating for the students to carry out some experiments in science courses. However, they have to learn experimenting, how to obtain good results, how to document their work and to write reports and how to work in groups. It takes a lot of time to reach these goals, but they seem worth for doing this additional effort.

Graphic pocket calculators like TI-92 from Texas Instruments help to visualise mathematical concepts, to plot graphs and to execute tedious and complicated calculations like determining regression curves. Experimental data can be investigated and visualised quite comfortable by using the TI-92. The main problem is to obtain a large set of experimental data of high accuracy.

CBL from Texas Instruments is a Calculator Based Laboratory which allows to collect data during physical and chemical experiments. Data are stored directly to a calculator e.g. the TI-92 for graphical visualisation and further manipulation. CBR from Texas Instruments is a motion detector which allows to gather a large amount of data points from an object in motion. CBL, CBR and TI-92 support data collection and manipulation. However careful experimenting is absolutely important for obtaining good quantitative results, which are necessary for functional modelling of experimental data.

We report about experiments being carried out in the years 1999 to 2001 at the Bundesrealgymnasium Landwiedstrasse, which is an Austrian Grammar school in Linz. In eight different classes consisting of approximately 100 students at the age of 16 to 18 experimenting with the CBL and TI-92 was integrated within regular science classes. About 50% of the students were girls. It was an important goal of these courses to train cross curriculum reasoning by the students. Most of the experiences mentioned in this paper concern to these science classes projects.

In one of these projects a group of 20 students at the age of 16 was testing the water quality of freshwater during regular biology classes using CBL and ion selective probes from Vernier. Their quantitative results of several water samples were compared to the official data obtained from the local government. So the students had a good feedback according to the accuracy of their experimental work.

During the last two years we introduced CBL, CBR and TI-92 to math and science teachers within several in-service teacher training courses promoting cross curriculum teaching. It was surprising to see that they had problems similar to the students when treating experimental data.

2. Experiments carried out

The experiments carried out by the students should lead to a better understanding of physical and chemical laws. On the other hand the students should learn to model and interpret data using mathematical methods. Cross curriculum reasoning was a major objective of the project. We have been inspired by [Holmquist, Randall, Volz 1998] and reports of Texas Instruments. In these articles and books we also found detailed descriptions of the experiments and handouts which could be used within lessons directly. The experiments were carried out by the students in groups of two or three.

We started with simple experiments measuring the temperature of endo- and exothermic processes to demonstrate how to use the CBL-system. In the next experiments the students had to determine melting heat and melting temperature of ice. Due to inaccuracies the students did not obtain 0°C exactly for the melting temperature. It was interesting that not all students wondered about that fact, some of them also documented very unrealistic values in their reports. Most of the students documented all the digits displayed on the CBL and did not care about the significant ones.

Investigating the laws of Boyle-Mariotte and of Gay-Lyssac the students used pressure sensors. In these experiments the students had to learn to model data by functions. They tried to find the parameters for the fitting functions by themselves as well as to determine regressions curves with the graphical pocket calculator.

By determining the concentration of an unknown solution using a colorimeter, plotting the curves of titration using a pH-probe or measuring the concentration of salty solutions using a conductivity probe the students had to solve some typical problems from chemistry.

In traditional physics courses it is very complicated or almost impossible to investigate the movement of a body in motion by measuring the distance of the body according to time. Using the CBR of Texas Instruments it was very convenient for the students to obtain big lists of (distance/time) - pairs describing the motion of a body. Fitting data points by functions lead to a mathematical description and analysis of several processes of motion.

Being familiar with the handling of the CBL and TI-92 the students analysed the quality of freshwater (see [Johnson, Holman, Holmquist 1999]) by using ion selective probes of Vernier in laboratory and outside. An intensive and very accurate calibrating of the probes was absolutely necessary for obtaining good results. This was completely new for the students. On the other hand having good calibration values it was really easy to measure the concentration of several ions in freshwater. Having only single point measurements there was no sense for a mathematical analysis. It was much more interesting to interpret the results and to compare them with official limits. Furthermore the students learned about the methods of how to analyse freshwater quality. Visiting the local institution for freshwater control the pupils learned that the same methods were used there.

During pre- and in-service teacher training courses we treated experiments concerning the cooling process of liquids and the unloading process of a capacitor in addition to the experiments mentioned above. It was surprising to see that they had problems similar to the students when treating experimental data, since they were used to operate with "exact data" solely.

3. Functional modelling and Interpreting

It was a major goal of the project to find fitting functions for the data obtained during the experiments and to give physical and chemical interpretations of the results. The students had to find functions fitting data by hand as well as to determine regression curves by the calculator. However, we preferred the method of varying the parameters of functions, because it required some mathematical reasoning of the students to choose suitable parameters for fitting functions. In case of determining fitting functions by the computers automatically the students had to interpret the parameters of the regression equations obtained from the calculator. We had to give the students a short introduction concerning the least square method and the meaning of the coefficient of correlation. In the following we discuss several types of functions being required for modelling data obtained in the experiments above.

3.1. Linear functions and direct relationship

Modelling data obtained during the experiments proving the laws of Beer and Gay-Lyssac we use linear functions.

According Lambert Beer´s Law light absorption is directly proportional to the concentration of the solution. In this experiment the students had to determine the concentration of an unknown green coloured solution. The students had first to make a sequence of different solutions from a stock solution of known concentration and to measure their light absorption using a colorimeter. Due to Beer´s Law the concentration/absorption data points lie on a straight line (see fig. 1). Now the students had to determine a regression line by hand or automatically by using the pocket calculator. Obtaining an almost homogeneous straight line indicates the accuracy of the sequence of different solutions produced by the students. Next the students had to measure the absorption of light of the unknown solution and to determine its concentration using the regression line by hand calculation or from the graph directly (see fig. 1). Note the steady change of mathematical reasoning, chemical interpretations and practical work in this example.

Homogeneous linear functions are typical for direct relationship. It was quite unusual for students (and even teachers) to combine the concept of relations with the concept of functions (see also the comments concerning rational functions). Of course, they were aware of the fact that a direct relationship can be illustrated by a straight line, however it was new that homogeneous linear functions indicate direct relationships.

Modelling data points obtained in the experiment (Gay-Lyssac) investigating the pressure p of a confined gas according to temperature T (measured in °C) the students used linear functions of type $p(T) = m \cdot T + b$. The vertical intercept b indicates the pressure at a temperature of 0°C. However there is also a physical interpretation of the intercept with the horizontal T-axis. It indicates the absolute zero of temperature (-273,15°C).

3.2. Quadratic functions

Modelling the motion of a bouncing ball or a ball rolling down a ramp quadratic functions were required. This was due to the fact that the motion of a bouncing ball as well as the motion of a ball rolling down a ramp were special cases of the free fall which could be described by the formula $\frac{a}{2}t^2 + v_0 t + s_0$, where s_0 denoted the starting point and v_0 the starting velocity of the ball.

For modelling the motion of a bouncing ball it was more convenient to use quadratic functions in perfect-square form $a \cdot (t-b)^2 + c$, where b and c denoted the coordinates of the vertex of the parabola (see fig. 2). Suited values of the parameter b and c could be found using the Trace mode in the graph window. Many students (and even some of the teachers within in-service teacher training courses) tried to match the graph with positive values for a at their first attempt. It took some time to find out that a suitable value was $a = -4.9$. The students had also to give a physical interpretation of the value of a ($a = -\frac{g}{2}$, where g is the gravitational constant of acceleration g = 9.81 ms^{-2}).

Some of the teachers within in-service teacher training courses tried to find the parameters in the following way: They chose the parameters b and c as the coordinates of the vertex of the parabola as described above. For determining the third parameter a they selected a sample point from the graph and tried to solve an equation depending on the variable a solely. However they had chosen a sample point very near to the vertex, so they obtained an unsuitable function. This was due to the fact that the data measured were not totally exact and even minor round-off error might cause wrong results. It turned out that even teachers were not used to handle "real data".

3.3. Rational functions and inverse relationship

Rational functions were required for modelling graphs of pressure p against volume V obtained by the experiment of Boyle´s Law. The students had to find suitable values for the parameter a within the functions $p(V) = \frac{a}{V}$ (see fig. 3). Functions of this type are typically for inverse relationships.

The inverse relationship between pressure p and volume V in the experiment above could also be proven by testing the relation $p \cdot V = const$. The students had to compute the product $p \cdot V$ in a data/matrix window of the TI-92 pocket calculator and to verify whether the product was constant. Although the students had done the calculation of the product they did not see the simple interdependency $a = const$ of the parameter a in the rational function and $const$ in the relation above by themselves. However, it was easy for the students to verify the transformation $p \cdot V = a \Leftrightarrow p = \frac{a}{V}$ algebraically. It was a problem for the students to combine the concept of functions with the concept of relations and algebraic manipulations.

3.4. Exponential functions

For modelling the decrease/increase of temperature during a cooling/heating process or the decay of voltage when unloading a capacitor exponential functions of type $a \cdot b^x + c$ were required.

Decreasing exponential functions of type $a \cdot b^x$ having the x-axis as an asymptote were suitable functions for modelling the decay of the voltage of a capacitor during the process of unloading. A typical graph of voltage against time of an unloading process can be seen in fig. 4. We had to choose suitable values for the parameters. For the parameter a we substituted the initial voltage 4.8. However it was not so easy to find a suitable value for the parameter b. Instead of this we chose an equivalent formulation of the type $a \cdot e^{-\lambda x}$ for decreasing exponential functions. For determining a value for λ we used the relation $\lambda = \frac{\ln 2}{\tau}$, with τ being half life which was the quantity of time for a decaying process to be reduced to half. We determined $\tau = 0.8$ from the graph (see fig. 4), i.e. after 0.8 s the voltage of the capacitor had decreased to 2.4 V which was half of the initial voltage. Since the unloading process started after 0.8 s we had to shift the graph of the exponential function $4.8 \cdot e^{-\frac{\ln 2}{0.8}x}$ in direction to the right by subtracting 0.8 from the x-values. Thus we obtained the following function $4.8 \cdot e^{-\frac{\ln 2}{0.8}(x-0.8)}$ fitting our unloading process.

Temperature vs. time graphs of cooling processes typically have horizontal lines $y = c$ as an asymptote, where c denotes room temperature. Similar to the modelling process above we had to find functions of the type $a \cdot e^{-\lambda x} + c$ shifting the graphs vertically by adding the constant c.(see [SCHMIDT 1995]) However, it was not so easy to determine regression curves for cooling processes by the calculator, since the TI-92 was able to compute regression equations of type

$y = a \cdot b^x$ solely. To manage this problem we had to shift data points first by subtracting the room temperature c, to compute a regression curve for the transferred data points next and finally to re-shift the regression curve by adding the constant c again. (see fig. 5)

3.5. Trigonometric functions

Trigonometric functions were required for modelling periodic processes, e.g. the motion of a pendulum (see fig. 6) or of a spring. The students had to find suitable values for the parameters of functions of type $y(t) = a \cdot \sin(\omega \cdot (t - b)) + c$, where a was the amplitude of the motion. b and c allowed to shift the graph of the function to left or right and up and down respectively. The value of c was easily found by measuring the initial position of the pendulum/spring. The frequency ω was determined by measuring the time T of a period according to the relation $\omega = \frac{2\pi}{T}$.

4. Experiences and comments

The students were really motivated. According to questionnaires and feedback forms they enjoyed practical work and felt free of the "pressure of learning". Some of the students also mentioned the importance of learning how to use technical instruments.

Experimenting with the CBL, CBR and the TI-92 (or comparable graphic calculators) required and trained several basic skills in different areas, e.g. mathematical skills, verbal skills, practical skills and social skills (see [Aspetsberger 2000 a]).

The main goal of the regular science class project concerning mathematics education was to recognise the functional interdependency of experimental data and to find suitable fitting functions [Aspetsberger 2000 b]. For this reason the students needed some knowledge about different types of functions. They should know the typical shapes of the graphs and how the graphs change if the occurring parameters are varied. Although the students had already learned the underlying mathematical knowledge in regular math classes, it was new for them to apply this theoretical knowledge in real situations. This was also relevant for even quite simple mathematical concepts like direct and inverse relationships. The students had to decide which relationship is applicable. E.g. there is an inverse relationship between the pressure p and the volume V of a confined gas if the product p*V is constant or there is a strong argument for a direct relationships if the data points lie on a straight line running through the origin.

A further mathematical goal was to confront the students with real data. From regular math lessons the students were always used to obtain exact results from their calculations. It was quite surprising for them to obtain from the experiments e.g. 0.576°C or 1.25°C for the freezing temperature of water instead of the expected 0.000°C. They had to learn that experimental data could be inexact and to see the need of statistical methods.

It was surprising how difficult it was for the students to read and carry out instructions stepwise without additional explanations of the teacher. However, it was much more unfamiliar for the students to document their work writing reports and interpreting the results obtained. This was really an important verbal skill, which the students had to learn. Writing summaries of the experiments they understood the background of the respective experiments and some of the students wished to repeat the experiments to obtain better results.

Concerning the use of the TI-92 during experimenting a secure handling of the various commands and features for the different representations of data would be very helpful however it was not absolutely necessary. In this case - at the beginning of the courses - we had to give more detailed instructions. At the end of the courses we gave only short descriptions of the experiments. Extensive instructions were sometimes confusing for the students.

One of the major problems was the lack of time. In regular lessons – lasting only 50 minutes - we had to explain the goals and the background of the experiments and afterwards the students had to do the experiments. However there was often no more time left to discuss the problems occurred during the experiments. The discussion and interpretation of the results obtained by the students had to be delegated to the next lesson, which sometimes was one week later.

Interpreting results was difficult for the pupils especially in the course of testing water quality. It was very hard to estimate the accuracy of the results obtained by the CBL-system. The students were not familiar with the necessity of calibrating the probes and they wrote in their protocols all (senseless) digits of the results indicated on the display.

It was also new for the students to work in groups. They had to distribute work to different group members in accordance to their abilities. The further problem for the group members was to accept a unique grade for the whole group. It was quite difficult to find a fair grading for the students according to their individual achievements.

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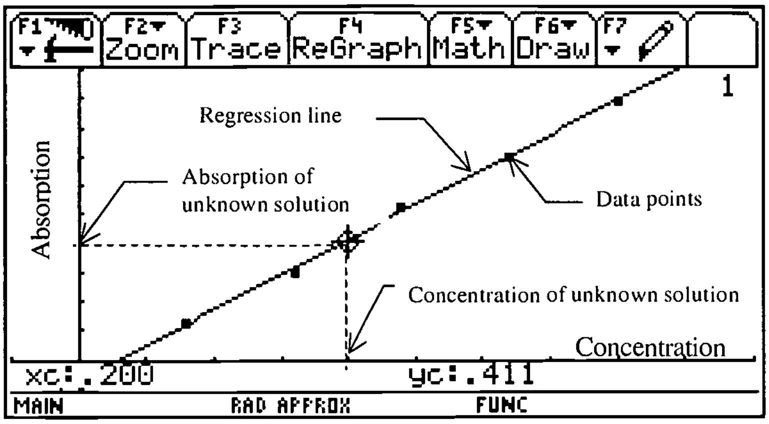

Fig 1: Determining the concentration of an unknown solution using Beer´s Law

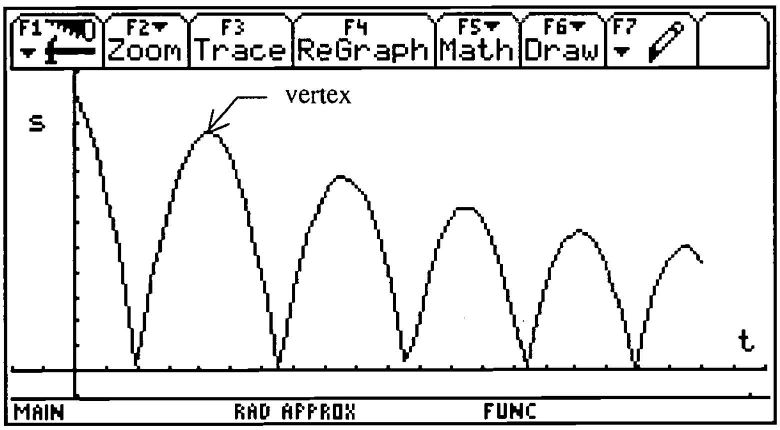

Fig 2: Motion of a bouncing ball. Quadratic functions in perfect square form $a \cdot (t-b)^2 + c$ are suitable for fitting the parabolas. The coordinates (b;c) of the vertices of the parabolas can be determined from the graph using the Trace mode.

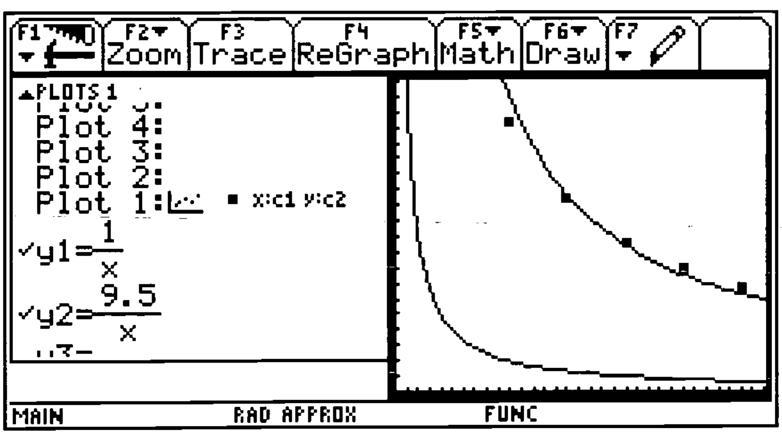

Fig 3: Using rational functions for modelling Boyle´s Law.

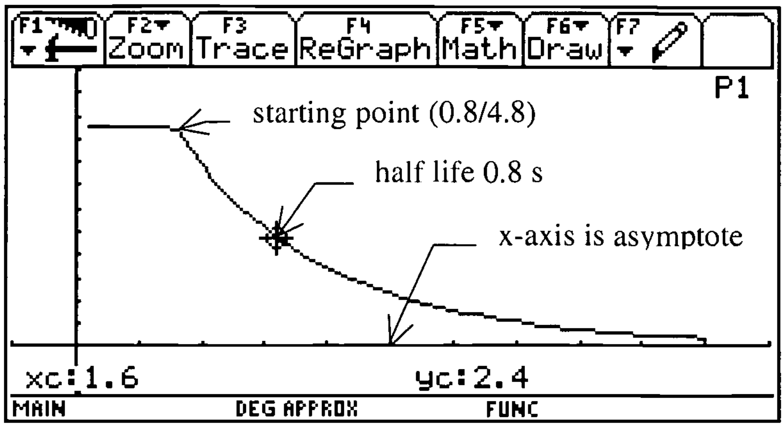

Fig 4: Using exponential functions for modelling the unloading process of a capacitor

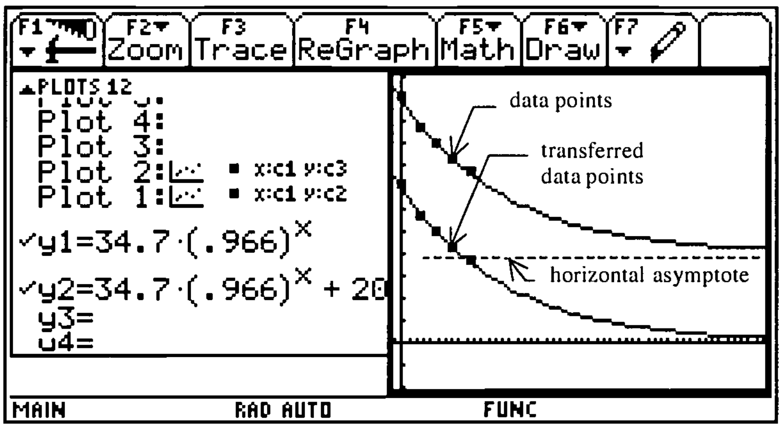

Fig 5: Using exponential functions of type $a \cdot e^{-\lambda x} + c$ for modelling cooling processes having a horizontal asymptote $y = c$.

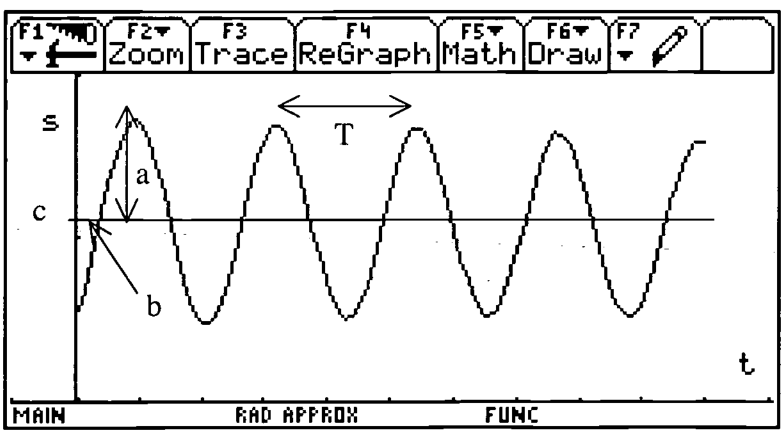

Fig 6: Modelling periodic motions of a pendulum using trigonometric functions of type $a \cdot \sin(\omega \cdot (t-b)) + c$.

DEVELOPING WALLIS:
A Web-based System to Enhance Mathematics Teaching

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ABSTRACT

Mathematics course-work is required for most science and engineering degrees. However, reports in the field of Mathematics Education address the growing deficiency in mathematical skills amongst students and the need for universities to take steps to moderate this problem. One of these is the development of material in an ICT format.

Based on our previous separate work we are developing a more generic tutoring environment. Our long-term goal is an authoring and tutoring system which will allow members of staff to design their own material and students to use it asynchronously as additional support to their conventional studies. It is hoped that interaction with the system will eliminate any misconception that they have from high-school and increase their motivation to study.

This paper presents our approach towards the implementation of the system and a preliminary pilot-testing which have so far raised significant issues.

1 Introduction

In the last few years researchers and university teachers are more concerned than ever with the evident problem of the growing deficiency in mathematical skills amongst science and engineering students, the so-called Mathematics Problem. Recent reports (for example, Hunt and Lawson, 1996; LTSN 2000) show a decline amongst students with apparently good A-level grades in Britain, and there are similar problems in other countries.

Apart from the grade inflation, which increases the complexity of teaching mathematics at university level, the other most common factor that universities have to face is the diverse background and level of the students (GCSE, A levels, Highers and different foundation courses for Britain and similar diversities in other countries). In addition, since these students do not come to university to study mathematics as a major subject, they are less motivated. Students, particularly in departments where they can postpone the module for a later year (as in Greek Universities), do not really understand the significance of studying mathematics at an early stage. This becomes evident to them later when they face problems with their other courses and although they may try to catch up, it is already too late. Finally, engineering (and other science) degrees cannot afford to spend a lot of lecture time in reinforcing basic mathematical skills. This should have been taken care of earlier, in school, and this is why reports (for example, LTSN, 2000; National Skills Task Force, 2001) urge a joint strategy involving schools, universities and government. Unfortunately, these efforts have brought no significant changes as yet. Consequently, universities need to take steps on their own to moderate the problem.

Based on the above, the design and delivery of an appropriate mathematical curriculum is of central importance to our department, which teaches some of these essential 'non-specialist' mathematics topics to many hundreds of students in science and engineering degrees. Currently the courses follow a more or less conventional structure with lectures, tutorials, weekly assignments as well as the final module assessment. As an additional support, much of the paper course material is also made available on the web. The university's role, on the other hand, is to respond positively to the use of a variety of strategies in order to improve the situation described above and to provide a more efficient solution. After a major restructuring of these courses, the time is now ripe for course materials to be developed in an ICT format.

2 Background

Lately, many researchers (such as Major, 1993; Battista, 1998; Cumming, 2000) argue that despite the efforts, and the resources spent, very few educational systems are in 'routine use' due to the tendency to develop them in the isolation of researcher's laboratories. One the other hand, there have been many reports (for example, Barron, 1998) portraying the web as a world-wide, efficient, easily integrated, interactive technology for learning, providing developers do not neglect its pedagogical features.

Clements and Battista, in (Kelly and Lesh, 2000) describe a model for integrated research and software development. Based on this as well as other researchers' views in the Artificial Intelligence in the Education (*AI&ED*) field (Conlon and Pain 1996;

Koedinger, 1993; Cumming 2000), we decided to design the whole system based on observations and on a careful and lengthy user study of real tutorial situations instead of simply on our intuition and creativity.

Therefore one of the authors already delivers and attends tutorial and lecture sessions that are related to the material being developed, in order to explicitly explore some of the misconceptions that student have, what kind of help would be particularly interesting for them and what to base the design on. This way, while exploring possible solutions for delivering the material, we simultaneously consider what would be appropriate for the students and how to make them actively participate in the learning process rather than merely delivering knowledge through a different source.

Furthermore, in our design we are careful to take account of several issues that many researchers (for example Kyriazis and Mpakogiannis, 2000; Boshier et al., 1997; Strickland, 2001) consider particularly important. These include the visual interface (accessibility, interactivity, attractiveness), the input tools for answers and assessment, the dialogues and feedback techniques as well as the goals of the software, the material's accuracy, and its proper evaluation. To successfully address all these issues we are collaborating with researchers, technologists, lecturers and more importantly students in order to maximize the software's contributions.

Finally, from reports about educational software (Underwood et al., 1996; Pelgrum, 2001), we take into consideration the fact that teachers (or lecturers) would like the opportunity to be more involved in the whole design process of computer-based environments for their students and that they comment favourably (Wood, 1998) on systems which were designed to help them monitor their students' progress and identify individual strengths and weaknesses. At the same time, such a system should not require advanced programming skills so as to be useful to all members of staff.

3 Design and Implementation

Based on our previous separate work we are now developing $W\grave{a}LL\hat{\imath}\int$ a Web Based Assistant for Learning in a Locally Integrated System. The project's main goal is to build a more generic tutoring environment together with an authoring tool which will allow members of staff to design their own material, and students to use it asynchronously as additional support in their conventional studies.

Many possible solutions were explored, but we decided to deliver the application through the Internet mainly because web availability will permit students to work on it independently and in their own time as extra support for their conventional studies. Consequently we employ the use of a Java Server to dynamically create some of the pages and Java applets for the interactive parts firstly because of their platform independence and, not least, because of the growing list of available mathematical applets which are freely distributed and which can be integrated easily into our system.

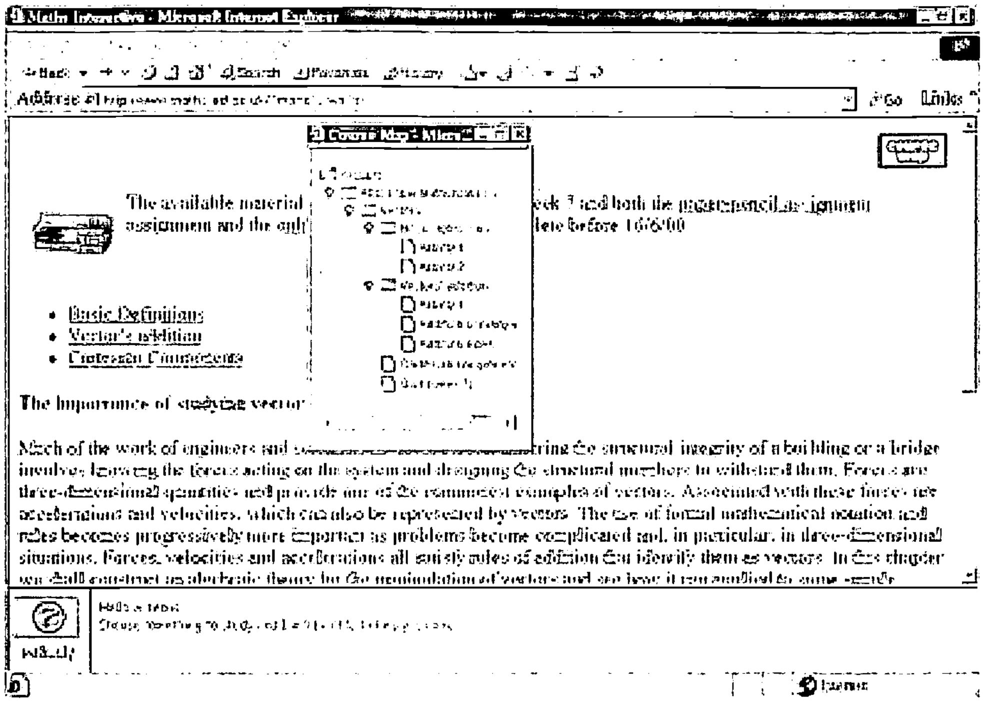

Figure 1: The Environment

3.1 The environment

The whole system is delivered through a web browser [1] and consists of two basic parts; the main frame and the feedback frame (see figure 1).

In addition to static text, the main frame contains, where applicable, interactive parts that are embedded into the whole HTML page. As students work through the material, they can interact with the insets to see various items such as, a different example, some special cases of a formula, animations on how to do things and material that cannot be seen in a static book or web page. In this way, aside from simply reading through some online text, which has proven to be very boring for some of them, they actively engage with it. The feedback frame consists of a text area where we provide help to the students when necessary and a button with which they can explicitly ask for help.

From early observations it was evident that we had to have an efficient way for the students to navigate through the material which was more sophisticated than hyperlinks between pages. Therefore we provide the students with a pop-up window (figure 1) which presents a tree-like map of the material and some other buttons which are activated according to the page the student is accessing. For instance the 'welcome page' as well as any separate interactive page contains a 'resume button' which takes the students to the last material they were studying. The use and necessity of this, as well as other buttons, is something that we need to test before continuing with the system's development.

[1] the only requirement being the Java Plug-in which is freely distributed by Sun (http://java.sun.com) and easily installed on any platform

3.2 Interactivity and Feedback

The system knows which pages the student is currently accessing and if she/he asks for help the system suggests what they should study afterwards. As more material is developed we will be able to build a more detailed description of the user's knowledge and have the system suggest to the student to study other parts of the site.

In addition we provide a more specific kind of help to the students who are working on their own. We follow previous work (Mavrikis, 2001) which observed that a particular type of feedback called affective (in the sense of targeting the emotional and motivational state of the student) can be facilitated effectively to increase students' willingness to work with the system and study the material. In this we targeted Dynamic Geometry Environments (DGEs) (like Cabri, Geometers Sketchpad etc) which were enhanced with a feedback mechanism that not only avoids the need for a teacher always explaining the task and supporting those working in a lab, but also helps students interpret their actions in a meaningful way.

In a similar way, the interactive applets communicate with the feedback frame and provide task-specific hints, regardless of how the students use the help system. In addition, by monitoring mouse activity and the user's interaction with the rest of the system's components (for instance, toolbar, navigational menus etc,) we provide students with lively directed comments on their actions (figure 2).

3.3 A Prototype

As we have already pointed out, observation of lectures and real tutoring situations provide valuable information for the design of the system. Based on these, we developed a prototype system which addresses the subject of vectors and particularly the introductory aspects of their study. This prototype enabled us to run a pilot test to see how to proceed with the system's design.

3.3.1 The material and the activities

Usually students do not understand that vectors are completely different algebraic objects from numbers. They tend to neglect to use a different notation to denote a vector and they do not describe them in terms of components. To cope with that effectively when students study the relevant HTML page the interactive embedded applets demonstrate some of those aspects of the theory by allowing them to manipulate a vector and change its size, its direction, visualise the unit vector and generally 'play' with vectors while reading about their definition and properties.

In addition, some weaker students have difficulties with the graphical interpretation of a vector in a three dimensional (or even two dimensional) coordinate system. To deal with this, we designed separate activities that are linked to the main pages (figure 2). These direct students to work on cartesian axes where, by manipulating the vectors and using some tools, we expect them to answer relevant questions (such as giving the magnitude, direction or unit vector of a given vector). The rest of the activities deal with addition, subtraction or component notation.

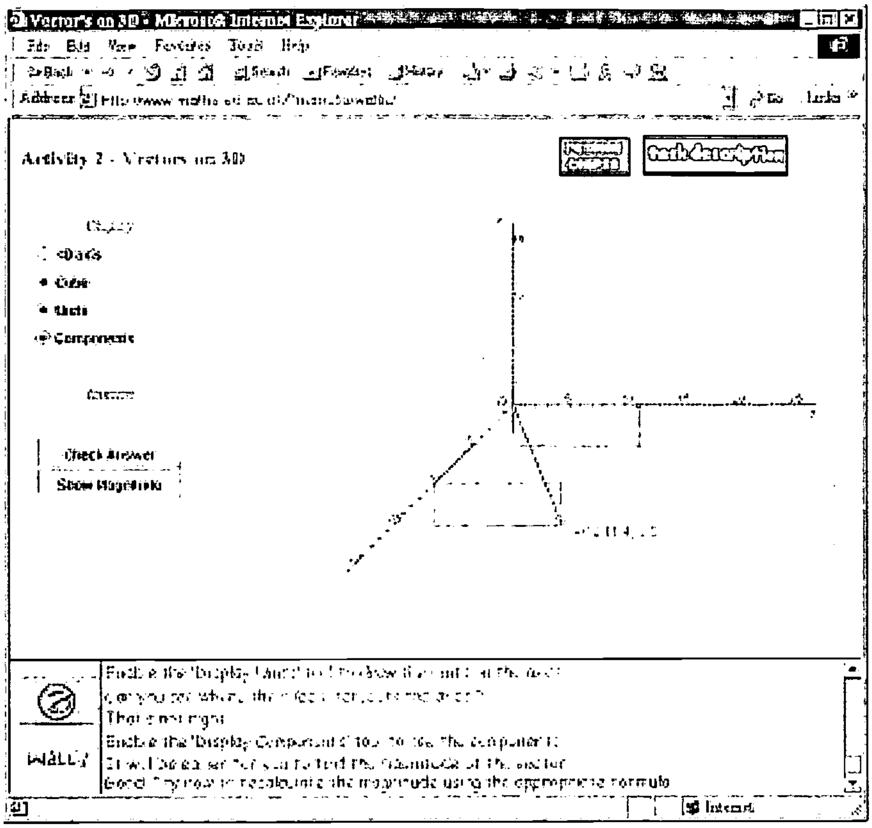

Figure 2: An example of a page with an interactive applet

4 Evaluation: a Pilot test

4.1 The Aims

It has long been argued that proper evaluation is an important aspect of the development of computer based teaching and assessment. As Conlon and Pain (1996) outline, only well designed steps based on research methodology can lead to effective software. One of the most important steps is that of the formative evaluation and in particular a phase which Clements and Battista (2000) call 'investigate the components' in the sense of testing the individual parts of the software.

In our situation, this means that we need to check if the students are indeed able to control the input devices, how they understand the screen design and actions they have to take, and where to put the various components on the screen. Apart from that, we need to know their general attitude to accessing material online as well as their view of online self-assessment or quizzes. This is the point of conducting a small pilot-test in such early stage of the project with so little material developed.

4.2 Participants and Methods

The participants were 9 female and 5 male first year students attending the module of 'Applicable Mathematics' consisting mainly of basic university level algebra. They come from diverse backgrounds and study for various degrees (Physics, Chemistry, Engineering). This small cohort of students had only taken GCSE or Standard grade mathematics which would not usually be sufficient to study numerate subjects in Edinburgh. But they may have had special circumstances which made it impossible to study more mathematics in school but there was still a reasonable expectation that they could pick up the necessary mathematics in a fast track course at the start of their studies. In fact, when asked how familiar they were with vectors, three of them ticked 'just familiar' on the questionnaire, all the rest said they were not familiar with them at all. The trial was run in a week when the students had no other contact hours.

The subjects were given an information and a questionnaire sheet which asked them to explore the site, study the online material and complete an online quiz in their own free time. They were asked to conduct the first session in a specific lab of the department so that an expert could be present to provide assistance, if necessary, either about the use of the site or about the material and the concept of a vector.

In addition to helping the students, the lab session would give us the opportunity to actually observe them in a real situation and see their reaction towards the system. Apart from one dedicated observer that the students were aware of (since they could ask questions of him) there were two more experts of whom the students were unaware. Although they were able to observe only five students in action, due to time constraints, their help together with the detailed log files, that were recording every action of the students, were particularly instrumental in reaching some important conclusions.

4.3 A Review of the Results

All of the subjects reacted favourably to having material online and although some had minor problems none were frustrated or confused to the point of discomfort.

Some problems that occurred with two of the first students that used the system had to do with a Netscape Navigator bug and how it handles applets. This frustrated the specific students and although the problems were resolved immediately, they were still unsatisfied, as was obvious from their comments. This event shows the significance of such a phase and the problems that can emerge; problems which otherwise might not have been found. The rest of the students, having not faced any problems, remarked favourably about the system both during their session and in the questionnaires. Moreover, eight of them used the site again after their introductory session, not only to complete the quiz but also to interact with the material again.

It was very interesting to see two of them to actually take notes using a notebook from the online material as they might do in a conventional lecture. Most of the students used a calculator for the answers, while some of the more experienced among them actually expected the existence of an online calculator.

In addition, from careful observation of their interaction (and from the detailed log files) it seems that all of them were capable of navigating through the site without many difficulties. The pop-up map proved useful to all of them, even the less literate ones. This is because from early observations we knew that students usually lose this kind of window. This happens when the main browser window regains focus, i.e., becomes the selected window. To avoid this we decided to force the pop-up window to remain always on top unless the student explicitly closes it. This caused some problems to those more familiar with computers who wanted just to minimize the window but found that the window kept bouncing back; something that hey found particularly annoying. Nevertheless, it is better to avoid the confusion for the less experienced ones since the others are always capable of finding their way through.

Other difficulties that students faced with the interactive applets interface were soon overcome either by discussing it with other students or after reading the task's description. The tools, although unfamiliar to them, were used appropriately. It seems though, that it was not always the system's hints that helped them to use the tools but rather the students' curiosities. The help button was used effectively by some of them but this is something that needs further analysis based on the log files. We need to see what their action was before and after the hint, as well as its effect. A preliminary analysis shows that most of the times that they explicitly asked for help they followed the suggestion accordingly.

More interesting results though, come from the experts who were observing the students without their knowledge. Apart from discussing how to do things (such as navigate or find particular material) they were all surprised by the feedback frame. 'Look it knows my name', said one and 'it talks back to you' were the first comments that students usually made.

Moreover, by watching their behaviour, the experts could see that they were not always reading the feedback but were just trying actions or asking questions of the person who was in the lab. The fact that they were able to ask questions of a person probably diverted them from seeking help from the system. When we suggested the use of the help button they usually followed the system's advice correctly. They were often surprised to find that their question had been anticipated in the system's help. This in conjunction with comments that they sometimes found the interaction boring probably means that we have to divert some important feedback to other sources such as a pop-up window, audio or even an animated agent, in addition to, or instead of, the feedback frame.

On the other hand, we implemented two applets that were not designed to give feedback at all. These were used deliberately, to observe the students' reactions. As expected, they asked more questions about their use as well as their meaning and a preliminary analysis of the log files shows that they sought help from the system more often, something that had, of course, no effect. This shows again how task-specific feedback can help the students interpret their actions and avoid the need for long descriptions or the advice of an expert.

Finally, the questionnaires that the students completed, provide us with information and ideas for further development of the system. For instance, students said that they would like to have increasingly more complex material, immediate feedback from the quiz, and the ability to print some material as well as a report of their achievements.

5 The Educational Impact

From a learning perspective the development of the system provides the students with an alternative supportive medium which they can use in their own time. With it, they

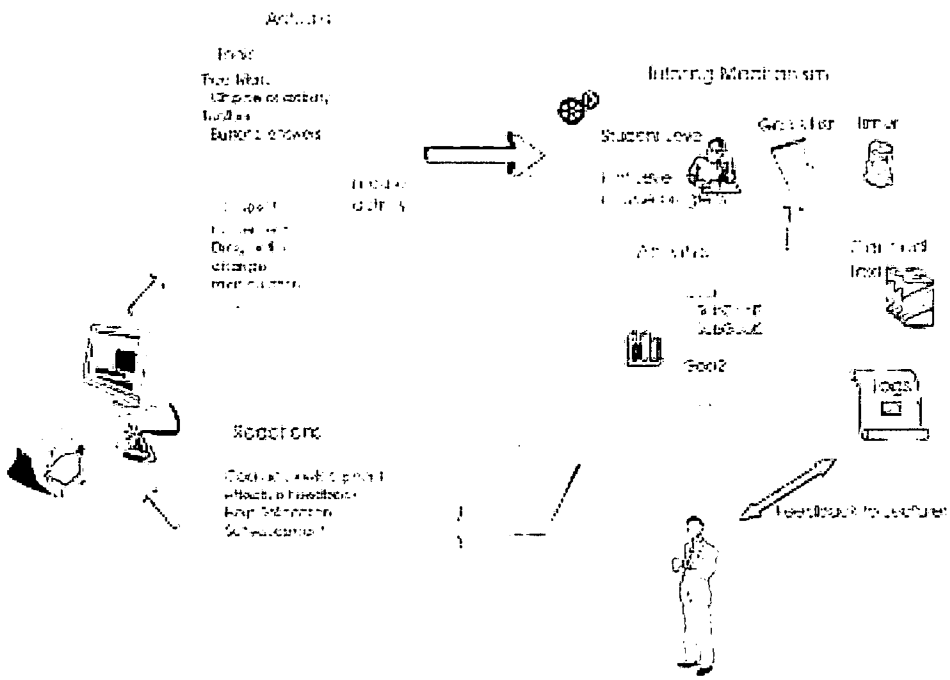

Figure 3: The system's impact

will be able to study on their own and assess themselves in a way which would otherwise consume too much of the department's resources and would be less motivating for the students.

The fact that our long term goal is to build an authoring tool for the lecturers will provide the latter with the ability, not only to create specific activities and online material, but also to monitor their students' progress and offer personalized help.

Students, on the other hand, engage in learning, explore and discover the subject (in a constructivist sense) instead of simply receiving information from a 'modern' lecture that uses computers for demonstrations. Moreover, by receiving tailored feedback specifically for each activity, as well as their actions they can rely on more information in addition to their own intuition and reflect on the knowledge they receive.

6 Further Work

Further development of the system will be based on direct feedback from the students and the lecturers and continuous loops of observations, changes, and development. For instance, from comments that other researchers we are collaborating with have already made, it is evident that we need to pay more attention to personalized fonts and the colours of the feedback and take into careful consideration disability issues. All these will lead to a more solid and effective system and to a product driven by the educational needs instead of a 'technological wizardry'.

By conducting further research on the development of the feedback mechanism, we will be able to build a more detailed user model that tackles students' actions as well as their 'help solicitation process' more effectively.

It remains to be seen if the interaction with the system will eliminate misconceptions that students have from high-school, how much they will use it, and if their initial motivation to work with it will remain active.

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USING TECHNOLOGY AND COOPERATIVE GROUPS TO DEVELOP A 'DEEP UNDERSTANDING' OF SECONDARY SCHOOL GEOMETRY

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ABSTRACT

In Fall 2001, the Conference Board of the Mathematical Sciences (U.S.A.) released a list of recommendations on the mathematical preparation of prospective teachers. These included recommendations that prospective teachers take courses that "develop a deep understanding of the mathematics that they will teach" and that "prospective teachers should develop the habits of mind of a mathematical thinker and demonstrate flexible, interactive styles of teaching."

In 1997, Ohio University significantly revised the 'Foundations of Geometry' sequence taken by prospective secondary teachers. The revised course uses a significant amount of group work and technology to plant the seeds for a deep understanding of the geometry taught at the secondary level. By using software programs and manipulatives, students begin building an understanding of non-Euclidean and Euclidean geometry from the outset of the course. The use of cooperative group work and written reports on group projects develop student's writing and oral communication skills.

A major goal of the course is to give the students the experience of 'doing mathematics'. During the course, the students use the experience gained using software programs and manipulatives to develop their own axiom systems and use these systems to prove theorems. This paper describes the overall structure of the course, how and where various learning aids are used, and discusses the effectiveness of the course in promoting a 'deep understanding' of the secondary school geometry curriculum. The assessment is based on student work and journals collected during the first four years the course was offered in its current form. The evidence suggests that the students improve their ability to prove theorems and develop a good understanding of models and axiomatic systems.

Keywords: Geometry, Non-Euclidean Geometry, Geometer's Sketchpad, Secondary and Middle School Teacher Preparation

Introduction

In its Summer 2001 report, *The Mathematical Education of Teachers*, the Conference Board of the Mathematical Sciences (U.S.A.) "calls for a rethinking of the mathematical education of prospective teachers within mathematical science departments." [CBMS, pg. 3] The report is sweeping in scope and makes eleven recommendations related to the mathematical training of prospective teachers, cooperation among the parties involved in teacher education, and the support of high quality school mathematics teaching.

The first recommendation is that "Prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach." (Recommendation 1) They note that K-12 teachers need a mathematical foundation that will help them assess errors, nurture talented students and recognize their students' level of understanding. Proof and justification are also emphasized. The fourth recommendation asserts that mathematics courses should ". . . develop the habits of minds of a mathematical thinker and demonstrate flexible, interactive styles of teaching." In a discussion of technology, the report states that prospective teachers should be given experience with technology with two goals in mind: the short term goal of using it in teaching and the long-term goal of helping them "become thoughtful and effective in choosing and using educational technology." [CBMS, pg 48]

In Fall 1997, Ohio University began to offer a revised geometry course that meets many of the criteria suggested in the Conference Board (CBMS) report. This course is taken primarily by prospective middle and secondary school teachers. Motivated by the desire to have a course aligned with the NCTM standards [NCTM], to have the students gain the experience of 'doing mathematics' and develop the material in a manner consistent with pedagogical styles that they will eventually have to adopt for licensure, the course rigorously develops much of the content of the secondary school curriculum using structured cooperative groups working in a computer lab.

In that the revised course used a significantly different teaching style and added some content, it was natural to wonder how students would respond to the course and whether or not the course was effective in meeting its goals. From the outset, the instructors kept copies of student work and journals with a view towards assessing the effectiveness of the course.

Course Description

The revised geometry course was designed to give students a strong understanding of the content of a standard secondary school geometry course and to build connections between geometry and other areas of mathematics. Although developed prior to the release of the CBMS report, it is consistent with many of the recommendations the CBMS report makes regarding geometry courses for prospective teachers. In particular, it provides a solid understanding of core concepts of Euclidean geometry; an understanding of the nature of axiomatic reasoning and facility with proof; multiple representations; an introduction to transformations; and uses dynamic drawing tools to conduct geometric investigations. [CBMS, pg.41] In that teachers who are taught using 'reform' techniques tend to use them more than teachers taught using traditional techniques ([Jo91], [MTH95], [SS94]), the revised course was taught in a computer lab using structured cooperative groups.

The course provides an in-depth discussion of the axioms used in some standard secondary geometry texts. In the United States, many secondary school geometry texts include a development of Euclidean plane geometry (loosely) based on a set of axioms developed by the School Mathematics Study Group ([WW98]) in the early 1960's. (It should be noted that these texts also often discuss transformational geometry and introduce non-Euclidean geometry.) In the revised course, we develop this axiom system and establish the standard results of Euclidean geometry.

There are several ways in which the revised course is meant to sharpen the preservice teachers understanding of Euclidean geometry. First, Euclidean geometry is developed in greater detail than the typical secondary text. In order to simplify the mathematics at the secondary level, secondary texts often incorporate theorems into the axiomatic system. For instance, it is not unusual for each of the ASA, SAS, SSS and SAA criteria for the congruence of triangles to be assumed as axioms (cf., [Sch01], [Ser97]). In the revised course, one of the group projects requires the students to establish that SAS implies ASA and SSS. Secondly, students explore the validity of familiar Euclidean propositions and concepts in non-Euclidean settings. For instance, they also justify that AAA is a criteria for the congruence of triangles on the sphere and in the Poincaré disc using Lenárt spheres and a software program.

Third, the course develops some topics from multiple viewpoints. Transformational geometry is introduced via MIRAs, using matrices and vectors, and from an axiomatic approach. The result that the composition of three reflections with concurrent axes is a reflection is discovered in a group project using a manipulative (the MIRA) and GSP, revisited as a result on matrices on $\mathbf{R}^2$ with the Euclidean metric, and then given an axiomatic proof which is valid in elliptic, hyperbolic and Euclidean geometry.

Lastly, the topics are developed from a constructivist viewpoint. At the beginning of the course, while working in groups, students are asked to develop their own axioms and definitions in order to establish some well-known results from geometry. This is consistent with the 'Necessity Principle' suggested by G. Harel: "Students are most likely to learn when they see a need for what we intend to teach them, . .", where the 'need' is an intellectual need [Ha00]. After class discussion, the students use their definitions and axioms as the basis for explorations into some models of non-Euclidean geometry. Most lectures are based on group projects that introduce the topics covered in the lecture.

The content of the course is introduced via group projects; approximately 70% of the class time is spent having the class work in structured cooperative groups. Each project consists of two or three progress reports, which require each group to write-up the results of their investigations. The progress reports are collected and returned with written comments. Although the comments address writing style, minor and major errors, points are only deducted for major errors. Each project ends with a final report in which the students rework some of the results of the progress reports and synthesize the results of the progress reports and lectures related to the project; points are deducted for both minor and major errors in the final report.

In addition to the group work, students are assessed via traditional exams (two midterms and a comprehensive final), individual quiz scores, individual homework and journal entries. Group work accounts for 40% of the student's final grade; the remaining 60% is based on individual work.

The first project introduces students to axiomatic systems by having them develop an axiom system that will allow them to establish the standard formula for the area of a trapezoid. The project combines the use of technology, lectures and cooperative group work in the following manner:

• Progress Report 1: Students describe a procedure for finding the area of a polygonal region assuming they know the formula for the area of a square and a triangle. They then use one of these procedures to 'justify' the standard formulae for the area of a triangle, square, rectangle and trapezoid.

• Lecture on axiom systems: Students are introduced to the idea of an axiomatic system and a model of an axiomatic system. In particular they are introduced to axioms, primitive terms, definitions, and theorems. Students do a homework set based on the material introduced in the lecture.

• Progress Report 2: Based on their work in progress report 1, the groups develop a set of axioms for a theory of area and definitions for rectangles, trapezoids, et cetera. They verify that GSP is a model of their axiomatic system and then prove the standard area formulae using their axiomatic system.

• Progress Report 3: Students are introduced to spherical geometry. Using Lenárt spheres, they explore the validity of their area axioms on the sphere. They modify their area axioms and use the modified axioms to obtain the formula for the area of a triangle on a sphere.

• Final Report: A class discussion leads to a consensus on a set of area axioms. The students write up their axiomatic systems for area in the plane, provide proofs of the standard area formulas, and prove a formula for the area of a triangle on a sphere.

The second project proceeds in much the same way as the first. Students are given a set of axioms to produce rays and measure angles and the groups prove that the angle sum of a triangle is 180. To do this they need to add an axiom equivalent to the Euclidean parallel postulate. They are then introduced to the Poincaré disc via a software program (NONEUCLID) and establish that it also satisfies the axioms used to construct rays and measure angles. In the second progress report they explore the Poincaré disc via several statements equivalent to the Euclidean parallel postulate. The defining characteristics of absolute, elliptic, hyperbolic and Euclidean geometry are then introduced in a lecture. In a final report, the groups establish that one can construct parallels in absolute geometry and, that if the Euclidean parallel postulate holds, the angle sum of a triangle is 180.

At this point, the students are about a third of the way through the two-quarter sequence. They are working with software models of Euclidean and hyperbolic geometry and a physical model of elliptic geometry. During the remainder of the sequence, they study congruence of triangles, similarity, circles, the ruler postulate and given an axiomatic introduction to transformational geometry using a similar pattern of progress reports and lectures.

In order to encourage student participation, each final report and progress report has a quiz associated with it. The 'correct' answers for the quiz are based on the group's work on the report and the total of the quiz scores consists of 30% - 40% of the grade for the report. The intent of the quizzes is to keep individual group members engaged in the project and to prevent one person from dominating the group and submitting work only he/she understands.

Student Response and Performance

Overall, the students respond to the course in a positive fashion. Initial concerns about group work and writing proofs diminish as the course progresses and, at the conclusion of the course, (anonymous) student evaluations for the course are nearly entirely positive.

As students enter the course, although they had taken an introduction to proof course as a prerequisite for the course, their journal entries contain spontaneous remarks indicating concern over their ability to create proofs and/or their ability to communicate their proofs to others. By the fourth week of the course (journal 2), some positive comments are made regarding proofs and by the end of the ninth week (journal 4), far more positive entries than negative entries occur. As the course continues into the second quarter, fewer entries regarding proofs, both positive and negative, occur.

One of the main goals of the course is to have the students learn to create and write proofs. The students generally view working in groups and using technology as having a positive effect on learning how to do proofs. In student journals from 6 two-quarter sequences, students made 134 comments on these issues; 106 were positive and 28 were negative. Student journals indicate that working in groups is beneficial in that it allows brainstorming, peer instruction, group checks of proofs and confidence building. Negative comments included that 'time pressure' sometimes did not allow individuals time to understand the entire project, that there was difficulty transferring skills from group work to individual work, sometimes the groups developed and learned incorrect arguments, and that group work slowed progress through the material. The primary positive theme regarding use of technology is that it provides a context to do explorations and build intuition with different geometries. The negative comments included that it was hard to move from the dynamic drawings to axiomatic arguments, and learning the programs took time away from the mathematics. Most student comments regarding the use of technology and group work are made before the tenth week.

The quality of proofs submitted during the group projects improved over time. In order to test whether or not proof creation and writing ability improved, three proofs were identified, each of them appearing in a progress report and final report. For each proof, 5 or 6 key steps or issues were identified and the submitted work was evaluated as follows:

1. In each proof and for each issue, it was determined whether the issue was partially identified or clearly *identified*; ½ point was given in the first case and 1 point in the second. These points were then summed for each issue over all of the proofs reviewed. (For instance, if looking at the work of 9 groups, 2 had missed the issue X, 4 had partially identified the issue X and 3 had clearly identified issue X, a total of 2x0 + 4(1/2)+3(1)=5 'identification' points would be associated with issue X.)
2. In each proof and for each issue, it was determined whether an issue was partially (1/2 point) or correctly (1 point) *resolved*. For each issue, the 'resolved' points were summed as above.

Figures 1 - 3 show the results for the proofs that appear in the progress reports. Each letter indicates an issue or step related to that particular proof. Note that at week 2 the groups first have trouble identifying issues and then resolving them. In week 5, they are better at identifying issues that need to be resolved. At week 12 of the 20-week sequence, scores for identifying and resolving issues are about the same. In addition to doing these proofs in a progress report, the same groups were asked to redo them in the final report for the project. Performance did not significantly increase and, in some cases, performance actually declined on the second attempt.

At the conclusion of the first course in the sequence, anonymous student evaluations often contain comments indicating that they found the combination of technology and active learning led to a better understanding of the material than a traditional lecture based course.

In order to test the validity of this perception, exam performance was compared on topics developed in lecture vs. topics developed in groups. Using 6 sets of final exams, it was found that the students earned 68.7% of points possible on group based questions and 72% of points possible on lecture based questions. (There were 248 responses to 25 questions, 134 responses to lecture topics and 114 responses to group topics.) The distribution of scores appears in Figure 4, which shows the percentage of responses that earned a particular score. For instance 47% of the responses to the lecture-based questions scored either 9 or 10 on a scale of 0-10. Note that the 'group' performance is slightly better in the middle scores.

Several topics developed in the revised course had also been covered when the course was taught in a traditional lecture format. These topics had been developed in class and stressed as important. Student performance was analyzed using final exams from the 'revised' course and from the 'traditional' course. Figure 5 shows the relative frequency of scores for 159 responses to 4 test questions, 68 from the revised course and 91 from the lecture course. As before, performance on the exam appears to be approximately the same for the traditional and revised course.

The final exams from the traditional and revised courses also had some 'novel' questions that had not been discussed in class; the intent of these questions was to test the students' ability to apply the content of the course to an unfamiliar problem. The students in the traditional course earned 47.1% of all possible points and in the revised course they earned 50.4% of all possible points. The performance of the two groups on these questions is shown in Figure 6; note that performance in the revised course is slightly better in the middle scores.

Discussion

The most remarkable aspect of the above analysis is that there does not appear to be a substantial increase in student performance when a topic is developed in groups instead of a lecture format. Students spend far more time with a topic when working in groups, have discussions with the instructor on parts they are having difficulty with, receive comments on their written work, and, in this course, eventually receive a solution sheet with a correct version of the argument. When a topic is developed in lecture, it is discussed once and lecture notes are distributed. That students appear to do as well in a traditional course as the revised course on both topics developed in class and 'novel' problems is equally remarkable; especially in light of student comments (and the instructors' impression) that student's in the revised course develop a superior understanding of the material in the revised course.

There are some possible reasons for this appearance. One is that students may have a clearer understanding of lecture topics than group topics in the revised course. Material developed in groups often contains a number of minor errors; once learned, students may not correct these errors and hence reproduce them on the exams. Topics developed in lecture, on the other hand, have fewer errors.

Also, since the lectures build on the group experience, the students are still getting the benefit of the group work during the lecture.

A weakness in the analysis of performance in the traditional and revised courses is the simplistic method of comparing performance. First, the analysis is based on the grades assigned at the time; the problems were not graded using a common rubric (over all sets of exams). Also, the analysis cannot distinguish between memorized proofs and proofs that the students genuinely 'understand'. Note, however, that this does not seem to explain the similarity of performance in the traditional and revised class on 'novel' problems.

Given the discrepancy between the above analysis and the impressions of students and the instructor, it seems that an in-depth qualitative study should be done. In particular, students should be interviewed regarding their work in groups and on exams. These interviews could indicate at what point in the course to collect quantitative data.

Conclusion

The revised course has many of the features of a course intended to lay a foundation for a deep understanding of curriculum content. In particular, while being centered on the secondary school curriculum, it expands on the content discussed in the secondary curriculum. It appears that students become more comfortable with the notion of proof and the proofs done in groups improve over the duration of the course.

The analysis discussed in this paper, however, does not suggest that the revised course is superior to the traditional course in helping students create and write proofs at the time of the final exam. This analysis will serve as the basis for a more rigorous study of the effectiveness of the course.

The revised course may offer a variety of benefits not discussed in the analysis. It is at least the instructors' impression that students leave the course with a good intuition for hyperbolic and spherical geometry and have a solid understanding of the wide ramifications of the different parallel postulates; it is the hope that this broader perspective of geometry will give them a context to think about axiom systems, models, and Euclidean geometry in particular. In the end-of-course student evaluations, the students report a strong increase in their appreciation of geometry; hopefully, they will convey this appreciation to their own students.

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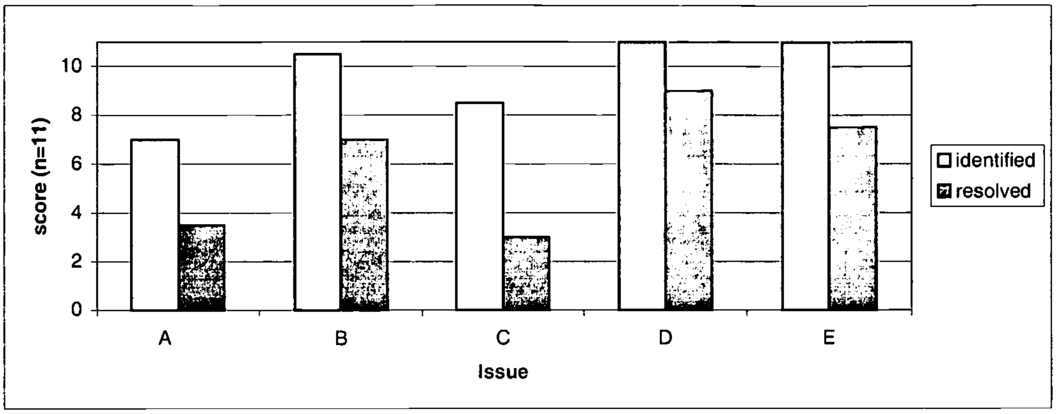

Figure 1: Proof Performance, Week 2

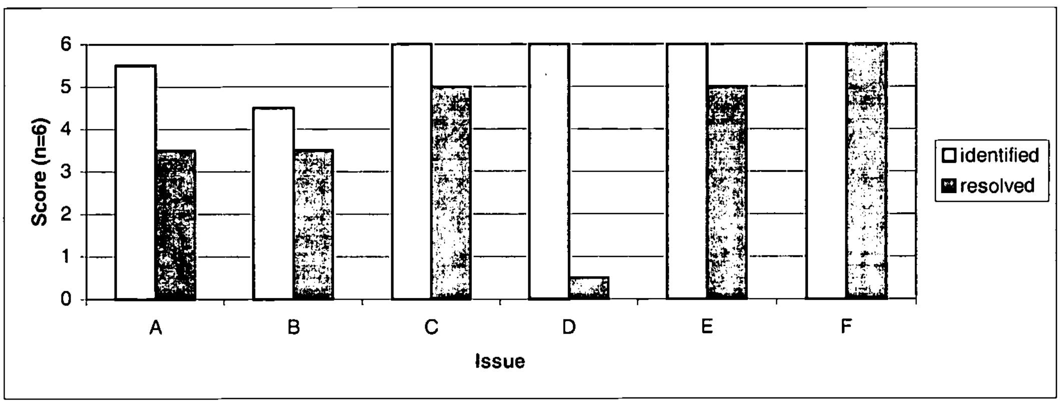

Figure 2: Proof Performance, Week 5

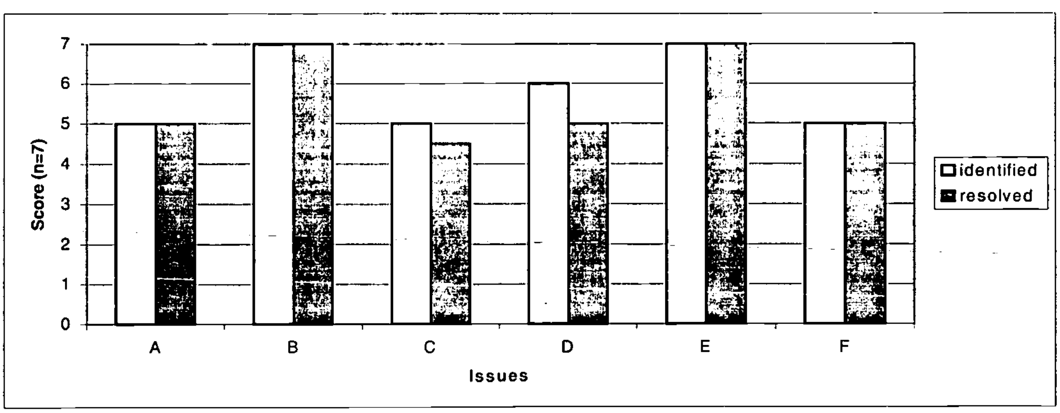

Figure 3: Proof Performance, Week 12

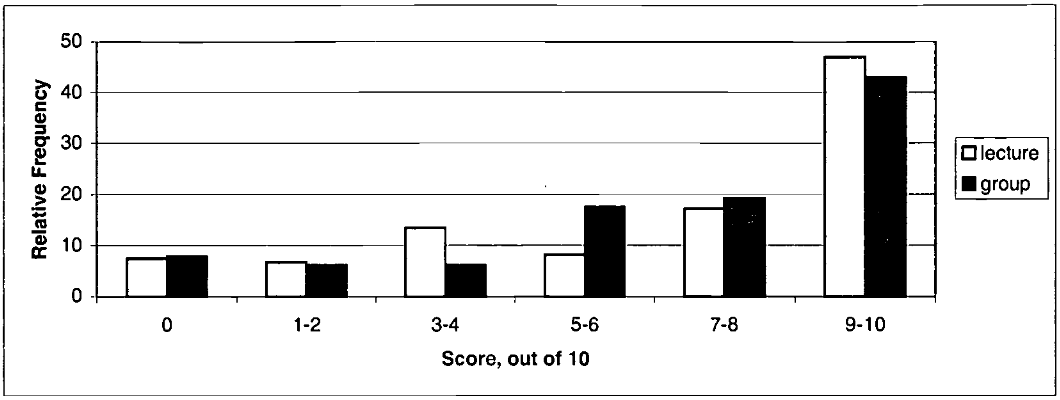

Figure 4: Exam Performance on Lecture and Group Topics

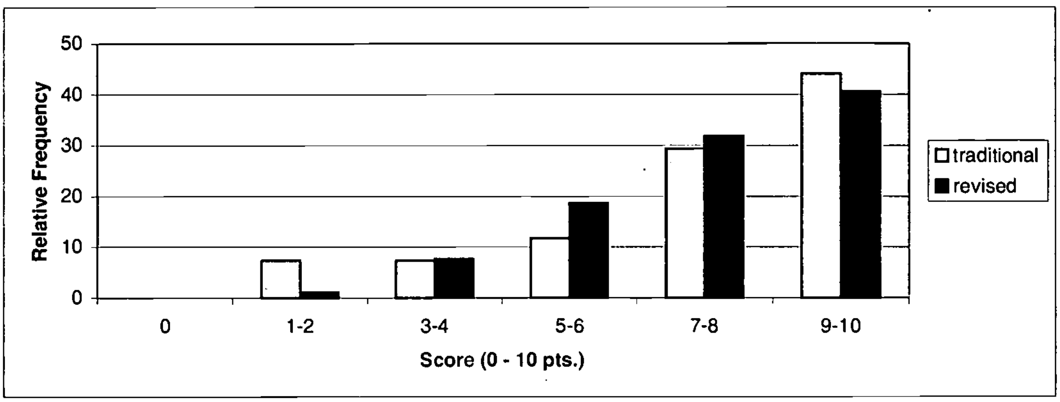

Figure 5: Exam Performance in Traditional and Revised Course

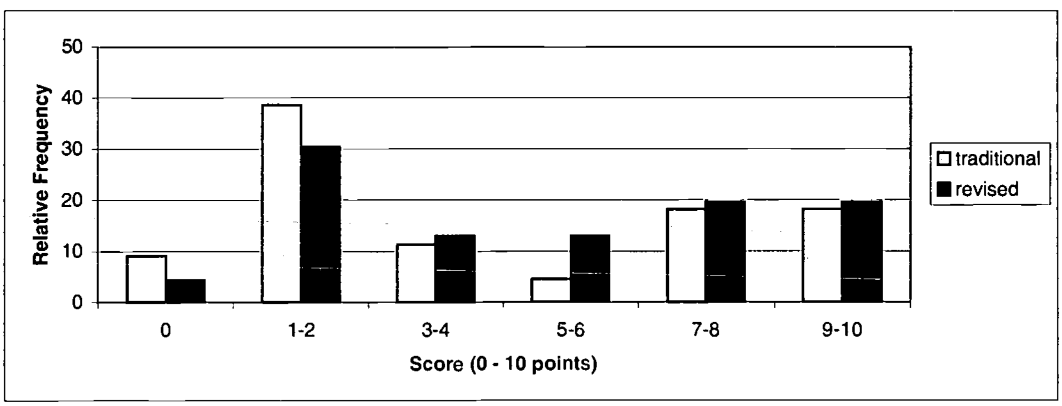

Figure 6: Exam performance on 'novel' items in traditional and revised course

TEACHING CALCULUS WITH DIGITAL LIBRARIES

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ABSTRACT

A few years ago I was shown some Java applets that were of sufficient educational interest to cause me to begin a modest search for material I could use in my elementary calculus class. The search was frustrating—a lot of widely scattered material and much more chaff than wheat, but some of the good material showed real promise. This lead, with a lot of help from my friends, to a successful proposal to the National Science Foundation's Digital Libraries Initiative for the Math Forum along with the Mathematical Association of America to create the Journal of Online Mathematics and its Applications, http://joma.org. Part of the goal of JOMA is to search out and peer review mathlets—applets and other interactive web-based teaching tools for mathematics.

The MAA and Math Forum soon received a further grant to create the MathDL site, http://mathdl.org, for which JOMA is the cornerstone. In this talk I'll examine the advantages to using digital libraries for finding calculus resources and how this material can be effectively used for teaching calculus. In addition to JOMA, a number of other digital libraries now contain mathlets and more extensive teaching material. These include Merlot and iLumina. Such libraries will be surveyed, and how new users might find them will be discussed. I have taught calculus using these resources and will be doing so this fall. My talk will update this rapidly changing area through June, 2002.

Keywords: calculus, digital library, interactive web

Teaching Calculus with Digital Libraries
Part 1: My Experience

Introduction

Technology brings exciting possibilities for visualizing mathematical concepts and for student interaction with meaningful calculations and images. But even though computer algebra systems have become simpler, most of us are unwilling to spend the time in a calculus class to develop student's necessary expertise unless our department or institution has a commitment to a particular system. Thus, the advent of very simple to use, interactive web-based material brings wonderful possibilities to the rest of us.

Additionally, the US National Science Foundation and other funding sources have for some years been financing the development of web-based educational modules aimed at providing supplementary teaching resources and student projects. Some are directed toward calculus. Some involve computer algebra systems and some are based on free software.

The problems lie in finding good quality and also appropriate material. Searching the web for a given topic produces a flood to wade through, almost every drop irrelevant, much of the relevant of unacceptable quality, much of the remainder redundant repetitions of the same few themes (check out Java function graphers, for example).

Digital libraries are an effort to collect and sort out the wheat from the chaff, to switch to a drier metaphor. Spurred on by visions such as Frank Wattenberg's in

http://www.dlib.org/dlib/october98/wattenberg/10wattenberg.html,

the NSF has now had several initiatives that have resulted in the construction of digital libraries and at least three have some mathematical content. Moreover, it has become common for popular texts to have web sites devoted to supplementary resources and commercial digital libraries have begun to spring into being.

What is out there and how can we harness it to the teaching of our courses? (Calculus has been chosen for focus and because it's what I'm currently up against.)

What this paper is about

This paper is concerned with teaching calculus making use of resources taken from digital libraries. It does *not* address online courses, it is about using online material to enhance one's non-virtual course.

How it will keep from being out of date

This paper is being submitted in January and by July when it is presented, the World Wide Web is likely to have changed dramatically. I wish to give would-be readers an up-to-date version and also seek their ideas and information on the subject—I plan to teach calculus in the fall and would like my resources and techniques to profit from this. Consequently, I am taking this paper and placing it at http://mathforum.org/wiki/CalcWithDL as a "wiki." This is a collection of easily editable web pages, so step right up and add your comments or change what I've said to match your ideas.

Background and possible biases

A few years ago I was shown some Java applets that were of sufficient educational interest to cause me to begin a modest search for material I could use in an upcoming elementary calculus class. The search was frustrating—a lot of widely scattered material and little of quality, but some of the good material showed real promise.

This lead, with a lot of help from my friends, to a successful proposal to the National Science Foundation's Digital Libraries Initiative for the Math Forum along with the Mathematical Association of America to create the Journal of Online Mathematics and its Applications, http://www.joma.org/. Part of the goal of JOMA is to search out and peer review *mathlets*—applets and other interactive web-based teaching tools for mathematics. Another goal is to publish high quality *modules*, online learning materials whose scope goes beyond that of the simpler mathlets. The MAA and Math Forum soon received a further grant to create the MathDL site, http://www.mathdl.org, for which JOMA is the cornerstone.

In this talk I'll examine the advantages of using digital libraries to find calculus resources and how this material can be effectively used for teaching calculus. In addition to MathDL, a number of other digital libraries now contain mathlets and more extensive mathematics teaching material. These include Merlot and iLumina. I will look at all these sources, and others below with dispassionate eyes, pointing out warts as well as virtues.

My Experience Using Mathlets in Class

In the fall of 2000 with the MathDL project just off the ground I was again scheduled to teach elementary calculus so I looked over what was available and decided to use some of the few applets relevant to my teaching. No other types of mathlets were then available but now other platforms such as Flash are showing some advantages in terms of loading speed and stability.

At first I explained to my students that they needed to be patient because the applets were slow to kick in and that some wouldn't even work with certain browsers (a little test information was available in JOMA). Then I demonstrated an applet showing an interpretation of the derivative as the slope of a surfboard as a wave the shape of the given function was surfed. (I didn't expect it would be necessary for the students to further play with this simple applet, but I gave them the url in case they wished to.)

For homework, I provided the url of an applet which gives a quiz on the shape of the derivative of a function, giving them some functions randomly chosen from a list and asking them to choose the corresponding shapes. Throughout, I put all the material on a course management system to which they had access, although a simple web page would have been sufficient. I warned the students that they would need to know this material in a few days for their first quiz. For that quiz I printed out some images from the applet. Perhaps half the class showed they had not put in enough time on the assignment to adequately master the concept. Student comments indicated that those having difficulty thought they had learned the concept on their own and had no need of the applet, or else they couldn't get the applet to work. Sigh.

I promised them another opportunity to demonstrate mastery of the material on the next quiz, and most had indeed come to grips with the underlying tricky concept. Students indicated that once they got going on the applet they found the experience both valuable and enjoyable.

I also used applets later in the course, for example for demonstrating Riemann sums.

Lessons (Re-) Learned

First of all, technology has its difficulties, not only for students but for classroom demonstrations as well as I (re-re-re- ...) discovered during my first presentation when computer-projector difficulties ate up more time than I had allotted. One should *always* go *completely* through a demo on one's own before subjecting others to it. Nor should we take for granted all students' ability to work with even very simple technology on their own—over the course of this experiment I had several come in and work with me before they developed real understanding of how to deal with the interactive applets.

Secondly, simple interactive tools can apparently meaningfully contribute to mathematics education. This statement is based on informal evidence, but it is backed by some later preliminary research the Math Forum has done with school students; we have more ambitious plans for research on the educational impact of digital library material, as well. My calculus class was quite small, one of the joys of teaching at the earliest morning hour at my institution, and I got to know most students reasonably well. The class was varied in ability and effort put forth. Grades ranged from A to D, with an average a bit above B-. Based on discussions with them, most students found the experience of using the applets a useful addendum to their regular work, and one that provided them with new understanding. One student, who had struggled all along regardless of the learning medium, found this experience frustrating and not worth her while.

I would also offer my subjective conclusion that based on quiz results, students ended up with a better mastery of the material where I used applets than they normally would have.

Conclusions

My conclusion here is that the problems and successes in using web-based technology is pretty much the same as the more familiar problems and successes in using computer algebra systems with students in labs. There is the added difficulty that the students encounter the material when not under supervision. There are also the added liberating factors that they can work flexibly on their own and after learning to use one simple interactive program they can then use most. Moreover, most of the resources are free, and they offer great variety because of the large number of developers and lack of program constraints.

Part 2: A Survey of Digital Libraries with Math Content

(A) "Early" Digital Libraries

Educational Object Economy (EOE) http://www.eoe.org/

EOE was founded in the mid-90's, not just as a digital library but to develop and distribute tools to enable the formation of communities engaged in building shared knowledge bases of learning materials. It lists some 82 applets devoted to calculus in its Java Applet Library. These are nicely browseable, often with good descriptions and images, but since these are freely exchanged resources, postable by any visitor, there are some very curious objects classified under calculus. Nonetheless, this collection was one of my original inspirations and their goals of developing an authoring community and of making tools that are reusable and interoperable remain laudable. (In fact, they are reflected in the Developer's Area of JOMA).

Math Archives http://archives.math.utk.edu/

The Math Archives predate the current NSF digital library initiatives, but they do have selected calculus resources as a very long but searchable list at http://archives.math.utk.edu/topics/calculus.html. Additionally, Larry Husch's *Visual Calculus* http://archives.math.utk.edu/visual.calculus/ is a collection of modules that can be used for studying or teaching calculus. They use a variety of media—LiveMath, Java Script, Flash, Java, Maple, etc., frequently with options. They are clearly done and presented and objectives are articulated. The Calculus Resources Online, http://archives.math.utk.edu/calculus/crol.html, features material available from various universities, along with some other material.

Math Forum http://www.mathforum.org/

According to The Calculus Page, http://calculus.org/, This "may be the most comprehensive, up-to-date calculus website anywhere on the internet." See

http://mathforum.org/calculus/calculus.html or http://mathforum.org/library/topics/svcalc/

for different entry points. There are many carefully annotated links to calculus material on the web. One of the distinguishing features of the Math Forum is that there are also human mediated interactive resources such as Ask Dr. Math and the Problems of the Week, both of which have calculus components. Moreover, these resources and the Internet Mathematics Library—the entire site, in fact—is developed as a whole with interconnecting parts.

(B) NSDL Libraries, that is to say Science, Technology, Engineering, and Mathematics Libraries funded by the National Science Foundation

In general, with a few lacunae noted below, all have these common features: peer review, searchable and browsable, detailed information about materials, not just lists of lists but genuine teaching material available.

MathDL http://www.mathdl.org/

MathDL is a digital library managed by the Mathematical Association of America and hosted by the Math Forum. It has a number of areas of potential interest to the calculus teacher. The Journal of Online Mathematics and its Applications (JOMA) features

- Mathelets, small interactive web-based tools, such as applets, for use in teaching mathematics
- Modules, larger teaching units and student projects that may require computer algebra systems

There is also a Digital Classroom Resources section that at the moment contains no calculus oriented material.

At the moment the more extensive mathelet material contains a browse structure, and a search engine for the whole site is expected in thc near future. The project has been highly selective and thus far out of some 900 mathlets examined, only around 16 have been published. In addition there are a dozen modules, some of which could be included in a calculus course. Many of the modules are available for a number of different computer algebra systems, see for example this table from an article in the current issue

http://www.math.duke.edu/education/ccp/materials/intcalc/index.html.

Information provided for each mathlet includes author, intended uses, appropriate courses, software specifications (results of some platform testing), author's statement, avalibility of code, and acknowledgements.

Merlot http://www.merlot.org/

The Multimedia Educational Resource for Learning and Online Teaching (MERLOT) is an international cooperative for high quality online resources to improve learning and teaching within higher education. It has some 47 objects classified as calculus materials. Each peer review is conducted by at least two higher education faculty members who, from their individual reviews, compose a "composite review" that is posed to the MERLOT website, with an Amazon-style star summary and possibility of user reviews.

The reviews focus on quality of content, potential effectiveness as a teaching tool, and ease of use. Resources vary from applet simulations through tutorials, quizzes, reference materials, and websites.

The brows structure is a bit coarse grained—one can obtain a list of all 541 mathematics materials, and sort by title, author, date, rating, or item type, but that's all. The entries also give web institution, and location. Searching for calculus produced 154 items, undifferentiated between single and multivariable. Advanced search allows you to make this distinction and also allows one to specify:

material type, title or name, content url, description, primary audience, technical format, language, whether cost or copyright, source code available, authors' name, email, and organization, whether peer or user reviews are available, date restriction, and assignment or advanced assignment search (which includes learning objectives and education level.)

iLumina http://turing.csc.uncwil.edu/ilumina/homePage.xml

iLumina is a digital library of sharable undergraduate teaching materials for science, mathematics, technology, and engineering. There are some 72 calculus entries. One can carry out quick and advanced searches, and also browse (first by "taxon path" which leads to a list giving date, title, author, resource type and data type; the latter not too clear to me). Browse entries and searches lead to especially thorough descriptions of resources:

title, authors, download location, description, keywords, taxonomy path, type of learning resourdce, level of interactivity, difficulty, end user role, structure, cost, copywright, data type, size, tech requirements, other platform requirements, "is part of", and contact information.

The Advanced Search allows one to search on most of this information, but doesn't allow one to search via type of software, e.g. applet, or for tech and platform requirements—key user needs in my opinion. The review process is not clear.

(C) Other Digital Libraries and Sites

Commercial Digital Libraries

There have been a number of commercial digital liberary ventures, most of which have yet to prove very successful. These include Questia, eBrary, XanEdu, NetLibrary, and JonesKnowledge. None appear to have any real math content, nor STEM content for that matter.

Publisher's Resources/Digital Libraries

Most publishers now offer some sort of online support for their texts and/or CD-rom material. I've not attempted to survey the latter but have attempted to look over what is available for some of the leading texts. At the moment my quest has not had much success. I've found the Internet arm of the publishing industry to be in a bit of disarray. One site, for the text I'm planning to use, had instructor materials for the several variable text for the single variable page. Some editor's did not have up-to-date information on their sites and some publishers are proving to be not forthcoming with necessary information. I will persist. I welcome your knowledge.

Some non-digital library calculus sites of note

It is worth remembering that there are calculus sites that do not aspire to the digital library format and nonetheless have much to contribute to some calculus teachers, even though they go beyond the scope of this paper. For example, for students seeking practice exams there is the venerable COW (Calculus on the Web) site of Gerardo Mendoza and Dan Reich of the Mathematics Department at Temple University, http://www.math.temple.edu/~cow/, and also Mike Gage's WeBWorK, http://webwork.math.rochester.edu/docs/docs/, of the University of Rochester.

An emerging site of some interest is Calculus@Intemet, http://www.calculus.net, a potpourri of everything calculus-related. At this time it needs some digital library organization and clarity to be really useful.

How are the NSDL digital libraries doing?

Imagine that you are preparing to teach a calculus course and are searching for help over terrain known to be rocky, or perhaps you are in the midst of your course and are looking around for first-aid when both you and your text have not got through to as many students as you think you should. Here are some issues that are likely to be confronting you:

1) hardware constraints; perhaps your institution uses Macs, perhaps you have ancient PCs.

2) software constraints; perhaps a particular browser is the only one supported, perhaps you have either a single computer algebra system available or no such system.

3) specificity; you need material that deals with a specific topic, say the chain rule, and need a search engine to take you directly to this.

4) general direction; you are willing to browse around in a general area to see what is available.

5) quality; you are only looking for really high quality material that gives students real insights.

6) quantity; you would like a number of items from which to choose.

How are the digital libraries doing to help you in your quest? (1) & (2): Alas, a mixed bag. No library tells you upfront about hardware or software constraints, nor at this time allow you to search on these, but MathDL and iLumna tell you when you get down to looking at individual items. (3) All will have search engines and iLumina already has a detailed search (although it doesn't give you exactly the choices I'd like). On the other hand at this time the value of these search engines is weakened because of (6): the libraries lack quantity so that you can't expect to find anything when you make a very specific search by topic. (4): None have particularly good browse facilities at this time, although MathDL plans to allow users to use a generic calculus text table of contents. Perhaps the libraries are doing best with quality (5), since all have given considerable thought to acceptance criteria and most are using peer review.

At this time the digital libraries can be likened to open stack libraries where it is necessary to browse to find good fits to your needs, rather than closed stack libraries where the "card catalog" contains enough information that you can confidently order from the circulation desk. Moreover, the shelves are not very full at this time, although the digital librarians are eager to fill them up.

Nonetheless, there *are* interesting materials in these libraries and careful thought has gone into the catalogued materials, the information gathered for users, and the user interfaces.

Thus, at this time each of the three main NSDL libraries discussed above have valuable features and lack certain things that my generic calculus teacher above would find important in looking for appropriate material. Roughly speaking, the libraries have three distinct and useful models that can be vastly oversimplified to

- journal publishing (MathDL),
- a super-Amazon with both peer and user reviews (Merlot),
- the best of current digital library notions about metadata (iLumina).

I repeat: this is a vast simplification and all the approaches are valuable.

Big questions for the digital libraries:

I. Will they package their material in a manner users find convenient?

II. Will they be able to fill their shelves? Is there enough good material out there? Will they be able to find it before search engines catch up? I tried "chain rule applet" on Google and it seemed to give back more interesting material earlier in the vast list of matches than heretofore, and the popularity measure it employs may mean that the cream will rise to the crop.

III. Will meta-digital libraries be necessary? That is, will it be necessary to have an über-library that one searches for the material one wishes in the various digital libraries? One hopes not. Fortunately, the NSF is funding attempts to allow easy searching across all digital libraries. Unfortunately, we're not there yet and if a digital library does not keep track of information the user finds important, such as necessary platform, it won't be searchable for.

IV. Is it possible for the digital libraries to give us genuinely valuably distinct look, feel, and material, or are they doomed to expend the immense amount of effort and time already put in to producing products that are so similar that users would have been better off if they combined forces?

Coda

What will I do in my calculus course this fall?

Ah, through this talk I have become much more familiar with the various libraries and how to use them. I've tried to impart this to you as well. By the deadline for this paper I have not been able to make detailed choices of particular learning materials, but I will continue to work on this and the paper I present will contain references to the material. Moreover, as mentioned, this paper is being set up as a wiki at http://mathforum.org/wiki/CalcAndDL, its existence will be made known to as many calculus teachers as I can find, and I hope that some will contribute their ideas and suggestions so that I can do a good job in my course.

ALGEBRA IN THE TRANSITION FROM HIGH SCHOOL TO UNIVERSITY

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ABSTRACT

In this paper we communicate different aspects relative to the design and implementation of a didactical proposal for the teaching of algebra in a first university course (*Elements of Algebra*) at the Universidad Nacional de la Patagonia Austral (UNPA) for students that require a solid formation in mathematics.

One of authors of this proposal is the teacher in charge of Elements of Algebra at UNPA. The other one develops her work at the University of Buenos Aires. To implement the changes planned in the class and in the kind of tasks the students will be required to solve, we decided it was essential to include a stage of work with the rest of the teachers of *Elements of Algebra*.

For the elaboration of a proposal we have considered various dimensions of analysis:

- A theoretical frame about the teaching and learning of mathematics, taking into account the theorization of G. Brousseau, G. Vergnaud and Y. Chevallard, among others.
- A reflection about the meaning of algebraic symbols in their use as a tool to solve problems, considering for this theoretical elements furnished by A. Arcavi and J. Drouhard, among others.
- A critical analysis of the selection of problems that previously formed part of the practical work, keeping those which could allow a work centered in the construction of the sense of mathematical objects and the particular methods of algebra.
- The knowledge of the characteristics of the population of students to whom this would be directed.

As a result of this work, a **new exercise booklet** was produced for a part of the course. Then, all the subject teachers attended a **workshop to discuss the problems proposed in the booklet** and some questions relative to its implementation. Finally, **changes have been made to the course and some episodes were registered** and analyzed.

The purpose of this paper is to explain briefly the four dimensions of analysis considered when elaborating the practical work and to describe and analyze the three instances of work mentioned in the previous paragraph. We will also try to show –from certain aspects of the effective realization- the difficulties that appear when a change is introduced, that requires the reformulation of the personal relationship of each student with the study of mathematics, as well as repositioning students and teachers in their roles in the classroom.

The students, the teachers and the mathematical activity are in the center of our study interests.

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[1] Teacher of Elements of Algebra at the Universidad Nacional de la Patagonia Austral - Province of Santa Cruz, Patagonia, Argentina -with a formation on Didactics of Mathematics: centered on Didactics of Algebra, she formulated and implemented a methodological change in the subject *Elements of Algebra* of the first year for the careers of System's Analyst and Mathematics Teacher. This implementation is the result of a project of pedagogical innovation that was selected in a Pilot Convocation by the Secretary of Superior Education of the Ministry of Culture of the Nation in November, 2000. which allowed this teacher to do an assistantship of two months at the Centro de Formación e Investigación en Enseñanza de las Ciencias of the University of Buenos Aires.

[2] Teacher of the University of Buenos Aires, director of the Centro de Formación e Investigación en Enseñanza de las Ciencias, directed the assistantship and also worked on the formulation of the methodological change..

1. Description of the problem

Characteristics of the population of students and representations of the university teachers

In general, the students that have finished high school have acquired skills in algebraic operations not linked to the situations in which they can be used. At the beginning of the course Elements of Algebra (annual) at UNPA, they find themselves facing a "different kind of mathematics" marked by the presence of a new transversal and fundamental element: proof. This is a rupture in the passage from one level to the other. There is a lack of balance between what the student knows and how what he knows is used, because there are familiar objects, but they do not "function" as they did in High School. For example, all the knowledge acquired about operations with polynomials is not sufficient to develop strategies that allows them to formulate and proof general statements about numbers.

After a short period of time at University the students have the impression that they were not taught anything in High School. And this impression is "somehow confirmed" by the university teachers. In general, the teachers of Elements of Algebra come to the conclusion that the remarkable failure of our students (usually, less than 10% of the students pass) "is due to" a deficient previous formation, which they reduce to "absence of some algorithms and lack of a study habit".

These teachers apparently identify the work in mathematics in University as heavily linked to language and the formal manipulation of the rules of the language to prove. They end up insisting more on the proving procedures than in the sense of the objects and the practices. The work is finally reduced to the acquisition of the rules of treatment of the formal language, showing a rigid and finished mathematical task. The structure of mathematics is lost.

In previous years we observed that some students could repeat a procedure of a demonstration, that is, they recognized a procedure and could apply it in another proof, which did not mean that they understood what they were doing, they were only doing what they were asked to.

Taking this problematic into account we decided to redesign de course of Elements of Algebra.

2. Theoretical Frame

- **Global problem: teaching and learning mathematics**

As we have said, our conception about the teaching and learning of mathematics is nourished by theoretical elements of the Theory of Didactical Situations of G. Brousseau, the Anthropological Theory of Y. Chevallard and the Theory of Conceptual Fields of G. Vergnaud.

To summarize their most important characteristics we will transcribe some paragraphs from the Curricular Design of the city of Buenos Aires[3].

> "There are many ways of knowing a mathematical concept, which depend on everything a person has had a chance to do in relation to that concept. This is a fundamental starting point to reflect on teaching.
>
> The set of practices that a student uses for a mathematical concept will construct the sense of that concept for that student.

[3] P. Sadovsky. Pre- Curricular Design for the General Basic Education. General Framework. Formative Sense of Mathematics in School. Secretary of Education. Government of the city of Buenos Aires. 1999.

We assume a position according to which the process of reconstruction of a mathematical concept begins with the set of intellectual activities that are used for a problem whose solution cannot be found with the knowledge available."

Problem solving must not be the only kind of work done in class. To make the work in class fertile, it is fundamental to include moments of reflection about what has been done, articulation of different strategies, discussions about the economy of certain procedures, confrontations of the perspectives of the students… This has to do with creating a space for debates, for establishing conjectures, to validate them.

- **Local problematic: the teaching and learning of Algebra**

There are many researches about the problem of the teaching and learning of Algebra. We have mainly considered the contributions of researchers Abraham Arcavi (1994/95), Anna Sfard (1991), Josep Gascón (2000), Brigitte Grugeon (1997), J.F. Nicaud (1994), Jean Philippe Drouhard et al. (GECO 1997), Mabel Panizza, Patricia Sadovsky and Carmen Sessa (1995).

We would like to highlight some considerations of J.F. Nicaud (1994) about the treatment of algebraic expressions. He says that the mathematical objects that represent algebraic expressions are partial semantic models, where calculations can be done or transformations performed over formal expressions can be justified, that is, algebraic calculations can be meaningful. He defines three semantic levels:

- **First level** (evaluation level): significance is given to an algebraic expression by replacing values in the variables and doing the corresponding calculations.
- **Second level** (treatment level): an expression is transformed into equivalent ones. It implies the knowledge of the transformations laws and how to justify them. This justification is based on the fact that an expression and its transformed coincide for every evaluation.
- **Third level** (level of resolution of problems): strategies are known that permit the choice of the necessities to solve a particular problem, giving sense to the calculations.

We believe that a freshman student at UNPA has not achieved the third semantic level in the treatment of algebraic expressions, that is, he "does not know" how to organize his activity to arrive at a conclusion.

We could say that the student is a "formal automaton" as described by GECO (1997), that is, a student that, when manipulating algebraic expressions of elementary algebra, does not take into account that by transforming an expression he must obtain another equivalent to it. In this case, the question of the validation of the result is not posed in terms of the equivalence of the writings obtained, but above all in terms of the conformity with rules and proceedings (for example, "what is subtracting passes adding").

It is on these aspects that we will center our proposal.

3. Changes proposed

According to what we have said so far, we have given priority in our proposal to the dimension of algebra as a tool for validation. We will there distinguish various levels:

- Algebra as a tool for generalizing numerical properties.
- Algebra as a tool for calculation, to find a result or to validate assertions.

- Algebra as a model for intra and extra-mathematical situations.

We can identify different dimensions in the changes proposed, even though they are naturally closely related.

i) **Changes in the problems of practical work number 2 of the course,** which deals with the field of real numbers.

In the previous version of this practical work students were asked to prove properties of real numbers based on the axioms. Our experience of many years reveals that this work had a great impact on the students: they could not grasp the logic of the task and felt they had no resources to do the required demonstrations. As a result of this, on one hand, they lost their self-esteem, and on the other, and according to the difficulties they encountered, the students were not sure of the use of this kind of task for their mathematical development. This causes a lack of confidence, which constitutes another ingredient of the atmosphere of the class and it does not contribute to the learning environment we wish to install.

In the reformulation we considered another scheme, pointing to the acquisition of "symbol sense" (Arcavi, 1994).

We will show as an example some of the exercises proposed. (Exercise 3 is inspired in an activity narrated in GECO (1997).

3. $(a + b)^2 =$ ....

a) Complete the right hand side with an algebraic expression so that the equality will always be true.

b) Complete the right hand side with an algebraic expression so that the equality will always false.

c) Complete the right hand side with an algebraic expression so that the equality will sometimes be true and sometimes false.

Give an example where it is true and another where it is false.

d) Describe the set of solutions of question c).

4. a) Invent two "formulas" that are always true.

b) Invent two "formulas" that are always false.

c) Invent two "formulas" that sometimes are true and sometimes false. For each of the "formulas" that you invented, give examples of values for which they are false and values for which they are true. Describe all the solutions.

These exercises were thought according to various purposes:

- To allow the student a personal work of creating expressions according to different requirements.
- To be able to consider the conditions a), b) and c) as possible for every equality between two algebraic expressions, breaking the dichotomy right/wrong.
- To make the algebraic rules manipulated by students and their solidity explicit to the teacher.
- To allow in a class discussion of the resolutions, a circulation of the different methods used and the knowledge of each student.

We knew that this task would be a challenge for the teachers because unexpected answers would force them to use their own algebraic abilities.

The dynamics of the class during the discussion of these problems was far from the one for the traditional problems of the kind "prove that...", that were solved showing the "correct proof".

For these two problems, as in others of the worksheet, we were also trying to recuperate conceptions, concepts and terminology seen in High School. The words "formula", "algebraic expression", "identity", "equation", "function", that generally coexist in the body of knowledge of students in an isolated way, are re-captured trying to enrich their senses.

Another example of the proposed changes is the following:

In the practical work students were asked to prove that if the product of two numbers is 0, then at least one of them is 0. This property was evident to the students, but impossible for then to prove. On the other hand, it was not available when, in another exercise, they had a product equated to 0. All this shows us how "useless" this exercise was at this stage. Instead, we proposed the following exercise:

Let a and b be two real numbers.

a) Find all the solutions of a.b = 10. Can you describe them all?

b) Find five solutions of a.b = 3. Can you describe them all?

c) Find all the solutions of a.b = 0. Can you describe them all?

with the idea that only after certain manipulation, that could eventually include the graphic representation of the solution, the teacher would pose the question: "How can we justify that, if a.b = 0 then a = 0 or b = 0"?

We will finally mention exercise 16:

16. Are there two real positive numbers a and b that verify that $\frac{1}{a}+\frac{1}{b}=\frac{1}{a+b}$?

Justify your answer.

This exercise appeared in the previous version of the worksheet, in the last place. Almost no one solved it, since they needed a strong guidance from the teacher to do so, resulting of little value for the cognitive advance of the students. Our hypothesis was that, in the new practical work the problem would be tried by the students, as a result of all the previous work, and so we decided to leave it.

In summary, the exercises that form the new practical work give sense to the mathematical activity of the student, favoring his independence and a posterior reflection that allows a discussion with his peers and teacher about the work being done.

We were worried about the student just trying to remember algorithms instead of using his common sense and creativity.

We were also interested in enriching the field of activities that the student recognizes as relative to the mathematical work, incorporating the following:

- establishing conjectures,
- validating results,
- finding counterexamples to invalidate a possible result,
- determining the domain of validity of a formula,
- analyzing different strategies of solution for one problem,

- explaining his solution to others,
- listening to and debating the solutions of his peers.

ii) Changes in the job of the teacher

As we pointed out in the description of the problem, most of the teachers of this course were centered mainly in the learning of algebraic manipulation of the rules of language to demonstrate. And, even though they failed at it, they blamed this failure to the lack of knowledge of the students.

Due to the exercises we presented and the kind of work we proposed, that always included a moment of reflection and discussion, an important change in the teacher's position was needed.

The activities we showed in the last paragraph of i) were also new to the teacher as activities for a mathematics class.

iii) Changes in the "job" of the students

We have enumerated different activities – inventing formulas, debating, establishing conjectures, explaining to others, listening to and debating other solutions – that imply a strong change with regards to what the students used to do in class – struggle alone to solve an exercise and then listening to the solution given by the teacher-.

Besides this, we asked them to work in groups to give them the chance of a first and more "private" moment of discussion, without the intervention of the teacher. We programmed that a possible teacher's intervention in the groups did not have to be an "evaluation" of what the students were producing, but one that would help them deepen their work and contribute to justify what they were doing. (This point would be, without any doubts, a big rupture for the teacher with respect to his traditional role).

At last, we planned that the groups had to present (in writing) the solutions to some problems that had not been discussed in class. This was to make them pay attention to the written formulation and it also gave information to the teacher about the advances of the group work.

The teacher corrected the exercises, making a mark in case of a mistake or something imprecise, without saying what the mistake was, without saying "the right answer".

The corrected exercises return to the group to be re-written and only when the students could not solve the problem, the teacher would intervene.

The periodicity of the assignments would also let the students and the teacher have an idea of the *evolution* of the work.

4. Difficulties and achievements in the implementation

To carry out the design previously reported, we had many meetings with the teachers of the course: we presented and analyzed the new proposal for practical work number 2 before it was used with the students.

Even though the teachers were asking for a change –according to the high failure rate- these changes we proposed were institutionally too far from what is considered as a "university mathematics course". At the same time, they alternated between trying to understand the object of each exercise and the type of class dynamics we proposed, and its rejection for considering it more appropriate for "High School".

As the new work was implemented, we had meetings to discuss what had happened and to plan the work in class.

Even though there were important changes in the kind of work of the students, the teachers found it difficult to manage the moments of collective recuperation of the personal work and to take advantage of the different solutions that would arise.

For example, teachers were very uneasy about exercise 3 because they could not anticipate what the students would do. In a class, a student suggested the following to complete a "false equality"

$$(a + b)^2 = (a + b + 1)^2$$

The teacher said in one of our meetings "I screwed up and I accepted it as correct". This teacher had no problem the following class to go back to this exercise and find, with the students, the set of solutions of the equation, but his words revealed that his new job was less "secure and comfortable" than the job he was used to do.

From the point of view of the student's work, we can say that it improved significantly. As we anticipated, they tried to solve exercise 16 and obtained different solutions that allowed a fruitful debate. This was a confirmation that the problems could somehow work in interaction with the "knowledge" of the students. This "knowledge" does not only include objects and procedures but also topics related to the kind of practices developed before. Our students showed that the work they had done up to that point "backed them up" to try exercise 16 without problems.

5. Final comments and future perspectives

It was clear to us that the greatest difficulties in the implementation were on the side of the teaching team.

The mathematical formation of the teachers is an important variable to take into account, because what is understood as mathematical activity is conditioning for what is considered that teaching mathematics is, and algebra in particular.

But what seems more important is that the institutional requirements do not prevent the teacher from listening to the students and work *from* their knowledge. Much more work has to be done to obtain this.

Teaching to prove with an increasing level of formality is not an exclusive task of the course Elements of Algebra: it is a long process that needs coordinated teaching actions (in periods that are longer than a course).

The starting point of this process is given by the state of knowledge of the students. In this sense, the criticisms we had received saying that the kind of work we proposed was more proper for High School is not pertinent because this type of work is actually absent from Argentinean High Schools.

Without any doubts, part of the work that we propose in the course of Elements of Algebra could be taught in High School[4] and when this happens, we will have to think in another kind of practice for the first year of University.

As for now, we think that the proposal made is realistic in its objective of improving the quality of the mathematical work of the students and of helping them in their start at University.

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USAGE AND USABILITY OF THE MATHEMATICAL WEB PAGES: An Example

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ABSTRACT

The main goal of this paper is to consider the usage and usability testing of a virtual mathematical interface. The main sections of the Laboratory of Applied Mathematics at Lappeenranta University of Technology new Web pages will be also introduced here.

The reason initiating the design of this interface was that, for some reason, the mathematical skills of new university students in Finland, as well as other countries, are not as good as one would expect. Students must revise high-school mathematics before starting their actual university studies, and computer-based study materials are one solution to this problem.

The primary intention of this project was to offer mathematically less skilled students basic material in an appropriate manner that they could use independently and that would help them grasp the basic concepts in mathematics studies, which they require later on in their studies.

In order to avoid reproducing, in the Internet already existing teaching material, the author decided to review the educational Web pages that had already been produced by other authors. Soon the author had found a sufficient amount of suitable educational Web pages for above-mentioned purposes. Finally, the Internet pages prepared here are ready to be checked, and usage as well as usability testing will be the methods used for testing.

1 Introduction

In 2001 the Eastern Finland Virtual University Network (ISVY) decided to grand funds for the distribution of basic-level mathematical material via the Internet.

First, several issues had to be taken into account before the new project could be initiated. One of these issues was related to the use of distributing educational material via the Internet, and below are some points that justify the use of the Internet as a medium for the dispersion of educational material [1]:

Material, which is on the Internet,

- is easy to reuse, transform and combine with other materials
- is fast to use at least with fast connections
- is accessible 24 hours a day
- can be accessed by many users.

After the material has been distributed and some kind of interface implemented to use this material, some questions remain to be answered. The usage and usability of this new interface are examples of such issues. Good usage roughly means that, for users, the pages are effective and pleasant to use. Pages are somewhat usable if the user is able to find exactly what they are looking for and immediately notice if the pages do not contain the necessary information. By good usability, we strive to achieve a user-oriented working environment. Usability is an important aspect in Web-based learning environments, since a poorly usable environment requires, of the user, a lot of time and effort which is away from the learning process.

This paper will shortly present an interface, which was implemented for the Lappeenranta University of Technology (LUT) and which uses a collection of materials [3]. This paper will also discuss the philosophy, the principles of planning, execution and future of this experiment. The core of the experiment consists of study packages based on hypertext, Web browsers, computer algebra systems and visualizations based on java applets. In addition to universities, the material delivered and categorized here can also be used as an additional reference at mathematically oriented schools. This paper will focus on mainly discussing the future of this project, which is strongly linked to the forthcoming usage and usability of the material.

2 Testing Usage and Usability

Why should usability be tested? One would expect that the content of pages, with good usability is easier and faster for students to adapt. The basic rules of usability state that Web pages have been well planned if they have: i) a consistent layout ii) a consistent design iii) a clear order of information iv) a clear way in which the information has been arranged v) been designed to allow easy and consistent navigation vi) been implemented in such a way that the pages and any graphics they contain are aesthetically comfortable [4].

Good usability can create better conditions for teaching, and improve the quality of learning. Of course good usability alone does not guarantee good learning results, since pages, which exhibit good usability, are only a tool for learning. Good teaching always supports the communication between the student and teacher, creates more interaction between students, supports active learning processes, provides accurate feedback, helps control the use of time, sets goals high enough, takes into account different talents and ways of learning. [5]. Obviously, a good learning procedure is such a complex area that no tool alone can guarantee it.

It is in any case good to estimate the usability of products in some way, since different usability tests will reveal the worst usability problems in an easy and effective way. Normally, usability tests also save a lot of money in the later phases of projects. There are mainly two approaches to usability testing. The first one is expert-oriented testing, i.e. the, so-called heuristic approach and the other end-user oriented approach.

2.1 Heuristic

An interface can be evaluated in a heuristic manner which means an expert-orientated evaluation, where an expert goes through an application from display to display, button to button and menu to menu using some well-known guidelines. The main advantage of this approach is that it is effective in relation to the time and money that it requires. The disadvantage of this method is that opinions normally do not come from the end-user, but from an expert.

The most well-known rules of usability evaluation for heuristics are perhaps Jacob Nielsen's ten rules [6]. This part of usability mainly shows how pages have been designed.

2.2 Usability Testing

Actual usability testing is a method which is intended for revealing the true problems experienced by the end user. It is preferable that usability testing be, to some extend performed already of the prototype phase of a project, which in this case, involved the author discussing the opinions of their colleagues and some students as to the design and contents of these pages. Real usability testing starts when the product is considered by an expert to be ready. The designer or expert user has to elaborate a list of the vital functions of the product as well as of some test operations with which to test these functions. The expert has to also perform pilot testing before end-user testing can actually begin.

Nielsen has written that *'Some people think that usability is very costly and complex and that user tests should be reserved for the rare web design project with a huge budget and a lavish time schedule. Not true. Elaborate usability tests are a waste of resources. The best results come from testing no more than 5 users and running as many small tests as you can afford.'* [7] and [8]. This is due to the fact that already the first test five users usually find as much as 85 % of usability problems. At this point, the designer is already bursting to fix these problems in redesign. After the new design has been prepared, tests needs to be run once again.

The case discussed here involves different categories of teachers and students at the university, which requires two different test series to be run. Nielsen recommends that between three and four users from each category be selected for testing when there are two categories. As a result, in each test case. As a result, in each test case, there will be between three and six participants. A normal test situation is videotaped and the participants speak as freely as possible while proceeding through the key sections of the interface. It is important that the supervisor of the tests not help the participants once the test has started, since the purpose of these tests is to study problems in the usability of the interface. Once the test has been completed, there is closing discussion in which possible problems in the product are revised as well as which suggestions for improvement or comment as to the appearance of the product from users are collected. It is however, the most essential for notes to be made as to how the user is actually using the product [9]. Finally, the test supervisor prepares a written report as the basis of the test results, considers the results and implements possible changes before initiating a new round of tests.

3 Realization of Virtual Mathematical Web Pages

The author, at first thought about using a design approach called the Mud-Throwing Theory of Usability which has recently become popular for the design at new websites and innovative Internet services with the idea of throwing a design at the wall and seeing if it sticks. The assumption is that speed is everything. If the initial design has weaknesses (i.e., drops off the wall), these weaknesses can always be fixed in redesign [10].

Actually, it is the author's opinion that in mathematical non-commercial web-design, speed is indeed not such an important matter. This is due the facts that

1. the information contents do not change radically over time,
2. students are very critical customers and if a design is bad, they will not come back to use the interface when its design has been improved. As Nielsen states, *'Once a user has had a bad experience on a website, it is very difficult to convince him or her to come back.'*,
3. while speed is not everything, customer (in our case student and teacher) satisfaction is [10],
4. launching a bad site with poor usability is a guaranteed way to waste money, since it will have to be redesigned more or less immediately [10].

According to the usability theory [7] it is only necessary to test with very few users in order to gain the vast majority of insights into the usability of a design. Usability feedback can be obtained at a very early stage of the design when nothing has been implemented yet and there are nothing but a few sketches of the proposed new service. Testing does not therefore need to delay implementation. In fact, testing with an early prototype of a future site will sometimes speed up a project and save time as the designer discovers that certain features are unnecessary or that things should be done in simpler ways than originally thought.

The basic philosophy of this project was to give the student an open study environment, which consists of information on mathematics, problems to be solved, computational tools, guidance, general help (both technical and mathematical) and visualizations. The student can study what they feel to be the most interesting. Students will hopefully begin to do their own research and form a view of mathematics. Thus, the view of the learning process is constructivist. In principle, the criterion of learning is the ability to solve given problems. On the other hand, students are not graded. The purpose is only to give them a study environment that will be interesting enough to make them study mathematics at their own pace. The design principles were set as follows:

- To create a mostly hypertext-based mathematical forum with visualizations, animations etc.
- The system has to work smoothly.
- Students should receive feedback and be able to submit it as well.
- The material has to be well organized.
- The material has to be relevant for students,.
- The distribution of the most important content of the material should be free.
- Request permissions for the usage of material in all cases.

Once the material is complete it is important that

- The content of the new interactive pages is ordered in a pedagogically meaningful manner.
- All the material is in order in the sense that all the authors have been notified and licenses requested for the use of their material.
- The system be verified to ensure that it functions as required.
- The opinions of other's as to the new pages (co-workers, students) be requested in addition to recommendations for improvements.
- Any required changes be made and the system verified once again.

4 Main Functions of the Interface

All the pages have a feedback form, which is intended for students to use to send the designers any information as to what they require of the pages. The pages, in the final outlay, consist of the following main parts:

4.1 Main Page

- Main pages in Finnish[3].
- Main pages in English[11].

Both these sets of pages have various links and their main mathematical features are shown below.

4.1.1 Introduction to University-Level Mathematics

By clicking this link [12], the student can find links to

1. The basics of mathematics [13].
2. The basics of high-school level mathematics [14].
3. Quizzes and games prepared by Franz Embacher [15].

All of these pages include a lot of mathematical theory, exercises, visualizations, quizzes and games for students.

4.1.2 Ordinary Differential Equations

This section consists of material [16] prepared by Simo Kivelä and his working group and includes theory, visualizations, exercises and examples of ordinary differential equations.

4.1.3 Virtual Mathematics

By clicking this link [17], the student can find links to

1. Examples [18] which include a lot of useful examples on java coding and mathematics.
2. Sorted applets [19] which will help visualize mathematics.
3. Ready material [20] which includes all kinds of useful mathematical material, such as the previously mentioned pages by Franz Embacher [15].

It should be mentioned that this is just the main part of the materials what we have placed on our pages.

5 Future

So far, the following items have been noticed as features that should be corrected in order to improve the functioning of the system.

- Improve categorization by, mainly changing some names of the menus.
- The feedback form should be altered.
- A chat and/or bulletin board should be added on to our pages in order to provide users with more interaction with the teacher.

The author will also use these pages in the second basic mathematics course which he will start lecturing on March. The author will implement a small interface for the needs of this specific course, collect data from the students while they use the interface and run usability tests for the students. The author will also run tests for mathematics teachers. After analyzing the results oh these usability tests, the author will make the necessary changes to the interface and run further tests. Finally the following actions will be taken:

- Active advertising of these pages will be initiated.
- When the package is used via the web, students will be able to send their answers to problems to the server. Here, the answers will be checked, some feedback given and statistics is collected. Thus, it will be possible to send the student a survey of the work done during several sessions (which problems have been solved, how many correct and incorrect answer were given, etc). It must be emphasized that the idea is not to grade the student, but to provide feedback for self-evaluation. On the other hand, the data collected in this way can be used for further developing the system.
- The aim is to develop a system for collecting data on work of students and to analyze the data.
- These pages will be maintained constantly.
- The materials used in mathematics courses will be linked and used here.
- Even more co-operation will be undertaken with other universities, high-schools etc.

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CONNECTIONS WITHIN MATHEMATICS – WHAT QUESTIONS SHOULD BE ASKED AND WHAT ANSWERS SHOULD BE GIVEN?

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ABSTRACT

Recognizing interconnections within mathematics is one of the main emphases of the NCTM's standards (2000): "Thinking mathematically involves looking for connections, and making connections builds mathematical understanding." From our experience we have come to realize that many teachers are not aware to the various interconnections exist within mathematics.

In this paper we describe a process in which the main purpose was to develop teachers' (pre-service and in-service) awareness to some interconnections and bring them to appreciate the importance of holding such view. The initial stages of the process were based on confronting the participants with questions regarding three concepts that have different representations in different contexts: 'a straight line', 'a parabola' and 'similarity'. The analysis of the data, obtained through a questionnaire and a discussion, showed that teachers tend to build in mind several isolated concept-images for a certain concept. Each concept image is formed in accordance with a specific context, and the dominant ones are those including algebraic properties of the related object. In advanced stages of the process we introduced answers to the questions in order to illuminate interconnectivity, and thus support the participants in creating a concept image that unites all the relevant aspects. Finally, the teachers were asked to reflect on the process they have gone through. The reflection showed that all the participants had acquired new mathematical and didactical ideas and that awareness to the importance of acquiring a connectionist view was formed.

KEYWORDS: teacher, connection, concept, geometry, algebra, definition, proof, straight line, parabola, similarity.

Introduction

Recognizing interconnections within mathematics is one of the main emphases of the NCTM's standards (2000): "Thinking mathematically involves looking for connections, and making connections builds mathematical understanding. Without connections students must learn and remember too many isolated concepts and skills. With connections, they build new understandings on previous knowledge".

We believe that in order to convey such a perspective it is essential that the teachers themselves possess a 'meta perception' of mathematics.

In this paper we describe a process in which the main purpose was to develop teachers' awareness to interconnections within mathematics and bring them to realize the importance of holding such view. The participants were two groups of mathematics teachers. One group consisted of seven in-service teachers (IST), each having at least five years of experience teaching high-school mathematics. The other one consisted of twenty-six pre-service teachers (PST) in their third year of academic studies towards the B.Sc degree in mathematics education. The process was based on confronting the participants with concepts that have different representations in different contexts. To our opinion, such concepts have the potential to emphasis interconnections within mathematics.

Theoretical background

In order to analyze the ways in which teachers comprehend various concepts we have found it constructive to gather two theoretical frameworks: the theory of concept image and concept definition and the theory of global and local coherence.

According to the theory of concept image and concept definition (Tall & Vinner, 1981), during the process of learning a certain concept, one builds a concept image and a concept definition in his mind. A concept image is the "*total cognitive structure that is associated with a concept*" and a concept definition is the "*form of words used to specify that concept*". One might hold a concept definition that does not correspond to its mathematical definition or is not linked to his or her concept image. Poor concept image means using a few prototypical examples of the concept while considering that concept (Hershkowitz, 1990). A somewhat richer concept image means basing judgment upon more prototypical examples and their mathematical properties. A full concept image includes a wide variety of examples associated with the concept and their properties.

Using the theory, we have found (Shriki & David, 2001) that teachers are able to demonstrate a full concept image while relating to a concept in a specific context (e.g. considering the parabola as a graph of a quadratic function). However, when they are asked to explain the connection between contexts, they are not always able to do so, and thus exhibit a poor concept image.

To explain this phenomenon we use the theory of global and local coherence. The theory relates to the way information is stored and retrieved from our memory. According to the theory, the tendency is to look for a lack of contradiction within a view (local coherence, LC) rather then for a lack of contradiction between possible views (global coherence, GC). One of the main questions derives from the theory concerns the factors that are the most influential in creating a certain view. Chi & Koeske (1983) found that the configuration and the structure of the associative network[1] determine whether or not one is able to utilize efficiently his or her knowledge. Shriki & Bar-On (1997) found that students' errors are not always a result of deficiencies in knowledge but can be sometimes attributed to a lack of GC view. The ability to create a GC view (and thus

[1]For more details see Anderson (1985)

producing correct answers) was influenced by the nature of the structures contained in the associative network: structures that united the required elements and the connections between them enabled generating answers from a GC perspective.

Unifying both theories, it can be argued that teachers might build in their mind distinct structures, each of them containing the required elements and connections for a specific context. They are capable of exhibiting a full concept image regarding each context in an isolated manner, but at the same time they exhibit a poor concept image regarding the concept as a whole. The reason for this apparent contradiction can be explained by the absence of a connection between those distinct structures.

Interconnections within Mathematics: The case of the straight line, the parabola and similarity

Various concepts and topics are taught in a cyclic manner in different contexts. Many teachers tend not to exhibit the connection between the contexts, and thus cause the creation of separate structures. In the following, we describe a four-phase process in which two groups of teachers (seven IST and twenty-six PST) were exposed to three examples of such concepts: the straight line, the parabola and similarity.

Phase I – The participants were asked to complete a questionnaire (Appendix A), and then to reflect on the task.

Phase II – Mathematical background was presented, and was followed by introducing questions. The questions were formulated in such a manner that would enable to initiate a conversation regarding the issue of interconnections.

Phase III – Answers were given and a discussion was held.

Phase IV – The participants were asked to reflect on the process.

SUMMARY OF THE ANSWERS RECEIVED FROM THE QUESTIONNAIRE

14 out of 26 PST suggested a 'geometrical definition' for the straight line. The most frequent suggestions were given: "The shortest path between two points" and "A collection of an infinite number of points". All the IST responded correctly.

The answers obtained regarding the concept of parabola were quite similar to our previous findings (Shriki & David, 2001)[2].

The distribution of answers to the question that dealt with parabola's similarity and the explanations that were given are summarized in Table 1.

[2] Twenty-one IST and thirty-three PST teachers participated in that study. They were asked to complete a questionnaire, which included questions 3-7 in appendix A. Analyzing the data gained from the description of the curves properties it was found that on average 38% of the PST and 48.82% of the IST teachers demonstrated a full concept image regarding each defined curve, in an isolated manner. When the teachers were asked to explain the logical connections between the definitions and to sketch a Venn-Diagram, only 1 PST and 2 IST were able to do it correctly, and thus the others expressed a non-connectionist view of that concept.

	Statement 3			Statement 2				Statement 1	Other
PST N=26	N=7 (26.9%)			N=11 (42.3%)				N=1 (3.8%)	N=7 (26.9%)
	N=3 They all have the same shape	N=3 They all have the same pattern $y=ax^2+bx+c$	N=1 It is possible to transform any parabola to another by means of translation, rotation, reflection and shrink/ stretch	N=4 Depends on the ratio between the coefficients of the algebraic pattern $y=ax^2+bx+c$	N=2 Depends on the points of intersection with the x-axis	N=2 Depends on the type of the extreme point	N=3 No explanation	No explanation	No reply
IST N=7	N=3 (42.9%)			N=2 (28.6%)					N=2 (28.6%)
	N=1 They all have the same pattern $y=ax^2+bx+c$	N=1 It is possible to transform any parabola to another by means of translation rotation, reflection and shrink/ stretch	N=1 No explanation	N=1 depends on the symmetry axis	N=1 No explanation				N=2 Depends on the definition of similar parabolas

Table 1

Summarizing the data we can argue that there is a strong tendency towards conceiving the three mentioned concepts in an **algebraic** manner, with almost no reference to the interconnections within each concept.

WHAT QUESTIONS SHOULD BE ASKED?

In order to emphasize the interconnections exist within each concept we confronted the teachers with questions we believe had the potential to bring them to rethink and rebuild their concepts images.

The concept of straight line

Mathematical background: Students first encounter the concept of a straight line as a fundamental concept in Euclidean geometry. Later on they learn to identify the graph of a linear function as a straight line when they learn basic concepts in analytic geometry. In other words, a fundamental concept in geometry appears as a 'defined' concept in algebra. As a result, we arouse the question of our 'right' to entitle this mathematical object by the same name ('a straight line') in both contexts.

What should be asked? Is it allowed to denominate a function like $y=3x+5$ by the name 'a linear function' and to entitle its graph 'a straight line'? Why?

The concept of parabola

Mathematical background: In ninth grade, students in Israeli high school learn the concept of quadratic function, to sketch its graph and to find 'special' points. By that time, the word 'parabola' becomes a synonymous with 'a graph of quadratic function'. As a consequence the statement: "the parabola is a graph of a quadratic function" develops into teachers' and students' dominant concept image regarding the parabola (Shriki & David, 2001). The definition of the parabola as a geometrical object[3] or as one of the conic sections is introduced only to eleventh grade students who learn in advanced classes of mathematics.

Based on the geometrical definition, teachers represent an algebraic pattern, but they restrict it only to the case in which the focus lies on the x-axis and the diretrix is parallel to the y-axis (the graph of an implicit function of the form $y^2=2px$, where $p\neq 0$; $p\in R$). The core of the instruction then becomes exclusively analytic, without any investigation of the geometrical characteristics of the parabola.

Translating the geometrical representation of the parabola into an algebraic representation, three distinct sets are constituted, in accordance with the 'direction' of the directrix (parallel to the x-axis, parallel to the y-axis, not parallel to the axis). The general equation is usually not presented in high schools[4].

What should be asked? Is the graph of quadratic function a parabola? Is the parabola a graph of quadratic function? Why? What subgroups of parabola are constituted by its algebraic definition? Is the graph of the function $y=x^n$, where n is an even number, n>2, a parabola? How can the parabola be built by using only geometrical means?

The concept of similarity

Mathematical background: In contrast to the two previous concepts, 'similarity' is not a mathematical object but a feature that connects between objects. Two objects can be either similar to each other or not. In Israeli schools the concept of similarity is introduced to students in ninth or tenth grade, but only in the context of triangles, thus teachers refer solely to the concept 'similar triangles'. As a consequence, there is no discussion regarding the various aspects of the concept 'similarity'.

What should be asked? How can we determine whether or not two polygons are similar? Is it possible to talk about similarity of other curves? Can two parabolas be similar? Two ellipses? Two circles? Two graphs of third degree functions?

WHAT ANSWERS SHOULD BE GIVEN?

This section includes the essence of answers we suggested to the questions above.

The concept of straight line

Why is the graph describing all the points that satisfy an equation of the form: $ax+by+c = 0$; $a^2 + b^2 \neq 0$ a straight line? Since 'straight line' is a fundamental concept in Euclidean geometry, answering the question one cannot use a definition. He or she should look for necessary and sufficient conditions for three points to lie on the same Euclidean line. Verification will focus on the case in which $a,b\neq 0$, since the other cases are much simpler. Figure 1 shows three points $A(x_a,y_a)$; $B(x_b,y_b)$; $C(x_c,y_c)$ which satisfy the equation $ax+by+c=0$. A line through point A, parallel to x-axis, and lines through points B and C, parallel to the y-axis, are drown. Points D and E designate their intersection.

We obtain:

$$\frac{BD}{AD}=\frac{y_b-y_a}{x_b-x_a}=\frac{\Delta Y_{AB}}{\Delta X_{AB}} \quad ; \quad \frac{CE}{AE}=\frac{y_c-y_a}{x_c-x_a}=\frac{\Delta Y_{AC}}{\Delta X_{AC}}$$

[3] See definition of curve Λ_1 in appendix A.

[4] An equation of the form $ax^2+2hx\ +by^2+2gx+2fy+c=0$ is called 'a second degree equation'. This equation describes a parabola iff $h^2-ab=0$.

The ratio $\frac{\Delta Y}{\Delta X}$ is constant (and equals $-\frac{b}{a}$) for every two points which satisfy the equation $ax+by+c=0$, and thus $\frac{BD}{AD}=\frac{CE}{AE}$. Using similarity of triangles or an inverse theorem of Thales, the three points A,B,C are on the same Euclidean line. The opposite direction is obvious, and is performed with the aid of an axis and by using Thales theorem.

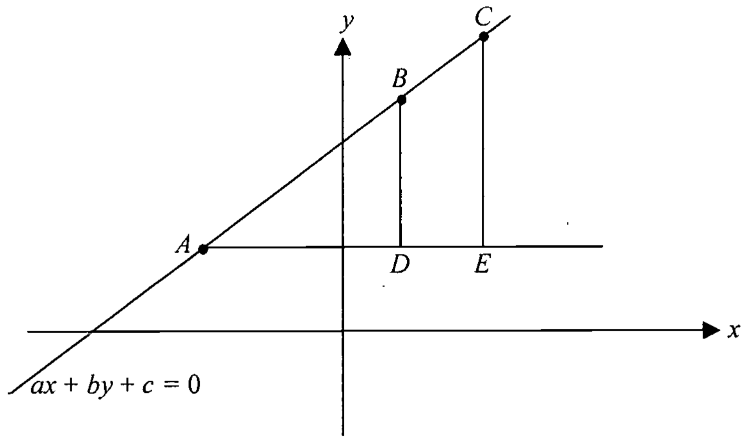

Figure 1

Discussing the need for justifying the 'right' to use the same name ('a straight line') for two concepts within two different contexts, and the proof itself, brought the participants to be aware of this obvious but somewhat neglected interconnections within mathematics.

The concept of parabola

In order to justify that the graph of a quadratic function is a parabola it is necessary to show that the graph of a quadratic function of the form $y=ax^2+bx+c$ ($a\neq 0$; $a,b,c\in R$) fulfills the requirements that are derived from the geometrical definition of the parabola.

The proof is exhibited in figure 2. Based on symmetry considerations of the graph of the function $y=ax^2$, suitable 'candidates' for the focus and the directrix are a point on the y-axis, $F(0,k)$, and a line parallel to the x-axis, $l: y=-k$, in accordance.

In order to find the value of k, we have to solve an equation that satisfies the geometrical condition

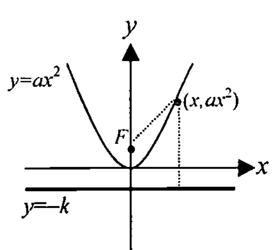

Figure A2

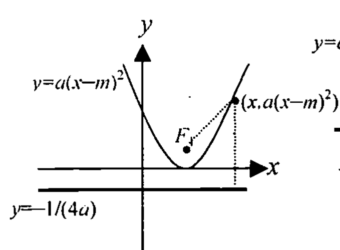

Figure B2

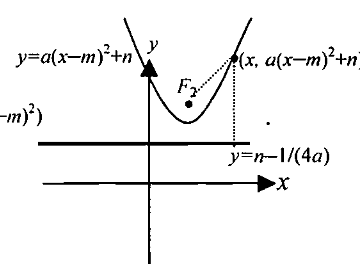

Figure C2

Figure 2

(figure 2a): $(x-0)^2+(ax^2-k)^2=(ax^2+k)^2$. It follows that $k=1/(4a)$ and thus l: $y = -1/(4a)$; $F(0,1/(4a))$. The rest of the proof is left to the readers (figures 2b, 2c).

We have shown that the graph of any quadratic function is a parabola. It is important to note that the inverse is not true.

A similar proof is introduced in figure 3 regarding the graph of an implicit function of the form $y^2=2px$ ($p\neq 0$; $p\in R$).

$(x+k)^2=(x-k)^2+2px \Rightarrow$

$F(p/2,0)$; l: $x = -p/2$

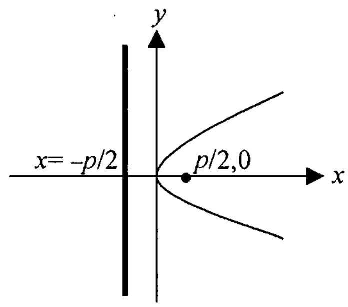

Figure 3

As we know, the graphs of functions of the form $y=x^n$, where n is an even number $n>2$, are all 'look like' parabola. Are they really all parabolas? Using the same process, we have to solve the equation: $(x-0)^2+(x^n-k)^2 = (x^n + k)^2$. We get: $x^2+x^{2n}-2x^nk+k^2=x^{2n}+2x^nk+k^2 \Rightarrow k=1/(4x^{n-2})$. Since the value of k depends on the selection of a point (x, x^n) on the graph of the function $y = x^n$, it can be concluded that those graphs are not parabola.

Following is a Venn-diagram, which describes the logical connection between the four sets of curves described in appendix A[5]:

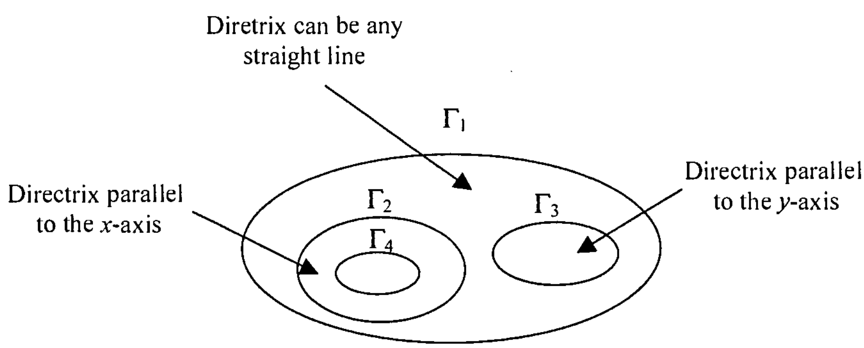

Figure 4

In order to emphasize the geometrical characteristic of the parabola we introduced two methods for receiving a parabola by using only 'geometrical tools' (see appendix B and C).

The concept of similarity

As it was mentioned, most teachers are not familiar with the definition of the concept 'similarity'. Furthermore, we have found out that they were not even aware to that fact, since they did not distinguish between 'triangles' similarity' and 'similarity'. The first step should be to

[5] A function of the form $f(x)=(ax+b)\cdot(cx+d)$; $a,c\neq 0$, $a,b,c,d\in R$, has at least one root, since its roots are determined by the roots of the two linear generator functions. Algebraically it means that if a quadratic function has no roots, it is not possible to write its pattern as a product of two linear patterns.

define 'similarity': Two figures are similar iff there is a transformation (translation, rotation, reflection and stretching/shrinking) or a composition of transformations that maps them into each other. If it is possible to map one figure into another by isometric transformations, those figures are congruent. According to that definition, it is obvious that all regular polygons with the same number of sides are similar, and that all the parabolas are similar. Algebraically, all congruent parabolas are obtained by manipulating isometric transformations on the graph of $y=x^2$, as shown in figure 2. For the graphs of the functions $y=ax^2$, $y=bx^2$, where $a\neq b$ a stretching/shrinking transformation should be manipulated (multiply the coefficient of the function $y=ax^2$ by b/a or the coefficient of the function $y=bx^2$ by a/b). The same can be done with any two parabolas.

Discussion

By the end of Phase I, the teachers' reflection expressed confusion, embarrassment and even frustration for their inability to answer the questions correctly.

A further reflection was conducted at the fourth phase. It is interesting to note that the PST responses emphasized mathematical aspects, while those of the IST focused on didactical aspects. Most of the mathematical ideas were new to the PST, and they indicated that they had learned many new facts. They expressed their fear from teaching those subjects, and admitted to feeling ashamed for not knowing them. Many PST expressed their surprise of the fact that their teachers had never asked them those questions, and were never concerned about introducing the interconnections in mathematics.

The IST said they have never thought about the connection within each concept in the various contexts, and thus they have never felt the need to 'justify' using the same name of objects within different contexts. They stated that definitions and concepts should not be taken for granted, without rethinking and investigating them.

The analysis of the data we collected through all four phases of the process described above reveals the tendency to yield answers from a LC view. Most participants demonstrated isolated concept images, and could reason only in the framework of a single context. By the beginning of the process each participant held a certain concept image and concept definition regarding the mentioned concepts in the various contexts. We have found that the most common prototypical examples of those concepts carried algebraic characteristics.

Relating to this finding it is interesting to refer to the finding of a study, which dealt with reading comprehension of mathematical proofs by undergraduate students at the course of advanced calculus (David, 1996). It was found that students, while reading mathematical proofs, focused mostly on their algebraic parts. As a consequence they misinterpreted and misunderstood the mathematical text. We believe a further study is needed in order to explain this 'algebraic tendency' phenomenon.

The fact that the participants were not bothered by the absence of links between contexts for a specific concept can be well explained by the LC theory: In each context the appropriate part of the mental image of that concept is retrieved. It is quite obvious that the concept image of each distinct part of the concept has its 'own independent life', without any interconnections between parts, despite the fact that those different concept images share many components. The remarkable point is that those common components do not function as motivators for unifying all the distinct concept images of a specific concept under a new comprehensive image. Such an image could successfully confront the complementary as well as the contradictory aspects exist between each isolate image held for each context.

We are confident that all the teachers had learned new mathematical facts, and had begun to establish some interconnections between them. It is hard to tell whether old structures were broken and new ones were formed instead, but we believe the teachers will continue to inquire the concepts introduced and the connections between them, as well as develop a new look at other concepts, and thus create a GC view of mathematics.

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Appendix A

1. How is a straight line defined in Euclidean geometry?
2. How is a straight line defined in algebra?

Following are four definitions of curves. Draw each curve and describe its properties[6]:

3. Given a line l and a point F not on the line. The curve Λ_1 is the locus of the points in the plane so that their distance from the point F equals their distance from the line l.
4. The curve Λ_2 is the graph of a function of the form: $y=ax^2+bx+c$, where $a\neq0$; $a,b,c\in R$.
5. The curve Λ_3 is the graph of an implicit function of the form; $y^2=2px$, where $p\neq0$; $p\in$ R.
6. The curve Λ_4 is the graph of a function which its pattern is a product of two non-constant linear patterns.

7. Sets Γ_i , i=1,2,3,4, contain all the curves of the form Λ_i, in accordance.
   Sketch a Venn-Diagram that describes the logical connections between the four sets of curves.

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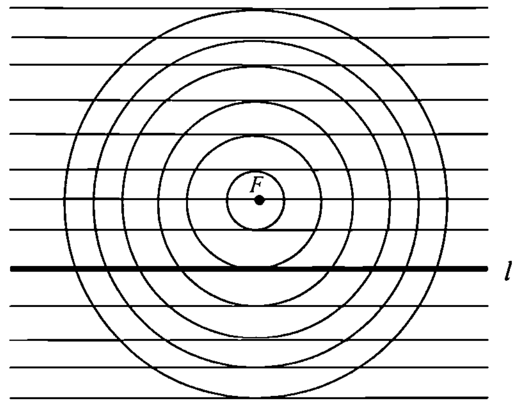
8. Mark the statement you agree with, and explain your choice:
   There are no two parabolas that are similar to one another.
   There are some parabolas that are similar to one another, and there are some parabolas that are not.
   All the parabolas are similar to one another.

---

[6] This assignment was taken from Shriki & David, 2001.

Appendix B

Below is a sketch of concentric circles with a center at point F.
The distance between each two adjacent circumferences is one unit as well as the distance between two adjacent lines. One of the lines is designated as l.
Draw points that their distance from the point F equals their distance from the line l.



Assume you could repeat this process an infinite number of times, what would you get?

This activity demonstrates a simple process of constructing a parabola based on its geometrical definition.

Appendix C[7]

a. On a rectangular paper mark a point C near the bottom edge, and a point D on its edge (figure 1).
b. Fold the paper so that point D will unite with point C (figure 2).
c. Make a crease, open the paper, and mark the crease (figure 3).
d. Mark additional points on the bottom edge, and repeat the process (point C remains the same).

Questions:
- Assume you could repeat this process an infinite number of times, what curve do you think would bound the area where there are no creases? Explain your answer.
- What is the connection between that curve and the points C and D?

[7] This activity is based on Scher (1995).

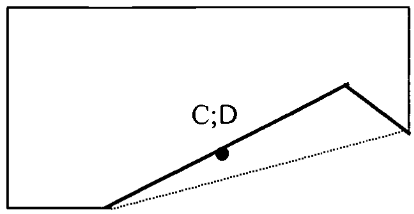


Figure 2 – folding the paper

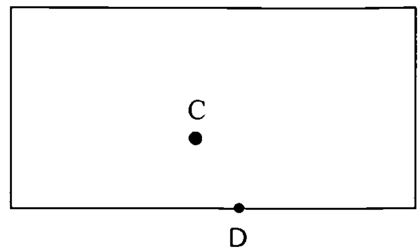


Figure 1 – marking points C and D

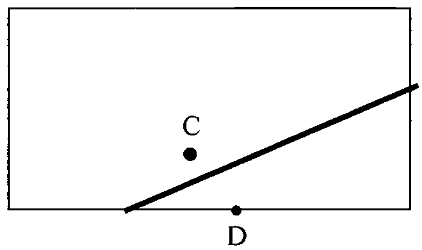


Figure 3 – marking the crease

What can you say about a line tangent to parabola?

The proof that the obtained curve is a parabola is beyond the scope of this paper.

A GRAPHICAL EXPLORATION OF THE CONCEPTS OF EIGENVALUE AND EIGENVECTORS IN $\mathbb{R}^2$ AND $\mathbb{R}^3$

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ABSTRACT

We study a graphical approach to the concepts of eigenvalue and eigenvector of a square matrix. This approach is based on an interactive computational environment created with *Cabri Geometry II*. The environment allows a simultaneous display of: a) a square matrix A of size 2x2 or 3x3, b) an arbitrary vector $\boldsymbol{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ and its image under the linear transformation defined by A and c) the graph of the characteristic polynomial $P(\lambda)$ associated with A. The entries A as well as the coordinates of vector $\boldsymbol{v}$ can be directly (this is, on-screen) manipulated, so providing a method for a graphical analysis of eigenvalues and eigenvectors of a given matrix. This exploration is guided by the graph of $P(\lambda)$

Key Words: Teaching with Technology, Eigenvalues, Eigenvectors

1. Introduction

The concepts of eigenvalues and eigenvectors constitute an important topic for a first year course in Linear Algebra and it is, at the same time, a source of difficulties for students. Some of these difficulties could be relates with the diversity of mathematical objects in which these notions rest and in the almost exclusive use of algebraic symbols in the process of teaching.

In this work we present a computational environment that uses graphical and numerical representations of a dynamic nature, designed with the purpose of facilitating the conversion among different representations of the same mathematical object. In the context of the theory of R. Duval [1] on Semiotic Representation of Registers, this conversion is a cognitive activity, needed to achieve a conceptual apprehension of the mathematical concepts.

The idea to create a computational environment based on dynamical representations has been taken from Sierpinska [2], but the representations here used are different, because the purpose of this work is considerably more modest regarding the level of abstraction with which the concepts are discussed.

We present the activities designed with Cabri Geometry II [3] to explore the concepts of eigenvalue and eigenvector of a square matrix and we describe the instructions to construct the files that are used. In the elaboration of these files we have tried to reduce to the minimum level the requisites on software needed to interact with them.

2. Definitions and Calculations

Even though the notions of eigenvalue and eigenvector can be defined in a more general way, in a first course in Linear Algebra the definitions can be presented as follows: If T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, defined by $T(\boldsymbol{v})=A\boldsymbol{v}$, where A is a matrix of size nxn, then it is said that a vector $\boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{v} \neq \boldsymbol{0}$, is a *eigenvector* of T if there exists a real number λ such that $T(\boldsymbol{v})= \lambda\boldsymbol{v}$. The number λ is called an *eigenvalue* of T. As it is usual, in this paper we will refer to λ as the eigenvalue of matrix A and to $\boldsymbol{v}$ as its corresponding eigenvector.

From these definitions it is possible to go directly to the calculation of eigenvalues and eigenvectors of the linear transformation. But the procedure of algebraic calculations is not simple at all. The diversity of mathematical objects and algorithms involved sometimes makes it difficult for the student their reproduction, and this induces to memorization. Many of the errors made by students when performing the calculations could be explained in terms of the reduced attention that the teaching activity dedicates to the comprehension of the significance of the concepts, symbols and operations that are involved in these calculations.

Figure 1 shows, in a schematic form, the concepts and operations that are involved in the calculation of eigenvalues and eigenvectors. (See [4])

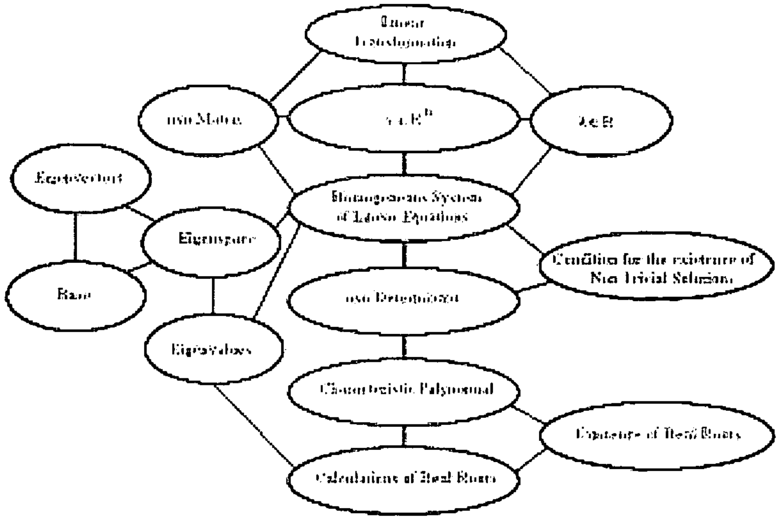

Figure 1

This procedure can be summarized as follows:

1. Set the vector equation $Av = \lambda v$, or, equivalently, $(A-\lambda I)v=0$, where I denotes the identity matrix of size $n\times n$;

2. Write the vector equation $(A-\lambda I)v=0$ as an homogeneous system of linear equations;

3. Calculate and set equal to cero the determinant of the matrix $A-\lambda I$ to obtain the characteristic polynomial, $P(\lambda)$;

4. Calculate, or find approximated solutions to, the real solutions of $P(\lambda)$, in case they exist. Each real root of $P(\lambda)$ is a real eigenvalue;

5. Find the set of non trivial solutions (subspace S_i) for the homogeneous system of equations obtained in Step 2, for each real eigenvalue λ_i;

6. Determine a base for each subspace S_i. The vectors that constitute a base for S_i are the eigenvectors corresponding to λ_i.

3. The Context of the Teaching Activities

The teaching approach described in this work is a part of a more general project, which pursues a reformulation of the teaching of the Linear Algebra course that is offered at the Universidad de Sonora, to science and engineering student. This reformulation combines the work in the classroom with the activities in a computational environment created with Cabri. The design of activities with Cabri is oriented by the two following general principles:

1. The computational environment shall allow the interaction of the student with the representations provided by the computer to the extent of permitting the student to perform modifications, with the objectives of detecting behavioral patterns and of formulating conjectures on the represented objects an their characteristics;

2. A first graphical approach to the mathematical concepts can be useful to create a more concrete base of significance, before examining these concepts in a more abstract level, and the manipulation

performed by the student on graphical-dynamical representations can help in the construction in this base of signification.

In the course on Linear Algebra, the immediate background to the topic discussed in this article is *linear transformations*, which more important properties are also explored with a computational environment similar to the one described here.

4. The Computational Environment

The environment that has been created with Cabri Geometry II allows the exploration of the concepts of eigenvalue and eigenvector for square matrices of sizes 2x2 and 3x3 and works with three simultaneous on-screen representations, namely:

1. The graphical representation of v and of $T(v)$, where v is a vector that can be directly manipulated, which, in turn, modifies vector $T(v)$;
2. The matrix A, whose entries can be varied, so modifying vector $T(v)$ and the characteristic polynomial $P(\lambda)$;
3. The graphical representation of $P(\lambda)$, which can be manipulated, not directly, but through the entries of matrix A.

These representations allow the student to perform explorations at two levels:

1. A vector v will be an eigenvector of A when v and $T(v)$ are collinear. Then the objective consists in "dragging" vector v until this happens. Once such a vector v has been found, the student can calculate the magnitude (i.e., the absolute value) of the corresponding eigenvalue by dividing $\|T(v)\|$ by $\|v\|$. Additionally, he/she can move v without changing its direction to verify that this motion does not alter the calculated eigenvalue. At this level the graph of the characteristic polynomial remains fixed and its real roots coincide with the found eigenvalues.
2. By varying the entries of matrix A the student can look for eigenvalues and eigenvectors of other matrices. This allows the exploration of the behavior of eigenvalues and eigenvectors for some interesting matrices, as are, for example, diagonal, symmetric, triangular, singular, etc.

5. The Teaching Activities

The main activities performed by students at first level are related with the search of eigenvalues and eigenvectors. In the case of square matrices of size 2x2 the student has to rotate vector v, around the origin, in search of a vector v_1 that is collinear with $T(v_1)$. Once vector v_1 has been found the student is asked to drag it, keeping its direction unchanged, to conclude that, in the direction of v_1, there are infinitely many vectors that are collinear with $T(v_1)$. All of these vectors are multiples of v_1 and, hence, it makes sense to choose one of them, v_1 say, as a base for the subspace to which them belong.

Since v and $T(v)$ are collinear for every vector in the direction v_1, then for every v_1 there is a real number λ such that $T(v_1)=\lambda v_1$. The student can now take any of these vectors and with the "Distance and Length" tool from the Cabri menu can calculate the norms of v_1 and $T(v_1)$. Number λ is then the quotient between the norm of $T(v_1)$ and the norm of v_1. This quotient has to be taken with positive sign if v_1 and $T(v_1)$ have the same direction and with negative sign if they have opposite directions. Figure 2 shows the eigenvector $v_1=c(1,1)$, $c\neq 0$ corresponding to the eigenvalue $\lambda=2$ of matrix $A=\begin{bmatrix}1 & 1\\ 3 & -1\end{bmatrix}$

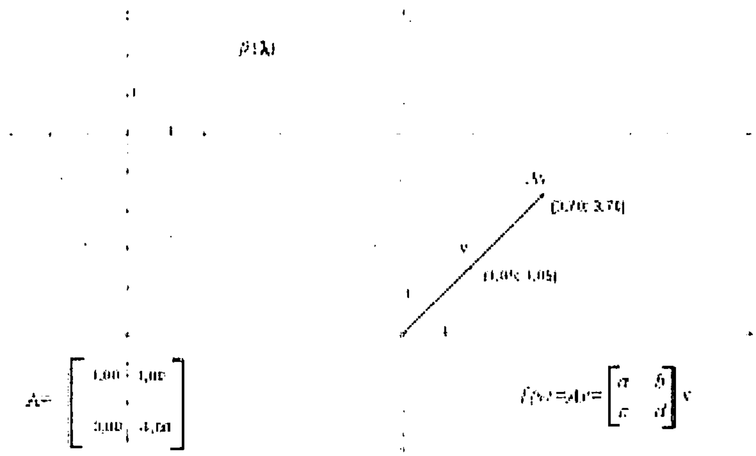

Figure 2. An approximated eigenvector in $\mathbb{R}^2$

The calculations can be repeated for several of the vectors found in the direction of v_1 in order to conclude that the number λ is the same for all such vectors.

The construction in $\mathbb{R}^3$ is manipulated in a slightly different way as compared with the one in $\mathbb{R}^2$. These differences are related with the problem of representing a three-dimensional vector on a two dimensional screen. In $\mathbb{R}^2$ the graphical search for collinearity between vectors v and $T(v)$ is exhaustive, since in rotating vector v around the origin, the whole plane is "swept". In $\mathbb{R}^3$, however, the search is not that simple. According to the design of the construction, students have to combine two ways of moving vector v in order to detect collinearity: one of them on the point P and the other on the point v. Point v is the end point of the vector and point P is its orthogonal projection on XY plane. See Figure 3 for an approximated eigenvector of $A=\begin{bmatrix}-1.43 & 1.51 & 0.55\\ 2.14 & -1.00 & 1.12\\ 1.11 & 1.49 & -1.09\end{bmatrix}$

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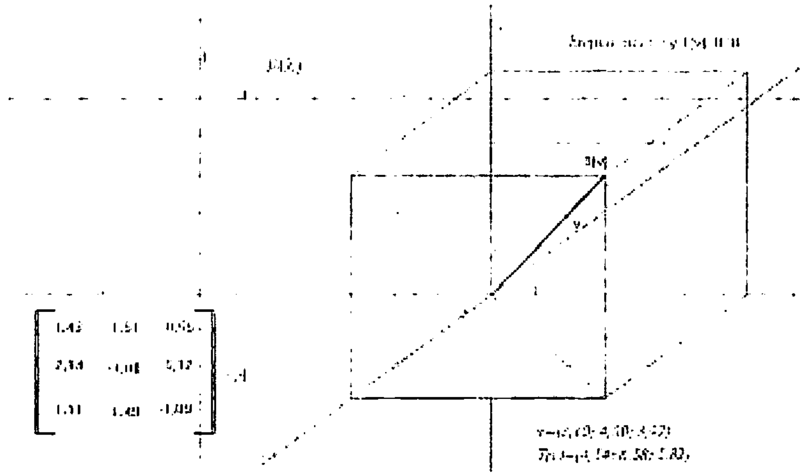

Figure 3. An approximated eigenvector in $\mathbb{R}^3$

Since vectors v and $T(v)$ in $\mathbb{R}^3$ may seem to be collinear without actually being, the environment shows on-screen the measure of the angle θ between v and $T(v)$ as a guide that orients the search. With this resource, the aim is to find a vector v such that the angle θ is either 0° o 180°. Given that Cabri does not provides a direct way for calculating the norm of a vector in $\mathbb{R}^3$, the environment also includes the coordinates of v and $T(v)$ in order to facilitate the calculation of $||v||$ and $||T(v)||$. With these quantities, the student can obtain the absolute value of the corresponding eigenvalue.

The exploration activities so far performed by students belong to what we have called Level 1 Explorations. Once familiarized with the calculations of eigenvalues and eigenvectors in this environment the student is asked to answer some questions, the answer of which requires changing the level of exploration activities. These questions are of the following type:

a) If the characteristic polynomial of a matrix A has no real roots, how many eigenvectors can A have?

b) What is the relation among the eigenvectors of a symmetric matrix?

c) How many eigenvalues does a singular matrix have?

d) How many eigenvectors does a singular matrix have?

e) If a matrix A is diagonal, how are the entries of A related to its eigenvalues?

f) If the characteristic polynomial of a matrix A has a real root of multiplicity two, how many eigenvectors can A have?

g) If the diagonal entries of a diagonal matrix A are all the same, how many eigenvectors does A have?

In order to answer these questions has to try several matrix entries and pay attention to the graphical behavior of $P(\lambda)$. All the questions involve activities belonging to Level 2 and they have turned out to be more difficult to answer as compared with those formulated at Level 1.

6. Conclusions

Students have been able to successfully perform the activities proposed in this environment and they have shown interest in the topic. The different answers given by students have generated a fruitful discussion about the meaning that the topics under study acquire in this environment. However, the supervision of the professor has turned out to be very important to give the activity the proper orientation, particularly on those areas where difficulties have been detected.

Some of these difficulties are the following:

1. Students have had problems to identify negative eigenvalues, partly because in this case the collinearity of v and $T(v)$ is not always clear in the environment and partly because sometimes they find it difficult to identify the effect of multiplying a vector times a negative number.

2. They have found it difficult to explain what happens with eigenvalues and eigenvectors for singular matrices and they have requested for help in this case to identify on-screen the eigenvector that corresponds to a zero eigenvalue.

3. In the last question, sometimes it has been hard for students to get the conclusion that the eigen-subspace has dimension 2.

These difficulties have been directly observed during the teaching development. A more detailed analysis on the achievements and difficulties to move from one register of semiotic representation to another that shall be based on the written answers given by students is in progress. Given the diversity of representations and registers that the teaching design involves, this analysis has not been simple at all.

Appendix: Instructions to construct the Cabri File to work in $\mathbb{R}^2$

In this appendix the Cabri instructions to generate the environment in $\mathbb{R}^2$ are given. The construction in $\mathbb{R}^3$ is similar.

1. With the "Show Axes" tool ask Cabri to plot a set Cartesian coordinates. Refer to this system as System I.

2. Draw an arbitrary "Vector" on the origin of coordinates.

3. With the "Label" tool denote with v the end point of the vector.

4. With the "Numerical Edit" tool write four numbers in a 2x2 matrix look-like configuration. These numbers will represent the matrix A associated with the linear transformation $T(v)=Av$. Figure 4 shows matrix $A = \begin{bmatrix} 1.5 & 1.8 \\ 2.2 & 1.2 \end{bmatrix}$, with a=1.5, b=1.8, c=2.2 and d=1.2.

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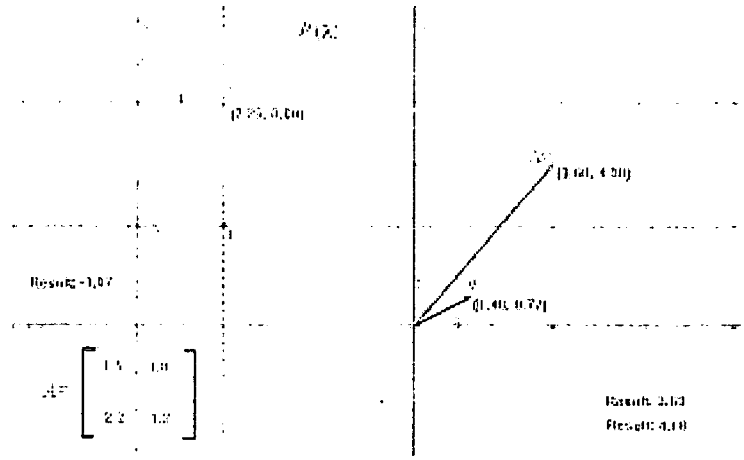

Figure 4: A detailed aspect of the construction in $\mathbb{R}^2$

5. Use the "Calculate" tool to compute the coordinates of vector Av and drag each of the corresponding coordinates from the calculator to the Cabri screen.

6. With the "Measurement Transfer" tool project the first coordinate of vector Av to the X axis and the second coordinate to the Y axis. Mark with a "Point" the corresponding projections.

7. Draw a "Perpendicular Line" to X axis through the point determined on this axis in Step 6 and then draw a "Perpendicular Line" to the Y axis through the corresponding point.

8. Draw the "Intersection Point" between the two perpendicular lines obtained in Step 7 and call Av this point.

9. Draw a vector from the origin of coordinates to point Av. If (x, y) denote the coordinates of vector v then the coordinates of Av will be $(1.5x+1.8y, 2.2x+1.2y)$.

10. Using the "New Axes" tool draw another system of coordinates. Refer to this system as System II. Fix the origin of System II sufficiently far away from the origin of System I.

11. Use the "Point on Object" tool to select an arbitrary point S on the X axis of System II. With the "Equations and Coordinates" tool ask Cabri for the coordinates of this point in System II. Draw a "Perpendicular Line" to X axis (of System II) passing through S.

12. Use the entries of matrix A (see Step 4) to calculate the number $P(\lambda)=(a-\lambda)(d-\lambda)-cb$, taking λ as the abscissa of point S.

13. "Drag" the number obtained in Step 12 from the calculator to the Cabri screen.

14. Use the "Measurement and Transfer" tool to project $P(\lambda)$ into the Y axis of System II. Draw a "Perpendicular Line" to Y axis through the point determined by $P(\lambda)$ on the Y axis.

15. Draw the "Intersection Point" between the perpendicular lines obtained in Steps 11 and 14. Call L such point.

16. Ask Cabri for the "Locus" of L when S moves along X axis of System II. This Locus is the graph of the characteristic polynomial $P(\lambda)$ of A. See Figure 4.

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THE IMPROPER INTEGRAL. AN EXPLORATORY STUDY WITH FIRST-YEAR UNIVERSITY STUDENTS[1]

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ABSTRACT

In this paper we analyse the answers of a group of first-year university Mathematics students to a questionnaire, with the aim of determining the difficulties they have when carrying out non-routine tasks related to improper integrals.

Among our research questions, we distinguish the followings: *How do students react when they have to face up to tasks of a non-algorithmic type, questions of reasoning and non-routine questions in the topic area we are involved in? In which system of representation do they feel more comfortable? Are they conscious of the paradoxical results they can achieve? Are they able to articulate different systems of representation in questions related to improper integrals? Do they establish any relationship between the new knowledge with the previous one, particularly the one related to definite integrals, series and sequences?*

The questionnaire consisted of nine questions including not only calculus tasks and determining the convergence of given improper integrals, but also intuitive questions and some paradoxical results too (for example, a figure with an infinite longitude which closes the same area as the unit circumference, or an infinite figure with a finite volume). We particularly asked the students to interpret most of the results they had obtained.

Answers given by the students to each of the questions were categorized, which allowed us to reach some partial conclusions to our research. The answers obtained also allowed us to decide on the selection criteria for choosing students to be interviewed.

From analyses carried out, we can conclude that there are students who have difficulties in articulating the different systems of representation, and have problems in connecting and relating this knowledge as a generalization of previous concepts, such as definite integrals, series and sequences.

KEYWORDS: Improper integral, registers of representation, articulation, transference.

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1. Introduction

The concept of improper integral is first came across by Mathematics Degree students during the second half of their first-year studies while covering the subject *Mathematical Analysis II*. From then on, our students continue to come across the concept, basically, for example, when calculating integral transforms and Fourier series.

The main aim of our research is to design a teaching sequence for Improper Integrals using a *Computer Algebra System* (CAS); we therefore considered it necessary to carry out an exploratory study to identify obstacles and difficulties faced by First-Year Mathematics students when learning the contents related to improper integration. We are also interested in detecting certain errors and difficulties, which arise when making conversions between the algebraic and graphic registers or when handling elements within a single register (Duval, 1993).

With these ends in mind, we drew up a questionnarie in which we included a group of non-routine questions in order to then discover the level of students' understanding, where understanding is taken from the point of view of Duval's theory of systems of semiotic representation (Duval, 1993). The research questions posed refer mainly to the handling of algebraic and graphic representation. They are as follows:

- How do students react when faced with non-algorithmic type questions, which entails reasoning and non-routine questions about improper integrals?
- Which system of representation do they feel more comfortable in?
- Do students make any geometric interpretation of the results they obtain?
- Can they articulate different systems of representation in questions related to improper integrals?
- Do the students relate this new knowledge to their previous knowledge, especially regarding definite integrals? And do they relate it to their knowledge of series and sequences?

2. Theoretical Framework

As stated, we made use of the theoretical framework designed by Duval (1993) to evaluate the levels of students' knowledge when both algebraic (formal) and graphic systems of representation come into play. With the help of this theoretical framework, and once the answers to the questionnaire were analysed, we were able to design a competence model that would allow us to classify the levels of understanding when only these two systems of representation come into play. We then selected six students to be interviewed.

The core of our theoretical framework (and the part used in designing the questionnaire) is based on Duval's ideas of construction of knowledge (1993): we consider it necessary to distinguish between a mathematical object and its representation to achieve Mathematical understanding. And in order to attain this aim, different semiotic representations of a mathematical object need to be used. Duval defines these representations as follows:

> Semiotic representations are productions made up of the use of signs that belong to one system of representation, which has its own constraints of meaning and function.
>
> A geometrical figure, a text in natural language, an algebraic formula, a graph are all semiotic representations that belong to different semiotic systems.

Duval goes deeper into this idea until finally defining a semiotic register of representation:

> A semiotic system can be a register or representation if it allows three cognitive activities related to semiosis:

1) Formation of an identifiable representation as a representation of a given register.

2) Treatment of a representation, which is the transformation of the representation within the same register where it was formed. Treatment is a transformation that is internal to a register.

3) Conversion of a representation, which is the transformation of the representation into another representation in another register where the whole or part of the meaning of the initial representation is preserved. Conversion is a transformation external to the original register.

Duval notes that, as each representation is partial with respect to what it represents, interaction between different representations should be considered absolutely necessary to form the concept.

However, several authors, such as Hitt (2000), feel that not only are the transformation tasks within a register of representation important, but also equally important is the opposition between examples and counter-examples.

As we are also interested in the connections students make to their previous knowledge, the concept of transference also plays an important role (see Hitt, 2000; 2002).

3. Methodology

Subjects

The questionnaire was completed by a group of thirty-one students- thirteen male and eighteen female – at the end of the second half of the 2000-01 course, and was undertaken during a class session. Participating students were taking all or some of the First-Year subjects offered in the Mathematics Degree Course, especially the subject *Mathematical Analysis II*. Few of the students were found to be studying First-Year Mathematics for the first time.

Students had one hour to answer the questions set.

The questionnaire

The questionnaire was made up of nine questions, including not only calculus tasks (Items 3, 4 and 5) and improper integrals convergence (Items 4 and 7), but also intuitive questions (Items 2, 8 and 9) and some paradoxical results (Items 3, 4 and 6). Especially, students were asked to interpret most of the results they achieved (Items 3, 4, 6 and 7). The texts of the questions are given in the Appendix.

Analysis of the answers to the questonnaire

We can now look at and analyse the results obtained:

***Item 1*:**

As in our study we also include the transference that might be undertaken from previous knowledge to new knowledge (improper integrals), we thought it fundamental to analyse what conception students had of definite integrals.

Students' answers were categorised into several groups of answers. Some of these groups are included in others. The most striking results for the answers to this item are:

- Twenty-nine students (93.54%) mention that this can be used to calculate areas, but only four students (12.90% of the total and 13.79% of the set) mention the sign of the function.
- A mere four students (12.90%) explicitly mention that the interval $[a, b]$ is finite.
- Five students (16.12%) speak of previous calculation of the indefinite integral in order to speak of the definite, and a further five (not the same) mention Barrow's Rule.
- Seven students (22.58%) appear to conceive of definite integrals as an operation.

- Five students (16.12%) refer to partitions and three (of these) to Riemann Sums.
- Four students (12.90%) refer to some condition of continuity for *f(x)* and only five students (16.12%) note that this is bounded.

We can see that a large percentage of students consider the integral to be an area, but it seems few take into account the circumstances when it is definitely an area and when not, thus confirming results from other research (e.g. Hitt, 1998).

***Item 2*:**

A graph of a function tending to infinite at one of the ends of integration is given. However, its integral is finite.

Our aim is to check whether the presence of a graph (graphic register) will confuse students as to their knowledge of criteria and theorems and make them think that the area is infinite.

Based on analysis of their answers we can note:

- Only eleven students (35.48%) state that nothing can be said a priori, recognising that it might be finite or infinite.
- Eleven students (35.48%) say that this integral represents the area beneath the curve.
- Five students (16.12%) say that it will be positive, and one of them adds that the area is always positive.
- Four students (12.90%) feel more inclined to think that it might be infinite, although they recognise a priori that this cannot be known.
- It is significant that thirteen students (41.93%) do not say anything about the value of the integral; they either describe the function or the integral, but do not say anything else.
- Two students (6.45%) state that they would have to solve it to know the result.
- Three students (9.67%) say that it cannot be solved because it fails to fulfil certain conditions.
- Only one student (3.22%) leaves the question unanswered.

We had believed that the brief text would produce concrete answers, but it seems that this brevity has generated a wide variety of answers. This should be taken into account if this question were to be used in another experiment.

A relatively low number of students (eleven) answered by affirming that nothing could be known a priori, while an equally low number of thirteen students (fourteen, including the student who failed to answer at all – to which should be added the students who affirm that it cannot be solved, among others) wrote nothing at all about the value of the integral.

Furthermore, a large number of students attempted to carry the text over into the algebraic register: six students classify it as an improper integral, thirteen describe the behaviour of the function of the integrand, two attempt to separate the integral into subintervals in order to calculate, while two attempt to solve it.

***Item 3*:**

This question was set to find out whether simple integral calculation leads students to forget the aim of the calculus. Furthermore, we wanted to see if they correctly use calculus techniques for improper integrals[2]. Our objective was simply to create a cognitive conflict by making them face an infinite surface with the same area as the unit circumference. The answers attained were as follows:

[2] Therefore, it was essential for us to choose a case which we considered simple.

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- Only fourteen students (45.16%) correctly solved the integral. Two of those students who managed to solve it (14.28% of the group) did not use symmetry.
- Only eight students (25.80%) clearly expressed that the area was π^3. And only one student (3.22%) related the area beneath the graph and the circumference.
- Seventeen students (54.83%) failed to write a single explanation about what they did and interpreted nothing. Two students (6.45%) did write something, but did not answer the questions.
- Ten students (32.25%) did not finish the calculus and, in total, twelve students (38.70%) did not manage to calculate the area[4].
- Three students (9.67%) seemed unable to integrate the function.
- And another three students (9.67%) integrated at intervals such as [0, 2] or [0, 4], perhaps misled by the drawing.
- Another three students (9.67%) wrote down the equation for the circumference or drew it, while one student considered its area as an integral.
- Four students (12.90%) did not answer the question at all.

***Item 4*:**

Given the form of the second integral, we believed students would easily work out that this was the volume that produced the surface determined by the first integral and that this volume would be infinite, as a consequence of the result obtained in the first integral.

As they needed to calculate two integrals, the types of answers were highly varied. The most striking results are as follows:

- Twenty-two students (70.96%) calculated the first integral right. However, fewer students (fourteen – 45.16%) calculated the second one right.
- A mere ten students (32.25%) express that one integral is the area of a function and the other the volume, and only three students (9.67%) clearly state that the area could be infinite and the volume finite.
- Five students (16.12%) were unable to solve the second integral, or attempted to solve it using incorrect methods.
- The two students (6.45%) who separated the integrals did so wrongly.
- Eleven students (35.48%) did not interpret anything; we include here those students who did not solve the integrals correctly.
- Two students (6.45%) concluded that there was no relation between both integrals. One of them had realised that one was the area and the other the volume of the same function.
- Only two students (6.45%) left this question unanswered.

It seems that even "simple" calculation of integrals causes the students problems. They also appear reticent when asked for an interpretation of the calculation carried out. This is mirrored in the fact that of the fourteen students who calculate both integrals, only ten express that one represents an area and the other a volume; and only three of these attempt to state clearly that it is a matter of a figure with an infinite area but which produces a solid of finite volume.

[3] In other words, they explicitly interpreted that the integral calculated was the area beneath the graph.

[4] We refer to students who in fact tackled the question. The students who left it unanswered are not included here. In total, sixteen students failed to attain the value π.

***Item 5*:**

In this case the original text of the question[5] was altered, as we were also interested in finding out how many students would think it was right.

This question can be worked on in two registers: in the algebraic, the function of the integrand can be said to have singularities at the interval of integration, so that the solution mechanism is faulty; and in the graphic register, it can be reasoned that the function is strictly positive at the interval of integration, so that the integral cannot be negative.

Our analysis of the answers is as follows:

- Twenty-two students (70.96%) clearly state that the function is problematic in the origin. But not all students realise that, if this is the case, the integral cannot be tackled as they did so. In fact, five of these (22.72% of the twenty-two students) believe that Barrow's rule has been well applied.
- Only sixteen students (51.61%) say that the integral should be split in two to calculate it correctly.
- Only twelve students (38.70%) are thought to have explained clearly how to solve the integral correctly.
- Three students (9.67%) correctly went about the solution of the integral, concluding that the area is infinite.
- A mere four students (12.90%) took note of the symmetry of the function. Of these, only two correctly solved the integral.
- Six students (19.35%) drew the function.
- Six students (19.35%) assigned absolute values to the result so that it was positive, as the integral "*is an area*".
- Three students (9.67%) believe that the trouble lies in the fact that it was necessary to add the integration constant.
- The answers of the eight students (25.80%) who stated that there was no error were considered contradictory, as they later looked for some mistake. In total, nine contradictory answers (29.03%) were found.
- Only one student (3.22%) left the question unanswered.

***Item 6*:**

Analysis of the answers gives the following results:

- Only ten students (32.25%) correctly went about the question and calculated the right result for the integral. Of these, only seven (70% of the group) interpreted the result (correctly or incorrectly).
- Five students (16.12%) failed to give any type of interpretation or explanation (three of them – 60% of the group – solved the integral correctly).
- Before calculating the integral, only one student (3.22%) stated that the figure does not have to enclose an infinite volume, although it increases when the figure is prolonged. And another student (3.22%) concluded that the volume would go on increasing, but did not specify whether it would reach a boundary or not.
- Six students (19.35%) concluded that the volume of the figure would be the same if we went on prolonging it.
- One student (3.22%) said that the figure did not enclose any volume because it did not cut the axis.

[5] Eisenberg & Dreyfus (1991)

- Fourteen students (45.16%) left the question unanswered. Perhaps it would be more reliable and clearer if the percentages given for this item were expressed as a function of the number of students who actually answered the question.

Item 7:

In this question a relationship between series and integrals is clearly shown; once more, students need to combine the graphic and algebraic registers to bring out all the richness of this relationship. Also, while in Item 4 both integrals are interpreted in space, in this case interpretation is made two-dimensionally.

The question was asked in order to check whether students would be able to transfer their knowledge of series to this new situation, and if they can do so, how do they do it.

The types of answers obtained are as follows:

- Only one student (3.22%) correctly interpreted each of the integrals, using a graph for the first and a similar graph for the second. This student was later interviewed.
- Three students (9.67%) calculated both integrals and concluded that the first diverged while the second did not.
- Another three students (9.67%) calculated the integrals but added no comments. In total, six students (19.35%) calculated them.
- Four students (12.90%) appear to have confused the behaviour of the function with that of the integral. For example, Student 8 calculated both integrals and wrote: "*As we can see when solving the integrals, the function* $f(x) = \frac{1}{x}$ *tends to* ∞ *at the interval* $[1, \infty)$ *and the function* $\frac{1}{x^2} = f(x)$ *tends to* 1 *at the interval* $[1, \infty)$".
- Twelve students (38.70%) left the question unanswered. Perhaps the percentages given for this item should only be given taking into accout the actual number of students who in fact worked on the question.

We can see how most students prefer to work in the algebraic register. When they are asked for graphic production, or asked to use a specific graph, few do so. Also, many students do not have a clear idea about the relationship expressed in this item between series and integrals.

Item 8:

This question was asked using only the graphic register. We intended to check whether students were able to enrich their knowwledge in this register by using what they already knew in the algebraic register (after solving the previous items).

Some of the results obtained were:

- Sixteen students (51.61%) say that it is false and put forward a counter-example or reasoned it out. One of the students (6.25% of the group) reasoned wrongly and one student says it is false, but writes nothing else. This student calculated both integrals in Item 4, but found no relationship between them.
- Nine students (29.03%) put as a counter-example Item 4; two students (6.45%) Item 7, and two students (6.45%) put forward a mistaken counter-example, the case of the function in Item 6.
- The nine students (29.03%) who said that text is true had not solved Item 4. There were also three students (9.67%) who calculated the integrals in Item 4, without any interpretation, but could not solve this item.

- Four of the students who said that it was true (19.90% of the total and 44.44% of the group) attributed the properties of the area to the volume; two (22.22% of the group) did not make any reasoning, and three (33.33% of the group) reasoned it out through integral properties. Furthermore, two of them (22.22% of the group) put forward an example to prove the property.
- Five students (16.12%) left the question unanswered.

It appears logical for students, in order to conceive of a figure with a finite volume, to think of this as closed and bounded. This obstacle seems to be strongly related to the lack of articulation between registers and will be the focus of further research work.

***Item 9*:**

This is the only question in which the numerical register was explicitly used. Once more, we attempted to check whether students would use what they knew in the algebraic register, as the area had been calculated under practically the same function in Item 2. We therefore used another scale for the graph so that the similarity was not so obvious.

Many of the answers were "brief", including little justification, and this might be because they had already calculated the area or because the numerical values given in the table might have been familiar.

Some data we can take from their answers are:

- Only two students (6.45%) clearly state that it is the same as Item 3.
- Only three students (9.67%) say that the total area is π. Curiously, only one of these students had solved Item 3, and one student carried out the calculation right, but had not solved Item 3.
- Thirteen students (41.93%) used a dynamic pattern and said that the area "tends", "nears" or "draws close to" π.
- Five students (16.12%) left the question unanswered; three of them (60%) had solved Item 3.

4. Implications for Interviews

Although from the very start, conducting interviews had been seen as useful in order to carry out this study, this idea became a necessity once the students completed the written tests and the answers obtained analysed. We are aware of the interest and richness brought about by a qualitative, rather than merely quantitative, analysis of this type of study.

Of the nine questions that made up the questionnaire, five were chosen for the interview, and a new one was added by means of which we intended to produce more transferences between knowledge of sequences and that of improper integrals, using the graphic register. Students were selected on the basis of the overall results to the questionnaire and/or on the basis of some answer, which we considered significant. A total of six students were interviewed, being almost a fifth (19.35%) of the original total.

In further studies, we will comment in detail on the results of the interviews and our quantitative anlysis will be complemented by a qualitative one.

5. Conclusions

After reading the answers to the thirty-one questionnaires, we can plainly see that students prefer to work in the algebraic register and that the use of non-routine questions, as well as those questions where they are asked to reason and justify their answers, disorientates them.

The main difficulties we detected were due to a lack of meaning or knowledge regarding previous concepts (such as convergence, sequences and definite integrals).

As noted (Item 1) we can see that, even as an area, students have no clear sense of the integral concept. Indeed, they appear to conceive of the definite integral ALWAYS as an area, so that they interpret this as the sum of the integral of the function in the parts where it is positive plus the absolute value of the integral of the function in the parts where it is negative.

Very few students have a clear idea of, or can explain, the conditions of having a finite interval and a function bounded in it. It seems they have learnt that the definite integral "is an area", although this is not made clear in their definitions. Also, many of the students make a merely algorithmic use of this concept.

Intuitive type questions lead to unclear answers (Item 2). Also, students attempt to focus on them formally (also Item 7). They also fail to interpret the results, although expressly asked to do so (Items 3, 4).

In general, we can see that students are not accustomed to combine several registers in order to interpret results (Item 4), or fail to use the graphic register when asked to do so (Item 7). Other difficulties which arise when solving the questions derive from a lack of coordination between both registers.

To answer our initial questions, we can state that students are not accustomed to non-routine type questions and that this type of question can disconcert them. Also, in spite of a lack of formal tools, evident among many students, they prefer to work in the algebraic register (or are limited to this), in spite of their sometimes being asked to use or produce a graph. Therefore, it appears that, generally speaking, they are unable to articulate information between these two registers. The tendency to restrict themselves to the algebraic register impedes graphic interpretation of many results.

Finally, with regard to transferences of previous knowledge, we can see that in general, no such transference takes place (for example, in Item 7, in the transferences between the series object and the improper integral object). Moreover, we have discovered that in many cases this previous knowledge is not altogether complete, thus limiting transference.

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APPENDIX

Item 1:

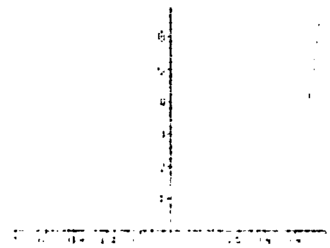
How would you explain to a classmate the meaning of $\int_a^b f(x).dx$?

Item 2:

The following graph represents the function

$y=\sqrt{\frac{1+x}{1-x}}$.

What can you say a priori about the value of $\int_0^1 \sqrt{\frac{1+x}{1-x}}dx$?

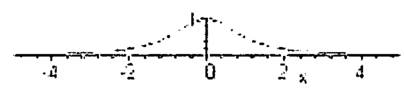


Item 3:

We know the circumference with radius 1 encloses an area of π squared units.

The graph we show you next is the graph of the function $y=\frac{1}{1+x^2}$. Check that the area this curve encloses with the OX axis is π too.

Interpret the result.



Item 4:

Calculate the value of the following integrals: $\int_2^\infty \frac{1}{x-1}dx \qquad \pi.\int_2^\infty \frac{1}{(x-1)^2}dx$

Interpret geometrically the results obtained. Is there any relation between both integrals?

Item 5:

Can you find any mistake in the following reasoning?

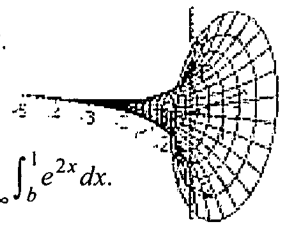
$$\int_{-1}^{1}\frac{dx}{x^2}=\int_{-1}^{1}x^{-2}dx=\left.\frac{x^{-1}}{-1}\right|_{-1}^{1}=\left.\frac{-1}{x}\right|_{-1}^{1}=\frac{-1}{1}-\frac{-1}{-1}=-2$$

Item 6:

The following figure shows the result of rotating the curve $y=e^x$ around the OX axis.

If we continued prolonging the picture towards the **left** (towards - ∞), which volume do you think it would enclose?

To check your intuition, we will calculate it. Each of the circular sections has a radius of e^x. Therefore, the area of each section is



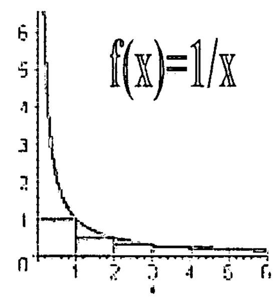
$$A(x) = \pi.(\text{radio})^2 = \pi.\ (e^x)^2 = \pi.\ e^{2x}.$$

If we sum all the areas we will obtain:

$$V(x) = \int_{-\infty}^{1} A(x)dx = \int_{-\infty}^{1} \pi.e^{2x}dx = \pi. \lim_{b \to -\infty} \int_{b}^{1} e^{2x}dx.$$

What happens? Interpret the result.

Item 7:



We know that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

In view of this results, what can you say about the value of

$\int_1^{\infty} \frac{1}{x}dx$ and $\int_1^{\infty} \frac{1}{x^2}dx$?

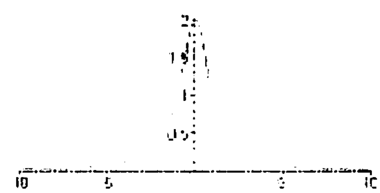
Use the graph provided.

Item 8:

Is the following reasoning true or false? Why?

" If a region has an infinite area, then the solid formed when rotating that region around one of the axis has an infinite volume".

Item 9:



The following graph represents the function $y = \frac{2}{1+x^2}$.

We provide the table with the value of the integrals

$\int_0^n f(x)dx$ for different values of n. What do you think it will happen if we continue increasing n?

N	100	200	300	400	500	600	700	800	900	1000
$\int_0^n f(x)dx$	3.121593320	3.131592736	3.134926012	3.136592664	3.137592658	3.138259324	3.138735512	3.139092654	3.139370432	3.139592654

ANIMATIONS TO ILLUSTRATE ILL-CONDITIONING AND AN INTRODUCTION TO MATRICES USING MAPLE

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ABSTRACT

Computer packages, usually Maple, are used in most of our undergraduate courses but less so in the lower level courses. Students of geospatial sciences (surveying) undertake the equivalent of first year engineering mathematics in a year and a half. In the fourth semester they undertake the course MA067 designed specifically for them consisting of two thirds numerical linear algebra and one third introduction to complex variables and conformal mapping.

In the third year the surveying students take a course "Geometry of Surfaces" which is a classical differential geometry course with one lecture and one lab session (using Maple) per week. All calculation and assessment is done using Maple in the computer lab. This course has run successfully for a couple of years. However the students find that it is difficult at the start of the course with the necessity of learning differential geometry and Maple concurrently.

In MA067, some Maple was introduced. Only one hour per week was available for the classwork practice sessions and the Maple lab sessions. This resulted in the availability of five sessions for Maple. These were devoted to a general introduction to Maple and then work with entering matrices (using the Matrix data type in Maple 6) and solving linear systems of equations (matrix equations). Students used Maple to solve exercises that they had solved the week before "by hand" in the practice class.

To add interest, following an expository lecture, a Maple presentation was given to reinforce the understanding of ill-conditioning. Maple animations illustrated ill-conditioning and this was contrasted with an animation of a well conditioned system. These animations were strikingly effective and appreciated by the students. Besides anecdotal feedback, we report on a feedback survey designed to investigate student attitude to the Maple component of the course.

1. Introduction

The Computer Algebra System (CAS) packages Maple and Mathematica have been used in teaching and research in our department for about a decade. Due to the success (from the perspective of both staff and students) of our CAS laboratory sessions, we have progressively increased the usage of these in our teaching and are fundamentally changing the way we teach. Since 1998, our mathematics program has included computer laboratory sessions (usually using with Maple) for almost every course. Teaching students of other departments, particularly from engineering, geospatial sciences and computer science, is a core activity of our mathematics department. In consultation with these departments, we have also increased the use of various software packages as appropriate to the course. Where the demand is for specialist packages (such as for Matlab toolboxes or commercial software Finite Element Method packages ...) they are used, otherwise Maple is incorporated in the course. This usage of Maple varies a lot from a little "tutorial" support, to major assignment work, to Web delivered material, to "immersion" with 100% of the assessment in an examination in the computer laboratory.

This paper discusses the introduction of Maple (in "support" mode) in the course MA067 for the surveying students in the semester before they undertake the Maple "immersion"course: Geometry of Surfaces. Geometery of Surfaces is a third year classical differential geometry course with one lecture and one lab session (using Maple) per week. All calculation and assessment is done using Maple in the computer lab. This course was initially developed by one of the authors using Mathematica and ran successfully in 1998 (Blyth 1998). With a site license for Maple, this course was rewritten using Maple and run from 1999 onwards. By the end of the course, students are quite positive about this approach. However the students find that it is difficult at the start of the course with the necessity of learning differential geometry and Maple concurrently.

Since MA067 is a prerequisite for the Geometry of Surfaces course, and a review of this course was required in 2001, we took the opportunity to provide an introduction to Maple in MA067. With an allocation of 3 contact hours per week for one semester, two hours were used for the classes (the lectures – with less than 50 students) and the third hour alternated with a tutorial / classwork exercise followed by a Maple laboratory session on (essentially) the same topic. We also presented Maple animations in a lecture demonstration to illustrate ill-conditioning (of matrix equations). The examination is a traditional one (with no Maple) and contributes 80% to the assessment.

Introducing students to Maple was an objective, but it was important to do so in a way that did not negatively affect their attitudes with respect to using Maple. This is clear since the Geometry of Surfaces course follows MA067, but a further incentive for us is that Geometry of Surfaces has a slightly insecure status (sometimes compulsory and sometimes an elective - which the students have been strongly encouraged by their home department to select). We designed and administered a comprehensive questionnaire to investigate the attitude of the students to the use of Maple. This survey provided evidence that most students had a positive to neutral attitude to using Maple.

This MA067 Maple Questionnaire was a modification of the Maple Questionnaire designed and used for the course MA910 that is offered in the first semester of first year. MA910 is primarily the first semester of a traditional calculus course taken by science students. It is closely related to the course taken by the engineering students, but one of the topics studied by the engineering students at the end of the semester has been replaced by a Maple topic. Because of the module structure used in this and related courses, the Maple module is dealt with in one block near the end of the semester (in contrast to the Maple work being distributed throughout the semester as in

MA067). The students enrolled in MA910 were in two groups, the first of which was studying a multi-major degree in science and will probably not take a major in mathematics. This student group had not already used Maple, so they were provided with an introductory program. The second MA910 group consisted of students studying our specialist mathematics degree or our Dual Award of our mathematics degree and a diploma in information technology. This second group of students took an additional mathematics subject and had been using Maple in two one-hour laboratory sessions per week. The more advanced work done with this second group of students, already experienced with using Maple, is described in Blyth and Naim 2001. A discussion of the Maple survey results for both of these different groups is provided in Saunders and Blyth 2001.

2. Maple and MA067

The course MA067 replaced a previous course: the content was the same but the number of contact hours increased from 2 to 3 hours per week. We used two lectures weekly to cover the same content as previously and introduced a new weekly "tutorial". Usually a topic was discussed in the lecture and then followed by a practice class (where students completed some problems by hand on a worksheet which was handed in for marking) and repeated a similar Maple worksheet (also handed in) the following week.

An outline of these weekly tasks follows:

Week 3.	Maple 1.	Introduction to Maple
Week 4.	Tutorial 1.	Error / relative errors / error bound & loss of significance
Week 5.	Maple 2.	Matrices and solving Systems of Equations
Week 6.	Tutorial 2.	Direct & Iterative Methods
Week 7.	Maple 3.	LU Decomposition
Week 8.*	Surveying Camp	
Week 9.*	Tutorial 3.	Residual Correction Method
Week 10.	Maple 4.	Residual Correction
Week 11.	Tutorial 4.	Interpolation and estimating derivatives
Week 12.	Maple 5.	Intro to Complex Analysis (Not marked)
Week 13.	Tutorial 5.	Complex problems and past exam

The Maple worksheets were handed out as hard copies to students in the lab. Students were required to enter all code themselves and were assisted whenever they required help. Assessment consisted of 10% for the "by hand" work, 10% for the Maple assignments and 80% for the examination (in which Maple was not available). With just as few sessions available for using Maple, the emphasis was placed on the numerical linear algebra (and an introduction to complex analysis) rather than having a (trivial) introduction to lots of topics.

Maple's new linear algebra package (LinearAlgebra) contains commands for matrix and vector manipulation. This was used, not the original (and still provided) Maple package, linalg. This led to some improvements, particularly with respect to display and manipulation of the matrices and vectors. However for the worksheet on the residual correction method, the default use of hardware floating point arithmetic made it difficult to change the precision for the calculation of the residual. Students were led through these issues with their Maple worksheet in the lab and asked to do the following Maple assignment:

The ill-conditioning assignment

Consider the 2 equations in 2 unknowns:

$$\frac{x}{1001} + \frac{y}{1003} = 0.2002$$

$$\frac{x}{1003} + \frac{y}{1005} = 0.1998 .$$

This can be written as the matrix equation A x = b where

$$A = \begin{bmatrix} \frac{1}{1001} & \frac{1}{1003} \\ \frac{1}{1003} & \frac{1}{1005} \end{bmatrix}, \quad b = \begin{bmatrix} .2002 \\ .1998 \end{bmatrix} \quad \textbf{and} \quad x = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Given an approximate solution $y = \begin{bmatrix} -100 \\ 310 \end{bmatrix}$, calculate the residual correction **y1** and two further iterations of residual corrections (giving **y2** and **y3**).

An additional Maple worksheet on vector and matrix norms was developed but not used due to time constraints. The students are not available during their surveying camp, but in 2002 we will start the Maple labs earlier and we will be able to include the norms worksheet.

3. The Maple animation illustrating ill-conditioning

Before the residual correction method was covered, an introduction to ill-conditioning was given in the lectures and followed by a Maple presentation in the next lecture. This used Maple animations to illustrated ill-conditioning and was contrasted with an animation of a well conditioned system. These animations were strikingly effective and appreciated by the students according to the anecdotal feedback specifically on this session.

The model problem considered was the solution of the 2 equations in 2 unknowns:

$$\frac{x}{2} + \frac{y}{3} = 1$$

$$\frac{x}{3} + \frac{y}{4} = 1 .$$

This can be written as the matrix equation A **x** = **b** where both off diagonal elements are 1/3. Finding the algebraic solution has the graphical representation of finding the point of intersection of the two lines. Ill-conditioning is exhibited by small errors in the off diagonal elements leading to larger errors in the solution: this was explained in terms of finding the intersection point of nearly parallel lines.

An animation to illustrate the ill-conditioning was run with the error in the off diagonal elements changing from 0% to 10% in steps of 1%. The changing lines were shown along with a blue dot at the solution for the exact case. Two of these frames are given in Figure 1.

Graphically, ill conditioning corresponds to the lines nearly parallel. So the best behaviour should be when the lines are nearly perpendicular. If the lines (the equations) are

a x + b y = k1

c x + d y = k2

the slopes of the lines are - a/b and - c/d. The lines will be perpendicular if c/d = - b/a (WHY?).

We construct an example: if we keep a=1/2, b=1/3, then c/d = - 2/3. We also keep c=b=1/3, then d= -1/2. We also choose the right hand side constants so that the solution point remains the same, to obtain A x = b where

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{2} \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$$

These animations were strikingly effective and appreciated by the students (as reported by them at the time).

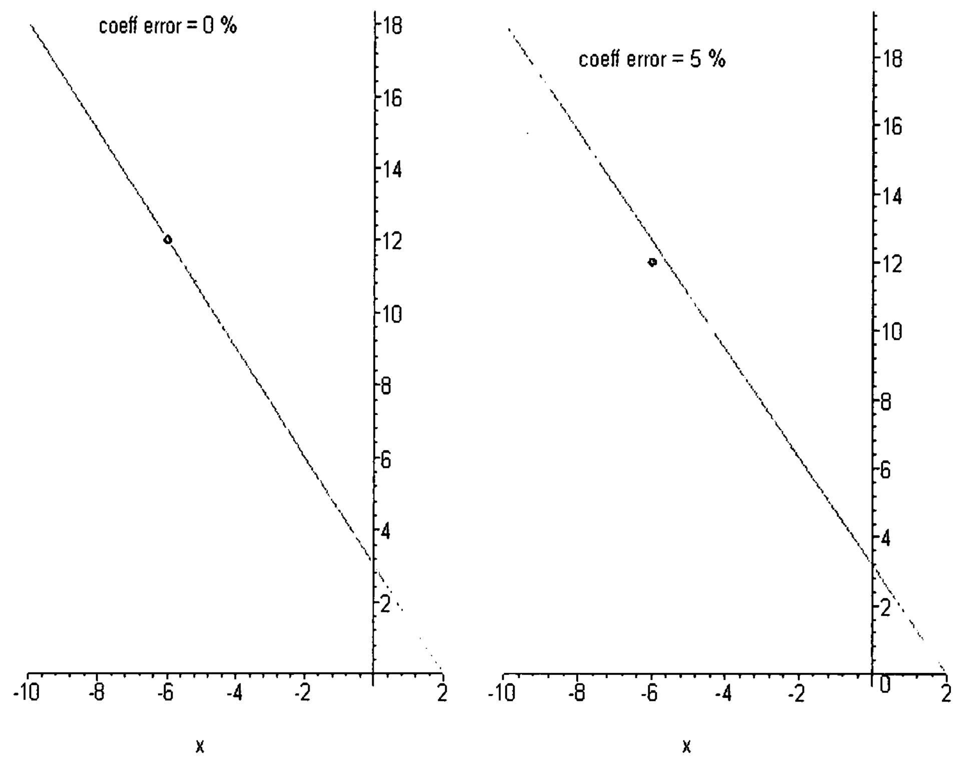

Figure 1: Two of the frames from the ill-conditioning animation.

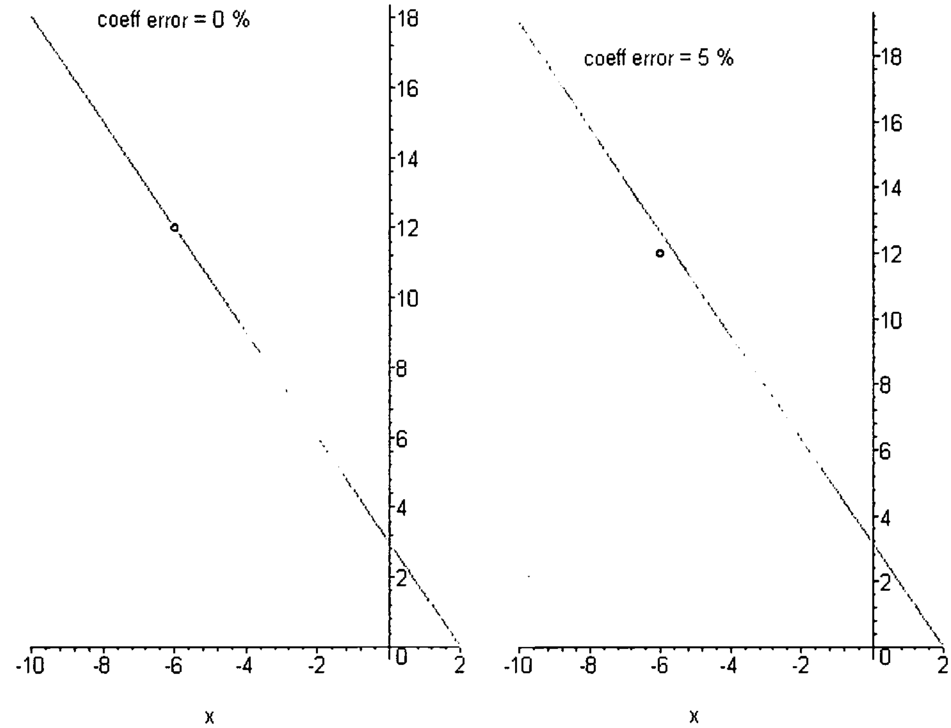

Figure 2: Two of the frames from the well-conditioned animation.

4. The Maple questionnaire

We designed a comprehensive questionnaire to investigate the attitude of the students to the use of Maple. For a report on related work, see Galbraith et al. 2001. Our survey was administered in the last teaching week and a copy of the questions is given in the Appendix. The questionnaire items were presented on a Likert scale with 5 points from strongly disagree to strongly agree with neutral in the middle.

Factor analysis of the responses led to grouping into the following five scales (where the responses are indicated as negative, neutral, positive):

The impact of the Maple component on motivation and interest (0, 18, 4)

The impact of the Maple sessions on mathematical understanding (3, 17, 2)

The level of appreciation of Maple's usefulness as a tool (2, 11, 9)

The ability to use Maple as a problem-solving tool (6, 10, 6)

The appreciation of the features of Maple (3, 13, 6)

It is clear that most responses were neutral or positive – so we didn't do much damage! Our objective was to introduce Maple without disaffecting students' attitude to the use of Maple (since the following course on Geometry of Surfaces uses Maple comprehensively in teaching and assessment). In this modest aim, we were successful, but we would have liked to see more positive responses. It appeared that the very positive anecdotal feedback after the animation demonstration in the middle of the semester was a dim memory by the time of the survey (at the end of the semester).

Reponses concerning aspects of teaching methods and delivery indicated that the students liked to receive the worksheets in hardcopy form and to be able to sit down (in the lab) and get on with the worksheets straight away. The students did not want Maple to be included in the exam but they did want the Maple component to be worth more of the total mark for the subject. There was some indication that the students would have liked more help in the Maple sessions (although the worksheets were not seen as being too difficult). The students did not find working in the lab difficult but there was a mixed response to the level of distraction there.

5. Conclusions

The introduction of Maple (in "support" mode) in the course MA067 for the surveying students in the semester before they undertake the Maple "immersion" course: Geometry of Surfaces achieved its objectives. The questionnaire provided useful feedback, which will lead to some minor changes such as some more tutorial help in the lab and increased marks for the Maple work. We will also start the "tutorials" earlier, which will allow for the inclusion of an extra Maple worksheet on vector and matrix norms. We will pursue further development of both the Maple work and mechanisms for student feedback.

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APPENDIX

The Maple Questionaire (the first page cover sheet and the boxes for the responses are omitted)

<u>Section B – Response to Maple</u>

Please place a tick against each statement according to the scale strongly disagree (SD) up to strongly agree (SA). Several items are used to gain an indication of your response to similar aspects of the Maple sessions so some items may seem to be repetitive. There are an equal number of positively and negatively phrased items.

1. I am glad we used computers as part of the course
3. Seeing how Maple solved problems helped me to understand methods better
4. I think that software like Maple would rarely be used in industry
5. Maple is too slow
6. Maple should be included in the exam
7. I would find it difficult to use Maple to solve a random set of problems
8. Maple was just one more thing to learn
9. Maple is not particularly useful as a Mathematics tool
10. The individual Maple sessions need to be longer
11. There were too many distractions in the lab
12. The Maple help files are good

13. It takes more than one semester to feel competent using Maple as a tool
14. I think Maple would be useful for Maths research
15. It was easy to save my work
16. The worksheets were too difficult
17. The graphs produced in the Maple sessions were helpful to my understanding
18. I would rather do the Maple classes in one block
19. I found it quite easy to edit commands to solve a new problem
20. I think I understand the topics we covered with Maple better than other topics
21. The computers were unreliable
22. I liked seeing how software could be used to carry out computations
23. I found the Maple sessions uninteresting
24. Maple is useful for doing long or complex computations
25. I was unclear what commands to use to get Maple to do what I want to do
26. I found that I was generally confused when using Maple
27. The Maple help files and examples were all we needed to learn the commands
28. Calculations done on paper are easier to check
29. I like being able to sit down and get on with the worksheets straight away
30. I understand better when I do the whole solution by hand
31. I would have liked more help in the Maple sessions
32. The Maple component should be worth more of the total mark
33. The Maple edit features are useful
34. I think that spending the time practising questions would be more useful
35. I could access the Maple work quite easily
36. Maple is useful as a checking tool
37. I like to see the Maple worksheets in printed form
38. I find working in a lab uncomfortable
39. I find Maple's language difficult to use
40. I can enter a method using Maple commands quite easily
41. I would have liked to have done more Maple work on other topic areas
42. The Maple graphics I have seen are good
43. I would rather have continued with normal lectures
44. I know what Maple needs to do to solve the problems

Section C Overall, how did you find the experience of using Maple this semester?

Thankyou for your time.

STUDENTS' ASSUMPTIONS DURING PROBLEM SOLVING

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ABSTRACT

This paper analyses the solving process of two undergraduate students on a non-routine mathematical problem. By comparing these students' work, it can be observed that their processes suggest different approaches in relation to the way a solution was sought. One student followed a process in which his principal activities were centred on *discovering* those key ideas that would allow him to tackle the problem. The other seemed to be more focused on *inventing* a way of dealing with the situation and building a solution to the problem. Since in order to invent a solution some useful facts have to be first stated or discovered, it may be speculated that a process which aim is to *invent* is more flexible than a process which aim is to *discover*.

Key Words: Mathematical Problem Solving, Approaches

1. Introduction

By analysing students' problem solving activities and the (externalised) reasons behind their courses of action it may be possible to gain some insight into their assumptions about the nature of the solutions they are trying to achieve. It may be speculated that a student whose aim is to *discover* a solution believes that the solution is something that already exists and that his or her duty is to uncover it. On the other hand, a student whose aim is to *invent* a solution either does not believe that a solution is *out there* or believes that if this is the case s/he still can create her/his own solution. The objective of this paper is to discuss these different approaches and their implications to problem solving.

2. Approaches to Mathematics and Problem Solving

In relation to teaching mathematics, what is meant by "mathematics" (i.e., the view held towards mathematics) affects the way in which mathematical problems are presented and the way in which problem solving is conducted (Shoenfeld, 1992; Goldin, 1998). For instance, the definition-theorem-approach to mathematics is a paradigm that has affected mathematics education by focusing attention to the logico-deductive activities carried out by the student (Davis and Hersh, 1986). Furthermore, studies related to what students believe is expected from them when doing mathematics suggest that they hold different views of what "doing mathematics" is about, and that this, in turn, affects their achievement (e.g., Alcock and Simpson, 2001; Hazzan, 2001).

In the case of problem solving, it may be speculated that a solvers' idea about the nature of mathematics may affect the way in which a solution is sought. For example, if a student holds a Platonist view of mathematics (see Hersh, 1997), his or her approach may suggest an attempt to discover those key entities required to solve the problem. On the other hand, if a student holds a view of mathematics as a human construction, his or her approach may be better defined by an attempt to "build" a solution.

The purpose of this paper is *not* to show that the solving processes analysed here fall into a "Platonist" or into a "Constructivist" approach. The scope is more limited in the sense that the aim is to discuss two observed approaches in relation to solving a mathematical, non-routine problem. These approaches suggest different assumptions about the nature of the solution that is expected to be found, and this will also be discussed.

3. Methodology

The written work analysed here belongs to two students from a group that took part in a ten-week, problem-solving course. The course participants were all doing undergraduate degrees in maths, physics or computer sciences. The course was structured with the objective of introducing students to vocabulary and concepts that could help them reflect on their own solving processes and share their experiences (based on Mason, Burton and Stacey, 1985).

During the course, students were required to solve problems and encouraged to develop a rubric for recording their ideas and experiences. As a final assignment, they had to choose one between two problems and were required to submit a script of their process. The solving processes discussed here correspond to this final assignment.

Students' solving processes (corresponding to the final assignment) were coded and the activities identified suggested two categories. On one side, some students' processes suggested that their objective was to *discover* a solution to the problem. Alternatively, other processes suggested that the main objective was to *create* a solution to the problem. An analysis of Martin and Kyle's solving processes is shown here in order to discuss these two categories identified. (Real names have been changed to ensure anonymity.)

The problem. The problem that the students had to solve was stated in the following way:

> These rectangles [see Figure 1] are made from 'dominoes' (2 by 1 rectangles). Each of the large rectangles has a "fault line" (a straight line joining opposite sides). What fault-free rectangles can be made?

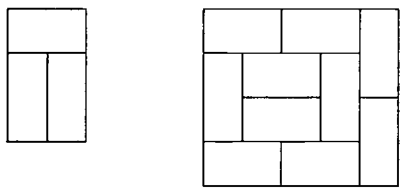


Figure 1

The problem was open in the sense that it allowed students to decide whether they wanted to approach it by assuming that "no fault-free rectangles" could be made, or the opposite. Another characteristic of this problem was its geometric nature and thus the fact that students worked (as expected) with geometric representations. In general, Fault-Free rectangles (FFR henceforth) *can* be built if their dimensions are 5 by 6 or larger, and not equal to 6 by 6.

4. General view of Martin and Kyle's solution process

In general terms, Martin's process may be described by the exploration of three main ideas. Firstly, he analysed the idea of "blocking and extending" in order to explain *why* faults appeared and how they could be blocked in order to build FFRs. Secondly, he looked at the possible "building blocks" of rectangles made with 2 x 1 dominoes. In relation to this, first he tried to justify that, since the basic structures that compose rectangles made with dominoes are inevitably faulty, FFRs cannot be built. However, he later found a FFR and decided to use the argument of the basic structures to describe the composition of FFRs. Finally, Martin's third idea was that of building a set of FFRs made of 3 x 2, 4 x 2 and 1 x1 "dominoes". From the latter approach, he expected to build a fault-free structure which "dominoes" could then be easily split and transformed into 2 x 1 dominoes.

In Kyle's case, his process is better described by three stages rather than by three basic ideas. Kyle's first stage appeared to consist of a process of systematic specialisation aimed at building a FFR; the product of this stage was, actually, a FFR. The second stage can be defined as the analysis of the newly-built FFR and the development of a way of extending it by 2 units in either direction (horizontally or vertically) and keeping it faultless. Since the FFR found by Kyle had dimensions 6 x 5, he showed that, by his method of extension, he could build any FFR with dimensions (6+2n) x (5+2n) with n = 1, 2, 3, ... Finally, during the third stage of his process, Kyle aimed at answering the question of whether FFR with even by even dimensions could be made at all. In order to do this, he tried to build an even by even FFR by combining his initial systematic specialisation with the new idea of systematically increasing the dimensions of a rectangle. The result of this last approach is shown in the next section.

It may be said that Martin's general approach was to look for potential key ideas for tackling the problem and then to verify whether these ideas were useful or not. On the other side, Kyle's key ideas seemed to have emerged from a process of experimenting with particular aspects of the problem and trying to make use of the results of this analysis. In Kyle's case, the results from one stage usually constituted the key ideas for the next stage. As for Martin, the different ideas explored were more or less independent.

5. Martin and Kyle's different approaches

Martin's solution process

As said before, Martin's solving process seemed to be guided by the exploration of three different ideas. The first idea was that of finding a way of "blocking [faults] and extending [the size of the rectangle]" in order to build FFRs. It is interesting to note that, in developing this concept, Martin seemed to be trying to develop a "systematic approach" that would allow him to control the situation and solve the problem. The following passage is from Martin's written description of his solving process:

> AHA! Will try a systematic approach of 'blocking' faults.
>
> INTRODUCE *concept of block*. Given a rectangle with [a] fault line a block is a single tile added to stop the fault. E.g.,

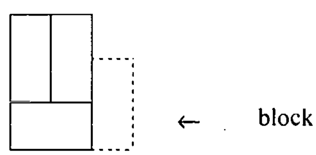


Figure 2

> INTRODUCE *concept of extension*. Once a block is made, the shape is extended to create a new rectangle by adding tiles, e.g.,

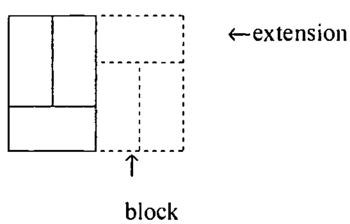


Figure 3

Conjecture 1. Method of blocking and extending will produce a fault-free rectangle.

Martin devised and presented his method of "blocking and extending" as one, which would allow him to systematically build a FFR. It may be said that he based this method on two simple ideas: (1) the idea that faults must be blocked and (2) the idea that this leads to the need of increasing the size of the rectangle. However, the fact that he was not being able to build FFRs using this method, suggested him that they could not be built at all. As Martin put it: "repeated failure put idea of 'no solution' in my head". In this way, he decided to modify his method and, later, to abandon it.

Martin's second approach was to look for the basic building blocks of rectangles made with dominoes. He believed, at a point, that it was not possible to build FFR from dominoes and aimed at proving this by showing that all rectangles contain sub-blocks that are faulty. For doing this, he first conjectured that all large rectangles contain either two vertical dominoes arranged side by side, or two vertical dominoes arranged one on top of the other, or both. In his solution process, he wrote the following:

AHA! Must test all ways of sticking

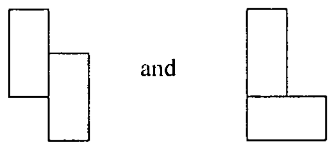


Figure 4

together (to each other and self). We must prove all ways of doing so necessarily imply.

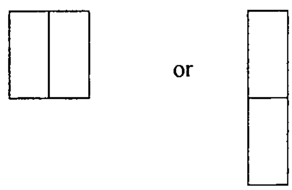


Figure 5

to complete rectangle. E.g.,

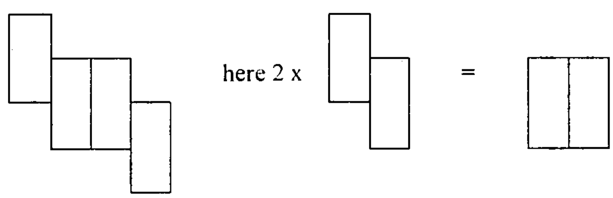


Figure 6

In this second approach, Martin tried to justify that every rectangle made up with dominoes would contain a basic (faulty) combination, and, therefore, that FFR could not be built. He found a "convincing" argument to the first part of this conjecture but, "with much disappointment and frustration", ended up producing a FFR. So he found himself in the position of having to modify his approach once more and to look for new ideas. The following figure shows Martin's FFR:

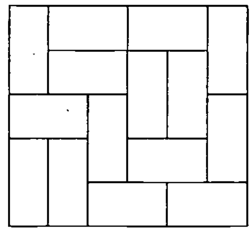
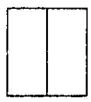


Figure 7

However, Martin did not abandon the idea of basic building blocks completely. Instead, he modified his approach by writing a new conjecture:

Conjecture 4 Faultless rectangles with



must have

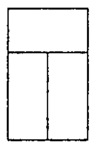


Figure 8

Conjecture 4 was modified several times, but the idea of distilling the basic blocks contained in rectangles and FFRs remained. Martin provided an argument to show that it is inevitable to use these basic blocks and concluded that the 3-domino shape shown in Conjecture 4 is "necessary for a faultless rectangle".

So in his first and second approaches, it can be seen how Martin seemed to be looking for ideas that could be useful for dealing with the problem. The idea of building FFRs systematically through a simple but "justified" method (as in his first approach) did not prove to be very effective. Then, the idea of distilling the basic building blocks in rectangles made with dominoes (and later in FFR) did not seem to provide much information as to "what fault-free rectangles can be made". Nonetheless, at this point, he had already accumulated two pieces of information: (1) that it is possible to build FFRs using dominoes and (2) that these FFRs contain 2 and a 3-domino, basic structures.

The third of the basic ideas explored by Martin was the following:

> AHA! Relax original question, allow any size and ratio for rectangles to create a fault free rectangle (with view to dividing up larger rectangles to 2 x1). Specialise randomly:

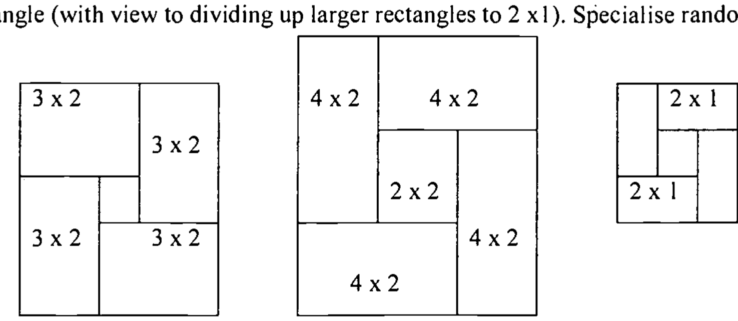


Figure 9

The idea behind this approach was that once a FFR rectangle was built with these 3 x2, 4 x 2, etc. rectangles, then these so-called "larger rectangles" could be split and transformed into 2 x 1 rectangles (dominoes). This did not turn out to be true in practice. However, by comparing these new FFR (made up of larger "dominoes") to his previously built FFR, he found that they had some basic structures in common. The result was that this took him back to the 3-domino, basic structure mentioned above. Namely,

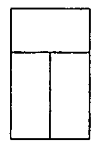


Figure 10

The reason why Martin's aim can be described as that of discovering lies in his overall approach. Martin's approach suggests that he was looking for a key idea (or ideas) in order to solve the problem. By comparing Martin's solving process to Kyle's (discussed in detail below), it may be said that the latter followed an approach that is better described by the way ideas were developed and transformed.

Kyle's solution process

Kyle's first approach was to "specialise systematically". At the beginning of his process he wrote: "[I] don't know the nub of the question yet! Specialise to understand what the question really wants first. Specialise systematically". In this way, he began by trying to build FFRs with 2, 3, 4, 5 and 6 dominoes. But he was not able to build any FFR and thus declared himself "STUCK! Not sure if it can be done". At this point, he decided to try to transform his last (faulty) rectangle into a FFR by adding dominoes to it (i.e., without restrictions on the number of dominoes used). The result was encouraging as he was able to build a FFR with dimensions 5 by 6.

A second important stage in Kyle's process was to analyse the way in which he had produced his first FFR and to devise a method for increasing its dimensions. As a result from the analysis, he distilled a list of important steps:

- Started off with basic 6-domino shape
- Eliminated horizontal fault by adding [blocks] "1", "2", "3" [see figure below]
- Added "4" to counteract vertical fault
- Swapped "5", "6" to vertical to counteract horizontal fault
- Built around to make it complete
-

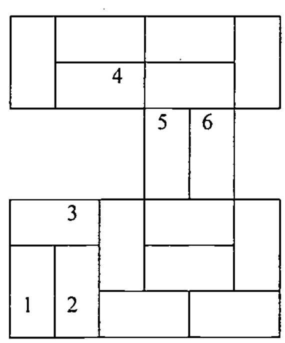


Figure 11

It may be said that it was the fourth step (swapping 2 horizontal dominoes for two vertical ones) the one that formed the basis for Kyle's method of extending his 6 x 5 FFR by two units in either direction, i.e., horizontally or vertically. This method consisted of taking a FFR and selecting a pair of, say, horizontal dominoes (if the size is to be increased vertically) placed one of top of the other. Then, all the dominoes on the level of the top domino or above (from the pair selected) had to be removed. The next step was to swap the selected pair of dominoes by a pair vertical ones and to add a full row of vertical dominoes to the top. Finally, the structure had to be "capped" again in order to return it to a rectangular shape. This would then produce a FFR larger than the previous one by at least two units (as layers of vertical rectangles—which increase the height by 2—can be added indefinitely). In order to justify why this method worked, Kyle added that:

> This new shape creates no horizontal faults and any vertical faults will be irrelevant because the existing rectangle (the "bottom part") cancels these out.

Using this method, Kyle showed that his FFR with dimensions 6 x 5 could be used to construct any N x M FFR with N = 6, 8, 10, ... and M = 5, 7, 9, ... As said above, Kyle based this method on the idea that, once a FFR is constructed, its dimensions could be increased systematically by "opening" the figure, inserting dominoes as required and, finally, "recapping" the figure in order to return it to a rectangular shape. The only condition was to make sure that, after the splitting, faults were "immediately taken care of".

The last stage in Kyle's solution was his sub-process of trying to decide whether N x M rectangles of even dimensions could be built. So far, he had only been able to construct FFRs with even by odd dimensions. Furthermore, his method for increasing their size only allowed him to add 2n (n = 1, 2, 3, ...) units (vertically, horizontally or to both dimensions) to already-built FFRs. So he suspected that even by even rectangles could not be made. For verifying this, he decided to begin by using the same strategy he had used before and that yielded his first even by odd FFR. I.e., he began with a faulty rectangle and added dominoes to it hoping that this would eventually lead to a FFR. Also, in order to increase the possibilities of this FFR having even by even dimensions, he decided to begin with 4 dominoes instead of 3. In Kyle's words:

> TRY... and find an n x m rectangle which is fault free [n, m even].
>
> Earlier method: start with a basic rectangle and extend it. Previously started with a 3-domino rectangle so start with a 4-domino, this in the hope of getting n, m even.

After experimenting with this approach and not being able to construct any FFR, Kyle concluded that:

> Staring with 2 dominoes together, it is impossible to cancel 1 fault at a time without producing an even by odd rectangle each time infinitely or until you find a fault-free rectangle.

But, apparently, he was not convinced of his own arguments and thus continued trying to build an even by even FFR by experimenting and, at the same time, trying to use the information he had already accumulated on the problem. He was not successful in this attempt and closed his solving process by providing an unclear argument to the following conjecture:

> Starting with an even by odd dimensioned rectangle, and performing a series of iterations, it is impossible to get an even by odd rectangle as a result.

As said before Kyle followed a solving process that can be described in terms of the strategies he developed for building and extending FFRs. Also, each strategy was built on the result of either previous strategies or previous sub-processes (e.g., the idea of building even by even FFRs using the concept of extension defined in a previous strategy). Comparing Kyle's solution process to Martin's, it may be said that Kyle's aim is better described as that of using emergent ideas to *build* a solution. Martin proceeded by developing key ideas and then testing them (hoping, maybe, to find one on which to base his solution); Kyle on the other side, noted useful ideas as he worked with particular aspects of the problem and then transformed these ideas into a strategy.

6. Conclusion

The analysis of Martin and Kyle's solving processes shows that, for dealing with the same problem, these students followed different routes. The nature of the differences between their processes suggests that their assumptions about the solution they were attempting to find may be different. Martin's approach could be described—from a researcher's point of view—as that of assuming that ideas are "out there" while Kyle's approach as that of creating solutions.

For Hersh (1997) a mathematician assumes the role of a Platonist when he works as if he believed that mathematical entities cannot be created and that they "exist whether we know them or not" (p. 73). On the other side, when a mathematician works as if mathematics is not discovered but created, s/he is, according to the author, working as a formalist or an intuitionist. But, in spite of this apparently clear-cut distinction, the author suggests that mathematicians may actually adopt these two roles at different times:

> When several mathematicians solve a well-stated problem, their answers are identical. They all *discover* that answer. But when they create theories to fulfill some need, their theories aren't identical. They *create* different theories. (Hersh, 1997, p. 74)

In the case of the students, trying to give an account of the assumptions held during problem solving is an activity that can help us gain insight into their understanding about mathematics. This can be done, as Hersh indirectly suggests, by looking at what students do (and say, and write) during problem solving. In terms of validity, qualitative research methods such as grounded theory (see Glaser and Strauss, 1967) provide means for producing valid results. In this respect, even though an account of a student's assumptions will inevitably be a researcher's construction, it ca also be a scientifically justified one.

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UNDERGRADUATE STUDENTS' PROJECTS WITH SPECIAL NEEDS PUPILS[1]

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ABSTRACT

The paper considers particular projects undertaken by undergraduate students' during the course of didactics of mathematics, which is part of the five year full time study for future teachers of special needs pupils. The projects involved the students doing experiments with special needs pupils in their schools. Following the work in school the students had to describe their experiments, then analyse them together with any work produced by the pupils to determine the pupil's abilities and/or thinking processes during the solving a mathematical problem. The student's own analysis of their experimental work with the pupils has subsequently been analysed by the tutor, who concentrated on the following aspects: how did the student work with the pupils?; how confident is the student in evaluating/analysing his/her pupils` work?. How can the student's own attitude towards mathematics and the understanding of mathematics be changed by having to prepare and present experiments, then work with pupils doing mathematics. Does evaluating/analysing his/her pupils' work also help this change? One project is considered from these points of view.

Keywords: student-teachers, student's project, pre-service of special needs pupils, experimental work, unconventional classroom mathematics, student's analysis of pupil's work, analysis of student's project

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1. Introduction

The projects of twenty-eight undergraduate students' were analysed, five of these became integral parts of broader research and have been/are to be submitted as diploma theses. As future special needs teachers the students undertake these projects when they are in year 3 of their study. The whole course of study lasts five years with mathematics being studied for two years, geometry and arithmetic in year 2 and mathematics education in year 3. These students have five weeks teaching practice in schools/other institutions for special needs. It sometimes happens that they do not teach mathematics during this practice because they are teaching other subjects or they are working in hospitals or penitentiaries where mathematics is not taught.

2. Theoretical framework

What do these students need in order to be able to teach mathematics? They are given lectures/seminars on theories of mathematics education and the tutor gives examples of the theory in practice from his/her own teaching experiences. However we believe that together with these lectures/seminars the students should have their own first-hand experiences of working with pupils whilst they are doing mathematics. Such experiences show the student-teacher that they need to have sufficient mathematical knowledge themselves to be able to satisfy the pupil's quest for knowledge, they need to be able to establish good teacher-pupil relationships, they also need a portfolio of teaching strategies which they can draw upon to meet the very different needs of a broad subject like mathematics. This belief is supported by writers such as Mason (1994), who writes "I see working on education not in terms of an edifice of knowledge, adding new theorems to old, but rather as a journey of self discovery and development in which what others have learned has to be re-experienced by each traveler, re-learned, re-integrated and re-expressed in each generation"; Sierpinska and Leman (1996) who state that "knowledge in relation to theory of instruction, should be regarded as 'potential of action developed through experience'"; whilst Tall (2001) states "In preparing students to be teachers of primary mathematics, I have advocated that they need to have a real insight into how mathematics develops cognitively. ...It means starting to reflect on one's own experiences to see why certain things were difficult, or even impossible, at the time."

3. The situation

A common reason why these students – future special needs teachers – wish to study at the university arises from their need to help children or people with special needs. From such a student's point of view, 'to help' means becoming a teacher so that they can improve the abilities of special needs pupils in all school subjects in order that these pupils can integrate into normal life. The majority of these students give this goal the highest priority. However one of the subjects they have to teach is significantly different from all other subjects for most of them and that is mathematics. Most of these students do not like mathematics because they have had bad experiences themselves whilst learning the subject. They are afraid of mathematics. They do not have the confidence to solve mathematical problems. They remember how they were taught and their template for teaching mathematics is to give pupils a set of instructions for solving a particular type of problem, to require the pupils to memorise algorithms and definitions and to apply these by rote. The students also feel that mathematics is not an appropriate subject for special needs pupils because they think it is too difficult for these pupils, especially for mentally handicapped pupils. The students have not had experiences that show that mathematics can enrich the intellectual life of anyone. They come to the university with belief that the subject - mathematics is an institutional obstacle, which it is necessary to overcome in order to be able to help pupils.

4. The necessity of a change

The situation described above shows the absolute necessity to change the students' attitudes towards mathematics. This implies a complete change of teaching strategies from the traditional way in which students are taught at university. We want the students to experience that mathematics can enrich everyone and to understand that mathematics is as a part of our culture, and not just drill exercises to enable the teacher to see who can recall algorithms from memory. We try to achieve this goal in the following ways:

• To build up the student's belief in his/her own mathematical abilities. We individualize the mathematical needs of each student so that they can find appropriate tasks for themselves, which are not too difficult and not too trivial, otherwise the tasks do not help to build their self-confidence in their mathematical ability. In our experience we have found that the best way to do this is to give the students sets of graded tasks so that they can find the task that enables them to experience success in solving process as a result of their intellectual work.

• To use constructivistic approaches in mathematics teaching.

- We want the students to discover their own strategies as they solve mathematical tasks.

- We want the students to discuss with each other and the tutor, their strategies of how to solve the tasks; whether the tasks are solvable or not; whether all possible solutions have been found; the moment in a solving process when the students felt hopelessness. We want them to learn from their mistakes.

- The tutor initially leads/chairs discussions on mathematics in the student group. Eventually the students suggest their own ideas in the discussion which may or may not agree with the tutor's point of view. The more reticent students see their classmates contributing their ideas and this makes them willing to offer ideas into the discussion.

• To experience mathematics through observing pupils doing mathematics.

The mathematics, which these students are asked to do in school, is not conventional classroom mathematics but comprises non-traditional problems and environments, for instance making buildings from dice with the pupils having to sum the visible dots on them. Some tasks involve tetraminoes or pentominoes, orientation in the plane, making buildings from cubes and recording their characteristics, addition triangles, triads, patterns, combinatorial problems and so on. The students prepare their own formulation of the problems for the pupils of a certain grade (not necessarily with special needs pupils), and usually express some thoughts on how they think the pupils will solve them. The students then do the experiments with the pupils. We use non-traditional problems and environments so that the pupils cannot use nor rely on memory/skills used and already met in the classroom. The pupils often ask unexpected questions and use unusual strategies and the students in the role of teacher/researcher have to react to these. The students observe the pupils working and then write down their experiences from experiments. This write up and analysis of their work is the main part of their projects.

5. The project: How many triangles are in the figure?

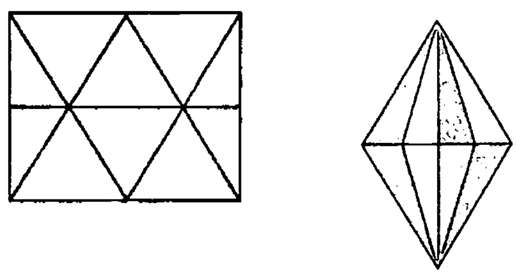
- Characteristics of the student Peter

Peter was a 2nd year student with average ability in mathematics. He was hard-working but sometimes he had difficulties with mathematics. His attitudes towards mathematics could be described as follows: He was afraid of doing mathematics individually. He was not sufficiently confident of his intellectual ability to believe he could solve problems on his own. He was very nervous when tests had to be written. But he liked discussing the problems or tasks in his student group. He usually offered some strategies how to solve it them. He often came to black board to describe the strategies.

- Description of the project

Peter made cards on which were geometrical figures (one figure on each card) consisting of white-black or colored triangles (16 cards with white-black triangles and 16 cards with coloured triangles, see Fig. 1). These cards (13x9 cm) were covered with film so that they would not be damaged when used. It is possible to wash the cards if a marker is used on them.

Figure 1



The aim of his project was to observe how pupils of different ages were able to distinguish triangles in figures. He asked pupils how many triangles there were in the figure and he measured the time taken for finding that number of triangles, which a particular pupil thought, was the total. Because he wanted to do a broader investigation, not only with pupils but also with students and adults he asked one classmate to help him.

Peter prepared tables for recording the given task and time needed for the pupil to discover the number of triangles.

He also prepared and used a computer program with the geometrical figures on the screen. The size and colors of figures were same as on the cards. The program was supported by sound-track instruction for the pupils and the tasks became more accessible for pupils with reading difficulties than when the tasks were on the cards only. It is worth noting that the pupils considered as game and not as task the computer version of the task.

The first investigation was carried out with white-black figures. He observed 34 pupils/students, aged 7-18. The second investigation was carried out with colored figures. He observed 59 people aged 6-41. The third investigation was carried out with figures on the computer. He observed 29 pupils/students aged 6-20.

All the results of his investigations were elaborated in four tables and eleven graphs (with the help of the computer). He then looked for correlations between age and success in the experiment for all 'pupils' and then male and female separately followed by the correlation between the time taken to complete the experiment and age.

His analysis of his results showed some of the figures had caused more difficulties than others for the pupils/students/adults. The pupils were the most successful when they solved the tasks on the computer. Young males were considerably more successful than females, but from the age of fourteen upwards, these differences decrease. He was aware that he could not make general conclusion from such small sample.

- **Analysis of the project**

Peter created his own non-traditional mathematical environment for pupils. In order to help pupils with reading difficulties he decided to computerize his original cards and used the computer program Quarelldraw, which was new for him and he had to learn to work with it. He also learned how to provide the computer version with sound-track instructions. It told the pupils what they had to look for and if their answer was right or not. He learned to work with different age groups (from child to adult age). He learned to make graphs and evaluate them. He learned the necessary statistical theory that he had to use when analyzing his results. On the basis of these facts we can see that Peter's self-confidence in his mathematical ability rose. He was not afraid to work on his own. He used mathematics not only as a tool for investigating thinking processes (that is, how pupils/students solved the tasks) but as a tool for statistical analysis of his investigation. At the end of year 3 of his study he decided to continue with this work and to use these initial results as a base for his diploma thesis.

6. Conclusion

Peter's project is an example of the student who gained confidence in his own ability by carefully looking at the teaching strategy he was using and deciding to put his task into a computer program to help weak readers, choosing a task in which he was interested in, particularly the effect of age on the results and his need to analyse the data he gathered. All this gave him the incentive to go much deeper into his own mathematical thinking, to research the necessary computer techniques and mathematics needed to present his tasks, to analyse the data he had gathered and to communicate his results in his project. This work in turn made him realise that mathematics was not the subject to be feared and each small success which he achieved as he went through this process helped to build up his confidence in his own ability. This experience will encourage him to try things in future, which previously he would have thought beyond his intellectual capacity.

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PRECALCULUS WITH INTERNET-BASED PARALLEL REVIEW

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ABSTRACT

An innovative precalculus course with an integrated Internet-based component will be described[1]. The course: (1) has an Internet-based "just in time" review component, (2) has Internet–based weekly tutorials, practice, and testing, (3) is designed for science and engineering students, (4) integrates the study of functions of two variables $f(x,y)$ and other basic three dimensional ideas, and (5) incorporates the use of symbolic algebra systems and other innovative pedagogy.

The study of multivariable functions is traditionally postponed until multivariable calculus. However, with the aid of a set of low cost manipulatives that we have developed to aid in the visualization of three dimensions, multivariable topics are being effectively incorporated into the precalculus curriculum. The goal is to build an early geometric intuition in three dimensions in students. This goal directly addresses a common concern voiced by our colleagues in engineering departments.

The Internet component of the project allows the establishment of a weekly practice, tutorial, and quiz system that helps students review the pre-requisites for upcoming material and review the material just covered in class. This component consists of a large and highly organized data bank of questions, a set of accompanying tutorials, and the software necessary for generating and administering quizzes on-line. The Internet component is being designed to facilitate its implementation in a wide variety of institutions. Interested faculty will be able to easily edit, contribute to, and adjust the data bank of questions to suit their needs.

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1. Introduction

This article addresses the precalculus course typically taught in many universities in the United States. The course is essentially an introduction to the function concept and a study of the main properties of polynomial, rational, exponential, logarithmic, and trigonometric functions. It usually also includes an introduction to vectors, matrices and linear systems, as well as several other elementary topics. Students who plan to follow careers in science or engineering and who do not have an adequate preparation in pre-college mathematics will take this course in their first year of university studies prior to enrolling in the calculus sequence.

The traditional precalculus course starts with a review of pre-college mathematics including properties of real numbers, algebraic expressions, solving equations, the Cartesian coordinate system, and lines. For an underprepared student, this introductory review attempts to teach in a very short time everything that the student did not learn in two or three years of pre-college mathematics. It is not surprising that it frequently fails to do so. Another problem with the traditional course is that it tries to cover too much material leaving little time for experimentation, collaborative learning activities, and other innovative pedagogies. Time constraints will frequently force professors, even those who would prefer alternative more student-centered approaches, to lecture a substantial amount of the time. In the past few years there have been several prominent attempts to redesign the course: by giving it a more modelling oriented approach, by incorporating graphing and sometimes symbolic computation, and by placing more emphasis on numeric and geometric representations while building connections with the traditional algebraic approach. Some of these course restructurings managed to introduce innovative elements into the course essentially by de-emphasizing algebraic computation skills. For students majoring in some fields this is a perfectly reasonable thing to do as one may argue that calculator and computer technology can be used for many of the routine calculations that students need to do in the course. But some students, particularly those planning to follow careers in engineering, do need a substantial amount of algebraic manipulation skills. Engineering and some professional licensing programs require these skills. The problem is how to restructure the precalculus course so that alternative pedagogical approaches can be incorporated into the course, and so that incoming students with a weak mathematics background can succeed in the course while still developing a substantial amount of algebraic manipulation skills. This is an important problem in universities where many of the science and engineering students start at the level of precalculus and universities wishing to open the doors to engineering careers to students with weaker mathematics backgrounds.

The course described below addresses the above problem by taking most of the basic algebraic computation skills out of the classroom but not out of the course. This frees enough class time for innovations without compromising algebraic computation skills. Basic skills are reviewed in parallel to the course using an Internet-based system of weekly quizzes.

2. Internet-based Component

Every week the precalculus students take a quiz, which is available via the Internet. Most of the questions have randomly generated parameters so the students may practice taking the quiz as much as they want prior to the "real" quiz taken for a grade. In fact, they must attain a professor-defined expertise level before the system allows them to take the weekly quiz for a grade. In the practice quizzes, the system tells the students which problems he/she answered incorrectly and allows the student to link to the appropriate web-based tutorial (see Figure 1).

Weekly quizzes have two parts (see Figure 2): a *Review Topic Section* consisting of the prerequisite material needed for the next week of classes and a *Precalculus Topic Section* consisting of the course material covered in the previous week of classes. The *Review Topic Section* of the quizzes allows distributing the review of basic material throughout the entire course, covering what is needed for the course just at the time it will be used in the course. This is in contrast to the common practice of attempting to do all the reviewing of prerequisites in a block of time at the beginning of the course. With the weekly quiz system, reviewing is done as needed throughout the course via the Internet and, more importantly, without taking any class time. The *Precalculus Topic Section* of the quizzes is meant to help students keep up to date with the course material.

Each week, every student in the precalculus program enters the principal page for the given week via the Internet (quiz.uprm.edu). They may enter this page from any computer laboratory in the university or from their home. This can make it possible for a department of mathematics to support the intensive use of computer resources from a large quantity of students without placing too much of a burden on already existing departmental resources. It also allows a more efficient use of computing resources campus wide. From there, they may enter either the *Review Topic* section or the *Precalculus Topic* section. Upon entering the *Review Topic* section, the review topics necessary for the following week's precalculus topics will be outlined with links to pertinent tutorials. Should they wish to review these materials, they may proceed to the online tutorials. If they feel that they already understand these review topics, they may proceed to a practice quiz which will evaluate how well they comprehend these topics. Upon submitting their answers for the practice quiz, the computer generates a report indicating which topics they did well on and which topics require further review. This report includes links to pertinent tutorials. When the practice quiz indicates a sufficient degree of understanding (exactly what is "sufficient" is at the discretion of each instructor), the student is cleared to take a real quiz which counts toward their final grade. In our case, the real quiz may only be taken in the Testing Annex to the Mathematics Tutoring Laboratory where their identity is verified. Also, in our case the weekly quizzes count for 15% of the grade. Of course, anyone adopting the system will choose to give whatever credit they deem reasonable.

The three major components of the Internet component are: (i) the highly organized databases of questions that may be used both for quizzes and practice; (ii) the tutorials of review material to which students are referred, and (iii) the software used to administer the quizzes and refer the students to appropriate tutorials. These components have been class-tested for one year and are presently at different stages of development.

In order to make the quiz system useful to a wide range of precalculus professors each having their own idea of how to teach and what to teach in the course, large and comprehensive databases of questions are being developed to cover the topics normally taught in the precalculus course per se, and the review topics needed to understand the course. The guiding principles in their development are:

- Professors will be able to edit the databases, adding or removing questions as they wish.
- Professors may contribute to a central data bank. This will be available to any educational institution and will be maintained by the project directors.
- Databases will exist for each topic of precalculus. The organization will be Topic_Focus_Degree-of-Difficulty. For example, Lines_geom_easy will be a database which will contain questions concerning lines which have a geometric focus and are easy.

- Databases of questions which require an extensive written response will be created; however, these will not be graded by the system. Student responses will be forwarded to a professor's chosen address. If professors do not want to use these questions, they need not include them. Other than the open response questions, multiple-choice questions are also available on the quizzes. While this presents somewhat of a limitation, it is perfectly adequate for most of the topics dealing with basic algebra skills.
- Databases of questions which require that the students have graphing and/or computer algebra system technology available will be available for each topic. Once again, if professors do not want to use these questions, they need not include them.

As an example, for an introduction to linear functions we would end up with perhaps 12 databases each of which would contain 100 questions. The titles of these databases might be

Lines_algebraic_easy, Lines_algebraic_int, Lines_algebraic_hard, Lines_geom_easy, etc. Professors may then use the quiz generating software described below to indicate how many questions from each of these databases should be placed in their quizzes.

The software works in the following manner: (1) professors fill out an electronic form indicating the content of the quiz; (2) students receive a practice quiz which has been randomly generated from databases of questions and which is unique to each student; (3) students complete the quiz and submit their responses; then (4) the computer corrects the quiz and generates a report for the student which contains (i) percentage score (ii) electronic links and/or references to the text where they may review topics that they did not pass, and (iii) the questions which were incorrectly answered. If they pass the quiz with a grade predetermined by the professor, they are cleared to take a real quiz that is administered in the Testing Annex to the Mathematics Tutoring Laboratory or any other specially designated place.

In the quiz system, after students take practice quizzes, they are directed to a tutorial. This will generally be in the form of an Internet link. However, as always the system allows professors to substitute any other link or reference of their choosing.

3. The Text

A textbook has been written and was pilot tested during the past academic year. Pilot testing will continue for two more years before commercial publication. The book offers a mix of traditional and what might be referred to as "reform" elements. Perhaps this is the major strength of the textbook. It bridges two visions of what a precalculus textbook should be. Topics are presented using simple common sense examples to build on the intuition of students. Multiple representations and the corresponding interconnections are explored throughout the book. The presentation is informal in style. Each section starts with a note on the prerequisite topics that students should review using the Internet-based quiz system. Practice problems are embedded in the topics and examples being presented so that students will have the opportunity to test their understanding while reading the main text. Science and Engineering oriented examples and problems are found throughout the book. To get a clear idea of the nature of the textbook one would need to examine the presentation in more detail than is reasonable to include in a short article. The content and order of the textbook is quite traditional except in the part of the book where a group of topics is presented in a three-dimensional setting.

4. Introduction to Three Dimensions in Precalculus

The need for geometric visualization of three dimensions is very pronounced in engineering. All engineering majors must take graphics, physics, and statics courses immediately following or concurrently with precalculus where the material frequently occurs in three dimensions. However the formal presentation of three dimensions does not occur until the third semester of the calculus sequence. The primary impediment to presenting three dimensions earlier in the students' academic program is that it requires that they think abstractly. The traditional presentation typically does not include means of concretely visualizing coordinates in three dimensions. Hence, when students contemplate lengths of vectors, slopes of lines, and other simple topics in three dimensions, they cannot actually see the vector or the line with which they are working. Our pilot testing experience shows that students are comfortable with concepts in three dimensions when points, lines, vectors, and curves can be represented with concrete objects in three dimensions. To do this a set of manipulatives, resembling something like a Lego for three dimensional explorations in precalculus, was developed. We call this tool, the 3-D Kit. The most fundamental element in the 3-D Kit is a suitcase, which opens to form three dimensions. A rough sketch is provided in Figure 3. The pole, which forms the z-axis, fits inside of the suitcase and when the suitcase is opened, there is a hole within which it can be placed to form three-dimensional axes. The exterior part of the suitcase which forms the x and y axes are covered with an erasable board which allows the user to write on the xy plane. There is also an attachment which provides raised x and y axes as shown in Figure 4. Also within the suitcase are multicolored balls, which serve as points; antennas, which serve as vectors; foam covered strips of wire which serve as curves and as contours; and the necessary props to sustain points, vectors, and curves in place.

By incorporating three dimensions into the precalculus program, engineering students are being prepared for topics they will shortly need in other science and engineering courses. However, this approach also strongly reinforces two-dimensional concepts presented earlier in the course. For example, the concept of a rate of change, or slope is presented early in the course in a two-dimensional setting. The concept is presented again in three dimensions when dealing with the slope of a plane in the x direction and in the y direction. This provides multiple situations and visualizations to reinforce the basic concept of a slope. All of the presentations in three dimensions support parallel themes in two dimensions. Hence, presentations in three dimensions serve both to expose students to three dimensions which is important for engineering students and to reinforce the basic concepts of precalculus as learned in two dimensions. The textbook includes an introduction to the Cartesian three-dimensional space, vectors in three dimensions, functions of two variables, linear functions of two variables and planes. Some of these ideas are useful later in the course when treating linear systems of equations in three variables, and in the presentation of conic sections.

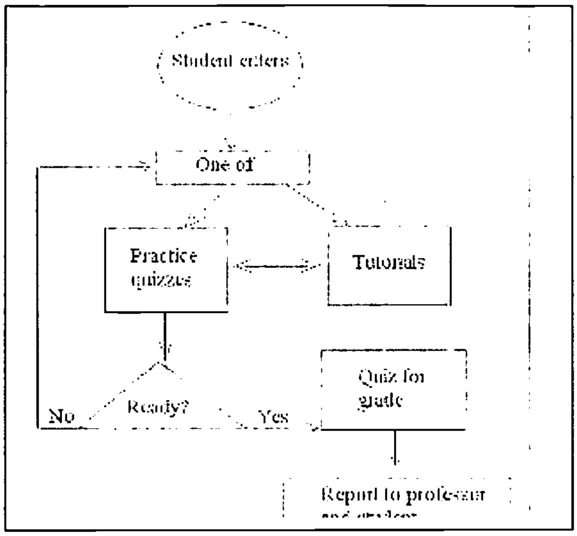

Figure 1

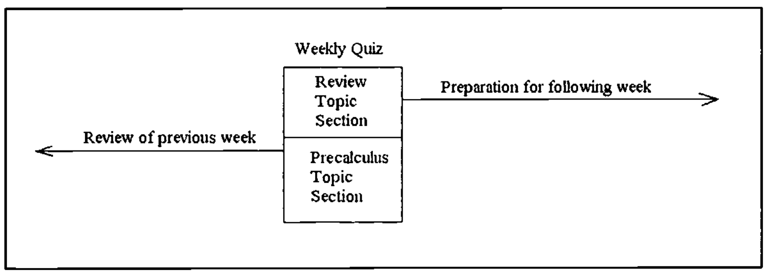

Figure 2

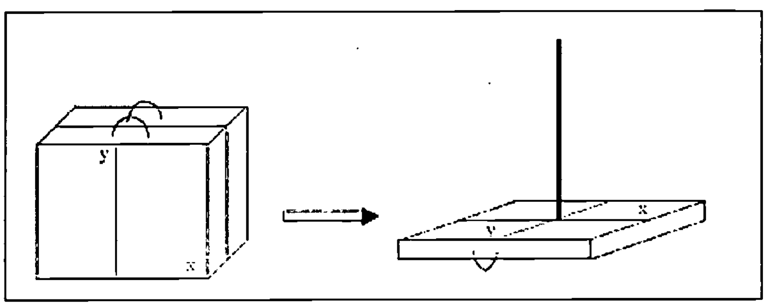

Figure 3

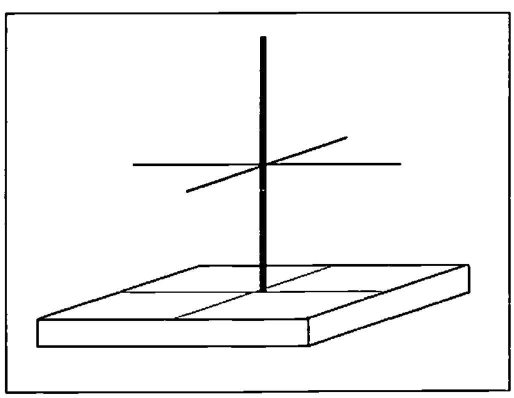

Figure 4

WHAT IS MODERN IN "MODERN MATHEMATICS"? HOW SHOULD MODERN TEACHING REFLECT THIS?

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ABSTRACT

It is commonly held that what distinguishes modern mathematics is the availability of high-speed electronic computers and pocket calculators with graphical capabilities. Consequently, mathematics is usually taught in schools and undergraduate courses as if Euler and Gauss were our contemporaries, with electronic gadgets replacing tables, slide-rules and sketching.

We discuss cultural changes in mathematics over the past two hundred years overlooked by such an approach, specifically the rise of rigour and algebra. These have altered the face of mathematics, providing a deeper understanding of many important results, by providing a unified, coherent setting.

At the same time, these developments have increased the power and scope of mathematics, by enabling it to deal with non-quantitative problems, by making many computations accessible to computers and by making it more applicable to other disciplines.

We use, in particular, the Fundamental Theorem of Calculus as an illustration and offer a programme for teaching calculus in a manner which accommodates these developments and eases the student's path to further studies.

1 Background

The purpose of this paper is to provide a sketch of cultural changes in mathematics over the past two hundred years, using in particular the Fundamental Theorem of Calculus as illustration. Brevity dictates that details be missing.

Such discussion as this paper intends to foster is urgently needed, for at the dawn of the 21st century undergraduate and high school mathematics are generally still taught as if Gauss and Euler were contemporaries, rather than historical figures whose bicentenaries have been all but forgotten.

Two main strands are discernible in mathematics, often perceived to be in conflict, namely *solving problems* and *constructing theories.* The history of mathematics clearly shows they are symbiotic. For trying to solve previously intractable problems has frequently led to advances in available theory or the development of new theory. Newton's Differential Calculus is a prime example. Equally, purely theoretical advances have frequently — and unanticipatedly — led to solutions of problems in areas to which no significant connection had been suspected. The application of the theory of fibre bundles to arbitrage, of cohomology to number theory and of Hopf algebras to theoretical physics (in the guise of quantum groups) spring immediately to mind. A propos the last example, in [2], Dieudonné was still able to write of the connections with the natural sciences of category theory (p.246) and homological algebra (p.180) "None at present". How dramatically and quickly that changed!

Moreover, we have witnessed the same theory has arise independently and contemporaneously from both practical needs and "purely speculative" considerations. For example, Russell and Whitehead's philosophical programme in *Principia Mathematica* and Dehn's interest in knots both led to the word problem in group theory.

However, this symbiosis does not mean that there is no clearly discernible trend in the history, development and culture of mathematics.

The overriding trend in mathematics over the last two centuries has been increasing rigour and increasing "*algebraisation*": There has been a systematic formalisation and axiomatisation of calculations and arguments. This is not due to a theological addiction to formalism, but the inevitable consequence of several developments: Disturbing paradoxes were found to be immanent in mathematics as practised, and several cherished expectations were dashed by the cold light of irrefutable reason.

The discovery of non-Euclidean geometry by Bolyai, Lobachevski and Riemann, the proof by Galois of the unsolvability of polynomial equations of degree higher than four and the discovery of the paradoxes of set theory shook mathematics to its very foundation, destroying the cherished certitude of established beliefs and confident expectations.

New phenomena have been observed which were unthinkable observed without the guidance of mathematically formulated theory. The discrepancy between the prediction from our theoretical model and the actual recorded observations of our planetary system led to the postulation of previously undiscovered planets, and the subsequent search, guided by the theoretical predictions, resulted in the discovery which confirmed the theory. Who would have thought of trying to measure the bending of light rays around Mercury had Einsteinian theory not predicted it?

Mathematics had provided coherent explanation of events, processes and observations in terms of elegant and insightful theory.

Instead of abandoning mathematics, a rigorous, axiomatic formulation was sought, devoid of the pitfalls while preserving the powerful theories and theorems. By and large, the scientific and philosophical community was convinced that the major results were correct, even though some of the reasoning used to justify them was flawed.

So, after centuries of primary concern with problem-solving, mathematics was forced to concern itself increasingly with constructing theories, not just in an ad hoc manner to justify particular techniques to solve particular problems, but in a considerably more serious and catholic manner.

An enduring legacy of this development is that by becoming more fundamental, mathematics today is more abstract and rigorous, thereby becoming more applicable and more broadly applied. Electronics, meteorology, modern "financial management", not to speak of computing and computers, quantum theory and relativity theory are all inconceivable in their current forms without modern mathematics.

Paul Dirac's observation in 1931 ([3] p. 368) has lost nothing of its aptness.

> *"The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected. What however was not expected by the scientific workers of the last century was the particular form that the line of advancement of the mathematics would take, namely, it was expected that the mathematics would get more and more complicated, but would rest on a permanent basis of axioms and definitions, while actually the modern physical developments have required a mathematics that continually shifts its foundations and gets more abstract. Non-euclidean geometry and non-commutative algebra, which were at one time considered to be purely fictions of the minds and pastimes of logical thinkers, have now been found to be very necessary for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advances in physics is to be associated with a continual modification of the axioms at the base of mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation."*

2 The Fundamental Theorem of Calculus Revisited

The Fundamental Theorem of Calculus serves well to illustrate Dirac's point.

If $f : [a, b] \longrightarrow \mathbb{R}$ *is continuous, then the function*

$$F : [a, b] \longrightarrow \mathbb{R}, \quad x \longmapsto \int_a^x f(t)\, dt \tag{1}$$

is continuous on $[a, b]$ *and differentiable on* $]\, a, b\, [$ *with*

$$F'(x) = f(x) \tag{2}$$

for $a < x < b$.

This was originally an astounding theorem, for it demonstrated that two apparently unrelated problems — finding a function whose derivative is a given function and finding the average value of a given function — have a common solution. It also provided a link between the then new differential calculus, and integration, which had been known since antiquity in the guise of the "method of exhaustion".

One immediate consequence is the use of results from differential calculus to provide strategies and techniques for integral calculus, as we discuss below.

One direction in which calculus subsequently developed is multivariate calculus, from which we quote some results — Stokes' and Gauss' theorems — in a convenient form.

Let $\boldsymbol{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be differentiable. Let S be a suitable oriented surface in $\mathbb{R}^3$ with boundary ∂S, which we take with the induced orientation. Let V be a suitable oriented solid region of $\mathbb{R}^3$ with boundary ∂V, which we take with the induced orientation. Then

$$\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_S \nabla \times \boldsymbol{F} \cdot d\boldsymbol{s} \tag{3}$$

$$\int_{\partial V} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_V \nabla \cdot \boldsymbol{F} dv \tag{4}$$

where $\boldsymbol{r}, \boldsymbol{s}$ and v denote line, surface and volume elements.

Thinking of F' as the gradient of F, and agreeing that integrating a function on a finite set consists of summing its values on that set, we obtain a reformulation of the Fundamental Theorem of Calculus.

Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable. Let I be an oriented interval whose boundary ∂I is taken with the induced orientation. Then

$$\int_{\partial I} F = \int_I \nabla F \cdot d\boldsymbol{r}. \tag{5}$$

Thus, Green's, Stokes' and Gauss' theorems are merely "higher dimensional" — more "complicated", as Dirac would say — versions of the Fundamental Theorem of Calculus. Alternatively we may view the Fundamental Theorem of Calculus as a special case or application of the others.

Subsequent developments took a different direction, with far more profound consequences. They consisted of the investigation, primarily due to Poincaré, of the word "suitable" in the above theorems. Poincaré studied *simplexes* and their *boundaries.* A k-simplex in $\mathbb{R}^n$ is the convex hull of $k+1$ points. However the boundary of a simplex is not a simplex, but rather a *chain* of simplexes.

Writing σ for a simplex, $\partial\sigma$ for its boundary, ω for a *differential form* and $d\omega$ for its *differential*, the above theorems all assume the form

$$\int_{\partial\sigma} \omega = \int_\sigma d\omega. \tag{6}$$

[Recall that a differential 0-form is just a smooth real valued function $f(x_1, \ldots, x_n)$ and its differential is the 1-form

$$df := \sum_j \frac{\partial f}{\partial x_j} dx_j \tag{7}$$

If $\omega = f dx_{j_1} \ldots dx_{j_k}$ is a differential k-form, then its differential is the $(k+1)$-form

$$d\omega := \sum_i \frac{\partial f}{\partial x_i} dx_i dx_{j_1} \ldots dx_{j_k} \tag{8}$$

with the convention that $dxdx = 0$.]

The work initiated by Poincaré involved the notions of *homology, homotopy* and *Betti numbers* to describe regions of $\mathbb{R}^n$ and he introduced a group, the *fundamental group*, to characterise when the integrals of a function along two paths between the same two points necessarily coincide.

As a result of the insights and influence of Emmy Noether, these notions were recognised to be best formulated in terms of homology, cohomology and homotopy *groups*.

Cartan was able to provide a sound algebraic foundation for what physicists and engineers had been practising with differential forms, often to the consternation of mathematicians.

Finally, as a consequence of the work of de Rham, we would now say that the differential forms form a *chain complex*, that is, a sequence of abelian groups $\{\Omega^k \mid k \in \mathbb{N}\}$ and homomorphisms $d^k : \Omega^k \to \Omega^{k+1}$ $(k \in \mathbb{N})$ with $d^{k+1} \circ d^k = 0$. In the language of differential forms used by physicists and engineers: "Every exact form is closed". [Recall that the differential k-form ω is called closed if $d^k\omega = 0$ and it is called exact if it can be written as $d^{k-1}\sigma$ for some $(k-1)$-form σ.]

In the case of $\mathbb{R}^3$, $d^1 \circ d^0 = 0$ is the familiar statement from vector calculus "curl-grad = 0" or $\nabla \times \nabla f = 0$, and $d^2 \circ d^1 = 0$ is the statement "div-curl = 0" or $\nabla \cdot \nabla \times \mathbf{v} = 0$.

Heuristically, the Fundamental Theorem of Calculus and its generalisations in Equation 6 state that the de Rahm complex is dual to the simplicial complex of suitable spaces.

The k-th homology group of a chain complex is the kernel of the k-th homomorphism factored by the image of the preceding one, so that

$$H^k(\mathbb{R}^n) := \ker d^k / \mathrm{im} d^{k-1}. \tag{9}$$

It provides a measure of the extent to which closed k-forms are exact. The Fundamental Theorem of Calculus yields the statement

$$H^1(\mathbb{R}) = 0. \tag{10}$$

It and the more general versions together yield

$$H^k(\mathbb{R}^n) = 0 \tag{11}$$

for all $n, k \in \mathbb{N}$ with $k > 0$.

This purely algebraic statement lands us where mathematics has progressed since Euler. The increased abstraction has necessitated and been made possible by algebraisation and the development of new, abstract and often non-quantitative theories such as the modern theories of abstract algebra, functional analysis, integration theory, harmonic analysis, axiomatic set theory, algebraic topology, differential geometry, algebraic geometry and category theory.

Rather than making modern mathematics more remote from applications and applicability, this abstraction has had the opposite effect: Mathematics is now applied to more disciplines than ever before, with previously intractable problems now solved. Fermat's Last Theorem furnishes the most recent example of a problem elementary enough for a layman to understand, whose solution has required the application of some of the most abstract mathematical theories, which, on the face of it, have no bearing on the problem and in ways which few had envisaged.

Moreover, because of this algebraisation has meant many calculations which formerly required skill and/or ingenuity have been reduced to routine manipulations and algorithmic procedures, suited to solution by computers.

3 How Does This Affect the Way We Should Teach Mathematics?

It would be patently absurd to appeal to de Rham cohomology in a first course on calculus! But it is equally absurd to present such a course as if de Rham cohomology belonged to the realm of science fiction.

The principal difficulty with many current "elementary" and "introductory" courses is that by hiding — or, at best, just ignoring — rather than revealing and emphasising the unity and coherence of modern mathematics, they impede both understanding and subsequent advancement.

Mathematics courses, especially "low level" ones, are all too frequently taught as collections of computational tricks and arcane formulæ to be remembered just long enough to pass the next examination.

Students often complain of needing to forget pictures and "intuitions" acquired in first calculus courses when learning multivariate calculus and having to start all over again.

Students typically perceive calculus and algebra as mutually exclusive, as must be expected from the way they are usually taught.

I believe that calculus is best taught algebraically, given the historical development outlined very briefly above. That history actually illustrates two different ways in which the algebraisation of mathematics in general, but especially calculus, has occurred.

Firstly, it has become clear that calculus is the study of $\mathcal{F}(X)$, the real algebra of real valued functions on the subset X of $\mathbb{R}^n$ and of certain of its sub-algebras.

Secondly — and this is the more recent development — we now assign to each of the subsets X of $\mathbb{R}^n$ a collection of algebraic invariants which reflect many of the significant properties of X.

Clearly this second aspect can hardly be introduced into a first course on calculus, although it might be alluded to in informal discussion of what a student would meet eventually, when pursuing mathematics far enough.

But I have tried basing the calculus courses on the first aspect, viewing calculus as the study of the real algebra, $\mathcal{F}(X)$, of real valued functions on the subset X of $\mathbb{R}^n$ and of certain of its sub-algebras. A first course in calculus typically considers only the case $n = 1$ with $X \subseteq \mathbb{R}$. It is the central notion of a limit which distinguishes calculus from "pure" algebra and calculus may be regarded fruitfully as the study of how limits interact with the algebraic structure of $\mathcal{F}(X)$.

This has the immediate pedagogical advantage of providing context for the central theorems, thereby dispelling much of the mysteriousness and lack of motivation students so often criticise. Moreover it leads naturally to other topics. Differential geometry, for example, may be fruitfully regarded as the study of the sub-algebra, $\mathcal{C}^\infty(X)$, of $\mathcal{F}(X)$ consisting of all smooth real-valued functions defined on the manifold X. While this seems to lack geometry, it is an amazing fact that under mild conditions, the manifold can be recovered from the algebraic structure of $\mathcal{C}^{\infty}(X)$! Of course computation in differential geometry relies on the algebra.

It also demonstrates that much of the work can be reduced to "mere" symbolic computation and done by machine, the crucial step being the determination of how limits behave and how they interact with the algebraic operations on $\mathcal{F}(X)$.

Of course, I do **not** commence teaching calculus by announcing to students that we shall be spending the course studying the real algebra $\mathcal{F}(X)$, just as I would definitely **not** say to pupils at the outset of learning elementary arithmetic at primary school that they will be learning about the ring of integers!

Rather, I introduce the structure piece by piece, establishing the relevant properties. The course is guided by the algebraic structure, so that **after** the course, a student meeting the notion of an algebra and homomorphism of algebras can readily recognise that the calculus course provided examples of algebras and homomorphisms between them. After all, upon being told what a ring is, any pupil who has a good mastery of elementary arithmetic should appreciate that the integers form a ring.

Specifically, we define a sum and product on the elements of $\mathcal{F}(X)$ "point-wise", that is to say, given functions $f, g : X \longrightarrow \mathbb{R}$, we define

$$f + g : X \longrightarrow \mathbb{R}, \quad x \longmapsto f(x) + g(x) \tag{12}$$

$$f.g : X \longrightarrow \mathbb{R}, \quad x \longmapsto f(x).g(x) \tag{13}$$

At this stage it can be fruitfully pointed out that in fact this also includes multiplying a function by a real constant, for we may identify each real number c with the constant function $X \longrightarrow \mathbb{R}$ which assigns c to each and every element of X. In this way we may regard $\mathcal{F}(X)$ as an extension of the set of real numbers, very much the way that the integers form an extension of the counting numbers, the rational numbers of the integers and the real numbers of the rational ones.

If the range of g is a subset of X, we can also define the *composition* of g and f,

$$f \circ g : X \longrightarrow \mathbb{R}, \quad x \longmapsto f(g(x)). \tag{14}$$

These operations provide the algebraic structure on $\mathcal{F}(X)$. Applying them to the functions

(i) $e_c : X \longrightarrow \mathbb{R}, \quad x \longmapsto c$

(ii) $p_1 : X \longrightarrow \mathbb{R}, \quad x \longmapsto x$

is sufficient to define all polynomial functions on X.

If we add

(iii) $s : X \longrightarrow \mathbb{R}, \quad x \longmapsto \sin x$

(iv) $p_{-1} : X \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{1}{x}$ as long as $0 \notin X$,

we have all the rational functions and trigonometric functions.

Finally if we include

(v) $exp : X \longrightarrow \mathbb{R}, \quad x \longmapsto e^x$,

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together with the inverse functions to all of the above whenever these are defined, then each elementary function — and therefore all those studied in calculus courses — arise by means of recursion. In other words, these few "basic" functions, together with the three algebraic operations *generate* all the functions met in a first calculus course. These functions are frequently referred to as *elementary functions* and we write $\mathcal{E}(X)$ for the set of all elementary real valued functions defined on X. Clearly, $\mathcal{E}(X)$ is a sub-algebra of $\mathcal{F}(X)$.

It is therefore enough to study the behaviour of these five functions as long as we know how the algebraic operations behave with respect to the other operations studied in calculus.

The behaviour of our basic functions with respect to taking limits is easily determined directly "from first principles" in the case of the first three and with appeal to the properties of the functions in the case of the last two.

As to the relationship between limits and our algebraic operations, it is easy to show that

$$\begin{aligned} \lim_{x \to a} (f+g)(x) &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \quad (15) \\ \lim_{x \to a} (f.g)(x) &= [\lim_{x \to a} f(x)].[\lim_{x \to a} g(x)], \quad (16) \end{aligned}$$

as long as both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

Moreover, if $\lim_{x \to a} g(x) = k$ and $\lim_{y \to k} f(y) = \ell$, then

$$\lim_{x \to a} (f \circ g)(x) = \ell. \quad (17)$$

Of course we must specialise to a sub-algebra of $\mathcal{F}(X)$ to ensure that the limits exist and this again illustrates a common procedure in mathematics: When something does not work in complete generality, restrict attention to the cases where it does and investigate the minimal restrictions required as well as the reason(s) for the failure in complete generality.

Furthermore, this leads to a very natural way of distinguishing continuous functions, for these are precisely the functions for which we can evaluate $\lim_{x \to a} f(x)$ by "plugging in", that is, by evaluating f at a. Such functions form a sub-algebra, $\mathcal{C}^0(X)$, of $\mathcal{F}(X)$.

The corresponding results in the case of differentiation are no more difficult in the case of our basic functions, and the rules for the behaviour of differentiation with respect to the algebraic operations are precisely the linearity, Leibniz rule and chain rule every student meets. Once again we must restrict the set of admissible functions to which we can apply this operation.

We have the following table for this more restricted set of functions.

Limits	**Differentiation**
linearity	linearity
rule for products	Leibniz rule
rule for composites	chain rule

We can therefore evaluate limits, differentiate explicitly any function which is generated by our five basic functions using the three algebraic operations. Thus we have, in effect, an algorithm for evaluating the limits of, or equally for differentiating, the class of functions generated by our basic functions and the algebraic operations:

1. Express the function in terms of our basic functions using only the three algebraic operations.
2. Write down the limit/derivative of each of the basic functions appearing.
3. Execute the algebraic operation on the appropriate limit/derivative corresponding to each algebraic operation on the functions.

It "only" remains to translate this into your favourite programming language.

The case of integration is more interesting. Our basic functions can be integrated directly with ease and, using the Fundamental Theorem of Calculus, we derive the linearity of the integral, integration by parts and integration by substitution corresponding to our three algebraic operations.

We can tabulate the relationships as follows.

Limits	Differentiation	Integration
linearity	linearity	linearity
rule for products	Leibniz rule	integration by parts
rule for composites	chain rule	integration by substitution

But there are differences as well between differentiation and integration.

Whereas we specialised as we went from just functions to continuous functions and again when we went from continuous functions to differentiable ones, we do not continue this line of specialisation by passing to integrable functions. On the contrary, while every continuous function (and *a fortiori* every differentiable one) is integrable, not every integrable function is continuous.

Moreover difficulties arise if we insist on explicitly expressing integrals purely in terms of our basic functions and the three algebraic operations. For while differentiation maps our special subset, $\mathcal{E}(X)$, of $\mathcal{F}(X)$ to itself, integration does not. The above algorithm cannot be used to calculate integrals instead of taking limits or calculating derivatives.

This inconvenience has several didactic advantages.

a. It provides a natural example of a problem which cannot be solved algorithmically.

b. It naturally raises questions leading to further and deeper study of mathematics and opens the way to more applications. It provides, for example, motivation for Taylor series and Fourier series as techniques for evaluating otherwise inaccessible integrals by using readily computable ones and limits, demonstrating once more the power of the central notion of calculus.

c. It illustrates another situation common in mathematics: We push our available techniques as far as we can and then seek to find other means when we are confronted with situations beyond the scope of our current techniques and methods. Moreover, the obstacles and difficulties we meet often prescribe specifications for the new techniques and methods we need to develop.

What more can we wish for as teachers of mathematics?

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GEOMETRY: BACK TO THE FUTURE?

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ABSTRACT

In this paper we consider the use of current technology to help us address fundamental difficulties in comprehending geometrical thought in a geometry course designed for Mathematics teachers. Our conception of geometry involves a full spectrum of activities, from concrete exploration and experimentation, through conjecturing, problem-solving, and on to formal proof. However, the full range of this spectrum requires very different qualities of thinking that seem to make it very difficult to implement in a mathematical curriculum. This was apparent in the work of Archimedes, who conceived of formulae for areas and volumes using a 'mechanical method' that relied on thought experiment, but published many of his results only in terms of 'proof by exhaustion', which severed the link between creative conception and formal presentation. This dichotomy has continued throughout history, with our present school curriculum seemingly oscillating between a focus on formal Euclidean proof on the one hand and a more empirical study of 'space and shape' on the other. Our approach faces both ends of this dilemma by using dynamic geometry softwares and a study of selected historical struggles to develop formal proof and to help students to reflect on the dialectic process between exploratory work with figures and proof elaboration. We outline our theoretical perspective, consider the issues involved, and report on the progress of different types of student who attended the course.

Introduction

Geometry seems to be firmly back in the curricula of undergraduate mathematics in an increasing number of leading Mathematics departments, as witnessed by a growing offer of new textbooks (Hartshorne's *Geometry: Euclid and Beyond*, is a particularly beautiful example). A somewhat related phenomena is the mounting stream of documents stressing the need for a greater emphasis on geometry in the school mathematics curriculum (see for instance NCTM, 2000 and Oldknow & al., 2001).

While in some countries, as it is the case of France, geometry has always taken an important place in the curriculum and is considered as a fundamental subject, in others (including Brazil), Dieudonne's war cry of "down with Euclid", back in the sixties, seems to have echoed for far too long and with particularly zealous fervour. In this paper we discuss ways of providing, both in undergraduate and continuing education teacher training programs, effective means for adequately preparing the teachers who must undertake the task of teaching geometry in secondary schools.

Our conception of geometry involves a full spectrum of activities, from concrete exploration and experimentation, through conjecturing, problem-solving, and on to formal proof. The importance of geometry in the curriculum goes beyond it's recognised content: it's acknowledged as fundamental for the development of the understanding of mathematics and science in general. *«Elle est un objet d'enseignement incontournable tant du point de vue de l'étude des situations spatiales que du point de vue de la constitution de la rationalité scientifique.»* (Bkouche, préface of the book Géométrie (Carral, 1995)). It is regarded as a good opportunity for the student to evolve from observation skills to hypothetical-deductive skills (Rauscher, 1993).

On the other hand, our experiences show that the majority of teachers working in Brazilian secondary schools has a less than adequate grounding in geometry. This assumption is confirmed in two separate ways:

-the knowledge of geometric fundamentals of samples of teachers attending continuing education courses at our institutions, whenever put to the test, has proved far from adequate. There seems to be a direct bearing in their ability to deal with basic notions, and with the concept of proof.

-the mediocre performance, in geometry related questions, by students in all university entrance examinations. As a consequence, our troubles in first year courses are not very different from what we see reported in dozens of accounts from other countries. For instance, if we consider the Linear Algebra courses, we know for sure that our students are not getting the grounding in spatial geometry that was to be desired before entering the University, and that has an influence in their initial difficulty to deal with the spanning of subspaces and the geometry of linear transformations.

A vicious circle is in place, whereby less than adequately prepared students start teacher training courses in Mathematics, finish their courses with a less than adequate proficiency, and go into the profession feeling less than secure about their own mathematical ability. To us, a good place to try to break this stalemate is a geometry course rich in opportunities to deal with deep questions which have a bearing in the school curriculum, specially as regards mathematical proof.

Geometry in the School Curriculum: the role of Dynamical Geometry Softwares

The recent change in the curriculum concerning geometry is that, maybe with the help of dynamic geometry software[1], geometry is taught nowadays with a bigger emphasis on experimental approaches. «*Les diverses activités de géométrie habitueront les élèves à expérimenter et à conjecturer, et permettront progressivement de s'entraîner à des justifications au moyen de courtes séquences déductives* » (Programmes de cinquième, BO, p. 24). Carral (1995) also says «*La géométrie élémentaire doit être considérée comme une science physique et son apprentissage doit se faire comme une science expérimentale...* ».

But even in countries like France, where geometry has always been an important part of mathematical teaching, it has also always been one of the hardest to teach, particularly when proof is concerned. Trouble with the concept of proof seems to be a feature not exclusive of Brazilian or French schools. As we write this we can find the following text, in the web site of the British Association of Teachers of Mathematics, as part of an apparent endorsement for a book soon to be distributed by ATM: "*This book argues the case for the use of proof based on 'seeing is believing'. Using a 'Tracing', on top of a 'Diagram', we can often show clearly the truth of an assertion. In other words we can prove it.*"

In general, the curriculum oscillates between more figure exploration/less formal geometry teaching and less figures/more proof elaboration. The dialectic process between exploratory work with figures and proof elaboration, which can be seen in the historical evolution of geometry, seems to give curriculum formulators a hard time.

Maybe more than just knowledge, we want the student to develop competencies in knowledge construction. Particularly, we want him to acquire skills in exploring figures, elaborating and experimenting with conjectures, , problem solving and proof formulation. But this set of skills, which seems so natural to the scientifically trained, does not come so naturally to the students. The concrete object does not have the same signification and is not explored in the same way by the mathematician and by the student: the way the concrete object is used strongly depends on the previous knowledge of who is using it. Even more important is that teaching based on the exploration of the concrete object makes the none evident assumption that the interaction with the concrete will effectively produce the construction of the desired knowledge (Balacheff, 1999).

In the case of geometry, the concrete object is often a diagram, and to understand the differences between the student and the teacher in it's exploration it, researchers in maths education often consider two different objects (Parzysz, 1988; Arsac, 1989; Laborde et Capponi, 1994; Balacheff, 1999):

- a concrete object, the drawing, which is a material representation of the figure,
- a formal object, called the figure, which corresponds to the class of drawings representing the same set of specifications.

One of the difficulties in the geometry classroom is that the student may be thinking in terms of the drawing instead of the geometrical object, whereas the teacher is using the abstract representation of the geometrical objects, the figure. Helping the student to read a figure in a geometrical way, and to use it as a tool to conjecture or understand proof, and not as the proof itself is part of the job of teaching geometry.

[1] As the authors include people involved in the development of two different dynamical geometry packages, the reference here is to the genre, as opposed to a particular implementation which, we feel, only strengthens the argument.

From this point of view, dynamic geometry softwares can have a specific contribution. They can provide new representations of geometrical objects which, in some ways, concretise the formal figure. We take as one of our assumptions that these softwares can provide new ways to learn geometry, and by way of consequence, new ways to teach the subject. Their use in programs of improvement of mathematical preparation of teachers of Mathematics provides us also with the opportunity to discuss with them how to integrate mathematical softwares in their teaching toolkit.

A course of Geometry for teachers: the choice of a historical reference

The course we are experimenting with Mathematics teachers (both at graduate and undergraduate level) discusses the contents and notes of the "Elementos de Geometria", by A. M. Legendre (1809), and some of descendants of this book. Incursions into more modern treatments and contemporary results are made when appropriate.

There were many different reasons for this choice, of which we mention only a few:

- the text was written by a mature mathematician, at a time close enough in history that we have a fair idea of what was known by him at the time of writing. Some results were new then, as the proof by Lambert that π is an irrational number (the proof that π is transcendent took a while longer, even though Legendre sounds convinced that this is so in his notes). The question on the ruler and compass constructible polygons was settled by Gauss a few years after the first edition was published (in 1794), and this information is included in the editions afterwards.
- and, of course, there is proposition XIX of book I, where Legendre tries to prove, unaided by the fifth postulate of Euclid, that the sum of the internal angles of a plane triangle equals two right angles. Throughout different editions he changes the proofs, each one of them beautiful, and each resorting to a hidden postulate (discussed by him later in a note). The use of apparently correct proofs to exercise the critical judgement of mathematics students was an established habit in soviet schools (see Bradis, Minkovskii & Kharcheva (1999) and references therein).

Legendre's proofs are a more subtle challenge than the geometrical examples in the last mentioned reference, where the absurdity of the statement can be made immediately apparent by a carefully drawn figure. Contrast this with Legendre's proposition XIX, where the statement refers to a result known as correct, the writer has the authority of a classical master and, if we try to check every step of the proof with a (Euclidean) plane geometry computer package, the software will (have to) confirm the truth of every statement. The history of the birth of hyperbolic geometry alone would justify von Neuman's ([1961], quoted in Artmann, 1999) careful choice of words, before saying *"that mathematical ideas originate in empirics, although the genealogy is sometimes long and obscure"*. Mathematicians had to know what geometry they wanted before they were able to devise models for hyperbolic geometry. It was almost irresistible to succumb to the authority of "the nature of the straight line", even for mathematicians of the calibre of Legendre.

The different attempts by Legendre to prove that the sum of the internal angles of a plane triangle equals two right angles, or to prove a postulate equivalent to the parallel postulate, provide good illustrations of the dialectic relation between figures and proofs. In his case, the figure cannot provide the geometrical information he needs, as this depends on knowledge not available to him when exploring the figure. Instead of helping, the figure he uses implicitly suggests information he then proceeds to use, and destroys his argument as a mathematical proof.

The role of Dynamical Geometry Softwares as a Tool for searching for solutions

Among the modern treatments covered in some detail during the course, we include transformations. In particular inversions, so we can construct the Poincaré model of the hyperbolic plane and make the flaws in Legendre's argument adequately clear. The treatment at this point is greatly aided by resorting to dynamic geometry software to aid in the visualisation of the meaning of theorem statements and proofs.

The guiding principles we try to enforce when using Dynamical geometry softwares in the course can be summed up in the words of Archimedes, contained in the foreword to his "The Method of Mechanical Theorems": "... *a certain special method, by means of which you will be enabled to recognise certain mathematical questions with the aid of mechanics. I am convinced that this is no less useful for finding the proofs of these same theorems. For some things, which first became clear to me by the mechanical method, were* afterwards proved geometrically, because their investigation by the said method does not furnish an actual demonstration. *For it is easier to supply a proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge*." [our italics] (Dijksterhuis, 1987, pp 312, 313).

The frustration felt by many commentators of the classical Greek texts on geometry, with the "*tantalising ... absence of any indication of the steps by which they worked their way to the discovery of their great theorems*" (Heath, introductory note to The Method, pg. 6) must be paralleled by many readers of geometry textbooks. That dynamic geometry softwares can be a nearly ideal tool to engage students in activities leading to formulation of useful conjectures has been said by many. Taken to extremes, the deformation of the usage seems to lead some to propose to do away with mathematical proofs altogether (for a sober discussion, see the introduction in de Villiers, 1999). Instead of adding to this, we present two examples. The first comes straight from Legendre's text (the appendix to book IV), but we use the English wording of Wentworth (1938), a formerly popular American textbook which went through tens of editions:

"Theorem: *Of all polygons with all sides given but one, the maximum can be inscribed in the semicircle which has the undetermined side for its diameter*"

As stated, the method of proof seems almost mandatory: suppose at least one vertex is not on a semicircle, and check that pulling it to comply will increase the area. Notice the format of the statement though, which models the majority of variational problems which cropped into geometry textbooks after Legendre. The reader is *given* the solution, and asked to *prove* that it is the right one.

That is also the case of our next example but, this time, instead of fully reproducing the statement in F.G.-M.(1920, pg. 768), we shall withhold part of the information:

"*Given that E is a fixed point in the interior of the convex angle $\angle A$, find the smallest segment BC, joining the two sides of angle $\angle A$, which passes through E.*"[2]

There is no tool in geometry comparable to the use of the derivative to *find* the conditions which must be satisfied by segment BC, and that is why the original statement goes something like "prove that the segment with the property so and so is the right one". In this case, though, the textbook tells us how the original author found the right segment: Newton proposed this problem

[2] The segment BC is often called Philo's line.

after having solved (through calculus) the more general case where, instead of being in an angle, E is a point in the region between two given curves.

But the lack of calculus is no reason to postpone the investigation of this type of problems. Dynamic geometry softwares can aid us in two crucial steps towards the right solution:

- disproving wrong conjectures we make along the way. The segment orthogonal to the angle bisector is a frequent guess in our course when students are trying to solve this last example, as is the segment tangent to the inscribed circle passing through E.
- helping to make apparent the features common to the solution of particular cases (E in the angle bisector, or E in one of the sides of the triangle, for instance).

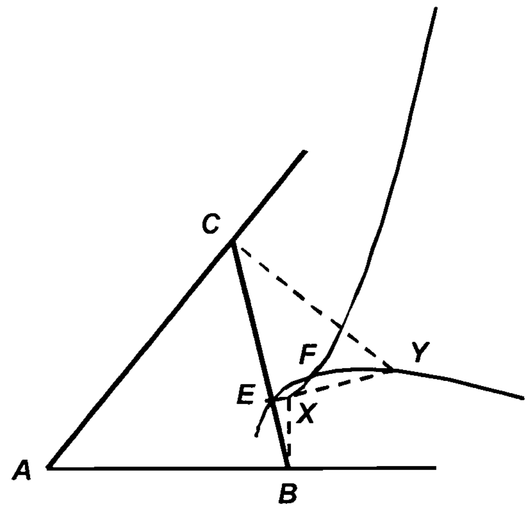

Figure 1: the smallest segment BC, joining the two sides of the angle, which passes through E is obtained when points X and Y coincide with F.

In this last example, the software might even steer us into a way to *construct* the right segment, which we would not get with the proof alone. As it turns out, to construct the right segment we have to construct first the point F in Figure 1. B is then obtained projecting F on the side of the angle. F is obtained as the intersection of two parabolas, the loci of points X and Y in the figure, point Y being the intersection of the perpendicular through E to segment BC with the perpendicular through C to the side AC of the angle. Point X is similarly constructed from the side AB. Point F it is not constructible with ruler and compass, but a paper folding construction is possible, and that in itself leads to interesting discussions. [3]

An Informal Evaluation of Outcomes of the Course

In assembling the course materials there were several issues to be resolved that themselves constituted worthwhile parallel teaching experiments. Not the least of those was the preparation of a new edition, both in paper and electronic, of Legendre's translation into Portuguese, published originally in 1809. All the retyping and language adaptation was executed with and by future mathematics teachers. The reflections sparkled in them by this work were so rewarding that we are

[3] See also Cuoco and Goldenberg (1997) for different examples and a complementary viewpoint.

now starting work of the same kind with other groups and other books. But a description of this would take us away from the main object of this paper.

Instead of this, let's report on the subsequent careers of two groups of students who took part on preliminary versions of the course. One group is formed by undergraduate students and the other is formed by three teachers: one had just then graduated, and the remaining two are experienced teachers, coming back to the university after fifteen years or more of practice, for a graduate course.

The three teachers will have completed their M.Sc. degrees by the time this is read. The two more experienced teachers, who work in highly respected schools in Rio de Janeiro, have decided to prepare their master dissertations as texts other teachers could use to complement their views on axiomatic approaches to geometry. One of the dissertations discusses the axiomatic required to treat paper folding as a mathematical object. It also includes a comprehensive research on published theorems that can be derived using this approach. The other dissertation studies the hyperbolic versions of traditional Euclidean geometry theorems.

The remaining teacher, the one who had just graduated, also decided for geometry as a theme for his dissertation. He studied the generalisation of results of Steiner on the Simson line when the triangle is inscribed in a conic, instead of in a circle. Projective geometry arguments are used throughout the work to generalise the intended results.

As for the undergraduate students, four of them (out of a group of twelve) have decided to write their final undergraduate essays in Geometry. One of them is tackling the solution of maxima and minima problems by geometrical methods, starting with Legendre's fourth book and its appendix. The other three are now involved in the editorial project mentioned above.

Conclusions

To speed up the process of educating teachers with the needed expertise in geometry, we propose to take advantage of the momentum provided by a stronger contemporary stimulus over the educational system: the need to incorporate ICT technologies into the school curriculum. That a mathematics teacher must have access to adequate preparation to cope successfully is true in the case of ICT as well as for geometry. We propose to deal with both needs in a single program, dedicated to prepare teachers to integrate ICT into their classroom through the device of placing them on an environment where they use ICT to learn geometry. In this work we endeavoured to present a strong case for two assumptions we made when starting this project:

- the benefits of using geometry softwares as an integral tool in undergraduate and continuing education geometry courses;
- the benefits, both cultural and mathematical, of revisiting, through the viewpoint of dynamic geometry, classical results in the geometry literature.

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TOOLS FOR SYNCHRONOUS DISTANCE TEACHING IN GEOMETRY

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ABSTRACT

To tackle the problem of teaching Mathematics, using Internet for synchronous communication, our group has developed a suite of three softwares designed to provide an adequate environment. This paper describes some of the characteristics of these tools, and preliminary results of experiments conducted to test how supportive they prove to be in a distance taught course.

The first of these, MathChat, is a general-purpose tool for mathematical communication. With it we can provide tools for

- creating mathematical symbols and equations on the fly,
- symbolic algebra facilities, and
- plotting of functions and surfaces,

this set of tools is integrated with the usual "chat room" facilities.

Tabulae, the second, is a dynamic geometry software with a built in communication server. With it we can instantly set up a virtual classroom, where each student receives, in real time, each step of a geometrical construction the teacher is realizing on his own machine. During the whole process the student is free to modify or add elements of his/her own, and to voice doubts or suggestions. Students can also direct their own work for instant check by the teacher or share it with the "classmates".

Finally there is Mangaba, our 3Dimensional dynamic geometry workhorse. Written in JAVA, it shares most of the communication features available in Tabulae. As an additional feature, it is capable of generating VRML code for any scene the user may construct. The primitives available include a comprehensive repertoire of construction and intersection primitives.

1. Introduction

Internet has opened a new dimension for distance education, where synchronous, long distance interaction and information interchange between students and teachers is relatively easy and affordable. Nevertheless, synchronous communication for distance education courses still occurs mostly through Internet Relay Chats, which are essentially tools for the exchange of text messages. This conventional chat room is inadequate for mathematics, where a special language of symbols, diagrams and text was, from very early times, developed to communicate. The combination of these elements is present even in the early texts of Euclid and Apollonius. That mathematical communication and text exceeds the capabilities of the standard tools available in the Internet will be apparent to anyone who browses through some of the many discussion lists specialising in different aspects of the subject.

Of course one might argue that, as a class, mathematicians are able enough to take maximum advantage of the media such as it is, and that we communicate well enough among ourselves, thank you. The long lifespan of some of the discussion lists mentioned above could be evidence of just that. But, much as a master chess player can mentally play several opponents at a time without resorting to a chessboard, it is doubtful that he/she would advocate this as the preferred medium for teaching beginners, especially at a distance.

The ever increasing use of TEX among the community, on the other hand, seems to indicate that, when provided with a more appropriate tool for the job, our average mathematician will prefer not only to read text with the proper symbols, but he/she will take the trouble to set his own text to make it more presentable to the circle of peers. So maybe our master would use a chessboard at home after all.

Diagrams and figures are another matter altogether. Sometimes we are suspicious of them, maybe most times we use them badly [C1] but, when using the blackboard for teaching we so impress engineers that when they try to think up better alternatives for us they come up with oddities such as the "geometer's workbench" [G]: a panel driven by several computers and several LCD projectors that tries to better us at producing diagrams. So we assume that, at least for teaching, most of us would be only too happy not to have to forsake our special symbols and careful use of diagrams when we are finally driven to use the internet to communicate with our students at a distance.

In doing this we cannot rely on complex apparatus such as a mathematician's workbench, even it was not unavailable. We must take into account real world limitations and, whenever possible, preserve the formats and tools we already use. For the remaining of this paper we envisage a situation like the scheme depicted in Figure 1, with each student possibly at home and connected to the Internet through a narrow bandwidth telephone modem.

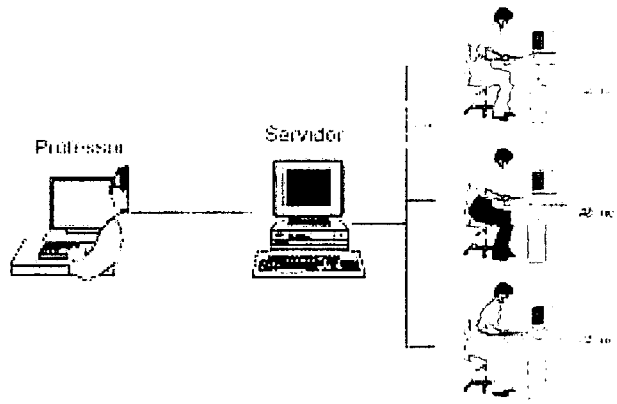

Figure 1: A simple scheme for a synchronous lecture.

2. A double edged approach to formulae and graphics: MathChat

The last ten years have seen an enormous increase in the use of Computer Algebra packages for teaching mathematics, to the extent of developing specialized interfaces to facilitate their use in specific courses and pedagogical applications (see for instance the Metric project, at – http://www.metric.math.ic.ac.uk).

Most educational experiments with CAS are intended for the student on campus, at special purpose facilities. Our approach with MathChat, on the other hand, is designed so that the student will have access to the same level of facilities whether on campus, at home or anywhere he/she can connect to the internet. It provides an environment where three complementary means of input are available: a text area similar in most respects to what is available in a "normal" chatroom, an area where command lines can be uploaded to the server, and a formula generation area, where the user can upload formulas and mathematical symbols. The result is displayed to the participants of the chat in a single window, much like a conventional chatroom, except that the messages will include a flexible array of mathematical expressions, graphs, and even animations.

An important constraint for the design brief of MathChat was the demand that all interchanges of information through the network were to be lightweight enough so that sessions are fully interactive, even through low bandwidth networks. That's why mathematical expressions are transmitted in Latex format, to be rendered locally at each client machine.

MathChat is designed so that it can use a variety CAS as mathematical engines. The current version uses Maple, running on a server machine. In this case the "speaker" uploads a Maple command, or sequence of commands. This is processed at the server, and the final result is transmitted to all participants. If the command generates graphical output, this is sent to the client as a GIF image. Again, mathematical expressions generated by Maple are transmitted to the clients as Latex code, and rendered locally using a version of the WebEq library. See Figure 2 for a screenshot of the work area of the software.

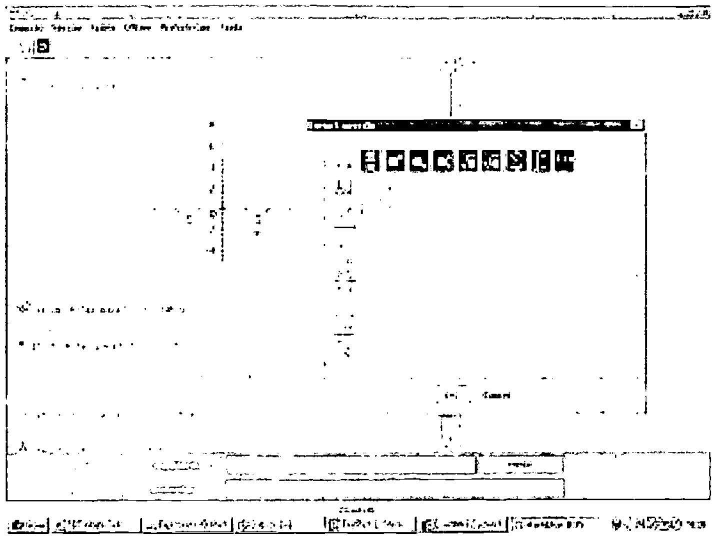

Figure 2: A screenshot of MathChat.

An important test bed for this software has been a variety of continuing education courses for high school teachers of mathematics. In this case the lecturer, a fellow mathematician who was also a highly experienced Maple user, would interact with groups of up to fifteen teachers at a time. What surprised us most in the experiment was the intensity of the exchanges. Contrary to what we were led to expect by the literature available on the use of chats for distance teaching, the lecturer could barely cope with the demand generated. The main bottleneck of the system was the viewing area available in the personal computer as the rate of messages grew, and great care was demanded to ensure that no question would go unseen.

As the MathChat project unfolds, we expect the software will be used in a variety of ways. Some groups of users currently running tests to start use in the near future comprehend:

-tutors in distance taught mathematics and physics courses;

-tutors and lecturers in a variety of campus based disciplines, as an aid in the task of helping groups of students with their assignments;

-study groups where students get together at set times in cooperative work centered in different course contents.

3. Tabulæ as a tool for distance taught plane geometry

When the issue of distance teaching is considered, synthetic geometry poses some interesting questions. Consider the examples proposed in [C1] and [C2] as samples of "good" use of figures in mathematical text. Their use is strongly reminiscent of the use of the blackboard by a successful teacher: the drawing is effected step by step, and the student is called in by the instructor to direct attention to the portion of the drawing that is relevant at a given moment. Animations that can be played step by step are an improvement over static figures, and it is quite easy to prepare routines that will do just that on the canvas of a dynamic geometry software. Or they can be written as Java applets, as in [M], where Casselman's prescription for the presentation of Euclid's proof of Pythagoras' theorem is beautifully transposed to an electronic media.

But we wanted an instrument which would give us, for geometry, the same facilities we have with MathChat to perform constructions on the fly. We wanted, in other words, a "blackboard" that would be available at the screen of each student in a lecture delivered live through the Internet, and where the instructor could write each construction step by step, not necessarily according to a previously determined script, but possibly as an impromptu response to a student's request.

We also wanted more than that: we wanted the student to preserve at all times the complete control of his/her machine, to be able to experiment and propose variations, or even to diddle, in the way he might do while taking notes in a classroom. Therefore a protocol were you surrender control of your machine at the same time you give another person the right to write in it at a distance was out of the question. That precludes the use of general purpose communication tools like NetMeeting.

What we did was to build, into a purpose written dynamic geometry software, an Internet communication server integrated with a "Chat" tool. Its range of uses comprehend:

- as standalone software, for individual use, with the full set of functionalities of a conventional dynamical geometry software;

- a communication vehicle for lectures delivered live through the Internet;

- a vehicle for collaborative and group study, with no constraints posed on relative geographical location of any participants or network bandwidth.

The tool allows a model of Web delivered lecture where, in the same manner as in a traditional geometry classroom, the teacher can gradually construct a geometrical object, while explaining the related content either through an integrated chat window or through a voice channel. A video link is also possible, if the connection bandwidth available to the students permits.

The model puts the teacher in full control not only of each step of construction, but also of the timing to be employed to effect it. Each step is instantly available to every one of the students in the class, who can even experiment with variants of their own while the construction is in progress. The teacher may get feedback from the students through the voice or chat channels, and can even receive a student's work for exam upon request.

During a problem solving activity a teacher might hand over to a given student the prerogative of transmitting to all his colleagues, for instance. Outside classes students might use the same tool to organize themselves in study groups.

The principle that makes a fast rate of communication viable even through slow networks is that, at no time, we transmit large packets of information. Images, for instance, do not have to be transmitted. Instead, the specifications of a given geometrical object are passed on to the recipient's copy of the software, which them proceeds to construct an exact equivalent on the local computer. To give only an example: to specify a perpendicular bisector on the student's machines, the size of the information we need to transmit is under 30 bytes. Even a complex object like a geometrical locus doesn't use anything significantly larger than that.

The objects constructed by the instructor at the student's screen are fully functional dynamical geometry constructions, which the student can modify, use as stepping stones for a more complex construction, etc. Of course they can also be saved, as well as the entire session, for later study. Figure 3 shows a screenshot of Tabulæ, including the message line.

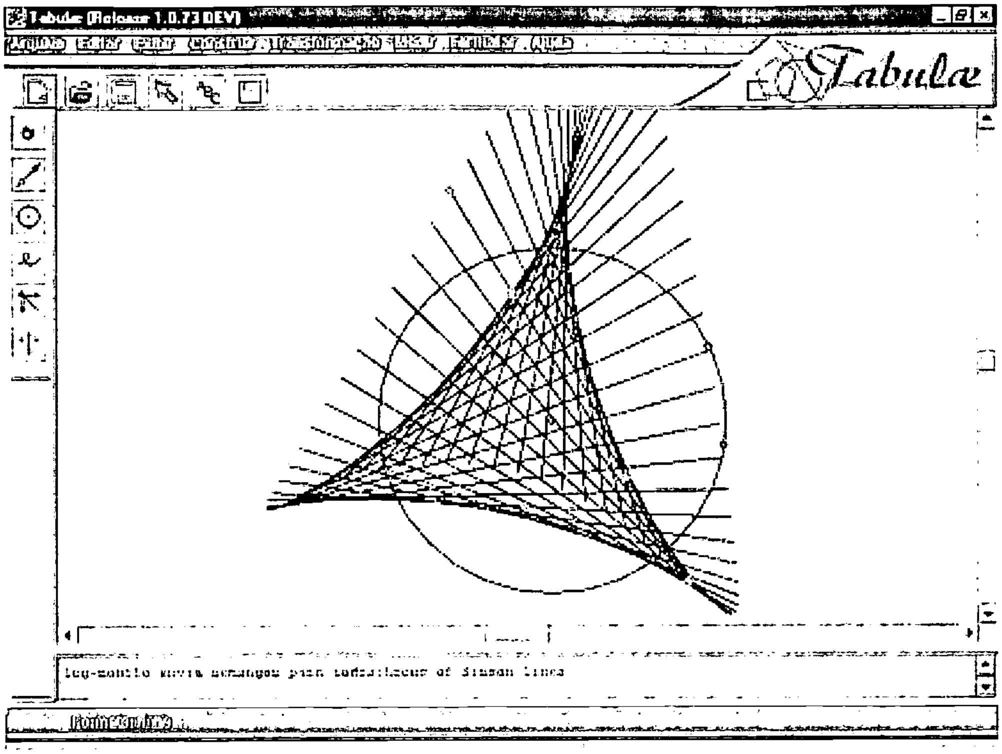

Figure 3: A screenshot of Tabulæ. Notice the message line.

4. Mangaba: a tool for 3Dimensional geometry

For 3Dimensional geometry, there are no established examples in the market as yet. The conception of a fully functional dynamic geometry software for space geometry is a complex undertaking, and it is only recently that we were able to progress to a point were we can confidently say that we were successful. A full description of the software will be given elsewhere in a forthcoming paper. Figure 4 shows a screenshot of Mangaba.

This is not the place to discuss the difficulties in learning and teaching space geometry. We direct the interested reader to [B], for an account of very basic difficulties of young children, and to [Ch], for a fine analysis of the evolution of the teaching of the subject in France during the last century.

To give an idea of what we are endeavoring to achieve with Mangaba we quote from Hadammard ([H] , vol. II, Book V, pg. 83):

"In all construction problems we assume, save express indication to the contrary, that we can

- *construct a line in space given two points;*
- *construct a plane given three points in space;*
- *find the intersection of two given planes, an of a straight line with a given plane;*
- *perform, in any given plane in space, all the known constructions of plane geometry.*

This assumptions are, of course, merely conventional, and there is no way to perform them in practice. Nevertheless, Descriptive Geometry teaches us to represent, by way of plane *figures, the figures in space and, in this mode of representation, the constructions referred above can be performed with straight ruler and compass.*

We shall call effective constructions those required to be performed with no recourse to the preceding convention."

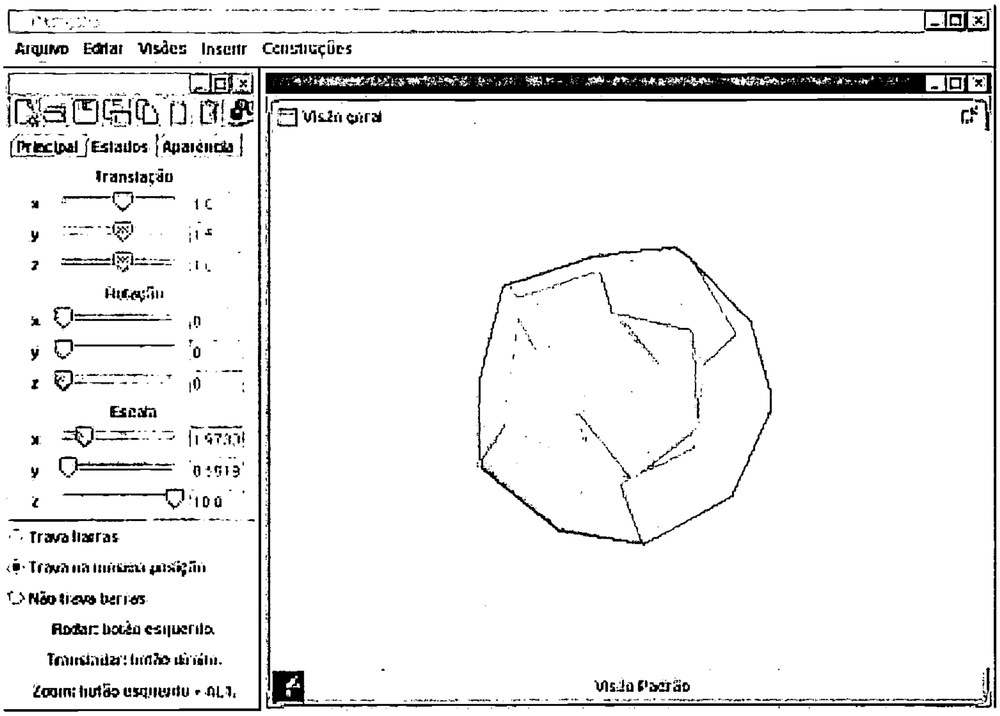

Figure 4: a screenshot of Mangaba.

The excerpt above gives us some indication of the level of abstraction we are dealing with when teaching space geometry, even in simple problems. By way of an example, think of the following exercise, found in any of the classical textbooks of the nineteenth century an, even today, in many textbooks, for instance in France:

"Suppose you are given two skew lines in space, and a point P not in either of them. Find a straight line through P that intercepts both given lines."

There are several interesting issues that can be raised by an exercise like that. Some have to do with the status of the diagram depicting the situation, and for that we refer the reader again to [Ch]. But let's instead think for the moment of the first steps towards solution to the problem: one could think of the data and come up with: the point P, together with one of the lines, gives us a plane. "Construct" (in the sense of Hadammard) the intersection of this plane with the other line. That, if it's well defined, gives you a point Q. The line joining P and Q is the sought for solution.

The point here is that the whole operation has to be conducted as a thought experiment. Even after you come to a successful solution, there is very little to show, as the "construction" you achieve is still a thought experiment, and cannot easily be made more concrete. Where you trying to teach this at a distance, there would be very little you could show to help a student come unstuck.

Mangaba deals very easily with simple constructions like this. At the same time, the repertoire of geometrical objects in space geometry, apart from being much larger than what you find in the plane, can include very complex objects. Just think, for instance, of the diverse stellated icosahedra. If we want them in the same footing as text in the available vocabulary for a synchronous interchange through the Internet, we clearly have to specify with care our exchange dialect. The procedure we have adopted is a little different from the one we use with Tabulæ, but it allows us to specify a complex object, or a geometrical transformation performed on it in the student's machines, with very little more expense in terms of bandwidth than for the case of the simplest object, i.e. the point.

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EFFECTS OF USING CALCULATORS (TI-92) ON LEARNING TRANSFORMATIONAL GEOMETRY

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ABSTRACT

The present study aims at finding out the effects of using the advanced calculator TI-92/Cabri in teaching transformational geometry on teacher students' attitudes, geometric thinking levels and achievement. The subject of study, i.e. the participants in the introduction to geometry course, were 78 students from freshmen elementary school mathematics education students at Hacettepe University, Ankara, Turkey. Three instruments were used in the present study to find out the relevant factors and effects of TI-92/Cabri aided/supported geometry teaching. The data were gathered by means of the designed instruments and analyzed by using PC-SPSS. In the analysis, several tests were used in order to understand the effects of various factors on the attitudes and achievement. This statistical analysis compares the mean scores of each group and reveals whether this differences significant or not.

Key words: Transformational geometry, TI-92/Cabri, Geometric thinking levels, Attitudes

Introduction

Among geometry topics, transformational geometry is the most interesting one, and it is one of the cores that support students' creativity in geometry courses. In this topic, students are expected to learn how to transform any regular or non-regular geometric shapes in 2-D space, and discover some features of shapes, rules, etc. Flipping, sliding, rotating, translating, and reflecting can occur transformation. Representations in this area of the geometry make an abstract interpretation, which can be easily understood in the real life applications. Students can have the opportunity to see connections within mathematics, between mathematics and the various areas of human activity and can develop an understanding of the types of reasoning that form the basis of mathematical thought (Natsoulas, 2000). They can also make connections between mathematics and art which makes the life livable. Furthermore, six mathematical activities all cultures participated in the past and today are counting, locating, measuring, designing, playing and explaining. Among all these activities, designing results in richest and most diverse outcomes, which is an outcome of transformational geometry. For example, in the pattern of a Turkish rug, one can see repeated, reflected, translated or rotated objects. Understanding those beautiful patterns by the help of transformational geometry make students gain positive attitudes towards both this topic and mathematics.

The main purpose of this study is how to teach an extracurricular geometry topic to the freshmen grade teacher students by the help of the advanced calculator TI-92/Cabri. Since the instruction of transformational geometry is almost impossible by just using blackboard and chalk, planning the help of an instructional technology and mathematical tools is essential. But we should be aware of the results of the introduction of the technology and the implementation. Therefore, at the end of the implementation, we tried to find out the effects of using TI-92 in teaching transformational geometry on students' attitudes and geometric thinking levels. Thus, the research question of this study is *"What are the effects of using TI-92 in teaching transformational geometry on teacher students attitudes, geometric thinking levels and achievement?"* One of the results of the investigation might be finding out the more effective ways of teaching geometry and the discussion of the implementation of the technology. Since the presentation of geometry topics in Turkish schools is far from the visual activities, which are vitally important in teaching geometry, the instruction should change. Although such instruction may not meet the requirements, students are expected to develop their spatial sense and geometric thinking levels.

2. Background: A Short Overview of Teaching Geometry in Schools and Use of Technology

Geometry is an interesting area of mathematics and can enhance the students understanding level in the other areas of mathematics. However, students generally do not like the geometry and are unsuccessful at it. There might be many reasons of this negative attitudes and underachievement. Here we review the issue very briefly and notice the use of technology in teaching geometry.

2.1. Teaching Geometry in Schools

Although the national mathematics curriculum of USA and UK strongly recommend that students should study transformational geometry, this topic is not offered in Turkish National Mathematics Curriculum (TNMC). Actually whether a topic exists in the curriculum or not, the most important criteria for gaining an attention for a topic is the university entrance examination in Turkey. This examination determines which topics are taught at the schools, what aspects of topic are emphasized and de-emphasized, and the instruction method in general. The topics, which are evaluated at the university entrance examination, have the great emphasize while the others are mentioned superficially, even sometimes skipped. Since transformational geometry neither exists in the TNMC nor is asked at the university entrance examination, this unit is totally undermined. Since the TNMC is so loaded, at first glance it is not seemed reasonable to teach that 'struggled' topic. However, mathematical reasoning is not isolated for every topic. The mathematical reasoning is a process rather than a product. This means if a person can reason in one mathematical concept, he/she can use that way of thinking in the other area of mathematics. Therefore, the progress a student makes in transformational geometry would affect his/her perception, attitude and achievement of both geometry and mathematics. Another possible reason why we do not offer this topic might be the difficulties of presentation of this topic in the classroom environment. In order to teach this topic by traditional method, the teacher has to draw the objects very clearly and carefully on the blackboard, which makes the presentation of this topic is so difficult and requires an additional skill from the teacher. As an additional help, some instructional technology like educational software or graphing calculator can be helpful for this situation.

2.2.Use of Technology in Learning/Teaching Geometry

In geometry lessons, students do the activities of constructing and drawing patterns and relationships. Many times these constructing and drawing activities are so difficult without any technological help. They need software programs for computers or advanced graphing calculators. Physical aspects such as speed, color, screen resolution etc. make the software programs preferable for individual use, but compromises have to be made when providing mass education. In addition to this computer programs and computers are generally much more expensive than calculators. Small size and easy usage are the other important preferable aspects of calculators over computers. They could be viewed as computers available to all students because of their low cost, ease of use, and portability (Waits & Demana, 2000). Calculators can play an important role in students' construction of mathematical relationships (Wheatley, 1990). Increased used of calculators in school, ensures that students' experiences in mathematics match the realities of everyday life, develops their reasoning skills, and promotes the understanding and application of mathematics.

There is growing evidence that paper-pencil manipulation skills or just blackboard instruction do not lead to better understanding of mathematical concepts. Indeed use of hand-held calculators can provide more classroom time for the development of better understanding of mathematical concepts by eliminating the time spent on 'mindless manipulations.' As Podlesni (1999) stated they remove the unnecessary, tedious and time consuming tasks, thereby allowing students to 'see the forest for the trees.' Therefore, the main advantage of using calculators during instruction is to help reduce the load on students working memory so those more significant problems can be enhanced. The use of calculators creates a computational advantage as well as helps them to improve their selections of appropriate problem solving strategies. Learning is fun and the changing technology gives students a change to watch their teachers share in that joyous adventure (Usnick, Lamphere, Bright, 1995).

3. Method and Implementation

Here short information about the method of the research and the instruments used in the present study are given.

3.1. Purpose, Problem and Hypothesis

Purpose: This study aims at to find out the effects of using advanced graphing calculator, namely TI-92/Cabri, in teaching transformational geometry on the freshman teacher students' attitudes, geometric thinking levels and achievement.

Problem: The research question is stated as "What are the effects of using TI-92/Cabri in teaching transformational geometry on teacher students attitudes, and geometric thinking levels.

Hypothesis: From this research question, we hypothesized the following two statements:

- **H0(1):** The mean score difference pre and post implementation of attitude scale of the group are not significantly different;
- **H0(2):** The mean score difference of pre and post implementation van Hiele geometric thinking level test of each group are not significantly different.

3.2. Instruments Adapted and Developed

Three instruments were used in this study. They are van Hiele Geometric thinking level test (VHL), Geometry attitude scale (GAS), and the instructional materials.

Van Hiele Geometric Thinking Level Test (VHL) (Usiskin, 1982): In order to determine students' geometric thinking levels 25-item VHL will be used. The items represent the five geometric thinking levels proposed by van Hiele[1]. Teacher students' total score will be considered out of 25 for this test. The content validity of the Turkish version of van Hiele geometric thinking level test were confirmed by a group of a mathematician and mathematics educators. A pilot implementation on 31 seniors of mathematics department and 61 freshman and sophomore of department of computer education and instructional technology was ensured its construct validity[2]. Reliability measures of the levels of the original VHL and the Turkish version of it ranged between 0.79-0.88, 0.51-0.88, 0.70-0.88, 0.69-0.72, and 0.59-0.65, respectively.

Geometry Attitude Scale (GAS): This scale will be used in order to determine the teacher students' attitudes toward mathematics. It consists of 37, 5 point Likert type items. These items represent 4 dimensions of attitude: interest, anxiety, importance and enjoyment. Factor analyses revealed that these 4 dimensions are valid and reliability coefficient of this scale is 0,89 for the first administration and 0,90 for the post administration.

[1] First five items represent level 1, second five item represent level 2, the items number 11-15 represent level 3, the item number 16-20 represent level 4 and the last 5 items represent level 5. This instrument was translated into Turkish in during a master thesis study (Duatepe, 2000).

[2] The mathematics majors got significantly higher score than the other students.

Instructional Materials for Calculator Aided/Supported Geometry Teaching (IMCA/SGT): They are a pile of lecture notes and worksheets which were either adopted and translated into Turkish from English and designed by the researchers.

3.3. Design of Research and Procedures

Sample: Participants were 78 teacher students from the freshmen grade level at the Department of Elementary School Mathematics Education, Hacettepe University (HU), Ankara, Turkey. Because of some missing data, data from 67 students (45 female, 22 male) were taken into consideration.

Procedure: In order to test the hypotheses stated above, a pre-experimental research design was implemented and the study lasted 3 weeks in a month. The freshmen teacher students from the HU in Beytepe Campus, Ankara were taught 3 transformational units: translation, rotation and reflection in 3 hours a week, totally 9 hours. Instruction was done by the use of the teacher unit of TI-92, i.e. calculator, view-screen and OHP and sample of IMCA/SGT. In order to do this, each teacher student was given TI-92 and followed the instruction with this powerful device while one of the researcher who was the instructor guiding them. The instructor gave the directions and the teacher students tried to follow these directions with the help of TI-92 and were worked as a group of three or four. The instructor reflected the right answer on the screen placed in the room by means of an OHP so that the teacher students could see what are they expected to see on the screen of their calculators. By this way they were going to be responsible for their own progresses.

The instructor accepted the students' autonomy and initiative as in the constructivist approach so that the teacher students were encouraged to engage in dialogue both with their classmates and the instructor, and they ask questions to each other frequently. The instructor encouraged them to think by asking thoughtful, open-ended questions and reflect their thinking on the subject. During the implementation they were received some worksheets to help them clear their ideas related with what they did by the use of calculator. These worksheets were prepared by the researchers to help the teacher students on discovering some important features, aspects and rules of transformational geometry.

To evaluate the success of the introduction and integration of TI-92, both instruments, i.e. GAS and VHL were implemented at the beginning of the semester, i.e. before the teacher students were received any instruction in order to determine their prior attitude and geometric thinking level and after the treatment.

4. Analyses of Data and Discussion of Results

The data were gathered by means of the designed instruments and analyzed by using PC-SPSS. In the analysis, t-test was used in order to understand whether the hypotheses are true or not. This statistical analysis compares the mean scores of each group and reveals whether this differences significant or not.

Table 1. Descriptive Statistical Analysis of Data about VHL and GAS

Variable	N	mean	mode	min	max	Std Deviation
PreGAS	67	131.67	145	98	169	44.64
PosGAS	67	145.10	150	116	177	17.87
PreVHL	67	14.04	15	10	22	5.05
PosVHL	67	15.96	17	12	22	2.57

As it is seen from Table 1, student teachers got higher scores from the post implementation of the measures. In order to see whether this differences are significant or not, t test was used. Independent t-test result revealed that post implementation of GAS (posGAS) ($M = 145.10$; $SD = 17.86$) is significantly higher than pre implementation of the GAS (preGAS) ($M = 131.67$; $SD = 44.64$). The t test result can be seen from Table 2. As it is seen from that table, this difference was significant at $\alpha = .05$, ($t(67) = 2.723$; $p<.08$).Therefore, the first hypothesis was rejected. In other words, there is a significant difference between pre and post implementation of Geometry Attitude Scale.

Table 2. The t- test Results of Student Teachers' Scores

Paired Difference	PDM*	Standard Deviation	Std Error Mean	95% Confi. Interval	t	df	Sig. (2-tailed)
PosGAS- PreGAS	13.433	48.034	-25.149	-25.15/-1.72	2.289	66	.025
PosVHL-PreVHL	1.910	5.744	-3.312	-3.31/-.51	2.723	66	.008

*PDM: Paired difference mean

On the other hand, independent t-test result also showed that post implementation of VHL (posVHL) ($M = 15.96$; $SD = 2.57$) is significantly higher than pre implementation of VHL (preVHL) ($M = 14.04$; $SD = 5.05$). According to Table 2, this difference was significant at $\alpha = .05$, ($t(67) = 2.289$; $p< .05$). This means that hypothesis H02 was also rejected. In other words, there is a significant difference between pre and post implementation of van Hiele Geometric Thinking Level test.

5. Concluding Remarks

As it is seen from the previous part, both hypotheses, H0(1) and H0(2) of the present study were rejected. In other words, using TI-92/Cabri in teaching transformational geometry has a significant positive effect on student teacher' attitudes, and geometric thinking levels.

Related with hypothesis H0(1) it is observed that classroom became a real learning environment. Student teachers were more active and problem solvers. Learning was fun and more exciting in that environment. Therefore attitude toward what is learned by calculator was increased as the previous researches (Dunham, 2000).

As hypothesis H0(2) rejected, it was stated that student teacher got significantly higher scores on VHL after the instruction. It means that the calculator has a positive effect on the VHL test score. However, if the result is investigated deeply, this significant effect will be meaningless to some extents. Table 3 shows the detail of the analysis of scores that the student teachers got on the pre and post implementation of VHL.

Table 3. Descriptive Statistical Analysis of Data about VHL

Variable	N	mean	mode	min	max	Std Deviation
PreLevel	67	2.134	2	0	3	1.028
PostLevel	67	2.433	3	1	3	0.783

It can be seen in that table that student teacherss' van Hiele Geometric thinking level is somewhere between 2nd and the 3[d] level on both pre and post implementation of the VHL. So it can be said that the scores on van Hiele Level Test increased significantly during instruction, but this increase is nor enough to get next van Hiele Geometric Thinking Level.

Moreover, it can be also concluded from the present study that teaching geometry by TI-92/Cabri technology is more effective. On the other hand, the presentation of topics in transformational geometry in the classroom environment was easier than by traditional method. The instructor/teacher did not have to draw the objects very clearly and carefully on the blackboard. Hence, this struggled topic can easily be taught in classroom environment in Turkey and elsewhere.

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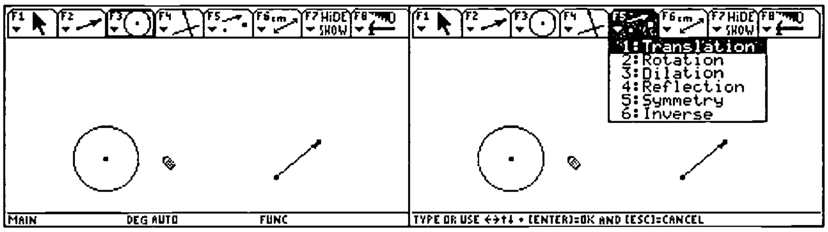
APPENDIX. ACTIVITY ON TRANSLATION

Aim: At the end of this activity series, students will be able to translate triangle, quadrilateral and circle by means of calculator.

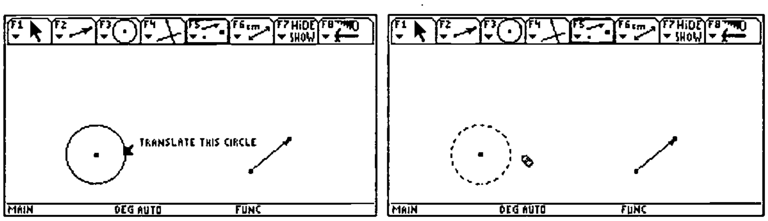
A: Translating a Circle
Objective: At the end of this activity series, students will be able to translate circle by means of calculator.

Script:

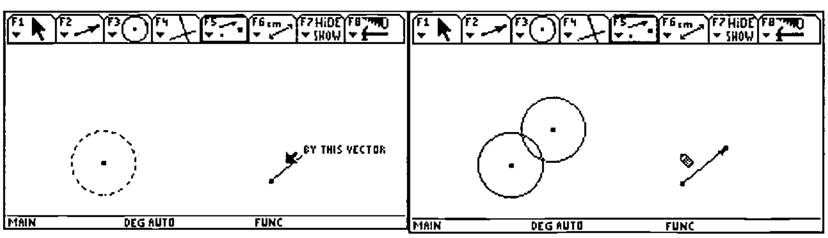
1. Construct a vector (F2, *7: vector)* and a circle (F3, *1: Circle*). Determine the radius of a circle by pressing the arrowsof the big blues button. Then select *Translation* by pressing F5.



2. Select the circle as an object of translation. Move cursor to see 'Translate this *circle*' on the screen. When you see this on the screen press '*enter*'. By this way you can select the circle as an object of translation. If you have done this correctly, the sides of the circle would turn into rounding dots.



3. Construct the vector which determine the direction and the magnitude of the translation. (Move cursor till seeing *'by this vector'* on the screen then press '*enter*') After that you can see the translated circle and your original circle in the screen as you see from below figure.



WRITTEN META- COGNITION AND PROCEDURAL KNOWLEDGE

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ABSTRACT

The ability to express one's mathematical thoughts in writing and computational proficiency can be viewed as reflecting different aspects of an individuals' understanding of mathematics. Computational proficiency is the primary means used by educators to assess student's understanding of mathematics and thus in the mathematics classroom cognitive development is measured for the most part through students' ability to apply their procedural knowledge in a problem solving environment. In contrast, in written mathematics one's thoughts are not so much involved with the application of procedural knowledge as with reflection upon the concepts and procedures themselves.

In this article we analyze the relationship between an individual's ability to apply their procedural knowledge and the ability for meta-cognitive reflection and conceptual thought during written mathematics, with writing exercises designed in accordance with the framework for conceptual development of Sfard. (Sfard,1992,1994) and graded according to the scoring rubric set forth in Countryman. (Countryman,1992) This teaching research was done at a community college with students enrolled in the remedial courses of elementary algebra and basic mathematics who frequently displayed difficulties with computational exercises.

Key words : conceptual, procedural, cognitive development, mathematics writing.

Anderson's model of learning.

Within cognitive psychology, thought or cognition is typically viewed as containing both procedural knowledge and conceptual knowledge. For Anderson, procedural knowledge is, "knowledge of how to do things," it is frequently unconscious, and is housed in task-orientated structures called production systems. In contrast, conceptual knowledge, what Anderson calls, "declarative knowledge," is "knowledge about facts and things," it conscious and fully housed in the hierarchical structures called schema, organized by degrees of generality. (Anderson,1995).

In Anderson' s model, learning begins with actions on existing conceptual knowledge and with increasing knowledge the individual begins to internalize the procedures involved, incorporating them into productions systems, leaving aside the conceptual knowledge upon which the procedures arose. A process called "proceduralization." Thus, the acquisition of procedural knowledge is dependent upon existing conceptual knowledge and the knowledge gained by the repeated use of procedures or actions. (Byrnes and Wasik,1991)

Piaget and Vygotsky

Sierpinska in (Sierpinska,1998) contrasts the positions that Vygostsky and Piaget have on the ability of writing to influence thought and development. She states that, "Piaget would not claim that the activity of communication can change the course of development. On the contrary, he would claim that development is a precondition for a person to express him or herself clearly in writing." In contrasts, speaking of Vygotsky she states: "Vygotsky was claiming that writing can have an actual impact upon development."

Piaget Model

One explanation for Piaget's position is based on the relationship between procedural knowledge and conceptual knowledge inherent in his model of learning and development. Piaget would essentially agree with Anderson that learning begins with actions on existing conceptual knowledge and for both Anderson and Piaget an individual's ability to internalize procedural knowledge is an essential component in learning. However, for Piaget the relationship between procedural and conceptual knowledge is more complex, because in Piaget's view, after the individual gains proficiency with and internalizes procedural knowledge he or she begins to reflect upon this process and as a result gains new conceptual knowledge. (Byrnes and Wasik, 1991) In particular, for Piaget conceptual knowledge and procedural knowledge are both integral parts of a single cognitive schema, they are not separate. Thus, with models of learning based upon Piaget, concepts are assimilated into cognitive schema. Furthermore, this assimilation occurs in the advanced stages of cognitive development , which are characterized by abstract, metacogntive reflection and conceptual thought and are dependent upon completion of the first stage, the internalization of procedural knowledge. (Sfard, 1992)

This characteristic of models of development based upon the work of Piaget has lead the authors of (Haapasalo and Kadijevich, 2000) to hypothesize that such models subscribe to the "genetic view," which states that, acquisition of conceptual knowledge is dependent upon efficiency with or internalization of procedural knowledge. Therefore, the genetic view supplies a good hypothesis for Piaget's position that, written mathematical thought is dependent upon cognitive development.

Vygotsky model

For Vygotsky writing and algebraic thought are similar in nature because they both require conscious reflection upon previously unconscious or intuitive thought. More specifically, for Vygotsky, both writing and algebraic thought, involve conscious reflection upon what he would call the "spontaneous concepts" of speech and arithmetic. (Vygotksy,1986)

Thus, for Vygotsky written thought and algebraic thought are linked together or related by conscious reflection furthermore, unlike Piaget's model, which requires procedural efficiency before metacognition and conceptual thought, for Vygotsky algebraic thought begins with conscious reflection upon existing unconscious or "spontaneous" conceptual knowledge. Specifically, for Vygotsky written mathematical thought is not dependent upon procedural knowledge or cognitive development rather, it is an active agent in promoting such growth.

Research Question

In contrast to previous research directed towards establishing the effect that written mathematics has on promoting conceptual development or mathematical maturity we analyze the relationship between a student's ability to apply their procedural knowledge and his or her ability to reflect upon such knowledge during written mathematics. (Bell and Bell, 1985), (Lesnak, 1989) (Ganguli, 1989) In order to analyze this relationship, we measure procedural knowledge by the students' course average and we measure meta-cognition and conceptual knowledge through students' scores on writing exercises through out the semester. These two measurements of knowledge are then used as independent variables in a multivariate statistical model with cognitive development, measured by the students' GPA as the dependent variable.

Our goal was to analyze the relationship between written mathematical thought and procedural knowledge in terms of the contrasting viewpoints offered by Piaget and Vygotsky. On the one hand Piaget's position interpreted as the "genetic view" that, meta-cognitive reflection and conceptual thought during the act of writing are dependent upon procedural knowledge. On the other hand Vygostky position that conceptual thought and meta-cognitive reflection during written mathematics are beneficial in promoting development. Furthermore, the benefit an individual derives from such reflection is independent of his or her ability to apply their procedural knowledge.

Theoretical Framework

In order to employ writing as a tool to both measure and promote conceptual development we employ the three-step model due to Sfard (1991,1992,1994), which is based upon the work of Piaget. Then, we follow Shepard (Shepard, 1993) who matches levels of conceptual development with the appropriate writing categories due to Britton (Britton et. al.,1975). In the model of Sfard, concepts are assimilated into the schema in the last stage of a three-step abstraction process.

"A constant three step pattern can be identified in the successive transitions from operational to structural conceptions: first there must be a process performed on the already familiar objects, then the idea of turning this process into a more compact, self contained whole should emerge, and finally an ability to view this new entity as a permanent object in its own right must be acquired. These three steps will be called interiorization, condensation and reification." (Sfard, 1992, pp.64-65).

Interiorization

According to Sfard a procedure is interiorized when it, "can be carried out through mental representations, and in order to be considered analyzed and compared it needs no longer to be actually performed." We match the interiorization step, with the late initial learning phase and generalized narrative writing category in Shepard's work. In this phase Shepard recommends writings that produce, "personal examples of concepts" or that explain, "definitions of procedures in one's own words." In this phase we continually asked our students to translate algebraic expressions and expressions back and forth between language and symbolic language.

Condensation

According to Sfard, "at this stage a person becomes more and more capable of thinking about a given process as a whole without feeling an urge to go into details." In describing condensation, Sfard makes an analogy to computer algorithms when she writes that condensation allows the individual to look at a procedure as autonomous, "from now on the learner would refer to the process in terms of input-output relationships rather than by indicating any operations." On the effect that condensation has on an individual's ability for abstraction she writes, "Thanks to condensation, combining the process with other processes, making comparisons, and generalizations become much easier."

We match the condensation step of Sfard with the intermediate learning phase and the low level analogic and analogic writing categories in Shepard. In this phase Shepard suggests, "explaining how to solve a problem" and further, "explaining how concepts are relates," or explaining why, "concepts and procedures either do or do not apply." Thus, at this phase students' were required to turn their meta-cognitive reflection away from the definitions or rules of the procedures and towards the conditions that govern their use, as well as the difference and similarities between procedures or conceptual objects.

Reification

In the words of Sfard (Sfard, 1992), "the condensation phase lasts as long as a new entity is tightly connected to a certain process." Of reification she notes, "the new entity is soon detached from the process which produced it and begins to draw its meaning from the fact of its being a member of a certain category." In Sfard's model of conceptual development, like all models based upon Piaget's work conceptual development takes place in the framework of a cognitive schema. Thus, the last step of reification is identified with structuring and organization of one's cognitive schema, a step necessary for conceptual development. As explained by Sfard, for an individual who has not organized their schema, "information can only be stored in an unstructured sequential cognitive schemata." In contrast for an individual with a structural understanding, their cognitive schemata has a "compact whole" thus through a process of ordering or restructuring it becomes a "hierarchical schema." Furthermore without such an ordering, "there is hardly the place for what is usually called meaningful" (Sfard,1992).

We match the reification step of Sfard with the early terminal phase and the analogic-tautologic writing category used in Shepard, who recommends writing categories that involve, "speculating about several different ways to solve a novel problem." More specifically in our work we required students to focus their meta-cognitive reflection not on the procedures and not on the rules that govern their use but instead on the strategies involved when applying procedural knowledge in problem solving. Our objective was to encourage students' organization and structuring of their cognitive schema.

Results

For slightly over 180 students (n=183) the correlation between course average and GPA was R = 0.398, which was significant at the 0.01 level (high degree of significance). The corresponding R -square value was 0.158 and thus approximately 15.8% of the GPA was determined by course average. The correlation between writing scores and GPA was 0.402, which was also significant at the 0.01 level. When we used both course average and writing scores as independent variables together to explain GPA the R-value was 0.455 and the corresponding R-square value was 0.207. Thus, approximately 20.7% of the GPA was explained using course average and writing scores. This represents an increase of 37% over the 15.8% explained using only course average. It is not to be expected that course average and writing scores in one class would explain most of a student's GPA through out their college career. However, the 37% increase of explained GPA when writing scores were added to course average is an indication of the important role written mathematical thought has in learning and cognitive development. The F-value of this multivariate model was 23.952, which had a 0.000 significance rating, thus the use of writing scores and course average resulted in a very significant model in which neither course average nor writing scores dominated the other. In particular, writing scores were not dependent upon the ability to apply one's procedural knowledge.

Analysis

We have argued that written mathematical thought by its reflective nature is predominately composed of conceptual thought and meta-cognitive reflection upon procedural knowledge, both of which characterize the more advanced stages of development in Sfard's model. In contrast, we have argued that computational proficiency is predominately composed of the ability to apply procedural knowledge, which epitomizes the initial stage of development. Our result that, written meta-cognitive reflection and conceptual thought are independent of an individual's ability to apply his or her procedural knowledge provides evidence against Piaget's position interpreted as the "genetic view," i.e., the more advanced stages of cognitive development are dependent upon completion of the first "interiorization" stage.

This result indicates that reflection upon procedural knowledge is not always a by-product of the repeated actions that characterize the "interiorization" stage. Instead, meta-cognitive reflection can proceed during the act of writing about mathematics as well as through the process of repeated actions. Moreover, this result provides evidence in support of Vygotsky's position that development can proceed through reflection, while writing, upon existing conceptual knowledge independently of the "interiorization" process, i.e., reflection due to repeated actions.

In obtaining this result we stress that we do not interpret this as evidence that, mathematical educators should ask students' to reflect upon a procedure before being asked to perform the procedure. Instead we interpret our result to be an indication that many students have the ability to reflect on a procedure during written mathematics before they are efficient in applying the procedure.

We conclude with reminding the reader that the setting of this research was a community college with a high percentage of students who had difficulties computational problems, i.e., application of procedural knowledge. This study was designed to test whether such students could use language reasoning skills during written mathematics to assist in developing their ability for meta-cognitive reflection upon procedural knowledge that would then carry over to procedural knowledge. Thus, as educators we were pleased with the results.

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A PROBLEM-BASED LEARNING APPROACH TO INTRODUCTORY LOGIC

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ABSTRACT

This paper describes the restructuring of a second year logic course at Trinity College, Dublin. The course aims to develop student skills in the propositional and predicate calculi and to encourage students to exercise these skills in applications that arise in computer science and discrete mathematics. This paper details how the teaching methods used for this course were realigned with these aims. The restructured course is being delivered for the first time in 2002 and its outcomes will be reported on in detail at the Conference in July.

1 Introduction

It is claimed in [1] that teaching students about proof using formal proof methods is superior to teaching them using "semiformal" methods. The calculational predicate logic of the text *A Logical Approach to Discrete Math* [2] has been proposed as one possible method of achieving this. This text has been recommended for the introductory logic course 2ICT5 for the last four years. This is a second year undergraduate course taken by Information and Communications Technology students in Trinity College, Dublin.

The aims of the course are to develop student skills in the propositional and predicate calculi and to encourage students to exercise these skills in applications that arise in computer science and discrete mathematics. It was felt that these aims were not reflected in the method of course presentation; too much emphasis was placed on the technical theory involved, and too little on the application of the material.

It was noticeable in both lab classes and exams that students tended to avoid questions requiring the very skills that the course tries to promote. This paper reports on major developmental work done on the course to realign the teaching methods with the course aims. The students work in small groups on substantial problem sequences, supported by the lecturer and postgraduate assistants. The students themselves, guided by the problems, construct most of the course theory. However, they also attend plenary lectures where recent themes are pulled together and coming themes previewed. Each student's work for the semester is collected in a portfolio which will form part of the continuous assessment for the course. They will also be required to submit a number of assignments and sit the usual end-of-term examination.

Major developmental work was required to produce course materials to implement the proposed restructuring. It was necessary to produce a workbook of "terse" notes, sample questions with solutions, portfolio questions and assignments. The increased emphasis on problem-based learning and problem solving should create an atmosphere where students engage with the course in a more meaningful and appropriate way. The restructured course is being offered to students for the first time in the second semester of the 2001/2002 academic year.

2 Aims and Objectives

This project aims to address a problem with the delivery of the second year undergraduate course "An Introduction to Logic". This is a 12 week course, with three contact hours per student per week. It is based on *A Logical Approach to Discrete Math* by Gries and Schneider [2]. This book employs a novel approach to the teaching of logic, teaching students to view formal logic as a fundamental and pervasive tool and encouraging them to use it in many different applications. For this purpose the authors use an equational logic, a formalization which the author of this paper has not seen in any other logic text. This lack of reference texts that use equational logic reduces the inputs to student learning significantly and is one of the areas addressed by this project.

In previous years the course was taught using the traditional method of two expository lectures and one tutorial per student per week. Reflection on this structure, course evaluation questionnaires and informal discussions with a postgraduate student who took the course during a previous academic year, have led to the conclusion that the emphasis of the course presentation needs to focus more on the application of the material. The existing emphasis of the course led students to take a surface approach to learning, and this was reflected in their preference for the more theoretical examination questions. Only 86% of students attempted the examination question based on the application of the material involved, while 100% of students attempted the purely theoretical examination question. Moreover, students scored twice as well on the theoretical question then on the application based question.

3 Implementation

The project is presently being implemented as part of the second year mathematics strand of the Information and Communications Technology degree program. A detailed description of each of the main components of the project is given below.

3.1 Course Delivery

Each one hour class involves a mixture of problem-based learning, problem solving and discussion, complemented by a small amount of blackboard teaching. Each week the students are given a sequence of problems to work through in class. They are encouraged to work on these problems with other students and to interact freely with the lecturer and postgraduate assistants.

3.2 Assessment

All students are required to keep their solutions to the class problems in a portfolio. The semester is split into two six weeks terms and the portfolio is submitted for assessment at the start of the second term. This forms half the overall continuous assessment mark for the course. Any portfolio problems which are not finished in class must be completed in the students own study time.

As well as the portfolio problems, the students are given two assignments to complete. These will generally contain questions similar to those done in class. These must be completed within a week and handed in for marking. Each week the students also receive a selection of extra problems, designed to be more challenging than the portfolio and assignment problems. The stronger students in the class are encouraged to attempt these problems.

3.3 Plenary Lectures

During the semester a number of plenary lectures will be given, these pull together the themes of the previous weeks and help to chart the way through the course.

A postdoctoral assistant was employed to assist in the constructive alignment of the course materials with the desired learning outcomes. The assistant's main task was to produce a workbook to be used to engage students in learning activities that are likely to achieve the desired learning outcome of increased skill in the application of the material being covered.

It is expected that the course restructuring will lead to students taking a deeper approach to their study of the course material. It is envisaged that this will have been facilitated by use of the course workbook and materials. In order to assess the impact of the course on student learning, two class surveys will be conducted at equally spaced intervals during the course. The feedback obtained from these will be analysed to evaluate the outcomes of the course, and to further refine the course for delivery in subsequent years.

4 Project Outcomes

The revised course objectives, along with the new methods of teaching and assessment, should create a learning environment that encourages students to engage more fully with the course. In previous years the level of understanding reached by many students taking the course can be described as multistructural (using the SOLO taxonomy, described in [3]): Students view the course as a "disorganized collection of items" and are unable to apply the concepts to problems of a similar format to those encountered during the course. Using the restructured course outlined above; students should develop a deeper understanding of how the concepts form an integral part of the theory of logic, and then be able to relate the concepts to the assigned problems.

The less academically committed students within the class should benefit from this project as the more active teaching methods employed should require such students to view the material at a higher level – relating, applying and possibly theorising about what is involved.

5 Evaluation

The main aim of the restructuring outlined above is to encourage students to engage more fully with the material to be covered. Both formative and summative assessment will be used to determine the success of the project in terms of learning strategies adopted by the students and examination results achieved.

Examination performance and survey results obtained during the project implementation in the 2001/2002 academic year will be compared and contrasted with with those obtained during previous years of the course. [4] outlines the link between attitudes toward mathematics and performance in undergraduate engineering mathematics courses, so this study will look for the existence of a similar link in undergraduate computer science courses.

Specific details of how the learning methods adopted by the students will be assessed are provided below:

A course diary will be kept by the lecturer. This will be filled in after each contact session, and will include brief descriptions of the material covered, as well as reflections on the teaching methods used. The students' level of interest, quality of understanding and the extent of retention of key points will also be noted. Any general feedback from the students will also be included in the diary.

A structured group feedback session will take place after six weeks of the course. This will be based on the methods outlined in [5] and involved asking each class member write down their answers to a number of questions, including:

1. What was the BEST feature of the course for you?
2. What was the WORST feature of the course for you?
3. What ways do you think the course could be IMPROVED?

The students will be asked to discuss their responses in groups of four, and to record points on which they are agreed. These comments will then be collated by the lecturer in front of the whole class. The feedback obtained will be used to evaluate student learning methods and to determine any necessary changes to the course structure. The lecturer will report to the class on how the information obtained will be used.

The postgraduate assistants will also be used to help determine the approach the students adopt to the course. The assistants will be able to assess the students grasp of problems they are tackling by asking questions on how they intend to approach problems. A comprehensive questionnaire that includes both numerical gradings and open-ended questions will be given to students during the last week of the course. This will assess a number of aspects of the course, including the effectiveness of the project in terms of student learning. The final examination will also be used to assist in the evaluation of the course. A study of types of questions tackled, as well as analysis of the final marks awarded, should provide evidence of the learning approaches adopted by the students.

6 Preliminary Evaluation

Preliminary results from an evaluation of the first six weeks of the course are only available at the time of writing. This is due to the fact that this course is being presented for the first time in 2002. This section includes both qualitative results obtained for the group feedback session outlined above and quantitative results from the annual Foundation Scholarship examination. Foundation Scholars of Trinity College are elected each year based on the results of these examinations. As these examinations are not compulsory, it is usually only the stronger students that choose to sit them.

6.1 Qualitative Results

Students who sat the course in previous years made the following observations:

- "I enjoyed reasoning about problems in English, although it was difficult."
- "I was recently asked for help by a second year student. I looked at the question and hadn't the faintest idea how to do it. Perhaps I learned a certain frame of mind for approaching problems, but not much else."
- "The tutorials should be made shorter so that it is possible to finish them within the time given."

Students sitting the course during the current academic year made the following observations:

- "I think it is a good idea to encourage the practice of the logical methods involved in order to help us understand the course better ."

- " The tutorials are very helpful but there is too much work involved in the portfolios."
- " The amount of formulae can seem overwhelming, but I'm beginning to understand how they all fit together."

6.2 Quantitative Results

In the Foundation Scholarship examination of 2002 it was noted that all students attempted the "applied" examination question, compared to 60% of students in 2001. The observed significance level associated with this difference is 0.01% and so we conclude that there is a significant difference in the proportion of students attempting the applied question in 2002 compared to those who attempted the applied question in 2001.

The difference between the average examination scores on this question were also compared. The null hypothesis used was that the average scores obtained in 2001 and 2002 were the same; while the alternate hypothesis was that the 2002 average result was significantly higher than that from 2001. It was concluded that the two values differ significantly as the observed significance level for the test was 0.8%. We may thus conclude that current students did better on the applied question on the Foundation Scholarship examination than those in 2001.

Students were surveyed in order to ascertain how they viewed the course objectives. They were asked to indicate if they felt the objective given related to the second year logic course. The table below give a list of objectives and the percentage of students who viewed them as being core objectives of the course. The observed significance levels given relate to the difference between the percentages shown in each row in the table.

Objective	2002	2001	Observed significance level
Manipulating Boolean Expressions	85.4%	78.4%	24.2%
Applying propositional calculus	74%	64%	20%
Translation of English statements into Boolean Expressions	65.4%	87.6%	1.79%
Developing different methods of proof	63.6%	63%	48%
Reasoning about variables other than Boolean ones	54.54%	67.6%	14.69%

Table 1: Student ratings of core course objectives

There is a significant difference in the perceived importance of translating English statements into Boolean expressions, with past students viewing this as the primary objective of the course. The data obtained from the current students suggest that they believe the course is focused on the manipulation of Boolean expressions. It should be noted that current students have not yet completed the course and that their impression of the course objectives may change over the final six weeks of the course.

These preliminary results suggest that the realignment of the course materials with the stated objectives is achieving the required results. More detailed data and analysis will be required to prove this is the case.

7 Summary

In this paper we have detailed a new teaching initiative being introduced to a second year logic course at Trinity College, Dublin. Preliminary results indicate that students are more willing to attempt applied examination questions and view the course objectives as being more than just the translation of English statements into Boolean expressions.

A full analysis of the impact of these initiatives on student learning will be completed by June 2002. A detailed report on the outcomes, including a full analysis of the data obtained along with details of problems encountered will be given at the Conference in July.

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COMPUTER SCIENCE STUDENTS NEED ADEQUATE MATHEMATICAL BACKGROUND

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ABSTRACT

Computer science is a relatively young discipline. The first computer science instructors were mathematicians and the first CS curricula were just modifications of mathematics curricula. As computer science has grown and matured, and some of its fields became independent disciplines of their own, the role of mathematics in it has faded significantly. The number of mathematical courses in the CS curriculum naturally has been decreased. The amount of mathematics required for a computer science majors has become dangerously small - so small, that it is starting to jeopardize student's ability to learn, understand and appreciate fundamental theory of computer science. This paper is devoted to studying the influence of computer science students' insufficient mathematical background on their ability to learn, understand and appreciate theoretical courses of computer science, particularly the Theory of Computing.

KEY WORDS: Mathematics in Computer Science, Computer Science Education.

1. Introduction

Throughout the history of computer science education there has been debate on what should be the appropriate mathematics background for CS majors. During the last decade this topic became more actual, since with the development of computer science and software engineering there has been a pressure to make the CS curriculum less mathematical. As a matter of fact the CS curriculum did become less mathematical. A number of studies have been carried out and surveys have been published during the last few years in favor of less mathematically rigorous CS curriculum [1-4]. Timothy Lethbridge, for example, surveyed a number of software developers on the importance of the knowledge they obtained at university in their jobs [5]. Mathematical knowledge ranked very low in this survey. The results lead Lethbridge to the conclusion that "Relatively little mathematics turns out to be important for software engineers in practice." Another argument in favor of cutting down the number of mathematical courses in the CS curriculum is to encourage more students to enter this field, since mathematics is one discipline that scares away many students interested in joining the information technology workforce [1,2]. Also, there is another viewpoint about why mathematics should be weighed less in the CS curriculum – it doesn't deny the importance of mathematics, but emphasizes, that if we were to include all the math that is useful for a CS major, it would result in a specialized math major, leaving us short on computer science itself.

The pressure for a less mathematical CS curriculum has alarmed a big army of computer science educators. A huge number of papers and articles have raised serious concerns regarding the role of mathematics in computer science and software engineering. A number of studies and surveys have been carried out to show that mathematics and mathematical thinking are central to computer science education [3,6-9]. The basic argument is the following: mathematics is a mindset that fundamentally improves one's ability to devise and implement algorithms. Mathematics is used to model the problem domain, to specify and design high quality software, develop correct and efficient algorithms. The main benefit CS professionals get from mathematics they learn at the university is the experience of rigorous reasoning with purely abstract objects and structures. "It is not what was taught in the mathematics class that was important; it's the fact that it was mathematical," states Keith Devlin [10].

The goal of this paper is not to repeat the importance of the mathematical knowledge for a CS major in a long run, but to emphasize its necessity while at school, when the student is taking certain required courses, namely, theoretical courses of computer science. As mentioned in Computing Curricula 1991 and repeated in Computing Curricula 2001 [9], "Theory is one of the three primary foundations of computer science. It depends on mathematics for many of its definitions, axioms, theorems, and proof techniques. In addition, mathematics provides a language for working with ideas relevant to computer science, specific tools for analysis and verification, and a theoretical framework for understanding important computing ideas." It seems natural to expect that by the time students get to the theoretical courses, they have been received the adequate mathematical background that will allow them to handle these courses without any difficulty. But is this true in reality?

This paper is to show how mathematically unprepared today's students are for theoretical courses of computer science, particularly the "Theory of Computing". Numbers and facts introduced in this paper are based on a study carried out at California Polytechnic State University in San Luis Obispo – one of well-respected public universities in the USA.

2. Motivation for the study

I have been teaching at the university level for 19 years: first 15 years at Yerevan State University (YSU), Republic of Armenia (a former USSR republic) and 4 recent years at California Polytechnic State University (Cal Poly), San Luis Obispo, USA. I taught different courses of computer science in both universities – programming courses, as well as theoretical courses. Comparing students in two different universities (with two different CS curricula – mathematically charged CS curriculum of YSU and significantly less mathematical CS curriculum of Cal Poly), it was hard not to notice, that having approximately the same level in programming courses, students' performance in and attitude towards theoretical courses of computer science is dramatically different. YSU students don't have any distinguished feelings about theoretical courses of computer science. Students at Cal Poly, on the other hand, find theoretical courses very hard and have a certain fear, if not hostility, towards them. This caught my attention from the very beginning of my career at Cal Poly. I started to observe closely students' performance in a senior level required class Theory of Computing (holding the reputation of one of the especially tough courses), trying to find out what is causing the irritation and difficulty. Throughout six quarters I gathered data representing students' performance in this course (12 sections, 315 students total – 2 sections per quarter, around 25-30 students per section), analyzed test results, compared student's grades in this course with their grades in other courses, surveyed students, talked to different professors teaching theoretical courses and got their opinions which were in absolute agreement with my observations and conclusions. The outcomes of this study are presented below.

3. The real picture

Theory of Computing is a heavily math-flavored course. Textbooks are written in formal mathematical language – all concepts are defined formally, all results have mathematical proofs, all algorithms and techniques are presented with the help of formal mathematical notation. To be able to handle this course, one must have an adequate mathematical background – first of all be very comfortable with mathematical notation to be able to read and understand the text; be knowledgeable in set theory, graph theory, combinatorics; understand very well functions and relations, proof techniques etc.

The six-quarter study of students' performance in Theory of Computing gave a clear picture of what is going on and why students don't like this course. Below we'll show some of the interesting outcomes of this research. Since the results for different quarters of investigation came out very similar and consistent, we'll bring the results for one quarter only – Fall 2001 (two sections, 56 students all together).

Results of the study show that for the majority of students the grade in Theory of Computing is lower than their average grade in other courses. Even in comparison with the majority of other "difficult" courses, students' performance in Theory of Computing is noticeably less satisfactory. For example in Fall 2001, the grade in Theory of Computing of 60.9 % of students was at least by one letter grade lower than their grade in Data Structures course – a relatively hard course, containing both programming and theoretical elements. More precisely,

19.6% of students had a decrease of two letter grades,

41.3% of students had a decrease of one letter grade,

34.8% of students had maintained the same grade,

4.3% of students had an increase of one letter grade.

The consistency of this situation leads to a necessity to find out why are students having difficulty maintaining their average grade in Theory of Computing. Where exactly lies the problem? What qualities do students lack that keeps them from being successful in this course? Here are some major flaws that have repeatedly surfaced during the period of investigation:

1) Many students are uncomfortable with mathematical notation – mathematical text is very incomprehensible for them. As a result these students are unable to read and understand the textbook on their own. The teacher has no choice but to spend a great deal of lecture time on interpreting what the book says in a more "human" language, instead of using that time presenting interesting results, techniques, examples. Needless to say, that these students are unable to present (reproduce) information with the help of formal notation as well. In the anonymous survey conducted at the end of the quarter, 45.5% of students admitted that they cannot read and understand the textbook on their own; 60% of students confessed that if teacher doesn't explain, they will not be able to understand the concept looking at its definition; 76.36% of students said that they cannot understand on their own the full meaning of the results stated in theorems; 47.27% believed that even after understanding the theoretical material (with or without help), they will not be able to reproduce it on their own.

2) Many students do not know basic mathematical concepts. As a result these students are unable to understand the new concepts represented in the course. Consequently, they are incapable to understand the course material at least at the theoretical, abstract level. For example, in the quiz on languages 53.98% of students answered "yes" to the question "Can a string contain infinite number of elements?" (a string is defined to be a finite sequence of alphabet letters), 62.33% of students answered "no" to the question "Is $\varnothing$ a language over the given alphabet?" (a language is defined to be a subset of the set of all strings over the given alphabet). In the final exam, defining a Pushdown automaton as a mathematical system, 46% of students failed to specify the domain and 68% of students couldn't specify the range of the function (called transition function) representing the work of the machine.

3) Many students do not have an understanding of proof techniques. Consequently, these students are not able to use major results of Theory of Computing to prove certain characteristics of objects they are working with. For example, in the final exam students were required to prove that the given language is not regular. To do this, one needs to use the "proof by contradiction technique" and the property established in Pumping Lemma for regular languages (the two Pumping Lemmas are fundamental theorems in the Theory of Automata and Formal Languages). The results of the final exam show that 72% of students failed to do this assignment due to their vague understanding and incapability of using the "proof by contradiction" technique.

4) Many students cannot remember the names of concepts (old or new), definitions, important results. Consequently, these students are not able to express their thoughts correctly and precisely, cannot formulate clear questions, and lead a mature conversation on the topics of the course. For example, in the final exam 48% of students failed to name the three basic μ-recursive functions, and 60% of students couldn't list the three operations that are used to create new μ-recursive functions from the basic ones. 56% of students were not able to state the Church-Turing Thesis – one of the most fundamental hypotheses of the Theory of Computing. And the average grade of the class for the essay question in the final exam was only 50.68%.

It is worth mentioning that professors teaching other theoretical courses in the department strongly agreed with these conclusions, and reinforced them with facts from their own experience.

On the other hand it will be only fair to mention, that when the turn comes to practical issues such as creating an automaton to accept a language, constructing a grammar to generate a language, designing a Turing machine to perform the required job, applying an algorithm to, for example, transform the given nondeterministic finite automaton into a deterministic machine, these same students are impressively bright and inventive. As a matter of fact, students' grades in tests on practical material is significantly higher than their grades in theoretical tests: in the final exam the class average for the practical part (exercises on constructing abstract machines, grammars, applying algorithms) was 76.2%, while the class average for the theoretical part (short questions testing their understanding of theorems, concepts, their knowledge of definitions) was only 62.86%.

4. Conclusions

Our study shows that quarter after quarter students keep making the same, almost exclusively mathematical, mistakes – a vivid evidence of insufficient mathematical training. According to Cal Poly's CS curriculum, students are required to take two quarters of Calculus (Calculus I and II) and one quarter of Discrete Mathematics (Discrete Structures) in their first year of education. Later, in their third year, they are required to take a one-quarter course in Statistics and two quarters of math or statistics elective courses. Theory of Computing is a senior level course that students take in their fourth year of education, and normally, by the time they get to it, they have taken all abovementioned math courses already. Well, looking at the results of our study, it is quite obvious that the amount of math courses that students take does not build a steady mathematical background. This limits students' ability to learn, understand and appreciate theoretical aspects of computer science. Particularly, one quarter of Discrete Mathematics is not enough to master such topics as Set Theory, Graph Theory, and Proof Techniques, which are used in different theoretical courses of computer science. And considering also that Discrete Mathematics is required to take in the first year of education, it is quite natural for students to forget in couple years all the knowledge received in this course.

We, professors who teach theoretical courses in the Computer Science department at Cal Poly, strongly believe that our students need more mathematical training. Discussions with professors in different universities give the impression that this is not an unusual situation for other American universities as well. Of course it would be easy to suggest adding a few math courses to the CS curriculum and taking care of the problem, but how realistic that suggestion is. The truth is that CS curriculum is heavily loaded already, and it would be naive to assume that new courses can be added to it easily. In spite of this we need to take actions and find an acceptable and reasonable solution to the problem. Obviously it will take a lot of efforts and few compromises, but if we don't do anything about this now, it will be hard to justify the existence of the word "science" in the title of our discipline in the near future.

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CRITICAL FACTORS AND PROGNOSTIC VALIDITY IN MATHEMATICS ASSESSMENT

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ABSTRACT

High school mathematics is traditionally more procedural than conceptual in character, as well as formally less rigorous, than is mathematics at the university level, and hence puts less demand on logical reasoning and conceptual understanding. To find an instrument to make a reasonably good prognosis for success in undergraduate mathematical studies, it is therefore necessary to look closely at the demands of the future mathematical activities rather than only more narrowly at what has actually been accomplished at the high school level in terms of content and methods. In this paper the development of a short test for prognosticating academic performance in mathematics is discussed, and the results from a group doing the test when entering university is related to the results on their first mathematics courses.

Based on research literature and an analysis of the demand of the courses, the design of the test was built upon ten factors that were found to be critical for passing the mathematics courses in the educational programme being considered: conceptual depth, control, creativity, effort, flexibility, logic, method, organization, process, and speed. The critical factors cut across the content -process distinction and are expressions of a holistic view of mathematical performance. To prognosticate academic performance it is necessary to identify important nodes of integration in the web of mathematical ideas, concepts, skills, forms, affects, and so on. The **critical factors** constitute vertices where the different dimensions of mathematical thinking meet.

In the paper the construction of the test is discussed, and the results show a strongly significant correlation to performance on the target undergraduate mathematics course. A notion of **prognostic validity** of the test is outlined and discussed. The paper shows ho w test construction, analysis and interpretation of the outcome, depends heavily on what the result is going to be used for, and how a mathematics assessment design by necessity leads into discussions about the nature of mathematics and the understanding/performance of mathematics. What seems to be typical in mathematical problem solving is that many of the critical factors are involved in one problem solving process and must be combined for success.

1. Introduction

School marks in mathematics alone may have limited value for prognosticating performance in mathematics at the university level. High school mathematics is traditionally more procedural than conceptual (cf. Hiebert, 1986) in character, as well as formally less rigorous, and henc e puts less demand on logical reasoning and conceptual understanding. To find an instrument to make a reasonably good prognosis for future success in college mathematics, it is therefore necessary to look more closely at the demands of the future mathematical activities than only more narrowly at what has actually been accomplished at high school.

The development of an assessment instrument to prognosticate academic performance in mathematics is discussed, along with test results, compared to results from the first university course in mathematics for one group of students. An underlying assumption is that some of the general problems of assessment in mathematics become visible through the window of an example.

In mathematics assessment it is common to make the distinction between content and process variables, thus forming a matrix of combinations of different aspects of these two objectives[1]. In the NAEP mathematics assessment there are five content and four process variables. To the framework of the APU secondary assessment further dimensions affecting the assessment outcome have been added to the matrix, such as the mode of assessment, context, and attitudes. The content and process categorization is used also in The National Criteria for Mathematics, where as much as 17 process objectives are listed. (See Ernest, 1989, for descriptions and references) Content and process knowledge, or domain-specific and general-strategic knowledge, are closely related or dependent of each other (Alexander & Judy, 1988; Perkins & Salomon, 1989), making it difficult to separate them in a meaningful way in an assessment situation.

During the work with the assessment standards in the USA it has been stressed that any assessment in mathematics should deal with **important** mathematics: "Answers to the question *What is the important mathematics here?* Should be reflected in: • the plans for the assessment, • each assessment task and activity, • the interpretation of students' responses, and • the intended uses of assessment results" (NCTM, 1993, p. 29). It is part of the nature of prognostic testing that the mathematics achievement one is trying to predict deals with content unknown for the students at the time of the testing. Therefore it is necessary to look at what *aspects* of mathematical thinking are important for the future studies, and then find relevant known content.

Important factors for doing mathematics successfully have been analysed for example by Krutetski (1976), and an increasing number of studies also of advanced mathematical thinking have appeared (e.g. Tall, 1991; Holton, 2001). The choice of such factors must be based on literature studies and on experienced teacher judgement, including the marking of exams protocols (cf. Webb, 1992, p. 672). The term **critical factor** has been chosen here to indicate that with low 'levels' of these factors students will (most likely) meet problems to pass the mathematics courses considered. Also belief factors influence study results significantly (Niss, 1993; Webb, 1992), but will not be considered here.

[1] The meaning of the term 'process' is here vague, as it could refer to a specific mathematical skill, or to a general cognitive strategy.

2. Critical factors

In the present study 119 civil engineering students were enrolled in a four-and-a-half years programme with four different branches: Computer science (D), Industrial engineering (I), Mechanical enginee-ring (M), and Applied physics and electrical engineering (Y). Ten factors have been found to be critical here.

Conceptual depth – That mathematical concepts and procedures have been learned by root is often observed in students' attempts to solve well chosen problems. Conceptual depth shows for example when solutions are "simple" and accurate, right to the point without unnecessary complications over a number of tasks, but is often hard to trace in protocols.

Control – There are at least two aspects of control that are critical in this context. One refers to the "looking back" process of checking up a result that has been obtained and the feeling that it is reasonable. The other aspect is more delicate to describe but may be captured by the phrase 'I know what I'm doing', I'm controlling the mathematical entities I'm working with because I'm familiar with their properties (cf. Bergsten, 1993).

Creativity – In school mathematics fantasy, or originality in mathematical thought, is seldom emphasised, but when it shows is an indicator of problem solving ability. In the international mathematics education community there are now special conferences on creativity.

Effort – It can sometimes show in a protocol that the student has tried hard to work out the problem. For weaker students effort is one of the most critical factors. However, as this is an affective factor, it can't always be judged from a written response protocol alone.

Flexibility – The ability to change to a thinking mode suitable for the particular problem, for example to alter between a numeric, graphic, or symbolic form of representing mathematical ideas (sometimes called *versatile thinking*; see Tall 1991), is important for solving a wide range of mathematical problems. Included here is the ability to view mathematical symbols representing either a mathematical object or a mathematical operation to be performed (Gray & Tall, 1991; Sfard, 1991).

Logic – There are many faces of logic involved in a problem solving process. One is *rigour*, i.e. the extent to which a conclusion in the solution process is logically valid, and the necessary assumptions pointed out. Another face is *consistency*, i.e. the absence of contradictions. *Completeness, accuracy*, and *generality* in reasoning may also be regarded faces of logic.

Method – There are 'natural' and easy ways to solve a problem, and there are 'clumsy' ways. Choice of method with respect to its *efficacy* is a critical factor. Student often loose track in producing an increasing amount of 'algebraic mess'. The degree of *simplicity* may be viewed as a logical factor but may be considered a factor of its own as it goes beyond logic.

Organisation – This factor refers to the 'layout' of the written student response to a given problem. Is the logic visible, or are different points made randomly, as it looks, over the page?

Process – Is the student response predominantly *procedural* or *conceptual* in character? Messy algebraic manipulations are often indicators of a procedural approach, detached from conceptual understanding. Conclusions based on such an approach are often mathematically incorrect or meaningless. Figurative components, such as diagrams, reveal the presence of imagistic thinking, indicating that a conceptual approach has been used (cf. Goldin, 1987). Another such indicator is the text inclusion of reasoning in words, or short algebraic solutions. An integration of the procedural and conceptual process aspects is often stated a characteristic of understanding mathematics (Hiebert, 1986; Gray & Tall, 1991).

Speed – In university mathematics exams the speed factor can of course be significant for both the amount and the quality of the outcome.

A point of discussion is how the critical factors relate to the content-process distinction. Now, the distinction in itself is fuzzy (cf. above, and e.g. Lerman, 1989), and Perkins and Salomon (1989) advocate a synthesis. In any reasonable meaning of the terms, clearly content is involved in the conceptual depth factor, and process for example in the logic and method factors. In fact, method is the outcome of a content-process integration. Thus, the critical factors cut across the content-process distinction, and are expressions of a synthesis of the kind just mentioned, i.e. of a holistic view of mathematical performance. To prognosticate academic performance it is necessary to identify important nodes of integration in the web of mathematical ideas, concepts, procedures, skills, and so on. The critical factors constitute vertices where the different dimensions of mathematical thinking meet. That is why they are considered critical for prognosticating mathematical performance.

3. Test construction and results

With the previous discussion in mind, how should a written test be designed to predict the degree of successful academic performance in mathematics, and how should the responses be analysed and interpreted? It should not be possible to solve an item by direct reproduction of memorised techniques only, excluding pure routine tasks (cf. Christiansen & Walter, 1986) in favour of more complex problem solving. It is also obvious that all the critical factors above cannot be 'covered' in each one of the items. Therefore the design and the interpretation of the results must be based on an integrated local-global analysis. The rationales behind the selection of the items of the prognostic test[2] will be briefly discussed.

Mathematics consists, among other things, of ideas and the formal representations of ideas (cf. Mac Lane, 1986). One important idea is that of generalisation, often formalised by using algebraic symbolism (item 8). In mathematical problem solving the input is often an algebraic expression. The problem can consist of reasoning in algebraic terms (items 1 and 7), using only procedural knowledge or a combined (integral) procedural-conceptual approach by using for example numbers or diagrams. Mathematical reasoning can start with a diagram that needs to be analysed (item 6) and/or linked to algebraic symbolism (item 5), or a diagram may be constructed as a support for reasoning (item 4, possibly also items 3, 7, 8 and 10). Solving quadratic equations (item 3) is an example of a skill that can become highly automated, and is here used to reveal the presence of the control function. Control is critical also in items 1, 2 and 7. Hypothetical thinking is implicit in most mathematical problem solving, and has therefore been chosen as the core of a problem (item 9), keeping the 'technical' parts at a low level of difficulty. Pattern recognition is often fundamental for finding a solution to a mathematical problem (items 3 and 10).

To ensure effort the level of difficulty has been kept rather high, considering the students in focus and the time constraint (the speed factor). A test with tasks where only little effort is needed will not capture the status of the critical factors.

[2] The DP test, with 10 items; see Appendix A.

The achievement level may be viewed as a product or as a synthesis of the critical factors and content knowledge. A point of discussion is if the levels of the critical factors can be quantified and scored separately, or if they should be integrated in the achievement score.[3]

The person constructing the DP test (i.e. the present author) did not teach the calculus courses in question (but has done so previous years), nor did collaborate with the examiners, nor did they take part in the construction or evaluation of the test.

Out of a total of approximately 600 beginners at the civil engineering programme of the university, one group (i.e. class) from each of the branches D, I, M, and Y was randomly chosen, making a total of 119 students doing the DP test. The test[4] was administered before the beginning of the first regular mathematics course (calculus). Calculators or mathematical tables were not allowed. On each item the scoring was 0, 1, 2, or 3, where 3 was indicating a correct solution, with high levels on the relevant critical factors, 0 or 1 an insufficient solution, with low levels of the factors. Thus the range of the total score (sum) was from 0 to 30. Group means and standard deviations are shown in ta ble 1, frequencies of different sums in table 2, and means of items in table 3.[5]

As can be seen from the tables some groups differ significantly in achievement, differences that are not explained by their school marks in mathematics. That the items 9 and 10 scored very low (table 3) cannot be explained by their difficulty alone but also by the time constraints. The correlation between item scores and total score (table 3) are relatively even (i.e. homogenous test), the lowest explained by the low variance of the item. A factor analysis (table 3) reveals only one dominating factor, possibly a general reasoning factor[6]. The second factor in size is related to items of an algebraic character. Item 5, with low variance, did not correlate with the other items.

The prognostic value of the DP test can be measured by its correlation to the results of the mathematics courses that the same students took during their studies. For this paper results from the calculus course that followed immediately after the DP test will be discussed. This was a one semester course with one mid term exam and one final exam. The courses and the exams were identical for the groups D and Y. Groups I and M had separate (similar, less demanding) courses. On all exams there were seven items of problem solving with a maximum score of 3 on each item. In tables 4a (mid term exam) and 4b (final exam) group means, standard deviations, and correlations with the DP test are shown.[7] All correlations in table 4 are significant or strongly significant.

4. Comments on response protocols

Some general comments to each item of the DP test are given below.

Item 1 – A vast majority of the students seemed to bring a purely procedural approach from high school when it came to dealing with inequalities. After 'simplifying' the conclusions were often incorrect, irrelevant, or nonsense. There were often low levels on the logic, method, and control factors.

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[3] An integral approach was chosen here

[4] Announced as a 90 minutes long diagnostic test

[5] See Appendix B

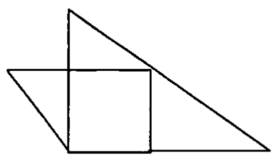
[6] Often labelled g in the literature (e.g. Gustafsson, 1988)

[7] See Appendix B

Item 2 – The most common mistake on this item was to not multiply the denominator outside and inside the parenthesis.

Item 3 – Only a third of the students observed that one of the roots of the equation they obtained was false (in most cases due to 'squaring' the equation). Here the control factor was indeed critical, process being purely procedural.

Item 4 – Most students based their solutions on the notion of similar triangles, and only a few applied the Pythagorean theorem. The direct solution obtained by transforming the triangle to a trapezoid with equal area (related to the creativity factor; see figure below) was not found in the response protocols.



Item 5 – Students used different identification methods, the most common checking up the value of y for one or two values of x. Only a few seemed to have argued on asymptotic behaviours.

Item 6 – Methods differed a lot in simplicity. Some students displayed a bunch of remembered area formulas, without knowing what to do with them.

Item 7 – Comments made on item 1 apply also here. The conceptual depth factor scored low on this item. 'Rules' from solving an equation were in some cases translated to an inequality. With low conceptual depth the control factor cannot work, and the procedural approach can lead almost anywhere. The inclusion of a parameter has put an extra load on the logic and the conceptual depth factors. As already noted, the absence of graphical solutions is here, most likely, an indicator of a low level of the conceptual depth factor.

Item 8 – This item puts emphasis on many aspects of the logic factor. Levels of explanation differed considerably (rigour), and difficulties of expressing the general formula in a simple form were frequent.

Item 9 – The problem was attempted by 46% of the students but solved (score 2) only by 8 %. Many students got lost in algebraic manipulations, an indicator of not understanding the logic, not really knowing what to do with all the algebra.

Item 10 – The problem was attempted by 51% of the students but solved (score 2) only by 3 %. In most of the attempts the equation $x^4 - x^2 = 0$ was solved, or a graph was drawn.

As an overall comment, what seems to be typical in mathematical problem solving is that many of the critical factors are involved in one problem solving process and must be combined for success.

5. Prognostic validity

The validity of a test depends on what the information (test result) is to be used for. For the kind of test discussed here it seems proper to talk about **prognostic validity**. This means that the analysis and interpretation of the results (i.e. the written test protocols) must be based on how well they may prognosticate academic performance in mathematics. The prognostic validity of the test may then be valued from the outcomes of this process, and is thus a function of such factors as design, selection of tasks, content specification, and rationale for protocol analysis (see e.g. Webb, 1992, p. 674).

In this paper, with prognostic validity in focus, a framework for the protocol analysis has been suggested consisting of ten factors that have been judged to be critical for future academic performance in mathematics. In a traditional achievement score, which within a mathematics department has a considerable reliability and validity[8], according to the standards of the faculty, critical factors may implicitly be evoked. There is seldom, however, a stated 'general manual' for the correcting procedure. The validity and reliability of the markings are normally based on teacher experience and judgement only.

A theory of how to find appropriate forms for analysing responses to an assessment situation is still lacking (Webb, 1992). One has to take into account not only the four components mentioned above[9] but also the interaction between them. This means that one must keep some kind of control of the whole assessment process, so that all its aspects are in alignment with the purpose of the assessment.

One preliminary quantitative measure of the prognostic validity of the DP test is given by the correlation between the total score on the DP test and the total score on the university mathematics exams. As can be seen from table 4, these ranged from .52 to .90 on the first calculus exam, and from .44 to .86 on the second. For the Y and D programmes, this prognostic validity of the DP test was quite substantial.

A fact that must be considered here is that the DP test and the first calculus exam both are written problem solving mathematics tests, given with a delay of only two months. Therefore a positive correlation between those tests was to be expected and explained maybe by the general cognitive ability factor g (cf. the factor analysis in table 3). However, the point made here is that the DP test, based on high school mathematics content only, was designed to have a strong correlation with the exams results, and it may well be the case that in basic university mathematics the g factor shows by its influence on the critical factors (cf Gustafsson, 1988). More data will be needed to further evaluate the prognostic validity of the DP test.

6. Discussion

It is becoming generally acknowledged that to provide a good picture of a student's mathematical ability an assessment 'package' is needed (Niss, 1993). It is, however, also necessary to ask the 'reverse' question: How much, and what kind of information can you get from only one written test? After all, written tests are often all you can get. What information there is hidden in a response protocol is a result of the interaction of the student with the test tasks and the assessment situation. The test will measure the kind of mathematical performance that the items and the situation will evoke.

The items of the DP test were constructed to show the students' levels on the critical factors. This influenced the analysis of the protocols in such a way that the scoring was made against the relevant critical factors. The judging of the levels of these factors from the protocols are related to what is expected from this group of students, which means that they cannot be objective or absolute but are socially referenced. During the work it was impossible to keep these factors apart from mathematical content or achievement. An attempt was made to mark one achievement score (locally on each item) and one global score on the critical factors, to give a summarised or adjusted score of mathematical

8 Validity in relation to course objectives

9 As there quoted from NCTM, 1993, p. 29

performance. However, the achievement score was, indeed, always based on the level of some critical factor(s). Therefore, only one integral score was chosen.

One shortcoming in all testing is that students may have the knowledge but don't use it (cf. Schoenfeld, 1987). To what extent the critical factors are related to the ability of evoking relevant knowledge in a problem solving situation is an open question. This is related to the cognitive status of the critical factors, which is beyond the present scope.

This paper illustrates how test construction, analysis and interpretation of the outcome, depend heavily on what the result is going to be used for. It also shows how a mathematics assessment design by necessity leads into discussions about the nature of mathematics and of doing and understanding mathematics. The prognostic test DP was designed to make the critical factors visible. The results indicated a substantial prognostic validity of the test, and further developments and experiences will show the degree of substance in this conceptualisation.

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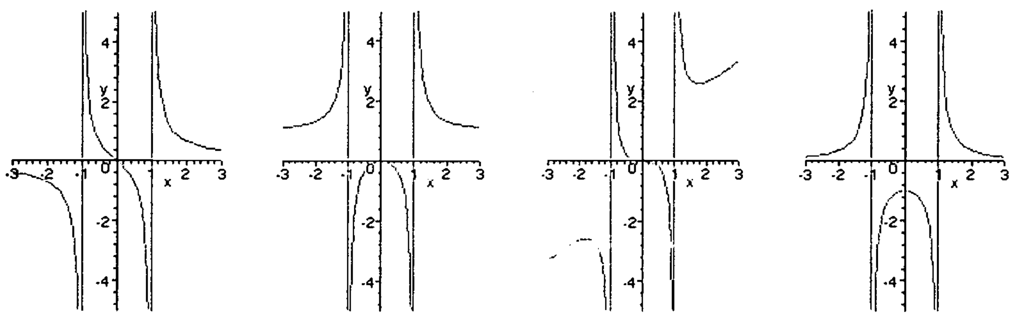
APPENDIX A – The DP test

1. For what positive numbers a, b and c does the inequality $\frac{a}{b} < \frac{a+c}{b+c}$ hold?
2. For the numbers $a_1, a_2, a_3, \ldots$ we have $a_1 = 0$, $a_2 = 1$, and $a_{n+2} = \frac{1}{4}(3a_{n+1} + a_n)$
   For all natural numbers n 1. Evaluate a_5 .
3. Find all real solutions to the equation $x = 1 + \sqrt{x}$.
4. Inscribe a square in a right-angled triangle so that two of its sides fall along the smaller sides of the triangle, and one vertex on the hypotenuse. Show that the inverted value of the side of the square equals the sum of the inverted values of the smaller sides of the right-angled triangle.
5. Match the function (a-d) to the corresponding graph (1-4):

   a) $\frac{1}{x^2-1}$ b) $\frac{x}{x^2-1}$ c) $\frac{x^2}{x^2-1}$ d) $\frac{x^3}{x^2-1}$



(1) (2) (3) (4)

6. A circle is inscribed in an equilateral triangle, which is inscribed in a circle that is inscribed in a square inscribed in a circle (with figure in test). What portion, in percentage, is the smallest circle area of the biggest circle area?
7. For what real numbers x is $x^2 < ax$? (a is a real constant)
8. A triangle has no diagonal. A square has two diagonals. A regular pentagon has five diagonals. How many diagonals are there in a regular

   a) hexagon ? b) n-polygon ? (n is a natural number 3)
9. Let a and b be any positive numbers. For the numbers $A = \frac{1}{2}(a+b)$, $G = \sqrt{ab}$ and H,
   where $\frac{1}{H} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$, we have (1) A G and (2) G H
   Show that (1) implies (2).
10. Find (without the use of derivatives) the minimum value of

    a) $x^4 - x^2$ b) $4^x - 2^x$

APPENDIX B – Tables

Group	m	s	min/max	n
D	10.9	6.4	2/30	29
I	8.6	4.4	3/22	31
M	8.8	2.5	3/12	31
Y	13.4	6.1	1/27	28
Total	10.3	5.3	1/30	119

Table 1. Means (m), standard deviations (s), minimum/maximum score, and group sizes on the diagnostic test DP

Sum	1-3	4-6	7-9	10-12	13-15	16-18	19-21	22-24	25-27	28-30
F	8	19	33	27	10	13	5	2	1	1

Table 2. Frequencies (f) of students with sums in given intervals (sum) on the diagnostic test DP

Item	Factor 1	Factor 2	Factor 3	r	m	%
1	.03	.65	.27	.58	1.0	87
2	.03	.03	.82	.42	1.9	92
3	-.04	.68	.18	.46	1.0	96
4	.43	.45	.39	.72	.9	77
5	.17	.15	.23	.34	2.6	96
6	.72	-.01	.26	.59	.9	64
7	.65	.27	.28	.65	.4	80
8	.69	.02	-.08	.45	1.1	85
9	.33	.72	-.19	.57	.3	46
10	.51	.49	-.23	.49	.1	51

Table 3. Factor loadings (varimax rotation) on DP items (eigenvalues 3.00, 1.09, and 1.08 respectively), and Pearson correlations item-sum (r). Means of items (m) and the proportion of students (%) that attempted items (n=119).

Group	m	s	min/max	n	r
D	4.4	5.3	0/21	25	.90
I	4.4	4.5	0/15	30	.57
M	5.4	3.4	0/11	28	.52
Y	7.7	4.1	2/17	27	.62
D+Y	6.1	4.9	0/21	52	.79

Table 4a.
Means (m), standard deviations (s), min/max scores, group size (n) and Pearson correlations (r) between mid term exam of calculus and the test DP

Group	m	s	min/max	n	r
D	5.7	5.3	0/17	23	.86
I	6.7	4.9	0/16	30	.44
M	7.0	4.8	0/15	27	.64
Y	9.1	3.9	2/19	27	.66
D+Y	7.5	4.9	0/19	50	.78

Table 4b.
Means (m), standard deviations (s), min/max scores, group size (n) and Pearson correlations (r) between final exam of calculus and the test DP

LEARNING TO TEACH ALGEBRA:
An Italian experience with reference to technology

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ABSTRACT

The Ministry of Education (MPI) and the Italian Mathematical Union (UMI) have carried out a project for in-service teacher training in algebra. It consists on three stages. First a group of 20 selected teachers attended a series of lectures. The lectures were videotaped. On the ground of these lectures the teachers produced didactic materials (forms, references, etc) recorded in a CD. The final products (videotape and CD) are sent to the schools all over the country to be shown to mathematics teachers. Four lecturers developed the subject (the teaching and learning of algebra) according to the following streams:

- general educational issues based on international literature in the teaching and learning of algebra
- algebra and information technology
- a new approach to algebra through number theory
- history and algebra.

The present work reports the technological part of this teaching project.

The use of a computer algebra system can improve the teaching of algebra, helping the teacher in several ways.

However several difficulties can show up in the classroom use of CAS: elementary "pencil and paper" algebra rules and procedures are not always the same as the tasks performed by a machine, some problems arise in the relationship between algebra and graphics.

The aim of the lessons and of material produced is:

1. to give the teacher a good knowledge of a CAS (Derive);
2. to explain the problems that can arise in the classroom use of symbolic math;
3. to give a hint for the solution of these problems.

Keywords: Algebra Teaching, Computer Algebra Systems.

1. Introduction

The Ministry of Education (MPI) and the Italian Mathematical Union (UMI) have carried out a project for in-service teacher training in algebra. It consists of three stages.

In the first one, a group of 20 selected teachers attended a series of lectures. Four lecturers developed the subject (the teaching and learning of algebra), according to the following streams:

- general educational issues based on international literature in the teaching and learning of algebra;
- algebra and information technology;
- a new approach to algebra through number theory;
- history and algebra.

The lectures were videotaped. On the ground of these lectures the teachers produced didactic materials (forms, references, etc) recorded in a CD. The final products (videotape and CD) are sent to the schools all over the country to be shown to mathematics teachers.

The present work reports the technological part of this project and describes, in the next section, how the lecturer organised the course, with the aim of explaining the fundamental ideas of computer algebra to the teachers; then some of the problems that arise in the use of computer algebra are presented; in the final section some material prepared by the teachers is described.

2. The fundamentals of a CAS

The use of a Computer Algebra System (CAS) is now quite common in the classroom; among the different types of software commercially available, Derive is the best known and widely used in Italian schools.

However the use of Derive, in most cases, is limited to the graphing of functions and the applications to analytic geometric and calculus. One of the aims of this project is to give a starting hint for a more widespread use of the software, stressing its computing and algebraic capabilities; the graphical part is not considered, being a common background for the teachers involved in the project.

It should be emphasised that the lectures do not want to be a comprehensive guide to Derive, but only an introduction. Based on this material, the teacher is invited to build his/her personal teaching material, more closely related to his/her project and tailored on the students background.

All in-service teachers in Italy in the late eighties were involved in the PNI (Piano Nazionale per l'Informatica), where Pascal was taught together with its applications to Mathematics; an important feature of this project was the necessity of emphasising what is different and new in Derive, compared to a Pascal-type programming environment; these differences are very noticeable when we use numbers.

For this reason it is not surprising that a strong emphasis was put on the Exact mode used by Derive in treating rational numbers and radicals; this feature, together with the use of "arbitrary precision" integers, opens up a new field of applications to prime numbers, to the production and verification of numerical conjectures; all these applications are impossible or very limited using the Integer or LongInteger type of Pascal.

The second topic is algebra of polynomials; here we must notice that Derive has different levels of simplification (the basic simplification, the EXPAND and the FACTOR command). The FACTOR command enables us to factor a polynomial at different levels according to the algebraic field we are considering. The treatment of algebraic functions and the operations on polynomials is also considered, introducing the QUOTIENT, REMAINDER and POLY_GCD functions.

The following step is very important: the use of variables and the definition of functions. Since Derive is a functional language, the user must be aware that the definition of a function is a "natural step" in Derive. Here again one of the main aims was to convince the audience, familiar with Pascal, to change to this new way of working: a bottom-up approach was used in showing how we can start with a simple function and, using composition, obtain a more complex one.

The complex field plays an important role in CAS; in this first step the basic operations on complex numbers are presented; but this it is not sufficient, since several options for computations in complex fields can affect also elementary algebra; for instance the choice of a BRANCH for complex non single-valued functions may give results in the computation of a root. Then it is necessary to go deeper in the subject and to study the choice of a branch for roots and logarithms.

Algebraic equations and inequalities are an important topic in high school algebra; here again the use of a CAS may be very useful; it is interesting to see how Derive can solve algebraically third and fourth order equations and that the results provided not always are easily understandable, again for a problem of representation of complex numbers.

Derive can solve equations algebraically and numerically, while using Pascal we can solve them only numerically; this capability opens up a new interesting application: how to teach the student the limitations of algebraic procedures and how to make him/her aware of the opportunity of switching from the algebraic solution to the approximate solution.

Vectors and matrices are fundamental tools in a CAS, that can help in performing boring tasks such as inverting a matrix or reducing it; moreover, they are a fundamental tool for programming, since some iterative programming structures are implemented in Derive by the definition of a vector; then the study of vectors and matrices is also a preliminary step to programming.

Derive (at least in Version 4, used in the course) has a limited programming environment; but its peculiarity of being a functional language may be very interesting in teaching; SUM, PRODUCT and VECTOR were introduced first; then a deep presentation of functional iteration (both on scalar and vector) follows. The last topic is recursion and the relation between the recursive and the iterative definition of a function and the related problems of computing efficiency.

3. The problem of using a CAS

The presentation of Derive described in the last section was mainly technical; it aim was to provide the teacher with the necessary background about the features of Derive, in order to use CAS in the classroom, knowing "what it happens inside"; however during the presentation several issues arose about the didactical problems of the use of CAS in the classroom.

When speaking about problems due to the classroom use of a CAS the general trend is to tackle the subject in a too wide sense, blaming the CAS for many difficulties that are not typical of CAS but that exist for all types of software.

Then we must concentrate on problems that really depend on the interaction between numerical, algebraic or graphical results produced by the CAS and the same results obtained with paper and pencil; we describe some of them.

1. While the student can easily master algebra of polynomials and the results are generally predictable and understandable by the students, the opposite happens when we consider algebraic functions; here it is difficult to understand the effect of the three different levels of simplification.

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2. The SOLVE function for the solution of equations involving rational functions does not control the compatibility of solutions; then we have non acceptable solutions, as shown in lines #1 and #2 of Figure 1. Moreover, Derive seems to work in extended real numbers; the results shown in lines #3 and #4 are easily explained using limits, but are very hard to understand for a 15 years old student of elementary algebra.

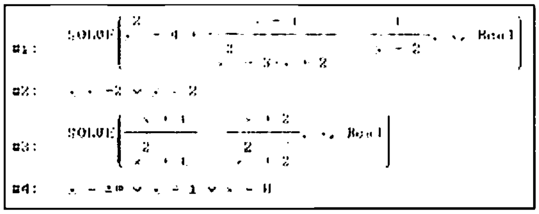

Figure 1

3. The use of complex functions in internal computations of Derive can create some problems; while algebraic computations can be controlled by carefully choosing the branches and the domain of variables, the graphs drawn by Derive may be quite different from the graphs that a student in basic algebra can expect.

Two examples are shown in Figure 2 and in Figure 3. In the first one, the function $y = \sqrt{x-1}\sqrt{x+1}$ is plotted, in the second example the function $y = |\ln x|$.

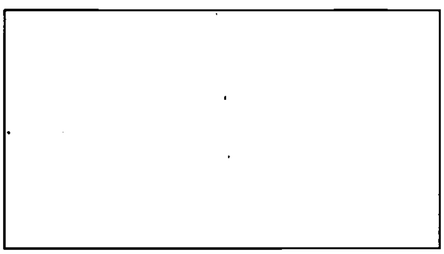

Figure 2

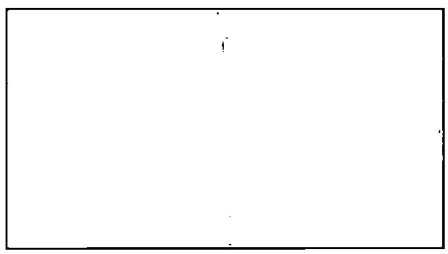

Figure 3

The problem of "wrong graphs" is important in teaching; as a matter of fact the use of computer in this context does not help the teacher and may cause false problems, generated by the use of a more advanced tool (complex numbers) by Derive, while the student is taught to work in the real domain; it is necessary to force Derive to work only with reals.

To do this some programming capabilities are required (as described in [Boieri, 1996]), since it is necessary to change the definitions of some built-in functions.

4. The teaching material

At the end of the course described in the last sections, the teachers worked at the production of teaching material. Here we give a brief review of a part of this material, with some remarks; we selected the teaching units more directly and closely related to algebra; for each subject we show the age of the students that are supposed to use it in the Italian school system.

1. Using numbers and variables in Derive; how to write an algebraic expression in Derive; the relation between handwriting of an expression and "linear writing" of Derive (age 14-15).

This unit is very important; the student is used to read and to write arithmetic and algebraic expressions in the standard multilevel form. Derive requires linear writing and the user must use correctly parentheses. The number of these parentheses can be quite large, even for simple expressions, and this can be very confusing for student.

To solve this problem, the unit emphasises how to transform an algebraic expression in a tree. Derive has a very useful tool for the interpretation of an algebraic expression (and then for the construction of the related tree): using the mouse or the keyboard arrows, the user can explore an expression at the different levels.

At the end of the unit the student could be able to switch from one to another way of writing an expression: standard textbook form, linear Derive form and tree.

2. How to check a numerical conjecture (age 14-15).

   Computer algebra allows working with integers of "arbitrary length" in exact mode; this technical feature can be used in teaching, opening up a new field of application: the verification or the refutation of conjectures. Some conjectures can be formulated using an elementary language and a student of 14-15 can tackle very interesting problems, starting from the easy ones, such as the sum of the first n integers, and ending with some conjectures about prime numbers.

3. How to solve an equation and a linear system using a step by step procedure (age 14-15).

   In this unit first and second degree equations are considered, together with equations involving rational functions or radicals, that can be reduced to first or second order equation.

   In Section 3 we pointed out some of the technical problems arising in the solution of algebraic equations in Derive. After a thorough analysis of these problems, the teachers agreed about the necessity of avoiding the "black box" approach, i.e. using the SOLVE function. They agreed about the opportunity of using a step by step procedure of solution, assisted by Derive, working on complete expressions or on subexpressions.

   When an equation must be solved, the student is stimulated to write down his/her solving strategy and then to use Derive in order to perform the steps.

   The "added value" of the use of Derive is not its computing power but the necessity for the student of organising very precisely the questions to be posed to Derive and the answers received by it.

   The problems of incorrect graphs arising from the use of complex numbers in internal computations were considered by the teachers too advanced to be presented to the students in high school; in this case the set of correctly defined functions is used as a "black box".

4. Graphing a function with Derive; the search for zeroes of a polynomial equation (age 15-16).

   This unit is closely related with Unit 3; after having solved first and second degree equations, we move to higher order polynomial equations.

   Using Derive, we can show to the student that it is possible to solve third and fourth equations (as pointed out in Section 2); some carefully chosen examples can show how Derive either computes a " readable" solution or computes an ugly expression of several lines; in any case we get a solution. With fifth order equations, sometimes we get a solution; sometimes Derive is unable to give an answer.

   These examples can motivate a presentation of some results about the solvability of algebraic equations, about formulas for third and fourth order equations and about non-solvability of equations of fifth or higher degree in the general case.

Moreover they can be the starting point for a discussion and for a study that can be continued throughout high school classes: the relation between algebraic and approximate solution of equations.

Derive is an ideal tool for this study, since it offers a purely algebraic microworld (the Exact mode) and an approximate computing microworld (the Approximate mode) in the same piece of software; we can try first an algebraic solution of the given equation, then move to an approximate one, when necessary.

We want to emphasise that this is a starting point; indeed, the approximate methods of solution of an equation involve the concept of sequence (bisection method) and some calculus (Newton method).

The teachers agreed about the opportunity of introducing the possibility of solving approximately in the first two years; it is reasonable to start using approximate methods as "black boxes", moving to a higher knowledge and to a wider use of them in the last three years of high school.

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THE HISTORY OF MATHEMATICS IN THE EDUCATION OF MATHEMATICS TEACHERS: AN INNOVATIVE APPROACH.[1]

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ABSTRACT

It is a widespread view that mathematics teachers should have a working knowledge of the History of Mathematics, for three reasons. First, so they can make lessons more lively and interesting by adding to the lessons stories about mathematicians of the past; second, to help students develop a sense of Mathematics as a human production, always evolving; and, third, to develop a better understanding of the foundations of Mathematics. We understand there is a fourth reason: through that study the teacher can develop an understanding of the process of meaning production for Mathematics that would allow her/him a much finer reading of the learning processes in the classroom, as well as an understanding of the possibility of different meanings being produced for the 'same' mathematical object, for instance, 'linear equation', function' or 'dimension'. We have developed and conducted a course on the History of Mathematics for undergraduate students in which we read and discussed, over 30 two-hour sessions, four texts: (1) C. Wessel's paper on the analytical representation of directions; (2) G. G. Granger's text on the philosophy of style (a section related to Euclid's 'Elements'; (3) a section of A. Aaboe's 'Episodes from the Early History of Mathematics' (part of chapter 2); and, (4) a section from R. Hersh and P. Davis' 'The Mathematical Experience' (on the Chinese Remainder Theorem). We went from a primary source (difficult reading for them) to texts which discussed 'style', 'interpretation' and 'different presentations', aiming at helping them develop an awareness of the processes involved in meaning production for Mathematics; as much as possible we related the current experience with the experiences they had as undergraduate students taking Mathematics courses. Data from the course will be presented and discussed.

KEYWORDS: History of mathematics, meaning production, education of mathematics teachers.

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Introduction

It is generally agreed by mathematics educators around the world that a working knowledge of the history of mathematics is important for mathematics teachers. The reasons given for that vary. Given the existence of a whole ICMI Study on the relation between history and mathematics education (Fauvel & van Maanen, 2000; see particularly chapters 4 and 5) it is not necessary to go any further.

To the current views, we want to add another. We will argue that a course on the history of mathematics should also be a place where students will discuss processes of meaning production, for both historical and present-day texts.

At our department

In this section we present our department's official view on why the history of mathematics (HM) should be in the curriculum and at which point. This will set the background on which our approached was developed.

In our curriculum a course on the HM is mandatory for both future teachers and future researchers; it is offered for seniors.

The course description says:

> OBJECTIVES (at the end of the course the student will be able to): have a general view of the historical development of mathematics up to the 18th century, as well as its relations to the social development. To establish relations between the development of mathematical concepts in ancient times and their theoretical development from the European Renaissance on. To identify the key mathematicians of the past and to link them to their work.

These objectives are clearly related to specific contents, not to formative processes. We think that the underlying idea is that "...as much as the mathematical content, the mathematics teacher needs to know its history, that is: the history of the content of mathematics" (Baroni & Nobre, 2000; our translation).

We searched the pedagogical project of our undergraduate program (Dept of Mathematics, 1992), looking for further explanations on the choices made; there we found:

> "Many of the difficulties and dissatisfactions regarding [the undergraduate program] are related to the lack of connection between different courses and to the lack of an organised development of learning." (p. 10)

And in this case one could think of the HM as offering some kind of 'stitching' of the different courses and contents, by working with their developments and relations along history. But a different solution was adopted,

> "[...] to group the contents in well defined paths, which we will call 'areas', with structure and extension such that they allow the student a more global and deeper understanding". (p. 10)

and the course on the HM was left to the last year.

The view that what you learn in a course includes the way, in which you have learned it, is becoming more widely accepted (Cooney at al., 1999). For instance, what you learn in mathematical courses that include a parallel discussion of historical and mathematical aspects is different from what you learn in traditional courses that do not do it. And the way in which history is approached would also make a difference.

In the case of teacher education, his/her mathematical education must, in our view, include a discussion of processes, which the future teacher will face in his/her professional life, and by that we mean the courses on Analysis and Algebra, for instance. In other words, if the future teacher is to leave the university better prepared to teach than when she/he entered it, it is not enough to get him to review/practice the content she/he is going to be teaching plus offering 'foundations' for those school topics.

As the authors of a recent and detailed survey on 'teacher preparation research' point out (Wilson et al., 2001), there is no sound evidence "[evaluating] the relationship between teacher subject matter preparation and student learning." (p. 6), and also that,

> "The conclusions of these few studies [that deal with that question] are provocative because they undermine the certainty often expressed about the strong link between college study of a subject matter area and teacher quality" (p. 6)

With that in mind we designed and conducted a course on the MH for undergraduates. The core of the course would be directed towards helping students to develop an awareness of meaning production processes, such as when one is 'interpreting' or trying to understand a primary source text, but also when studying mathematics in present-day ('live') textbooks or attending lectures.

On Meaning Production

Probably the most repeated phrase in situations like the ones indicated above is "what is he talking about?" Similarly, teachers at all levels could ask about their students, "what do they think I am talking about?" Unfortunately, our guess is that this question is much less frequently asked than the other one.

Our central objective in the design of the course was that students could develop an awareness of those processes in his/her own thinking but also, as we are primarily interested in teacher education, that they developed an awareness of being on both sides of the meaning production process, that is, an awareness that not all that is natural or familiar to him or her is natural or familiar to the students, a fact which has important implications for the classroom activities.

What was needed to guide our work was a model, which dealt with meaning and knowledge production. But that model would have to deal primarily with processes, as the students' thinking has always to be read "during the flight".

We decided to adopt the Theoretical Model of Semantic Fields (TMSF), developed by one of us as a tool to support teaching and research in mathematics education (Lins, 1992, 2001).

Its central notions are those of 'knowledge' and 'meaning'. 'Knowledge' is characterised as a statement in which a person believes (a statement-belief), together with a justification she/he has for making that statement. 'Meaning' is characterised as what a person actually says about an object, in a given situation. It is not everything that a person could eventually say about that object. Meaning production and knowledge production always happen together, and objects are constituted through meaning production.

A third notion on the TMSF is relevant here, that of 'interlocutors'. It has to do with why a person thinks she/he can make a given statement in a given activity. We understand interlocutors as modes of meaning production that a person internalises as legitimate during his or her life; they are cognitive elements, not real people. In other words, to believe we can say something we must also believe that 'someone else' would say the same thing with the same justification.

The course

We decided that the starting point would be an excerpt of a text by Caspar Wessel, a Norwegian surveyor who read, in 1797, the paper "On the analytical representation of direction; an attempt. Applied chiefly to the solution of plane and spherical polygons" (Wessel, 1959). The text was translated to Portuguese, from the English version available to us.

Wessel says about his paper that,

> "This present attempt deals with the question, how may we represent direction analytically; that is, how shall we express right lines so that in a single equation involving one unknown line and others known, both the length and the direction of the unknown line may be expressed."

Wessel was a surveyor, interested in solving the problems of his profession, and what he does is simply to develop a representation of lines that allows him, using complex numbers in the calculations, to "solve polygons" (that is, given some of the elements of a polygon to determine all the others, a most common problem in surveying).

To anyone this is not a simple text and trying to read it with present-day eyes made some passages completely obscure or meaningless. That was what made the text seem appropriate: we would be able to discuss meaning production for that text as the production of a plausible account of what Wessel was talking about, which were the objects he was dealing with.

The main question that came up from them during the reading of Wessel concerned the kind of reading we were asking them to do: were we producing an interpretation in the sense of a particular (more, or less, correct) reading of the text, or were we in fact simply producing a plausible account for it? That led us to choose as the second text a section of G. G. Granger's book, "The philosophy of style" (Granger, 1974), in which he approaches the differences in the thinking of several mathematicians from the point of view of different styles:

> "[Our purpose is] to distinguish the plurality of modes of expression and of construction of a concept, and to produce an understanding of how this plurality is linked to distinct ways of practicing and even, if one wants to adopt this expression, to live the symbolism." (our translation) (p. 35)

We proposed the reading and discussion of a section on what Granger calls the Euclidean style. It is particularly interesting because there is plenty of discussion about the relationship between number and magnitude in Euclid; at some points Granger seems to suggest that Euclid could not associate both notions, while at other points he suggests that Euclid did not do so because it would be against his style.

The main question that emerged from that reading was a great difficulty, for the students, to conceive an actual separation between geometrical magnitudes and numbers. So we proposed the reading of a section of A. Aaboe's "Episodes from the early history of mathematics" (Aaboe, 1964), where he presents some of the work of ancient Greek mathematicians.

The fourth and last text was a section of "The mathematical Experience", by P. Davis and R. Hersh, on the Chinese Remainder Theorem. There they give seven presentations of the theorem, ranging from an old Chinese one, to one from computer science and a generalisation to structures other than **Z**, found in a book on algebraic number theory. They analyse the differences among them, but without assuming each refer to different objects. We used this text as a model for their final assignment.

The students during the course

In this section we illustrate the types of questions and comments coming from the students.

During the presentation of the course it became clear that the students' expectation was that the course would consist of lectures on chronologically organised 'stories', involving mathematicians of the past, their mathematical work and their personal lives, a kind of "the rich and famous in the history of mathematics".

Difficulties appeared soon. The students were struggling because the words were being used in ways unusual to them. For a phrase like,

> "[For as we pass from arithmetic to geometrical analysis, or from operations with ordinary numbers to those with right lines, we meet with quantities that have the same relations to one another as numbers, surely; but they also have many more." (Wessel, op. cit., p. 56)

we wanted the students to produce their own understanding; we tried to help them with questions like "what could 'relations between numbers' be?"

On the second day of the course, however, one of the students asked us "whether the rest of the course would be like 'that'". When we asked what the 'like that' meant, she said, "well, you know, like tripping [the slang]..." and our answer was "yes", but with the addition that in respect to their intellectual capacity we would not 'water down' the course, despite being aware of their difficulties. This was a highly relevant passage to us, because it made us aware that at least some of the students were actually frustrated by having to think instead of sitting passively at a lecture; it also gave us the opportunity to introduce a discussion about how we saw the process they were going through as completely similar to studying, say, Analysis from Rudin's textbook.

Two exchanges can be seen as typical here. The first happened as we were discussing § 3, where Wessel says,

> "If the sum of several lengths, breadths and heights is equal to zero, then is the sum of the lengths, the sum of the breadths, and the sum of the heights each equal to zero." (Wessel, op. cit., p. 59)

This is a most intriguing passage of the text, because it does not relate directly to anything said before it in the text. When we asked the typical "what is he talking about?" there was a deep silence and then one student said,

> "Well, he is saying that if the sum of the length, the breadth and the height is equal to zero, then the length, the breadth and the height are each equal to zero."

This is a kind of situation in meaning production that in most cases goes unnoticed. It is quite common that the teacher will simply say that repeating the phrase is not enough that the student has to 'explain it'. What could happen in a teacher's mind that would make him or her see that student's statement as a repetition of the original one? We argue that the teacher could complete the student's statement to make it identical to the original one. Why? Possibly because the teacher was not aware of the possibility that the student could not, in that specific situation, produce any meaning for the original statement, and that means that cognitively he could not see the plurals, and that is why his statement is all in the singular.

We asked the student to read aloud, for the whole dass, § 3, and he did so, reading it in the plurals. When asked again to explain what Wessel was talking about he said,

> "Uh? He is saying that if the sum of the length, the breadth and the height is equal to zero, then the length, the breadth and the height are each equal to zero."

Exactly the same statement, only this time spoken slowly and with a different intonation, as teachers many times do when a student says she/he did not understand something... The crucial point here is that the student was clearly convinced that what he was saying was right and that he was not even aware of the removal of the plurals. But what about the 'objective' text in front of him?

Some of his colleagues said they understood his 'explanation' and that they agreed with him. We were actually puzzled, as we could not produce any plausible meaning for what had happened, and we told them so. That led to more exchanges and we finally understood what was happening. For those students, lengths, breadths and heights were measurements of line segments and at best (or worst, because it is a pretty weird situation) they could be zero. To talk about a sum of lengths being equal to zero is already talking about all lengths being zero and that is what the student was talking about, using, instead of, say, three lengths, a length, a breadth and a height.

But Wessel was, in our understanding, talking about completely different objects. For him lengths, breadths and heights were directed line segments in three orthogonal directions (he does not say this, though). We came to this interpretation precisely because we wanted to produce a plausible meaning for the original statement, without having to change it.

Best of all, this 'incident' gave us a great opportunity to deepen our discussion of meaning production: how many times in their lives, studying 'present-day' mathematics, similar situations might have happened? How many times they (and us) might have read what was not written, simply because we could not produce meaning for what was actually printed?

A second, very brief, exchange, illustrates a different point. Struggling with the text, a student asks the professor:

> "Would this guy have ever spoken to a mathematician while writing this?"

and explained her question:

> "Because it seems he doesn't know what he is talking about..."

She transforms her own difficulty in producing meaning for the text into Wessel's ignorance of mathematics. But she knew from the beginning he was a surveyor, not a mathematician. Would she say something similar if reading a mathematics textbook, written, supposedly, by a mathematician? We think most likely she would not; she would place the difficulty on herself. But in Wessel's case she felt it was legitimate to question his mathematical understanding.

On the section "On meaning production" we spoke of interlocutors we internalise during our lives and which are the sources of legitimacy for what we anticipate we can say. There is, in the situation above, an explanation using the notion of interlocutor: the student had internalised, along her life and experiences that only mathematicians know how to talk about mathematics. Maybe that included remarks by mathematics professors about physicists and engineers.

The crucial process, however, that we want to emphasise was that she transformed her own difficulty in producing meaning for the text into Wessel's ignorance of mathematics. Had she done that before or while studying mathematics, perhaps saying that such and such author does not 'write well'? (but not that the author does not know mathematics...). Also, the discussion generated by her question, about legitimacy and meaning was quite useful, in again, establishing a link between 'interpreting historical texts' and 'studying present day texts', and the link was meaning production.

The reading of the second and third texts followed the pattern of the first, although they were different in the kind of content. We did not stay too long on the fourth text, as it was to be used mainly as a model for the final assignment.

We shall now discuss some general aspects of the final coursework produced by the students. The assignment was to choose a mathematical concept/notion/idea, to find several different presentations (as in Davis and Hersh) and to comment on how each presentation would constitute different objects.

The themes chosen were: the plane; square root; 2nd and 3rd degree polynomial equations; logarithms; irrational numbers; Pythagoras' theorem; fractions; parabola; ellipsis; integer numbers; parallel lines; rule of l'Hôpital; the fundamental theorem of Calculus; differentiation; complex numbers; the number ð.

The criteria for marking was to give full marks (10) if both different presentation and comments were given and correct, to give half of full marks (5) if only the presentations were given and correct; intermediate marks could be given in accordance to this.

Four papers were given full marks: the plane; the fundamental theorem of Calculus; complex numbers; parabola. One was given near full marks, differentiation (9.0), and the remaining eleven were given 5.0's or 5.5's.

These results seem to suggest that there is a clear cut between those who accept that different presentations do constitute (or suggest the constitution of) actually distinct objects, and those who could only see the same objects through different presentations (and for that reason could not say that different objects were there).

Final remarks

This was a first attempt at producing a course on the HM for undergraduate students (particularly future teachers).

It was clear that these students' understanding of history and of the role of studying history was stereotyped and superficial; we think this is an issue to be addressed at future versions of this course.

Two objectives were achieved: to get students to participate reflectively in processes of meaning production and to establish a first link between interpreting a historical text and studying present-day mathematics.

Having only a third of the students reached a sufficient understanding of the process of meaning production, in this particular course, might be taken as too low. But the course was indeed ambitious from the beginning and we knew it was a difficult task. For that reason the one-third success achieved was, we think, very encouraging, and the level of interaction of the students in the discussions is another source of support for the further development of this approach to HM in the education of future teachers.

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COMPUTER ALGEBRA SYSTEMS IN A MULTIVARIABLE CALCULUS COURSE AND CENTER OF GRAVITY PROBLEMS

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ABSTRACT

In teaching a multivariable calculus course, two main difficulties facing the student are to draw surfaces in three-dimensions, and to setup and calculate tedious triple integrals. The added third dimension causes great difficulty even to the well prepared student who has successfully finished a two semester single variable calculus course. The student must now suddenly think in three-dimensions. Visualizing and drawing the corresponding three-dimensional surfaces pose a significant challenge to the novice. To alleviate the problem, the student and the instructor must resort to the modern technologies. Computer algebra systems (CAS) such as *Mathematica* and *Maple* are well equipped to handle such tasks. The paper has two goals: One goal is to demonstrate the usage of the CAS *Mathematica* to learn some standard topics of a multivariable calculus course, such as vectors, partial derivatives, graphing of three-dimensional objects, and multiple integrals. As the second, but most important goal of the paper, we will consider the special topic of the center of gravity of solid objects. This topic was chosen because it uses very many of the techniques learned in a multivariable calculus course. We will show how to use the CAS *Mathematica* to evaluate tedious triple integrals arising in calculating the center of gravity. *Mathematica* can also be used as a visualization tool to draw the graphs of three-dimensional solids under consideration. Usually a standard multivariable course only considers the center of gravity of fixed solids. However, with the aid of *Mathematica*, the students are in an ideal position to consider variable solids as well. Thus, the paper introduces the novel concept of the locus of the center of gravity of certain types of variable solids. The paper also illustrates several facets of A CAS in undergraduate education – the usage of a CAS as a computational tool, visualization tool, experimentation tool, and a conjecture-forming tool.

1. Computer Algebra Systems in a Multivariable Calculus Course

Some of the topics of a standard multivariable calculus course include vectors, partial derivatives, directional derivatives, surfaces in three-dimensions, extrema of functions of two variables, cylindrical and spherical coordinates, multiple integrals, and variable transformations (see [15] and [17]). The following diagram illustrates some of these different facets of a multivariable calculus course:

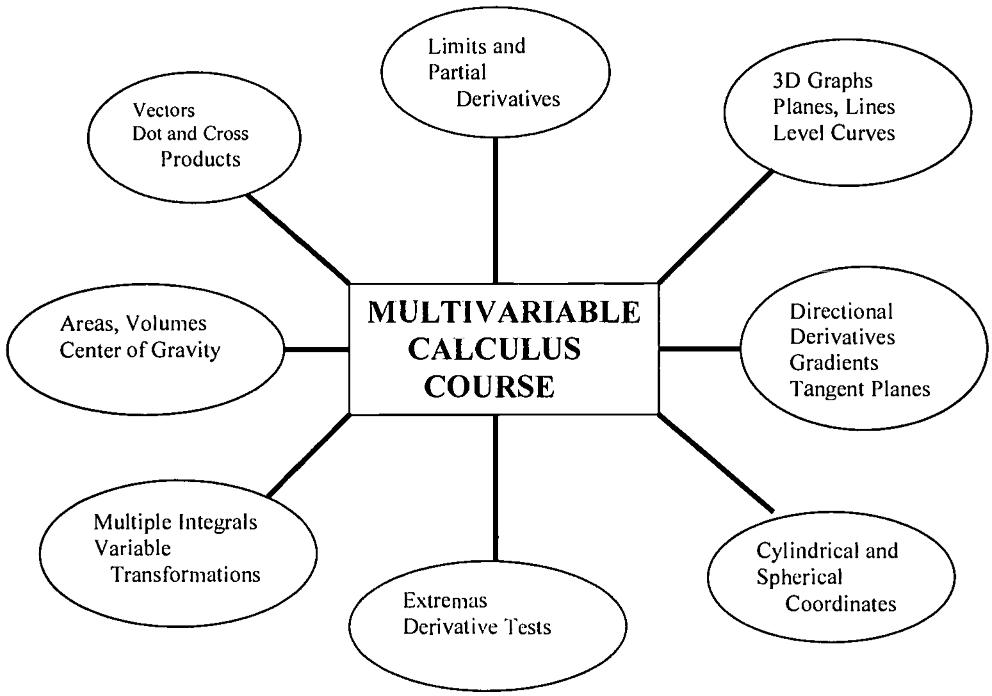

Figure 1.1 Different Facets of a Multivariable Calculus Course

The above diagram is not meant as to represent a complete exhaustive list of topics covered in a standard multivariable calculus course. In this section, we will discuss how to use the CAS *Mathematica* to learn some of the above topics. *Mathematica* is a general purpose CAS. It can be used as a numeric or symbolic computational device, a tool to draw two or three dimensional graphs, a visualization system to analyze data, or even as a multimedia studio to combine sounds and animation. The built-in programming language of *Mathematica* makes it an excellent tool to investigate mathematical or physical problems. Some good references on *Mathematica* are [2], [13], [18], and [19]. For the usage of *Mathematica* as a visualization tool, the reader can refer to [6], [8], [9], [11], and [12]. For the usage of *Mathematica* as a computational or a conjecture-forming tool, refer to [3], [4], [5], [7], and [10].

1.1 *Vectors*

(a) Dot Products:

The *Mathematica* command **"Dot"** can be used to calculate the dot product of any two vectors, two-dimensional or three-dimensional (see [19]).

Example 1.1 Find the dot product of the vectors $u = <1, 2, -3>$, and $v = <-4, 1, 2>$.

Vectors are represented by objects in *Mathematica* called lists. For example, the vector u above is given as the list $\{1, 2, -3\}$. To find the dot product of the two given vectors, use the following *Mathematica* command:

Input: **Dot[{1,2,-3}, {-4,1,2}]**

To execute the command, press **"Shift-Enter"**. The output is -8. Therefore, $u.v = -8$.

Note: Another way to input the dot product command in *Mathematica* is to use the **"."** operation directly from the keyboard. For example, the following command "**{1,2,-3}.{-4,1,2}**" yields the same result as before.

(b) Cross Products

The *Mathematica* command **"Cross"** can be used to compute the cross product of any two three-dimensional vectors (see [19]).

Example 1.2 Find the cross product of the vectors $u = <1, 2, -3>$ and $v = <-4, 1, 2>$.

The following *Mathematica* command achieves the task:

Input: **Cross[{1,2,-3}, {-4,1,2}]**

Press **"Shift-Enter"** to execute the command. The output implies that $u \times v = <7, 10, 9>$.

(c) Triple Scalar Products and Vector Identities

One important result on the dot and cross products is the following vector identity, where u, v, and w are any three-dimensional vectors (see [15] and [17]):

$$u.(v \times w) = (u \times v).w \tag{1.1}$$

Either side of the equation (1.1) is referred to as the triple scalar product of the vectors u, v, and w.

Example 1.3 Use *Mathematica* to establish the vector identity (1.1).

The following *Mathematica* program perform the required task:

Input:
u = {u1,u2,u3};
v ={v1,v2,v3};
w ={w1,w2,w3};

```
expr1=Dot[u,Cross[v,w]]
expr2=Dot[Cross[u,v],w]
Simplify[expr1-expr2]
```

Press **"Shift-Enter"** to execute the program. The first three lines of the program define the three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$. The next two lines will compute the left and right-hand sides of the equation (1.1). The last line of the program will compute and simplify the difference between the two sides of equation (1.1). As an output of the program, one can observe that this difference is zero, establishing equation (1.1).

The following example shows the famous connection between the triple scalar products and 3X3 determinants:

Example 1.4 Given that $\boldsymbol{u} = < u_1, u_2, u_3 >$, $\boldsymbol{v} = < v_1, v_2, v_3 >$, and $\boldsymbol{w} = < w_1, w_2, w_3 >$, use *Mathematica* to show that the triple scalar product $\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w})$ is given by the following 3X3 determinant:

$$\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{1.2}$$

We can use the *Mathematica* command **"Det"** to evaluate the determinant of any square matrix (see [19]). Consider the following program:

Input:

```
u={u1,u2,u3};
v={v1,v2,v3};
w={w1,w2,w3};
expr1= Dot[u,Cross[v,w]]
expr2=Det[{{u1,u2,u3},{v1,v2,v3},{w1,w2,w3}}]
Simplify[expr1-expr2]
```

Press **"Shift-Enter"** to execute the program. The fourth and the fifth lines of the above program compute the left and right-hand sides of the equation (1.2) as expr1 and expr2, respectively. According to the output corresponding to the last line of the program, the difference between expr1 and expr2 is zero. This verifies the above equation (1.2). The importance of equation (1.2) is that it provides the volume of the parallelepiped formed by the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ with a common initial point (see [15] and [17]).

Example 1.5 Use *Mathematica* to calculate the equation of the plane passing through the point (x_0, y_0, z_0) with the normal vector $< a, bc >$.

Here is the idea: Let (x, y, z) be an arbitrary point on the required plane. Then $\boldsymbol{u} = < x - x_0,\ y - y_0,\ z - z_0 >$ is a vector lying on the plane. Therefore, the vector $\boldsymbol{u}$ must be perpendicular to the normal vector $\boldsymbol{n} = < a, b, c >$, so $\boldsymbol{u}.\boldsymbol{n} = 0$. Thus, the equation of the plane can be can be found by setting the dot product of the vectors $\boldsymbol{u}$ and $\boldsymbol{n}$ to be zero. So, consider the following *Mathematica* command:

Input: **Dot[{x-x0,y-y0,z-z0},{a,b,c}] = = 0**

Press **"Shift-Enter"** to execute the above command. The output confirms the following well-known equation of the required plane (see [15] and [17]):

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{1.3}$$

One can of course experiment with the command by assigning numerical values for x_0, y_0, z_0, and a, b, c.

1.2 *Partial Derivatives*

The *Mathematica* command for differentiation, **"D"** can be used to compute all types of partial derivatives of multivariable functions. For example, the command **"D[f[x,y],{x,n}]"** computes the nth partial derivative of a function $f(x, y)$ with respect to x (see [19]).

Example 1.6 Given that $f(x,y) = x^2 + y^2 + x\,Sin(xy) + 5$, compute the partial derivatives $\partial f / \partial x$, $\partial f / \partial y$, $\partial^2 f / \partial x^2$, $\partial^2 f / \partial y^2$, $\partial^2 f / \partial x \partial y$, and $\partial^2 f / \partial y \partial x$.

The following commands perform the required task:

Input:

```
f[x_,y_]:= x^2 + y^2 + x*Sin[x*y] + 5
D[f[x,y],x]
D[f[x,y],y]
D[f[x,y],{x,2}]
D[f[x,y],{y,2}]
D[f[x,y],x,y]
D[f[x,y],y,x]
```

Press **"Shift-Enter"** to execute the above commands: The first line of the program defines the function $f(x, y)$. The other lines are self-explanatory. The last two lines compute the mixed partials $\partial^2 f / \partial x \partial y$ and $\partial^2 f / \partial y \partial x$. The six results are respectively $2x + xyCos(xy) + Sin(xy)$, $2y + x^2Cos(xy)$, $2 + 2yCos(xy) - xy^2Sin(xy)$, $2 - x^3Sin(xy)$, $2xCos(xy) - x^2ySin(xy)$, and $2xCos(xy) - x^2ySin(xy)$.

Example 1.7 Use *Mathematica* to show that the function $f(x, y, z) = 1/\sqrt{x^2 + y^2 + z^2}$

satisfies the following Laplace's Equation (see [15] and [17]):

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0 \tag{1.4}$$

Here are the commands:

Input:

```
f[x_,y_,z_]:=1/Sqrt[x^2+y^2+z^2];
a=D[f[x,y,z],{x,2}];
b=D[f[x,y,z],{y,2}];
c=D[f[x,y,z],{z,2}];
Simplify[a+b+c]
```

The commands are executed by pressing **"Shift-Enter"**. An output of zero indicates that the given function satisfies the Laplace's Equation (1.4). A function such as $f(x,y,z)=1/\sqrt{x^2+y^2+z^2}$ above, satisfying the Laplace's Equation (1.4) is called a harmonic function (see [15] and [17]). The first line of the above commands can be modified to discover other types of harmonic functions.

1.3 *Three-Dimensional Graphs and Level Curves*

(a) Three-Dimensional Graphs

Mathematica provides an excellent system to visualize three-dimensional graphs. Among other methods, one can use **"Plot3D"** or **"ParametricPlot3D"** commands to plot such graphs (see [13], [18], and [19]).

Example 1.8 Use *Mathematica* to graph the hyperboloid $f(x,y)=x^2-y^2$.

Here is the **"Plot3D"** command to perform the task:

Input: **Plot3D[x^2-y^2,{x,-7,7},{y,-7,7},PlotPoints->25]**

Output:

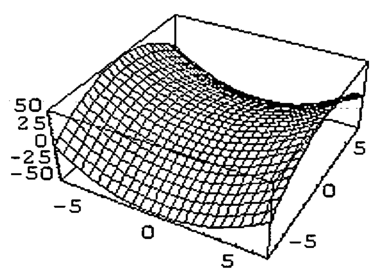

Figure 1.2 The graph of the saddle $f(x,y)=x^2-y^2$

One can use the *Mathematica* command **"ViewPoint"** to look at the surface from different camera angles (see [13], [18], and [19]). Execute the following command to look at the surface from the point (2, 2.5, 0.1):

Input: **Plot3D[x^2-y^2,{x,-7,7},{y,-7,7},PlotPoints->25,ViewPoint->{2, 2.5, 0.1}]**

Output:

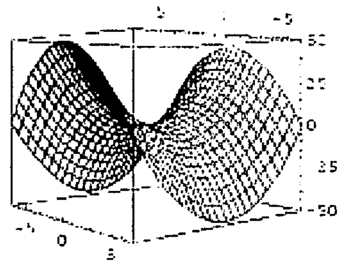

Figure 1.3 The saddle from a different viewpoint

In this paper, we have used *Mathematica* version 3.0. However, note that the *Mathematica version* 4.0 also enables the user to rotate 3D graphs in real time using the **"<<RealTime3D"** command (see [13]).

Example 1.9 Use *Mathematica* to graph the sphere $x^2 + y^2 + z^2 = 4$.

Here it is more convenient to use the **"ParametricPlot3D"** command of *Mathematica*. One can parametrize the sphere using the spherical coordinates (see [15] and [17]): For example, $x = 2Cos\theta\, Sin\phi, y = 2Sin\theta\, Sin\phi, z = 2Cos\phi$ where $0 \le \theta < 2\pi$ and $0 \le \phi \le \pi$ represents any point on the sphere. We use these coordinates with the **"ParametricPlot3D"** command:

Input: **ParametricPlot3D[{2Cos[theta]*Sin[phi],2Sin[theta]*Sin[phi],2Cos[phi]},**
{theta, 0,2Pi},{phi,0,Pi}]

Output:

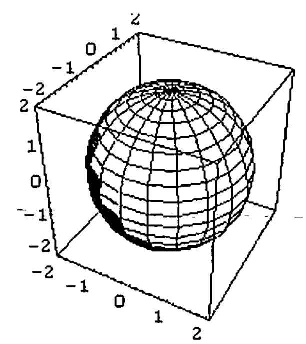

Figure 1.4 The graph of the sphere $x^2 + y^2 + z^2 = 4$

One can restrict the parameters θ and ϕ to see an appropriate portion of the sphere. In other words, this provides a way of "cutting open" the sphere to see the inside: For example, execute the following *Mathematica* command to see what happens:

Input: **ParametricPlot3D[{2Cos[theta]*Sin[phi],2Sin[theta]*Sin[phi],2Cos[phi]}, {theta, 0, 3Pi/2},{phi, 0, Pi}]**

Output:

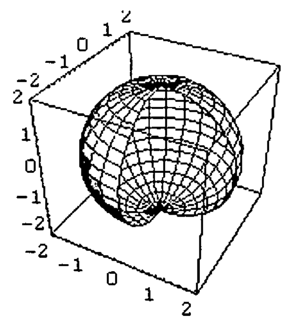

Figure 1.5 The sphere $x^2 + y^2 + z^2 = 4$ with an opening

(b) Level Curves of a Two Variable Function

In general, the level curves of a surface $z = f(x, y)$ are the curves in the *XY*-plane given by $f(x,y) = c$ where c is an arbitrary constant (see [15] and [17]). The **"ContourPlot"** command of *Mathematica* enables one to draw different level curves of a given surface (see [13] and [19]).

Example 1.10 Draw level curves of the surface given by $f(x, y) = x^2 + y^2$.

Input: **ContourPlot[x^2+y^2,{x,-5,5},{y,-5,5},Contours->10,PlotPoints->20]**

In the above, the option **"Contours->10"** will draw 10 contour lines corresponding to ten different heights (c-values):

Output:

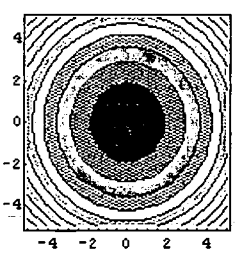

Figure 1.6 The level curves of $f(x, y) = x^2 + y^2$

For the above example, the level curves are all circles. This is because any equation of the type $x^2 + y^2 = c$ where $c > 0$, represents a circle.

Example 1.11 Draw level curves for the surface $z = x^2 - y^2$ corresponding to the values $c = 1, -1,$ and 0.

Mathematica can also be used to draw a specific level curve corresponding to a given c-value as this example requires. For example, to draw the level curve corresponding to $c = 1$, one can use the option **"Contours->{1}"**:

Input: **ContourPlot[x^2 - y^2, {x, -2, 2}, {y, -2, 2}, Contours ->{1}, ContourShading->False]**

Output:

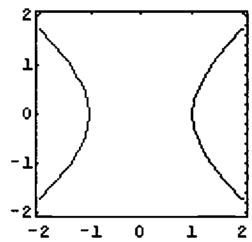

Figure 1.7 The c-level curves of $z = x^2 - y^2$ for $c = 1$

The level curve corresponding to $c = 1$ is a hyperbola, because the equation $x^2 - y^2 = 1$ represents a hyperbola in the XY-plane. One can also plot all three level curves corresponding to the values $c = 1, -1, 0$ together in one diagram. In this case, use the option **"Contours->{1,-1,0}"**:

Input: **ContourPlot[x^2-y^2,{x,-2,2},{y,-2,2},Contours->{1,-1,0}, ContourShading->False]**

Output:

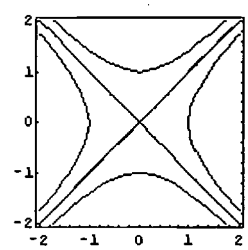

Figure 1.8 The c-level curves of $z = x^2 - y^2$ for $c = 1, -1,$ and 0

It is true that *Mathematica* computes and plots with amazing efficiency, but unfortunately this might set a dangerous trend in students' minds. For example, some students might tend to believe that getting the final answer or the graph is the only objective, and might fail to see beyond this point. Therefore, it must be repeatedly stressed the importance of interpreting the answers obtained by a CAS. For instance, the students must be questioned as to why there are two straight lines in the above level curve diagram. The reason is that the level curve corresponding to $c = 0$ is given by $x^2 - y^2 = 0$, which is equivalent to the pair of straight lines $y = \pm x$. If a CAS is not used with a very open and inquisitive mind, it can create permanent damage in the mathematical upbringing of the students!

1.4. *Directional Derivatives, Gradients, and Tangent Planes*

(a) Directional Derivatives and Gradients

Example 1.12 Find the directional derivative of the function $f(x,y) = x^2 + y^2 + Sin(xy)$ in the direction of the unit vector $\boldsymbol{u} = < 3/5, -4/5 >$.

Note that the directional derivative of a function $z = f(x,y)$ in the direction of the unit vector $\boldsymbol{u} = < u_1, u_2 >$ is given by (see [15] and [17])

$$D_u f(x,y) = \nabla f(x,y).\boldsymbol{u} \tag{1.6}$$

where $\nabla f(x,y)$ is the gradient of the function $f(x,y)$ defined by

$$\nabla f(x,y) = < \partial f / \partial x, \ \partial f / \partial y > \tag{1.7}$$

Thus, using the previously described *Mathematica* commands for the dot products and partial derivatives, one can compute the required directional derivative as follows:

Input:
```
u={3/5,-4/5};
f[x_,y_]:=x^2+y^2+Sin[x*y]
gradf={D[f[x,y],x],D[f[x,y],y]};
Dot[gradf,u]
```

Press **"Shift-Enter"** to obtain the required directional derivative as
$D_u f(x,y) = -(4/5)(2y + Cos(xy)) + (3/5)(2x + yCos(xy))$.

It must be noted that the two partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ are special cases of the directional derivative. For instance, when $u = < 1, 0 >$, the equations (1.6) and (1.7) imply that $D_u f(x,y) = < \partial f / \partial x, \partial f / \partial y >.< 1, 0 > = \partial f / \partial x$. The above program can be used to observe these facts as well.

Note: It must be noted that *Mathematica* has a built-in function **"Grad"** to compute the gradient of a function. However, before using this command, one must separately load the *Mathematica* vector analysis package by using the command **"<<Calculus`VectorAnalysis` "**. This package also has

other built-in commands such as **"Div"**, **"Curl"**, and **"Laplacian"** (see [19]). Also refer to Example 1.13 below.

(b) Tangent Plane to a surface

Consider a surface given by $f(x,y,z)=0$. Suppose f is differentiable at the point (x_0,y_0,z_0) on the surface. Then a normal vector to the surface at (x_0,y_0,z_0) is given by the gradient $\nabla f(x_0,y_0,z_0)$ where

$$\nabla f(x,y,z) = <\partial f/\partial x,\ \partial f/\partial y,\ \partial f/\partial z> \tag{1.8}$$

Here we are assuming that $\nabla f(x_0,y_0,z_0) \neq \mathbf{0}$. Therefore, under these conditions, the equation of the tangent plane to the surface at (x_0,y_0,z_0) is given by the following dot product equation (see [15] and [17]):

$$\nabla f(x_0,y_0,z_0).< x-x_0, y-y_0, z-z_0 > = 0 \tag{1.9}$$

Example 1.13 Use *Mathematica* to find the equation of the tangent plane to the surface $z=4-x^2-y^2$ at $(1,1,2)$. Also graph the surface and the tangent plane together.

The following *Mathematica* program uses the equation (1.9) to compute the equation of the required tangent plane. The *Mathematica* vector analysis package was used to calculate the gradient conveniently as mentioned before.

Input:

```
<<Calculus`VectorAnalysis`
f[x_,y_]:=4-x^2-y^2
phi[x_,y_,z_]:=f[x,y]-z
{x0,y0,z0}={1,1,f[1,1]};
tgpl=Simplify[ Dot[Grad[phi[x,y,z],Cartesian[x,y,z]]/. {x->x0,y->y0,z->z0},
                                    {x-x0,y-y0,z-z0}]]
p1=Plot3D[f[x,y],{x,-3,3},{y,-3,3},DisplayFunction->Identity];
p2=Graphics3D[{RGBColor[1,0,0],PointSize[1/60],Point[{1,1,f[1,1]}]}]
p3=Plot3D[tgpl+z,{x,0,2},{y,0,2},DisplayFunction->Identity]
Show[{p1,p3,p2},DisplayFunction->$DisplayFunction]
```

As the output, one obtains the equation of the tangent plane as $6-2x-2y-z=0$. The program also produces the following graphs of the surface together with the tangent plane at the point (1, 1, 2):

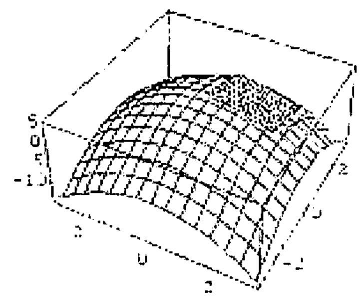

Figure 1.9 The tangent plane to the surface $z = 4 - x^2 - y^2$ at (1,1,2)

1.5 *Multiple Integrals*

The *Mathematica* command **"Integrate"** can be used to compute many multiple integrals (see [19]):

Example 1.14 Calculate the triple Integral $\int\limits_{x=-1}^{x=1} \int\limits_{y=2x}^{y=x^2} \int\limits_{z=x-y}^{z=x+y} xy^2 z(2xy + y^2 z)\, dz\, dy\, dx$.

Input: **Integrate[x*y^2*z(2x*y+y^2*z),{x,-1,1},{y,2x,x^2},{z,x-y,x+y}]**

Press **"Shift-Enter"** to obtain the output as –256/45.

In the next sections of the paper, we will consider an important topic of a multivariable calculus course, namely the center of gravity of solids. This topic uses several fundamental concepts of a multivariable calculus course, such as partial derivatives, three-dimensional graphing, multiple integrals, variable transformations and Jacobians, normal lines and tangent planes. Thus, the center of gravity problems provide us an excellent opportunity to present the usage of a CAS in a multivariable calculus course.

2. The Center of Gravity of Three-Dimensional Solids

Consider the continuous function $z = f(x, y)$ defined on a region R in the *XY*-plane. We will assume that $f(x, y) \geq 0$ for any $(x, y) \in$ R. Let S be the solid under the graph of *f*, directly sitting above the plane region R. Then the center of gravity $G(\overline{x}, \overline{y})$ of the solid S is given by (see [1], [14], [15], [16], and [17]).

$$\overline{x} = I_1 / I \quad (2.1)$$

$$\overline{y} = I_2 / I \quad (2.2)$$

$$\overline{z} = I_3 / I \quad (2.3)$$

where the integrals I_1, I_2, I_3 and I are defined by

$$I_1 = \iiint_D x\,dV \tag{2.4}$$

$$I_2 = \iiint_D y\,dV \tag{2.5}$$

$$I_3 = \iiint_D z\,dV \tag{2.6}$$

$$I = \iiint_D dV \tag{2.7}$$

In the above integrals (2.4)-(2.7), D denotes the three-dimensional region defined by the solid S.

Mathematica can be used to calculate the above integrals efficiently. The following examples illustrate how to use *Mathematica* as a computational and a visualization tool to understand the basic center of gravity problems:

Example 2.1 Find the center of gravity of the solid bounded by the graphs of $z = x^2 + y^2 + 1$, $x = -1$, $x = 1$, $y = -1$, $y = 1$, and $z = 0$.

As discussed in section 1 of the paper, the following **"Plot3D"** command of *Mathematica* can be used to visualize our solid.

Input: **Plot3D[x^2+y^2+1, {x,-1,1}, {y,-1,1}, PlotPoints□40, ViewPoint ->{2.081, -2.552, 0.779}]**

Output:

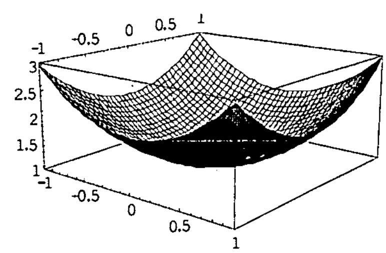

Figure 2.1 The solid bounded above by $z = x^2 + y^2 + 1$

One can now setup and calculate the integrals (2.4)-(2.7). For example,

$$I_1 = \int_{x=-1}^{1}\int_{y=-1}^{1}\int_{z=0}^{x^2+y^2+1} x\,dz\,dy\,dx = \int_{x=-1}^{1}\int_{y=-1}^{1} x(x^2+y^2+1)\,dy\,dx = \int_{x=-1}^{1}\left[x^3 y + x\frac{y^3}{3} + xy\right]_{y=-1}^{1} dx$$

$$= \int_{x=-1}^{1} (2x^3 + \frac{2}{3}x + 2x)\,dx = \left[\frac{x^4}{2} + \frac{x^2}{3} + x^2\right]_{x=-1}^{1} = 0$$

As the problem gets more complicated, the above types of triple integrals become more tedious to do by hand. Thus, A CAS becomes very helpful with the calculation. As discussed in section 1, one can use the **"Integrate"** command of *Mathematica* to compute the above integral (see [13] and [19]):

Input:

```
Integrate[x,{x,-1,1},{y,-1,1},{z,0,x^2+y^2+1}]
```

The output is zero, which means $I_1 = 0$. Similarly, one can use *Mathematica* to obtain $I_2 = 0$, $I_3 = 266/45$, and $I = 20/3$. Then the equations (2.1)-(2.3) imply that $\bar{x} = 0$, $\bar{y} = 0$, and $\bar{z} = 133/150$. This means that the center of gravity of the solid is given by $G(0,0,133/150)$. At this point, the students must be questioned as to why the first two coordinates of the center of gravity are zero. The reason is that our solid is symmetric about the *z*-axis, so its center of gravity must lie on the *z*-axis, which implies $\bar{x} = 0$ and $\bar{y} = 0$.

Example 2.2 Find the center of gravity of the solid bounded by the cylinder $x^2 + y^2 = 4$, the planes $x + y + z = 5$, and $z = 0$.

The **"ParametricPlot3D"** command can be used to graph the cylinder, while the **"Plot3D"** command can be used to graph the plane. The first two lines of the following program graphs these two objects, and then suppresses the output using the option **"DisplayFunction->Identity"**. The final line of the program combines the cylinder and the plane using the **"Show"** command, and displays back the combined output using the option **"DisplayFunction->$DisplayFunction"** (see [13] and [19]).

Program 2.1

```
p1=ParametricPlot3D[{2Cos[theta],2Sin[theta],z},{theta,0,2Pi},{z,0,8},
        PlotPoints□40,DisplayFunction□Identity];
p2=Plot3D[(5-x-y),{x,-2,2},{y,-2,2},PlotPoints□40,DisplayFunction□Identity];
Show[{p1,p2},DisplayFunction□$DisplayFunction, ViewPoint->{1.416, -1.191, 2.833}]
```

The output is as follows:

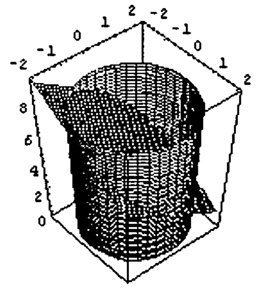

Figure 2.2 The solid bounded by $x^2 + y^2 = 4$, $x + y + z = 5$, and $z = 0$

After visualizing the solid, one can now setup the integrals corresponding to equations (2.4)-(2.7). Note that the base of our solid is a circular region given by $x^2 + y^2 \le 4$. Therefore, it is better to use cylindrical coordinates to evaluate our integrals. In other words, consider the variable transformation $x = r\,Cos\theta, y = r\,Sin\theta,$ and $z = z$. The Jacobian J of the transformation is given by the following determinant of a 3X3 matrix (see [14], [15], [16], and [17]):

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial z \end{vmatrix} \tag{2.8}$$

The *Mathematica* command **"Det"** can be used to calculate the above 3X3 determinant:

Input:
```
x=r*Cos[theta];
y=r*Sin[theta];
j=Simplify[Det[
   {{D[x,r],D[x,theta],D[x,z]},{D[y,r],D[y,theta],D[y,z]},{D[z,r],D[z,theta],D[z,z]}}]]
```

The last input line above evaluates the Jacobian (2.8). According to the output, the Jacobian is given by $J = r$. Therefore, the integral (2.4) becomes $I_1 = \int\limits_{r=0}^{2} \int\limits_{\theta=0}^{2\pi} \int\limits_{z=0}^{5-rCos\theta-rSin\theta} (r\,Cos\theta)(r)\,dz\,d\theta\ dr$. The following *Mathematica* command evaluates it:

Input:
```
Integrate[r*Cos[theta]*r,{r,0,2},{theta,0,2Pi},{z,0,5-r*Cos[theta]-r*Sin[theta]}]
```

The output is -4π, which means that $I_1 = -4\pi$. Similarly, it can be shown that $I_2 = -4\pi$, $I_3 = 54\pi$, and $I = 20\pi$. Then equations (2.1)-(2.3) imply that the center of gravity of

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the solid is given by $G(-1/5, -1/5, 27/10)$. Observe that unlike the Example 2.1, the present solid is not symmetric around the z-axis, so the x and y-coordinates of its center of gravity are not zero.

In both of the above examples, we considered the center of gravity of fixed solids. But what will happen to the center of gravity if the solid is changed gradually? Let us consider the solid in Example 2.2 again. Recall that the upper boundary, or the roof, of this solid was given by the plane $x+y+z=5$, and it passes through the point $(0,0,5)$ on the z-axis. The equation of this plane can be rewritten as $(x-0)(1)+(y-0)(1)+(z-5)(1)=0$. Therefore, it follows that the normal vector to the plane at the point $(0,0,5)$ is given by $<1,1,1>$ (see [1], [14], [15], and [17]). One way of changing our solid is to change this normal vector gradually. One can imagine that as this normal vector is changing, the roof of the solid starts tilting around the fixed point $(0,0,5)$. As the solid changes, its center of gravity changes. It is of interest to track down this center of gravity in three-dimensional space. This leads to a series of interesting locus problems in three-dimensions (see [11] and [12]).

Before the CAS became popular, investigating the problems such as the center of gravity of variable solids was a nontrivial task. Such problems were never dealt with in an undergraduate curriculum because of the complexity of the calculations. However this situation has completely changed due to the wide availability of fast computers and CAS.

In the next section, we will utilize *Mathematica* to investigate the center of gravity of certain types of variable solids. As mentioned in the second to the last paragraph, our motivation comes from Example 2.2.

3. The Center of Gravity of a Class of Variable Solids

Consider the elliptic cylinder in three-dimensions given by the equation

$$x^2/a^2 + y^2/b^2 = 1 \tag{3.1}$$

where a and b are fixed positive constants. Consider the plane through the fixed point $(0,0,c)$, $c>0$, with variable normal vector $<s,t,1>$ where s and t are real parameters. We will assume that c, s, and t are such that the plane will intersect the elliptic cylinder in the upper-half space $z>0$. It is clear that the equation of the plane is given by (see [1], [15], and [17])

$$z = c - xs - yt \tag{3.2}$$

Let S_1 be the solid bounded by the elliptic cylinder (3.1), the plane (3.2), and the plane $z=0$. See the following figure:

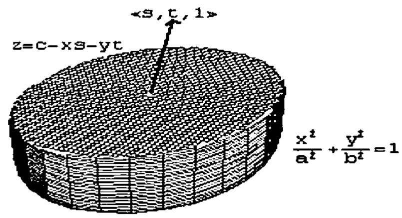

Figure 3.1 The solid S_1 with the normal vector $< s,t,1 >$ at the point $(0,0,c)$ on its roof

Let $G(\bar{x},\bar{y},\bar{z})$ be the center of gravity of the solid S_1. As the parameters s and t change, the roof of the solid changes. Therefore, the center of gravity G of the solid S_1 changes. We want to investigate the locus of G in three-dimensions for changing s and t (see [11] and [12]). The following *Mathematica* program is written using the ideas described in Example 2.2. This program calculates the coordinates of the center of gravity G of the solid S_1, and the locus of G for changing s and t. It then plots the graphs of the elliptic cylinder (3.1), and the locus of G in the same set of axes. Finally, the program makes an animation of G and the solid S_1 in the three-dimensional space.

Program 3.1

```
Clear[x,y,z,r,theta,a,b,c]
x=a*r*Cos[theta]; y=b*r*Sin[theta];                    (* Defines the variable transformation *)
j=Simplify[Det[{{D[x,r],D[x,theta],D[x,z]},{D[y,r],D[y,theta],D[y,z]},
          {D[z,r],D[z,theta],D[z,z]}}]];                        (* Calculates the Jacobian *)
ix=Integrate[j*x,{r,0,1},{theta,0,2Pi},{z,0,c-x*s-y*t}];
iy=Integrate[j*y,{r,0,1},{theta,0,2Pi},{z,0,c-x*s-y*t}];
iz=Integrate[j*z,{r,0,1},{theta,0,2Pi},{z,0,c-x*s-y*t}];
i=Integrate[j*1,{r,0,1},{theta,0,2Pi},{z,0,c-x*s-y*t}];
{x0,y0,z0}=Simplify[{ix/i,iy/i,iz/i}]                                  (* Calculates G *)
Clear[x,y,z]
expr=z/.Solve[Eliminate[{x,y,z}=={x0,y0,z0},{s,t}],z][[1]]        (* Calculates the locus of G *)
p1=ParametricPlot3D[Evaluate[{a*Cos[theta],b*Sin[theta],z} /.
     {a->1,b->2}],{theta,0,3Pi/2},{z,0,10},DisplayFunction->Identity]     (* Defines the cylinder *)
p2=Plot3D[expr/.{a->1,b->2,c->5},{x,-2,2},{y,-2,2},Mesh->False,
                     DisplayFunction->Identity]                        (* Defines the locus *)
Show[{p1,p2},PlotRange->{{-2,2},{-2,2},{0,10}},
            DisplayFunction->$DisplayFunction]     (*Plots the cylinder and the locus together)
p3:=Plot3D[c-x*s-y*t/.{a->1,b->2,c->5,t->s/3+Sin[s]+Cos[s]},
      {x,-2,2},{y,-2,2},PlotRange->{0,10},DisplayFunction->Identity]      (* Defines the roof *)
Do[Show[Graphics3D[ {PointSize[1/40],RGBColor[1,0,0],Point[{x0,y0,z0}]}]/.
           {a->1,b->2,c->5, t->s/3+Sin[s]+Cos[s]}], p2,p1,p3, PlotRange->{0,10},
              DisplayFunction->$DisplayFunction], {s,-2,2,0.2}]
```

The program can be executed by pressing **"Shift-Enter"**. As the first output, one obtains the following coordinates of the center of gravity G of the solid S_1 :

$$G = \left(-\frac{a^2 s}{4c}, -\frac{b^2 t}{4c}, \frac{4c^2 + s^2 a^2 + t^2 b^2}{8c} \right) \tag{3.3}$$

As the second output, one obtains the equation of the locus of G, as given by equation below:

$$z = \frac{c}{2} + 2c\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) \tag{3.4}$$

In fact, one can obtain the equation (3.4) manually by eliminating the variables s and t from the three equations $x = -a^2 s/(4c)$, $y = -b^2 t/(4c)$, and $z = (4c^2 + s^2 a^2 + t^2 b^2)/(8c)$, which arise from equation (3.3). However, the Program 3.1 does this automatically, using the **"Eliminate"** command of *Mathematica* (see [19]). Note that the equation (3.4) represents an elliptic parabolid opening up with z-intercept at $(0, 0, c/2)$. It is interesting to observe that the locus of the solid S_1 is an elliptic parabolid.

The third output of the program plots the graphs of the elliptic cylinder (3.1), and the locus of G as given by equation (3.4) in the same set of axes, showing their relative positions:

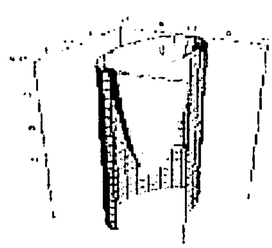

Figure 3.2 The graphs of the elliptic cylinder and the locus of G

The final output of the program produces an animation of the solid S_1 with its tilting roof, along with its center of gravity. The animation can be run by grouping the graphic cells generated by the program into a single cell, and then by double clicking on this single cell. One can observe the different positions of the center of gravity G of the solid S_1 as a moving red dot. Note that the red dot always lies on the elliptic parabolid (3.4), which lies inside the elliptic cylinder (3.1). A few frames of the animation are given below:

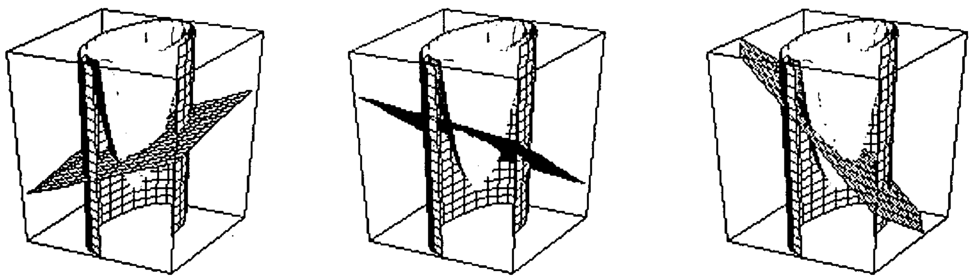

Figure 3.3 An animation of the center of gravity G of the solid S_1

Example 3.1 Observe that the equation (3.3) providing the coordinates of the center of gravity G indeed agrees with the results of Example 2.2. Recall that in Example 2.2, the equation of the cylinder was $x^2 + y^2 = 4$, while the equation of the roof of the solid was $x + y + z = 5$. By comparing these with the equations (3.1) and (3.2), one finds that $a = 2$, $b = 2$, $c = 5$, $s = 1$, and $t = 1$. For these values, the equation (3.3) implies that $G = (-1/5, -1/5, 27/10)$, agreeing with the final result in Example 2.2.

One can summarize the findings of this section in the following Theorem (see [11] and [12]) :

Theorem 3.1 Consider the solid S_1 described in this section, bounded on the sides by the elliptic cylinder (3.1), bounded on the top by the plane (3.2), and bounded below by the plane $z = 0$. Then the center of gravity G of the solid S_1 is given by
$G = (-a^2 s/(4c), -b^2 t/(4c), (4c^2 + s^2 a^2 + t^2 b^2)/(8c))$. For changing parameters s and t, the locus of G is an elliptic paraboloid with equation $z = c/2 + 2c(x^2/a^2 + y^2/b^2)$.

Proof. The student is encouraged to write a proof independent of *Mathematica*, using the methods discussed in Examples 2.1 and 2.2.

4. Conclusion

In this paper we observed how to use a CAS to understand several aspects of a multivariable calculus course, with the emphasis on the topic the center of gravity of a solid. This topic has inherent computational difficulties not just because of the third dimension, but also due to the tedious triple integrals and variable transformations. *Mathematica* can be used as a powerful computational tool to calculate those triple integrals involved. We also observed how to use *Mathematica* as a very effective visualization tool – not just static visualization, but also as a dynamic visualization tool. The

Mathematica programs we have used can also serve as a medium to experiment and a form conjectures on the center of gravity problems. Thus, the paper uncovers different facets of a CAS in undergraduate education. The paper also introduces a novel aspect of the center of gravity of solids, namely the study of the locus of the center of gravity of variable solids. This particular topic is not covered in calculus texts, traditional or otherwise. By introducing such nonstandard topics in a calculus course in conjunction with a CAS, one can take the undergraduate mathematics instruction to a new level. The paper uses *Mathematica* as the choice of CAS, but most of the ideas described here can be implemented by using other CAS such as Maple.

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ON THE PRODUCTION OF MEANING FOR THE NOTION OF LINEAR TRANSFORMATION IN LINEAR ALGEBRA: KIKA AND VIVIAN SPEAK[1]

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ABSTRACT

In our study, based on the Theoretical Model of Semantic Fields (Lins, 2001), we analysed the production of meaning for 'linear transformation' (LT), aiming at producing elements to support a further reflection on the teaching and the learning of Linear Algebra. As part of the study we have conducted interviews with two students of a first Linear Algebra course (undergraduate mathematics degree), seeking to elicit the meanings they were producing for that notion while engaged in trying to 'talk about' particular (and non-usual for them) LT's presented to them. Two of the aspects considered in the analysis were the meanings being produced (and the kernels thus involved, see Lins 2001) and the texts being produced (notations, diagrams, writing, speech, gestures). For instance, we have found out that the students always tried to find a way to visualise the LT's in question (as one may visualise the usual R^2 as a geometric plane). This study is part of a broader project ('A framework for the mathematics-content courses in the university preparation of mathematics teachers') and aimed at producing elements that allow an adequate reading of the process of meaning production in the classroom, leading to new approaches to deal with students' difficulties and to new approaches to the classroom practices of mathematics professors engaged in mathematics teacher education.

KEYWORDS: linear transformation, linear algebra, meaning production, semantic fields

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Introduction

The study reported here is a section of a larger study on the production of meaning for the notion of linear transformation in Linear Algebra. In this section of the study we wanted to examine the meanings two undergraduate students would produce for linear transformation in specific situations.

Other sections included a study of mathematical texts (historical and present-day) in which it is possible identify ideas related to our present characterisation of linear transformation, for instance in the work of Vieta and in the work of Peano.

The theoretical support for the study comes from the Theoretical Model of Semantic Fields (TMSF; see, for instance, Lins, 2001). Its central notions are those of 'knowledge' and 'meaning'. 'Knowledge' is characterised as a statement in which a person believes (a statement-belief), together with a justification s/he has for making that statement. 'Meaning' is characterised as what a person actually says about an object, in a given situation (activity); but it is *not* everything that a person *could* eventually say about that object. Those two notions are naturally useful for producing a dynamical reading of what people are thinking in given situations, a reading of processes rather then of states.

Our primary interest here was not to identify patterns of thought (in terms of its content) which would be immediately generalisable and, in this sense, it was not relevant that the students chosen be in any sense 'typical'. Rather, we wanted to elicit the extent to which it would be possible, with the support of the TMSF, to identify the meanings being produced by particular students in particular activities, for the notion of linear transformation; for this reason, a larger sample was not required. Nevertheless, it is clear that a larger scale study, conducted with the support of the TMSF and in a way similar to the present one could reveal more generalisable patterns; in this sense we think this study suggests a possibly fruitful line for future research. In this direction, the data gathered in this study is consistent with data gathered in other studies of our larger research project, and it suggests that natural and naturalised objects (such as a 'natural' notion of 'space') might play a central role in the production of meaning for mathematical objects and also that they are quite 'resistant' to the usual mathematics courses at university.

What we understand as generalisable, coming from this study, is the approach to the reading of processes of meaning production, which we adopted, and its usefulness in revealing details that would otherwise be missed. Based on what we have learned through the interviews we can suggest that such an approach is indeed required in the classroom if one wants to interact productively with students and if one wants to organise teaching as to be effective. With respect to the latter, our particular interest is in the mathematical education of mathematics teachers.

In the case of the students we interviewed, it is safe to say that the meanings they produced for linear transformations were not in line with what professors teaching Linear Algebra expect the students to produce, particularly in the sense that the notions of (vector) space, vector and linear transformation remained strongly linked to natural notions of space (as the physical space), vectors (as arrows) and linear transformations as operators that 'change' vectors into vectors of the same kind.

This paper focuses on four of a set of five interviews conducted with each of two undergraduate mathematics students. The interviews were designed with different purposes but in all cases there was the intention of getting them to speak as much as possible, to tell us as much as possible, in an explicit way, of their ideas and understandings. Our interest was on what the objects they were talking about actually were *for them* in those specific situations.

The interviews: Kika and Vivian speak

A set of five interviews were conducted separately with each of two mathematics undergraduate students, Kika and Vivian. At the time of the first four interviews they were taking a course called 'Introduction to Linear Algebra', focusing mostly on $\mathbf{R}^2$ and $\mathbf{R}^3$ with the usual structure; at the time of the fifth interview they had already finished the introductory course but had also successfully finished a second course on Linear Algebra which focused on abstract vector spaces.

Interview 1 was set to elicit what they would spontaneously say about some notions from Linear Algebra. Interview 2 was set to elicit some of the relations they established between these notions. Interview 3 was set to elicit how they would talk about a given linear transformation (given by the explicit transformation rule). Interview 4 will not be discussed in this paper. Interview 5 was set to elicit the extent to which their understanding of linear transformations depended on visualisation and on 'the natural space'. A delay between interview 5 and the others (11 months) allowed us to examine the extent to which visual-geometrical meanings had 'resisted' to the work with non-visualisable, abstract vector spaces.

Our observations will 'track' what Kika and Vivian said related to 'linear transformation'.

INTERVIEW 1

Interview 1 consisted of presenting them with a list of notions from Linear Algebra (matrix, linear transformation, sets of linear equations, vector space and vector), asking them to write down what they had to say about them and we subsequently spoke with them (separately) about what they had written.

With respect to 'linear transformation', there was a marked difference between what Kika and Vivian said. Although both mentioned it is a mapping with two special properties, while Kika gave this as the only characterisation, Vivian seemed to associate linear transformations with 'transforming', in the sense of doing something to the vectors:

> [VIVIAN] "Linear transformation I think is a mapping that has some *transformations*, like, like the rotation." (our emphasis)

This initial impression was later confirmed by the other interviews, when Kika too spoke of mappings as 'acting' on vectors and 'doing something'.

A particular aspect of what they said was quite important to us. Both of them referred to 'vector space' as being,

> [VIVIAN] "...all places where the vectors live, like, where they act. And also you can multiply a vector by a real number and it remains in this same, this same little place there, where they live, this same little house."

> [KIKA] "...the space where the vectors act, where we operate with them."

This natural notion of space as 'a place' will be present in all interviews and this has an important consequence. For both of them it is crucial that it be possible to *visualise* the space where the vectors are to be and this has to do with visualising the vectors; without this a vector space does not make sense and talking about a linear transformation involving such space becomes talking about a mapping only, as we will show on Interview 5, when they are faced with a space of matrices. Coherent with this, both understood vectors as [KIKA] "...an oriented line segment... [also used for] representing speed...".

At this point they were dealing only with $\mathbf{R}^2$ and $\mathbf{R}^3$ in the introductory course and it would be reasonable to associate them with the physical space around us but, as we will show, those understandings 'resisted' the second course they took, involving spaces and vectors which could not be easily or at all visualised that way. In the final section we discuss a possible implication of this for teacher education.

INTERVIEW 2

The following 'names' were each written on a card: matrix, basis, set of linear equations, determinant, linear transformation, vectors, linear combination, system of generators, dimension, mapping, vector space and linear independence. Nineteen random draws of three cards each time (the same draws for both students) were made and the students asked to group for each draw the two they saw as more closely related. Then each student was interviewed, about their choices.

There are two remarkable aspects in the groupings, particularly because in all these cases the answers of Kika and Vivian coincide. First, that every time 'linear transformation' and 'mapping' were on the same draw they were grouped together. Second, that every time 'linear transformation' and 'vector space' were on the same draw they were *not* grouped together; in the two draws containing both but not 'mapping', each was left out once.

The combined suggestion is that they understood a linear combination as a mapping only and took the operations in a naturalised way; being simply "places where the vectors live", vector spaces were not part of their understanding of linear transformations. That can be seen when they are asked to group 'vector space', 'linearly independent vectors' and 'linear transformation'. Kika groups the first two and excludes 'linear transformation', and explains:

> [KIKA] "The LI vectors live isn't it in the vector space and the transformation acts on the vectors of the vector space. But first they would be there and then it would act."

Vivian has a similar explanation for the same choice:

> [VIVIAN] "Because... the vector space is like the vectors' little house and then [it] has to be together with the vectors."

It is interesting to notice that in Kika's statement the fact that the vectors are linearly independent has no relevance, although she mentions it, and that Vivian does not even mention it.

When the draw was 'vector space', 'set of linear equation' and 'linear transformation' both grouped the last two and excluded the first; here the reason is probably straightforward, as their professor (following Banchoff's book) had defined linear transformations as given by a set of linear equations.

Summarising the relevant insights from this interview, it seems that those students had a natural understanding of 'space' ('space' as the space we live in, even if presented as $\mathbf{R}^2$ and $\mathbf{R}^3$) and that the teaching had not addressed this fact properly. That given, it was coherent in their thinking that linear transformations were seen as mappings 'only', particularly in the sense that 'mappings do something', as one finds in school mathematics.

INTERVIEW 3

Interview 3 consisted of presenting each student, separately, with a sheet with the question:

> How would you describe the transformation $T(x, y) = (-y, x)$ from $\mathbf{R}^2$ into $\mathbf{R}^2$?

They were allowed to think about the question for about 20 minutes and asked to write down their ideas. After that each student was asked to go to the blackboard and present her conclusions to the interviewer.

In both cases the first statement is that T is a linear transformation. But while Kika actually begins with the 'calculations' (as they called the algebraic verification of the properties) to verify that T is a linear transformation, Vivian never writes or says anything that shows she had actually done them.

The key to understanding Vivian's thinking seems to be in her answer when asked why she thinks that T is a linear transformation:

> [VIVIAN] "Because it satisfies the properties of linear transformation, both [properties]. *And it's called a reflection, this is going to be a reflection on the x-axis. I will draw.*" (our emphasis)

Somehow she convinced herself of the underlined statement (which is not correct) and our interpretation is that assuming that reflections are linear transformations she got to the conclusion on the first statement. in fact all her interview is clearly dominated by making drawings and talking about what the mapping 'does':

> [VIVIAN] (making drawings on the blackboard) "Let's give an example. I'll take a little vector here. Let's suppose with coordinates x, y, any. Here. By the transformation T it'll be *taken* here [...] it will be simmetrical in relation to the x-axis. The angle here is the same, everything simmetrical. This size here, this lenght, will be the same as this [comparing the original vector and the image]. And, yes, it is a linear transformation."

and after realising she was mistaken about T being a reflection on the x-axis,

> [VIVIAN] "Let me think something. [silence] *A linear transformation is something*" (our emphasis)
>
> [...]
>
> [VIVIAN] "Going back. It's a rotation."
>
> [...]
>
> [VIVIAN] "It's a rotation. It's a rotation of pi over two [...]"

Although stating she had done the calculations, they *never* materialise in *any* form (and she gets terribly messed up when trying to talk about them); all the time it is clear she is trying to determine 'which' transformation T is (among the 'prototypes' available to her) and *that* is what will tell her whether T is or not a linear transformation.

Kika, on the other hand, went straight on to the calculations (for the properties) and continued to show that T is injective (by showing that ker T = {(0,0)}). She then considered geometrical aspects (lenght preserving, angle preserving, by drawing particular vectors and their images under T), applied T to the canonical basis and determined the matrix associated to T. Only then, looking at the matrix, she decided T was a rotation of 90^{0} (as in the original written protocol).

Two aspects are more relevant in these interviews. First, the clear difference in the meanings produced by the two students for 'linear transformation' and the consequences of this on the way they deal with the task. Kika is dealing with a mapping that may or may not have some given properties (including being injective), while Vivian is dealing with a mapping which does something to the vectors and it is this 'something' that is central in characterising the mapping.

Second, that Vivian's thinking seems based (again) on a naturalised notion of space, while Kika's seems much less dependent on that. But after interview 5 it became clear that Kika's thinking here was, in fact, quite particular to the task, as the relation between $\mathbf{R}^2$ and the naturalised space was not problematic, that is, the effect of thinking with a naturalised space was not visible.

INTERVIEW 5

Interview 5 had two questions. Each of them was presented to the student on the blackboard and whatever they wished to write or sketch had to be done directly on it. One question was presented first and then discussed; when the researchers were satisfied with the discussion the second one was presented and discussed:

1) How would you describe the mapping

$$f:\{ax+b; a,b \in \Re\} \to \left\{\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}; a,b \in \Re\right\}$$

given by

$$ax + b \mapsto \begin{bmatrix} -b & a \\ 0 & b \end{bmatrix}$$

and

2) How would you describe the mapping

$$g: Z_5^2 \longrightarrow Z_5^2$$

given by

$(x, y) \mapsto (-y, x)$

These interviews were, in at least three aspects, quite different from the previous ones: (i) the vector spaces involved were not $\mathbf{R}^2$ or $\mathbf{R}^3$; (ii) in one question the mapping was not an operator and in the other the field was finite; (iii) at this time the students had already taken with success a second course on Linear Algebra, after the introductory one. As we had already said, the interview was set to examine the extent to which their understanding of linear transformations depended on visualisation and on 'the natural space', that is the extent to which visual-geometrical meanings had 'resisted' to the work with non-visualisable, abstract vector spaces.

Because of the limit imposed on the size of this paper, we will focus our attention on one student (Kika) and on the first question; we chose Kika here because on interview 3 she had preferred to verify the linearity algebraically (something necessary on both questions of interview 5), rather than visually as Vivian did. Further ahead we briefly comment on the other student and on the other question.

When asked about it, Kika stated that *f* is a linear transformation and sketched the calculations to support her statement. However, she is greatly disturbed by the fact that she cannot 'see' what the space of matrices is:

> [KIKA] "...It's difficult to talk [about *f*] because it is a pretty weird mapping."

[...]

[KIKA] "...because it takes a straight line [sic] into a matrix so, it is not something you can describe too clearly. [...] There is no way to describe the *form* of a space of matrices [...] when I think of mappings I always think of [the] domain [being] the real [numbers] [...] it can take to the space of the real [numbers], can take to a circle. I imagine like how the image set would be, right, how I would be describing it and quite frankly in this case I don't know how to describe its *image*." (our emphasis)

'Image' here is clearly used by Kika in a visual way.

To make visual sense of the domain she immediately described its elements as "straight lines" and later said, explaining the calculations she had done to show that f is a linear transformation:

[KIKA] "...if you take two distinct straight lines and you say they take you [sic] to a matrix of this form and if you apply f first to each of them and add the matrices, add the images, I think it would be the same thing if I take the two straight lines, put them together into a single straight line and apply the function. So in this case I think it's right..."

It could seem that she is only using the *name* 'straight line' given the strong link she establishes between the polynomials and the associated polynomial functions. But when she talks about her understanding of the domain there is little doubt that this is not the case:

[KIKA] "The domain is all the straight lines, right, it would be R two. [...] Because here it would be the equations of the lines."

[...]

[KIKA] "...as I vary the a and the b over all the real [numbers] I will be getting distinct lines, like this, this, this [drawing lines on a diagram with two orthogonal axis].... All ways. So they will occupy the whole plane, all R two. That's how I'm thinking. Because I can vary them over all of R, then it would be the whole of R two. All the sets [sic] of possible straight lines [...] Easier to *imagine* than the matrices." (our emphsais)

We think there is a strong suggestion here that Kika thinks vectors must have a visual image that somehow corresponds to arrows (oriented line segments) and that this is related to a naturalised space. She finds a way with the polynomials, but once she cannot produce such a visual image for the space of matrices, the mapping is treated only as a mapping and not as a mapping between vector spaces (she talks about f being or not injective and surjective, and uses the expression 'homomorphism', which does not belong to typical Linear Algebra courses in Brazsil). When one of the interviwers presents her with the statement "this mapping will take a given straight line of the first space to a straight line of the second space", Kika gets in trouble; after an exchange on what could straight lines be on the second space she said:

[KIKA] "Only if I took that matrix and multiplied it by a vector, I don't know, x, y, and it would be like that, the matricial multiplication..."

That indicates, we think, the extent to which she was not able to produce meaning for the statement presented to her, looking very much as a desperate attempt to make any sense of what had been presented to her. When working on question 2 she said that she could not think of rotations unless she could see the straight line that was being rotated and actually laughed when she was told it was a rotation of 90^{o}:

> [INTERVIEWER 2] "I will tell you: that is a 90^{o} angle. Do you want me to prove that the cosine of that angle is zero?"
>
> [KIKA] (laughs)
>
> [INTERVIEWER 2] "Do you?"
>
> [KIKA] (laughs)
>
> [...]
>
> [INTERVIEWER 2] "If I proved to you in a way you could accept..."
>
> [KIKA] "...see..."

She actually *corrects* the interviewer to say that what she needs is not convincing, is *seeing*.

On the first question Vivian, the other student, had a similar difficulty in accepting the space of matrices because she could not visualise them as 'vectors'.

On the second question both students represented the set of vectors as points on a Cartesian diagram, so they could *see* what the transformation 'does', and although in interview 3 both ended up talking of a transformation with the same rule $[(x, y) \mapsto (-y, x)]$ as being a rotation, they did not do so here. In interview 3 Kika had verified algebraically that the transformation was linear, but not here, suggesting that on question 2 of the fifth interview visualising the vectors was necessary *before* it made sense to engage with the algebra. Vivian, as she had done on interview 3, depended almost completely on what she could see the transformation 'doing'.

Conclusions and implications for the classroom

Overall we think that the data gathered through the interviews suggests that there might be a huge gap between successfully taking the two Linear Algebra courses they took and developing a mathematically sound understanding of the objects of Linear Algebra, particularly those of vector space (as a structure), of vectors (as elements of the base set of a vector space) and of linear transformation (as a homomorphism between vector spaces, which are structures). It also shows that the role of a naturalised space remained, in their case, considerably untouched by the ideas discussed during the courses.

The metaphor we have been using to describe this situation is that as the students go into the classroom they leave their natural ideas outside, then try their best to succeed inside the classroom and as they leave the classroom they leave the mathematical ideas inside, take the natural ideas back and go home. Instead of saying that as people finish their schooling they forget the mathematics (for instance, at the end of high-school), we say they do this *everyday*.

The mathematical education of mathematics teachers is the main object of our larger research project, and we strongly suggest that the situation above is highly undesirable in this case, for two reasons. First, because almost all the benefit for her/his mathematical development is reduced to practicing bits of school mathematics that appear during the 'advanced mathematics' courses, for instance, calculating with matrices or doing some analytical geometry, in the case of Linear Algebra.

Second, and more harmful, because the future teacher does not develop an awareness of the process described in our metaphor and for that reason s/he is unable to become capable of dealing with this situation in her/his professional life. To promote such an awareness is what we call 'to educate *through* mathematics' and we consider it to be a key component of the mathematical education of mathematics teachers.

On the basis of our analysis of the interviews there were two questions: (i) which are the objects the students are thinking about?; and, (ii) what are they saying about those objects? Technically speaking, those two questions must be seen as one (Lins, 2001).

We focused our analysis on three objects constituted by Kika and Vivian: space, vector and mapping/function/linear transformation. A naturalised space preceeded the others, as the place where things are, can be, and that does not depend of anything else for it to be conceived of, much like the physical space. Then there were vectors as arrow-like objects, which only made sense as visualisable (inside some naturalised space). Finally, there were functions which almost only made sense as literally transforming a vector into another vector *of the same kind* (operators), as if a vector itself was streched or rotated 'by hand' (again a naturalised space and naturalised operations underlying this possibility).

Those understandings are coherent among them and, as far as we could probe, they resisted to many hours of talk on abstract vector spaces and their properties; moreover, without assuming—at adequate times—a different understanding of those notions, much or all of Linear Algebra is quite useless in the education of future teachers.

Also, the approach we adopted to read the meanings being produced by Kika and Vivian proved to be quite adequate and useful, as it allowed us to go much beyond simply stating that they did not 'know' what a linear transformation 'is', that 'actually' they had not learned. It allowed us to produce a positive understanding of their thinking which showed a consistent set of objects (space, vector, transformation) at the kernel of their thinking, and allowed us to understand how their actual thinking did not correspond to what their professors likely expected from them. A crucial question to raise here is how they could be successful at the two courses they took on Linear Algebra.

Evidence such as that presented in this paper suggest the need for developing teaching approaches that treat natural ideas explicitly, by giving students a chance to talk about them, discussing them in their relation to abstract, non-natural, ideas. In the particular case of the education of mathematics teachers we believe this would require major changes in the way it has been traditionally conceived (mathematics courses plus pedagogical complementation).

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EFFECTIVE INTEGRATION OF COMPUTING TECHNOLOGY IN MATHEMATICS AT U.S.T.L.

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ABSTRACT

During the last four years, I have been engaged in the production of multimedia units for the French national program *On line University* which wants to cover the two first academic years in science.

This paper presents the creation of multimedia software in mathematics, its use for teaching and that of an e-learning platform with students. This experience was both enriching and difficult. In the mathematics department, each year, new progress has been made in the integration of new technology. An analysis of the successes and of the difficulties will be made. In conclusion, I shall suggest some of the chances offered by the links between mathematics and computer science in the field of e-learning.

1 On line University

1.1 The renewal of the first academic years

The development of universal education and the increasing duration of studies have led a large number of students to university and have increased their diversity, in terms of social background and of previous training. Most academic teachers are not prepared to cope with the problems caused by this diversity. Universities were urged to modify their teaching in order to reduce the failure rate. This was done by the setting up of tutorial sessions, whereby initiatives were taken by motivated local teams to develop modern teaching methods and that led to interesting successful results, especially in allowing large numbers of students to see experimental work using visual support.

An academic science network of self-learning centers, the R.U.C.A. was set up in 1987, with eleven universities : Aix-Marseille1, Bordeaux 1, Grenoble 1, Lille 1, Nancy 1, Nice, Paris 6, Paris 7, Paris 11, Toulouse 3, Tours, (and now ten more universities). The R.U.C.A. centers were first concerned with continuous training and used custom made educational material. In 1994, they decided to create their own resources, and the students were to use these resources in their initial training. During the 1990's, the R.U.C.A. teams gained experience by systematical gathering of educational resources and developing training material for the centers. In 1995, the R.U.C.A. launched the P.C.S.M project, *premier cycle sur mesure* meaning "tailor made classes" in order to cover the first two scientific academic years.

1.2 What is *On Line University* ?

In 1998, a general model for the R.U.C.A.'s productions was adopted by the network and the P.C.S.M took the name of *On-Line University** [1]. This project is financially supported by the Ministry of Research and Education.

1.2.1 The contract conditions

The *On-Line University* is a collaborative work and a collective property [2] of its creators. The national site of *On Line University* is on the Internet and, since october 2001, the access to the products are free of charge for consultation without limitation of time. Maintenance and a regular updating of resources are provided. [3] All public establishments in France have the possibility of downloading the products freely, without charge. Public establishments who sell formation will pay a licence. Outside, private or foreign establisments will contract with the CERIMES* and have to pay a commercial licence.

1.2.2 The Specifications

Products must be platform compatible. The choice of standards was decided, with the use of multi-platform languages (*HTML*, *javascript*, *Java*). These constraints are strong because total browser compatibility of languages does not exist. Anyhow, the

[1]* for the Web site in reference

[2]In France, the intellectual right of property of the authors is inalienable

[3]by the U.S.T.L. and soon, by the CINES

R.U.C.A. keeps track of technological progress and further evolutions are already in preparation (XML ...). The staff of the *On line University* program decided that it cannot require plug-in, nor a software package on the client computer. These decisions aimed to have a self sufficient program transferable to foreign countries. So formal computation[4] cannot be used for the *On Line University* program. The paradox is that many mathematicians working to produce units use formal computation during their academic work. The main tool in mathematics to introduce experimentation is not allowed and this indicates a strong limitation of this program in mathematics.

1.2.3 The Structure

Pedagogical structure has to be made flexible. The *On-Line University* program is conceived as a juxtaposition of modules. Each unit can be altered and modules can be reorganized by teachers for their own pedagogical needs inside their university. The graphical framework reflects a classical teaching structure, with two entries :
Activities : learning, practicing, simulating, observing, evaluating.
Themes : with the set of activities available on a given theme.
This teaching structure shows that, mainly, the producers of resources are scientific academics who work to integrate technologies but ignore[5] the researches in educational sciences and cognitive psychology. The ideas of distributed cognition and collaborative work are important, and I shall illustrate them in this paper.

1.2.4 *On Line University* in 2002

Already on line, there are now 885 hours of teaching ressources, in 21 units of 45 h :
— mathematics : 7 units of 45 or 30 hours = 255 hours
— physics : 7 units of 45 hours, 315 hours
— chemistry : 5 units of 45 hours, 225 hours
— biology : 2 units of 45 hours, 90 hours.
Units planned in 2002 represent 120 hours, (mathematics, 30), (chemistry, 45) , (biology, 45). *On Line University* will gather at the end of 2002, 1005 hours of ressources : mathematics, 285 ; physics, 315 ; chemistry, 270 ; biology, 135.

1.2.5 A national, cooperative realization

On Line University is an innovative creation based upon a network of teams of creators. How is it possible ? A unified piloting committee created in 1997 organizes yearly work distribution inside the network. The realization is done by multidisciplinary teams : academic teachers as far as didactic content is concerned and engineers providing technical realization. There is a link with software industries with appeals to companies, for the model, for audits and for specific computer problems. The validation of the contents is done inside the R.U.C.A. the resources have already been assessed by twenty five academic institutions.

[4] Mathematica, Maple, Matlab, Scilab, WIMs ...
[5] with few exceptions

1.3 The Problems in the R.U.C.A. network

1.3.1 Structural Problems

The cooperation inside such a large network is not gained at first. It has always to be created and maintained by discussion and common positions emerge sometimes only after long and fierce debates. Debates about the statutes are necessary but time consuming for the producers. The discussion about the contracts between the different universities implies not only the producers of resources but also the staff of the universities and the interests of all the participants are not always convergent. This is a huge problem for all the producers of e-learning resources in the world.

1.3.2 Integration of Multimedia Resources

For the students, these multimedia resources mean working at their own pace, possibilities of visualization and simulation in science for the discovery of concepts and the development of better intuition. Personalized services for the students should be created. The problems are linked to the size of the realization of the program that challenges the authors with all the problems of innovation and of the different models of teaching presently being debated.

The Guides : the modes of use should be thought about and explanations should be developed by the teachers. How to give the students help adapted to their various learning strategies ? How to reconcile guides and develop the students'autonomy ?

Adaptation of the resources : the possibility for a teacher to make a partial use of resources, to modify them and to integrate them into his own courses is an essential point. The *On-Line University* modules will be completely effective if they are used as a tool by teachers and if they give students precise work to do using them.

Small *grains* : two ways of creation of *grains* to be used by the teachers to create their own teaching material with *On Line University* will be provided inside the R.U.C.A.
— one is to share the units in small independent and self contained grains, without any external link. The teacher will use these grains inside his own creation of resources.
— the other is to provide a hypertexte structure that allows partial integration of units and several ways of use adapted for different kinds of students.
The future will be the use of XML for indexation of all the productions. A following up of metadatas for educational purposes is done by members of the R.U.C.A. but convenient tools are not yet at the disposal of the authors.

E-learning platforms : the integration of resources to combine face to face work with teachers and remote access to resources for personal work is a new problem in France. Each University of the R.U.C.A. chooses its e-learning platform and this software in itself is not sufficient. Researches in educational sciences shown that there is a risk that these tools create more isolation for the students. How then to create collaboration between students for learning ?

2 Technologies in Mathematics

I shall detail the problems that I have encountered as author of *On Line University* and of a book[6]. I feel that producing multimedia resources is different than writing books. One can see that many resources on the Web are just paper material put on line as *pdf* or *html* files.

2.1 Mathematics writing

Exploiting the possibilities of multimedia is not a part of the current culture of the authors. Animations and visualizations bring deep changes in the creation of resources. The possibility of experiment in mathematics is still in its initial phase. Significant progress can be seen with the use of *Java applets* in the unit *Differential Equations*[7].

2.2 Creation of teaching material in Lille

For many years, we have produced and shared pedagogical material inside the mathematics department of the U.S.T.L. We have an exercice data base, printed courses and experimental software using *dos*, each of two hours teaching :
— linear algebra and Gauss pivoting method for linear system;
— visualization of integration methods;
— qualitative study of differential equations;
— ε, δ definition of convergence of a sequence.
Last year, new versions of this software was developped using *java applets*. One can be seen in the exercice part of the *Integration* module of *On Line University*.

The first module of *On Line University* created in Lille is a transition program between secondary school and university, including *Logics and Naïve Set Theory*, *Elementary Arithmetics* and *Geometric Introduction to Linear Algebra*. The main objective was to provide students many exercises from elementary ones to more conceptual ones, with immediate self-assessment. This was done using *javascript* and many *multiple choice questions*. Written model answers are also analysed. Each exercise has a link with a lesson and a return button prevents disorientation inside the hypertext. A printed *pdf* version of the course material is also available inside the centers of multimedia resources. The unit about Taylor series and *Limited Developments* of functions[8] relies on the same objectives. We have used many graphs created with Maple and integrated in ordinary *html* pages, a limited attempt to exploit the possibilities of formal computation. A beautiful *java applet* for experimental work is provided inside this unit.

2.3 Mathematical typography

The whole set of mathematical symbols is not yet implemented in the browsers. Generally the teams of the R.U.C.A. do not use *LaTex* nor *Tex*; the choice of creating pictures for mathematical symbols was natural and the researches about mathematical typography are unknown by the staff of the program. In Lille, there are specialists of

[6] La fabuleuse histoire des nombres, Diderot édition, 1998
[7] by V. Gautheron (Paris 7) and E. Isemberg (Paris 11)
[8] a separate chapter in French books, but not in English ones

mathematical typography. Here, our resources were in *LaTex* and we knew the state of the art about the use of *LaTex* for the WEB[9]. While waiting for the implementation of *mathML* in the browsers, several attempts were made to use *LaTex* and *pdf* and rejected by the staff of the program[10]. With the software package *LaTex-for-html*[11] *gifs* were created for every mathematical symbol, four size *gifs* for each. The same *gif* is used in all the files and the next change, when *mathML* is available in all the browsers will be easy. *MathML* will solve the problem of composition of symbols in the formulas.

2.4 Multimedia material using *LaTex*

All the possibilities of *LaTex* are not always exploited by mathematicians. With *LaTex*, you can create slides, conference material, insert pictures, graphs and hyperlinks. The conversion to *pdf* files is easy. For my work in history, I make a systematic use of *LaTex*. I teach the *History of Mathematics* both to students and to teachers. Some of my papers and teaching material are on the WEB on the site of the LAMIA* laboratory, in the I.U.F.M.[12]. For *The History of Pythagoras' Theorem*, I have many many *java applets* using *cabri*[13]. I have a *pdf*, a printable file but on line, you can click on the pictures open a pop-up window with an applet and use the hyperlinks. A *html* version was created with *LaTex-for-html*. I train my students and the trainees to use Internet resources by providing in my teaching material hyperlinks to many websites and I show them how to analyse and make a critical use of such material. So, the mathematicians have many possibilities to use their professional word processor *LaTex* to easily create multimedia resources.

3 Integration of Technology

3.1 Important Efforts

These last few years, a very large financial support, often in association with European funds was made to equip universities, primary schools and secondary establishments. Altogether, the equipment of the establishments has grown very fast. Many colloquiums are organized at several levels for management staff and teachers. A national portal for the visibility of educational resources on the Internet has just been created for primary and secondary school teaching* and another one for academic teaching*.

At the U.S.T.L., seven resource centers are equipped with about two hundreds computers and organized by the S.E.M.M.*; Service d'Enseignement Médiatisé et Multimédia Fourteen young people are employed for supervision of these centers during opening hours. Now, the main use of these computers is mail, chat and forum on the WEB. Interesting use for personal researches and use of pedagogical multimedia exist but are not principal. The students are very fond of these resource centers that are full all day long. This in turn causes teachers who see this interest of the students to start

[9]see: The *LaTex* Web Companion, Integrating *Tex*, *HTML*, and *XML*, Michel Goossens and Sebastien Rahtz, Addison Wesley, (1999)

[10]see Cousquer's paper

[11]on the C.T.A.N, Comprehensive *Tex* Archiv Network

[12]A training college where I am head of the multimedia laboratory LAMIA

[13]a dynamic geometry software with possible conversion to *java applets*

to realize the importance of these centers. How to avoid the situation met some years ago, where many establishments were equipped with computers which were used by some colleagues keen on computer science among the general indifference of the others?

3.2 Developments in Cognitive Psychology

The lack of previous products is explained partially by the state of research in cognitive psychology and by software package performance. At the early stage of its development, computer-assisted training was linked with behaviour theories and led to many pieces of software being produced where students were strictly guided on a path of questions and answers. This aspect quickly found its limitations in view of the difficulty for researchers to elaborate a *model of the pupil*. It had the same limits as the underlying conceptions of teaching. With the development of artificial intelligence, some expert systems were created and training intelligently assisted with computer was developed, but resulting products remained marginal. The current state of computer software development with systems based on hypertexts and the use of Internet, introduces a qualitative change which makes a larger use of these tools in training possible. With the development of user friendly tools, the problems becomes different. Even if techniques are very important, didactical content becomes essential. The main objective is the integration of these technologies in teacher training and in the teaching of pupils.

3.3 Collaboration

Network-based learning is now well developed, especially in the U.S.A. and U.K. and it is possible to examine progress and draw conclusions. These three last years, I have animated a workshop in the I.U.F.M. about collaborative learning. We have studied many e-learning experiences [14]. A good synthesis can be found in the paper of Anderson and Jackson and in the book of P. Dillenbourg. Several laboratories of the North of France cooperate in *Formascience* program based on these ideas [15]. We all share the point of view of Scott Grabinger about the necessity of *REAL Rich Environments for Active Learning* and I have tried to apply the same ideas in my own teaching.

3.4 Experience in the maths department of the U.S.T.L.

We have made progress in the integration of multimedia. The department decided an experimentation in 2001-2002. A CD-rom of resources available for the first academic year has been created, gathering units of *On Line University* and other resources; it has been distributed to 800 students in the first academic year [16]. Each group has received from his mathematics teacher a CD-rom and organized the diffusion. So the resources act as help for personal work. Several teachers use the practical software in resource centers and new teachers are engaged in creation. A network of mathematicians animate a workshop for formal computation and its use in teaching.

[14] cf the ieee journal of 2000 July

[15] piloted by Alain Derycke and Chantal d'Halluin (U.S.T.L.)

[16] 27 groups : 15 deug Mias, (Mathematics and Computer Science), 8 deug SM, (Physics and Chemistry), 4 deug Mass, (Mathematics and Economy)

In 2000-2001, I have tried two uses of an e-learning platform inside my teaching, in a center of resources. The first experience was not convincing and I understood that, without new teaching methods, these tools are not interesting for attending students. So I prepared a new experiment in the *History of mathematics*. This experience was very positive. First, I did not explain the functioning of *Campus virtuel*. The students discovered it progressively with new tasks to fullfil. The forum was used by teams of students to solve open problem given without answers. They had to find collectively the answers; each team had the responsibility for one subject and the task to organize a structured discussion to find the solutions. The students were motivated and some discussions were interesting and rich.

4 Conclusion

Mathematics has very close links to computer science both for fundamental research, (logic, algorithms, codes, geometry, formal computation) and for applications. This link is now increasing and the new technological tools are going to change deeply the teaching of mathematics, with the possibility of simulation, visualization and experimentation. A French committee*(*Commission Kahane*) is examining the future trends of mathematics teaching from primary school to university. A strong idea is to develop laboratories of mathematics in secondary schools.

The Mathematics department of the U.S.T.L. is engaged in a reflexion about teacher training : the new technological tools rely upon mathematics and mathematic teachers can have a determinant role in giving an impulse to the use of technologies by pluridisciplinary teams of teachers. So the question is : how to create a high level of training both in mathematics and in the use of new technologies ?

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USING TECHNOLOGY TO IMPROVE THE CURVE LEARNING OF BASIC NOTIONS IN ALGEBRA, CALCULUS AND GEOMETRY

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ABSTRACT

One of the main problems facing mathematics teachers in scientific and technical disciplines (Physics, Chemistry, Engineering, etc.) at universities or engineering schools when receiving first year students is the need of providing them with the capabilities required to understand advanced notions from the early beginning in order to be able of following the initial explanations of teachers talking about Physics, Mechanics, Chemistry: usually the first explanation starts by writing down a differential equation in the blackboard when students hardly understands correctly what a real number is !

The objective of this paper is to report how, firstly, a proper combination of technology (distance web learning through **WebCT** plus the Computer Algebra System **Maple**) and, secondly, a different way of presenting difficult notions concentrated more on the ideas than in the formalisms have been extremely useful in order to:

- Give the students the capability of understanding the initial explanations of teachers talking about physics, engineering, etc.;
- Reduce the gap between the mathematics explained at the secondary school and the mathematics expected to be known by a student when entering at the university (a critical problem in Spain from several years ago); and
- Provide to the students, in a very fast way, with a more solid set of math foundations to be used as an initial stratum.

This experience has been organized around a course of 60 hours (27 hours the first month, 21 the second one and 12 the last one) delivered at the very early beginning of the first year for Physics students at our university. It consists in ten modules of six hours each with three hours of explanations devoted to motivate and illustrate concepts and techniques plus three hours of practical problems with one of them including the using of Maple.

The tool used to control the individual progress of each student was **WebCT** through the realization of several questionnaires containing multiple-choice questions trying to identify initial misunderstandings or to detect unexpected difficulties.

KEYWORDS: Experimental Mathematics, Computer Algebra Systems, Basic Mathematical Training

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1. Introduction

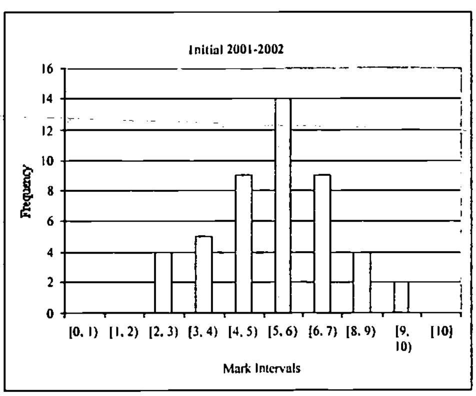
This paper is devoted to report an experience trying to solve (or alleviate) the problem of improving the mathematical stratum (in contents, in abilities and in comprehension) of first year university students in technical or scientific disciplines such as Physics, Chemistry or Engineering.

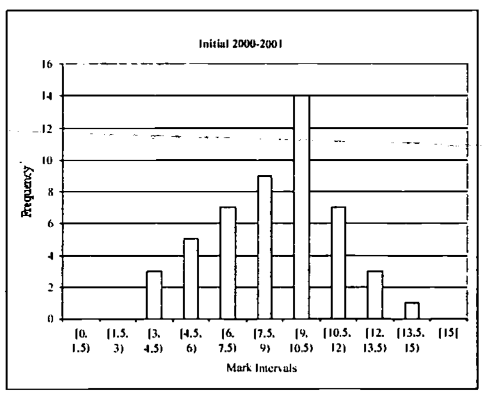
It is a common point, at least in Spain, agreed by almost all those professors teaching Mathematics to first year university students that they bring less mathematical notions (most of them assumed known by professors at university level), their ability to manipulate correctly (and, what is most important, in a coherent way) mathematical expressions is very poor and that the level of their understanding of basic notions is below the one required to be used without a previous remembering.

For the academic course 1999-2000, since the curricula of the Physics studies was updated and modified at that moment, it was decided to create a 60 hours course entitled *Laboratory of Mathematics* organized around ten modules of six hours each covering each module a concrete (and relevant) topic (see below for the concrete list of modules). Each six hours module has the following structure:

- The first three hours are devoted to present and motivate the relevant concepts mainly with examples and avoiding, if possible, complicated notations or mathematical language abuse such as $\forall\varepsilon>0 \; \exists\delta>0$.
- In the next two hours two professors per group assist several groups of at most 30 students where they do, alone or in-group, a set of selected exercises (easy manipulation tasks) or problems (more complicated questions involving usually the joint use of several notions).
- Last hour, with the help of the Computer Algebra System **Maple**, is devoted to re-do some of the exercises or problems considered into the previous two hours or to illuminate and clarify through examples with a computational flavor some of the concepts regarded in the considered module.

This course has been already delivered twice (for 1999-2000 and 2000-2001) and it is compulsory for first year students of Physics studies. Upon arrival, and in order to adequate the course content to the new students, they answer a questionnaire in **WebCT** with between ten and fifteen multiple-choice (very elementary) questions aimed to detect unexpected misunderstandings or new non-known concepts. Next tables present the results obtained by showing a big proportion of students do not manage concepts such as line/point/plane relative position in 3D space or relative to the distribution of rational/real numbers in the real line.





Students are evaluated through the answering of two one-hour questionnaires of multiple-choice questions in **WebCT** plus the realization of a three hours written exam containing a set of selected problems involving each one the manipulation of several concepts and some capability of manual (and correct) manipulation of mathematical expressions.

According to the initially detected problems and to the requirements of other non-math professors involved into the first year of the Physics studies the ten modules were defined in the following terms:

1) Numbers and equations:
   - Representation and manipulation of numerical and algebraic entities: integer, rational, real and complex numbers; polynomials; algebraic fractions; equations and inequalities involving the absolute value.
2) Matrices and linear systems of equations:
   - Matrix and vector operations; rank; determinants; linear systems of equations (Cramer's rule, Rouche criteria, Gauss algorithm).
3) Sequences and limits:
   - Arithmetic and geometric progressions; convergence; limit calculus; series; sum ability (hypergeometric, arithmetic-geometric, the number e).
4) Functions and continuity:
   - Function characteristics (domain, graph, symmetries, periodicity, extremes, inverses, asymptotes; elementary functions (trigonometric, logarithms, exponentials, etc.); limits of functions; continuity.
5) Derivatives:
   - Geometrical and physical definition; derivatives calculus; max and min computation; Taylor series; computation of the graph of a function.
6) Integrals:
   - Geometrical and physical definition; primitive calculus; area, volume and length computations; numerical integration.
7) Differential equations:
   - Solution of a differential equation; exponential of a matrix; homogeneous ordinary differential equations with constant matrix.
8) Analytic geometry:
   - Points and vectors; coordinate frames; transformations; lines and planes; incidence and parallelism.
9) Euclidean geometry:
   - Scalar and vector product; distances and angles; polygons, solids, areas and volumes; conics and quadrics.
10) Data manipulation and visualization:
    - Interpolation; least squares; curves and surfaces (parametric, implicit, visualization).

Topics in blue represent those concepts completely new to the students.

All the generated material can be consulted by visiting the web page (in Spanish):

http://gesacapc22.gestion.unican.es:8000/public/lab201/index.html

where:

- the lecture notes together with the selected exercises,
- the Maple worksheets corresponding to the selected exercises, and
- several questionnaires

are available.

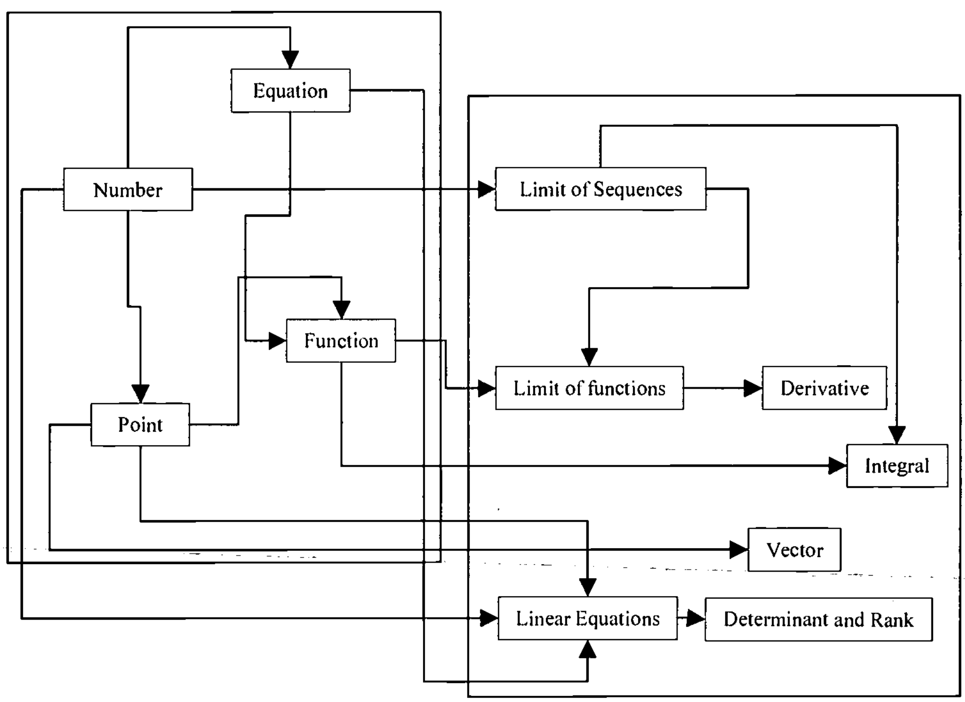
2. How to teach in an easier way difficult mathematical notions?

When arriving to the university students of scientific or technical disciplines have already heard about hard to understand mathematical concepts and, in fact, they have been evaluated in order to demonstrate their understanding and ability of manipulating such notions. Usually two different problems can be identified, with a non-very clear border, dealing with this question: either the concept is not correctly understood but its manipulation is rather acceptable or the concept is understood but not adequately manipulated. A third point to be addressed is the ability of using several concepts, initially not connected, to solve a concrete problem whose resolution requires the combined use of several techniques.

For example, it is very usual to find students with the ability of computing correctly derivatives or limits but without any clear idea about the meaning of what they are computing or without knowing why they are doing the computations in that way.

Our approach is concentrated around four building blocks: the notion of Number, the notion of Equation, the notion of Function and the notion of Point. If these concepts are not very well understood then the student will find big difficulties in order to follow not only other mathematical courses but also any other topics where Mathematics is the language and the tool (mechanics, dynamics, chemistry, etc.).

Around these four building blocks the different main concepts to be considered rotate as shown in the next diagram:



A first basic principle during all the course is to make explicit mention of when the basic building blocks are being used: for example when defining limit L of a sequence a_n the process where each a_n is closer to L than some $\varepsilon>0$ is presented as the concrete solving of a equation (inequality in this case) or the definition of determinant appears as an intelligent way of automatically solving a linear system of equations. In the same line students are shown that all the considered notions are strongly interconnected and thus, for example, the limit of a function f at a point α is introduced by considering the sequence $f(x_n)$ for any sequence x_n converging to α or the definition of definite integral appears as the limit of the sequences of areas approximating the desired to compute area below the graph of the considered function.

The second principle is devoted to provide motivations allowing the student to reproduce in many cases a formula or technique when it has been forgotten but it is needed: it is very easy to motivate how to compute the length of a curve by a very simple argument involving only the notion of integral as infinite sum plus Pythagoras Theorem.

Apart from the use of the building blocks as starting point to consolidate or introduce other notions, another fundamental objective of the course is to provide and improve the student's abilities concerning the formal and correct manipulation of mathematical expressions. For example, limit calculus is presented as a rewriting process where the initial sequence or function is presented in an equivalent form, where to read easily the value of the limit (in case it exists):

$$x_n = \sqrt{n^2 - n} - n$$

$$\Downarrow$$

$$x_n = \frac{(\sqrt{n^2 - n} - n)(\sqrt{n^2 - n} + n)}{\sqrt{n^2 - n} + n}$$

$$\Downarrow$$

$$x_n = -\frac{n}{\sqrt{n^2 - n} + n}$$

$$\Downarrow$$

$$x_n = -\frac{1}{\sqrt{1 - \frac{1}{n}} + 1}$$

This is done by the usual supervised mathematical training through exercises and problems plus the repetition of the latter with the help of the Computer Algebra System **Maple**. Next **Maple** session shows how a basic problem can be solved analytically, but visualizing at each stage what is going on, which is much more difficult to do (and time consuming) if a Computer Algebra System is not available.

Problem

Prove that if

$$f(x) = x^2 - 3x$$

then

$$\lim_{x \to 1} f(x) = -2$$

by computing for any $\varepsilon>0$ an interval around $x=1$, $(1-\delta,1+\delta)$, such that $f((1-\delta,1+\delta)-\{1\})$ is contained in the interval $(-2-\varepsilon,-2+\varepsilon)$.

```
> f:=x->x^2-3*x;
```

$$f := x \to x^2 - 3x$$

First the interval around 1 where the condition is verified is computed for $\varepsilon=1/10$.

```
> solve({f(x)>-2-1/10,f(x)<-2+1/10},x);
```

$$\{\mathrm{RootOf}(10\_Z^2-30\_Z+19, .9083920217) < x, x < \mathrm{RootOf}(21+10\_Z^2-30\_Z, 1.112701665)\}, \{$$

$$\mathrm{RootOf}(21+10\_Z^2-30\_Z, 1.887298335) < x, x < \mathrm{RootOf}(10\_Z^2-30\_Z+19, 2.091607978)\}$$

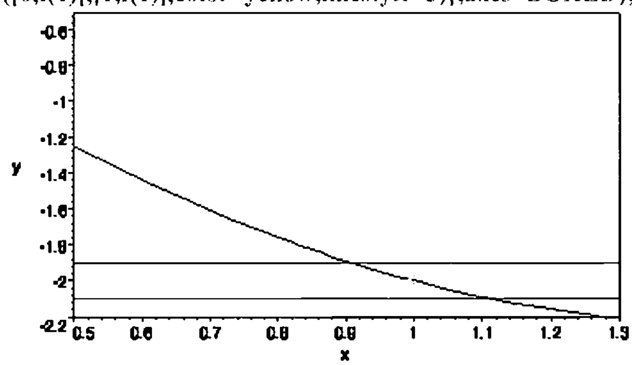
Next the graph of f is displayed together with the lines $y=-2-\varepsilon$, $y=-2+\varepsilon$, $x=\alpha$ and $x=\beta$ with α and β the endpoints of the interval around 1 and verifying the required condition.

```
> map(evalf,[solve(f(x)=-2-1/10,x)]);map(evalf,[solve(f(x)=-2+1/10,x)]);
```

$$[1.887298335, 1.112701665] \quad [2.091607978, .9083920217]$$

After solving these two equations, which are α and β?

```
> display({plot(f(x),x=0.5..1.3,y=-2.2..-0.5,color=black,thickness=1),
    plot(-2-1/10,x=0.5..1.3,y=-2.2..-0.5,color=blue),
    plot(-2+1/10,x=0.5..1.3,y=-2.2..-0.5,color=blue),
    line([1.112701665,0],[1.112701665,-2.2],color=green,linestyle=1),
    line([.9083920217,0],[.9083920217,-2.2],color=green,linestyle=1),
    point([1,f(1)],color=magenta),
    line([1,-2.2],[1,f(1)],color=yellow,linestyle=3),
    line([0,f(1)],[1,f(1)],color=yellow,linestyle=3)},axes=BOXED);
```

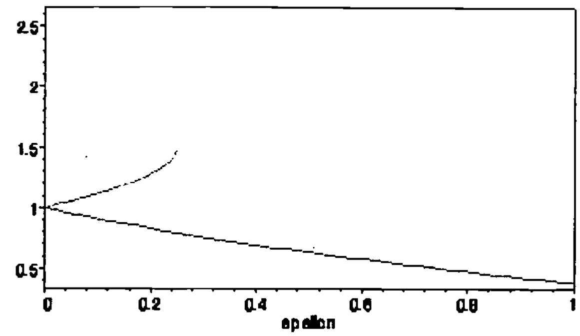


Finally the general case is solved. Study the roots in terms of ε giving the endpoints of the interval around 1.

```
> sol1:=[solve(f(x)=-2-epsilon,x)]; sol2:=map(evalf,[solve(f(x)=-2+epsilon,x)]);
```

$$sol1 := \left[\frac{3}{2}+\frac{1}{2}\sqrt{1-4\varepsilon}, \frac{3}{2}-\frac{1}{2}\sqrt{1-4\varepsilon}\right] \quad sol2 := \left[\frac{3}{2}+\frac{1}{2}\sqrt{1+4\varepsilon}, \frac{3}{2}-\frac{1}{2}\sqrt{1+4\varepsilon}\right]$$

```
>sol11:=plot(sol1[1],epsilon=0..1,color=yellow):
 sol12:=plot(sol1[2],epsilon=0..1,color=red):
 sol21:=plot(sol2[1],epsilon=0..1,color=green):
 sol22:=plot(sol2[2],epsilon=0..1,color=blue):
 display({sol11,sol12,sol21,sol22},axes=BOXED);
```



Solution: Use the previous computations to give a solution to the considered problem.

If ε is in the interval $(0, \frac{1}{4})$ then

$$f\left(\left(\frac{3}{2}-\frac{\sqrt{1+4\varepsilon}}{2}, \frac{3}{2}-\frac{\sqrt{1-4\varepsilon}}{2}\right)-\{1\}\right)$$

is in the interval $(-2-\varepsilon, -2+\varepsilon)$.

If $\frac{1}{4} \leq \varepsilon$ then

$$f\left(\left(\frac{3}{2}-\frac{\sqrt{1+4\varepsilon}}{2}, \frac{3}{2}+\frac{\sqrt{1+4\varepsilon}}{2}\right)-\{1\}\right)$$

is in the interval $(-2-\varepsilon, -2+\varepsilon)$.

3. About the mathematical impact of new technologies when used for teaching Mathematics

This section is devoted to show how the decision of using **Maple** (or any Computer Algebra System) to help the students in order

- to assist the understanding by providing an experimentation framework with visualization facilities; and
- to easily perform complicated computations

has several side effects that need to be taken into account as described later in this section.

The use of **WebCT** has also another implications derived from the use of internet for teaching but with a smaller impact concerning the mathematical contents of the course but remarking that the facilities provided by **WebCT** allows the teacher to easily control the individual progress of each student or to detect in advance unexpected misunderstandings.

For the material concerning the practical Maple sessions we consulted several texts available (see the references section to see a selection of the consulted textbooks) finding that

- Either there is no introduction to **Maple** (knowledge already assumed by the students) or, data structures & algorithms are freely used without providing the students with a minimal background to these Computer Science notions.
- When dealing with the computation of roots of polynomial equations (choosing the first significant example) Numerical Analysis enters immediately into the game; sometimes it enters before since LU or QR decompositions are explained for solving linear systems of equations. Of course our first year students do not know anything about floating-point numbers, errors (backward and forward), stability, etc. It is to be noted that those books using **Matlab** for Linear Algebra (for example Hill et al (1996),

Marcus (1993) and Smith (1997)) are more courses of Numerical Linear Algebra than even introductory Linear Algebra courses.

Note that no textbook was found fulfilling our requirements concerning, first, students entering into the first year at the university with a poor mathematical training and, second, without a previous knowledge of Computer Algebra.

Going from Mathematics+Technology to Mathematics

The decision of using a Computer Algebra System when teaching an introductory course of Mathematics to students of scientific or technical disciplines allows introducing the mathematical experimentation into the classroom. In many cases the using of **Maple** helps to the students to discover by themselves the definition of a mathematical concept: two canonical examples of this situation are the introduction of the derivative definition as a way of computing tangent lines to curves or the introduction of the integral definition as a generalization of the area concept to general curved domains.

Going from Mathematics+Technology to Numerical Analysis

Invoking the function *solve* in **Maple**, easily provides examples where no analytical solution can be computed: for example, to solve of a degree five polynomial equation does not have, in general, a closed form solution. **Maple** help automatically sends the user to invoke the function *fsolve* in order to get an approximation of a root for the considered equation in case this solution exists. Thus, in order to use properly Maple, students must have a (very basic knowledge) of what a floating-point number is and what it represents. Next step is the performing of elementary error analysis arising from, mainly, the solving of nonlinear equations in one unknown in order to check the goodness/badness solution provided by **Maple**.

Going from Mathematics+Technology to Computer Algebra

Due to the lack of previous training in Computer Algebra the first computer assisted sessions are devoted to learning the basics of **Maple**: numbers, polynomials, expressions, basic operations, vectors, matrices, etc. Very quickly, students started to use **Maple** as a powerful calculator able to solve problems otherwise impossible to solve by hand but also they are faced to what Computer Algebra is. Initially they separate two different kind of problems: those where the involved computations are purely symbolic (such as polynomial manipulations as the greatest common divisor, polynomial factorization or primitive determination) or numeric (such as root approximation or numerical integration). Special mention is made to the fact that both approaches must be used in a coordinated way since they are tools that Scientific Computing offers to the scientist or engineer to solve their problems.

Representation problems, which are a classical cornerstone in Computer Algebra, appear very often: if **Maple** is asked to compute the cubic root of -1 the answer always shocks the students:

```
> simplify((-1)^(1/3));
```

$$\frac{1}{2} + \frac{1}{2} I \sqrt{3}$$

If their first impression is to conclude that **Maple** has a bug (the statement that any software package has indeed bugs is clearly made explicit at the beginning of the course), this is not the case. This is the typical example where the reason why **Maple** returns this initially surprising result allows to introduce the discussion about how to define n-roots over the complex numbers or, without explicitly mentioning it, that many interesting complex valued functions are multi-valued. And that this is a very important problem in a discipline whose name is Computer Algebra.

Going from Mathematics+Technology to Data Structures, Algorithms & Programming

After the second or the third **Maple** session it is the right moment to explain several things that have been used implicitly: the notion of data structure (we have already used sequences, lists, sets, arrays, tables, strings, etc), the notion of algorithm (every worksheet is in fact the skeleton of one or several algorithms solving a particular problem) through the using of several **Maple** operators and functions and the different kind of tools **Maple** offers to the user in order to implement a procedure corresponding to a concrete algorithm (iteration, recursion, conditionals, etc).

It was initially planned that the computer assisted practical sessions must change their structure and no more prepared worksheets would be distributed: a concrete problem related with the current topics being discussed at that moment is distributed and the students may generate a worksheet containing the implementation of the algorithm solving the concrete problem proposed. But timing constraints have avoided up to this year to apply this initial plan.

4. Conclusions

The decision of using **Maple** into the practical sessions of the *Laboratory of Mathematics* course considered here has implied, first, a different way of presenting an introductory course of Mathematics for students of scientific or technical disciplines plus the inclusion and/or consideration from the early beginning of three new items into the curricula:

- An introduction to Computer Algebra through **Maple**.
- A short introduction to Numerical Analysis.
- An elementary introduction to Data Structures and Algorithms.

From the positive point of view it is important to remark that these three new items are inserted in a natural way since they are explicitly needed in order to make possible the using of the computer and **Maple** to solve some of the problems proposed and very closely related with the mathematical topics considered in the course. From the negative point of view it is clear that the time devoted to these three new topics is not used to deep inside some of the concepts of the course: it would be optimal if the students arrived in advance with the required knowledge of **Maple** and thus to avoid the spending of time in the first and third items. But this is difficult to achieve since this introductory course is taught at the very beginning of the first year of studies at the University.

From our experience, since it is very easy to motivate (and justify) the *soft* introduction of Computer Algebra, Numerical Analysis and Data Structures and Algorithms inside this introductory course, it seems to be a very convenient deal to include these topics as regular material but with a timing increase estimated in one more module. An extra advantage of this option would be the early introduction of these tools, which can be later used, into the teaching of other topics into the curriculum. It is worth to remark that it seems to be unavoidable the consideration of several topics (not usually classified as basic mathematical topics) from Numerical Analysis, Data Structures and Algorithms if a Computer Algebra System is to be used in the classroom.

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AN ANALYSIS OF THE PRODUCTION OF MEANING FOR THE NOTION OF BASIS IN LINEAR ALGEBRA[1]

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ABSTRACT

In this study we have investigated the production of meaning for the notion of basis in Linear Algebra, supported by the Theoretical Model of Semantic Fields, proposed by R. Lins (2001). It was conducted in three parts: (I) a historical-critical study, based on secondary sources, in which the key question was 'in which semantic fields were operating the mathematicians who constituted the notion of basis in the historical process of emergence of the elementary notions of Linear Algebra?'; (ii) an analysis of Linear Algebra textbooks, to investigate meanings which could be produced for the notion of basis from their reading; and, (iii) interviews with students of a first course on Linear Algebra (undergraduate mathematics degree), aiming at eliciting the meanings actually produced by them while engaged in solving proposed problems. The study allowed us to identify several and distinct meanings for the notion of basis being produced, coming from our many 'informants'; it had as a general objective to gather information which could help us and other professors a better reading of the classrom dynamics in a Linear Algebra course.

KEYWORDS: meaning production, basis, linear algebra

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Introduction

In this study we investigate the production of meaning for the notion of basis in Linear Algebra. It was conducted in three parts: (i) a historical-critical study; (ii) an analysis of textbooks on Linear Algebra; and, (iii) a case study with undergraduate students (mathematics degree) taking a first course on Linear Algebra. We adopted as a framework the Theoretical Model of Semantic Fields (TMSF) as proposed by R. Lins .

The central notion taken from the TMSF was that of 'meaning', characterised as "that which a person can and actually says about an object in a given activity" (see, for instance, Lins, 2001).

In the meaning production process some statements are locally taken as true without the need of further justification; a set of those 'local stipulations' taking part in a given meaning production process we call 'kernel'. Finally, we will call 'Semantic Field' to an activity of producing meaning in relation to a certain kernel; we will say, for instance, that a person is operating on this or that semantic field.

Just to illustrate, a local stipulation in a given activity (a child solving an arithmetical problem) could be related to whole-part relations, for instance, that 'if a whole has two parts and one is removed, the other part is left', although in another situation it could be recessary to justify this statement. Another general example is that one might use a mathematical result without thinking of why it is true, but in other situations it might be necessary to consider that.

Kernels are, then, sets of (locally) absolute truths that one uses in the process of, say, solving a problem, and semantic fields are characterised by the fact that whatever meaning being produced in a given activity, the person is 'using' those local stipulations as the firm ground to produce new statements, to justify them. For instance, one can speak of a semantic field of whole-part relation or a semantic field of a scale-balance, among others, when someone is producing meaning for a linear equation.

The historical-critical study

In this section we present what was found while we were trying to elicit the semantic fields in which mathematicians of the past were operating as the notions of basis was being constituted, during the emergence of the basic concepts of Linear Algebra.

We examined the work of mathematicians looking for the objects they constituted and for what justified what they were saying about them or doing with them (we call this the 'logic of the operations'). We included authors who did not constitute the notion of basis but were of interest because of some ideas present in their work, related to what we wanted to elicit. We took Crowe (1967), Dorier (1990) and Granger (1974) as central references.

Our first informant was L. Euler (1707-83). He speaks of objects such as functions, homogeneous and non-homogeneous linear differential equations, general and particular solutions of differential equations. Dorier (op. cit) and Bourbaki (1976) suggest he knew, for instance, that the general solution of a homogeneous linear differential equation of order n is a linear combination of n particular solutions. But he did not give evidence of having constituted the notion of linear independence of a set of solutions and that suggests he did not constitute the notion of basis for the set of all solutions.

Our second informant was F. G. Frobenius (1849-1917). Our reading of his work on a general theory of solving systems of linear equations led us to think that he constituded the notion of basis for the solutions of a homogeneous system, as he seems to use as local stipulations, in that respect, the notions of linear combinations of solutions, linear independence of solutions and considered the maximum number of independent solutions.

From W. R. Hamilton (1805-65) we were interested on his work on quaternions. He operated with them both geometrically and algebraically, and they were understood as linear combinations of the four units (1, i, j, k) using coefficients in **R**. Around a kernel which had objects such as complex numbers, vectors, ordered pairs, triples and quadruples of real numbers, Hamilton developed a notion of basis for the quaternions.

The work of H. Grassman (1809-77) was not understood by his peers, like Gauss and Moebius, at the time they were presented (Dorier, 1990). He worked with objects like extensive, derivable and elementary magnitudes, units (primitive, relative and absolute) and systems of units. From those he developed notions of linear dependence and independenc, linear combinations (for instance, speaking of derivable magnitudes), dimension, real vector spaces and of basis.

G. Peano (1858-1932) took to himself the task of making Grassman's work comprehensible to more people but, in doing so, he made original contributions. In particular he shows that if the product of three vectors, in the form of a determinant, is different from zero, then any vector can be written as a linear combination of those three. Peano, like Grassman, constituted the notion of a basis, and gave, for the first time an axiomatic presentation of vector spaces. Dorier thinks his understanding of dimension of a vector space is not completely clear.

The analysis, very briefly presented here, of the work of those authors, showed that, operating on different semantic fields, those mathematicians constituted objects that we could identify as precursors of our notion of basis. But it also indicated the extent to which the meanings produced for basis in each case were strongly related to the overall construction of each mathematician, to the problems they were trying to solve and the ideas they were trying to clarify/organise. It is in this sense that we say that the production of meaning can be only understood inside specific activities and not in an ideal, general, sense; that was also taken into account when we examined textbooks and when we interviewed students.

The study of textbooks

Our selection of authors here did not follow any specific principle. We simply looked at a considerable number of textbooks on Linear Algebra, and collected different definitions or characterisations for basis.

The question guiding the analysis here was "what statements can be made about 'basis of a finite-dimensional vector space' following each of those definitions or characterisations?"

We will consider here only two characterisations found in textbooks:

1) a basis for a vector space is an ordered set of vectors that both is linearly independent and generates the whole set of vectors

2) a basis for a vector space is a linearly independent set of vectors such that the number of vectors in it is equal to the dimension of that space.

It is worth noticing that both characterisations are possible, given that different authors organise the presentation of ideas differently.

To illustrate how assuming one of those meanings for basis could affect the thinking about the same problem, we present an example. Two students are presented with the question:

> "Consider $\mathbf{R}^3$ with the usual vector space structure, and A={u=(1,0,0), v=(0,1,-1), w=(0,0,2)}. Is A a basis for $\mathbf{R}^3$?"

Student 1 says 'yes' and offers this justification:

> "the set A is linearly independent because none of the vectors can be written as a linear combination of the other two. Also, A generates the whole space."

Student 2 says 'yes' and offers this justification:

> "$\mathbf{R}^3$ is a vector space and dim $\mathbf{R}^3 = 3$. As the vectors in A are [...] linearly independent, A is a basis for $\mathbf{R}^3$"

In this case it is pretty obvius who tought which way, but one consequence might remain hidden: student 2 depends, at this point, of some way of determining the dimension of the space s/he is working with. In many situations this leads students to some form of naturalised notion of 'space' with consequences we discuss elsewhere.

Also, from the statements of each student it is clear that the local stipulations are different in each case, that is, their thinking relate to different sets of objects. For the professor, it is crucial that s/he be able to read such processes and be aware of the implications of choices s/he makes in preparing a course and teaching it. We believe the TMSF offers a framework which supports that reading and usefully guides course preparation and teaching.

With students

What meanings would be produced for the notion of basis by a student taking a first course on Linear Algebra? That was the question guiding the case study.

Among the students who volunteered to be interviewed (undergaraduate mathematics degree), we chose two; let's call them Mark and Eli. The introductory course they were taking consisted of: matrices, systems of linear equations, determinants and finite-dimension vector spaces.

Four tasks were designed, and presented to them on four different sessions.

In our analysis we focused on: (i) the objects they were thinking with/about; (ii) the local stipulations being taken; and, (iii) the logic of the operations, that is, how those local stipulations and objects were supporting what they were saying.

We will discuss one of the tasks:

> "Consider the plane π given by $x - 2y + z = 0$ in $\mathbf{R}^3$. Find two basis for π "

Mark writes $x = 2y - z$ and considering the vector $(2y\text{-}z, y, z)$ he he obtains the vectors $(2,1,0)$ and $(-1,0,1)$. From that he concludes that they form a basis for π and that the dimension is 2. Then he isolates y and repeats the process. He said that,

> [MARK] "From the start I realised it was not the space $\mathbf{R}^3$, because of the text of the question. Because there it says that π is a plane. So, if it is a plane, there are only two vectors [sic] and the dimension is 2. So, to be $\mathbf{R}^3$ the dimension must be 3. From start I knew it was a subspace of $\mathbf{R}^3$. So it's enough to find two vectors."

He is clearly working with the assumption that the dimension of that subspace is 2, but because he does not verify whether the two vector are linearly independent, we understood that the dimension 2 was simply a property of a naturalised plane, a plane like the surface of a wall; it is simply too usual that everybody knows that the surface of a wall is bi-dimensional. That is also supported by his statement that 'there are only two vectors': those would correspond to the usual representaion of a plane, in analytical geometry, as a system of two Cartesian axis.

Eli has a different solution. She says,

> [ELI] "To be a basis, a set has to be LI and generate the space. For the plane π we have the generic vector (2y-z, y, z)."

From there she finds the generators by taking y=0 and then z=0 and verifies they are LI. When discussing her solution with the interviewer, Eli says that the set "has to generate $\mathbf{R}^3$". When the interviewer asks her about this, she says she got confused and that,

> [ELI] "[...] a difficulty, also, that I think I have, we have. That thing of using numbers in an equation to see what it generates in terms of, like, plane, solid, straight line. Like, sometimes we even know, but when we have to imagine, like..."

Her difficulty seems to be associated with not identifying directly and immediately from the equation what the subspace 'is'. Maybe this is the reason why she does not think like Mark, with the 'natural' dimension 2. In any case, our point here is that her thinking was different from Mark's and that means that the objects they were thinking with were different and that the meaning of basis for each of them was different.

As we have said before, in a classroom situation the professor must be aware of those processes and be able to handle them if teaching is to be effective.

After an exchange with Eli, Mark realises that,

> [MARK] "[...] I forgot to verify whether one vector is independent of the other [sic]. Because if they are dependent they won't generate a plane [...] they'll generate a straight line..."

But as the conversation continued he returned to his idea that it was enough to know that π was a plane. In terms of the TMSF, the interpretatioon is that in that specific activity 'linear independence' was not constituted into an object nor was a local stipulation. The interaction with Eli shows that Mark *could* produce meaning for it in that context, but also that *actually* that object did not belong properly to his thinking in that situation.

Mark thought with: equations, variables in an equation, generic vector, vector as a directed line (he uses drawings), (natural) dimension, subspace, $\mathbf{R}^3$. We think there is the suggestion here that naturalised objects (dimension and vector, here) are more likely to become (unnoticed) local stipulations than notions which are unfamiliar (linear independence, in this case).

Eli thought with: generic vector, equation, subspace, set, generate, linear independence. No evident naturalised notion seems to be centrally present in her thinking. Differently from Mark, who sees the equation as being the plane, for her it is more likely a relationship between the variables.

What each does in the course of solving the problem is based on what those objects *are* (for them); this is what we referred to as the logic of the operations (on the objects). Eli's plane, for instance, did not have the Cartesian axis attached to it (in this activity), so she has to think with linear independence; because the role of the axis is to provide a system of coordinates, it is clear that if one has the Cartesian diagram of the plane in mind it does not even make sense to have axis that are not 'independent'.

Just to make a relevant point . One could be strongly tempted here to say that Eli is thinking algebraically while Mark is thinking geometrically. As a local description it might look useful, but from the point of view of the TMSF it is misleading, as the static nature of such description (referring to *states*) makes it insufficient for the reading and understanding of *processes*.

Final remarks

Overall, our study highlighted a broad set of meanings that can be produced for the notion of basis in Linear Algebra, working with a varied group of informants. Those meanings reach from the ones found in textbooks, through the ones found in the work of mathematicians of the past, to the ones we found in the thinking of students.

By no means we wanted to produce a 'catalog' of meanings; what we wanted was to highlight the fact that it is not sufficient, from the point of view of mathematics education, to treat present-day definitions and characterisations as the (true) essence of something that is also to be (sometimes implicitly and many times incorrectly) found in the past and in students. We suggest that the complexity of meaning production can be only dealt with properly in mathematics education if we make *processes* our central object of study and understanding.

One of the students in our study said:

> "It's as the name already says, to be a basis [foundation, stepping stone] for you to know about whatever a person asks you, or an exercise, right?"

Particularly in the mathematical education of future teachers, we think it is necessary to raise the awareness of the existence of those processes and to help them to develop ways of dealing with such situations. And it does not seem plausible that normal courses on mathematical subjects (the same taken by future researchers in mathematics) are adequate. On the contrary, the results of our current research project suggest that the education of future teachers would benefit from a different approach in the classrooms.

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FACTORS AFFECTING STUDENTS' PERCEPTIONS OF DIFFICULTY IN CALCULUS WORD PROBLEMS

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ABSTRACT

The purpose of the research project detailed in this paper was to ascertain, if possible, the answers to the research question that can be broken down into the following subquestions:

1) What factors contribute to students' perception of the difficulty level of a word problem?
2) How do students rank word problems in order of difficulty?
3) Are there differences between experts and novices in the ranking of problems?

The data collection instrument was designed in such a way that relative effects on difficulty level between characteristics could be determined, although no absolute effects, such as a quantitative measure of difficulty on an independent scale. For instance, it appeared that the context of the problem (concrete or abstract) had a greater effect on perceived difficulty level than the presence of a diagram. It was not possible to see, however, whether the difference was a subtle one or a clear and consequential one. The results of this study are informative, and it is the aim of this paper to summarise the central ideas on which the study was based, to outline briefly how the data was collected and to draw conclusions on the analysed data.

KEY WORDS/TERMS: mathematics education, word problems, hierarchy of difficulty, modal vectors

This paper outlines the motivation behind the study as well as several similar studies and their influence on this research project. Word problems are defined in terms of a problem classification framework, which is then used to draw up several problems for use in a survey of students and lecturers of mathematics at university level. Conclusions drawn from the data obtained from the survey are detailed as well as implications for further studies in this area.

Motivation behind the study

Students are known to find word problems difficult (Gerofsky, 1999; Craig & Winter, 1991/92 among others), yet experts in the form of mathematics lecturers or postgraduate students tend to find them easy, even mechanical (Schoenfeld, 1985; Larkin et al, 1980). As a mathematics educator, I hoped that this study would provide me with a new insight into why students find them difficult, and thus be able to teach them in a more accessible, less opaque, manner.

Research question and research design

The research question this project was designed to answer was essentially "What affects the perceived difficulty level of a word problem?" This question was subsequently expressed as three separate questions namely

1) What factors contribute to students' perception of the difficulty level of a word problem?
2) How do students rank word problems in order of difficulty?
3) Are there differences between experts and novices in the ranking of problems?

The approach taken to answering these questions was to run a survey in which first year university students, as well as lecturers, completed a questionnaire. This questionnaire consisted of five word problems that the person completing the questionnaire was required to rank in order of difficulty, without attempting to solve the problems first. The survey results were analysed to determine what characteristics of these word problems affected their relative difficulty.

Other studies have been carried out that compare the relative difficulties of word problem characteristics, such as context, arithmetic operations, readability, presence of diagrams, and whether the problems are algorithmic or interpretive. Several of these studies are outlined below along with their implications for the present study.

Similar studies and their relationships to this study

The problems in this study have been divided into the categories *algorithmic and interpretive*. Algorithmic problems are defined as problems that require the problem solver to carry out some calculation, the numerical solution to which is the aim of the problem. Interpretive problems, in contrast, require little or no calculation and require the problem solver to draw a conclusion drawn from some information given and his/her knowledge of mathematics. Galbraith & Haines (2000) made similar divisions in their study involving first year undergraduate students. Their problems are primarily involved with graphs and functions, such as factorising quadratics, or reflecting and translating a graph. The problems are divided into mechanical (equivalent to algorithmic), interpretive and constructive, where constructive can be understood to be a combination of the two former categories. The difficulty level of each problem was measured by the level of success that the students experienced when attempting to solve the problems. The results of Galbraith and

Haines (2000) show clearly that mechanical problems are easier than interpretive problems, which, in turn, are easier than constructive problems. A limited number of problems appeared on the questionnaire of this study, none of which were interpretive, due to constraints on the length of the questionnaire.

Caldwell & Goldin (1987, 1979) carried out a similar study at junior school level (1979) and secondary school level (1987). The problems that they presented to schoolchildren were all word problems categorised as *concrete or abstract*, and hypothetical or factual. Concrete and abstract problems are defined in terms of the realism of their context, that is concrete problems are set in a realistic context and abstract problems have no immediate real world analogy. Hypothetical and factual problems differ in that factual problems simply describe a situation, while hypothetical problems suggest a possible change in the situation. In the Caldwell & Goldin (1979 & 1987) studies, the difficulty level of a problem was measured by the number of students who successfully solved the problem. Caldwell & Goldin (ibid.) found that abstract problems were significantly more difficult than concrete problems, a finding which is reflected in this study.

Smith et al (1994) measured the *readability* of problems on a university statistics examination paper according to number of words, number of clauses, and two measures of lexical density. Lexical density is measured as the ratio of lexical words to grammatical words, either in total, or per clause. They accorded each problem a difficulty level by recording how many students successfully completed the problem. They found no correlation between the readability and difficulty level of the problems. The findings of this project are in agreement with those of Smith et al (1994). The readability of a word problem does not appear to affect the difficulty level, either perceived or actual.

Threadgill-Sowder & Sowder (1982) compared the difficulty level of problems presented in verbal format versus those presented with detailed *diagrams* and minimal wording. The difficulty level was measured by the number of students (in junior school) successfully carrying out the problem requirement. The results showed that students found the problems presented almost entirely in diagrammatic form significantly easier than those presented in verbal form only.

The studies listed above all required the students to carry out the problems and measured difficulty by the percentage of students solving them correctly. This study was intended to be rather different, in that the students were not required to complete the problems. Indeed, the students were given no opportunity to do so. Difficulty ranking was to be affected by their perceptions of the problems alone. The students were required to read the problems and rank them in order of the *perceived level of mathematical challenge* represented by each one. Individual students could therefore judge this difficulty level in different ways, such as number of variables, expected time required to solve the problem, the geometric shapes involved, etc. The students were free to decide for themselves which problems they expected to require the most cognitive effort to solve. The problems had to be chosen very carefully, therefore, according to strict criteria, to allow a comparison of which characteristics of the problems affected this perceived level of cognitive demand.

Defining word problems

Different theorists have defined "word problems" in various ways. Some mathematics educators define word problems by their structure, appearance and the inbuilt assumptions behind them (Verschaffel et al, 2000; Gerofsky, 1996; Pimm, 1995; Janvier, 1987; Lesh et al, 1987). Word problems have an easily recognisable structure and some assumptions are always made (by students and teachers), such as assuming that information not mentioned in the problem statements will not be required for successful problem-solving (Gerofsky, 1996). A definition of word problems by their use as a tool, rather than by their characteristics is often used (Boote, 1998; Schoenfeld, 1989). Word problems can be very useful as a means of illustrating practical uses of an algorithm, or as a modelling tool in physics or statistics. A third method for defining word problems is by creating a framework in which multiple types of mathematical problem can be placed, of which word problems are only one. Dowling (1998) constructs one such framework, and Craig & Winter (1990) construct another. It is the framework of Craig & Winter, strongly influenced by the three-level cognitive model of Kitchener (1983) that is the framework used in this research project. Kitchener's cognitive model suggests that real-life problems (ill-structured problems) cannot be modelled by school taught word problems (well-structured problems). If this model is correct, it calls into question the widely accepted belief (Verschaffel et al, 2000) that word problems are taught in order to teach techniques that can be applied to real-life problems. This belief is allied to the concept of transfer of technique from one sort of problem to another, a subject hotly debated (Evans, 1999; Lave, 1988; Walkerdine, 1988). Despite the arguments against word problems being included in the syllabus, however, they need to be defined if they are to be studied, and hence a problem classification framework was developed.

The problem classification framework

Using this framework (see Figure 1 below), word problems are defined as *disguised well-structured* problems, which can be divided and subdivided into various categories. *Algorithmic and interpretive* problems are defined where algorithmic problems require calculation, and interpretive problems require applying knowledge to interpret information practically. These categories were also used by Galbraith & Haines (2000). Both of these can be divided into *concrete and abstract* problems (although abstract interpretive word problems are rare). Concrete problems are set in a context that is non-mathematical and realistic, whereas abstract problems are set in a mathematical context with no immediate real world connection (see also Caldwell & Goldin, 1987, 1979). Finally, the problems can be categorised as having a single form of *representation*, or having multiple representations, such as diagrams and tables. Much work has been carried out on representations and the translations between them (Pimm, 1995; Wood, 1995; Buxkemper & Hartfiel, 1995; Lesh et al, 1987).

The problem typology and the data collection instrument

A list of problems was drawn up with at least one problem in each category defined by the framework. An example would be algorithmic – abstract – no diagram. In this way a typology of 14 problems was drawn up, a full description of which, with more detailed discussion, is available in Craig (2001). Five problems were selected from this typology to appear on a questionnaire, which was then distributed to students registered for one of three first calculus courses at the University of Cape Town. The problems were chosen carefully to be different, yet closely related enough that useful comparisons could be made. The characteristics of the problems chosen are illustrated in Table 2. An example of a problem (Problem D) on the questionnaire is

A hollow cylindrical container has a circular base of radius 5 cm and vertical sides. A solid sphere of radius r is placed inside and at the bottom of the cylinder, and water is poured in until the sphere is covered. What value of r will maximise the amount of water needed?

The collection of data

The survey was carried out at the University of Cape Town, South Africa. The students were all attending one of three first year calculus courses, designed for science and business science students, commerce students and engineering students. 660 responses were received from the students surveyed and 20 responses from the experts, who were postgraduates and lecturers in the Department of Mathematics and Applied Mathematics. The questionnaires were distributed during tutorial sessions and the students were allowed ten minutes during which to complete them. The tutors were carefully instructed in the requirements of the survey and did not allow the students to carry out the calculations, nor to discuss the problems amongst themselves.

Figure 1 Problem classification tree-diagram

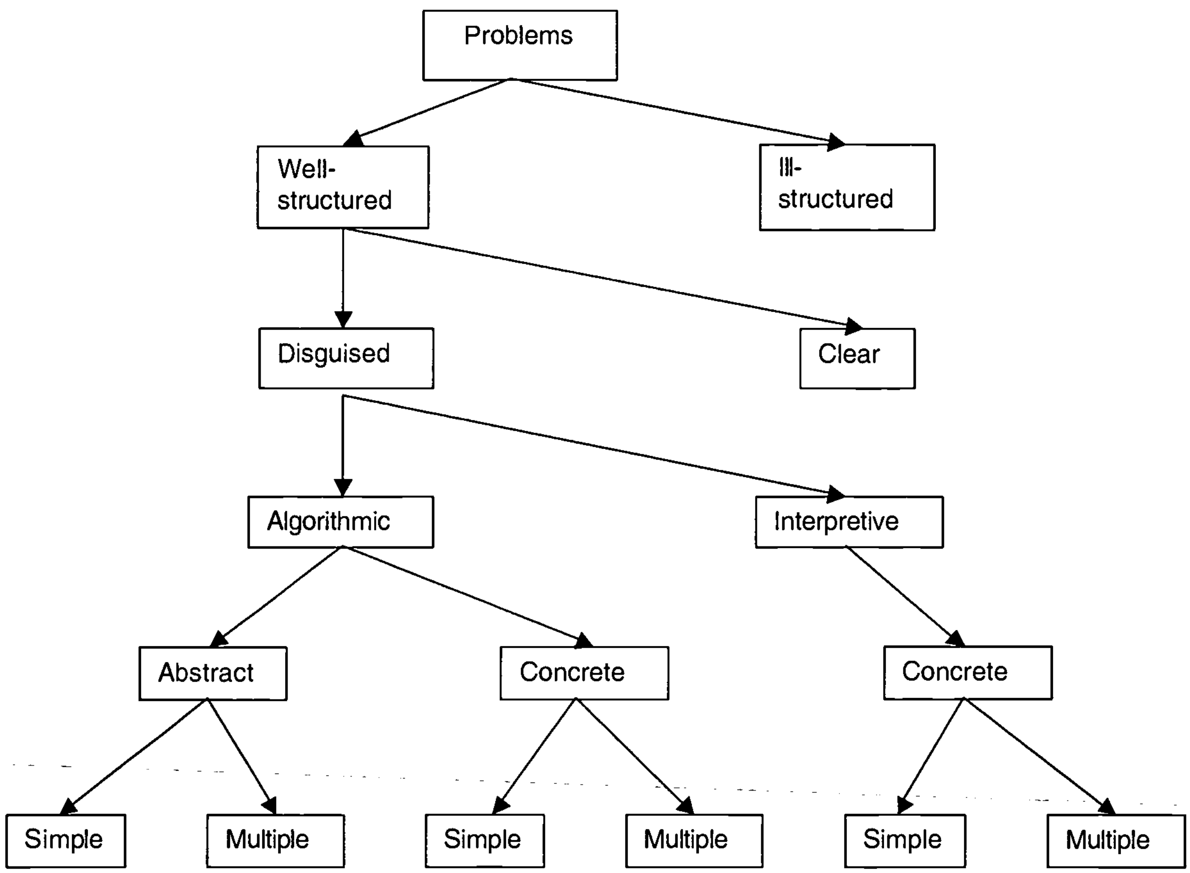

Table 2 Characteristics of word problems in questionnaire

	Context	Task requirement	Visual representation
A	Concrete	Algorithmic	No
B	Concrete	Algorithmic	Yes
C	Abstract	Algorithmic	Yes
D	Abstract	Algorithmic	No
E	Concrete	Algorithmic	Yes

The data

The information that the students had to complete on the questionnaire included a ranked list of the problems in the order easiest to most difficult, determined by their perception of the degree of mathematical challenge provided by each problem. This information was treated as a vector (for example BADEC) and a modal vector was calculated for each subpopulation, where the subpopulations were defined by gender, degree, or first language. The modal vectors were calculated in three ways, the most descriptive of which was the method by preference matrix (Siegel & Castellan, 1988), which measures the percentage of students ranking any one problem as preferred to any other (see Table 3 below). The modal vector was the same for every subpopulation except one that contained only 29 responses. This vector (AEBCD) ranked the problems in the following order (from easiest to most difficult):

Algorithmic - Concrete - No diagram (note: very common word problem)
Algorithmic - Concrete - Diagram (involving rectangles)
Algorithmic - Concrete - Diagram (involving circles)
Algorithmic - Abstract - Diagram
Algorithmic - Abstract - No diagram

Table 3 Absolute and relative preferences in the total population *

	A	B	C	D	E
A	•	620	629	614	542
B		•	401	448	264
C			•	427	191
D				•	142
E					•

	A	B	C	D	E
A	•	93.9	95.3	93.0	82.1
B		•	60.8	67.9	40.0
C			•	64.7	28.9
D				•	21.5
E					•

* Illustration by example of how to read the preference table: The first entry of 620 refers to the 620 students preferring problem A to problem B. The corresponding entry of 93.9 in the second table interprets the 620 students as 93.9% of the total population size.

Conclusions drawn from the data analysis

The ranking of the problems reveals preferences that are reflected in the studies mentioned earlier. A measurement was taken of the readability of each problem according to a lexical density test and the Flesch-Kincaid index. No correlation was found between readability and perceived difficulty, which correlates with the work of Smith et al (1994). Caldwell & Goldin (1987, 1979) observed that students find abstract problems harder than concrete problems, with which observation this study concurs.

A suggested hierarchy of difficulty, obtained from exhaustive analysis is:

- Familiar problems preferred to less familiar problems
- Concrete problems preferred to abstract problems (in agreement with Caldwell & Goldin, 1979, 1987)
- Problems with diagrams preferred to problems without diagrams (in agreement with Threadgill-Sowder & Sowder, 1982)
- Problems with rectangles preferred to problems with circles
- Readability indices (in agreement with Smith et al, 1994)

This hierarchy is apparent for every subpopulation within the novice population. The expert population tended towards this hierarchy as well, but not as clearly. The responses of the experts were widely spread with 14 different vectors from 20 questionnaires. The presence/lack of a diagram seemed to have less of an effect on expert perception of difficulty and unfamiliarity was less of a deterrent. The small number of experts taking part in the survey (20) did not allow for statistical analysis, but a descriptive analysis suggested that there was little correlation between the responses of experts and those of novices (students).

Contribution to the field and limitations of the study

The problem categorisation framework indicated in this paper is designed to apply to any mathematical problem, not solely word problems. The framework, along with the typology of problems developed with it in mind, can therefore be used to enable similar studies that can be considered as extensions of this one. For instance, a comparison can be made between the relative difficulties of clear and disguised problems. Another example would be to consider algorithmic and interpretive clear problems, rather than disguised problems. It is possible to extend the framework to include standard and non-standard problems (Craig & Winter, 1990; Yerushalmy & Gilead, 1999) that is, problems in their simplest form and ones that require simplification. This extension would be a difficult task, and one possibly open to debate.

In summary, the factors that appear to affect student perception of the difficulty level of a word problem are familiarity, context and visual representation in that order. Familiarity, particularly, plays a large role. Experts, in the form of mathematics lecturers and postgraduate students, do not have as clear a response. The varied responses from the experts suggests that there is no "correct" ranking of factors affecting difficulty, but that, as one gains in mathematical experience, one develops one's own preferences for different types of problems.

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COLLABORATION AND ASSESSMENT IN A TECHNOLOGICAL FRAMEWORK

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ABSTRACT

Following one year's experience of lecturing Calculus to undergraduate students at Università Bocconi, Milan, Italy, we have investigated the way students collaborate among themselves and with the lecturers when using an e-learning software (for further information see also the papers by M. Impedovo and G. Osimo); then we studied several approaches to the problem of the assessment of students' knowledge.

In the first part we have focused on the subjects (which are the preferred topics among students and why), the way the discussions are brought on (which kind of discussions are more popular and how the students discuss the subjects) and the impact of the discussions on the performances of the students (are they related to the way the students are involved in the collaborative environment?).

In the second part of this research, three methods to assess students' performances have been compared: a particular mathematical software, the evaluation sections of an e-learning software and a software, developed by the author, specifically designed for lecturers. In the latter case, the technological framework is explained in detail. In particular we discuss: a) the choices made for the interface; b) the intranet set up to guarantee maximum security before, during and after the examination and c) the modularity of the software developed. We consider these aspects interesting by themselves and because the problems they pose are too often neglected.

Keywords: assessment, collaboration, software.

The analysis reported in this paper is related to the first semester of one year's Calculus course to undergraduate students at Università Bocconi, Milan, Italy. The way students collaborate among themselves and with the lecturers has been studied. Students have used at the same time an e-learning software and a mathematical software. In addition, several approaches to the problem of the assessment have been compared and a sustainable solution is outlined. The pedagogical issues involved in this framework are described in depth in another article presented at ICTM 2002 by Michele Impedovo [1]. A complete description of the introduction of collaborative software at Università Bocconi has been published [8].

The goal of this project is a survey of the student online activity in the computer assisted Calculus course.

1. Online behaviour of students

A classroom of 140 students has been monitored during a Calculus course. With respect to analogous courses in other Italian universities, two novelties have been introduced:

a) students used an online collaborative software (OCS) and a particular mathematical software (MS) at the same time;

b) three interim examinations and the final exam were completely electronic, i.e. students had to answer questions and prepare solutions to problems using only the above-cited software.

The students were completely new to OCS and MS. While OCS is very user-friendly, MS requires some skills to be used efficiently. For this reason, a number of lessons on MS were given to students.

Students and teachers in the discussion area wrote 390 messages. Among the 390 messages, 260 messages (66%) were about mathematics, 76 (20%) about software issues, 43 (11%) about the organisation of the course and the last 11 (3%) were either private messages between teachers and students or messages about netiquette (see Fig. 1).

There were 331 discussion threads. Teachers initiated 96 discussions and students initiated 235 discussions. Only 72 out of 140 students created discussion threads. In the average each "computer active" student created three threads. In addition, 28 threads (10%) originated and were answered only by students.

Interestingly, among the 260 messages on mathematics, 69 (26%) originated neither in teacher's questions nor in elementary mathematical questions or comments, but by students themselves. All of these messages contained an attached file created with MS. There were also 12 more messages with attached files as electronic solutions to homework. These messages showed a great confidence in the use of mathematics. Several applications to economic subjects, e.g. consequences of tax reform, were autonomously found and deeply analysed by students.

Among the 76 messages about MS, 10 (13%) showed an advanced use of this software and investigations into numerical issues. It happened that students even helped university IT staff to solve installation problems of MS. Students generally like to work with computers and therefore they have an in-depth knowledge of the software they work with. This often makes students ask questions about MS not directly related to the course. Obviously, students expect teachers to answer all these questions with the same competence they show in the relevant mathematical issues.

As expected, the distribution of attached MS files was not uniform during the semester. At the beginning of the course students learnt to write mathematically in ASCII (e.g. the meaning of "3^4" is 3 raised to the 4th power) as this is the only way to convey concepts in OCS. The more they used MS the more they choose it to express mathematical ideas and doubts. At the end of the semester all the mathematical threads contained an attached file written with MS.

In Figure 2, exam grades are plotted against the total number of messages written in OCS. There seems to be no correlation between participation in online activities and exams grades. However, as expected, it is seen that those students who electronically answered some of the questions of their colleagues generally obtained good results).

2. The problem of assessment

The issue of technology becomes critical when one considers student assessment. In the move towards new technological courses, attention should be given to whatever is related to examination. In fact, there is no innovation if, after a bunch of technological lessons, examination takes place in a traditional way, i.e. using pencil and paper. However, if changes in the classroom practice are slow, the way teachers prepare and deliver tests is even slower.

In this framework, the choice of a Computer Assisted Assessment (CAA) model is crucial. This section focuses on the technological issues that arose when CAA was elected to be the only method of assessment in the Calculus course given at Università Bocconi.

2.1 Assessment with OCS and MS

Usually, OCS has a section devoted to assessment. As general-purpose software, OCS is not fully case-sensitive (e.g. limited type of questions, limited way of grading answers). Consequently, OCS fits only the needs of an average teacher. In addition, there is no way to embed an MS worksheet with mathematical formulas, even if without active content. In the discussion area of any OCS the issue of writing mathematical formulas is usually solved using a specific ASCII code. However, this practice diminishes the readability of the text that is very difficult to correct. This process is also unsatisfactory from a more general point of view, as students are *writing* and not *doing* mathematics.

On the other hand, no MS has assessment capabilities. In fact, using MS, there is no way to design mathematical questions allowing students to interact and be graded by the system. Despite this limit, it is common practice to assign a problem to be solved creating a worksheet with MS. Then, the file is submitted to OCS in an appropriate section. Other solutions were proposed (e.g. embedding Java applications in assessment software [2]) but they require programming skills and therefore are not practical.

A mixed solution was chosen in the course held at Università Bocconi. Students were given a set of 8 multiple choice questions (MCQs) plus a problem to be solved using MS for each examination. Students wrote mathematics and performed calculations, both symbolic and numerical, only using a computer. Questions were graded automatically by the system while the problem answers were evaluated manually by teachers. This solution has proved to be efficient and reliable. However, teachers are only partially satisfied for the limited flexibility of the system. Hence, in order to evaluate other solutions, existing commercial assessment software (AS) has been reviewed.

AS controls the assessment process in all phases. Commercial AS is in general very powerful in designing questions in many different types such as multiple choice, true or false, etc. An underlying database holds all the information about the exam. Some products even allow the inclusion of Java applets in the question. The design process usually ends with the creation of a comprehensive exam-file (generally of proprietary format) which is submitted to the server and, at the right date and time, to students as html (single or multiple) page. After answering questions, students submit the page via a normal html-form mechanism. The exam is then partially graded by the system and reviewed by teachers. It is also possible to statistically analyse answers. Apparently everything is fine but it is worth taking into account the following issues:

- Question types offered by commercial AS do not always satisfy teachers. Teachers do not make the transition to a CAA if forced to change the way they usually assess their students;
- Mathematical questions often require specific pre- and post-processing e.g. parameterisation;
- PCs sometimes crash. If this happens, there is no way to recover data;
- A unique identification for each student must be assigned. This is often already provided by the university information systems, but assessment software (which is proprietary) can not be integrated with foreign databases. The identification thus relies only on the assessment software security model, which is generally not well designed. Using paper and pencil there is automatic authentication of the writer. However, in a technological environment, all the cares must be taken to ensure one is who he/she claims to be. This is especially true in schools and universities where graduation has a legal value;
- To transfer files from the teacher's PC to the exam server raises security concerns. AS security models are generally very basic and, thus, not adequate. The only solution would be to set up a secure channel between the teacher and the exam server but this relies on IT staff and could be difficult to maintain;
- AS uses proprietary protocols and databases. However, in a few cases and under particular circumstances, some work to integrate AS with other information systems can be done (a detailed description of an actual experience similar to that described below is reported [3]);
- AS is usually paid on a per user basis and therefore is affordable only by few institutions.

In 1998, Università Bocconi started the project named EVEREST whose goal was to adopt a standard system of CAA. All the above issues were considered and it was decided to develop an open source solution that began working in June 2000 and now is adopted in several courses and master classes at the university.

2.2 A sustainable solution

The conclusion of the EVEREST project was that only a completely custom CAA system could address all the issues raised above. XML (Extensible Markup Language [4]) has been elected the common language of the whole application. This choice also allows including particular languages such as mathematics in the description of questions. Finally, it was decided to write the server side of the application using only open source software whose benefits are widely known. This solution was also sustainable: it can be easily extended, maintained and debugged; besides these great advantages, its modularity allows contribution from other people. There is a short description of the application flow below.

The exam is designed using a custom Windows application written in Microsoft Access. An XML file (with documented structure) is the final output. This file contains every detail of the exam: the text of all questions, their single grading and the way questions are to be put together and when. At present, the application allows multiple choice questions with 3 or 5 answers and open questions in which students are required to write a brief essay. The XML file is uploaded to an exam server using a protected https connection, which is similar to those used for e-commerce and grants security. In all phases, the exam server remains hidden from the rest of the academic network. In fact, in order to access the examination, a certificate of authentication is needed.

At the examination day, the system manager locks the PC classroom with a two-click operation. From this moment on, the PCs in the classroom can only connect to the exam server. This also prevents students from using the Internet to communicate to one another. Then, the teacher opens an "exam manager" page in a browser from which he/she can start the exam procedure, close it and see how students are dealing with their questions (see Figure 3). The teacher has complete control over their access to the exam (e.g. he can have them re-enter the exam if they had mistakenly submitted the assignment).

A single exam is subdivided in as many pages as questions (see Figure 4). Each answer is recorded on the server as soon as it is filled. A single PC crash is not a problem: the student can use a different PC without loosing data (obviously, there are more PCs than students).

After closing the exam, the teacher can automatically correct the multiple-choice questions. If more than one teacher is involved in the correction, the application composes as many html files as the teachers and send them via email. As far as the author knows this feature is not available in any of the commercial AS.

The server application has been written in PERL and a new major release, written in PHP and MySql, entered its alpha testing in early 2002. This new release is more modular and allows more types of questions to be accepted and processed by the application. In particular, it permits the embedding of images into questions.

The software interface to the students has continuously changed to reflect students needs and habits in browsing the web. There have been several changes in the layout, buttons and login procedures.

Finally, it should be noted that other researchers are studying the problem of the organisation of CAA sessions. For example, in late 2001, the British Standards Institution issued a guide (BSI 7988) to introduce minimum requirements for any organisation that uses computers to make assessments in the UK [5].

2.3 Future work

CAA is a fast pace moving subject in online learning, often underestimated in its importance in the university organisation. For mathematics its importance is even greater, because two different programs must coexist and work together. CAA requires great care as traditional assessment practices are completely changed. Therefore, a careful investigation of the needs of the institution is necessary.

The use of commercial AS produces several benefits but it also obscures some points in the whole assessment management process. This process is critical in an educational institution and people involved in (e.g. teachers and students) would like to have control on it. A custom solution can be one of the possible answers to this problem. This solution has some advantages: it can be customised to the need of each teacher, fully controlled and monitored, is as secure as the institution computer network and is always open to modifications.

Obviously, this solution can be deployed only with a distributed effort and therefore Università Bocconi plans to release the core of the application under some open source licence.

Among the possible extensions, which are now under investigation, two are interesting for mathematics teachers:

a) the incorporation of some TeX to HTML translator in the application. This incorporation permits teachers to write questions in TeX and students to view them in HTML within a browser window. As TeX is also partially structured, it is possible to pre-process TeX written questions in such a way that, e.g., "a+b" can become "3+4" for one student and "2+5" for another. This approach has previously been used [6].

b) the new MML (Mathematical Markup Language [7]) 2.0 standard from the W3C Consortium provides a further way of translating math into text. Mathematical questions can be written using the most recent versions of MS that can save their output in MML format. For displaying math, the Amaya browser, developed by the W3C Consortium, is capable of displaying MML. Amaya is currently being tested for use as the browser of choice for mathematical display.

3. Conclusions

The analysis of online behaviour of students and computer assisted assessment leads to the following conclusions:

a) a mathematical course can only benefit from the use of MS together with OCS. OCS becomes the area of discussion and exchange of MS files that, after a short period of training, are the way chosen by students to write mathematics. According to the author, the teacher has to take great care in the choice of the MS; an inappropriate (incorrect???) decision might mean that students keep on doing math in a non technological way;

b) the area of discussion is indeed effective: students pose questions and discuss subjects, sometimes autonomously. It is well known that clever students often pose difficult questions. For example, students were taught exponential growth is greater than the polynomial one. A student investigating the behaviour of two sequences, n^{100} and e^n, discovered that even if the latter grows faster than the former, it can be difficult to find for which n e^n is greater than n^{100}. This observation gave birth to two interesting threads, one concerning numerical mathematics and the other computer programs (two students used their programming skills to approximately solve the equation $n^{100} = e^n$);

c) due to the relatively large amount of messages about MS and their advanced level, teachers should be good at using MS and, in general, computers. In addition, a solid programming background helps teachers, as MS usually has programming capabilities. According to the author's experience, students are generally more confident with programming than with mathematics, thus they often try to solve problems via computer. It is anachronistic to tell them that this solution is not correct.

d) online participation seems to have no effect on students' performances. Online activities require confidence with computers but not necessarily with mathematics itself. In addition, high participation rates may originate in a high number of questions or in a high number of answers or both. In the author's opinion, the lack of correlation between students' participation and performances simply means that mathematics should not be confused with the use of any software even if a mathematical one. The collaborative environment is a mean not the goal of mathematical education.

e) The use of MS forces students to learn it . In order to consolidate this practice, schools/universities must have at least adequate laboratories. As students become aware of the computer evaluation, they ask more and more laboratory lessons. This is indeed the main goal of the author (and of the others involved in the project): to turn mathematical lectures into laboratory sessions, having students experience mathematics before they learn it;

f) finally, the technological issue is to be taken seriously. There is a common feeling among mathematicians that computers and software are easily run. In the author's experience this is not true. The use of computers and software cause a wide range of problems. Teachers should be able to face them in the most efficient way. This requires, for example, an in-depth testing of the chosen software and some support from the information technology department and even other staff.

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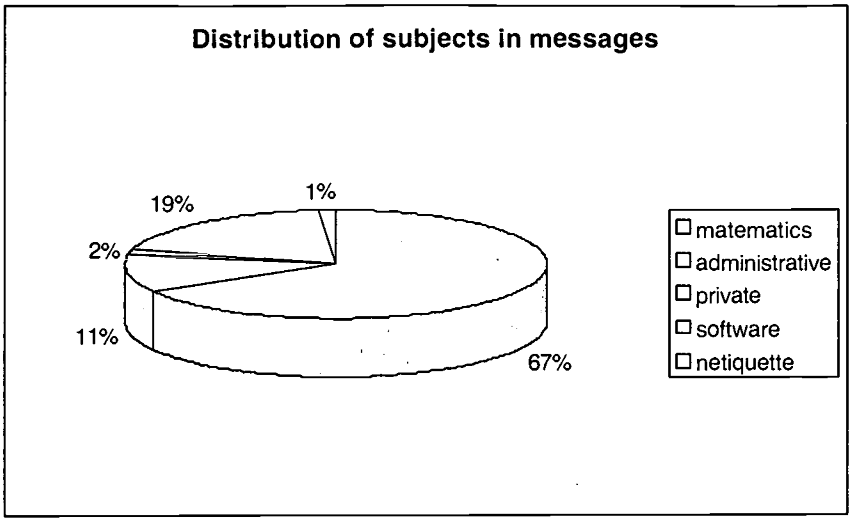

Figure 1: Distribution of subjects in messages in the collaborative area.

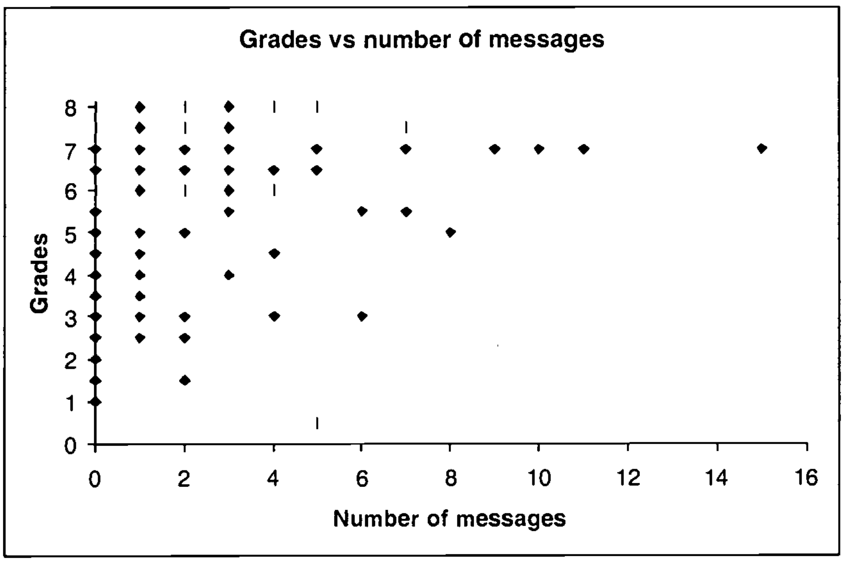

Figure 2: Each point represents one (or more) student. Its abscissa is the number of messages posted by the student to the collaborative area in OCS; its ordinate is the grade obtained by the same student in the mid-year examination. Grades range from 0 (minimum) to 8 (maximum) at a 0.5 step.

MAKING RELEVANCE RELEVANT IN MATHEMATICS TEACHER EDUCATION

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ABSTRACT

One of the features of the reform in school mathematics is that school mathematics should be made relevant to the learners. The incorporation of mathematical modelling in school mathematics is one of the ways that is offered to realise the relevance ideal. Ostensibly the inclusion of mathematical modelling will provide school learners opportunities to develop mathematical power i.e. the ability to make sense of the world and of mathematics. A key question in this regard is: How prepared are practising mathematics teachers to incorporate mathematical modelling in their teaching? This preparedness entails that mathematics teachers be knowledgeable with mathematical modelling as content.

In this paper this mathematical modelling content is elaborated upon and reports on a study which investigated secondary mathematics teachers' knowledge of mathematical modelling as content in South Africa and Eritrea. The one major finding of this study is that these teachers deem their experience with mathematical modelling as motivational and that they do find mathematical modelling problems dealing with social issues relevant. The second major finding is that in developing mathematical models for social issues teachers utilise very low levels of mathematics which is in essence against the intention of school mathematics reform. It is argued that this disjuncture—the engagement with low level mathematics and the personal expression and experiencing of the modelling as motivational and relevant—that requires attention in mathematics teacher education programmes aimed at assisting teachers to realise the relevance ideal in their teaching. These programmes, it is suggested, should not restrict the mathematical content knowledge to concepts, facts, procedures and proofs but it should also include mathematical modelling as content and at a minimum this would mean that teachers have to experience all the components of the mathematical modelling process.

Keywords: Mathematical modelling; relevance of mathematics; mathematics teacher education; content knowledge

Introduction

One of the demands made on school mathematics is that it should be socially, environmentally, culturally and academically relevant. Varied meanings are attached to relevance although the most popular meaning is that learners should be able to use the mathematics that they learn in real-life situations. Notwithstanding the difficulties associated with the notion of "real-life situations" it is generally believed that the relevance ideal could be achieved through the incorporation of applications and modelling of school mathematics. The burden to implement applications and modelling of school mathematics to realise the relevance ideals is left to teachers. In many countries teachers are underprepared to effect this implementation since they themselves did not experience the applications and modelling of mathematics in a meaningful way during their preparation as mathematics teachers. The discussion that follows deals with mathematics teachers' experiences of mathematical modelling and how, amongst other things, relevance is manifested during these experiences.

The problem of reconceptualisation and representation of content contextually

A common notion associated with relevance of mathematics is that mathematics should be represented in some context. Dowling (1996) and others have developed the notion of reconceptualisation of mathematics to index the representation of school mathematics in some contextual format. By this re-representation of school mathematics they mean that in order to make mathematics relevant designers of school texts reconceptualise mathematics and present it in a form different from what the canon is supposed to be. In this body of literature the canon of mathematics remains mostly undefined. The undertext, however, reveals that the mathematics is the content of school mathematics stripped of it designed contextual trappings. This decontextualised mathematics is in essence an elementarised version of what the French didactical school calls institutional mathematics. Notwithstanding the criticisms of the reconceptualisation movement, it is so that in order to make mathematics teachable, designers of texts and mathematical activities do develop and devise contexts, which can serve as carriers from which mathematical concepts, procedures and justifications can be developed. An unfortunate development in this movement is that the protagonists work from the assertion that this is a demonstration of mathematical modelling. The origin of this claim is well known and understandable. The consequences are more far-reaching. One such consequence is the absurdity of non-relevant context, which is the sugar-coating of normally calculational work by absurd contexts. Nevertheless, the intentions of the "concept-carrying-contexts" protagonists are laudable. No matter how fierce the critique decrying the sometimes whimsicalness of the concept-carrier-context notion, school mathematics is always a watered-downed version of institutional mathematics and is always reconceptualised to particularly make it teachable. It needs to be borne in mind, however, that whatever claims are made to embedding mathematics in context the purpose of this embeddedness is not the construction of mathematical models but rather the use of context and sometimes mathematical models as vehicles for the learning of mathematical concepts, procedures and at times justifications. Mathematical modelling should not only be a vehicle for these mathematical ideas. Remaining at this level conceals the "behind-the-scene" work and intricacies involved in the construction of a mathematical model. Julie (1992), for example, illustrates the intricacies involved in presenting a division problem in the context of a grandmother sharing money amongst her grandchildren. Jablonka (2002) takes this demonstration further in her study of the contextual representation of school mathematics by investigating the epistemological claims behind the context in school mathematics texts. In trying to come to grips with the behind-the-scene work and intricacies involved in mathematical model construction, it is necessary that mathematical modelling should also be experienced as content. Mathematical modelling as content entails the construction of mathematical models for natural and social phenomena without the prescription that certain mathematical concepts or procedures should be the outcome of the model-building process. It also entails the scrutiny, dissection, critique, extension and adaptation of existing models with the view to come to grips with the underlying mechanisms of mathematical model construction.

Mathematical modelling as content as a way into relevance

In our quest to address the issue of relevance we inserted into our teacher education (pre- and in-service) mathematics courses a section on mathematical modelling. These courses have evolved over the past 7 years and include teachers (i) studying existing mathematical of social and economic situations in a guided way, (ii) assessing hypothetical exemplars of learner modelling work, and (iii) constructing models in an immersed way. In all these activities the teachers were exposed to the normal cycle for mathematical modelling. The underlying assumption, particularly with (iii) is that teachers should experience mathematical modelling as content. This means that teachers' experiencing of mathematical modelling should be as near as possible to the way it is done in the practice of mathematical modelling. A major characteristic of this practice is that the actual problem is initially vaguely formulated although the ultimate outcome--an artefact to realise a particular objective as specified by a client--is known to both the model-developer and requester.

Some of the situations that teachers were required to develop models for over the years were: A salary system to bring about equity based on the principle of "equal pay for equal work" taking into account years of service, promotion criteria and qualifications; the Human Development Index and other social indexes such as a community development index; school enrolment projections and garbage accumulation.

Data were systematically collected. This data comprise of observations and video-recordings of teachers at work; the rough work that was produced during the model construction process; the final reports on the models, formal and informal interview conversations and post-activity questionnaires.

The analysis proceeded by reading and rereading the data pieces of an entire work session. A description of the insights gained from the read-reread process pertaining to the broad research project—teacher behaviour when engaging mathematical modelling—was formulated and presented as a summary narrative of the session. This summary narrative was studied and commented on. The emerging comments were related to the broad question. For the analysis of subsequent sessions, the summary narrative was compared and contrasted with previous commented summary narratives in order to identify statements which could be similarly or differently commented and eventually coded.

Major Findings

Dominance of model as vehicle

One fairly consistent finding the analysis rendered is that the model-as-vehicle paradigm dominates the model-construction activity. This is seen as the search for a formula to describe the situation under investigation as illustrated in a teacher's response, figure 1 below, of her experiences with the salary scale activity.

> [This is a translation of the teacher's response which was written in Afrikaans]
> It was a struggle to understand the problem. The many principles, variables had your head spinning. We started by trying to get a formula from the table—excited! Oh...the equation/formula does not satisfy all the conditions. The struggle starts again from the beginning or a different strategy is sought. Decide first to work with one post level only to simplify the problem—point of departure gets a ceiling—highest position and highest number of years in post level. Build other formulae around the norm. Process is much "trial and error." Again and again and doing things over and over. Fit and measure/Test/verify. A possibility? OK. You get some confidence because there is no right or wrong—tension of criticism is gone. It was nice to prostitute your brain and test your limits. The end is sweet.

Figure 1: Teacher's description of experiences during mathematical model construction

This notion of the existence of a formula that can be found from the data dominates most of the both initial collective and individual deliberations. This is even so when the problematic the teachers were engaged with was assumed to be of immediate relevance to the teachers. One would, for example, expect that after a teacher strike on salary increments an activity on the development of a model of salary increases that would lead to realisation of the "equal pay for equal work" principle would entice teachers to first discuss the issues related to this principle as it pertains to their own respective situations. This, after all, is one of the tenets of relevance: what does it mean to me personally. It was hoped that "what does it mean to me personally" would engage teachers in a critical discussion and analysis of the situation. However, the formula-seeking behaviour dominated over the situation-analysis, which could have led to the development of a model based on derived assumptions. We contend that this formula-seeking is related to teachers' major exposure to mathematical modelling as a vehicle in which they are in a major way required to lead learners to identify patterns based on formulae. Essentially there is nothing wrong with this behaviour. The division of approaches to modelling into empirical modelling—fitting formulae to data—and axiomatic—developing a model from a set of assumptions—requires knowledgeability of formula-seeking. However, a deeper issue is at stake. This relates to the flexibility and robustness of teacher knowledge of mathematics. There is an emerging literature corpus, which reports on the lack of flexibility of teacher school mathematics knowledge and the desirable mathematics content for teaching as it pertains to primarily the concepts and procedures found in school mathematics. The emerging finding related to school mathematical modelling extends these findings to an essentially neglected area, which is being deemed important to realise the relevance ideal to teacher knowledge about school mathematical modelling.

Immediate perceived usability

When constructing models practising teachers seem to express a preference for a kind of relevance, which is immediate to their work circumstances. Consider the excerpt, figure 2, of fieldnotes made during teachers' engagement with the school enrolment model.

The teachers had to particularise a model for planning the supply for mathematics teachers to their schools based on the number of pupils at their school and a school enrolment model provided by Gould (1993). They presented their particularisations to the class. At the end of the presentations we engaged in a conversation around their work and the experiences with the activity. Mr K started the discussion and he said: "This was one of the first pieces of work I did where I can see how I can use it in my situation. We know that the number of teachers for a year is determined by using the enrolment of the year before. Now I can actually use this model at school and we can determine the number of teachers a few years in advance."

Figure 2: Excerpt of fieldnotes on school enrolment model

This contrasts with the data on teachers' reaction to the model building activity on garbage accumulation given in figure 3 below.

PLASTIC SHOPPING BAGS

The Minister of Environmental Affairs and Tourism has recently raised the issue of plastic shopping bags contributing towards filthiness and unsightliness of township areas. The plastic bags are blown around and get attached to fences presenting a sore sight of schools as dirty environments. It has been said that "School fences appear to be constructed from plastic shopping bags."

Develop a mathematical model to describe the accumulation of plastic shopping bags against a school fence over a period of time.

Figure 3: The garbage accumulation activity

A similar discussion on their experiences with this activity produced nothing about the usability of the models that the teachers developed. Even the enticement of producing a letter to the municipal authorities about garbage collection intervals and a concomitant enticement of educating the immediate local community about the inherent health risks associated with the dumping of garbage in open spaces did not have subjective appeal to the teachers.

What engages teachers and what not is a complex issue. Immediacy in terms of what I can use in my situation as it is currently is emerging as a facet of teacher behaviour regarding relevance. One aspect of this facet is that this immediacy will be unequally distributed across teachers. Regarding the enrolment model, for example, it is so that teachers directly involved in the administrative matters of a school would find work with this model usable. Those not directly involved will have a different kind of attachment which might at times border on hostility given the status related to job security in instances when a decrease in enrolment will result in a decrease in staff leading to retrenchment. For the garbage accumulation activity, on the other hand, attachment is linked essentially to political convictions. The more teachers are convinced that they should participate in the day-to-day struggles of the communities from which their learners come, the more they will perceive model-construction of this nature as an immediate usable activity.

Settling for the simplest and the non-activation of deeper Mathematics

One of the rationales for the lobbying for the inclusion of modelling and applications in school mathematics is that it will play an activating role. Modelling and applications will, in addition to its usability features, be a catalyst for thinking about mathematics that learners (and teachers) did not think about before. This can be viewed as relevance to mathematical development. The excerpt in figure 4 is from a transcript where teachers were requested to extend the Human Development Index (HDI) after they had studied the construction of the HDI. They were to extend the HDI by adding a fourth factor, satisfaction with the government of the day, to the HDI.

T(4):	I need to know what is the satisfaction of the government.
T(1):	We could mention many things. To educate the people, to have fair tax on domestic production and increase the domestic production.
T(2):	That is clear.
	...
T(3):	The basic factor for satisfaction of the government could be education. Everyone has right to education.
T(1):	Right. To attain satisfaction, the government could develop clear and consistency national policy of education.
T(2):	In our discussion, we agreed that there could be change with HDI. The emphasis we made was on education index. We related satisfaction of the government with education. We tried to double the education index to attain increase in HDI for the satisfaction of the government.
T(1):	Okay, we need to come up to a conclusion. We could say that we will do change with education index. Therefore, we can write the revised HDI in the form of $$\text{HDI} = \frac{\text{L. Expec.} + 2\ \text{Educ. index} + \text{GDP index}}{4}$$

Figure 4: Excerpt of discussion on the extension of the HDI

A few things can be observed from this narrative. Firstly, for extending the index the teachers were contend to simply work with the categories involved in the HDI. Although their discussion included references to fair taxation; domestic production and increase in domestic production, they were contend only to double the education component of the index and view this as equivalent to a fourth factor. The teachers remained as near as possible to the model that was studied and their extension of the model was confined to same categories contained in HDI.

Closely linked to first issue is the teachers' tendency to use only simple arithmetic. The above example is illustrative of this phenomenon. Although the HDI appears on the surface to be a simple weighted additive model, there is much deeper mathematics underlying the eventual model. As indicated the teachers tend to use fairly simple arithmetic: the doubling of one of the factors. This was not confined to the construction of indexes but was also the case for garbage accumulation and the teacher salary increments.

Lastly, there is closure on the further exploration of mathematics. Across the data the consideration of deeper level mathematics was not observable. For example, the garbage accumulation problem can lead to the mathematics involved in stocks and flows. The seeds for a development in such a direction is clearly discernible in a group of teachers' description for the construction of the plastic bag accumulation model in figure 5 below.

> How can the accumulation of plastic bags be worked out? We can say the Accumulation (A_{pb}) is the proportion of learners that litter, times the proportion of the community that, multiplied by the amount of days and times the amount of plastic bags used per day gives the total accumulation. We then subtract the attempts made to clean up, found by multiplying the amount of cleaning up attempts by cleaning staff, divided by the amount of time spent on cleaning up.

Figure 5: Teachers' description of model for plastic bag accumulation

Although the above description can be faulted, the seeds for moving this description and the resultant model to the mathematics involved in stocks and flows is clearly observable if, for

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example, one such formulation (Bartholomew, 1976: 162) for stocks and flows given below is considered.

$$n_j(T+1) = n_j(T) + n_{0j}(T+1) + \sum_{\substack{i=1 \\ i \neq j}}^{k} n_{ij}(T) - n_{j,k+1}(T) - \sum_{\substack{i=1 \\ i \neq j}}^{k} n_{ji}(T)$$

Where:

$n_i(T)$—(i = 1, 2, ...k) denotes the number of persons in grade i.

$n_{ij}(T)$—number of people moved to grade j by time T + 1

$n_{i,k+1}(T)$—number staff leaving the university altogether.

$n_{0i}(T+1)$—new entrants

[Note: Bartholomew's models is for teaching staff at a university where they get promoted from one grade level to another under conditions specified by the institution.]

Nowhere in the teachers' deliberations, even during informal discussions, were anything said or done which would point that this kind of mathematics was activated. Teachers seem to be fixated on what they perceive the task at hand to be and hence resolving this perceived task is for them the point of closure.

Conclusion

It has been related above how fairly qualified teachers of mathematics deal with and experience mathematical modelling and how the notion of relevance was manifested in the work and experiences of the teachers. Relevance revealed itself as complex and variegated. In some instances it was manifested as immediatist in the sense of usable for the work situation where teachers found themselves. In other instances it is assumedly closely tied to the socio-political orientation of the teacher. And in still other instances it is linked to the teachers' awareness or not of possible deeper level mathematics embedded in the descriptions and models that they construct. This last issue is closely linked to the use of fairly elementary mathematics in the teacher's model-building activity.

It is contended that these behaviours of teachers are related to their exposure, in no small manner in South Africa at least, to mathematical modelling as a vehicle. Uncritically it is assumed that this modelling as a vehicle will satisfy the realisation of the expressed ideal of relevance of school mathematics. This is not necessarily the case particularly if the incorporation of the relevance ideal in school mathematics is ostensibly the only ideal which directly addresses the development of a mathematical temper—a spirit of dealing rationally with the desirable and undesirable effects mathematical installations in society. There is no doubt that this realisation can only be effected through mathematics teacher education programmes which, in addition to developing mathematical modelling pedagogical content knowledge, aim at developing mathematical modelling as content. After all, it is during the engagement with mathematical modelling as content that windows of opportunities are opened for dealing with relevance relevantly.

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USING COMPUTER ALGEBRA TO ENCOURAGE A DEEP LEARNING APPROACH TO CALCULUS

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Keywords: computer algebra, deep learning, concepts, calculus

ABSTRACT

The underlying concepts and proofs of introductory calculus involve difficult and abstract ideas that present a mountainous obstacle to many students. A tempting 'solution' for lecturers is to focus the teaching at this stage on techniques. This may have the advantage of ensuring acceptable pass rates but helps neither the students nor the teaching staff in the long term. Computer algebra systems offer both an opportunity and a challenge to present new approaches that assist students to develop better understanding of the basic concepts. They can be used to change the emphasis of learning and teaching of calculus away from techniques and routine symbolic manipulation towards higher-level cognitive skills that focus on concepts and problem solving.

Two of the key indicators of deep learning and conceptual understanding are the ability to transfer knowledge learned in one task to another task and the ability to move between different representations of mathematical objects. Computer algebra systems are multiple representation systems, that is, they have the ability to facilitate graphical, algebraic and numerical approaches to a topic.

The author will describe how carefully structured worksheets are used with Derive to ask questions then let the students provide the answers in such a way that they can construct their own knowledge. This allows learners to 'discover' rules, to make and test conjectures and to explore the relationship between different representations of functions and other mathematical objects using a blend of visual, symbolic and computational approaches. Students enjoy the power and versatility of computer algebra and are encouraged to become reflective, deep learners.

1. Deep and Surface Learning

Mathematics is used in real life to model systems and analyse data arising in science, engineering and the management sciences. In order to tackle real problems it is necessary to acquire skills of formulating a problem in mathematical terms, interpreting the solution and analysing its sensitivity, all of which require a good understanding of the underlying concepts of the topic. An exploratory attitude and expertise in investigative techniques are crucial to successful modelling or problem solving.

Conceptual understanding, flexible thinking and an exploratory approach are all indicators of deep learning. Deep learning is generally held to be at one extreme of the spectrum of approaches that students adopt towards learning. The other extreme is surface learning. In practice, most students fit somewhere in between.

Surface learning, as the name implies, involves simply 'scraping the surface' of the material being studied, without carrying out any meaningful processing of the content. Students who adopt such an approach are characterised by:

- concentrating solely on assessment requirements ;
- accepting information and ideas passively;
- memorising new ideas as a collection of rules without any attempt at integrating with the old ideas;
- failing to reflect on underlying purpose or strategy.

Students who adopt a deep approach, on the other hand, want to make sense of what they are doing and to build their own personalised knowledge structures. They tend to follow the general pattern of:

- endeavouring to understand material for themselves;
- interacting critically with content;
- relating ideas to previous knowledge and experience;
- examining the logic of arguments and relating evidence to conclusions.

Most mathematics educators would argue that they want their students to adopt a deep learning approach. They want students to develop mathematical insight and the ability to solve problems. Students need to be able to make sense of answers, to manipulate expressions mentally and to anticipate the likely outcome of a range of possible approaches. This kind of mental agility is an indication of deep understanding.

But experience shows otherwise, particularly when the topic being taught is calculus. The failure of large numbers of mathematics students to grasp the basic concepts of calculus is well documented. The underlying concepts and proofs of introductory calculus involve difficult abstract ideas, which perhaps explains why, traditionally, the focus at this stage is often on techniques. With sufficient practice, the majority of students can become competent users of the rules that enable them to differentiate or integrate a range of standard functions. They can also learn to apply these techniques to problems such as finding extreme values or the area under a curve. Far fewer students, however, can explain the underlying ideas of a limit, the differentiability of a function or integration as an infinite summation. Faced with the prospect of a large proportion of their students failing the final examination, many teachers abandon the attempt to develop conceptual understanding. The goal of the pragmatic lecturer is often for the student to develop skills in computational procedures, to apply the correct procedure in a given problem and to achieve good examination grades.

2. What's wrong with conventional teaching and learning methods?

Clearly the traditional style of teaching whereby calculus is presented as a logical exposition of definitions, proofs, techniques and applications isn't working as a means of encouraging deep learning. The reason it doesn't work is that presenting mathematics in a way that develops from formal ideas is not a sound approach, pedagogically. It does not attempt to build on the students' current knowledge structures.

Dubinsky (1991) asserts that, in his experience, students do **want** to understand concepts but if they don't understand what is being taught they lose interest and resort to surface learning techniques:

> *"As long as there is something for the student to think about, as long as he or she perceives that cognitive activity is leading to some sort of growth that could, eventually, lead to a solution of the problem, then there is little difficulty in maintaining the students' interest."*

The experiential model of learning developed by Kolb (1984) stresses that learners must be actively involved in the learning process. He presents it as a four stage cycle - planning, doing, thinking and understanding. In mathematics learning this cycle can be interpreted as being involved in planning the learning outcomes, carrying out appropriate learning tasks and activities, and reflecting on what has happened leading eventually to relational understanding. The final stage involves linking actual learning experience with the underlying theories and thus integrating the new concepts into existing knowledge structures. It may, of course, take many loops of the cycle to reach the desired deep understanding of the abstract concepts.

Research has shown that appropriate sequences of learning and teaching designed to help the student actively construct concepts in this way can prove highly successful. Dubinsky (1991) describes the outcome of a an introductory course in calculus in which students spent the first twelve weeks focussing on the underlying concepts before starting to practice techniques.

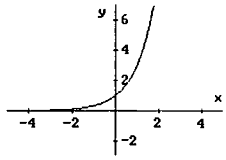
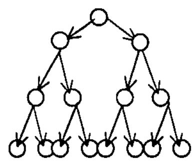
> *"By encouraging the students to think for themselves and to construct their own ways of handing concepts, it became apparent that they had integrated the ideas into their own knowledge structure."*

Tall (1986a) argues that it is not necessary for the student to complete a large number of different examples of a new idea in order to develop relational understanding, as is popularly thought. It is the third and fourth stages of Kolb's learning cycle, reflection and understanding that lead to the conceptual understanding and abstraction. This can be achieved effectively by using a few carefully chosen examples to illustrate and explore important aspects of the concept and nurture the required reflective abstraction. Tall refers to *"The single, representative example which so often seems to be in the mind of the mathematician who understands a particular concept"* as the 'generic example'.

3. Multiple Representation

In the mind of a mathematician, the 'generic example' frequently exists in several different but linked representations. For example, an exponential function may be thought of in symbolic, graphical, geometric and numeric forms (figure 1) but, once the abstract concept has been grasped, the user can switch between the different representations with ease in order to retrieve the one most appropriate to the current problem.

Figure 1: Representations of an exponential function

Graphical	Geometric	Symbolic	Numeric
		$f(x) = a^x$ or $\dfrac{dy}{dx} = kx$	x | 3^x 0 | 1 1 | 3 2 | 9 3 | 27 4 | 81

Mathematicians use symbols both to **think** about mathematics and to **do** mathematics, to communicate ideas and to express results. Seeing a symbol or symbolic expression conjures up in the mind a mental image of what the symbol represents but students' and teachers' mental representation of a mathematical symbol can be very different. The teacher is able to select the mental image most appropriate to the particular task. A student might think of a function purely as an algebraic formula (process) whilst the teacher is thinking of a function as an object to be transformed by some operation such as differentiation or indefinite integration (Tall, 1991). This situation can lead to confusion and leave students unable to understand the teaching thus creating an obstacle to learning. Another confusing situation arises when a student tries to cope with several competing mental representations of the same concept.

Students who can only work in one representation often fail to solve a problem correctly. For example, even when they are required to sketch the graph of a function in one part of a question, they may ignore the 'evidence' of their own sketch in a subsequent part of the question in which they are using an algebraic representation of the function. Teaching and learning should aim to integrate competing representations into a single representation, sometimes called a 'multiple-linked representation' (Tall, 1991). This allows a person to use several different representations at the same time, switching from one to another when it is appropriate to do so.

According to Tall, there are four stages to the learning process:

- Using a single representation;
- Using more than one representation in parallel;
- Making links between parallel representations;
- Integrating representations and flexible switching between them.

It is the recognition of links between parallel representations and their common properties that leads to the formation of an abstract concept of the mathematical object or process. Once the abstract concept has been formed, its 'owner' can link back to any one of its concrete representations when required. This 'multiple-linked representation' state of thinking is an essential pre-requisite to deep understanding and underpins the flexibility required for problem solving.

4. How computers can help

With the traditional undergraduate curriculum, students do not often regard themselves as active participants in mathematical exploration. Rather they are passive recipients of a body of knowledge, comprising definitions, rules and algorithms. Computers offer a number of didactic advantages that can be exploited to promote a more active approach to learning. Students can become involved in the discovery and understanding process, no longer viewing mathematics as simply receiving and remembering algorithms and formulae. (Shoaf-Grubbs, 1995). In particular, the computer provides opportunities for dynamic visualisation. Students can explore basic mathematical concepts from new geometrical and graphical perspectives. Several research studies have concluded that the visualisation features of computers can be used successfully to encourage multiple linked representations of mathematical concepts (for example Tall, 1986a, Schwarz, Dreyfus and Bruckheimer, 1990).

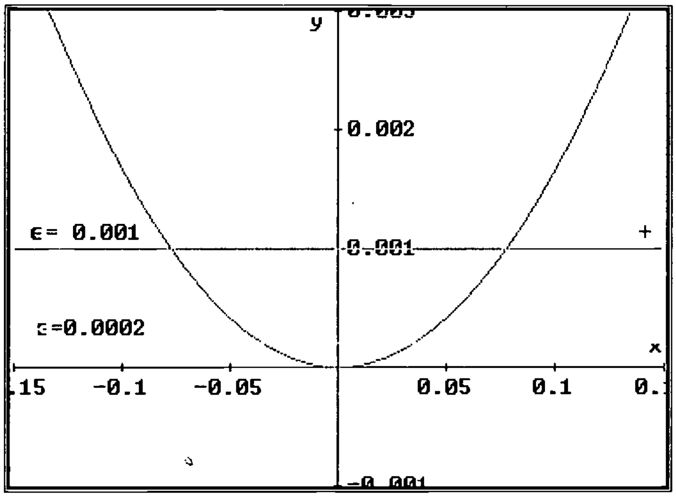
One software environment designed to promote the dynamic visualisation and exploration is 'Graphic Calculus' (Tall, 1986b) which uses visualisation to explore the concept of differentiability using the notion of 'local straightness'. If a function is differentiable at a point then, by magnifying the graph of the function on the computer screen, it eventually becomes like a straight line around the point. This can then be linked to numeric and symbolic approaches to give the notion of a derivative a computable meaning. Tall argues that the idea of local straightness is a more natural starting point for a student to understand differentiability than the concept of a limit. Using this approach, graphical methods 'lead' the analysis.

5. Computer Algebra

Visualisation with the aid of software such as Graphics Calculus or a graphics calculator can lead students to a greater understanding of concepts but this approach does not necessarily help them to cope with the corresponding symbolic representations. A computer algebra system (CAS) is a multiple-representation system; it has the ability to facilitate graphical, algebraic and computational approaches to a topic. It is therefore an ideal tool for directing learning towards multiple-linked representations of mathematical concepts. Through carefully designed activities students can investigate the links between different representations of objects, recognise their common properties and begin to construct their personal structures of mathematical knowledge. Student activities have to be designed with very detailed cognitive steps in mind. Appropriate teacher intervention will usually be required to ensure that the students follow through the required learning stages, in particular, the reflective thinking.

As an example, consider how students could use the computer algebra system, DERIVE to investigate the limit of $\frac{\sin x}{x}$, as x approaches 0, from various perspectives, which use different representations of the limit. They can tabulate values of $\frac{\sin x}{x}$ for smaller and smaller values of x, plot a graph of the function over an interval around x = 0 and use the LIMIT command in DERIVE to evaluate $\lim_{x \to 0} \frac{\sin x}{x}$. Having demonstrated convincingly that the limit is 1, the rigorous definition of the limit can be explored. The definition states that if $\lim_{x \to 0} \frac{\sin x}{x} = 1$, then, given any positive number ε, there is a corresponding number δ such that $\left|\frac{\sin x}{x} - 1\right| < \varepsilon$ whenever $0 < |x| < \delta$. In order to investigate this graphically, plot $\left|\frac{\sin x}{x} - 1\right|$ for $-1 < x < 1$. Then, by superimposing the graph of y = 0.001, say, the points where the two graphs intersect show the value of δ when $\varepsilon = 0.001$, as shown in figure 2. The meaning of 'given any positive number ε, there is a corresponding number δ .....' can be clearly demonstrated by experimenting with different values of ε. Each stage in the investigation should be followed by opportunities for recording results, reflective discussion and, if necessary, further exploration.

Figure 2: Exploring the Definition of a Limit



CAS activities in which students are asked to construct examples that satisfy certain constraints can also be used to encourage exploration of concepts, to focus on the links between different representations and to develop reflective thinking. The following simple example helps students to make connections between the derivative of a function and the slope and shape of its graph:

Define 4 non-linear functions, two that have a positive first derivative and two that have a negative first derivative in the range −10 to 10. Plot all 4 functions on a graph and describe the shape of each curve. How can you tell from the graph where the derived function is positive?

Research has shown that the ability to transfer what has been learnt from doing one mathematical task to doing a similar one is more likely when the learner has been helped to discover the rule for doing the first task. (Sotto, 1995) A computer algebra system is an ideal tool for allowing the learner to 'discover a rule' or to make a conjecture and then prove or disprove it. The tedious repetitive manipulation is automated and the learner can concentrate on the results. One obstacle to learning can be the teacher who is determined to 'teach' or at least to tell the student everything so that there is nothing left for them to construct for themselves. Support materials should concentrate on providing activities and asking questions then letting the student provide the answers through reflection. In the following exercise, DERIVE is used to help the learner discover the rule for differentiating functions such as sin(kx) and cos(kx) as an introduction to the Chain Rule for differentiation.[1]

Use DERIVE to obtain the derivatives of sin(x), sin(3x), sin(6x), sin(-2x) and sin(-2.5x). Deduce the general formula for the derivative of sin(kx), where k is a constant. Use the ***Limit*** *command in DERIVE to investigate* $\lim_{h\to 0}\frac{\sin(k(x+h))-\sin(kx)}{h}$. *Does the answer agree with your own rule?* (Similarly for cos(kx)).

[1] Many similar examples can be found in Learning Mathematics through DERIVE, (Berry, Graham and Watkins, 1996)

Write down the derivatives of the following functions and check them with DERIVE: sin(2x), sin(-4x), cos(3x), cos(0.5x)

6. Development of problem solving skills

A typical student approach to problem solving is to find a suitable worked example to mimic then carry out the computation. Clearly this strategy is limited by the extent of the students' memory bank of similar problems and inhibits flexible thinking. A better approach is to consider alternatives, experiment, conjecture and test, then analyse the results. A computer algebra system can be a major factor in developing an exploratory approach to learning mathematics and, in particular, investigating problems from multiple representational perspectives. Using the computer to produce graphs, carry out calculus operations or perform repetitive calculations, students can be encouraged to make and test conjectures, to consider alternative solutions and to tackle open-ended problems. Removing the burden of manipulation and computation allows students to spend time on these other activities. This approach can make the study of mathematics more enjoyable, more relevant and more rewarding (Mackie, 1992).

Most elementary courses concentrate on closed form solutions; approximation methods are only mentioned briefly. This is due to algebraic limitations and the tedious and extensive calculations required to obtain or analyse approximate numerical solutions. As a result, modelling is usually restricted to artificially constructed examples and leads students to question the applicability of what they are doing. The use of computer algebra removes these restrictions and allows students to solve real problems using a combination of numerical and algebraic techniques.

7. Disadvantages

The ideas expressed so far in this paper present a very positive role for CAS in mathematics education. Are there any disadvantages? Norcliffe (1996) warns that most revolutions are double-edged and that the computer revolution brings many challenges, concerns and dangers to mathematics. One of Norcliffe's concerns is that students' algebraic manipulation skills will deteriorate if they are allowed to rely on computer algebra but that these skills are an essential foundation for mathematics. Many other educators would agree that there are fundamental mathematical skills that are essential even when technological tools such as computer algebra systems are available. There is, however, no widespread agreement on what exactly these core skills are. It seems clear that students must still spend time developing manipulation skills. The potential of computer algebra lies in its ability to improve conceptual understanding and problem solving.

In a report of a project involving the use of DERIVE with A-level students to investigate the link between infinite summation and anti-differentiation, Terence Etchells (1993) describes how a lesson can go wrong if the students have an insufficient understanding of the underlying mathematical concepts:

> *"A very didactic problem with CASs is that they can produce meaningless expressions. Students are punching keys and performing operations on expressions that have no meaning; they are producing mathematical nonsense."*

A similar example from the author's own experience occurred with a class of students using DERIVE to investigate the effect of the constant k in the exponential function e^{kx}. Two students did not use the symbol ê for exponential e (despite a clear reminder in the worksheet) and thus could not display the graph. At least these students realised that something was wrong. A more 'serious' mistake was the failure by one student to use brackets correctly and she was therefore investigating the functions: $\hat{e}^x$, $\hat{e}^2.x$ and $\hat{e}^{0.5}.x$ without realising her mistake.

Teacher support and appropriate intervention is crucial to correct mistakes of this nature and enable students to make the important links. Judging the right amount of help at the right time is a skill acquired through practical experience. Students must be allowed sufficient time to learn the language and features of DERIVE before using it to enhance their learning.

8. Conclusion

"It would be a mistake to incorporate CAS into our courses primarily as exercise solvers, whilst continuing our present orientation towards carrying out algorithmic computations. Our goals, expectations, assignments and classroom instruction need to change in order to maximise the opportunities offered by modern technology." (Small & Hosack, 1976)

Although written some time ago in the early days of computer algebra in education, the view of these authors is still valid. Many of the changes, which they predicted, have not yet happened but are still necessary. Students measure the importance of an activity by the amount of time devoted to it. At present most of their time is spent practising routine skills. Perhaps it is not surprising that students view mathematics as a collection of formulae (to be memorised) and "to do maths" is to compute. If more routine computation is done on a computer more time is available for concentrating on concepts, motivation, applications and investigations. Computation will be seen as a means rather than as an end in itself.

The power of computer algebra goes beyond routine computation. It has the potential to facilitate an active approach to learning, allowing students to become involved in discovery and constructing their own knowledge, thus developing conceptual understanding and a deeper approach to learning.

A computer algebra system is a tool not a self-contained learning package or encyclopaedia of mathematical knowledge. It is the way in which it is presented to and used by students that determines its ability to influence learning. Much emphasis these days is placed on student-centred learning and less on the teaching but teaching and learning are equally important. It is necessary to first understand the learning process and then design teaching and learning activities to achieve these. Only then will students become deep learners.

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OF COURSE R^3 IS BLUE! DEVELOPING AN APPROACH TO TURN A MATHEMATICS COURSE INTO A MATHEMATICS EDUCATION COURSE[1]

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ABSTRACT

This study had as its objective to investigate the possibility of offering a mathematics-content course (in this case Linear Algebra) that is adequate to the professional development of mathematics educators (teachers, teacher educators and researchers); for reasons we will present and discuss, we started with the assumption that traditional content courses (in many cases the same as those presented to future mathematics researchers) were not adequate. The study consisted in the analysis of the transcriptions of videotaped lessons and other protocols collected at a four-months Linear Algebra course, taught to postgraduate students in a mathematics education postgraduate program in Brazil. We will focus on the presentation and discussion of the processes generated by the students' attempts to solve a mathematical problem, particularly on those relating to the production of meaning for the notion of space, and how the approach we took as professors (for instance, only to intervene to call their attention to recurrent statements or to divergences in the whole-group discussions) opened up a 'magic window' to the meanings they were producing for the notions involved, despite several Linear Algebra textbooks being available to them at all times. We will also argue that such a reading of the meaning production processes not only produces a very useful material for reflection during the course, but that it is in fact a necessary condition if we want content courses to be mathematics education courses that actually contribute to the professional development of our mathematics education students (teachers, teacher educators and researchers).

KEYWORDS: linear algebra, vector spaces, dimension, physical space, mathematics teacher education, meaning production

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Introduction

The education of secondary and high-school mathematics teachers follows, in Brazil as in many other countries, what we call the 3+1 format: the equivalent of three years of courses in mathematics, mostly the same courses a future researcher in mathematics would take, plus the equivalent of one year of courses in pedagogy related content. Many people in the mathematics education community have voiced their opinion that this is not adequate, but the issue has not yet been sufficiently studied (Wilson et al., 2001).

Since the second half of 1999 our research group has been working on developing an approach to the mathematical courses for future teachers that takes a view different of the 3+1

This paper reports an attempt to develop a *mathematics* course which would become, because of the way it was structured, a *mathematics education* course; one way to understand this is to think of this course as a service course directed to mathematics teacher education, as much as a Calculus course for engineers or computer scientists is not (or should not) be the same as that for the future mathematics researcher.

The course design

The core idea of the course was that while discussing mathematics (what we called 'the content') the students would also be discussing the processes they were going through (interaction, meaning production, obstacles/limits: 'the meta-content').

The subject chosen was Linear Algebra for postgraduate students (masters and PhD) at our postgraduate program. Many of them were at the time teaching mathematics at university level, for future teachers, some were school teachers and some were only working for their postgraduate degree; altogether 17 students. All sessions were videotaped.

We began the course proposing an investigation:

> " Let $\mathbf{R}^2 = \{ (a,b) ; a,b \in \mathbf{R}\}$
> Is it possible to exist a real vector space in which the vectors are the elements of $\mathbf{R}^2$, and such that its dimension is 3?"

One of the reasons for choosing this problem was that just to understand it one needs to take into consideration all the basic concepts of linear algebra, possibly leading, in our case to a revision of those concepts, that being part of their mathematical development. As we shall see, that did not happen so.

The students worked in groups of three and we said they could use any books they wanted to. One of the groups agreed to move to a separate room where their discussions were videotaped.

We told them that for the time being we would make no mathematical comments or answer mathematical questions to them; if they wanted to say something to us, we would listen, but they should not expect replies. Our silence was certainly responsible for the richness of the experience we were all going to go through. As the sessions went by the students were more and more open and making statements that could finally reveal to us what we were looking for: processes of meaning production for mathematics that were not only directed towards the mathematical meanings.

They had acess to all the information on the books, they had already taken Linear Algebra courses and many of them actually taught mathematics at university level, but almost all the discussion that followed would not be seen by a mathematician as 'mathematical'. That is when the title of this paper may make more sense: there was actually a discussion about whether or not, and why, $\mathbf{R}^3$ is blue.

The underlying theoretical framework

The crucial difficulty in a situation like this is how to understand the students' statements, in a very specific sense: how to categorise them without using primarily our own categories? Much more often than not, statements are interpreted according to categories which do not structure or organise the other person's thinking. A superb example of this is presented and discussed in G. Lakoff's book "Women, fire and dangerous things" (Lakoff, 1987).

The difficulty is precisely this: when a student makes a statement it is not sufficient to 'understand' the statement and to see whether or not we think it is a correct statement. It is also necessary to know what meaning *the student* produced for that statement (and the objects it refers to) and this is crucially related to why *the student* believes s/he can make that statement.

The support for this kind of reading in our work comes from the Theoretical Model of Semantic Fields (TMSF), developed by one of us as a tool to support teaching and research in mathematics education (Lins, 1992, 2001).

Its central notions are those of 'knowledge' and 'meaning'. Knowledge is characterised as a statement a person makes and in which s/he believes (a statement-belief), together with a justification s/he has for making that statement. Meaning is characterised as what a person actually says about an object, in a given situation (activity); it is not everything s/he person could eventually say about that object. Meaning production and knowledge production always happen together, and objects are constituted through meaning production.

A third notion on the TMSF is relevant here, that of 'interlocutors'. It has to do with why a person thinks s/he can make a given statement in a given activity. We understand interlocutors as modes of meaning production that a person internalises as legitimate during his or her life; they are cognitive elements, not real people. In other words, in order to believe we *can* say something we must also believe that 'someone else' would say the same thing with the same justification. The notion of interlocutor is relevant because it allows us to explain why sometimes meaning production does not occur (typically expressed by statements like 'I did not understand' or by silences) or to explain divergent meanings being produced by two or more people. In this sense, communication is understood as the sharing of an interlocutor ('speaking in the same direction', rather than 'speaking in each other's direction').

Those three notions, knowledge, meaning and interlocutors, allowed us to keep the classroom activity as open as possible at the same time we could reliably trace the processes of meaning production in their dynamical character.

With the students

The typical first approach was to assume that "$\mathbf{R}^2$ *is* the plane" and to try to see *this* plane as a plane in *the* space ($\mathbf{R}^3$), so that the points in it would have three coordinates. Some students argued that this was fine, while others argued that the plane was still a plane and *its* dimension was still two. Although most of the groups browsed the textbooks available they did not effectively engage in finding the relevant definitions and discussing the problem in relation to them.

Quickly it became clear to us that for those students the initial question could only be asking for some kind of construction through which to place *the* plane on *the* space. From that moment we started to use (among us, who were conducting the course) the expressions 'natural plane' and 'natural space' to refer to the association between $\mathbf{R}^2$ and $\mathbf{R}^3$ to the cartesian, geometrical (perhaps physical, but we could not tell at that point), plane and space, respectively.

It also became clear that those natural spaces had intrinsically bound to them the natural operations of addition of vectors (points) and multiplication by a (real) number. Those students did not, at that point, produce meaning for $\mathbf{R}^2$ and $\mathbf{R}^3$ as (structureless) sets.

Besides the mathematical discussion, some important meta-questions were being asked: what had been done to the linear algebra they had studied in the past? Why was it that the books did not seem to be of any help?

The group that was in the separate room also tried this approach, but they went a bit further ahead with an interesting idea; working on the previous idea of immersing the plane into the space, combined with the idea that an answer depended on the operations, they tried to find an addition of vectors in $\mathbf{R}^2$ that would produce a vector in $\mathbf{R}^3$, something like,

$$(a,b) + (c,d) = (e, f, g)$$

That is when we want to come to the discussion of the colour of $\mathbf{R}^3$.

We were having a whole group discussion of their approaches. One of us (let's call him 'Green') asked to speak. He said:

> [GREEN] I have always found the [sic] $\mathbf{R}^3$ much more interesting than the [sic] $\mathbf{R}^2$. I always see colour with the [sic] $\mathbf{R}^3$; for me it is blue. I was thinking of saying this. Perhaps because of the space?

He said it and left, excusing himself that he had some photocopying to do.

The first exchange we will look at happened immediately after Green left and has no visible relation to his statement. The second exchange happened at the end of that session, as Green asked to say a few more words.

At this point the students had already presented several ways of trying to make *the* space from planes, following their initial idea that what our original question actually asked was that.

> [DIVA] What is the plane for you?
>
> [ADES] Two linearly independent vectors. The union of the two linearly independent vectors.
>
> [...]
>
> [ADES] I am understanding the plane as an infinite paper sheet.
>
> [DIVA] That infinite sheet of paper, it has how many dimensions? [sic]
>
> [ADES] Two. Only defined by the two LI vectors.
>
> [...]
>
> [DIVA] But [...] if it does not have the third dimension as you place one over the other it would make no difference. So it would have to have a minimum of thickness [...] [2]

But Diva also admitted, when asked, that although she could not think of thickless planes inside the current activity, she would never mention thick planes to her students. How to make sense of this?

One interpretation is to say simply that Diva does not *really* know calculus or geometry or linear algebra. Our alternative interpretation is that Diva 'has' at least two 'planes': one is thickless

[2] At another point it becomes clear that Diva is not just discussing Ades' idea: she actually cannot think of a thickless plane as she discusses our original problem.

and belongs to her Calculus lessons; the other is thick and belonged to our sessions. To be more precise, Diva had internalised (different) interlocutors and as she spoke in the direction of each of them it became legitimate to speak either of thick or of thickless planes.

We now move to the second exchange.

> [GREEN, near the end of a session] [...] I would like to return to that question.
>
> [MILA] No, he said like, look guys, I think the $\mathbf{R}^3$ is blue. He speaks and leaves, he leaves things like that. [...] it upset me a lot [...] how can you see a colour like, blue? Why not yellow? Why not pink?

Mila took Green seriously and she was seriously questioning him; it is clear, by the tone of her voice, that she was actually disturbed by Green's statement. Mel said that "...he imagines the $\mathbf{R}^3$ blue, like", possibly because she believed that Green was not actually saying that the $\mathbf{R}^3$ *is* blue, but only that *he* imagines it so. There is a subtle but relevant difference between Mila and Mel. Mila believes Green's statement is about what $\mathbf{R}^3$ is and she cannot produce meaning for that (why *blue*?), while Mel is happy with Green imagining something about $\mathbf{R}^3$, having his own, idiosincratic, view, unrelated to mathematics. From a mathematical point of view, Mila leaves open the possibility that the mathematical object $\mathbf{R}^3$ have a colour, while Mel apparently does not, she seems to treat Green's statement as extra-mathematical. What could be the consequences of Mila's belief as she tries to understand and answer our initial question?

The exchange continues.

> [GREEN] Right, for me it is so natural that I can't even understand when you ask me.
>
> [BLUE] Because blue is a harmonious colour.
>
> [GREEN, talking to Maria Luiza] Do you also find [Mila's question] strange?
>
> [MARIA LUIZA] No, I was just about to ask what is the colour of the $\mathbf{R}^2$...
>
> [lots of laughs]
>
> [MILA] What about $\mathbf{R}$? [more laughs]
>
> [MUIARA] We associate the $\mathbf{R}^3$ to the space and the space for us, at school, at home, is the sky. It's just an association, nothing more.
>
> [BLUE] [...] In a general way the $\mathbf{R}^3$ is this, it is the space. What's the space? When you look to the sky, the most beautiful thing to look at is the blue.
>
> [GREEN] Yeah, I had really thought of the space thing. But does it seem unreasonable to anyone [here]?

Blue seems to have a more aesthetical approach: the colour blue is harmonious and beautiful. Muiara seems to follow Mel's view. And Mila remains concerned with how to determine the colour of a space.

The crucial aspect, though, is that they all seem to be thinking, as they try to make sense of Green's statement, with a naturalisation of (the) $\mathbf{R}^3$ as the (physical) space, and the fact that we were supposed to be trying to solve a *mathematical* problem does not seem to disturb them; that suggests that for those students the mathematical $\mathbf{R}^3$ is really simply *the* space and that imposes to it all the natural properties of the physical space.

The first exchange, involving Diva and Ades had shown that also $\mathbf{R}^2$ was seen in a naturalised way (to the extent that Diva could not conceive a thickless plane), and the mathematical consequence is clear: from the outset the students *knew* that the answer to our problem should be 'no', because they *knew* that *the* dimension of $\mathbf{R}^2$ *is* 2, and the original problem could only produce a search for some tricky immersion of $\mathbf{R}^2$ into $\mathbf{R}^3$ or a search for an argument to show what they already knew.

Our silence allowed for all that: planes in the space to become three-dimesional, rotating planes to produce the space, displacing planes to produce the space, thick planes, the harmony and beauty of the sky. The interesting question is: had we not remained silent, had we directed the process of meaning production through intervention, through the correction of 'wrong ideas', where would those things be? Our answer is: in the students' street-smart backpack, that would have been left outside the classroom as they entered it for each session. And as they left the classroom they would leave the mathematical folder inside and take the backpack again, and hit the road (either to go home or to work). But some of the 'street' ideas might well remain in their pockets and who knows what the effect they will have on the possibility of them thinking mathematically.

We continue to follow the second exchange.

> [MEGA] You [Green] identify yourself with the blue, like thinking the $\mathbf{R}^3$ is blue. You think the $\mathbf{R}^3$ looks like blue. I'm not joking, I'm serious, for instance, I simpathyse with the number seven, I like the number seven, I think the five is too fat and the four has a big nose. [...]
>
> [MUIARA] When I talk about space I find it difficult to understand it [in] $\mathbf{R}^1$ and $\mathbf{R}^2$ although when we are writing on a notebook, like that, you [leave, as as you write] a space [...also] you are on a plane and [leave] a space [between paragraphs] So I have to make an effort to think of a space [sic] in $\mathbf{R}^2$. [...] It is much more natural in dimension three for us to think of the space because there he occupies the whole volume.

Nosy numbers; space as in writing, as 'spacing' between words or paragraphs; a blue $\mathbf{R}^3$. This is the kind of material that made our course fruitful, through later discussions of what had been said by them, of why we remained silent and of what was requrired to understand the mathematical solution of the original problem.

The overall picture of the classroom is clear: a natural notion of space as something that is there to be occupied by something, a place where things (vectors, for instance, but also and equally, chairs and tables) can be, and a naturalised notion of $\mathbf{R}^3$ as *the* space; a natural notion of a plane as a smooth 'right' surface in which one draws and writes and rests objects, and a naturalised notion of $\mathbf{R}^2$ as *the* plane. The natural ones developd through the ordinary experience and other sign systems, the naturalised ones developed through experience in school (including university).

And what is wrong with that? For the vast majority of people we would say, nothing is wrong. For a few specialised professionals (some physicists working with superstrings, for instance) it would be a problem. But our main concern here is teacher education, so we must address the question from that point of view.

The evidence gathered in this course convinced us that through the discussion of their natural/naturalised notions of space and dimension, and by confronting those with the ones in linear algebra, we had achieved two objectives which are, from the point of view of mathematics teacher education, important. First, the students had the opportunity to reflect on their natural/naturalised ideas (which are almost always hidden in the background, and frequently conflicting with their mathematical 'correspondents') and on how these might affect their mathematical thinking, opening the door for real mathematical learning. Second through reflection on their own, lived, experiences, our students came to conceive the classrooms where they teach as places in which those processes are constantly happening, that is, as they face their own students they will be aware that similar processes might be happening even though with different objects as, for instance, when a school student encounters the notion of an infinite non-repeating decimal.

We think it was possible for our students to achieve both mathematical development (improved mathematical lucidity) and professional development (improved mathematical education lucidity).

Final remarks

Through a quite unexpected process the group arrived, reluctantly, at a solution: yes, it was possible to exist a real vector space in which the vectors are the elements of $\mathbf{R}^2$, and such that its dimension is 3; actually, infinitely as many (inducing the structure of the usual $\mathbf{R}^3$, using the fact that both $\mathbf{R}^2$ and $\mathbf{R}^3$ have the same cardinality). We discussed the basic notions involved (basis, dimension, the operations, inducing a structure using a bijetion) as well as how convinced they were of that (mathematically sound) answer.

Their reactions alone would generate a paper much longer than this one, but it is perhaps sufficient to say that one of the students, who had been very participative all along, said he was in shock, that he felt as if the ground had suddenly vanished from under his feet. His reaction was the starting point of a third phase of the course: "now you know that in mathematics there may be worlds quite different from our natural ones. So, keeping that in mind, let's look at some more of vector spaces". It had been *their* experience of surprise and 'shock', so they took it very seriously when we got to see families of paraboli associated to straight lines in the $\mathbf{R}^3$ with the usual structure.

All the time we made it very clear that our aim was not to correct their previous views, but to *add* a new possibility for meaning production and to help them to understand that sometimes one kind of meaning was more adequate, sometimes the other. And we stressed that everytime they had a student in front of them they should remember that maybe, just maybe, what was natural for the teacher was not natural at all for the student.

Evidence gathered in our other ongoing studies support the suggestion that what we met during this course is far from some kind of uncommited discourse (supposedly happening because we did not intervene to 'put an order' in it). Quite on the contrary, we think that discourse was committed and sound *in their terms*; we got a quite good understanding of what was supporting those students' thinking as they engaged with a *mathematical* problem.

A number of insights were gained during the course, and we will focus on three.

1) natural and naturalised objects have a high influence and a low visibility in students' thinking;

2) mathematics courses in the education of mathematics teachers should be designed as to create opportunities for those natural and naturalised objects to appear, and for two reasons:

(i) to help students to understand that natural and naturalised objects are not, in most cases, what we are talking about, in mathematics, thus improving their chances of learning; and,

(ii) to offer students the opportunity to discuss meaning production processes in a highly reflective way, as they were themselves the subjects of the processes being discussed.

3) it is possible that 'the mathematics of the mathematician', that is, mathematics as structured by the mathematician (calculus, analysis, linear algebra, algebra, and so on, and internally, inside each of those 'blocks') is not a suitable basis for the mathematical

education of mathematics teachers. Rather, we suggest that it is possible that the structures of natural and naturalised objects (space, measurement and quantity, for instance) might provide a more adequate basis.

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A STUDY OF CLASSROOM PROCESSES RELATED TO THE PRODUCTION OF MEANING FOR 'FUNCTION': THE CONTEXT OF REAL ANALYSIS VS THE CONTEXT OF DUAL VECTOR SPACES[1]

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ABSTRACT

Usually we expect our students to produce meaning for functions in Real Analysis as 'a correspondence between sets of real numbers'. In Algebra we generally start with function as 'a particular subset of a cartesian product', but when working with dual vector spaces we expect them to understand functions as elements of the base set of an algebraic structure. While our teaching experience had already confirmed the results of previous studies showing that students remain attached to the 'analytical' understanding of function, we decided to conduct a study that could further our understanding of this process. This study happened in the context of a regular Algebra course (undergraduate mathematics degree) particularly the section on duality of vector spaces. The data we will present and discuss come from transcriptions of lessons and from tests applied during the course. The theoretical support comes from: (i) EP ('Enseñanza Problémica', in Spanish), a didactical model developed in the former Soviet Union during the second half of the 20th century, based in the historical-cultural theory of Vygotsky, and which provides us with a set of categories that allows us to organise in a dynamical way professor-students interaction; and, (ii) the Theoretical Model of Semantic Fields, developed by R. Lins, an epistemological model that allows us to 'read' the processes of meaning production as they happen, 'on the fly'.

Keywords: function, meaning production, semantic fields, dual vector spaces, problematising teaching (enseñanza problémica)

Introduction

Usually we expect our students to produce meaning for functions in Real Analysis as a correspondence between sets of real numbers. In Algebra we generally start with function as a particular subset of a cartesian product, but when working with dual vector spaces we expect them to understand functions as elements of the base set of an algebraic structure.

What we mean by 'analytical understanding' of function is a function as a correspondence between variables, given or not by an expression or a formula. It's not common in Analysis to consider as different objects a function and the function obtained by a restriction of the domain. Similarly, in Analysis the difference between a function f and the image f (x) of an element of its domain is not generally emphasised.

Differently, in Abstract Algebra one has to consider those aspects carefully. In Algebra, not taking a function and its restriction as different objects can hinder the understanding of important theorems of Linear Algebra. Also, it would not be possible to find an 'inverse' for a not invertible function, by means of its restriction to an appropriate subset of the domain. Considering this 'inverse' is a situation that frequently appears in Algebra and its applications.

Not making distinctions between a function f and the image f (x) of an element is a serious problem when one is working with vector spaces duality, where the student has to deal with functions whose domain is a set of functions. This is the cases of the transpose of a linear map, or the isomorphism (in finite dimension) between a space E and its bidual space E**.

'Algebraic understanding' of functions, in the case of vector spaces duality, will mean for us the acceptance of a function as an element of the base set of a vector space structure.

After trying several approaches to the teaching of duality in Linear Algebra, students' difficulties with an algebraic understanding of functions persisted.

As our teaching experience had already confirmed the results of previous studies [3], [4], [6] showing that students remain attached to the analytical understanding of functions (which is essentially the one found in school mathematics), we decided to conduct a study that could further our understanding of this process.

We decided that in this study we would change the focus of our analysis from looking to what was missing in the students' conceptions to eliciting what they really where thinking about functions. This shift in our approach implied not only to consider a didactical model to support the organisation of classroom processes, but also an epistemological model to support the 'reading' of those processes as they happened.

The didactical model chosen was the 'Problematising Teaching' (from the Spanish 'Enseñanza Problémica'; PT on what follows in this paper); the epistemological model chosen was the Theoretical Model of Semantic Fields (TMSF on what follows). They are described in the next two sections.

Based on PT and on TMSF a course on Linear Algebra for undergraduate mathematics students was conducted at the University of Havana (Cuba). The lessons relating to vector space duality were audio taped and analysed; the fourth section of this paper has a discussion of one classroom episode and of the results of a test.

The 'problematising teaching' model

PT was developed during the second half of the twentieth century mainly in the former Soviet Union. It integrates principles, categories and methods, which support a coherent didactical strategy. It is based on the Historical-Cultural Theory of Lev Vygotsky, in particular on the psychological thesis that "cognitive activity always grows from conflict between the known and the sought" (see, for instance, [7]).

Conflict is established in a situation in which what the subject knows or believes does not match what is presented to him or is not sufficient to explain it. Such situation is called 'Problemic Situation' (SP) and it is the main category for PT; it 'rules' all the other categories in the model.

From the assimilation of the conflict by the subject results the 'Didactical Problem', which is the form in which the PS is actually going to be approached by the students; it points out the directions in which we are going to conduct the search to solve the SP. The other categories of PT are: 'Didactical Tasks' (which point out the ways in which we are going to conduct the search); 'Didactical Questions' (they help to solve specific conflicts, which remained concealed when the Didactical Tasks were posed); and, 'Problemic Complex' (defined as an structuring of the previous elements that is established from the ways in which they relate to a given concept, or by considering how an element can be derived from the preceding ones).

PT allows the professor to organise classroom processes in a dynamical way, combining the categories provided by PT and taking into account students' answers, reactions, comments or conclusions. He does not need to remain attached to preconceived ideas about what is going to happen at the classroom.[2]

The theoretical model of semantic fields

This epistemological model was developed to provide a basis for a sufficiently fine reading of the process of meaning production, particularly in classrooms (see [2]).

Its central notions are those of 'knowledge' and 'meaning'. 'Knowledge' is characterised as a statement in which a person believes (a statement-belief), together with a justification he has for making that statement. It departs radically from other characterisations of knowledge by assuming that the justification is a constitutive part of it, not simply a part of the process of that person being said to know something. However, in line with many other authors, it does not work with the notion of implicit knowledge, a quite problematic one; instead, 'implicit knowledge' is at best described as third-person knowledge, that is *I* am producing knowledge *about someone else*. 'Meaning' is characterised as what a person actually says about an object, in a given situation. It is not everything that person could eventually say about that object. Meaning production and knowledge production always happen together; at the same time objects are constituted through meaning production.

From those two central notions a third one is characterised, that of 'semantic field', which is the activity (in the sense of Activity Theory) of producing meaning in relation to a given set of local stipulations (statements locally taken as true by the person without requiring any further justification).

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2 As far as we know, there are not well stablished terms for those categories, so we will use our own translation into English.

A number of other notions related to why and how meaning production occurs, and to explaining why it is necessarily a social process are characterised (interlocutor and legitimacy, for instance) but they will not be presented here.

From the point of view of our interest in this study, two questions guided our reading of what was happening: (i) which are the objects the students are thinking with? and, (ii) what are the meanings being produced for those objects, that is, what are they saying about those objects? The two questions must necessarily be understood as a single one, as there are only objects as long as meaning is produced for them. It is important to notice that according to the TMSF the answers to those questions have to be taken as they come, in the sense that one must avoid 'completing', with his own meanings, what the other has said.

The study

As we have said, the study partially reported in this paper happened in the context of a regular Linear Algebra course (undergraduate mathematics degree).

Dual spaces were introduced inside the study of Inner Product Spaces (finite dimension), as a tool for studying the endomorphisms of such spaces; at the moment of the introduction of dual spaces we would normally expect the students to have mastered the basic theory of finite dimensional vector spaces as well as linear maps and their matrix representations. We reached the introduction of duality following a path traced through the use of the categories of PT and TMSF.

The excerpt we will discuss now happened at a point in which we were engaging in the study of the relation between hyperplanes in a vector space E and straight lines in the dual space E*. This relation involves a possible conflict between the students' previous understanding of vectors and functions, and the fact that vectors in the dual vector space are linear maps. According to the historical-cultural theory, in particular considering the concept of internalisation as developed by Vygotsky, teaching and learning can only be understood as a single process, so teacher intervention is not, in our analysis, a component strange to the process (as it would be seen from other theoretical perspectives).

After identifying the kernel of a linear form as a hyperplane, the fact that there is more than one non-zero linear form sharing the same kernel was established.

At this moment the following Problemic Situation was posed: "Is it possible to give a geometric interpretation of the relation between an hyperplane H in E and the linear forms in E* having H as kernel?"

This Problemic Situation was transformed into the Didactical Problem of comparing two linear forms y*, z* having H as kernel. Then, using the categories of PT, we organised the process so the students could move from comparing the images of y* and z* in a point x (not belonging to H) to comparing y* and z* as maps, as vectors in E*.

In what follows P is the professor and the S are students.

> P. Let us consider two non-zero linear forms y* and z* such that they have the same kernel H, Is it possible to find some relation between them?
>
> S_1. Yes.........(in a low voice)
>
> P. I need to compare y*(x) and z*(x) for all x in E?
>
> S_2. (Whispering) One x

P. There is $x_0 \in$ H such that E= H $\oplus$ $\langle x_0 \rangle$. Isn't it?

S_3. Ah, because x= x_1+k x_0 and this decomposition is unique

P. Then y*(x)= y*(x_1) +k y*(x_0)

S_3. y*(x_1)=0

S_1. For z* is the same thing!

P. And H is a?

SILENCE....

The professor continues the calculation until the following statement is reached:

y*(x)= z*(x) + α x, for some α in K, and she remarks that α does not depend on x.

P. What is the relation between y* and z*?

SILENCE....

P. y*= z* isn't it? Then if we have a hyperplane in E what do we have in E*?

SILENCE....

P. A straight line isn't it?

S_2. A family of straight lines, a vectorial straight line!

Examining the transcription from the point of view of the TMSF and considering the students' answers (silences included), it seems the students did not produce meaning for function as a vector. Silences came out at the moments of shifting from point-wise equality (the analytical understanding of function) to the equality of functions as vectors. The same happened when, after having established the equality y*= z* the professor asks for a geometrical interpretation of it. Only after the professor gives the interpretation of the equality y*= z* as representing a straight line in E*, one student repeats what the professor had said.

We will now analyse the student's answers to two questions:

I-Let E and F be vector spaces over K and y* and z* non- zero, non-proportional linear forms in e-mail. Prove that dim (Ker y*$\cap$ Ker z*) = n-2

With the exception of two students who wrote y* = z*, the others wrote y*(x) = z*(x) (some of them without specifying that this last equality holds for all x in E) or they went directly to consider kernels as hyperplanes and tried to apply formulas for dimension of subspaces.

II- If f: E$\rightarrow$F is a linear map, with E and F vector spaces over K.

Consider ^{t}f: F*$\rightarrow$E* given by ^{t}f (y*)= y* $\circ$ f

Prove that: (a) ^{t}f is a linear map from F* to E*; (b) Ker ^{t}f = [Im f]$^{\circ}$

The answers to these questions can be categorised into two groups:

Group1: Students who identified the function f to the image f (x).

S_4: y* $\circ$ f defines a map that goes from E$\rightarrow$K. It is, it belongs to E*. As the following is a composition of linear maps, ^{t}f (y*) = y* $\circ$ f is a linear map.

(Identifying ^{t}f to ^{t}f(y*), proving that ^{t}f (y*) is linear, but not that ^{t}f is linear)

S_5: To prove that ^{t}f is linear:

^{t}f(y*) (α x + β y) =α ^{t}f(y*)(x) + β ^{t}f(y*)(y)

(Identifying ^{t}f to ^{t}f(y*), proving that ^{t}f (y*) is linear, but not that ^{t}f is linear)

S_6: [Im f]$^{\circ}$ = {y* in F*| y*(y) = 0, $\forall$ y$\in$ Imf}

[Im f]$^{\circ}$ = y*(y) = y* $\circ$ y = y* $\circ$ f (x) = 0 = Ker ^{t}f

(Identifying y (=f (x)) to f)

Group2: Students who felt a "need to evaluate"

S_7 : Let $y_1^*, y_2^* \in F^*$

${}^t[f(y_1^* + y_2^*)] = {}^t[f(y_1^*) + f(y^*_2)] = {}^t[f(y_1^*)] + {}^t[f(y_2^*)]$

(Interpreting t as a function and evaluating)

S_8: ${}^tf(\alpha y^* + \beta z^*) = (\alpha y^* + \beta z^*) \circ f = (\alpha y^* + \beta z^*) \circ f(x) = (\alpha y^* + \beta z^*)(f(x))$

(An x appears!)

S_9: Ker ${}^tf = y^*$: ${}^tf(y^*) = 0 = y^* \circ f = y^*(f(x))$, $x \in E$, $f(x) \in F$

(Again...)

Using the TMSF we would say that those students were operating in a semantic field in which the analytical understanding of functions was central (the 'evaluating' behaviour).

Conclusions

Being able to reveal that the difficulties faced by the students were not due to something missing (the algebraic understanding of functions), but rather due to something strongly present (the analytical understanding) clearly suggests, we think, that it is not enough to present the new object and its properties; it is also necessary to bring forth the 'old' object and to promote the explicit discussion of how they relate. We also suggest that this is a quite widespread phenomenon in mathematics classrooms; our research group is currently conducting other studies and the findings strongly support this suggestion.

In order to promote such a discussion it is necessary that classroom processes be organised in an open and flexible way, so the students can voice their understandings. That means that the professor must be capable both of handling the didactical task and of dealing with the meanings being produced by the students; in both cases one is dealing with what is emerging, rather than simply guiding the students through a pre-established path and helping them somehow to overcome the hurdles.

We think the association of PT and the TMSF has proven to be a quite useful and effective way to help professors to move towards more efficient approaches in mathematics courses.

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AIM - A PARABLE IN DISSEMINATION

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ABSTRACT

AIM is an initiative in interactive mathematics, which exploits the power of the computer algebra package Maple V in an extremely flexible way that can be applied to a variety of curricula.

The authors present the results of a project concerned with replacing parts of core first year materials at the University of Birmingham, UK.

Most software for computer-based assessment has limited use in mathematics. Common problems are:

- Poor display of mathematical expressions. (despite MathML and plug-ins like IBM TechExplorer),
- Restricted choice of question types,
- Failure to recognise mathematically equivalent solutions,
- Difficulty of assigning partial credit,
- Inability to test students' creativity (eg give an example of a function which satisfies XXX but does not satisfy YYY)

Effective integration of computer algebra has made it possible to address these issues. The ability to monitor students' progress in more detail has allowed us to provide individual students with tailored advice on suitable additional learning opportunities (e.g. the use of appropriate learning packages) and to efficiently mount support activities (e.g. targeted small group sessions). This has enhanced and made more focussed support for our students.

Pilot studies have been very encouraging. Students find the software easy to use (97% agree/strongly agree), like the immediate feedback (100% agree/strongly agree), and find it helpful (87% agree/strongly agree).

Our paper:

outlines the genesis and nature of AIM,

reports and elaborates on the above results,

offers an indication of the range of applicability of this shareware - from widening participation to honing advanced specialist skills.

Of particular interest is a parallel study, which explores the factors, which determine whether an innovation is likely to be easily transferable. We look to distil principles of value to innovators in the learning and teaching of mathematics.

1. Introduction - What is AIM?

AIM is a system for computer-aided assessment (CAA) in mathematics and related disciplines. It has been tries and tested in both summative and formative assessment, with the emphasis leaning towards the latter. The acronym, introduced by the original developers at the University of Gent in Belgium, stands for **Alice Interactive Mathematics**. From there it has rapidly been embraced by academics around the world and is now undergoing further development in the UK. The original Belgian site is to be found at:

http://allserv.rug.ac.be/~nvdbergh/aim/docs/

Further information and examples for English speaking users is at:

http://www.mat.bham.ac.uk/aim/

and the most recent revisions, and documentation and downloads are available from:

http://aim.shef.ac.uk

Each of these sites offers ample opportunities for visitors to log on as a "guest" and interact with AIM in a number of mathematical situations.

2. What is distinctive about AIM?

Certainly there is no shortage of CAA systems that have been developed for mathematics. See for instance the articles appearing in the on-line periodical:
http://ltsn.mathstore.ac.uk/articles/maths-caa-series/index.htm
and the archives of MATHS-CAA@JISCMAIL.AC.UK at:
http://www.jiscmail.ac.uk/lists/maths-caa.html
We feel therefore that it is worth outlining what are seen to be the distinctive features of AIM.

AIM Exploits Computer Algebra In the case of AIM, the underlying computer algebra package is Maple. The full power of the mathematics programmed into Maple is therefore available to be called upon in both the authoring and the checking procedures.. It also generates a very high degree of flexibility for authors and participants. For example, if a student gives a correct answer in a form different from that supplied, AIM can still determine that it is correct.

- If a student solves a system of equations incorrectly, then AIM can substitute the incorrect answers back in to the equations, and show student that they do not work out.
- If a student integrates an expression incorrectly, then AIM can differentiate the incorrect answer and show the student that it is not same as the original function.
- More detailed feedback is available for certain common errors. For example, one can set a standard integration question asking students to integrate $\sin^2(x)$. Without any explicit action by the question setter, AIM will examine the form of the integrand and recognise that student be tempted to answer $\sin^3(x)/3$, or possibly $\sin^3(x)/(3\cos(x))$ etc. AIM can automatically generate explanations to cover these common errors.
- More generally, questions can be set up to give immediate and detailed feedback depending on mathematical features of the answer offered. It has to be admitted that this is easier for some (types of) questions than others but then this is probably the case for all CAA systems.
- Deeper understanding can be tested by asking the student to give an example of a mathematical object - say a function, a matrix, or a vector - with some specified properties. Through this kind of exercise one can begin to test, via CAA, the so-called higher mathematical attributes. See (Sangwin, 2002) for a further discussion of this issue. Clearly

a CAA system can only deal with questions of this type if it has enough mathematical "intelligence", such as Maple or a similar engine, behind the scenes.

Freedom of Ownership The source code for AIM is and always has been freely available within the academic community. From its inception in Belgium, a growing community of active mathematicians in Belgium, Canada, Australia, the US, the UK and elsewhere has adopted the common aim to develop a freely available resource for those with an interest in using technology to enhance learning and teaching in mathematics. Being owned by these academics, AIM is highly responsive to the interests of the discipline and its students. It is also inexpensive to set up; one copy of Maple running on one server, allows a department to mount a number of AIM sessions on the web. There is no further cost and the system makes no demands upon the student other than they access the web via a conventional browser.

Freedom of Expression AIM can perfectly well present questions in any of the usual CAA formats - multiple-choice, multiple response, numerical/text input. But AIM is at its most powerful when students are required to enter their answers as free text using Maple syntax. For example, an answer of $\mathbf{sin^3(x)/(3cos(x))}$ would be entered as **sin(x)^3/(3*cos(x))**. There is little doubt that questions of this type have a greater pedagogical value than, say, the commonly used multiple-choice vehicles. On the other hand, critics of AIM point to this need for syntax as a disadvantage of AIM. Advocates would point out two things:

1) For students with little or no facility in CAA environments, we would (as in any CAA system) restrict ourselves to multiple-choice and other "easy" formats.
2) There is a substantial help system that has been designed to support students as they negotiate the syntax. In particular,
   a) Answers that cannot be parsed are discounted without penalty. Students can ask AIM to parse their answers without marking them and to check that they are interpreted as intended.
   b) If a student enters an answer with mismatched brackets, then AIM can indicate graphically which brackets match against which other brackets, and which bracket is causing a problem.
   c) If a student forgets the syntax for multiplication (eg **5x** in place of **5*x**) then AIM will generally indicate the omission and report back the student's answer with the suggestions highlighted.
   d) Similar feedback is given for a variety of other common errors, such as "**t^-2**" in place of "**t^ (- 2)**", cosx in place of cos(x), and so on.

Of course the "help" in AIM is, as any other system, largely heuristic and based upon matters of judgement. Users will vary in their reactions to items. Nonetheless, as it evolves, students generally find it increasingly helpful. (See Section 4 below.)

Some see that the currently popular graphical interfaces for building up and checking mathematical expressions as the "solution" to the "syntax problem". However, many users of AIM see genuine educational benefits in asking students to come to terms with the syntax. For such educators the argument is that all mathematics graduates should be able to enter reasonably complex mathematical expressions into some kind of computer system and should have some awareness of how a system such as Maple operates. Some would go further and argue that interaction at a programming level really test whether a mathematical concept has been mastered. Dubinsky (2000) in another context has argued a similar case in the use of technology in testing understanding in areas such as group theory and other "advanced" areas of pure mathematics. AIM is a very good vehicle for encouraging these skills. It has more detailed and user-friendly help files than Maple itself and is flexible enough to allow authors some control over the level of rigour in syntax expected of students.

3. Authoring in AIM

Questions are authored using a simple web interface to edit a plain text file containing lines of code.

```
1. t> Give an example of a differentiable function
2. t> <i>f(x)</i> which has a turning point at <i>x=1</i>
3. v> 2
4. ap> <i>f(x):=</i>
5. s> [(ans)->`aim/Testzero`(subs(x=1,diff(ans,x))),x^2+2*x+3]
6. end>
```

Each line begins with an AIM "flag". For example, line 1 begins with the flag "**t>**" which instructs Maple to display this line as **text**. The body of the question is written in a simple mark up language, which may include standard HTML commands. The six lines above will generate the following:

Give an example of a differentiable function $f(x)$ **which has a turning point at** $x = 1$**.**

(followed by an answer entry box)

Full details on authoring in AIM may be found in Klai et al (2000) or at the web sites listed in Section 1, but one more example will be useful in illustrating the ease with which AIM can introduce powerful randomisation and feedback facilities.

```
1. h> p_ :=rand(1..3)();
2. h> q_ :=p_+rand(1..2)();
3. t> Give an example of a cubic polynomial <I>p(x)</I>
4. t> with the following properties
5. t> <OL> <LI> <I>p(0)=1</I>
6. t> <LI> <I>p(x)=0</i> at <I>x=$p_</I> and <I>x=$q_</I>.</OL>
7. v> 4
8. ap> <I>p(x):=</I>
```

The lines above generate the following question:

Give an example of a cubic polynomial $p(x)$ **with the following properties**

1. $p(0) = 1$,

2. $p(x) = 0$ **at** $x = a$ **and** $x = b$

At each presentation of the question, the parameters a and b are randomly selected as indicated. The flag "**h>**" represents an instruction to hide from the student the randomisation process. The flag "**v>**" sets the "value" or number of marks for the question and "**ap>**" generates an "answer prompt" for the student.[1] Conventional HTML commands have been incorporated in lines 3, 5, and 6 to generate an ordered list and italic display.

However, a few more lines of code can allow immediate tailored feedback to student. Lines 9 to 24 provide the marking regime and detailed feedback. Here we see how each of the four conditions is checked. If the student's answer fails then feedback is given and marks deducted.

[1] Note the line numbers are not part of the code. They appear here simply to facilitate reference.

Line 23 ends this procedure and provides a correct answer, in terms of the random variables, which will be substituted in before being displayed the system.

```
9. s> [ proc(ans) local marks_; marks_:=1;\
10. if not `aim/Testzero`(subs(x=0,ans)-1) then\
11. printf("Your polynomial fails to satisfy <I>p(0)=1</I>.<BR>");\
12. marks_:=marks_-0.25; fi;\
13. if not `aim/Testzero`(subs(x=p_,ans)) then\
14. printf("Your polynomial fails to satisfy <I>p(x)=0</i> at <i>x=%g </i>.<BR>",p_);\
15. marks_:=marks_-0.25; fi;\
16. if not `aim/Testzero`(subs(x=q_,ans)) then\
17. printf("Your polynomial fails to satisfy <I>p(x)=0</i> at <i>x=%g </i>.<BR>",q_);\
18. marks_:=marks_-0.25; fi;\
19. if not `aim/Testzero`(degree(ans,x)-3) then\
20. printf("Your polynomial is not a cubic.<BR>");\
21. marks_:=marks_-0.25; fi;\
22. marks_;\
23. end ,(1-x/p_)*(1-x/q_)*(1-x)]
24. end>
```

The powerful randomisation features built into Maple allow authors a great deal of flexibility. For example, the next few lines generate a question testing whether students can integrate a monic polynomial of degree 5 with many of the coefficients randomly varying within a pre-selected range (here integers from -9 to 9).

```
1. h>p_:=x^5+randpoly(x,degree=4,coeffs=rand(-9..9));
2. t> Evaluate the following integral:
3. p> Int(p_,x)
4. s> [proc(ans) 'aim/Test'(diff(ans,x),p_) end,int(p_,x)]
```

For a new user the lines 9 to 24 above may seem a little daunting, but a novice may start authoring at the level of simply editing existing freely available questions such as the four lines immediately above. With a little more confidence, new questions can be written *ab initio*, using the language and notation students have already seen, and linking if desired to a course or departmental web site.

4. What students think?

Over the last few years, colleagues have moved on from installing and running AIM to a process of evaluation. A major study (involving around 180 students) at the University of Birmingham is assessing through questionnaires and student focus groups, student reaction to AIM. The results of this study are still being compiled and analysed and will be reported in detail later. However, the following quotes from students indicate that they understand and appreciate the underlying purpose.

"A: Question 1 was certainly asking us stuff that we had to think about.

Interviewer: In what way?"

"A: You didn't give us an equation and then say "solve it". You have got to really think about what it means. You have to get a solution and then you think, OK that's the answer. Doing a question like that you think, argh, right, that is the shape of the graph."

Generally the students demonstrated a mature understanding of the rationale behind these questions. There was widespread agreement that such questions

"B ... test your understanding of the subject, rather than your ability to turn a handle."

At the same institution, AIM was used for part of the assessment of an advanced FORTAN course. When asked students (around 40) found the software easy to use (97% agreed/strongly agreed), liked the immediate feedback (100% agreed/strongly agreed), and found it helpful (87% agreed/strongly agreed).

At the University of Sheffield AIM was used with over 200 first year students. An analysis (Strickland, 2002) of their opinions revealed *inter alia* that they most liked:

- The fact that you could try again if answers incorrect (this was extremely popular).[2]
- Instant feedback
- The ability to do questions from home, in their own time, without any pressure.

Least popular were:

- Difficulties with syntax
- Lack of "method marks"[3]

Finally, in Belgium we find that in a class of around 45, 69% preferred assignments conducted and marked on the web by AIM compared to 31% who preferred paper assignments. Furthermore,

- 94% used their home computer (of those 66% used home computer exclusively)
- 25% used library computers
- 15% used computer university computer labs.

Overall 88% rated AIM as good or very good.[4]

5. Dissemination - what colleagues think?

This is perhaps the most interesting intriguing feature arising out of the genesis of AIM. Given its relative youth and the lack of organised sponsorship or marketing, AIM has spread remarkably easily among the academic community. There is already an active email discussion list serving the world-wide community of users. This contrasts with a number of other perfectly good and better-promoted systems. Through their connection with the LTSN Maths, Stats and OR Network in the UK the authors are aware of a number of worthy developments and initiatives in this area. But there is a noticeable inertia against these systems spreading to new users. None of these comments is intended to imply that these packages are intrinsically poor or defective. However the observation did prompt a discussion of exactly what features might assist in the dissemination of initiatives in learning and teaching. When asked, users of AIM quote the following reasons for adopting AIM.

- The ability to develop an "intelligent" system which exploits the power of computer algebra to address the mathematical needs of the course
- That it is an open source project developed and owned by the academic community
- The high degree of flexibility in authoring and the freedom to customise questions
- That it is deliverable over a standard browser with no additional plug-ins required.

[2] AIM allows lecturers to control whether students may try again. It also allows the imposition of a penalty (say 10% or 15% of the available marks) for second and subsequent attempts.

[3] This has been addressed to some extent by authors who have split questions and allowed say 50% for the subsequent correct manipulation of a wrong procedure or function.

[4] The complete results of the Belgian survey are presented in Table 1

These features (and others mentioned in earlier sections) are not necessarily unique to AIM. AIM is not the only system that meets these criteria. However taken as a whole they explain why AIM has been so readily adopted by international mathematicians.

6. Conclusions

As the debate about the relevance of CAA in the assessment of higher mathematical skills unfolds, it is clear that the needs of the discipline will not be met by many of the existing vehicles, relying as they do upon methods such as multiple-choice questions. At the same time encouraging reluctant practitioners to embrace new technological initiatives will be hampered by even the slightest obstacles be they financial, technological or pedagogical. In this paper, the authors wish not so much to advocate AIM (history shows that it needs no special pleading) but to open a discussion on the principles that will underpin easy transfer of initiatives in learning and teaching mathematics. Within the context of CAA, the four points in Section 5 have been distilled out of our observations. Too often initiatives thrive locally, fed perhaps by the charisma or commitment of their authors, but do not travel well. If we can identify the design features that will assist transfer and dissemination of such initiatives, then the whole mathematics community will benefit.

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There was sufficient instruction/resources provided to start using AIM:	72	25	3	0	0
AIM was easy for me to learn to use:	63	38	0	0	0
The opportunity to re-attempt questions in case of a wrong answer helped me to better understand the material:	56	34	6	3	0
The feedback from AIM was helpful:	19	44	22	6	9
The penalty system (deduction of 15% for every wrong answer) was fair:	38	16	19	25	3
The questions asked in AIM were capable of testing material that required a deeper understanding of the course:	34	41	9	13	3
The time available to complete the assignments was adequate:	69	22	6	3	0
Give your overall rating for AIM the way you experienced it, on the scale from +2 (good) to -2 (bad):o	47	28	22	3	0
Give your overall rating for the potential that AIM has for use in future math courses.	44	44	6	6	0

Table 1 - Views on AIM from Belgium (courtesy of http://allserv.rug.ac.be/~nvdbergh/aim/docs/)

KEPLER'S WINE BARREL PROBLEM IN A DYNAMIC GEOMETRY ENVIRONMENT

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ABSTRACT

The story of Kepler's Wine Barrel Problem seem to be worthwhile for mathematical research. Formulating the problem with the aid of Dynamic Geometry Software provides it with an additional dimension, that of visualization. Once we build a construction utilizing Dynamic Geometry Software, the exploration of the problem is accessible to variant levels of students, even those who are not familiar with calculus tools. The analysis of the construction, posing the problems to explore and being engaged in its investigation, requires skills of decision-making, conjecturing, reasoning, and problem solving.

In this paper there is a description of mathematical research conducted by the author. As an instructor in an advanced course in a program aimed to qualify in-service teachers as teachers'-educators, It seem to be important to be involved in mathematical research. Hence, I decided to expose the teachers to the problem and let them share my enthusiasm of the mathematical research in a Dynamic Geometry Environment. The message was to call them to experience in mathematical research as an important tool for becoming a teachers' educator.

KEYWORDS: dynamic geometry software, geometry, maximum, volume, problem posing, history, connected mathematics, teacher.

1. Introduction

Attending a workshop for mathematic teachers at the Haifa Technion, I was exposed to Kepler's wine barrel problem and its history background.

As Kepler noted in his book *New Solid Geometry of Wine Barrels* (in Tickomirov (1990)) he was inspired by an event in his life that occurred in the fall of 1613,:

In December of last year...I brought home a new wife at a time when Austria, having brought in a bumper crop of noble grapes, distributed its riches...The shore in Linz was heaped with wine barrels that sold at a reasonable price.

...That is why a number of barrels were brought to my house and placed in a row, and four days later the salesman came and measured all the tubes, without distinction, without paying attention to the shape, without any thought or computation. Namely the copper point of a ruler was pushed through the filling hole of a barrel, across the heel of each of the wooden disks which we refer to simply as bottoms, and as soon as the length to the point at the top of one board disk was the same as the length to the point at the bottom of the other, the salesman stated the number of amphoras contained in the barrel after merely noting the number on the ruler at the spot where the length in question ended.

I was astonished...

The key result in the book *New solid geometry of wine barrels* is Theorem V [Part Two]:

Among all cylinders with the same diagonal, the largest and most capacious is that in which the ratio of the base diameter to the height is $\sqrt{2}$.

Figure 1 presents a cylindrical wine barrel lying on its side. The barrel entry is in the middle point of its side (E); λ is the distance between the entry and the bottom heel (EH).

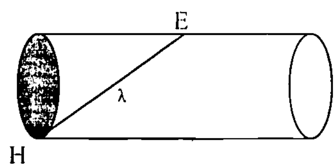

Figure 1

The problem seem to me as an interesting one and worthwhile for introducing to students. Since it demands a lot of mathematical knowledge it is not appropriate to all students. Looking for approaches to adjust for students in various levels, I tried to model the problem with the aid of a Dynamic Geometer Software (DGS)[1]. The computer educational software available today can shed light on the problem by presenting it in graphical form and therefore help the students visualize it better.

The following chapters (2-4), provide a description of the problem, its representation in DGS and suggestions of their investigation. In chapter 2 the problem is formulated in the plane, then in chapter 3 refers to the barrel assuming it is cylindrical and then chapter 4 there is an approach to describe the barrel as composed of two truncated cones having a common base. Chapter 5 provides a didactical analysis of the potential of teaching and learning Kepler's Wine Barrel Problem with the aid of DGS. Suggested tasks for teachers in a workshop appear in chapter 6.

[1]The Geometry Inventor(1994). (Logal Ed.),Software and Systems Ltd was used to this purpose. Of course any other Dynamic Geometry Software can do as well.

2. Kepler's planimetric problem in a dynamic geometry environment

Formulating the problem

Lets start with the plane version of Kepler's problem – the planimetric problem that fits the cylinder section:

Among all rectangles (ADBE) *with the same diagonal* (λ), *find the rectangle of maximal area* (Fig.2).

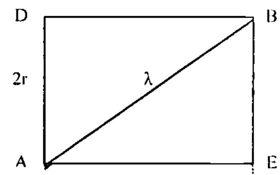

Figure 2

For a given λ, the locus of vertices D is a circle with AB = λ as its diameter (Fig.3).

Kepler's planimetric problem is then formulated as: *to inscribe in a given circle a rectangle of maximal area.*

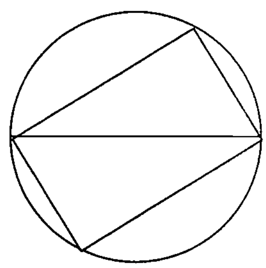

Figure 3

Construction of the problem with the aid of DGS

It is clear that a rectangle inscribed in a given circle is well defined by the ratio between its sides: $t = \frac{DB}{AB}$. Therefore, the rectangle area is a function of the ratio t. utilizing the DGS construction tools, it is possible to build all the rectangles inscribed in a given circle and then to build the graph of the area as a function of t, with the help of the software measure tools.

Description of the construction

1. Build a segment AB.
2. Find the midpoint C of the segment AB.
3. Build a circle with C as its center and with radius CA.
4. Place a point D on the circle.

The triangle DAB is half the rectangle DBEA (Fig.2).

We can use the measure tools to display the following measures on the screen:

- AB=λ
- $t = \frac{DB}{AB}$
- S = DB·DA the rectangle area.

The Geometry Inventor also includes an option to build a function graph by defining the variables of a function. We define the area S as a function of the variable t. Then, by dragging the vertex D along the circle, we create all possible triangles. For each one the appropriate measures appear on the screen, the graph is derived as shown in Figure 4 and the answer is revealed: the maximum value of the area is obtained on the graph for $t = 1$. The rectangle of maximum area inscribed in a given circle is a square.

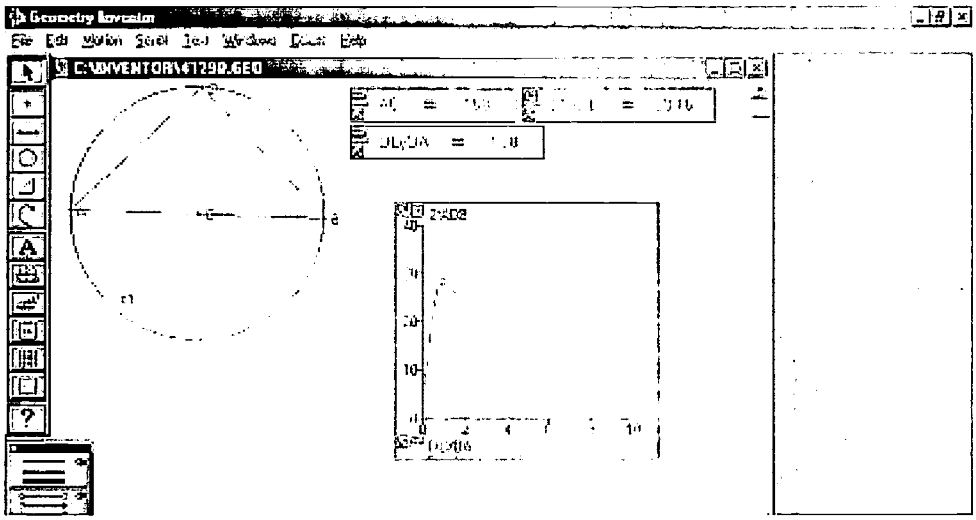

Figure 4

3. Solution of Kepler's wine barrel problem with the tools of geometry inventor: the case of a cylindrical barrel

In addressing the Wine Barrel problem, we first assume that the barrel is a cylinder and that the length of the segment connecting the barrel entry with its bottom is given (Fig. 2). This segment is the diagonal of half the cylinder.

The volume V of the cylinder is calculated as:

$$V = \pi r^2 \cdot H = \pi\left(\frac{AD}{2}\right)^2 \cdot 2DB = \frac{\pi}{2} \cdot AD^2 \cdot DB$$

We can use the software measure tools to compute the value V of the volume and the value t of the ratio between the height of the barrel and its bottom diameter. Using the approximation 22/7 for π we get V=11/7*DA*DA*DB. As in the planimetric example, as the vertex D is drugged along the circle the corresponding values of t and V, for each rectangle created, appear on the screen and a graph of V as a function of t is plotted (Fig. 5).

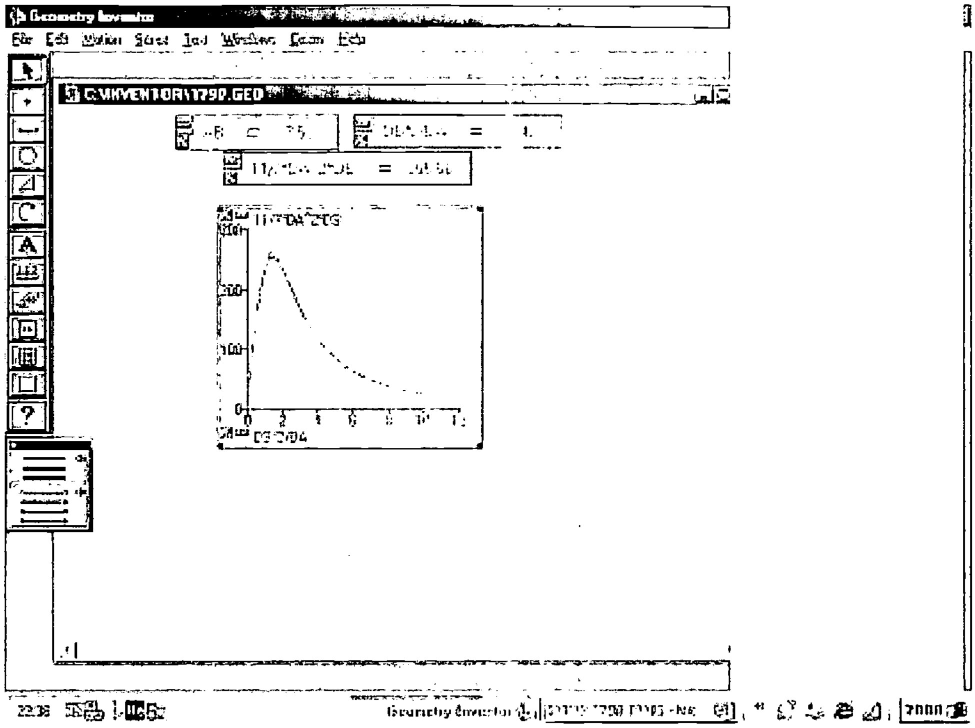

Figure 5

The student is therefore able to find the graph maximum point by simply dragging the vertex D until it reaches the peak on the graph. Figure 6 shows how easy it is to change the value of λ (which is AB) and get different graphs. The students might be surprised to learn that the maximum value of the volume V is always reached for the same value of t=1.42. This result fits the value $\sqrt{2}$ that is obtained when the function $V(t)$ is investigated with calculus tools (Shacham & Smukler(2000)).

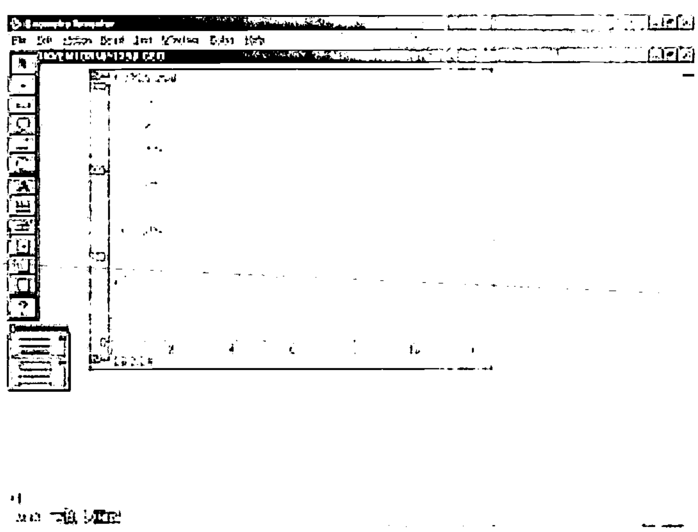

Figure 6

4. Solution of Kepler's wine barrel problem with the tools of geometry inventor: the case of a barrel composed of two truncated cones

Formulating the problem

The section of a barrel, which is composed of two truncated cones having a common base, is a hexagon divided into two trapezoids by one of its diagonals as shown in Fig. 7:

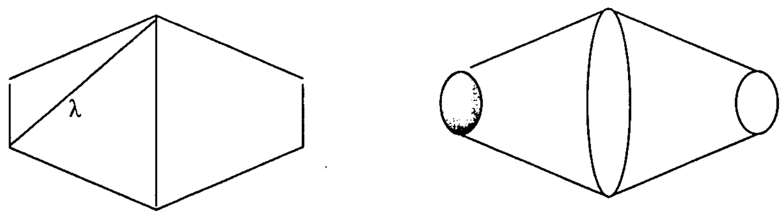

Figure 7

To investigate the problem, it is important to formulate it in the language of the geometry construction, that is, to decide which measurements are given and which are variable.

As in the case of a cylindrical barrel, λ denotes the length of half-barrel diagonal and t is the ratio between the height of the barrel and the diameter of its bottom. But, now there is an additional parameter influencing the barrel volume: the ratio between the diameter of the common base of the cones and the diameter of the barrel bottom. If we look at this section of the barrel we see that this ratio is between the lengths of the trapezoid bases. Lets designate it as b.

First, we shall look at all barrels with the same values of λ and b and ask how the value of t influences the barrel volume V. We can concentrate on one of the two trapezoids, and ask the following question:

A truncated cone is generated by an equilateral trapezoid rotating about its central symmetry axis. Among all equilateral trapezoids having the same diagonal λ *and the same ratio b between its bases, which trapezoid generates the truncated cone of the largest volume?*

Again, the DGS and its tools will be used to solve the problem. A suitable construction is needed for this purpose.

Analysis of the construction

Let say the trapezoid is $AHBG$ (Fig.8) and the given diagonal is $AB = \lambda$.

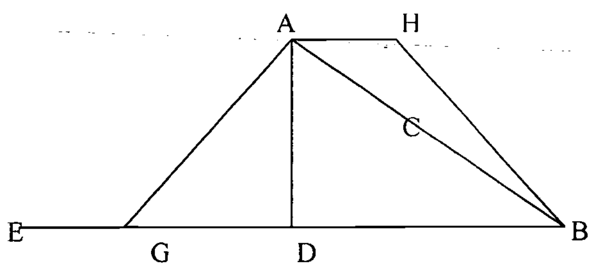

Figure 8

Looking at the altitude AD from A to the trapezoid base GB, we see that the locus of points D the heels of the altitude, is a circle with AB as a diameter. BD equals to the trapezoid midline, so if we extend BD beyond the point D ($DE=DB$), then BE equals the sum of the bases: $BE = AH + BG$ and therefore $EG=AH$. Consequently, the point G divides BE in the relation $\frac{GB}{GE} = \frac{GB}{AH} = b$.

Description of the construction

1. Build a segment AB.
2. Build a circle with AB as its diameter.
3. Place a point D on the circle.
4. Extend BD beyond D to E, so that $DE = BD$.
5. Place a point G on the segment BE
6. Draw a parallel line from A to BG.
7. Copy the segment GE on the parallel line starting at A, so that $AH=GE$.

$AHGB$ is the trapezoid.

Solving the problem with DGS tools

Display on the screen the measures of AB=λ, the ratio $\frac{BG}{GE} = b$ and an expression for V according to the formula (Shacham & Smukler(2000)):

$$V = \frac{2\pi}{3} \cdot \frac{\lambda^3 \cdot t}{\left(t^2 + (1+b)^2\right)^{\frac{3}{2}}} \cdot \left(1 + b + b^2\right)$$

With aid of the DGS graph tools build a graph V-t. Once again by simply dragging the vertex D around the circle we can generate all possible values of t while preserving λ. By dragging the point G along the segment BE, we get all values for b. Fig.9 shows the graph describing the function $V = V(t)$ as presented above.

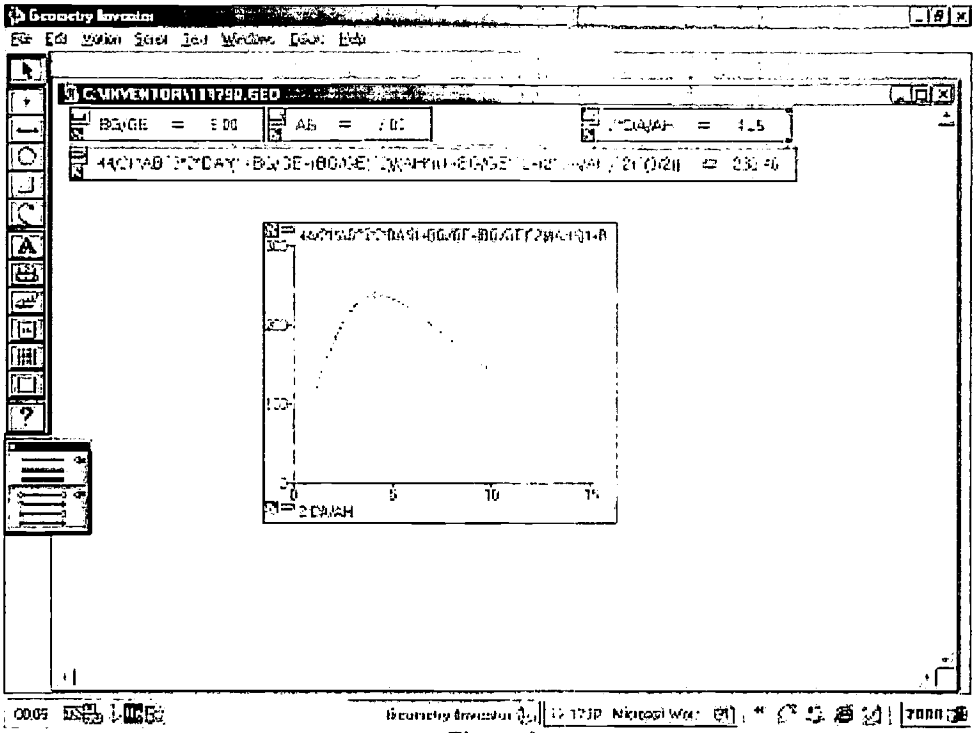

Figure 9

From the graph it is easy to find the maximum value of the volume corresponding to chosen values of λ and b. In the example shown in Fig. 9, we see that in case $\lambda = AB = 7$ and $b = 5$, the maximum value is reached for $t = 4.25$.

According to computations with the calculus tools ((Shacham & Smukler (2000)), the maximum value for the barrel volume is obtained for $t = \frac{1+b}{2} \cdot \sqrt{2}$. By substituting $b = 5$ we get $t = 3 \cdot \sqrt{2}$ or $t = 4.24$, which is very close to the value obtained from the graph.

Verification of the Graphical Solution

Once we have built a solution for given values of λ and b, it is easy to use the same construction and solve the problem for any values of λ and b; we need only change the givens. Table 1 below, describes results of constructions for different values of b and λ.

The value of t of the barrel with the largest volume as derived from the graph for each value of b, appears in the third column of table 1. The exact value of t, is $\frac{(1+b)\sqrt{2}}{2}$, appear in the right column of the table. We can see that the two values are relatively close.

λ	b	V_{max}	t (experimental)	$t = \frac{(1+b)\sqrt{2}}{2}$
10	2.01	627.43	2.11	2.12
3	5	18.71	4.22	4.24
10	5	695.29	4.24	4.24
10.02	9.98	743.67	7.76	7.76
15	5	2344.97	4.25	4.24
10	50.74	791.17	36.39	36.58
20	5	5550.4	4.25	4.24

Table 1

5. Investigating by problem posing: what if...?

Brown and Walter (1990) provide a wide variety of situations implementing the strategy of "what if and what if not?" questions and their importance in mathematics in general and in mathematics education in particular. The next step is to pose questions about the model.

For example: What if the parameters change?

1. *How does the maximal volume change when the parameter b is increased from b=1 to infinity?*

2. *How does the maximal volume change when the parameter λ increases?*

It is very easy to investigate with these questions with the aid of the construction in the geometry software environment.

5.1 The influence of the parameter *b* on the maximal volume

In order to investigate the influence of b on the maximal volume, we keep the value of λ constant and increase the value of b by dragging the point G (Figure 8).

Figure 10 below shows the graphs of the barrel volume as a function of t for $\lambda=10$ as b increases starting from 1.

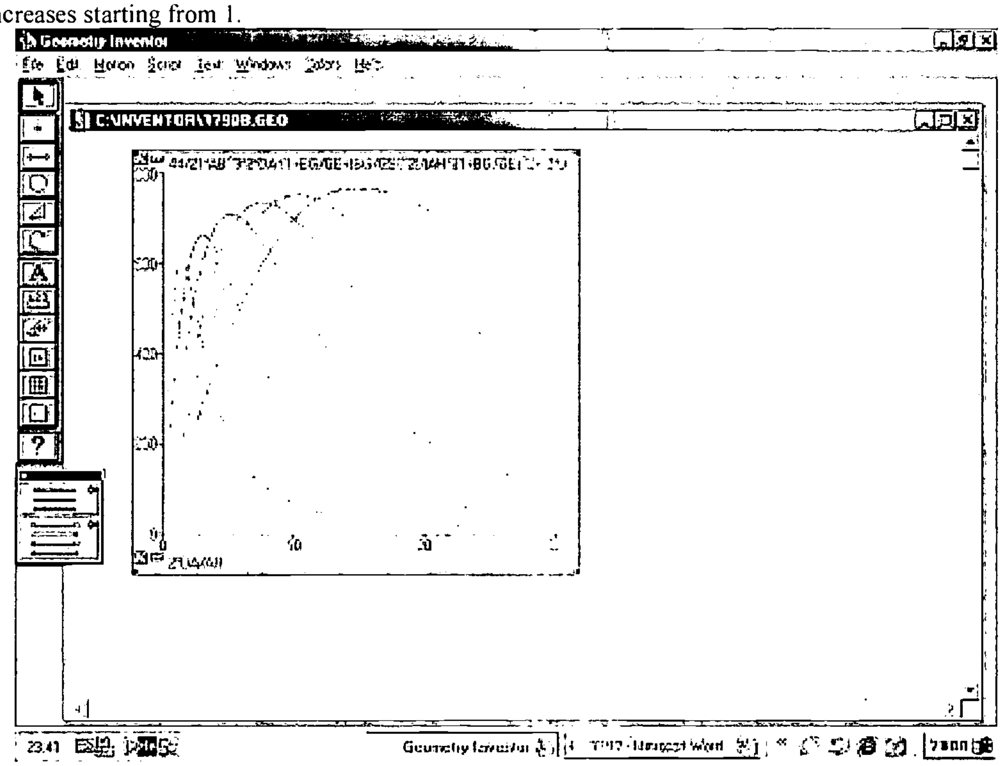

Figure 10

Table 2 presents the maximal volumes and the corresponding values of t ratios between the volume of non-cylindrical and cylindrical barrels when $\lambda=15$ and b increases from $b=1$.

b	t	$V_{max}(b)$	$V_{max}(b) / V_{max}(1)$
1	1.41	2044.07	1
1.1	1.49	2044.98	1.0004
1.5	1.77	2069.55	1.0124
2.01	2.12	2118.63	1.0365
3.02	2.84	2213.9	1.0830
5	4.24	2344.97	1.1472
10.12	7.88	2500.38	1.2232
26.01	19.12	2626.10	1.2847
52.33	37.77	2673.10	1.3077
110.09	78.36	2698.90	1.3203
226.64	161.11	2711.29	1.3264
279.68	198.25	2713.54	1.3275

Table 2

In case b=1, the barrel is cylindrical and so its maximal volume equals 2044.07. From Figure 10, as well as from Table 2, one can see that the barrel volume approximates the cylinder volume when b is close to 1. From the right column in table 2, we learn that as the values of b increase, the barrel maximal volume also increases. However, this increase is limited and the ratios between the different maximal volumes to the cylinder volumes of cylinders for the same value of λ tends to converge to 1.32. This result matches the result 4/3 obtained by analytic investigation ((Shacham & Smukler(2000)).

5.2 The influence of the parameter λ on the maximal volume

Let us keep the value of b constant and just change the value of λ. Figure 11 shows the graphs of the barrel volume V as a function of t for b=5 and for values of λ= AB from 5 to 12. From these we see, that the maximal barrel volume increases as a result of the growth of λ, with no limit.

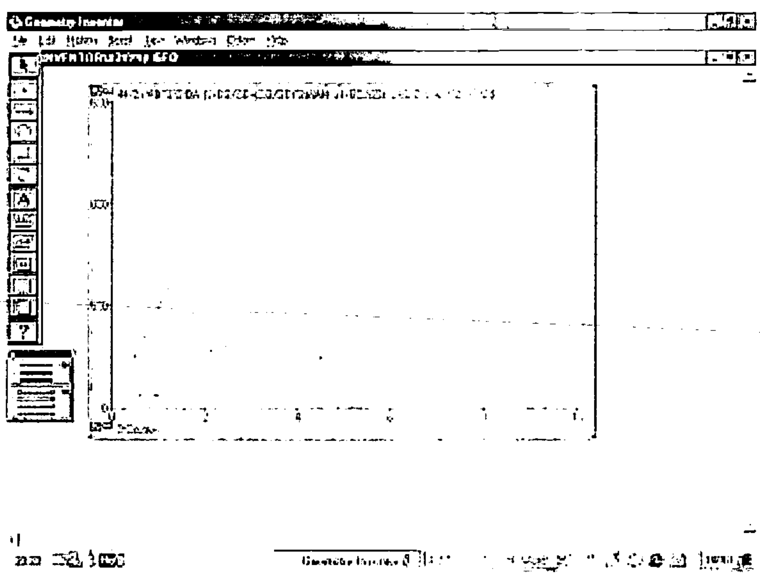

Figure 11

The results shown in Table 3, lead to the conclusion that the growth of the maximal barrel volume is proportional to λ^3.

Λ	V_{max}	t	V_{max} / λ^3
3	18.66	4.33	$18.66/3^3$=0.691
8	355.69	4.26	$355.69/8^3$=0.694
10	695.29	4.24	$695.29/10^3$=0.695
15	2326.06	4.75	$2326.06/15^3$=0.689
20	5494.42	4.83	$5494.42/20^3$=0.689

Table 3

Another conclusion of the investigations presented in Table 3 is that the values of t providing the maximal volume of the barrel are not influenced by the values of λ.

6. Achieving educational goals

We can concentrate now, in some important educational principles as follows: motivation; problem posing; visualization; reasoning; connected mathematics; mathematics for all students. In the next sessions there is a description of them as well as an analysis of their place in the suggested approach to Kepler's Problem.

6.1 Motivation

The NCTM. Standards recommend using worthwhile mathematical tasks to introduce important mathematical ideas and to engage and challenge students intellectually: Well-chosen tasks can stimulate students' curiosity and help them develop an interest in mathematics.

Another recommendation is to utilize electronic technologies such as DGS, as useful tools for imposing worthwhile problems (NCTM (2000)

Posing the historical background of Kepler's wine problem is a worthwhile activity that can motivate the students. The use of the DGS allows students to solve the problem without being distracted by complex computations. The students can focus on decision-making, reflection, reasoning, and problem solving.

6.2 Problem posing

One important educational goal is to let students pose new questions using the strategy of "what if and what if not?". Students should then be encouraged to pose questions, then to investigate their own conjectures. (as in chapter 4).

6.3 Visualization

In the activities described above, we used DGS, in order to furnish visual images of the mathematical ideas' rules and concepts. The graphical tools of DGS enable visualizing connections between various parameters.

Purdy(2000) talks about the advantages of DGS as an aid in visualizing maxima in volume problems. In our example the students also benefit from the graph which is built with the aid of the software.

6.4 Reasoning

NCTM standards(2000) call for investigating mathematical conjectures and developing and evaluating mathematical arguments and proofs. As Bruckheimer and Arcavi (2001) write on the potential of DGS to promote links between empirical and deductive reasoning. They show how DGS can support the development of a proof by using empirical evidence as the source for insight and inspiration for a deductive argument while they build the proof of Morley's theorem. For many students the proof cannot be understood in another way.

In order to build the construction with the aid of DGS, students must first analyze and understand Kepler's Problem. This demands logical and geometric considerations and thus students are involved in reasoning.

6.5 Connected Mathematics

Paul Goldberg(1996), calls in his project "Connected Geometry", for a curriculum that helps students engage in meaningful mathematical activities which will offer them a chance to understand and appreciate the relationships between variables; unifying themes within mathematics. Furthermore such activities should help them and to connect their previous experiences to mathematics and develop and use mathematical habits of mind. He describes the benefits of teaching connected mathematics as helping students to become experimenters, describers, tinkerers, inventors, visualizers, conjecturers, and guessers.

The Kepler's Wine Barrel Problem may be seen as a typical example of connected mathematics. As follows:

(*i*) Building the *algebraic model* of the problem, i.e. expressing the barrel volume V as a function of t, requires understanding the relations between variables and building the proper function.

(*ii*) Building the *geometric model* of the problem and representing it with the aid of DGS, requires an understanding of geometry and analyzing – the kind of ruler – compass construction. This subject is not included in Israel's curriculum.

(*iii) creating* the appropriate graph of the function enables one to investigate how its shape as defined by the parameters, influences the barrel maximal volume.

Thus there are connections within mathematics like: function and graphs; geometry; technology and calculus, as well as to every-day life and to the historical background.

6.6 Mathematics for all students.

Solving Kepler's wine problem with the DGS, enables teachers to adapt the task to different students. The more able students can be exposed to a advanced task such as analyzing the geometry construction and translating it to use the software, while less able students can lean upon these results, build the graph of the function, watch it on the screen and read the solution from the graph. Students who are easily distracted may focus more intently on computer tasks, while those who have organizational difficulties may benefit from the constraints imposed by computer environment. Therefore students who have trouble with basic procedures in basic mathematical concepts can still develop and demonstrate an understanding of mathematics, which in turn can eventually help them learn the basics.

7. Suggested Activities for Teachers

As an instructor in an advanced course in a program aimed to qualify in-service teachers as teachers'-educators, I find it important to be involved in mathematical research. Hence, I decided to expose the teachers to the problem and let them share my enthusiasm of the mathematical research in a Dynamic Geometry Environment. The message was to call them to experience in mathematical research as an important tool for becoming a teachers' educator.

After the teachers were exposed to the tasks that were modified to them (see example in the appendix), they were asked to develop, each of them, a mathematical project. They chose problem from the curriculum and started a mathematical exploration. They are in an on-going process so it is pre mature to have any conclusions.

Appendix

Activity 2

Among all cylinders with the same measure of the diagonal (λ) *of half the cylinder, find the cylinder of maximal volume* (Fig.12).

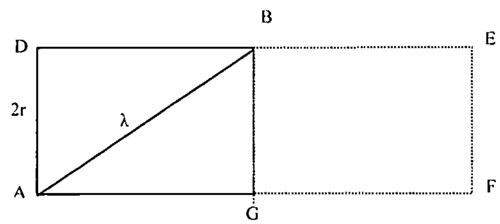

Figure 12

1. Analyze the construction
2. Present the problem utilizing DGS.
3. Describe the construction.
4. Plot a graph of V (the volume of the cylinder) as a function of t – the ratio between its sides.
5. Answer the question above using the obtained graph.
6. Verify your solution with other mathematical tools.

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ALGEBRA, COMPUTER ALGEBRA, AND MATHEMATICAL THINKING

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ABSTRACT

Mathematical symbolism in general—and symbolic algebra in particular—is among mathematics' most powerful intellectual and practical tools. Knowing mathematics well enough to use it effectively requires a degree of comfort and ease with basic symbolics. Helping students acquire symbolic fluency and intuition has traditionally been an important, but often daunting, goal of mathematics education. Cheap, convenient, and widely available technologies can now handle a good share of the standard symbolic operations of undergraduate mathematics: differentiation, integration, solution of certain DEs, factoring and expansion in many forms, and so on. Does it follow that teaching these topics, and even some of the techniques, is now a waste of time?

The short answer is "no." On the contrary, as machines do more and more lower-level symbolic operations, higher-level thinking and deeper understanding of what is really happening become more, not less, important. Numerical computing has *not* made numerical viewpoints obsolete; neither will computer algebra render symbolic mathematics obsolete. The key question is how to help students develop that bred-in-the-bone "symbol sense" that all mathematicians seem to have. What really matters is that students use mathematical symbolism effectively to pose worthwhile problems in tractable forms. Once properly posed, such problems are well on the way to solution, often with the help of technology. The longer answer, explored in the paper, concerns choosing mathematical content and pedagogical strategies wisely in light of today's technology.

Introduction

What does it mean to know and do mathematics effectively at the tertiary level? How do the answers reflect the present and future, when mathematical technology, including symbol-manipulating technology, is already widely available, and will probably soon be ubiquitous?

What should college-level (tertiary) students in particular know and what should they be able to do, in order to be mathematically educated in a technology-rich environment? How can we teachers help bring students to this kind of knowing?

I approach these questions from a perspective that's fairly common in the United States: I'm a generalist mathematician who teaches reasonably pure mathematics to North American college students. About one-third of my students in an average class intend, with varying degrees of intellectual seriousness and interest, to complete a 4-year mathematics major. Only a small minority (10% or fewer) of students plan postgraduate study in mathematics. A more typical student plans to work after graduation in a technical but not university-level academic job, such as software engineering, database management, or high school teaching.

I am a practitioner of, not an expert researcher in, mathematics education, and so will not presume to offer advice on the education research agenda or how it should be carried out. What I hope to contribute is a teacherly and mathematical perspective on some content, techniques, and ideas related to symbolic mathematics that I think are mathematically important to today's tertiary students, and how I think students can be helped—sometimes with technological assistance—to acquire these advantages.

1 The technology background

Disputes over educational uses of mathematical technology have been around as long as the technology itself. Years ago one heard the "desert island" argument from opponents of instructional technology: Students who are permitted to use, say, calculators for school arithmetic might suffer disproportionately if later shipwrecked on low-tech islands. This argument is seldom heard anymore; it was killed either by the rising availability of cheap calculators or by the worldwide decline in passenger marine travel. In any event, there's no doubt that many students can now afford and keep readily to hand the technology needed to perform a huge share of the algorithms encountered even in tertiary mathematics. It's well known, for instance, that the TI-89 handles integrals, derivatives, partial fractions, and much more. But did you know that the TI-89 can also handle many of the residue calculations given as exercises in complex analysis texts? With powerful computer algebra systems such as *Maple* and *Mathematica* also becoming more affordable and available to students, the technology background has shifted markedly.

With the desert island argument no longer tenable, technology opponents resort to other arguments. Technology takes too much time to learn; students can't think in the presence of machines; technology use is just a post-modern cover for dumbing mathematics down—another nail in the coffin of civilization. I find these arguments unconvincing at best and dishonest at worst. How much do you think your students really struggle with technology as they pirate music files from the Internet? The dumbing-down argument is worst of all: it is simple "calumny" (as Tony Ralston observes in [**2**]) to equate technology-based reform with lowered intellectual standards or expectations.

This is not to deny, on the other hand, the existence of good, important, and (in my opinion) still open questions surrounding pedagogical uses of technology. Owning a calculator that "knows" how to expand rational functions in partial fractions does not necessarily obviate the need to understand something of the idea—and perhaps even of the process—by hand or by head.

At the school level, arguments over technology use often touch on the role and importance of paper and pencil arithmetic (PPA) in technology-rich environments. At one extreme are calculator abolitionists, asserting (with perhaps more vehemence than evidence) that calculator use is somehow inimical to reason—children, in this view, can either push buttons or think, but not both, and certainly not simultaneously. At the opposite end of the spectrum are other abolitionists, such as Tony Ralston, who advocate abolition not of calculators but of PPA itself, at least as an explicit goal of K-12 mathematics education. (One should hasten to add that Ralston also recommends greatly *increased* emphasis on mental arithmetic (and perhaps also on mental algebra) to replace PPA. His eloquent paper [**2**] is well worth reading.)

Beyond with this clash of opinions is, I believe, an important basic agreement on ultimate goals. In the end, most of us care far more about whether students can pose and solve novel and challenging problems than about what technology they may use along the way. What counts most is effective mathematical thinking, which comprises such elements as "symbol sense" and facility with mathematical structures; both are discussed in more detail below. What is mainly at issue, I believe, is whether technology can help, or must hurt, the cause of teaching students to think well mathematically.

2 Number sense and symbol sense

At the elementary level, what may matter less than PPA facility *number sense*, that intuition for numbers that includes such things as an ability to estimate magnitudes, an eye for obviously wrong answers, and an instinct for choosing (rather than necessarily performing) the arithmetic operation needed to solve a given problem.

At the secondary and tertiary levels, the mathematical symbols under study become much more general than numerals (which are, of course, symbols in their own right), and the degree of abstraction rises as students progress. The objects symbols stand for in more advanced mathematics might be unknown numerical quantities, functions, operators, spaces of various sorts, or even more abstract objects. At these higher levels of study the analogue of number sense is *symbol sense*, as defined by Arcavi [**1**] and others. Symbol sense is harder to define and delimit than number sense—appropriately enough, given the greater mathematical depth and breadth of, say, polynomial algebra as compared to integer arithmetic. (Arcavi lists at least seven aspects of symbol sense—only one of which involves actual symbolic manipulation.) Arcavi links symbol sense closely to *algebra*, asserting that acquisition of symbol sense is the proper goal of teaching algebra.

A student with good algebraic symbol sense should *see* that something is amiss with an "equation" like

$$(a+2b)^4 = 17a^4 + 8a^3b + b^3a + \sqrt{ab}.$$

BEST COPY AVAILABLE

She should also *know*—without any calculation—that of

$$a^2 - b^2 = (a - b)(a + b) \qquad \text{and} \qquad a^2 + b^2 = (a + b)(a + b),$$

one is right and one is wrong. Similarly, one sees rather than computes that equations of the form

$$a^3 - b^3 = (a - b) \cdot (\text{something}) \qquad \text{and} \qquad a^3 + b^3 = (a + b) \cdot (\text{something})$$

can be arranged to hold, while

$$a^4 + b^4 = (a + b) \cdot (\text{something})$$

probably cannot.

In this paper I take broad views of both "symbol sense" and "algebra." By symbol sense I mean the general ability to extract mathematical meaning from and recognize structure in symbolic expressions, to encode meaning efficiently in symbols, and to manipulate symbols effectively to discover *new* mathematical meaning and structure. By "algebra" I mean symbolic operations in general, including not only algebra in the classical sense but also such things as formal differentiation and expansion in power series.

Definitions may differ, but *whatever* one means by "symbol sense", it's clear that tertiary-level mathematics takes a lot of it. Tertiary mathematics is a symbol-rich domain, and doing mathematics successfully at this level requires considerable comfort and sophistication with symbols. Above all, students need a clear sense of the things symbols represent, and how to extract meaning and structural information from symbolic expressions.

Perhaps this should all go without saying—who can doubt that symbols ought to mean something to students? In practice, however, we've all seen students floating untethered in the symbolic ether, blithely manipulating symbols but seldom touching any concrete mathematical ground. For example, many students struggle to make sense of a symbolic expression such as

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} = \infty.$$

This is hardly surprising; after all, the statement's truth or falsity is far from obvious to a newcomer to infinite series. But a more basic source of difficulty, I believe, is that the expression's meaning—let alone its truth or falsity—is highly compressed in the symbolic representation. "Unpacking" the symbolism to reveal meaning and structure can be a daunting challenge in its own right, as we see often as our students confuse or conflate the terms and the partial sums of infinite series.

This brings me to my main questions:

1. How can we use technology—and symbol-manipulating technology in particular—to help students acquire symbol sense in the broad sense discussed above?

2. Where does better symbol sense lead? How can students use better symbol sense to understand mathematics more profoundly?

3 Building symbol sense

Technology can be used in many ways to help students make sense of symbols and symbolic expressions. We give two brief examples.

Example: Unpacking symbolic expressions

One approach to making sense of the densely packed symbolic expressions students encounter at the tertiary level is to use technology to "unpack" them and investigate their parts. (This is the essence of *analysis*.)

For the infinite harmonic series discussed above, for instance, the *Maple* command

```
> s := n -> evalf ( sum(1/k, k=1..n)  ) :
```

defines the partial sum function $s(n)$. Evaluating $s(n)$ is now easy for specific inputs n:

```
> s(10), s(20), s(30), s(40), s(50), s(60);

              2.929, 3.598, 3.995, 4.279, 4.499, 4.680
```

The results show $s(n)$ increasing, although slowly, with n.

That's a good start, but it leaves open the deeper question of convergence or divergence. Further experimentation (and perhaps some hints) might eventually suggest successively *doubling* inputs to s:

```
> s(10), s(20), s(40), s(80), s(160), s(320);

              2.929, 3.598, 4.279, 4.965, 5.656, 6.347
```

The situation is now much clearer; successive doubling of n causes essentially *linear* increase in $s(n)$ (by about 0.7 each time), and a useful analogy with logarithmic growth (which can lead to a rigorous proof of divergence) begins to appear.

Example: Looking closely at squares

Another technology-aided approach to giving meaning to symbols is to look very closely, from several viewpoints, at apparently familiar symbolic objects. Almost every American college student "knows," for instance, that

$$(x^2)' = 2x,$$

a fact that, while undeniably true, is almost entirely valueless without some deeper sense of what the symbolized objects and operations really mean. Here, too, students might use technology to help de-crypt the symbols, perhaps by plotting appropriate functions, zooming in on graphs, or calculating related derivatives.

For variety, let me suggest *another* approach to looking more "structurally" than usual at the squaring function, this time beginning from a numerical perspective.

What structure should a student see in the following list?

1 4 9 16 25 36 49 64 81 100 121 144 169 196 225 ...

The first answer is obvious—even the dullest student with any recent memory of mental or paper-and-pencil arithmetic sees the squares of successive integers.

So far so good, but let's keep looking. Taking successive *differences* in the first list reveals the simpler pattern of successive odd numbers:

1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 ...

Taking differences *again* gives an even simpler list:

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 ...

And so on. (Taking further differences soon loses its fascination.)

Starting from these basic structural ideas, students can move in many possible directions to explore—and perhaps solve—new but related structural questions:

- What happens if our original list arises by sampling not the basic quadratic function $f(n) = n^2$, but some *other* quadratic, say $g(n) = n^2 + 2n + 3$? Are the first differences *still* in arithmetic progression? Are the second differences still constant?
- How do differences behave if the original list samples the cubic function n^3? Or the exponential function 2^n?
- What happens if we move in the "opposite" direction, finding successive sums rather than differences? How does the "constant of summation" affect the results?

Quite different structural questions could also be explored. Students might notice, for example, that successive squares alternate between *exact* multiples of 4 and numbers of the form $4k + 1$. Or they might see pattern in the last decimal digits of successive squares:

0 1 4 9 6 5 6 9 4 1 0 1 4 9 6 5 6 9 4 1 0 1 4 ...

And so on, perhaps, into areas of modular arithmetic.

4 Beyond symbolics: exploring structures

We have argued that technology can help students build better symbol sense for tertiary mathematics. But why is symbol sense *worth* working to acquire? Where does it lead?

We should acknowledge first that, in actual practice and despite the presence of technology that could enable better things, a lot of tertiary mathematics still boils down to performing symbolic algorithms. As Ralston [2] says about college calculus in the USA:

> ... despite so-called calculus reform, the aim of most college calculus courses still seems to be to create a student-machine in which functions are fed to its maw and derivatives and integrals emerge at the other end.

In mathematical reality, of course, tertiary mathematics is about much more than algorithm performance, and technology may help us refocus attention where it belongs. The calculus, for instance, can be about mathematical objects and ideas—function, limit, derivative, differential equation, integral, infinite series—not just about formal calculations with these objects.

In my opinion, the true Holy Grail at the tertiary level is mathematical structure. Some italics may be in order:

> *Understanding basic mathematics profoundly means proficiency at detecting, recognizing, and exploiting structure, and at drawing useful connections among different structures.*

The preceding example illustrates most of these points: The basic structure of successive squares, once recognized and slightly manipulated, leads naturally to simpler or more complex structures, and to new, deeper, and more interesting questions.

There is nothing new about this focus on mathematical structure. Mathematics is frequently described, in one way or another, as the science of pattern. What may need emphasis, though, is the special importance of mathematical structure in *tertiary*-level mathematics. Here is where students meet new structures, and relations among them, in rich but potentially bewildering variety, ranging from abelian groups to planar graphs.

Quadratic polynomials: symbols reinforcing structure

We close with a final illustration of a pedagogical strategy—looking closely (perhaps using technology) at familiar objects—that focuses attention both on symbolics and on structures.

Quadratic polynomials are an excellent source of simple but not trivial examples; students should know them intimately and handle them often. The following example, although not particularly "technological", illustrates the value of studying familiar examples carefully, using symbolics, to reveal somewhat hidden structures.

(Before proceeding, we acknowledge in passing the good question of whether students should learn to manipulate quadratic polynomials *mentally*, as well as on paper and by machine. Ralston **[2]** recommends at least some mental manipulation. My hunch is that if quadratics are emphasized appropriately the question will become effectively moot: students will *automatically* acquire some mental facility with them. In any event, and whatever the medium of calculation or recording, students should *know*, not calculate, that $x^2 - 9$ factors as $(x-3)(x+3)$.)

In calculus, quadratic polynomials illustrate several important notions, including local linearity and "quadraticity", global nonlinearity, the meaning of the second derivatives, and geometric convexity. Quadratics also illustrate the possibility and the advantage of algebraic factoring, and more generally of the value of having convenient algebra formulas. One sees, easily, for instance, that the vertex of a quadratic polynomial lies midway between its roots, and that one root of a quadratic polynomial with rational coefficients is quadratic if and only if the other root is rational.

Example: Pythagorean triples and rational points

The rational roots property of quadratic polynomials has an interesting and perhaps unexpected "structural" consequence: there are infinitely many Pythagorean triples, and they correspond in a natural way to rational points on the unit circle.

The idea is as follows: Given a nontrivial Pythagorean triple (a, b, c) of integers, with $a^2 + b^2 = c^2$, we divide both sides by c^2. Renaming $x = a/c$ and $y = b/c$ gives a *rational* point (x, y) on the unit circle

$$x^2 + y^2 = 1.$$

Since the process can (essentially) be reversed, hunting for Pythagorean triples amounts to finding rational points on the unit circle. A few solutions are obvious; one is the "north pole" point, $(0, 1)$.

An ingenious way of finding other (indeed, essentially all) rational points is to find intersections of the unit circle with lines through $(0, 1)$ that have rational coefficients. Each such line that is not vertical has an equation of the form the line $y = mx + 1$, where the slope m is a rational number. Such a line intersects the unit circle at a simultaneous solution of

$$y = mx + 1 \qquad \text{and} \qquad x^2 + y^2 = 1.$$

A little algebraic work (by hand or by head) now produces the one-variable quadratic equation

$$x^2 + (mx + 1)^2 = 1.$$

This equation is easily solved for x. But we needn't bother, at least for the moment. Because all coefficients and the root $x = 0$ are all *rational* numbers, so is the other root. Because *every* line through $(0, 1)$ with rational slope cuts the unit circle in a rational point, we see that infinitely many rational points, and hence infinitely many Pythagorean triples exist. A little more work shows, moreover, that our recipe produces *all* rational points. Combining symbols, algebra, and various mathematical structures, we have solved a modest but nontrivial problem—and suggested methods of attack on many others. (Are there rational points on the circle $x^2 + y^2 = 3$? On $x^2 + 2y^2 = 3$)?) Somewhere, far in the distance, even the faint glow of elliptic curve theory can be detected.

5 Conclusion

As modern technology handles more and more of the algorithmic aspects of mathematics, even at the tertiary level, the importance of higher level mathematical thinking—symbol sense and facility with mathematical structure—become relatively *more* important. Used properly, high-level computing technology can help tertiary students see beyond the mechanics toward what matters most: mathematical structure.

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THE INFLUENCE OF THE "ENVIRONMENT" CREATED BY THE SOFTWARE SCILAB IN THE LEARNING OF LINEAR ALGEBRA

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ABSTRACT

This paper compares the results attained by a control group working with traditional methodology with those of an experimental group using an application software program called SCILAB. The focus is on linear algebra (matrices, determinants and linear equation systems) which forms part of "Mathematics II", one of the core subjects in the B. Sc. econ. course at the Faculty of Statistics and Economic Sciences, National University of Rosario, Argentina.

This survey comes under the "Teaching Mathematics with Computational Tools" project — P.I.D. 4-202-93-004. It is financed by Program 202 for the promotion of Scientific and Technological Investigation set up by the National University of Rosario, Argentina.

The conclusion is that the experimental group proved more successful.

Introduction

Efficient learning involves both time and effort at university level. In mathematics, students spend much time on routine calculus and fruitless operations before entering university. They will now have to concentrate more on shaping concepts and solving problems.

Administration and Economic sciences in particular, are based on advanced mathematical theories. Economic analysis, for example, includes comparative statistics, optimisation problems and dynamic control; all of which require the methodology of linear algebra, calculus and combinatorial mathematics. In engineering, mathematics does not only infuse essential academic discipline; it is also needed as a tool for overall use.

Because of the way mathematics courses are currently being developed, students have to struggle against abstractions and do not get to see any of the applications. The very nature or high level of some topics simply cannot be exemplified with graphic and numerical calculus. Inversely, the computer allows the immediate numerical verification of property as well as graphical representation in two and three dimensions. Theoretical results can therefore be used to solve concrete problems in real situations and the immediacy of the processor's answer indeed helps with the inductive exploration of knowledge.

This approach should lead to a more consistent view of the links between mathematics and its applications. But first, the students must be taught mathematic concepts. For the operative part, the computer can be used to meet that objective, as long as THE USER KNOWS WHAT HE WANTS AND CAN UNDERSTAND the result supplied to him by the computer.

Computational tools are currently restricted to fairly specific areas in many professions, e.g.; computation and numerical analysis, data processing, programming etc. Computational tools often contribute to working out problems of numerical calculus, but wider use is being hampered by the complexity of programming languages. In higher education, mathematics is generally taught without the help of any computational tool, although teaching mathematics could be made definitely more effective if suitable computational methodology and software were introduced. It is the argument that underpins the *Teaching Mathematics With Computational Tools* project, which was set up by the National University of Rosario. The experiment described in this paper is an integral part of the project.

This paper explores the achievements of two different groups. The first group, named control group, used traditional methodology, when the second experimented with a computational tool, namely an application software called SCILAB.

It can be inferred from the results obtained by the two groups that the new methodology has a positive impact on teaching linear algebra as it improves symbolic representation and manual algebraic operations skills as well as conceptualization and the mathematical representation of reality. It has been carried out an investigation design with quasi-experimental methodology.

Objectives

The objectives of the experiment were as follows:

- assessing how far a computer tool can help with learning linear algebra at university level.

- determining the impact of computational tools on university students of linear algebra's competence in solving problems. Further effects of the new computational environment on learning linear algebra were also considered.

These objectives were completed by comparing the levels attained by both groups, control and experimental alike.

Description of *scilab*, the software system used

SCILAB is a software system developed by France's Institut National de Recherche en Informatique et en Automatique, INRIA. It has been conceived to provide experts in applied mathematics with a powerful calculus tool. It uses the syntax of the MATLAB system. This system is kept as the interpreter and offers the greatest possible similitude to ordinary mathematical writing. It allows the manipulation of mathematical objects such as vectors, matrices and polynomials. It is also an open system because it allows the user to create new functions in a simple way.

Within the framework of the above-mentioned project, a compatible version for MS-DOS IBM PC was set up. It includes on-line help and the Spanish translation for the error messages of the English original version.

SCILAB may be obtained 'anonymous' at:

ftp.inria.fr (internet # 192.93.2.54) Directory:INRIA/Projects/Meta2/Scilab

ftp.unr.edu.ar (internet # 200.3.120.67) Directory:pub/soft/scilab

Development of the experiment

The experiment was carried out in *Mathematics II*, one of the core subjects in the B. Sc. econ. course at the Faculty of Statistics and Economic Sciences, National University of Rosario, Argentina. The focus was on matrices, determinants and linear equation systems. Five hours were allotted weekly to the experiment, three hours for theory and two for practice. The working hypothesis was that, within the teaching time normally allotted to the traditional course, it should also be possible to teach how to use a computational tool together with the standard contents of the course, and yet obtain better results and rationalised knowledge.

The experiment was started in the following conditions:

- The class (two hundred students) was divided into two random groups comprising an equal number of students (statistically equivalent). -. Each group was divided into sub groups that had around thirty five students in the practical part.
- The same amount of time, i.e. two hours per week, was allocated to work practice on the course subject in both groups.
- Both groups were taught an identical level of theory in the same time slot.
- Both groups were taught by highly qualified teachers of linear algebra.
- The control group worked in a traditional classroom with the traditional practical work guide.
- The experimental group worked in the computer room of the School of Statistics; two students being set per computer. The task in the computer were performed in the classroom. No time out class was needed. The course book used was parts of *Laboratorio de Análisis Matricial — Sistema Scilab (BASILE). Módulo 1 (4)*.

Note: To determine the equivalence of the groups, it was used like pre-test the students scores in the subject Mathematics I. With these data, the chi-squared test was done with a 95% confidence interval. To evaluate both groups, a single test was given immediately after the end of the course.

Test given to the students

Evaluation in the topics: Matrices, determinants and systems of linear equations with non computational operative.

1) Solution and analysis of a system

Solve by Gaussian elimination and determine the values of k for which the system of lineal equations is:

- compatible with only solution
- compatible indeterminate
- incompatible

$$
\begin{aligned}
x + 2y + 3z &= 1 \\
3x - 2y + z &= 2 \\
2x - 4y - 2z &= k
\end{aligned}
$$

2) Symbols, matrix operative and geometric interpretation

2.1) In the following system

$$
\begin{aligned}
2x + 4y + z \quad &= 11 \\
6y + 12z &= 24 \\
3x + \ y \ - 2z &= 4
\end{aligned}
$$

- Solve the system by inverse matrix if it is possible
- Interpret the solution geometrically
- Verify

2.2) Write explicitly the matrix defined by:

$$A = \{a_{ij}\}_3, \quad a_{ij} = (-1)^{i+j}$$

3) Representation of reality

A builder has to make the construction of five houses rural style, seven houses Cape Cod style and twelve houses colonial style. The builder knows, of course, the materials that each house type demands. These materials are steel, wood, glass, painting and manpower. The numbers of the following matrix represent the quantities of each house type, expressed in appropriate units.

	Steel	*Wood*	*Glass*	*Painting*	*Manpower*
Rural	5	20	16	7	17
Cape Cod	7	18	12	9	21
Colonial	6	25	8	5	13

The builder has two providers that give the following prices for material unit.

	provider 1	*provider* 2
Steel	15	10
Wood	8	7
Glass	5	4
Painting	1	1
Manpower	10	6

The following problems are presented:

- How many units will he need of each material?
- How much does it cost each house type, according to each provider?
- Which is the total material cost for all the houses that it will build according to each provider?

Note: Respond to the three questions using matrix products.

4) Determinants.

4.1) If

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 3$$

Calculate

$$\begin{vmatrix} a_1 & a_2 & a_3 + 2a_1 - 3a_2 \\ b_1 & b_2 & b_3 + 2b_1 - 3b_2 \\ c_1 & c_2 & c_3 + 2c_1 - 3c_2 \end{vmatrix}$$

4.2) Complete

i) If $A = \{a_{ij}\}$ is a triangular matrix then $D(A) =$

ii) If $A = \{a_{ij}\}$ is a diagonal matrix then $D(A) =$

iii) $D(I_n) =$

Results

1) Although they were in no way bribed with any extra incentives, e.g., partial exemption from exams or even easier exam passes and special grade awards; most students reacted both actively and co-operatively to the new methodology. They were not unduly concerned about the 200% increase in the workload induced by their need of becoming proficient in linear algebra and the SCILAB system.

2) The assessment of both groups consisted in a comparative analysis of their skills in the following variables:
- algebraic routine operations
- conceptualization
- matrix use of symbols and routine operations
- modelization, i.e., mathematical representation of reality.

When selecting these assessment criteria, both the students' required background knowledge and the specific objectives of the work set were taken into account.

The following items were included in assessing the results:
- solution and analysis of a linear equations system.
- basic operations by row of a matrix
- determination of the rank of a matrix.
- Gaussian elimination.
- interpretation of systems with three variables.

So that a figure could be assigned to the variables, both groups were simultaneously submitted to a test in which students had to operate without the calculus help of a computer.

The results are expressed as a percentage of the number of correct answers given by the students:

	CONTROL GROUP	EXPERIMENTAL GROUP
ALGEBRAIC ROUTINE OPERATIONS	73.9	73.3
CONCEPTUALIZATION	62.5	66.7
MATRIX USE OF SYMBOLS AND ROUTINE OPERATIONS	56.5	70.8
MODELIZATION	26.7	55.6
OVERALL ASSESSMENT	63.5	70.3

When assessing each item, the skills each group's students acquired for a given item was averaged out separately. The figure is expressed as a percentage of the number of correct answers given by each student.

For "algebraic routine operations", both groups scored almost equally well, even though it could have been assumed that the computer-assisted student would lose the skill of solving problems manually.

For "conceptualisation", the experimental group achieved a slightly higher score.

For "matrix use of symbols and routine operations" and "modelisation", the experimental group obtained markedly better results. As far as "matrix use of symbols and routine operations" is concerned, the fact that the computer cannot accept the writing of mistakes may explain the significant difference between both sets of results. In addition, as the software makes use of symbols with a strong resemblance to those used in mathematics, the student benefits from additional training in that variable.

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LEARNING WITH TECHNOLOGY: SIMILARITIES IN MATHEMATICS & WRITING

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ABSTRACT

In this paper, we will identify similarities in the learning and the creative process using technology in mathematics and writing at the college level and draw a parallel. Specifically, we will examine the parallel on learning symbolic representations at different levels with special attention to how controversial technologies--such as numeric, graphic, and symbolic calculators in math or word processors, spell checkers, grammar checkers, and graphic organizers in writing--help learning. We identify comparable learning variables in writing and mathematics using a theoretical model. Finally, we present specific parallel examples in solving problems in mathematics and in the writing process using technology.

Keywords: symbolic representation, learning variables, writing

Introduction

Learning mathematics and English composition is, on the surface, two very different enterprises. In this study we examined whether the use of technology makes learning these two subjects more similar in some fundamental ways. Do we teach the students how to use the existing technology to further their learning and creativity? We will identify similarities in the learning and the creative process using technology in mathematics and writing at the college level and draw a parallel.

Basic steps in the learning process such as constructing, exploring, and experimenting can be personalized and accelerated using technology. We will oversimplify and omit many obvious technical and discipline-specific differences for the sake of the comparison. Many other simpler parallels with examples from the natural sciences and engineering can be made. We compare writing to mathematics to show learning commonalities across the wide range of subjects and across developmental ages.

This inquiry draws on three sources of information. Reflecting the experiences of authors in both disciplines, they include a review of the literature on learning and technology, an ongoing study of undergraduate students in honors mathematics classes, and clinical casework with children and adults with learning disabilities. O'Donnell headed a federally funded, two-year, writing team with a jury of national learning specialists who reviewed the research on learning and technology, including content areas of writing and math. Gavosto has used the symbolic capabilities of a graphing calculator in teaching multivariable calculus, linear algebra, and ordinary differential equations in honors courses at the University of Kansas. Her observations of the students in her classes motivated part of this work.

This paper is organized as follows. We start by discussing the common features of the learning process between mathematics and writing, followed by examples. Finally we draw some conclusions and propose areas of future study.

General Similarities in the Learning Process using Technology

To talk about the similarities, we start by describing the technology considered. In writing, the technology refers to a word processor like Microsoft Word with a built-in spelling and grammar checker, a dictionary, and the capability of inserting graphics from a file in the text. It also includes cognitive mapping options or graphic semantic organizers, such as *Innovation* software commonly used in the brainstorming and conceptual structuring phases of writing. In mathematics, the technology considered is a TI-89, that is, a graphing calculator with symbolic capabilities. Similarly, a computer algebra system like *Mathematica* or *Maple* could be used.

During their first two years of college students typically demonstrate basic knowledge of the technology used in classes and have covered material with equivalent content in lower level courses. While we acknowledge significant individual differences in both content and technology skills, our comments here refer to students with basic knowledge of the technology and the content taught.

Learning, as we use it in this discussion, refers to the evolving science of human learning as described by the National Academies of Science in three recent works. The first, *How People Learn: Brain, Mind, Experiences, and School* (National Research Council, 1998/2000), reviews the current empirical research base. The second, *How People Learn: Bridging Research and*

Practice (National Research Council, 1999a), addresses the relevance of that research base to classroom practice and teaching. The third report, *Improving Student Learning: A Strategic Plan for Education Research and Its Utilization* (National Research Council, 1999b), proposes a 15-year plan of changes, now underway, to advance the learning of students and teachers. In brief, learning involves the change from not knowing to knowing. This essential "change," which manifests learning itself, results from experiences with technology merging with content of mathematics or writing.

What does the technology offer to the students? First, the technology gives multiple representations of the concepts. For example, many mathematical concepts consist of three different representations: numerical, symbolical, and graphical. The calculator or computer software allows the student to generate these representations easily. The corresponding paradigm in writing could be the semantic, syntactic, and graphical representation with words, picture, or graphic organizers in the form of concept mapping. Areas in which we note similarities between learning to write and learning mathematics are represented in the learning variables research grid (see Table 1).

Table 1. Learning Variables Research Grid

	Association of Elements	Rule Systems	Reasoning & Comprehension	Cognitive Levels
Math & Science	Symbol-Referent (& properties)	Numerical Rules Geometric Rules Formulate Rules	Problem Solving & Scientific Comprehension	**Features Analysis** **Discriminate vs. Generalize**
Reading	Grapheme-Phoneme (& properties)	Phonic Rules Syllabic Rules	Reading Comprehension	**Recognize vs. Recall**
Language	Phoneme-Referent (& properties)	Features of Lexicon Transformation Grammar Rules Phase-Structure Rules	Language Comprehension (written & oral)	**Attain Concept vs. Form Concept** *Among many others*

The cells in Table 1 describe different components of mathematics and written language within the categories of association elements, rule systems, underlying comprehension components, and multiple cognitive levels. For example, in the first column, associative elements, the various symbol systems link to content-specific referents. Technology may help with the number symbol associations in mathematics and the letter symbol associations in writing or the musical notation associations in musical composition by providing the learner with a way to manage the rule systems. We mention the musical notation to emphasize that, in addition to the three rows on mathematics, reading, and language (written and oral), one could add other rows whenever a new symbol system would fit in the first column. The musical notation symbol system (association elements in column one) is genuinely different from these symbol systems listed here. As a result, music would merit its own row; the row would link across for music with its own rule systems, reasoning and comprehension, and cognitive levels.

In the second column, rule systems, technology offers a functional perimeter within which allowable responses can be permitted. Disallowed responses can be prevented within the rule systems in math and writing using technology. This extraordinary active learning feature assists the mechanistic aspects of written language by means of spelling and grammar checkers for easier editing, and the mechanistic computational aspects of mathematics.

In the third column, reasoning and comprehension, a fundamental commonality crosses subject content areas. For example, in both mathematics and language, the reasoning and comprehension demands must link to stored prior knowledge in memory. When no existing connections are possible, the learner may cognitively seek this connection to *assimilate* incoming information or to *accommodate* a restructuring of the internal information. Building on prior learning means a vast collection of errors and correct information continues to be gathered by the learner. Comprehension efforts tap into to this cognitive storehouse. Understanding of the content in math may require problem solving with this knowledge, for example, while understanding of written language may involve composing complex written ideas with this knowledge.

All subject content areas (in rows 1, 2, and 3) share the cognitive levels (in column four). They represent intelligence, learning, memory, and cognition, which change rapidly across the developmental years from birth to adulthood. The leading explanations of the revolution in behavioral and cognitive psychology in the 1960s and the new science of learning in the 1990s include structural/behavioral models of development (Horowitz, 1994; Sameroff, 1983), cognitive/information processing models (Anderson, 1983; Anderson, Reder, & Simon, 2001; Newell & Simon, 1972); and the connectionist/ neurocognitive models (McLeod, Plunkett, Rolls, 1998). Composing text through the writing process demands linking the symbols and ideas through behavioral, cognitive, and connectionist processes.

Several analyses have been published describing the relationships between cognition and mathematics (relevant to the fourth column of Table 1). A more extensive explanation includes, for example, the work by Tall (1992) about the mathematical processes and symbols underlying undergraduate mathematics education. In addition, Anderson et al. (2001) analyze the "applications and misapplications of cognitive psychology to mathematic education (p. 1)."

An extensive research literature on learning and technology suggests that technology can facilitate the learning experience thorough diverse symbolic representations, though not always in the ways people expected. For example, achievement gains did not show up as expected. Perhaps better clinical trials in longitudinal studies will be needed to demonstrate expected gains. However, results are clear that students' motivation increases in mathematics and writing with technology tools. Students spend more time on the task. Universal design of instruction and universal design of classrooms, as well as accommodations with technology, provide access to students with disabilities who might otherwise be unable to participate in math and writing. Collectively, these results benefit learning (Anderson & Horney, 1997; Applebee, 1984; Bangert-Drowns, 1993; Cochran-Smith, 1991; O'Donnell, 2001; O'Donnell, Alexander, Jensen, 1999; Okolo, Hinsey, Yousefian, 1990; Woodward & Rieth, 1997). A National Academies of Science study concluded that technology impacts learning by: "(1) bringing exciting curricula based on real-world problems into the classroom; (2) providing scaffolds and tools to enhance learning; (3) giving students and teachers more opportunities for feedback, reflection, and revision; (4) building local and global communities that include teachers, administrators, students' parents, practicing scientists, and other interested people; and (5) expanding opportunities for teacher learning" (National Research Council, 1998/2000, p. 207).

At the college level the technology may help students either write a better text or give a better solution to a problem. Some of the crucial elements of the learning process include experimentation, exploration leading to discovery, and construction. By using technology, students may accelerate these steps at their own pace in the following way.

Experimentation with technology allows students to try many different investigational approaches, as text and formulas can be freely manipulated without the worry of correctness. Different versions may be "cut and pasted" repeatedly in writing and mathematics. Word processors give the text great mobility and so do calculators with mathematical expressions. Speller checkers, grammar checker, and symbolic capabilities of the calculator reassure students and help them persevere on difficult tasks.

Exploration with technology opens up many possible capabilities. A complicated problem or challenging essay can be simplified to a manageable problem. In the same way a finished product may be generalized. The length of the final product can be easily changed, shortening or lengthening the solution.

Construction with technology, particularly the input-output nature of the technology, provides an ideal tool to construct an essay or the solution of a problem. Building blocks (text and mathematical expressions) can be saved and combined in many different ways, yielding multiple levels of text and multiple levels of computations.

Our observations lead us to conclude that the technologies of writing and mathematics are especially well suited for learning that involves experimentation, exploration, and construction. The learner benefits from the multiple forms of numeric, graphic, and symbolic representation with more fully articulated understanding of concepts. Three core attributes of learning selected by the National Research Council (1998/2000), with which we think the technology helps, are (a) learning with understanding is essential, (b) learning builds on pre-existing knowledge, and (c) learning is an active not a passive process. The multiple representations created with calculators and computerized word processing and cognitive mapping graphics reveal concepts that enrich understanding, build on prior learning, and necessitate active participation in learning. Through the technology's reiterative editing features, through symbolic representations, and through successive approximations of the correct final work, the word processor and calculator generate learning episodes across the K-16 experience.

Examples

In this section, we give examples of the similarities between learning mathematics and learning writing described above. We will draw a parallel using the classical approach of Polya (1957) for problem solving with the addition of technology. The concrete example in mathematics to illustrate the use of the technology is a word problem from an ordinary differential equations course. The problem is an application of Newton's Law of Cooling. The data are the constant ambient temperature and the temperature of a body at two different instants of time. The question is to determine the amount of time needed by the body to reach a certain given temperature. We describe how this problem could be solved using graphing calculator with symbolic capabilities, like TI-89, by students working in small groups.

1. *Understanding the problem*: defining the unknowns, data, and conditions. In our problem, the wish to use the technology forces the definition of the variables. For instance, the two known points of the graph of the temperature function can be plotted after identifying the corresponding variables. Conjectures about the possible solutions and the model can be made.

2. *Devising a plan*: identifying what type of differential equation the model gives and how to solve the equation. The calculator can help in setting up the solution since the functions that solve the equations symbolically will only accept input in a certain order. This feature will not set up the equation, but it may help detect very rough conceptual errors like the dependence of the variables.

3. *Carrying out the plan*: finding the solution. The calculator can be used to solve the equation and can help find the two constants involved. It can also be used to check algebraic and numerical computations.

4. *Looking back*: examining the solution. After obtaining the solution, plotting its graph with the calculator translates the symbolic solution to a graphical one. The graph provides ample qualitative information of the temperature. In particular, this information verifies the validity of the model and the answer to the problem.

Gavosto observed that her students were able to solve problems like this using a symbolic calculator without any previous experiences with comparable problems. With the calculator, the students demonstrated more confidence in handling all the variables and constants involved in attacking the problem. The calculator also helped communication among the students about the steps involved in the computations. The best students were able to be analytical and critical of the feasibility of the solution obtained.

Examples in writing parallel those in mathematics. The steps of the writing process have been delineated through a compelling body of empirical research, ushered in by Janet Emig (1971) and Donald Graves (1983), and expanded in the work of many others such as MacArthur and Graham (1987) and Cochran-Smith (1991). *Process writing* features the steps of planning, prewriting, writing or composing, and revising or editing. Students share their work with peers and seek editorial response to their work in each prepublication step of the writing process. The number of parts in the writing process and their names have changed over time. Now, experts in the field have settled on five: prewriting, drafting, revising, editing, and publishing. The arrows in the process model represent how writing steps operate in a recursive, not sequential linear process (see Figure 1).

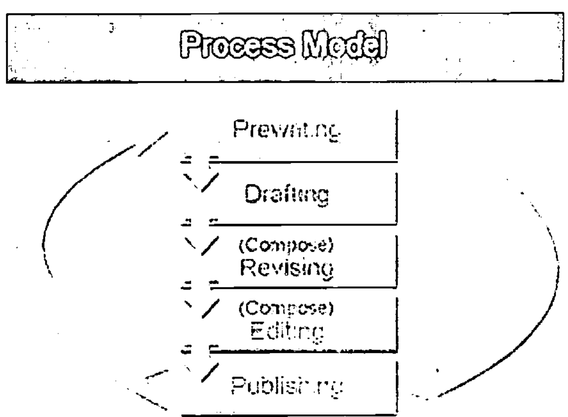

Figure 1. Writing Process Model (O'Donnell et al., 1999).

1. *Prewriting* involves brainstorming discussions and divergent thinking about alternative approaches to the writing problem or content to be written. This is analogous to Polya's first step of understanding the problem. The technology designed for this step includes computerized cognitive mapping and graphic organizing tools (such as *Innovations* software).

2. *Drafting* refers to the initial step of generating and structuring the ideas in written words. The outlining tools on the word processor sometimes help in this stage, but more often drafting is the initial attempt to put the ideas into rough connected written text. Several versions of drafts may show significant change during this stage. Since writing is a recursive process, the writer(s) may return to the previous prewriting step and back to drafting interchangeably several times as the text takes shape. This step may be most nearly analogous to the Polya's "devising a plan."

3. A. *Revising* is the first part of composing text. This step grooms the ideas and structure of the work. It fleshes out the text to full length, taking different shapes depending on the type of writing. For example, in an essay it involves the introduction to the topic, the analysis supporting the focus, delineating the arguments, providing evidence, and drawing conclusion from the discussion. Writers employ the full range of word processing options. Some authors use connectivity tools through the Internet to gather information, or e-mail to engage in shared writing projects with advanced track changes (such as *Microsoft Word Track Changes* for multiple authors). Revising processes resemble Polya's "carrying out the plan."

B. *Editing*, the second part of composing, consists of bringing the digital text into finished form. At this point spell checkers and grammar checkers, plus language tools like Thesaurus and Dictionary, are the features of word processing most commonly employed. This component may still be part of Polya's "carrying out the plan." It corresponds to checking the solution in mathematics, and is a major step in the writing process.

4. *Publishing* refers to the final step of making the work available in its finished form and receiving feedback from the intended audience. This takes many forms, in addition to the formal publishing well known to academics. It is the end-stage for all writing. For young children it can include sharing work with classmates and family. It may involve students writing in newsletters, yearbooks, journals, and multimedia digital alternatives with web sites, PowerPoint presentations, and Access Grid (supercomputing or Internet2) presentations. Publishing means entering into a dialogue about the written work with the intended authentic audience as a way of examining the validity of the approach taken. In this way the last step in the writing process may be similar to the last step in Polya's problem solving process of "looking back" after obtaining the solution to verify the validity of the answer to the problem.

Conclusions

We have not attempted a full comparison of both writing and math technology applications. Rather, we selectively described writing technology, such as the word processing and concept mapping of ideas in written work, to illuminate the potential advantages of improved learning. The learning of mathematics using technology such as numeric, graphic, symbolic calculators, visualization software, and computerized modelling tools suggests many areas yet to be researched.

Empirical evidence demonstrating which approaches work best with which aspects of writing and math learning, for which students, should lead to changes in the approach of faculty who still prefer traditional lecture format without technology. Researchers initially mistakenly presumed that the mechanistic aspects of technology would make the word processor a good teacher of rote and simple learning, helping the poorest writers do better. However, results consistently show that the best writers benefit most from using the technology, while the poorest writers still need significant help to compose text. This lends credence to the observation that technology advances learning of complex and abstract ideas in situations involving problem solving and experimentation, especially for very bright students. Better understanding the learning process will help widen the benefits of the technology to a larger number of students. If longitudinal clinical trials research supports the observations noted by Gavosto and O'Donnell, then applying these ideas in university classrooms would change the way faculty members help their students learn about writing and mathematics.

The similarities described here in the learning process across subjects should be studied further. Faculty across disciplines can learn from each other how students learn using technology. Pointing out the parallel with their preferred discipline can motivate students who like one of the disciplines but not the other. A research challenge ahead will be to analyze the developmental learning variables of symbolic representations in mathematics and in writing using technology.

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WEB-BASED TEACHING AND LEARNING WITH math-kit

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ABSTRACT

In this article we present the concepts and first results of math-kit, which is being developed at the universities of Bayreuth, Hagen, Hamburg and Paderborn. The research and development is part of the program 'Zukunftsinvestitionsprogramm' sponsored by the German government to introduce new media into university teaching.

math-kit is a web-based construction kit, which provides professors and students with multi media support for central topics in undergraduate mathematics. Moreover, its development is intended to close the gaps between the education of mathematics students and the training in technical disciplines such as computer science or mechanical engineering. math-kit tools combine mathematical algorithms with examples from other disciplines and vice versa. Furthermore, elements for student motivation, exploration, applications and visualization are contained. With the possibility of combining elements to support different learning objectives, professors are able to employ math-kit to compose individual teaching units.

¿From the technical point of view math-kit is based on the Sharable Content Object Reference (SCORM) standard and uses XML as the implementation language. The basic elements of math-kit are called assets, small highly interactive components like Java applets. Learning units can be built from assets; complete courses consist of different learning units. Further technical highlights of math-kit are the accessibility via the web and the possibility of using the computer algebra system MuPAD as the mathematical engine. In contrast to other systems, the elements of math-kit cannot only perform numerical computations, but the assistance of MuPAD also makes symbolic computations possible. The mathematical power of MuPAD can be used through the web without forcing the user to learn its complex programming language.

An outline of the system structure and some examples will be given.

1 Introduction

math-kit is a web-based consruction kit which is being developed at the universities of Bayreuth, Hagen, Hamburg and Paderborn. The research and development is part of a program called 'Zukunftsinvestitionsprogramm' (Future Investment) sponsored by the German government. Within this program, the German Ministry of Education and Science financially supports 160 projects which develop, integrate and evaluate the use of new media in university teaching [2] with a fundamental capital of 200 million Euro between 2001-2003.

Nowadays most university teachers agree that the use of computers is very helpful in specific learning situations, in particular when teaching mathematics to undergraduates and non-mathematics students. However, the use of computers remains mostly concentrated on isolated laboratory work and is not common in standard lectures. Here most lecturers still prefer traditional teaching methods such as the blackboard or OHP, as the integration of computer-based interactive teaching materials into lectures continues to require a larger degree of technical knowledge and the development of units is time-intensive.

Consequently this is the starting point of the project math-kit. In contrast to many existing computer based learning materials math-kit is neither a complete learning unit focusing on a specific subject nor a lecture or textbook equipped with hypertext and multi media support. Moreover, it is not intended to replace the human teacher. Instead, math-kit is a web based construction kit, which provides professors and students with small, multi media tools for central topics in undergraduate mathematics that can be combined in different ways to create individual learning units.

This project is mainly targeted at university staff who teach mathematics. In general, lecturers are not experts in the use of new media and clearly math-kit will only be integrated into individual teaching material if it is easy to use by non-specialists. Therefore a major focus of the project is to develop a technical platform that structures the elements of math-kit, and an interface which allows a non-expert to find and combine different elements into learning units. Supporting those who teach will be beneficial for students in the long term.

Another focus of the project is its evaluation. Elements of math-kit are currently being implemented used and evaluated at: the University of Bayreuth in lectures for teachers students; at the far distance teaching University of Hagen in courses for beginning mathematics students; at the University of Hamburg in lectures for computer science students and at the University of Paderborn in lectures teaching mathematics to engineering students.

2 Elements of math-kit

math-kit is designed to support different aspects and different settings of learning. Therefore various categories are distinguished and realized.

2.1 Exploration

In order to apply mathematical methods successfully to a given problem is it not sufficient to know the respective algorithms but to have a deep understanding of the underlying concepts. In this context, successfull learning is often regarded as an active and constructive process rather than a passive storing of information. Therefore, it is crucial to offer resources for self-controlled and explorative learning (see [3] for instance). This concept of learning is therefore a core idea within math-kit and reflected in its so-called elements of exploration. These are interactive elements, usually realized as Java Applets, which can be used by students in open learning settings. Often these elements allow and ask for direct as well as indirect manipulation.

2.2 Drills and Exercises

It is crucial for successful learning to give students the resources to practice mathematics and to offer them the opportunity to control their success in understanding. This applies in particular to students in a system of distance education. For this reason, exercises and drills play an important role in math-kit, especially when considering that for success in computer-based learning, the emphasis often lies in the necessity of constructive feedback. In this context, the computer has to give the learner as much freedom as possible, allowing them to choose alternatives, make as many attempts as necessary to solve the problem (see [3], [5]) and subsequently see their results. These ideas were taken into account when developing math-kit elements for exercises. math-kit uses the computer algebra system MuPAD in the background allowing the student to generate as many examples as needed, to receive help and to check the results. A few examples and more details about the implementation of these drills and exercises can be found in [6].

2.3 Application

Another basic principle of math-kit is the idea that mathematics should be taught and learned together with its applications. Mathematics is crucial for the understanding of many scientific fields, especially the technical sciences. Therefore, it plays a fundamental role in natural sciences and engineering courses as well as in the studies of computer science. However, the abstraction level of mathematics poses considerable problems for many undergraduate students - applications and connections are not always clear for them. Application elements of math-kit are intended to provide this link. 'Real life' problems, for example in connection with electrical circuits or economic models are explained, the hidden mathematics is revealed and it is furthermore shown how mathematics helped to solve the original problem.

Another difficulty facing teachers and learners in higher courses for example system theory in computer science or electrical engineering is that students may not remember the mathematics behind certain applications. With the application elements in math-kit, these gaps are intended to be closed.

2.4 Motivation

To a large degree successful teaching depends on the methods employed to motivate students. The already mentioned elements like explorations or applications can be used for motivation. Moreover, special teaching aids for instance videos are classical tools for increasing motivation and will be included in $\mathsf{m}\alpha\mathsf{th}$-kit. Furthermore, historical information such as biographies of famous mathematicians or the history of a mathematical problem can serve as motivational tools and are thus a part of mαth-kit.
In addition, we are especially interested in motivating female students to enroll in a technical science and to bring it to a successful end. Examples from the social sciences or art instead of examples of engines or electrical circuits are regarded as useful for addressing women. Therefore we will integrate such applications in mαth-kit.

3 The computer algebra system behind mαth-kit

3.1 Advantages of the use of a computer algebra system

Numerical calculations are widely known and can already be realized on a pocket calculator. The algorithms and their results are interesting in teaching mathematics and can also be easily implemented for use on the web. However, their use is restricted. For learning and teaching mathematics, it is crucial to deal with symbolic computations. A typical example is proving the identity $sin(2x) = 2sin(x)cos(x)$ or to differentiate or integrate functions symbolically. For such functionality, it is nearly impossible to use standard programming languages such as Java, however, so-called computer algebra systems (CAS) give answers to these problems. CAS are powerful software systems that combine numerical and symbolic computations and incorporate algorithms for nearly all kinds of mathematical fields. Within the project mαth-kit, the computer algebra system MuPAD is used. MuPAD is a modern CAS which has its roots at the University of Paderborn [4]. MuPAD is also very useful in generating exercises. In this way, exercises and drills in mαth-kit allow students alternatives to arrive at and enter their solutions. Each student has the possibility to individually make as many attempts as necassary to solve a problem as MuPAD generates as many exercises as needed. As MuPAD can analyze mistakes and assist with problems, it can be used for direct feedback to students (see also [6] for a more detailed discussion of this topic). Hence, the elements of mαth-kit not only combine numerical and symbolic calculations but also use MuPAD as a mathematical expert system in the background. Tools for calculating the row echelon form of an arbitrary matrix, an applet for the calculation of the symbolic and numeric value of an infinite sum or a tool for checking and calculating derivatives already exist [6]; others are about to follow. In these tools the use of the CAS is hidden from the user. Hence, the student does not need to learn the MuPAD language in order to work with mαth-kit. However, if explicitly needed, one can also integrate a complete MuPAD session into mαth-kit elements. This can be very useful when explaining the language and the possibilities of a CAS.

3.2 Computer algebra via the web

Usually computer algebra systems need to be installed on local computers and cannot be accessed through the web. Today concepts for web-based computing with CAS are being developed such that all their functionalities can be used in internet based applications. Integrating this into $m\alpha$th-kit has the advantage that no local copies of a computer alegebra system must be installed. Hence, all students use the same version of the system without needing to consider installation or local incompatibilties. In $m\alpha$th-kit we chose MuPAD as the algebra engine behind our applications because the pricing of MuPAD is attractive for universities. The main components of this engine are a Java client applet and a MuPAD Computing Server with JavaScript serving as means of communication between input/output components of the web pages and the client applet. More details are described in [8].

4 Technical outline of the system

4.1 Demands

As already mentioned, the main goal of $m\alpha$th-kit is to provide multi media support for lecturers. Professors like to use their own notation, have their own structure in their lectures and focus on different aspects in mathematics. Many of them are interested in using multi media tools as long as they are easy to handle and easy to adapt to their own lecture. Therefore, the main design concepts of $m\alpha$th-kit are the flexibility and adaptibility of elements together with an ease of use and the combination of elements. To achieve these principles, it is necessary to be highly granular and to give the user the possibility to build up learning sequences from small elements. One fundamental element covers only a very specific topic for instance transforming a matrix into row echelon form. This method is one of the basic algorithms in mathematics needed in different fields. The goal of $m\alpha$th-kit is to make this method available as a web-based tool, to make it easy to use and to integrate it into different lectures. With this flexibility, it is also possible to integrate elements of $m\alpha$th-kit into different learning contexts and to support different learning objectives.

It is obvious that not all interesting or difficult topics in undergraduate mathematics can be covered within the three year period of the project. Therefore, $m\alpha$th-kit is being designed to be extendable by authors outside of our group. Guidelines for other developers will be published.

To simplify the publishing of new elements of $m\alpha$th-kit, we plan to develop an authoring system. Authors are supposed to provide keywords for their elements in order to make all elements of $m\alpha$th-kit searchable.

4.2 Realization

The structure of all learning units and $m\alpha$th-kit itself are based on the SCORM (Sharable Content Object Reference Model) standard version 1.2 ([7]), which was proposed by the Advanced Distributed Learning Initiative [1] in 2001. This standard not only guarantees that all elements are searchable but also reflects the fine granularity of the system as well as the possibility to combine different elements. The atomic elements of $m\alpha$th-kit are small highly interactive components like Java applets called assets. Learning units can be created from assets; complete courses consist of different learning units. XML is the programming language chosen to implement the elements. With XML the content of elements is independent of the representation. It can be translated into any other document format that uses a hierarchical organisation like HTML and PDF. Hence, elements of math-kit can be adapted to personal needs or preferences.

5 Summary

In this article we presented the concepts and first results of math-kit, which is a highly flexible and adaptable web-based construction kit for multi media support in central topics in undergraduate mathematics. During our presentation several examples will be given. All examples and supplementary material will be published on our web site www.math-kit.de at the beginning of April 2002.

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TO WRITE A PROGRAM = TO FORMULATE ACCURACY

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ABSTRACT

Mathematics required completely accurate formulation. The control of accuracy could, however, be difficult. The control by our own thinking could be inadequate because even when the formulation is not sufficiently correct, we know what we meant to say and forgive ourselves, the inaccuracy we commit. One possibility, how to prevent this is to control the formulations by a program. Thus will be delegating the checking of our expressions to a computational technology, which will curry it out accurately. In my contribution I will therefor introduce several simple programs and demonstrate how to use them for checking accurate formulations of some problems - in particular solutions of examples from secondary-school level Math. To demonstrate this idea I have chosen a graphing calculator (esp. TI-83), which offers an easily managed, simple, easily understood programming language, which fulfil the requirements of structural programming. At the same time its' capability is sufficient to enable the solutions of practically all problems dealt with in secondary-school Math. The contribution will include examples of algorithms for simple tasks, as well as examples of more complicated problems, where the solution requires an accurate construction of algorithms for partial tasks. The issue of dealing with the verification of accurate formulation is an integral part of the subject "Computational Technology for Teachers of Math", which is included in the study program for students, prospective teachers of Math in the Faculty of Math and Physics at the Charles University in Prague, Czech Rep. This subject was included in study program about 5 years ago as part of modernization of this program with respect to increasing use of Computational Technology in the teaching of Math at the secondary school. In addition the contribution also demonstrate non-standard application of the graphing calculators in teaching Math.

KEYWORDS: Algorithm, programming, program, subprogram, accurate formulation, control of accuracy, secondary-school mathematics, graphing calculator

1. Introduction

In teaching mathematics we often find that although the students can solve mathematical problems, they cannot adequately describe their solution process step-by-step. Mathematics requires completely accurate formulation. The control of accuracy could, however, be difficult. The control by students' own thinking could be not adequate because even when the formulation is not sufficiently correct, they know what they meant to say and forgive them the inaccuracy they commit. One possibility, how to prevent this is to control the formulations by program. Thus will be delegate the checking accurate formulations of some problems – in particular solutions of examples from secondary-school level Mathematics. To demonstrate this idea I have chosen a graphing calculator (esp. TI-83), which offers an easily managed, simple, easily understood programming language, which fulfils the requirements of structural programming. At the same time its' capability is sufficient to enable the solutions of practically all problems dealt with in secondary-school Mathematics. The contribution includes examples of algorithms for very simple tasks, as well as examples of more complicated problems, where the solution requires an accurate construction of algorithms for partial tasks. In addition the contribution also demonstrate non-standard application of the graphing calculators in teaching Mathematics.

For those not familiar with the GC TI-83, let me just mention that its programming language has, in addition to the usual "input-output" procedures, the following basic commands (among others):

- the conditional test "If Then Else"
- the incrementing loop "For"
- the conditional loops "While" and "Repeat"
- the end of a block signification "End"
- a program as a subroutine execution "prgm"

(and only "one-letter" identifiers - name of variables, are available).

2. Examples

Four very simple examples – programs – are introduced for the purpose of demonstration. For each of them detailed commentary is given concerning those commands, where accurate formulation in the description of the relevant calculation algorithm is essential. Such accurate commands are necessary preconditions for the generated program to perform the calculation of a given problem correctly.

Example 1:

Write a program for solution of equation $a*x^2 + b*x + c = 0$.

Let me offer you my solution with comments:

PROGRAM:QE	
:Real :ClrHome	*These 3 commands illustrate, that we'll work in real mode, start*
:Promt A,B,C	*with clear display, and 3 values of coefficients a,b,c are asked*
	(their values are necessary, when we start the solution process)
:Fix 2	*The command for results' edition (2 decimal)*
:If abs(A) < 10^(-6)	*Very important part of program. We have to discuss all*
:Then	*possibilities of the coefficients (a,b,c) values and have to finish*
:If abs(B) < 10^(-6)	*solution in each of branch.*
:Then	*As a "side effect" in this discussion is test, if a value of each of*

:If abs(C) < 10^(-6)	*coefficients (a,b,c) = 0. Because they are stored as a real*
:Then	*variables in calculator, it is not good idea to use "direct" test*
:Output(5,1,"ALL X")	*(If "variable = 0"). For real variables is better to test*
:Else	*"if variable ∈ or ∉ of ε- neighbourhood of 0".*
:Output(5,1,"NO X")	*This part of program also shows, how to include If ... Then ...*
:End	*Else ... End command into the other If ... Then ... Else ... End*
:Else	*command*
:-C/B–>X :Disp X	
:End	
:Else	
:B^2–4*A*C–>D	
:If D<0	*We must respect then if D<0 the complex mode is necessary.*
:Then :a+bi :End	*And there is shown If ... Then ... End command (without Else)*
:(-B–√(D))/2/A–>X	*Very important command in this form; it illustrates that equal priority of operations must be respect.*
:(-B+√(D))/(2*A)–>Y	*And in this command there is illustrated how the brackets influate priority of operations and help us to use for formulas "pretty" notation.*
:Disp X :Disp Y :Stop	

Example 2.

There is shown, in two following examples, how it is useful to choose "more sufficient" of two conditional loops with different philosophy:

"while" loop – at first the condition is tested; if "true", all commands inside loop are executed and the loop is repeated, if "false", no command inside loop is executed and the following command after "while" loop is executed

"repeat" loop – commands inside loop are executed and when the execution of these commands is finished, the condition is tested; if "true", "loop" is finished and the following command after "repeat" loop is executed, if "false", loop is repeated.

This moment is very important indicator, if we understand what we wish to do, what we wish to achieve.

Problem 2.1 Two integers A and B are given. Use the Euclid algorithm to evaluate the greatest common divisor.

The program is very simple, of course:

PROGRAM: GCD	
:Prompt A, B	
:While A≠B	*In this case it is better to use the "while" loop, because if given*
:If A>B	*values of A and B are equal, we do not have to do anything, "while"*
:Then	*loop gives us the result directly. On the other hand it would be*
:A–B–>A	*difficult to formulate the condition for the "repeat" loop.*
:Else	
:B–A–>B	
:End	
:End	
:Disp A:Stop	

Sometimes, if "repeat" loop is used when it is not suitable, one "back-step" is necessary,.

Problem 2.2 Two integers A and B are given. Compute the $A : B$ (= quotient Q with reminder R, $R \in \langle 0,A)$), using only operation + and – and tests.

If we use "while" loop, all is without problem:

```
PROGRAM DIV1
:Prompt A,B
:0->Q
:While A≥B
:A-B->A                         I hope, all is clear and easy without any comment
:Q+1->Q
:End
:A->R
:Disp Q
:Disp R
:Stop
```

If we use "repeat" loop, there are two different situations:

```
PROGRAM DIV2
:Prompt A,B
:0->Q
:A->R
:Repeat R<B
:R-B->R
:Q+1->Q
:End
:Disp Q
:Disp R
:Stop
```

For 13 : 5 e.g. is all OK, the results in individual steps of the loop are:
1) $R = 13 - 5$ (= 8), $Q = 1$, $R < B$ (8 < 5) is "false" (test is the last provided command in "repeat" loop !)
2) $R = 8 - 5$ (= 3), $Q = 2$, $R < B$ (3 < 5) is "true", "repeat" loop is finished in this moment
Command :Disp Q gives correct quotient = 2 and in R is correct reminder 3.

But if we use this program for 2 : 5, the results in individual steps of the loop are:
$R = 2 - 5$ (= -3), $Q = 1$, $R < B$ (-3 < 5) is "true" , "repeat" loop is finished, but the result is not convenient (reminder R isn't from required interval $\langle 0,A)$ and Q=1 is wrong quotient). In this moment it is necessary to do "back-step", mentioned before, i.e. we have to repair it by following commands after :End of "repeat" loop:

```
:If
:R< 0
```

```
:Then
:Q-1->Q
:R+B->R
...
```

But we know only just now, why this "artificial" group of commands was included into the program. Be sure, it will be forgotten during very short time, and for the "second" person, who will use this our program, it is only as a puzzle! Similar situation is very helpful indicator, that "while" loop was better for this problem.

In this moment also let me mention, that if we need the values of variables A and B saved in original (e.g. for the output: "The GCD of A and B is ..."), it is necessary to move them into two auxiliary variables and all commands provide with these auxiliary variables. Their values are changed during calculation, but the values of variables A and B are saved. By the way it is illustrated in Problem 2.2. Value of variable A there is moved into variable R, its' values are changed and A has original value when work of program is finished.

For "Repeat" loop let me offer the next example:

Problem 2.3: Evaluate the sum of the series $\Sigma\,(1/2)^n$ ($n = 0, 1 \dots$) with given accuracy e.

The program is very easy, again:

```
PROGRAM: SUM
:Prompt E
:0–>S :0–>N :1–>D
:Repeat D≤E
:N+1–>N
:(1/2)^N–>D
:S+D–>S
:End
:Disp S:Stop
```

It's clear in this example, that we have to go through the block of commands inside "repeat" loop at least once, to obtain required accuracy.

There is another very important moment. It isn't necessary to evaluate in each step (1/2)^2, (1/2)^3 etc. When we are evaluating $(1/2)^i$ it is better to compute it as $(1/2)^{i-1} * (1/2)$; in the program we cancel the commands :0–>N and :N+1–>N and the 5^{th} line we substitute by command :1/2*D–>D)

In the last example I would like to illustrate, how it is helpful for description (and understanding) of solution of large, complicated problem, if that solution is split into several solutions of partial, simple problems.

Example 3:

The line p and two different points A, B in the same half-plane are given, non of which lies on the line p. Find such a point P on the line p, that the sum of lengths of segments AP and BP is minimal.

Solution.

What we have to do, when we solve this example "with pencil and paper":

- we have to find an image point A' of the point A in symmetry with respect the line p
- we have to draw line q, passing though points A' and B

- we have to find the result point P as an intersect of lines p and q

But the first step isn't elementary, there are three steps inside, in fact:

- we have to draw perpendicular line k to line p passing through point A
- we have to find intersection of lines p and k – point Q
- we have to find point A' on line k, in opposite half-plane than point A is, to be AQ and $A'Q$ equal segments

Now, when the example is analysed, we can start to write program or subprograms for each elementary step. Because we solve this example in plane, a point is described by two coordinates x and y and a line is described by general equation $a*x+b*y+c=0$. There is very easy solution for this (and similar) situation:

Let me suppose that we have written the following subprograms for partial, simple calculation:
prgm PLP – subprogram returns three coefficients of equation of line,
perpendicular to given line (by three coef. of eq.) and passing
through given point (by two coord.)
prgm PPL – subprogram returns three coefficients of equation of line, passing
through two given points (by two coord.)
prgm LLP – subprogram returns two coordinates of point, intersection of two
given lines (by three coef. of eq.)

and two subprograms for "input point" and "input line":

prgm RP – input of point (two coordinates, x and y)
prgm RL – input of line (three coefficients of general equation, a, b, c).

The last subprogram for the construction of the point A', symmetric point with given point A with respect to the line p, is missing. This problem we can solve, using vectors (vector AA' = 2* vector AQ). It's useful to write subprogram not for "2*vector" but for "n*vector" and use n = 2. Let us assume that we have written such a subprogram

prgm NV – subprogram returns two coordinates of end point E of
vector AE = n*vectorAB where A is starting point of both
vectors and B is end point of given vector

Sometimes the students are very surprised, why there are written two subprograms for "input point" and for "input line" separately and why the operation "input point" and "input line" are not included into subprograms LPP, RP and RL. The answer is very easy (and reason is very important and strong!) – sometimes we use for these subprograms points or lines, these are the results of previous calculation. In that moment these values are automatically disposable and on the other hand, it may be very inconveniently to have to input these values, in fact, once more.

And now we can write very easy final program. It is sequence of subprograms:

… :prgm RP

```
:prgm RP
:prgm RL
:prgm PLP
:prgm LLP
:prgm NV  (for n=2)
:prgm PPL
:prgm PLP ...
```

In fact, the solution of "large, complicated" problem there is described very precisely by solution of partial, simple problems

And having these subprograms (solution of partial, simple problems), we can describe precisely solution of a lot of other problems (triangle's midpoint or centroid, etc., etc).

3. Short resume

An important indication of the need to improve the description of a mathematical procedure is to find that the program leads to incorrect results. In such a case it is quite likely that step-by-step description used in programming has not been correct.

SVG – A NEW DIMENSION IN PRODUCING INTERACTIVE NETBOOKS

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ABSTRACT

SVG - Scalable Vector Graphics is a 2D-graphics markup language based on XML. It is compatible with other web standards: HTML, XML Namespace, Xlink, Xpointer, CSS 2, DOM 1, Java, ECMA/JavaScript, Unicode, SMIL 1.0, ... It allows us to include in HTML documents pictures described by their structure – composition of curves and shapes. Since the SVG viewer is not integrated yet into web browsers we need, to view SVG pictures, to install it as a plug-in. An excellent SVG plug-in was produced by Adobe.

The SVG pictures are not static (as standard bitmaps GIF, JPEG, PNG). The SVG viewer provides options to zoom in (to see details) and out (to see global/overall view), to move the picture, to search for text ... Besides this, using built-in animation capabilities or JavaScript program support, the pictures can be made alive and interactive. We can partition a SVG picture to several parts. Changing their attributes we can control their visibility. Using JavaScript this can be done interactively allowing to the user to select the parts to be displayed. We can also dynamically add or delete the elements of the picture and change their properties.

SVG pictures can be produced by drawing tools. But special web applications and programs for visualization of obtained data/results will produce most SVG pictures.

In the paper we present the main features of SVG and discuss their potential educational usage. Some our SVG based solutions are also listed.

Keywords: authoring, interactrive, dynamic, visualization, scalable vector graphics, standards, internet.

1. What is SVG ?

In 1985 Adobe presented ***Postscript*** that, combined with a laser printer, produced a revolution in publishing. Since the Postscript was not completely appropriate for the use on the web they, at Adobe, developed a new format PDF – ***Portable Document Format.*** PDF established itself as a leading format for publication of closed documents (reports, manuals, papers, tutorials...). Several other companies produced in nineties their own graphics description formats with plug-in viewers for them: QuickTime (Apple), CMX (Corel), Flash (Macromedia) ... Their main drawback is – they are not an open standard.

In 1998 two groups of companies submitted to the ***World Wide Web Consortium*** - W3C their proposals for web graphics format based on XML. The first group (April 1998 / Adobe, IBM, Netscape and Sun) proposed PGML - Precision Graphics Markup Language; and the second (May 1998 / HP, Macromedia, Microsoft and Visio) proposed VML - Vector Markup Language. Both groups merged into SVG - Scalable Vector Graphics development group that published already in October 1998 the requirements on SVG and in February 1999 the first draft. Several improved versions followed. The last version was published on 4th September 2001 as a W3C Recommendation.

SVG - ***Scalable Vector Graphics*** is a 2D-graphics markup language based on XML. It is compatible with other web standards: HTML, XML Namespace, Xlink, Xpointer, CSS 2, DOM 1, Java, ECMA/JavaScript, Unicode, SMIL 1.0 ... It allows us to include in HTML documents pictures described by their structure – composition of curves, shapes, text and also bitmaps.

To view a SVG picture we need a special viewer. In the latest versions of the most popular web browsers the viewer is already integrated. If we use an older browser, we need to install SVG viewer as a plug-in. An excellent SVG plug-in for Windows and Macintosh was produced by Adobe.

The SVG pictures are not static (as standard bitmaps GIF, TIFF, JPEG, PNG). The SVG viewer provides options to zoom in (to see details) and out (to see global view), to move the picture, to search for text, ... Besides this, using built-in animation capabilities or JavaScript/Java program support, the pictures can be made alive and interactive.

SVG pictures can be produced using drawing tools. On Windows we can use Adobe Illustrator 10, Corel Draw 10, WebDraw (by Jasc) and Mayura. But special programs for visualization of obtained data/results will produce most SVG pictures.

The main applications of SVG are data visualization, presentations (like Power Point), maps (GIS), technical layouts and educational pictures (illustrations).

2. SVG and HTML

Here is a simple example of picture description in SVG.

```
<svg>
 <circle cx="120" cy="65" r="30" style="fill:yellow;stroke:black;"/>
 <text x="100" y="55" style="fill:red;">Mathematics</text>
</svg>
```

It creates yellow circle with black border containing red inscription "Mathematics".

To insert a SVG picture into a HTML document we use the EMBED (or OBJECT) tag. For the picture from our simple example we have:

```
<EMBED SRC="simple.svg" NAME="simple"
  WIDTH="300" HEIGHT="100" TYPE="image/svg-xml"
  PLUGINSPAGE="http://www.adobe.com/svg/viewer/install/"
>
```

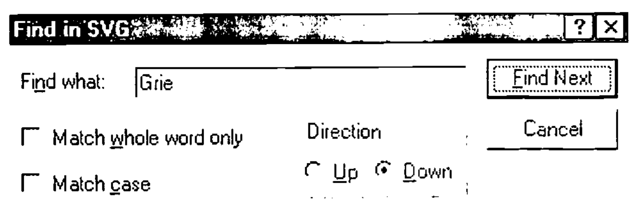
The attribute **SRC** determines the location (URL) of the SVG file; **NAME** becomes important in advanced applications using JavaScript or Java. The attributes **WIDTH** and **HEIGHT** are obligatory and determine the size of rectangle in which the picture is rendered. The value of **TYPE** is the MIME-type of the file - for SVG file it can be **image/svg+xml** or **image/svg-xml**. The attribute **PLUGINSPAGE** directs the user that has not a SVG viewer installed on his computer, to the web site from which he can obtain a viewer.

3. Advantages of SVG format

In this section we present an overview of possibilities offered by SVG found in different applications on the internet. Since one of the strongest features of SVG is interactivity and dynamics, in the paper medium, because of its limitations, we can give only some snapshots. The reader is invited to visit the original pages on the web. The web version of this paper with some additional links is available at http://www.educa.fmf.uni-lj.si/izodel/dela/SVG/ .

Zooming and moving: in SVG viewer we can zoom-in and -out the picture thus obtaining a detailed view of the selected part or an overall view of the picture. We can also **move** the viewing window. In Figure 1 some snapshots from *OECD Europa Atlas* (the page is in German, http://www.carto.net/papers/svg/eu/oecd.html) are presented. We first see the overall view of Europe. Then we zoom-in to Aegean region and further into Athens and Crete part. Note that, since in each case the picture is produced from its description, the quality of the picture depends only on the resolution of the output device (screen, printer).

Searching: Since in SVG textual data (for example, labels) are stored as text we can search for selected string. On the left bottom part of Figure 1 the result of searching for string "Grie" is displayed.



The viewer locates the label containing given string, moves the viewing window there and marks the string in the label.

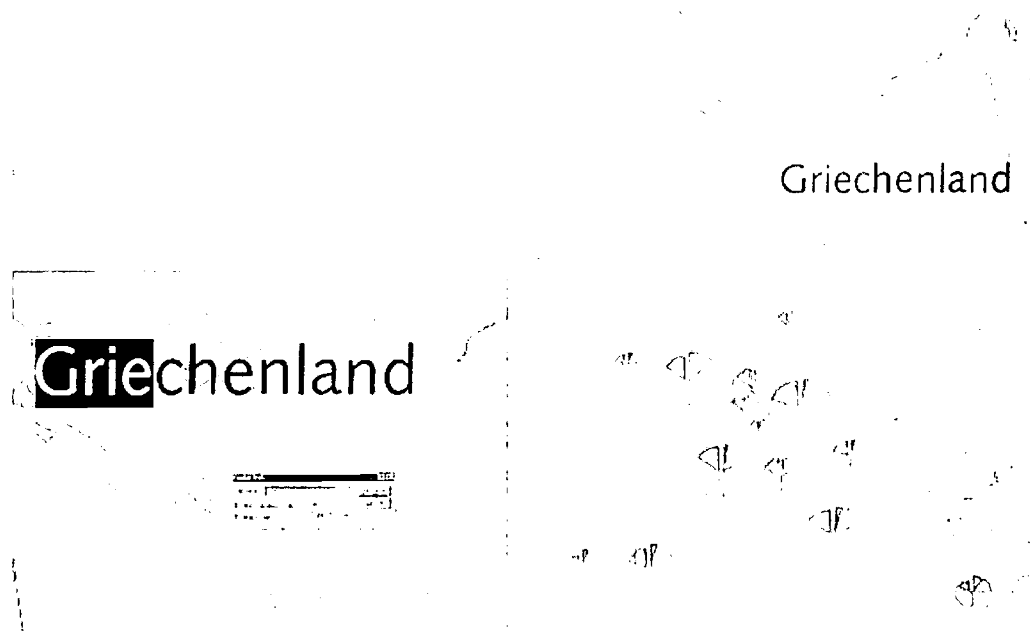


Figure 1: Zooming, searching and control of visibility

Controlling visibility: Using the **`g`** SVG tag we can partition a SVG picture into several parts. In the **`g`** tag two attributes **`display`** and **`visibility`** are available to control the visibility of the corresponding part. The difference between them is explained in SVG documentation. For example, setting the **`display`** attribute to **`none`** we switch the visibility of the part off; setting it to **`inline`** (or some other value) we switch the visibility of the part on. Using JavaScript this can be done interactively allowing to the user to select the parts to be displayed.

In Figure 1 in the overall view, the display of the relief and rivers is switched off.

In Figure 2 the sequence of states in the construction of the perpendicular from a given point to a given line is presented. To produce a dynamic visualization of the construction in SVG we first draw the final picture with some SVG editor. Then we put each picture increment elements into a separate group with its own visibility control, which is triggered by the user from the picture control HTML page using JavaScript.

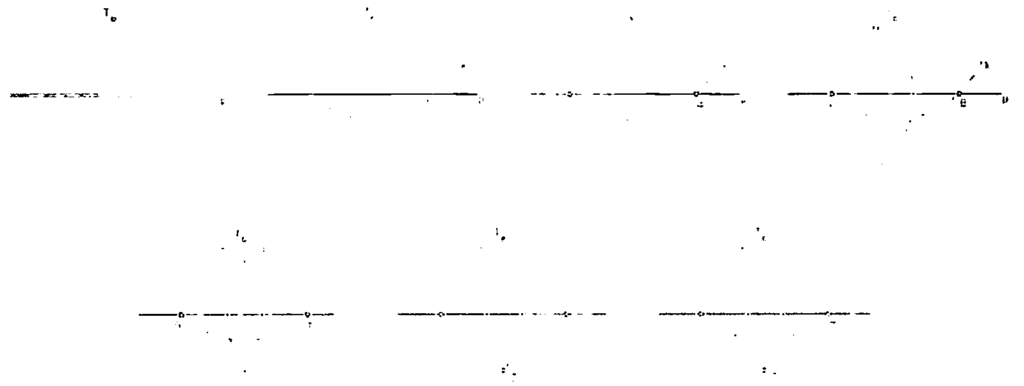


Figure 2: Construction of the perpendicular line

The same technique can be used to display user selected pictures in the same place on the page http://vlado.fmf.uni-lj.si/pub/CONF/DS1.01/kuk.htm , or to connect the picture elements with their descriptions/explanations http://www.usbyte.com/index_SVG.htm (select LU (load/unload) type > Interactive Drawing).

Dynamic changes of display elements: Using JavaScript or Java we can dynamically add or delete elements of the picture and change their attributes (size, location, style). These changes can be triggered by the user from the control HTML page.

On the overall view of Europe on Figure 1 we selected UK – the selected country is colored with a darker color. In the right bottom picture on Figure 1 we selected the display of pie charts representing selected economic data. The picture was augmented with pie charts.

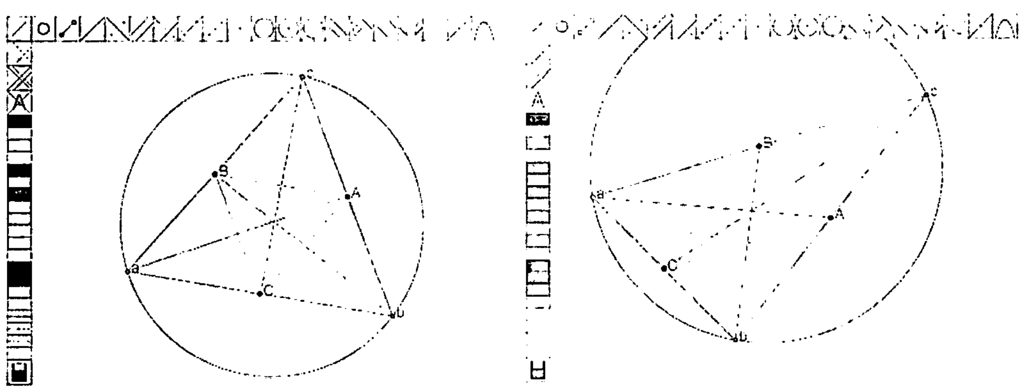


Figure 3: Geometric construction with SVGeom

The most interesting and rich site on the web with educational SVG contents is the ***Pilat Informatique Educative*** http://perso.wanadoo.fr/pilat/ that contains several nice mathematical examples and systems SVGeom and SVGFonc.

The system SVGeom is essentially a SVG/JavaScript based dynamic geometry system (similar to Geometry, Cabri or Geometric Sketchpad). In Figure 3 a geometric picture produced interactively in SVGeom is presented on the left. As seen on the right this picture can be interactively transformed preserving the structural relations among elements.

Animation: SVG provides, based on SMIL, basic animation capabilities. To get an impression visit the following examples: ***Kaleidoscope***: http://www.burningpixel.com/svg/Kaleid.htm , ***Lissajou curves***: http://www.mecxpert.de/svg/lissajou.html , ***PC Technology in motion***: http://www.usbyte.com/index_SVG.htm (select CD optical pick-up system in action).

The SVG format has several additional features: the SVG files are relatively **small**, in addition - **compressed** files are also supported; the picture can be built from ready-to-use components from libraries of **reusable** objects; **extensibility** by combining SVG with problem-specific XML solutions; **internationalization** – SVG supports Unicode and provides elaborate typography options (for example, the direction of writing); the picture description is **independent** of output devices and computer platforms, it can contain **metadata**; SVG provides excellent **color control**.

4. Our support for SVG based visualizations

To ease the preparation of visibility controlled SVG visualizations we prepared the ***SVGplayer*** - a collection of JavaScript functions for controlling the value of **`display`** attribute in parts of the SVG picture. At http://sio.edus.si/list/1/svg/svg04.htm you will find the page with SVG visualization of the construction of the perpendicular using the ***SVGplayer***. A ZIP with the last version of the ***SVGplayer*** is available at http://vlado.fmf.uni-lj.si/pub/SVG/SVGplayer/.

Based on the approach used in ***Logo2PS*** we prepared also ***Logo2SVG*** that allows user to save a trace of the Logo turtle as a (visibility controled) SVG picture. For details see http://vlado.fmf.uni-lj.si/educa/logo/logo2svg/.

We prepared also a SVG based ***function drawing*** page and a SVG/JavaScript version of ***EulerGT*** (http://www.educa.fmf.uni-lj.si/izodel/dela/Euler/).

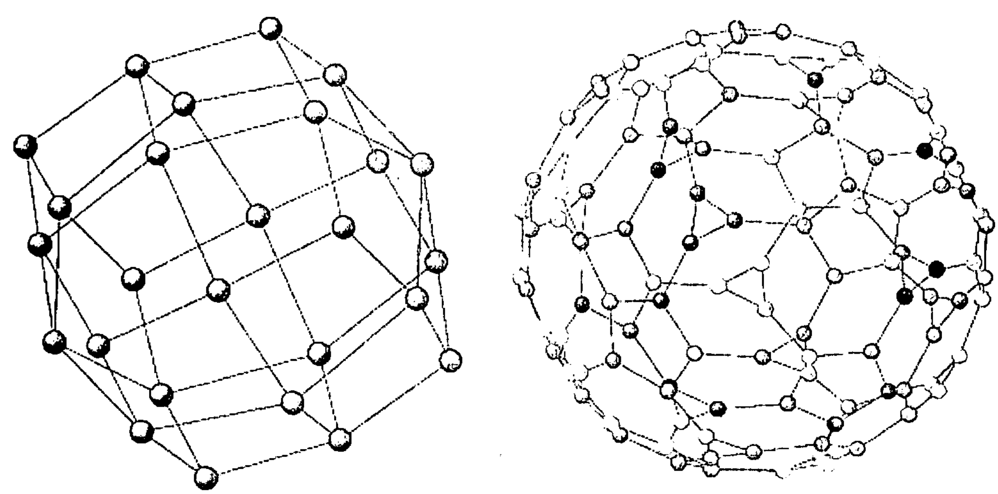


Figure 4: Graph visualizations from Pajek

For an example of the program generating SVG visualizations you can look at our program ***Pajek*** (http://vlado.fmf.uni-lj.si/pub/networks/pajek/). It is aimed for analysis and visualization of large networks. One among several graphics output formats supported by ***Pajek*** is also SVG. On Figure 4 you can see two SVG pictures of 3D embeddings of graphs produced by ***Pajek***.

5. Conclusions

We hope that the presented examples convinced the reader about the broad new space opened by SVG for the development of educational materials. Since visualization is an important component also in mathematical education we expect that SVG will get broader acceptance also in this field in the near future.

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SAVING MATH JOBS:
Keeping Math Courses within Math Departments

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ABSTRACT

There has been a trend for post-secondary math courses to move to other departments (statistics being taught in the business school, for example) and for math requirements to be reduced. Since math jobs are at stake, how can we stem or reverse this trend? In this paper, we talk about a successful curriculum innovation project involving calculus for business students, and some lessons learned about working with other departments and colleges. The project involved collaboration with the business school and members of all of its departments from the beginning. We first listened carefully to the needs of our client disciplines, both in terms of overall philosophy as well as specific topics. Then we looked to see what course concepts and texts already existed that might meet our needs, but soon realised that nothing really fit well and that we would have to craft a new solution ourselves. Our concept was to make a two-semester course with integrity that was problem-driven, and relate it to students' other courses, careers, and personal lives as closely as possible. We applied for and received grants from FIPSE, NSF, Villanova University, and Prentice Hall, which helped give us the time needed to develop new materials and the foresight and discipline to organise evaluations of the new course sequence. We worked extensively and sometimes agonisingly with an Advisory Committee from the business school as well as their Curriculum Committee and the math department, but made sure everyone was on board. We were careful to provide gradual and plentiful training and development for our math colleagues. The bottom line is that the new course has been a great success at all levels (student learning and attitudes, business school enthusiasm, and math faculty satisfaction). In this paper we will discuss details about our process and lessons learned.

Introduction: Storm Clouds on the Horizon?

Mathematics as a discipline is centuries old - quite old, as current post-secondary disciplines go. Probability/statistics, operations research, and computer science were developed initially mainly within mathematics, then later often evolved into separate disciplines within academia (and so, separate departments and programs within colleges and universities). Thus there is a fairly common pattern of subject areas being sired and developed within math, but then "moving out of the house" to strike out on their own, with the effect of reducing the size of the "household." This means there is a natural ebb and flow to the size of a math department over time, swelling to add new topics, then shrinking as they split off.

At the present moment in history, math departments seem to be a bit on the "ebb" side of the cycle. Many are still in the process of having computer science and/or statistics split off, are having statistics or discrete math courses taken over by other departments, or are seeing math requirements reduced, often to allow for requirements in new areas such as computer science, writing, or diversity. The loss or reduction of a single semester required math course in a program can mean the loss of several full-time faculty positions in a math department. Currently this requirement reduction seems to be happening in liberal arts and business programs most notably. But it is not happening at all institutions. Is there anything that a math department can do to prevent or minimise such losses?

In this paper, we will describe a project that we have been working on for the last decade to re-engineer the 2-semester first-year math service course sequence for our business school at Villanova University. The course incorporates most of the topics from courses usually called Finite Mathematics and Business Calculus in the U.S., including single variable calculus (both differential and integral), probability, matrices, partial derivatives and multivariable optimisation, including Lagrange multipliers and linear programming. We will describe our process in working with the business school, our math colleagues, and social science departments within our college of liberal arts and sciences to totally rethink this course sequence, implement the changes, and evaluate and monitor the results. We will also share a number of lessons we learned along the way, and give our advice for our math colleagues at other institutions around the world who wish to do all they can to keep from losing faculty positions in their departments. At Villanova, about 2/3 of the classes we teach in the math department are service courses, and nearly 1/4 of our classes are in the business calculus sequence (with comparable enrolment proportions). We will not focus here on issues of trying to increase the number of math majors, but on the provision of math service courses for other departments and colleges.

Recognising Symptoms: Houston, We Have a Problem!

About 10 years ago, some of us in the math department realised that we didn't really enjoy teaching our business calculus courses. One major reason was that the students really hated the courses. The students seemed to see the courses as pure torture, like a "hazing" ritual required to be inducted into the "fraternity/sorority" of business majors, to be tolerated and forgotten as soon as possible afterwards. On their evaluation forms they always wrote comments such as "When am I ever going to **use** this stuff?" Another problem was that the topics in the course felt very disjointed: it was a mishmash of unconnected fragments with no unity or flow to it.

We decided to check in with our business school to see how they felt about the course sequence. (Interestingly, and fortuitously, they later claimed that **they** had initiated the contact, so

both main actors felt ownership of the process. This is ideal, if you can swing it!) They expressed some concern that their students did not know the math that the business school really wanted and needed them to know. In some cases, the areas of deficiency were already on our math syllabus, but the students would often claim they had never seen the material before. This is not uncommon in such service courses, but we wanted to try to minimise the phenomenon. In our discussions with the business school, we realised that one factor in this disconnect could be a difference in notation and terminology in the two fields (math and business), so we tried to find where this occurred.

Our Solution

At this point, it seemed clear to all concerned that something needed to be done to modify the course, so an Ad Hoc Committee of math and business faculty was created to study the problem. This group decided that everyone's needs would be best served by making the course **problem-driven** rather than abstract and theoretical. We decided a good starting point would be to ask faculty in all of the business departments for **examples** of mathematical problems they used in their courses. This turned out to be very difficult. We got many lists of topics for different disciplines, but very few colleagues were able to give us concrete specific examples. Eventually we did get representative problems from each department.

Next we did a search of existing texts and courses to see if anything existed to do what we wanted to do. The closest we could find was the *Calculus Concepts* text out of Clemson University, in its early stages of development. This text focused on using real-world data and fitting curves to the data, and came much closer to what we wanted than anything else that existed at the time. We decided to adopt it in several experimental sections of our course sequence. Unfortunately, this text did not cover matrices and linear programming, which our business colleagues still wanted us to cover, so we realised that we would have to develop supplementary textual material on these topics ourselves to fit the style of the other topics.

As we used the Clemson text this first time, we realised that it was a great improvement over the traditional texts, but that it didn't go into as much detail about the process of math modelling as we wanted. It opened the door, but didn't walk all the way in, so to speak. At around the same time in our discussions with the business school, they expressed a preference for covering all of the single-variable calculus material in one semester. That way, students with AP credit could place out of that part of the course, but get the rest of the content in the other semester. Up to this point we had covered through derivatives in the first semester, then did integrals, partial derivatives, matrices, and linear programming in the second semester. As we discussed specific topics that the business school wanted and did not want, we realised that in fact we could cover all of the needed single-variable calculus in the first semester. This was possible because there were a number of topics we had been teaching that they did not care about, including implicit differentiation, related rates, the Mean Value Theorem, and most techniques of integration.

We then realised that we *could* put all of the single-variable topics in the first semester and the multivariable topics in the second semester. We weren't sure where to put the topics of compound interest and net present value, but saw that they could be thought of as involving functions of several variables (interest rate, time, etc.) and put them into the second semester. Now we started to see that we could go beyond the idea of math modelling, and could think of the course sequence as a course in **problem solving**: single-variable in the first semester and multivariable in the second semester. More specifically, we would be **teaching the entire process of solving real-**

world problems using math modelling, calculus, and technology. We could use the Clemson approach of using **graphing calculators** in the first semester to fit single-variable functions to data. After going to the first Harvard Consortium Conference on the Teaching of Calculus in 1992, we also realised that **spreadsheets** would be very helpful for the second semester, both for matrix calculations and to fit multivariable functions to data, paralleling what we did with the graphing calculators. This focus on problem solving and least squares regression would help give the course the **integrity** that we were looking for.

As we spelled out the entire process of solving real world problems, we realised that the very first step in the process is identifying and defining your problem in the first place. We knew that we wanted students to learn about and *experience* the *entire* process of problem solving. We had already done some experimenting with the use of **student-generated projects** (projects related to the course, but where the *student* chooses a topic based on their own life and interests) in this and other courses. We realised that a semester-long student-generated project was a perfect way to help students *learn* the entire process of problems solving and to see the **relevance** of the math as well. We like to use the analogy that the traditional approach to this course was like teaching students to fly at 5000 feet, but we wanted to teach them how to take off and land as well. Making those **connections** between the real world (the ground) and the world of math (up in the clouds) was exactly what the student-generated projects could do.

In the process of rethinking the course sequence from this perspective, we realised that there were also ways that we could make it flow more naturally and logically, and not feel so disjointed. We realised that the two semesters could be somewhat parallel in structure. They could start with defining functions, then focus on the process of formulating models, both from verbal descriptions and from data, then show how to take derivatives and optimise functions (with and without constraints), and talk about post-optimality analysis (including verification, validation, sensitivity analysis, and estimating margins of error). For the multivariable semester, we realised that we could cover matrices just before optimisation, just in time for solving the systems of linear equations that you obtain when setting partial derivatives of quadratic functions equal to zero. We could also cover Lagrange multipliers after partial derivatives, *after* which we could discuss shadow prices and linear programming, to give students a deeper understanding of shadow prices. For both semesters, it worked out conveniently that lower priority topics (integration and linear programming) came at the end of the semester, which meant that students would know all they needed for their projects about two-thirds of the way through the course. This meant that they could hand in drafts of their project reports, get extensive feedback and suggestions, and then hand in a revised report at the end of the semester, making it possible for them to produce a work of extremely high quality.

At around this time, we held discussions with the Math Curriculum Subcommittee of the business school's Curriculum Committee, to work out the details of topic coverage and course structure and philosophy. This was the hardest part of the entire process. There were several areas that turned out to be quite tricky and delicate to negotiate. One was what to do about integration. Economics and some Finance faculty wanted it covered, but not in great depth. We decided it should be covered, motivated largely by continuous probability (which the students would be using implicitly in statistics later) and Consumer and Producer Surplus. As mathematicians, we felt very strongly about covering the Fundamental Theorem of Calculus if we were going to teach integration, for its inherent beauty and importance in the history of ideas. They were convinced by our intellectual argument, happy that we were willing to drop techniques of integration and other topics they considered arcane.

The most difficult discussion involved the overall organisation of the course sequence. One member of the business curriculum math subcommittee felt very strongly that the course should be organised by having all of the **linear** topics in the first semester, and then all of the **non-linear** topics in the second semester. This was how he had learned it, and it is a very common structure, often broken up into Finite Math and then Business Calculus. We argued that this would totally destroy our idea of making the course about problem solving and using student-generated projects to reinforce the material and make it come alive for the students. The business school did like the idea of the student-generated projects very much. We told them that in our experience, there were almost no good project topics that were linear (a breakfast diet mix problem being about the only one), which would mean we couldn't really do the projects until the second semester. This would represent a huge loss in potential student motivation. After extensive discussion, we finally got approval for our structure by making some concessions in other areas that were not as critical to us.

Another difficult discussion revolved around technology, and this is quite common when working with faculty from other disciplines. Everyone agreed that spreadsheets were perfect for the second semester. But our business colleagues did not like the idea of using graphing calculators. Some of them required financial calculators for their students, and they felt this extra calculator was unnecessary. We talked about using spreadsheets in the first semester, but having taught with the Clemson text using graphing calculators, we knew that they were pedagogically far superior, since Villanova at the time did not have computer classrooms and the students did not have laptops. In fact, a very high percentage of the students came to college with a graphing calculator, so it was not a great burden. After demonstrating the power of the graphing calculator, and giving a free sample to each of the subcommittee members to see for themselves, they reluctantly agreed.

A final discussion with the subcommittee involved the topics for the student-generated projects. We said that examples of projects included finding the optimal amount of exercise in a day to maximise your energy level or the optimal amount of time warming up before a performance or athletic activity to maximise performance. They expressed concern that the topics were not of a business nature. We said that one of the main reasons for this was that the primary reason for the student-generated projects is **motivation**: to **connect** the math to something each student cares about in their own life. Secondly, we said that, while the topics do not appear strictly "business" in nature, most of them do involve optimal allocation of resources (such as time), which *is* very fundamentally an *economic* problem, quite analogous to many business problems. We did also say that we always encourage students to do real business examples (such as pricing T-shirts for a fund-raiser or cottage industry crafts, etc.). The combination of these arguments was strong enough for them to approve the idea, if somewhat reluctantly.

Implementation

Now that we had our concept clear, we knew that we had to do at lest some of the creation of materials ourselves. We talked with the Clemson authors to see if they were interested in working together, and they expressed openness to the possibility. We proceeded to write grant applications from the U.S. Department of Education's Fund for the Improvement of Post-Secondary Education (FIPSE) and the National Science Foundation (NSF), and in fact received both grants. These were especially helpful in giving us course relief to work on the course materials and in forcing us to develop a plan for evaluating our work. They also require us to form an Advisory Committee of business faculty, which was extremely helpful throughout the process.

In pursuing our discussions with the Clemson authors, we realised that our concepts were too different to work together, and so we struck off on our own. Several publishers expressed interest in our project, and we signed on with Prentice Hall, who also gave us a grant to fund laptop computers and other hardware and software that were all very helpful in the development process.

The first year of our FIPSE grant, the two of us taught experimental sections of the course, while everyone else continued with the traditional text and approach. During that first year, we gave several workshops for the math faculty, explaining our concept for the course, teaching the technology on the graphing calculator and using spreadsheets, and going over the processes of real world problem solving, math modelling, and student-generated projects. The summer after that first year, we led the first of many annual summer workshops for faculty from Villanova and other colleges and universities, and videotaped them. Because of all this faculty development, we were able to offer the new course in *all* sections during the second year of our grant. In fact, we wanted to keep a few sections using the traditional approach for evaluation purposes, but the business school wouldn't *let* us! They liked the new course so much, they didn't want to *deprive* any students from being able to take it. We were sorry in some ways to not be able to complete our evaluation as planned, but on the other hand, the strong endorsement was a form of evaluation in itself.

Our approach was radically different from before. To follow the spirit of being problem-driven, we even changed the pedagogy. Instead of a deductive approach teaching abstract concepts, then numerical examples, then simplistic applications (if there was time), we use an inductive approach giving realistic problems to motive material, then working with specific numerical examples using students' intuition, then generalising the patterns to present concepts. We want to give the students an intuitive conceptual understanding of the material. They should be able to solve simplistic problems by hand to understand the processes, but then be able to use technology to solve real problems.

Because of these radical changes, we had initially expected resistance from some of our faculty. It never materialised. Everyone who came to our workshops was extremely enthusiastic, and no one else in the department ever objected. Initially, some wanted to make sure the mathematical level of rigor was adequate. Many of them later said that they believe the students of this new course understood multivariable calculus better than the students in our engineering calculus, and have brought over some of our concepts to those courses. The first year that all sections were using the new approach, we had weekly voluntary discussions about the course. These were a wonderful experience, many of us discussing teaching together for the first time in a significant and regular way. Everyone commented on how much more lively the students were in class, and how much more satisfying it was to teach.

Results and Conclusions

Based on our statistical evaluation results, where students were randomly placed into control (traditional) and experimental sections, students learned significantly more of what the business school wanted. Based on 19-question pre- and post-tests they helped us construct, the experimental group scored about 5 points higher on average out of 19 points, with $p = 0.01$. Furthermore, the students rated our course sequence as significantly more relevant to their other courses, their careers, and their personal lives (the experimental group rating each category about

0.5 points higher or more on a 5-point Likert scale, all with $p < 0.04$). Instead of asking "When will I ever *use* this?" students now often say "I never knew that math could be *useful* before!" Our faculty enjoy teaching the course much more than before, although the student-generated projects can take more time to grade. Our business school has held up our project as a model for other curriculum reform efforts, and has been extremely supportive and enthusiastic.

What are the lessons for trying to maintain requirements and service courses in math departments? One is to **be on the lookout** for feelings of dissatisfaction with a course. If a course is getting stale, something is needed to revitalise it. **Get together with the appropriate client disciplines**, and assess what is working and what is not. **Determine the needs** of the client discipline (these can and do change over time, especially as technology changes), and **reconcile these with maintaining mathematical integrity** ***together***. **Look for existing texts** and materials to meet the needs and goals of the course, and **if necessary create your own**. If you need to create your own materials, **look for grant support** to be able to do it well. You cannot *over*do **faculty development** to train people for curriculum changes. Finally, **evaluate and monitor your results** and **maintain communication** with all concerned.

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INVERSE NUMERICAL FUNCTION CONCEPT AND COMPUTER SOFTWARE LEARNING
Or the necessity of distinguishing between D# and Eb

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ABSTRACT

We are in the context of a course on hypothesis tests and computing for French students taking plant biotechnology option for a professional degree.

It reports on a study in this context about statistical and numerical concepts and the tools used to determine them.

The question raised in this paper is:

how do the teacher's instructions to systematically use the graphic frame for determining the critical threshold with a table or with a spreadsheet, and the rejection area, explained as such, affect the students' work?

Keywords: obstacles, spreadsheet, inverse numerical function

1. Introduction

During the testing of a hypothesis or when looking for the critical value of a given risk threshold, two functions are involved: the random variable which follows a known distribution for the zero hypothesis and the inverse of the distribution function. Students appear to see the function for a given distribution as a 'black box' which transforms real numbers (values of the random variable) into probabilities which are also real numbers taken between 0 and 1. In this case, difficulties in doing from discrete to continuous representations (a point becomes a surface area) are encountered (Schneider 1991).

This paper recounts part of a study of phenomena produced when teaching the construction of statistical tests to third year university students.

2. Theoretical context

The work combines the following theoretical frameworks :

1 – The cognitive approach to instruments developed by Pierre Rabardel (Rabardel 1995). This especially involves investigating the instrumentation[1] of the spreadsheet for determining the significant value in a hypothesis test.

2 – The didactical engineering framework developed by Michèle Artigue (Artigue 1990). There is a dialectic between students' availability spreadsheets and their mathematical knowledge.

3. Discussion

The work is based on the hypothesis (developed during the author's work on her doctoral thesis) of the importance of teachers reflecting on and structuring knowledge discourses for learning purposes. The work is thus part of a wider study of meaningful reflection in order to clarify the concept of a probability distribution. An investigation of the kinds of obstacles[2] that have been encountered by students studing the Plant biotechnology leading to a professional university degree, gave the following main results:

-mathematical obstacles: there is some confusion between the availability of the direct function and the inverse numerical function.

-Didactical obstacles: some students do not use the trial and error possibilities offered by spreadsheet software.

-Computing obstacles: there is some confusion between the Gauss law inverts and the Gauss law spreadsheet function. Some students cannot manage without the teacher's help.

The conclusion was that it seems students are more likely to accomplish the task if they both have some theoretical knowledge **and** use the trial and error strategy.

A special course on this difficulty was designed to clarify the 'confusion' phenomenon when teaching the distribution function and its inverse function (both in mathematics and with the spreadsheet).

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[1] Instrumentation is the aspect of the instrumental genesis process when the subject is developing an instrument. The instrument is assumed to consist of two elements: an artefact and one or several corresponding user 'schèmes'

[2] Chiocca 2002.

The question raised in this paper is:

how do the teacher's instructions to systematically use the graphic frame for determining the critical threshold with a table or with a spreadsheet, and the rejection area, explained as such, affect the students' work?

4. Observation protocol and description of the research setting

The experimental situation is build to enable observation of a few schemes for spreadsheet software, taken as an artefact. A clinical, rather than a statistical approach, to the experiment and the observation was chosen as this would provide a sufficiently rich data set to reveal the students' errors.

The study used the students' final examination work (paper and diskette), video recordings and transcriptions of these recording. The data collection protocol enabled us to focus on tasks and techniques (Chevallard 1992). The filmed session completed the analysis of tasks and techniques used by students.

The population: consisted of 12 university students studing Plant biotechnologyat the universitéy level. The students have a school graduate and two years of higher education. They are on average 22 years old. They had previous experience of applying inferential statistics during their studies. For two years they had been steeped in conformance and adjustment tests and variance analysis. Since the official instructions of the Ministry of Agriculture excluded teaching theoretical developments, they had no knowledge of statistical theory and only a little of probability. They will have to use such statistical tests in their professional work, which for most of them will begin as soon as they have obtained their degree (95% of successfully qualified students find work within one year). They were thus very keen to learn how to use spreadsheet software for implementing the test.

5. The experimental situation

1 – *Elements of didactical engineering*

The teaching sessions 18 hours in all, were divided between work in the classroom using the tables found in most statistics manuals, and work in the computer room using spreadsheet features which appear to eliminate the need for paper tables.

first, experimental exercises with paper, pencil and spreadsheet software on the distribution law were given to the students to work on direct and reverse reading of a paper table and also on the availability of spreadsheet functions: Student's law and Student's law inverts.

When constructing a test of a hypothesis, the diagrams representing the density curves of usual laws, the abscissa of critical thresholds for a given probability (a surface area beneath a curve) and reject areas (IR intervals) are institutionalised[3].

2 – *Task*

The task to be accomplished by the student was as follows:

The examination consisted of four test constructions: two tests for comparing means with the same variance values and two independence tests using two random variables.

The researcher (teacher) asks the students to draw the reject areas for the zero hypothesis. The point was to see if they could produce such a drawing after 18 hours of class.

[3] Cf. the institutionalised diagram are shown in appendix

The students were allowed to use any documents they wanted, plus a pocket calculator, their course notes, statistical manuals and particularly the two artefacts: paper tables and the spreadsheet software.

3 - *A priori* obstacles

Mathematical:

The numbers written in the cells of the paper table are probabilities, in other words, surface areas under the density curve and are thus obtained by means of the distribution function. Given the probability, they are then told to look for the abscissa value. The work involves using the inverse function.

The risk threshold is the image obtained by the distribution function of a certain value called the critical threshold, which is thus the antecedent obtained, by the distribution function for the risk threshold. Learning these two concepts (antecedent, image, for a numerical function) is difficult. In France, students are taught the concept of image and antecedent for a numerical function three years before the final leaving examination and it is not explicitly reviewed in the official programme for later classes. However, teachers do review these concepts even if they do not do any explicit work on them.

Didactical:

Several critical values had been calculated when students were learning how to construct a test. Difficulties were encountered when, in the same session, the direct function was used (for instance when calculating the actual risk) and its reciprocal function (for finding the critical values which determined the reject area). It is thus likely that the same difficulty will occur for the learning task. Arguments frequently used in favour of spreadsheet software are that it eliminates the need for paper tables. A table is no longer necessary for finding a critical value and a reject area. But, without the paper table, there is no longer any graphic representation.

Since students are only able to use a little probability and statistical theory[4], assuming that they have any[5], it is difficult to explain meaningfully the search for reject areas and critical thresholds.

The word *zone* in French usually represents a surface area rather than a segment. It is thus likely to cause confusion between the reject area and the risk threshold (an interval or a set of IR intervals and probability, area under the curve). In English, the use of the word area would probably cause even more confusion.

Computing:

The spreadsheet help button provides instructions but no graphic representation.

In French, the spreadsheet specifies that LOI.STUDENT.INVERSE (the same for LOI.NORMALE.INVERSE) returns the value of a random variable according to Student's law as a function of the probability and the number of degrees of freedom. In French, the value of a random variable (which is a numerical function) corresponds to the numerical value taken by the random variable as a numerical function (for a continuous law) and not to the antecedent of a value for the distribution function.

The LOI.STUDENT INVERSE function requires giving the total risk even in the case of a bilateral test.

The names of laws or their reciprocal laws are not homogenous. The spreadsheet offers KHIDEUX.INVERSE and INVERSE.LOIF and not LOI. KHIDEUX.INVERSE and LOIF.INVERSE for the same model as LOI.NORMALE. INVERSE or LOI.STUDENT.INVERSE.

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[4] As used by Aline Robert: a knowledge, which can be used, is knowledge that a student can look for to use without it being suggested explicitly or implicitly.

[5] As used by Aline Robert: available knowledge is knowledge, which a student knows how to use if told that it is that knowledge which he needs to solve the problem.

6. Some results

Depending on the individuals, the task took at the least half an hour to solve with no assistance from another person.

The students produced two sorts of diagrams: a category of 'right' diagrams and another showing that the risk threshold and the critical threshold are confused. In other words, for image and antecedent by means of the distribution function, 6 out of the 12 students suggested false diagrams, 4 of them 'right' diagrams, in other words, diagrams which matched the expectations of the teacher-researcher, 2 of them suggested no diagrams in spite of the explicit instructions of the examination. The two students who did not produce any diagrams only attended 6 out of the 18 course hours. One may conclude that they did not spend enough time in the course to understand the didactical contract concerning these diagrams.

All of the students found the right value for the critical threshold, 10 out of 12 students found it by reading the paper table (by reading it backwards is what they said) whether the diagrams were right or wrong. One of the two students who used the spreadsheet made a false diagram, the other no diagram. Is this because the spreadsheet, while eliminating the need for the paper table, blocks out the graphic diagram at the same time?

The student who did not make any diagram appears to be in some kind of transition with respect to the use of the two artefacts: paper table and spreadsheet. Indeed, in the spreadsheet, he displayed the critical value by using the Student's law inverts function while at the same time specifying 'after reverse reading in the table'.

7. Concluding comments

Systematic use by the teacher of the graphic frame for determining critical thresholds and reject areas in this didactical framework, leads to one out of two students producing false diagrams.

On the other hand, most of the students were not able to use the spreadsheet features to look for critical thresholds.

There is a need to work on the concepts of antecedents and image. In the 'succession' of functions between the random variable and the distribution function, a distinction should be made between the image by the random variable and the antecedent by the distribution function. This would then make it easier to understand when distinguishing the real risk from the critical threshold.

These two conceps could be compared to the concept of coma in music: on certain instruments such as the violin, D# et E*b* are not confused.

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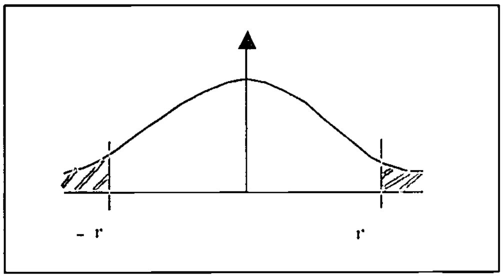
Appendix

Figure: bilateral test



The most frequent error consists of writing the following comment:
the rejected area is in the shaded area.

CONCEPT MAPPING AS EVALUATION TOOL IN MATHEMATICS

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ABSTRACT

In this study we initially discuss in brief some theoretical issues related to the notion of concept mapping, a scoring rubric for concept mapping assessment and a rationale for its design. Then, we examine two groups of eleventh grade students from a public school, who were taught the same textbook of mathematics, although they were targeting the entrance to different university schools and we investigate whether :

1.There is a difference on performance between these two groups of students in conventional written tests.

2. There is a difference between the cognitive structures of these groups concerning PMI.

3. Students' misconceptions in maths are clarified by the process of concept mapping.

4. There is a correlation between concept mapping ability of students and their performance in mathematical achievement test exists.

Finally we present the outcomes of our research which provide evidence that concept mapping is an essential supplementary tool for the evaluation in mathematics.

Keywords : Concept maps, evaluation, scoring rubric system, PMI.

Introduction

The teaching of mathematics must contribute to the development of both procedural and the conceptual knowledge. On the other hand, good learning of mathematics requires not only the knowledge of the different procedures and concepts of the subject matter, but also of the adequate relations among such concepts which lead to the construction of the right mathematical meaning.

In order to achieve a teaching that conduces to the desired learning we seek for a didactic methodology and a theoretical setting provided by educational psychology.

Behaviorism learning theory (Thorndike,1922; Watson,1970; Skinner,1974; Wittrock,1984) was focused on the presentation of the information and its transfer from the teacher to the learner with the latter seen as an empty vessel to be filled with Knowledge. However, the transfer of knowledge in the bipolar model teacher – learner did not work and the problem of inadequate learning remained.

Facing this situation cognitive scientists focused on the learners' side and formulated the contemporary *Constructivism theory* (Piaget,1959 ; Vygotsky, 1978 ; Classersfeld,1995 ; Cobb &Yackel,1998) whose main assumption is that Knowledge does not exist in an objective reality and is actively constructed from within by the learner.

The Constructivist model has been widely accepted aiming at the conceptual understanding for which Kinnear(1994) says : "Conceptual understanding is influenced by the prior knowledge brought by students to learning situations. This prior knowledge is labeled as *preconceptions, naive theories, alternative frameworks* or *misconceptions* " (p.6)

For constructivism, goals of instruction are, deep understanding and concept development and not behaviors or skills (Fosnot,1996). Accordingly teachers must aim to "establish explicit linkages for students between new information taught in class and students' past and future experiences... summarize, review and link main concepts at critical points through and at the conclusion of units and lessons'' (Ennis,1994,p.167).

Within this framework , it is very important that teachers know in the beginning and after a course cycle, whose dimensions are laid down by the curriculum, their students' conceptions about the subject matter of the instruction in order to design the appropriate activities for a conceptual change. The usual practice for students' knowledge assessment are conventional tests. These tests are perhaps suitable for the assessment of behaviorist skills, such as rules, formulas and algorithms, which concern the procedural knowledge(*) but they are not functional for the students' conceptual structure detection on a certain topic. Conceptual knowledge, generally called declarative knowledge, is the knowledge of facts, the meanings of symbols and the concepts and principles (Posner,1978) of a particular field of mathematics. It demands a conscious effort from both students and teachers and in this direction Steffe(1990) points out the need for a curriculum design as a network of mathematical concepts and operations that could deepen, unify and extend conceptions of mathematics.

Research suggests that understanding can be viewed as a connection between two pieces of information (Ginsburg,1977) and the degree of a student's understanding is determined by the number, accuracy and strength of connections (Hiebert & Carpender,1992)

A very useful tool for explicitly stressing mathematical connections is concept mapping. Concept mapping (Novak & Gowin,1984 ; Novak,1990) is a visual representation of an individual's knowledge structure on a particular topic. This representation takes the form of a finite graph with nodes that depict the mathematical concepts and links (lines or arcs) which in turn represent the relationships among them. Crosslinks are links that merge subnodes. Nodes, subnodes, links and cross links are labelled and arrows can be placed on the linking lines to indicate the direction of the relationship between concepts. Two nodes with the labeled link in a concept map are called propositions. Basic attributes of concept mapping according to Novak (Novak & Gowin, 1984) are: Hierarchy, Progressive Differentiation and Integrative Reconciliation. In sum "concepts maps are two-dimensional representations of cognitive structures showing the hierarchies and the interconnections of concepts involved in a discipline" (Martin,1994,p.11).

Concept maps are used to evaluate how students organize their knowledge and give an observable record of their understanding. Several researchers like Ausubel(1968), Novak & Gowin(1984),

Malone & Deckers (1984), Markham and Mintzes(1994), McClure et al(1999) have recognized the advantages of this form of information presentation and have used concept mapping strategies in order to see how the individuals structure their knowledge as the subject matter.

Theoretical background

The theoretical background of concept mapping refers to constructivist epistemology which was briefly mentioned above and in Ausubel's theory of meaningful learning that involves the assimilation of new concepts and propositions into existing cognitive structures. The cognitive scientist Ausubel(1966) distinguished meaningful vs rote learning and developed the *Meaningful*

(*) This definition of procedural knowledge refers to Cohen(1983)

Learning or *assimilation* theory. *Meaningful Learning* occurs when :

- New knowledge is integrated into the existing network of concepts and propositions in the cognitive structure.
- New knowledge incorporates into specifically relevant existing concepts or propositions
- There is the ability of explicit delineation of similarities and differences between related ideas

On the contrary, *Rote Learning* occurs with the arbitrary verbatim incorporation of new information into cognitive structures.

According to the meaningful learning theory, students obtain successful learning by establishing relations between the new concepts to be learned and the ones they already grasp. Prior knowledge is of great importance and Ausubel(1978) underlines that : *"If I had to reduce all of educational psychology to just one principle .I would say this : The most important single factor influencing learning is what the learner already knows" (p.163).*

Another support of concept mapping originates from systemic theory which asserts that meanings and concepts are not sums but organised physical systems of behaviours (Paritsis,1986 ; Dekleris,1986) and similarly confirmed by the association memory theory (Deese,1995).

Finally, neurobiologists' researches into the function of human brain, emphasize the important part of links and the connection with the conceptions, images and meanings (Changeaux,1988 ; Posner & Raichle,1994)

The use of concept maps has been founded in the suggestion that their structure parallels the human cognitive structure, as they show how learners organize concepts. Since we can not have a direct view of our cognitive structure, we use indirect methods as their indicators. One of these indicators is concept maps which researchers interpret as measures of this cognitive structure (Novak & Gowin,1984 ; Fisher et al,1990 ; Wandersee,1990 ; Lederman & Latz ,1995).The more meaningful connections an individual can put on a map ,the better understands the subject matter.

Objectives of the study

To my knowledge, there is little published work on concept mapping in mathematics, less in formal concepts and especially in Greek publications rather none. Thus, this study was organized on the basis of the following objectives :

1. To investigate whether a difference on students' performance exists in conventional written tests between two groups of students who attend the same advanced mathematical eleventh grade school program but they are targeting the entrance to different university schools.

2. To investigate whether a difference between the cognitive structures of groups concerning PMI exists.

3. To investigate whether students' misconceptions in maths are clarified by the process of concept mapping.

4. To investigate whether a correlation between concept mapping ability of students and their performance in mathematical achievement test exists.

Methodology

In this study the subject matter "Principle of Mathematical Induction" (PMI) was chosen. This concept constitutes a part of teaching material both in eleventh grade secondary school and in mathematics oriented courses in universities. Two eleventh grade classes which attended the unit for PMI from the same textbook were picked out for a week. The students received a training on how to construct a concept map in order to become familiar with this technique. In a week's period, one hour per day, a series of concept mapping examples were presented to the students and they constructed their own paper and pencil based maps.

The sample

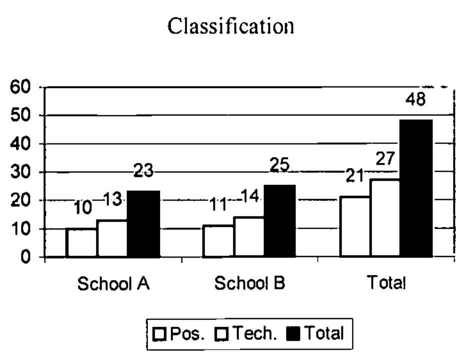


Figure 1

The sample for this study was comprised by forty eight secondary eleventh grade students from two different public schools. (Figure 1) Twenty one of them were targeting the entrance to university departments with advanced courses of mathematics *(Positive group)* and twenty seven of them were targeting the entrance to polytechnic schools *(Technological group)*.

Instrumentation

- After instruction in PMI, data were gathered with the following assessment tools :
    1. The conventional written tests.
    2. The concept maps that students constructed with the Key-Concept List method.
- ***Concept map scoring rubric***

The evaluation of the concept maps has been carried out by using both quantitative and qualitative methods. For the quantitative assessment, a scoring rubric (S.R) was constructed attempting a synthesis of three concept map scoring systems. Essentially, the *Relational Scoring System* (R.S.S) or scoring system for a concept network (McClure & Bell,1990 ; McClure et al.,1999) was employed (Appendix I). With this method, three parts of the proposition are scored :

a) The existence of a relation between the concepts

b) The accuracy of the label

c) The direction of the arrow indicating either a hierarchical or causal relation between the concepts.

In this method raters score individual maps by evaluating the separate propositions identified on the map. The score for each proposition ranges from zero to three in accordance with a scoring protocol (Appendix.I) that considers the correctness of the proposition.

This scoring system was modified as follows :

1. More points were assigned for *branchings* (Markham et al, 1994). One point was assigned to the first branching and one to three points for each successive branching depending on the differentiation level.

2. According to *structural scoring system* (S.S.S) (Novak & Gowin,1984 ; Novak,1990), more points were assigned in this way :

   One point for each valid *concept*. Zero to three points to each *cross link* as proposition (R.S.S) and two to five more points depending on the significance of linked domains. Two more points were assigned for each valid example and up to two examples were used.

The total score for each map is the sum of the above scores.

In addition taking into consideration the *comparison* rule (Novak & Gowin,1984) :

1. The 'master' or criterion map for PMI was rated with the above rules.

2. The student's map score was divided by the master map score in order to give a percentage for comparison.

- ***Rationale in designing the rubric***

Typically, the Novakian S.S.S. is used to evaluate maps. However, this system coming from biology is limited to hierarchical maps. Mathematics in eleventh and twelfth grades as well as, in mathematics oriented university departments, deal with formal concepts which are identified by their connections with other already known concepts like a network. There is no 'hands on' familiarity with these concepts resulting in a laborious effort in concept mapping. Besides, as Primo & Shavelson (1996) point out, imposing a hierarchical structure regardless of the content domain, is inadequate because an accurate concept map representation of hierarchical domain will be hierarchical itself.

Thus, we gave priority in propositions like R.S.S, which emphasizes networks and takes into consideration the hierarchy, as it appears in propositions.

The epistemological background for S.R. system and specifically the differentiation and integration of concepts, are founded in the Ausebelian Meaningful Learning and the Constructivist epistemology as previously noted.

Procedure

In order to avoid the influence of teacher's parameter the classes were taught by the same teacher.

After the instruction of PMI was completed the subsequent procedure was followed :

Students, with a consensus level about 0.9, generated the following list for the most important elements in PMI concept :

PMI, MMI (Method of Mathematical Induction), Simple induction, P(1), P(n)⇒P(n+1),One step, Two steps, Three steps, Infinite steps, Least element, Natural numbers (N),Ordering, Axiom, Theorem.*

The following day, a university professor, a secondary school advisor in mathematics and two experienced secondary school teachers of mathematics, taking into consideration the textbook in the domain of PMI and students' above list, created the Key-Concept List and the master – criterion map for PMI. (Appendix.II). The consensus level of the Key-Concept List was about 0.95 and comprised one more concept concerning the students' list : *The M.Ponens.*

A day after, students took a two hour conventional closed type written test which is the usual practice in Greek school examinations and generated in one more hour a concept map with the above Key-Concept List. The validity of the tests is ensured because of the analysis of PMI as the subject matter

Results and discussion

The four developers of the Key-Concept List and the criterion map, rated the written tests and the concept maps. The data were analysed separately with SPSS using t-test, ANOVA test and Pearson correlation coefficients.

A. *Conventional tests*

The Pos.Group was more homogeneous than Tech.Group since the range of scores in written tests were 9 vs 15. The average score for the Pos.Group was 13,52 vs 11,66 of the Tech.Group which means that the Pos.Group scored better than the Tech.Group (Figure 2). However, the Pos.Group's performance was not significantly better, since the t-test resulted in the comparison of means t-value = -1,905, which was not significant at 0.05 level (P-value = 0,063).

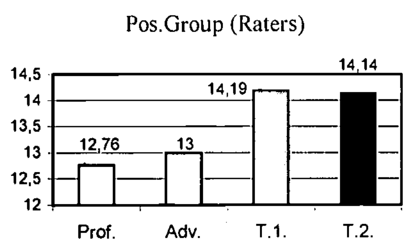


Figure 2

The secondary school teachers (T_1,T_2) rated higher than the professor and the advisor, but the differences between the raters were not significant at 0,05 level. For example, the ANOVA test results for the rating of the Pos.Group, were : F =1,55, P-value=0,209.

B. *Concept mapping*

The maps were rated according to the designed scoring rubric system, separately for the five different attributes of the concept maps (Concepts, Propositions, Cross links, Branching , Examples).Then, total scores were assigned which were obtained by summing the weighted partial scores and finally percentages were computed for comparisons, taking into consideration the total score of the criterion map on PMI. The secondary school teachers rated lower than the professor and the advisor, but again the differences between the raters were not significant at 0,05 level.

1. Attributes' scores

The Pos.Group scored higher than the Tech.Group. The presentation of the groups' learning profile in a graphical format (Figure 3) which depicts the scores of the various attributes, reveals that :

The Pos.group uses more concepts from the Key-List and makes more valid connections within these concepts than the Tech.group Thus insertion provides an indication that the Tech.group had a greater difficulty than the Pos.group in :

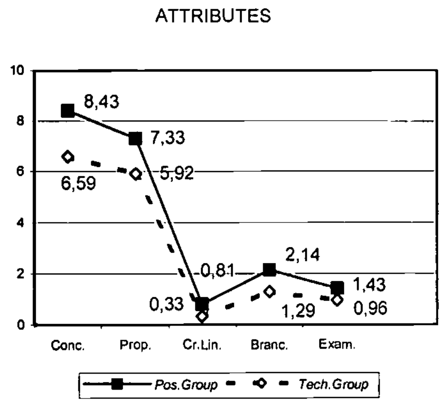


Figure 3

- Recognizing the important terms connected with PMI
- Recognizing, denoting and signifying the relations – links within concepts of PMI.

This difficulty is interpreted as a lower ability of the Tech.group to differentiate component concepts of PMI.

Further more, the lower score for branchings and crosslinks indicate a lower progressive differentiation and a lower knowledge integration in PMI as for the Tech.group in comparison with the Pos.group. The qualitative analysis of students' concept maps confirms this indication, since students tend to identify PMI as MMI and similarly the Simple Induction.

2. Total scores

The average in concept mapping scoring was 41,43 for the Pos.group vs 29,3 the Tech.group which means that the Pos.group performs better than the Tech.group in the task of concept mapping in PMI. This is indicated by the t-test which resulted in t=4,633 which is significant at 0,01 level (P-value = 0,000). Consequently the Pos.group has a significantly better understanding of PMI. The Professor and the Advisor rated higher than the secondary school teachers but differences between the raters, as in the conventional tests assessment, were not significant at 0,05 level. It can be suggested that these differences are related to the concept mapping scoring familiarity.

3. Criterion map and percentages

In applying the scoring rubric, the total score for the criterion map (Figure 4) is 100 and the percentages vary from 23%-53% with an average of 41,43% for the Pos.group and from 17%-50% with an average of 29,3% for the Tech.group. Although the criterion map was constructed from concepts which the students selected, considering them to be the most important for PMI, it is lineament of students' inability to grasp the meaning of PMI the fact that they score much lower than 50% with regard to the criterion map score.

C. *Detection of correlation between written tests and concept mapping*

The Pearson's correlation coefficients between written tests and concept mapping are low for both the Tech.group and the Pos.group. In particular, this coefficient is 0,022 for the Tech.group/Tests and 0,408 for the Pos.group/Tests. These coefficients indicate that there is insignificant correlation for Tech.group and a weak one for Pos.group. Nevertheless, the results are consistent with studies which examined a similar correlation. For example, McClure and Bell(1990) reported correlations about 0,50 between concept map scores and the final examination score (Science) and Novak, Gowin and Johansen(1983) found that relative correlation was 0.02 (Maths).

Generally, the students had great difficulty with the PMI concept as it was revealed by their concept maps. The written conventional tests provided evidence that the students :

1. Identified PMI as MMI
2. Believed that MMI was identical with the simple induction
3. Could not differentiate that $P(n) \Rightarrow P(n+1)$ is one proposition

This evidence was confirmed by the concept mapping procedure.

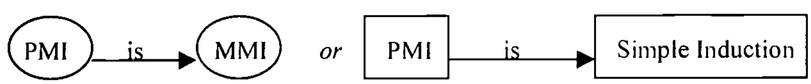
Conclusions and implications

From the above results and in relation to the research questions we set, it could be concluded that :

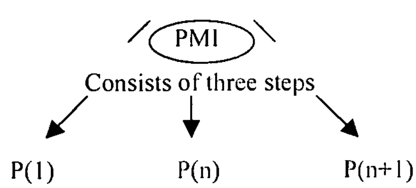
1.The students of the Pos.group performed better than the ones in the Tech.group in conventional tests but were not significantly better (a=0,05). Both groups scored in average above 50% of the scoring scale. Although the exercises in written tests regarding PMI were in the type "Show that" and required the mechanical implementation of mathematical's induction steps, the students did not score high. This is an indication of the difficulties they faced with the PMI concept.

2. The usual written tests detect mainly the procedural knowledge. Involving the concept mapping tool in the assessment tasks revealed, that :

- Both groups scored below 50%. However, the Pos.group performed better than the Tech.group in concept mapping and the difference was significant in favour of the Pos.group (0,05). Specifically the range of differences between the groups was greater when the students used the concept maps than when they used the conventional written tests.
- The students had great difficulty in making connections among the PMI concept components.
- The majority of the students failed to differentiate well the Key-List concepts, as there were indicative findings like :



- The majority exhibited the misconception that $P(n) \Rightarrow P(n+1)$ is not a single proposition. This was indicated in their concept maps by shapes like :



3. There was no substantial correlation between written tests and concept maps. This was interpreted as evidence that :

- Conventional tests can not differentiate well between procedural and conceptual knowledge

• Concept mapping is an essential supplement of conventional tests which reveals a different view of students' cognitive structures

Our sample of course, is small to generalize under a quantitative approach. However the outcomes encourage the use of the concept mapping technique and provide evidence that it is not only a useful but also a necessary supplementary tool for the evaluation in mathematics. Besides, although the designed scoring rubric (S.R) for concept mapping in mathematics must be further tested for reliability and validity, it seems that S.R. is effective as for concept mapping in secondary mathematics education and we strongly suggest its implementation in undergraduate mathematics courses.

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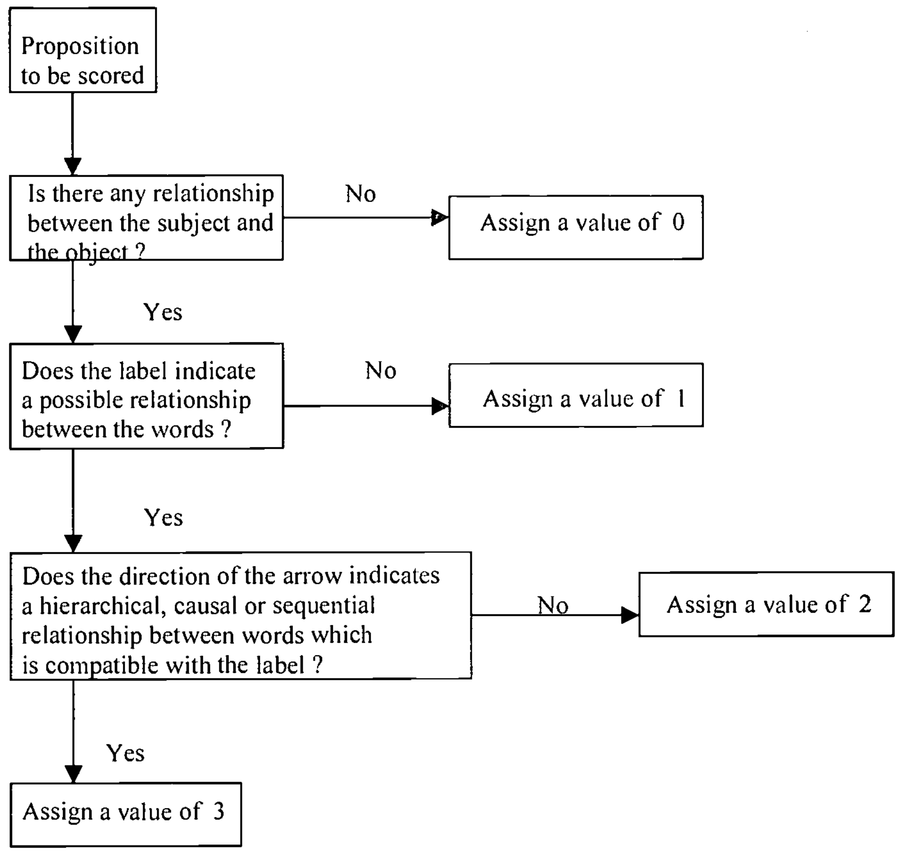
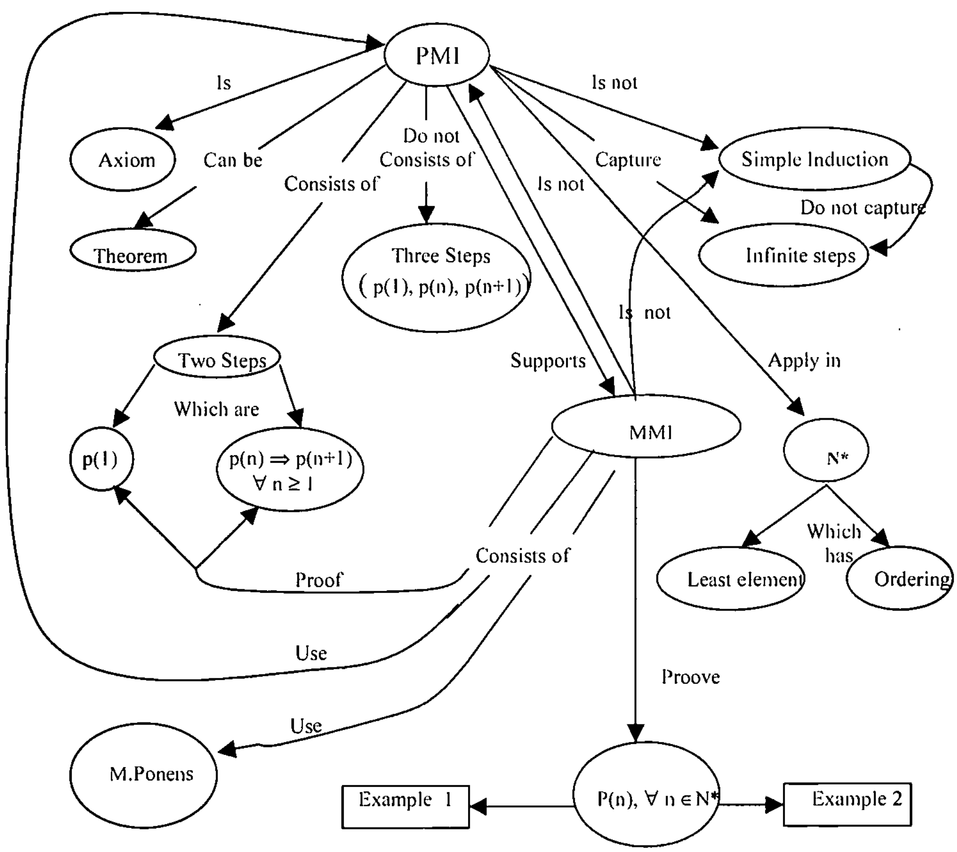
Appendix.I.



Protocol for the Relational Scoring System (R.S.S)

Appendix.II



Master-criterion map (Total score: 100)

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ABSTRACT

The definite integral is a major topic in Calculus with many student difficulties. In [6], we have traced the development of understanding as it progresses and found a curious phenomenon. When unable to proceed along a particular schema, students introduce a heuristic that helps them bridge the gaps in their understanding. In this particular situation such a gap is filled a change of a unit from that of a rectangle to that of a line segment of 0 width, the indivisible. They follow with the computation involving the sum of heights of infinitely many line segments to obtain the area under a curve - the definite integral. We suggest approaches to channel their thinking - a guided heuristic that confronts students with concrete physical scenarios where similar manner of reasoning leads to a contradiction. Using Zeno's paradoxes of the race between Achilles and the tortoise, we begin the process of introducing students to a directed heuristic. We follow with the mathematical context, using a construction by John Wallis [14], to provide a mathematical framework within which the intuition of indivisibles can connect with the notion of the area of two dimensional regions.

1. Introduction

The literature abounds with student difficulties on the Calculus topics of limits and definite integrals [4, 6, 7, 8, 11, 13]. Difficulties with limits in fact manifest themselves in topics such as the definite integral, the derivative of a function, etc. We will focus our attention in this paper on the students' difficulties with respect to one key concept, viz., that of estimating the area under an irregular curve by using the method of Riemann sums. The reason for the focus on the definite integral is an interesting phenomenon observed by us in [6], which is briefly described as follows.

In addition to the standard idea held by some students of approximating the area by sums of rectangles inscribed under a given curve, other students also saw the area as a sum of line-segments from the abscissa to the curve. More precisely, instead of seeing the area as a limit of the process of taking partial Riemann sums, they see it as a sum of the ordinates of the function in question. A historical study of the concept of the area under a curve reveals that the image held by the students corresponds in its general outline to the viewpoint presented by Archimedes in *The Method*, Cavalieri [3] in the *Geometria Indivisibilibus* and John Wallis in *Arithmetica Infinitorium* [14]. It is fascinating that this intuition of area has persisted on its own, without formal instruction, till the present day.

The method used by students' is their "heuristic" just as it was Archimedes' heuristic. A *heuristic* in our context is an approach used by students that makes sense to them. This approach need not necessarily be the way taught in class, nor need be mathematically precise but from the point of view of the student it is a way to see the solution to the problem under consideration. The dictionary meaning of a heuristic is "relating to exploratory problem-solving techniques that utilize self-educating techniques (as the evaluation of feedback) to improve performance".

Heuristics are commonly used in computer science, sometimes as the only solution to an existing problem. A heuristic solution is used in practical problems when no other known solution exists, or when a complete or more exhaustive search is too expensive. In the embracing of mathematics education reform in the past ten years however, heuristics have played a very important role. Most of the Computer Algebra Systems used in almost all "reform" mathematics classrooms are an outgrowth of the heuristic principle. Heuristics have been used by mathematicians since or before the time of Archimedes. It was Archimedes' heuristic proof that led him to the more rigorous mathematical proof of finding the area under a parabola [1].

Now, in the situation under consideration namely that of finding the area of a region by means of the method of Riemann sums students betray an interesting confusion while explaining the construction (excepts below), the confusion between two different types of "units" to measure that area, the rectangle and a line segment. A similar kind of confusion, is responsible according to Grunbaum [15], for creating the Achilles and Tortoise paradox of Zeno.

In general, it is difficult for students to notice the fact that they have indeed confused two entirely different quantities, when they are actually carrying out the process, or when they reanalyze their work. However, if the situation is posed to them in terms of a physical scenario such as a race between Achilles and a tortoise, clarifying the confusion becomes much easier.

2. Statement of the problem.

Consider the student excerpts below.

(In the excerpts below, I indicates an interviewer, and C, D are students)

Excerpt 1

[1] I: How do you go from a Riemann sum to make it equal to the 2/3 we got here?

[2]C: Make these rectangles infinitely small, smaller, smaller and smaller, I mean almost until
[3] they're a line, they're a unit and then you are just adding up these units and the smaller—this
[4] empty area is the more exact estimation until you get to a point where there is no empty space
[5] to be accounted for that gives an exact number.

Excerpt 2

[6] I: How would you get the closest . . .

[7] D: Um...

[8] I: . . . possible area?

[9] D: Closest possible area would be by taking the length of a line segment from the x axis to the

[10] function itself. And that would give you an infinitely many ...and many areas to add up. And

[11] that's what the definite integral gives you. It just allows you, you know, to be able to work **[12]** with basically a rectangle with no width, just height.

In the first excerpt, the student starts from the interval on which the function is defined and progressively reduces the width of the subintervals, and suddenly makes a jump to an entirely different unit, the line segment without width, or the ordinate of the function. That corresponds to the conceptual jump between two different intuitions, from the continuous one to the discrete one (lines 2, 3). This student conducts the process of infinite subdivision. The means to carry out this infinite subdivision starts out by using the rectangles as "units", however, a conceptual jump occurs, after which the units used are the line segments or ordinates of the function. The aim, of course, is to reduce the error between the estimated area and the actual area. The conceptual jump is not apparent to the student at the time it is carried out or later. In the world of paradoxes, an analogous situation is the race between Achilles and the tortoise.

In the second excerpt above, the student, also in an attempt to reduce the error between the actual area and the estimation, starts out by explaining that the error would be reduced by taking the infinite sum of the heights of line segments to get a two-dimensional area. Here, the summation of units of one kind, namely the line segments of 0 width are assumed to generate area of a two dimensional region.

We build our instruction based on the heuristic created by students to guide them toward a solution that is in agreement with their heuristic but is at the same time mathematically rigorous.

Notice the problems of the student C. He departs from the Riemann construction, most probably because the student's grasp of the limit concept and of its role in the construction is weak. That leads him to abandon the summation of two dimensional units of rectangles to get two dimensional area and to use an inappropriate unit - the line segment or the indivisible. Next he sums these one dimensional units to get a two dimensional area. Therefore, on one hand, we have to discourage the jump between the different intuitions and we use the analysis of Achilles and

Tortoise for that purpose (Section 3). On the other hand, we have to provide the student with the mathematically correct path to express the process of infinite subdivision [16]. Finally, we can also provide the student with the mathematically correct path between the summation of the lines and the area under the curve using Wallis-Cavalieri construction [Section 4].

3. Two Paradoxes

Let's look at the steps taken by students. Students' construction could be stated in the following steps:

1. The region whose area is to be determined is irregular.
2. Fit regular shapes (rectangles) in the region.
3. Regular shapes result in over-fitting or under-fitting, and involve an error.
4. Reduce width of regular shapes.
5. Error is reduced but still exists.
6. To eliminate error, consider line segments to be the regular shape.
7. Add heights of regular shapes.
8. Sum of the heights is the area of the irregular region.

Students proceed to determine the area systematically until they arrive at the end of step 5. At this point the concept that they should use is the limit of the numerical sequence corresponding to their visual image (steps 3 and 4). However, instead of applying the limit, they resort to using line segments. It is step 6 above that results in a paradoxical situation, in which the spatial units of rectangles are replaced by the discrete units of line segments of 0 width - the indivisibles. The danger in inappropriate coordination of the continuous and discrete elements is well illustrated by

the Paradox of Achilles and the Tortoise, which states: Suppose Achilles runs ten times as fast as the tortoise and gives him a hundred yards start. In order to win the race Achilles must first make up for his initial handicap by running a hundred yards; but when he has done this and has reached the point where the tortoise started, the animal has had time to advance ten yards. While Achilles runs these ten yards, the tortoise gets one yard ahead; when Achilles has run this yard, the tortoise is a tenth of a yard ahead; and so on, without end. Achilles never catches the tortoise, because the tortoise always holds a lead, however small. But – as we know full well – Achilles will soon pass the tortoise. We agree here with Grunbaum who [15] locates the source of the paradox in the imposition of the discrete structure of locations of Tortoise onto the continuous structure of the motion of Achilles. For if we restate the problem in a familiar language of "word problems", saying, find the distance x from starting point and the time when Achilles passes the Tortoise, there is no paradox. Instead we solve the familiar equation $\frac{100+x}{V_A} = \frac{x}{V_T}$, where V_A is the speed of Achilles and V_T is the speed of the Tortoise. The only discrete element here, the moment when Achilles is passing the Tortoise, is not imposed from outside but is an intrinsic element of the structure of the problem. Thus, the importance of the proper conceptual framework to avoid the paradoxical situation becomes, we hope, apparent to the student.

4. Wallis - Cavalieri construction

Pedagogically, the discussion of the paradoxical elements inherent in students' reasoning has as its goal to warn them of possible conceptual and epistemological difficulties along the taken path. An alternate route is to provide (prior to treatment of the definite integral), students' familiarity with the concept of the limit of a sequence and by coordinating that concept clearly with the geometric construction of Riemann sums [16]. Here, on the other hand, we provide a mathematically correct framework in which a bridge can be established between the intuition of lines and areas of two dimensional regions.

The Wallis-Cavalieri technique [17] is a development based on the work of John Wallis, of a method to calculate the area bounded by $f(x)=x^2$ on [0,1] [14]:

We construct a sequence of the Wallis-Cavalieri ratios		which are decomposed by the Wallis method into the estimated limit and an additional term
$W_1 = \frac{0^2+1^2}{1^2+1^2} =$	$\frac{1}{2}$	$=\frac{1}{3}+\frac{1}{6}=\frac{1}{3}+\frac{1}{6\times 1}$
$W_2 = \frac{0^2+1^2+2^2}{2^2+2^2+2^2} =$	$\frac{5}{12}$	$=\frac{1}{3}+\frac{1}{12}=\frac{1}{3}+\frac{1}{6\times 2}$
....	....	....
$W_n = \frac{0^2+1^2+2^2+\ldots+n^2}{n^2+n^2+\ldots+n^2}$	$\frac{2n+1}{6n}$	$= \frac{1}{3}+\frac{1}{6\times n}$

Therefore $\lim W_n = \frac{1}{3}$ *as* $n \to \infty$ and is the area of the required region.

The ratios $W_n = \frac{0^2+1^2+2^2+\ldots+n^2}{n^2+n^2+\ldots+n^2}$ can be seen as resulting from the expressions $\frac{\sum_{i=0}^{i=n}(\frac{i}{n})^2}{n+1}$ where the numerator represents the sum of squares of the ordinates of the function at n+1 points, while the denominator represents the sum of n+1 corresponding coordinates of the function $g(x) = f(1) = 1$ (recall that $f(x) = x^2$). Thus visually, the formula represents the ratio of the sum of n+1 lines under the curve to the sum of the corresponding lines in a circumscribing unit square.

In a slightly more general case when the domain of the function is [0, b] we will have the expressions $W_n = \frac{\sum_{i=0}^{i=n}(\frac{bi}{n})^2}{f(b)(n+1)}$. A corresponding Riemann sum on that subdivision is $R_n = \frac{b}{n}\sum_{i=1}^{i=n}(\frac{bi}{n})^2$, and the relationship between the two is $W_n = \frac{n}{b\times f(b)(n+1)}R_n$.

Therefore, as n→ ∞, lim W_n =

$$=\lim\frac{n}{b\times f(b)(n+1)}R_n = \frac{1}{b\times f(b)}\lim\frac{n}{n+1}\lim R_n = \frac{area\,under\,the\,curve}{b\times f(b)}.$$

Since $b\times f(b)$ has the interpretation of the area of the circumscribing rectangle, indeed we have here the ratio of areas of two continuous regions. Thus, in some metaphorical way, the intuition which sees the area as the sum of lines acquires a credibility of its own in a mathematically correct framework.

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ALGEBRAIC CALCULATOR TECHNOLOGY IN FIRST YEAR ENGINEERING MATHEMATICS

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ABSTRACT:

Algebraic calculators have made minimal inroads to most Engineering mathematics courses in Australia. Indeed, many still forbid normal graphics calculators in assessment despite their wide usage in the school systems, which feed the undergraduate courses. This is curious as even the algebraic calculator technology is no longer very new and reminds us of the resistance to change in undergraduate mathematics teaching.

Currently we are developing an engineering course in product design, which combines traditional course objectives with handheld CAS. For several years now, our Engineering students have used Mathematica from second year of course (although not in tests) and normal graphics calculators are used in all work in first year. The emphasis on facts and skills in the extant course means that over 60% of examination questions previously given in the first year course could be solved much more simply using an algebraic calculator.

The transition period requires that the traditional course be essentially maintained, partly to ensure student mobility between engineering courses, but some topics are modified for the new course and assessment is independent. Current engineering textbooks usually restrict themselves to traditional algebraic and calculus approaches, although graphics calculators are now more commonly used. Indeed many of these textbooks explicitly state opposition to the extension of CAS within the framework of the traditional course. This forces the provision of resources in-house to service a CAS approach to engineering algebra and calculus.

In this paper we discuss the introductory course and its implementation problems, illustrating how algebraic calculators can solve basic questions in a normal course, and how the calculators may be used in the future.

Keywords: Technology, CAS, Engineering

Introduction

Algebraic calculators have become quite common over the past decade, yet they have had minimal impact on either senior secondary or entry level university courses in Australia. This is despite increased use overseas – especially within Europe. This may seem surprising in view of the lead-time of taking up technology, which occurred in education with scientific calculators, graphics calculators and personal computing. Some of the controversy associated with the widespread adoption of scientific calculators and, more recently, conventional graphics calculators is outlined in Tobin, (1998a). In particular the ability to extend these conventional graphics calculators to algebraic work is noted. One consequence of this is students already can have very differential access to algebraic assistance with their present calculators, an existing equity consideration.

However there are some significant issues raised by this new algebraic technology which either did not arise before or which differ in degree.

Three particular issues stand out. Algebra is a pet love of the mathematics community, especially the teaching component, and linked inextricably with the issue of understanding in mathematics. To ease a student's path in algebra is seen by some as the ultimate 'cop-out'. The actual manipulations afforded by algebra appear to many as the essence of mathematics and skill in these manipulations marks true comprehension in the subject. In fact it is algebra which provides the mysterious and powerful language of mathematics and ensures its rituals are the preserve of the cognoscenti. Mathematical thinking and algebraic manipulation are too often seen as equivalent. This perspective ignores the very different levels of ability in algebra which are called for in everyday use of mathematics. There is always a need for some experts but functional ability in others is often sufficient. In any case solution of equations in algebra often requires good understanding with or without CAS as noted in Ball, (2001).

A second critical issue is that algebraic learning and use has grown over time to suit a curriculum which was very different. Thomas (2001) makes a distinction between two important aspects of algebra – its role in process of solution and its role in conceptual understanding. The CAS can assist very directly in the former, yet much of current algebra teaching is directed to this. The challenge is to exploit CAS in expanding student skills in understanding the basic concepts. This is a nontrivial issue. Despite widespread hopes that the new technology options would enhance understanding by what Kaput (1992) called multiple, dynamically linked representations of a concept, no hard evidence exists yet that this has been achieved (However see Boers and Jones, 1994). One reason may be that the tools are just grafted onto an extant course, which is invariably necessary in political terms, or that the tools merely increase the gulf between the able and weak students (differential skill enhancement).

Another issue is the saturation level of options. On one hand the conventional graphics calculators – especially with dedicated programs installed – provide their own level of algebraic assistance at new levels. On the other hand algebraic software in computers, particularly Mathematica, Maple and Derive, provide powerful assistance in algebra for the able students in real terms. Ball (2001) notes that different algebraic packages provide quite different solution options to simple tasks like equation solution so even the type of CAS can be an issue. However, these software packages can all certainly be used to obviate much unnecessary algebraic manipulation and calculus in the same way as the classical tables of formulae and integrals did in previous decades.

Students in Swinburne engineering courses already use Mathematica from second year and a pilot program is introducing algebraic calculators into first year. In this context, it is interesting to consider what the real impact these tools have on existing assessment and how might first year engineering students benefit from access to these algebraic calculators. In the test examples supplied here we see clearly how the algebraic processes are radically simplified by use of a CAS. This is a report of work in progress and necessarily raises more questions than it answers at present.

There exist several brands of algebraic calculators on the market suitable for use in mathematics courses. Suitable models are available from Hewlett – Packard, Casio and Texas Instruments. These include the TI 92 and its relative TI 89 from Texas and the samples in this paper are drawn from this latter model.

Test Examples and Discussion

Engineering Mathematics 1 is a common subject for all engineering students in first year at Swinburne University of Technology. The students enter with a background in calculus and algebra from a base level subject, Mathematical Methods, in final school year (or its equivalent) and most also have studied an advanced level subject, Specialist Mathematics. They have been previously been allowed normal graphics calculators and given a brief formula sheet.

Topics include some error analysis and multi-base arithmetic, but the focus is on functions using algebra, graphs and calculus. There is a brief unit on statistics. In second semester discrete mathematics, matrices and vectors are studied along with differential equations, curves and calculus of functions of several variables.

The 2001 semester 1 paper contained 23 questions for a total of 180 marks. Questions include many involving traditional algebra and calculus. The calculator could help substantially in many questions on this examination – especially in the calculus area. The detail of some questions, contrasting the suggested traditional answer with the calculator is given as follows.

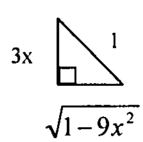
These questions totalled 120 marks – about 66% of the marks on the paper. The calculator could also have been used in other questions either to complete or check calculus or in traditional graphing and statistics. We begin by looking at sample questions from the previous examination with their classical and new solutions.

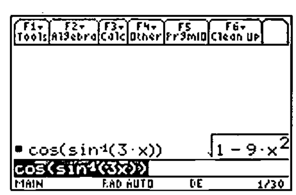
Topic: Trigonometric Algebra

Sample: Express $\cos(\sin^{-1}3x)$ as an algebraic expression in terms of x.

Let $\theta = \sin^{-1}3x$. so that $\sin\theta = 3x$.

We can draw a right triangle with $\sin\theta = 3x$





From the triangle, $\cos\theta = \cos(\sin^{-1}3x) = \sqrt{1-9x^2}$.

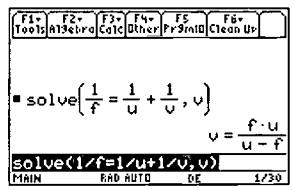
Discussion:

These questions have been designed to relate various trigonometric and inverse trigonometric functions. There are actual applications of course – a classical model of the daylight hours at a given latitude is a case in point – but its arguable whether this traditional approach is critical. Notice a diagram is the main aspect of solution here – the calculator cuts to the algebraic form directly. Students generally find these types of problems hard, possibly because they draw on various skills at once.

Transposition of Formulae

Sample: According to the lens formula, used in optics, $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$ where f is the focal length of a lens, u is the distance of an object from the lens, and v is the distance of the object's image from the lens. Rearrange the formula to make v the subject of the formula.

$$\frac{1}{f}=\frac{1}{u}+\frac{1}{v} \Rightarrow \frac{1}{v}=\frac{1}{f}-\frac{1}{u}$$
$$\Rightarrow \frac{1}{v}=\frac{u-f}{fu}$$
$$\Rightarrow v=\frac{fu}{u-f}$$



Discussion:

This is a necessary activity for formula use as it is too inconvenient to give every version of a formula. Algebraic manipulation in transposing formulae like this is legendary for generating errors as discussed in Tobin, (1998b).

If formula use is critical as well as knowledge of the meaning of formula elements then the algebra here has likely hindered weaker students in the past. In this optics example the solution is very direct. However use of a calculator for algebraic manipulation in practice can be a daunting effort at times as well. Consider this real example (drawn from the course notes) using the Darcy-Weisbach equation for turbulent flow. The students are asked to rearrange the formula to find the head loss due to friction, h_F.

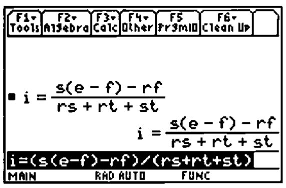
$$Q=-2{,}222D^{\frac{5}{2}}\sqrt{g\frac{h_F}{L}}\log_e\left(\frac{k}{3.7D}+\frac{4.1365}{\left(\frac{Q}{vD}\right)^{0.89}}\right)$$

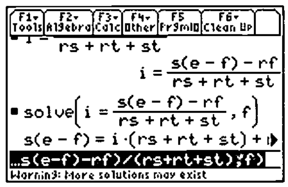
To enter such a formula takes time and substantial care on using real multiplication symbols not just expecting adjacent symbols to be read as multiplying. In such a case it is easier to do the transposition by hand! Another example on algebra gives the following equation from an electric circuit.

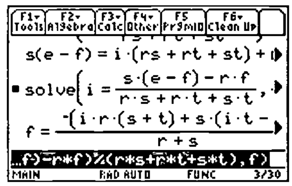
$$i_2 = \frac{R_2(\varepsilon_1 - \varepsilon_2) - R_1\varepsilon_2}{R_1R_2 + R_1R_3 + R_2R_3}$$

The students were asked to rearrange this to find ε_2.

Clearly we need to use new symbols and keep track of their one-to-one correspondence here. Suppose we use r for R_1, s for R_2 and t for R_3. We can use e for ε_1 and f for ε_2. Then using i for i_2 we may enter the equation in the calculator as shown. It all looks fine for use so we may try to solve for f. The result is shown in the middle screendump – the calculator has not picked out the appropriate multiplications. Expressly including the multiplication gives the correct result shown far right. This reminds us that relapsing into 'natural' writing can produce serious errors especially for a casual user!







These examples certainly remind us of Ball's remarks that understanding is needed to use algebraic computing technology efficiently! Students need to know the essence of 'undoing' operations, and their place in equation solving. They also must add a new layer of understanding on the fine detail of the CAS itself and how variables may be defined and used.

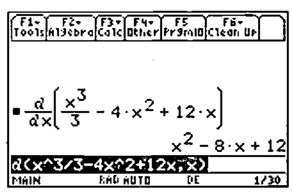
Differential Calculus

Sample: Find the gradient and second derivative at any point on the graph of the function y = $\dfrac{x^3}{3} - 4x^2 + 12x$ and hence find maximum and minimum points of inflection on the graph.. Sketch the graph, showing all maximum and minimum points, point of inflection and intercepts.

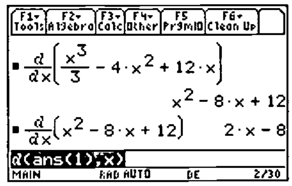
The components of interest here are the derivatives.

$$y = \frac{x^3}{3} - 4x^2 + 12x$$

$$\Rightarrow \frac{dy}{dx} = x^2 - 8x + 12 \text{ and } \frac{d^2y}{dx^2} = 2x - 8.$$

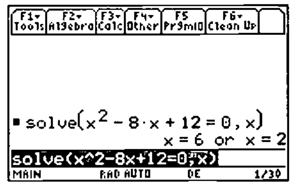


To locate maximum and minimum we set $\frac{dy}{dx} = 0$.



We can also use the calculator for the algebra.

$x^2 - 8x + 12 = 0 \Rightarrow (x-6)(x-2) = 0$ so $x = 2$ or 6.



(or solve by formula)

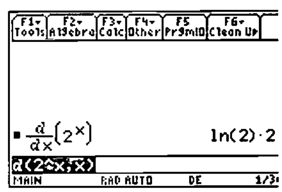
The rest of this question could also be solved by assistance of a conventional graphics calculator, although all those features exist on the CAS as well.

Discussion:

This problem requires students understand notions like gradient and point of inflexion and how these link to derivatives. This is not a simple black box - a calculator can only solve the components of the problem, with a good student using these to put together the elements in a solution. This is a suitable illustration of the calculator as tool and assistant rather than a replacement for thinking.

Sample: Find $\frac{dy}{dx}$ given that $y = 2^x$.

$y = 2^x$. Take natural logs both sides.



$\ln y = \ln 2x = x\ln 2$.

Differentiate with respect to x:

$$\frac{d\ln y}{dx} = \ln 2 \text{ so } \frac{d\ln y}{dy}\frac{dy}{dx} = \ln 2$$

$$\frac{1}{y}\frac{dy}{dx} = \ln 2 \text{ so } \frac{dy}{dx} = y\ln 2 = 2^x \ln 2.$$

Sample: Given that $y = [\sin^{-1}(1-x)]^2$, find $\frac{dy}{dx}$.

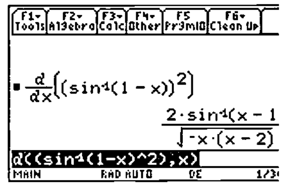
$y = [\sin^{-1}(1-x)]^2$.

Let $u = \sin^{-1}(1-x)$.

Then $y = u^2$ and $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u\,\frac{du}{dx}$

Let $v = 1 - x$ so $u = \sin^{-1} v$.

$$\frac{du}{dx} = \frac{du}{dv}\frac{dv}{dx} = \frac{1}{\sqrt{1-v^2}}(-1) = \frac{-1}{\sqrt{1-v^2}}.$$



Hence $\dfrac{dy}{dx} = 2u\,\dfrac{du}{dx} = 2(\sin^{-1}(1\text{-}x))\left(\dfrac{-1}{\sqrt{1-(1-x)^2}}\right)$

$$= \frac{-2\sin^{-1}(1-x)}{\sqrt{1-(1-x)^2}} \text{ or } \frac{-2\sin^{-1}(1-x)}{\sqrt{x(2-x)}}$$

Discussion:

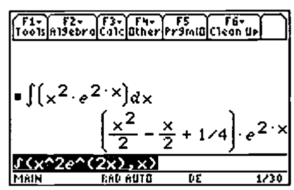
These provide a black box solution – one for a problem usually solved using tables and one for a chain rule derivative where sign problems can also occur. Problems such as this have often been set but there will likely be a decreased need for elaborate differentiation by hand in future. This is even more true of students in 'applied' rather than 'advanced' courses and reminds us that the algebra needs of all students need not be the same, bearing on the first issue raised in this paper.

Antidifferentiation

Sample: Find $\mathrm{I} = \int x^2 e^{2x} dx$

Let $u = x^2$. Then $\dfrac{du}{dx} = 2x$. Also let $\dfrac{dv}{dx} = e^{2x}$ so $v = \dfrac{e^{2x}}{2}$. By integration by parts we have

$\mathrm{I} = \dfrac{x^2e^{2x}}{2} - \int 2x\dfrac{e^{2x}}{2}dx$. Repeat integration by parts with $u = x$ and $\dfrac{dv}{dx} = e^{2x}$ so $v = \dfrac{e^{2x}}{2}$. Now

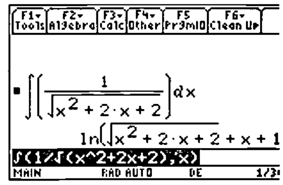


$$\mathrm{I} = \frac{x^2e^{2x}}{2} - \left\{\frac{xe^{2x}}{2} - \int\frac{e^{2x}}{2}dx\right\}$$

$$= \frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{e^{2x}}{4} + c$$

Sample: Find $\mathrm{I} = \int \dfrac{dx}{\sqrt{x^2+2x+2}}$

$$\text{I} = \int \frac{dx}{\sqrt{x^2+2x+1+1}} = \int \frac{dx}{\sqrt{(x+1)^2+1}}$$



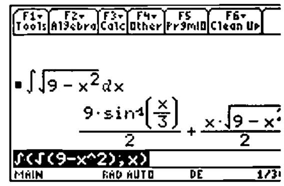
Let $u = x + 1$. Then $\dfrac{du}{dx} = 1$ or $du = dx$.

So $\text{I} = \displaystyle\int \frac{du}{\sqrt{u^2+1}} = \sinh^{-1}u + c$

$= \sinh^{-1}(x+1) + c$ or $\ln\left[(x+1)+\sqrt{(x+1)^2+1}\,\right] + c$

Sample: Find $\int \sqrt{9-x^2}\,dx$, and hence or otherwise find $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\,dx$

$I = \int \sqrt{9-x^2}\,dx$. Let x = 3sinθ so that dx = 3cosθdθ.



Then $I = \int \sqrt{9-9\sin^2\theta}\,3\cos\theta d\theta$

$= \int \sqrt{9(1-\sin^2\theta)}\,3\cos\theta d\theta$

$= \int 3\sqrt{(1-\sin^2\theta)}\,3\cos\theta d\theta$

$= \int 3\sqrt{(\cos^2\theta)}\,3\cos\theta d\theta$

$= \int 3\cos\theta\, 3\cos\theta d\theta$

$= 9\int \cos^2\theta d\theta$

$=9\int \frac{1}{2}(1+\cos 2\theta)d\theta$

$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + c$ for c a constant

$= \frac{9}{2}\left[\sin^{-1}\frac{x}{3} + \frac{2\sin\theta\cos\theta}{2}\right] + c$

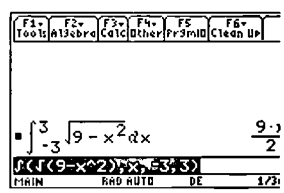
$= \frac{9}{2}\left[\sin^{-1}\frac{x}{3} + \sin\theta\cos\theta\right] + c$

But $\sin\theta = \frac{x}{3}$ so $\cos\theta = \frac{\sqrt{9-x^2}}{3}$

Hence $I = \frac{9}{2}\left[\sin^{-1}\frac{x}{3} + \frac{x}{3}\frac{\sqrt{9-x^2}}{3}\right] + c$

Then $\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{9}{2}\left[\sin^{-1}\frac{x}{3} + \frac{x}{3}\frac{\sqrt{9-x^2}}{3}\right]_{-3}^{3}$

$= \frac{9}{2}\left[\sin^{-1}(1)+0\right] - \frac{9}{2}\left[\sin^{-1}(-1)+0\right] = \frac{9}{2}\left(\frac{\pi}{2} - (\frac{-\pi}{2})\right) = \frac{9\pi}{2}$



Discussion:

These problems illustrate more difficult antiderivatives. Students find integration by parts particularly hard. In both cases there is a black box approach afforded by the algebraic calculator. As with the previous derivative problems, it is arguable how much time we need to have students learn these manipulation techniques in more general level courses. This is particularly true when students have been commonly given extensive tables to assist them in the past - the calculator merely extends this help.

Polynomial Approximation

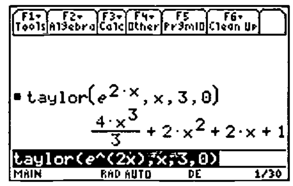
Find the Taylor polynomial of order 3 which approximates the function $f(x) = e^{2x}$ about $x = 0$.

$f(x) = e^{2x}$ so $f(0) = e^0 = 1$

$f'(x) = 2e^{2x}$ so $f'(0) = 2e^0 = 2$

$f''(x) = 4e^{2x}$ so $f''(0) = 4e^0 = 4$.

$f'''(x) = 8e^{2x}$ so $f'''(0) = 8e^0 = 8$.



Then for a third order Taylor expansion,

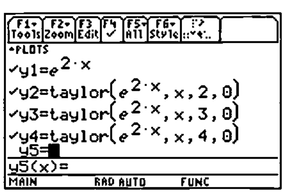
$$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) \text{ approx.}$$

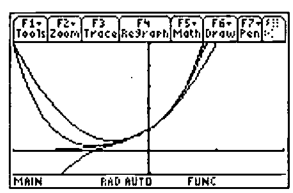
$$\text{or } p_3(x) = 1 + 2x + 4\frac{x^2}{2} + 8\frac{x^3}{3!} = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

Discussion

Polynomial approximations give some insight into function behaviour, which can be useful. The actual need for approximations is now diminishing as technology enable the original (possibly transcendental) functions to be used more readily. Consideration of when and how we apply approximations could actually be extended of course. Current courses usually focus on polynomial approximations but other approaches such as rational function approximations (Pade approximations) have been useful in the past but given little syllabus time. These could be included in a future syllabus.

It is also possible to use a number of easily generated Taylor polynomials to illustrate the power of higher order models in extending the range of use. This is illustrated following where Taylor polynomials of degree 2, 3 and 4 are used to approximate the function and graphs are provided.





Partial Fractions

Express $\dfrac{5x^2-9x+2}{(x-1)^2(x-2)}$ as partial fractions.

Let $\dfrac{5x^2-9x+2}{(x-1)^2(x-2)}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x-2}$ (1)

Multiplying (1) by $(x-1)^2(x-2)$ we get

$5x^2-9x+2=A(x-1)(x-2)+B(x-2)+C(x-1)^2$ (2)

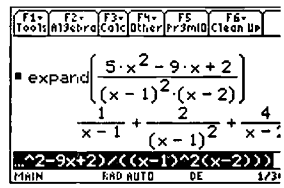
Substituting $x=1$ in (2):

$5-9+2=-B$ so $B=2$ (3)

Substituting $x=2$ in (2):

$20-18+2=C$ so $C=4$ (4)

By (3) and (4) in (2)



$5x^2-9x+2=A(x-1)(x-2)+2(x-2)+4(x-1)^2$

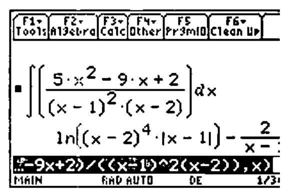
Substituting for $x=0$: we get $2=A(-1)(-2)+2(-2)+4(-1)^2$

So $2A-4+4=2$ and $A=1$.

Hence

$$\frac{5x^2-9x+2}{(x-1)^2(x-2)}=\frac{1}{x-1}+\frac{2}{(x-1)^2}+\frac{4}{x-2}$$

Hence find $\displaystyle\int\frac{5x^2-9x+2}{(x-1)^2(x-2)}\,dx$



Although the question asks that this be done as a consequence of part (a) the integral can be solved directly and our usual recourse to partial fractions becomes redundant.

Discussion:

Partial fractions have been commonly taught to provide for integration of rational functions ...this technology radically simplifies the integration. There do not seem to be valuable modelling issues lost by this simplification - at least at this level. Few problems call for summing rational expressions and fewer still for re-expressing in such sum forms.

Curriculum Issues

The preceding examples demonstrate that tests on facts and skills in mathematics will be substantially affected by access to algebraic calculators. The use of these tools in calculus is much more significant than prior access to tables or conventional graphics calculators could achieve. The issue of how much understanding a student needs to have becomes important as many traditional problems take on 'black box' solutions. A student needing a three-year mathematics course and using this in other subjects has different needs from a student who has a one-year course, which does not directly feed too many higher year subjects thereafter.

The examples suggest that the calculators can make the most direct impact in calculus. There is a strong case for reducing the time spent on generating every type of derivative and antiderivative from rules and focussing more on the meaning and application of these notions. This is not very different from making tables of formulae available, and is particularly justifiable for students in more general courses. On the other hand, the algebraic work in manipulating equations can be so convoluted that the calculator may be a source of difficulty for the student rather than an assistant as the examples reveal. Of course, this requires that the students can perform algebra efficiently by hand. It is likely that an able mathematician will commonly do better than a CAS machine user on a range of algebraic and calculus features but this may not be true of the average user.

Thomas (2001) has discriminated the use of CAS in two forms - as a conceptual process representation tool (CPRT) and a conceptual object representation tool (CORT). The examples given in this paper illustrate the CPRT form, which uses the CAS to ease a process. This is common for assessment problems. Use of the CAS in CORT form requires more focus on the object. In this context it can be more a learning tool than assessment tool. We can for example, demonstrate features such as symmetry in a graph, or show how successive Taylor approximations perform on modelling a function locally, rather than directly obtaining such an approximation, as illustrated in the discussion on the Taylor polynomial example.

Collectively these conceptual representation tools or CRTs may provide the engine for a new approach to learning higher mathematics. In practice what we aim to achieve will remain the same – we examine common function forms in graphical, algebraic and numerical terms and use the results for modelling real situations and solving abstracted problems. A CAS tool has access to all these three aspects of course, so students can examine all aspects of a problem at once. Conventional graphics calculators have limited analysis directly to graphical and numerical aspects, although these have often fed algebraic interpretations too of course. However we can now see that some calculus procedures at least may be better automated which impinges on curriculum issues as a substantial time is currently spent on learning techniques a CAS machine can do instantly and accurately and with likely no loss of understanding of the basic notion. At this level of operation we are doing no worse for calculus than removing the square root process does for arithmetic!

This curriculum goal has always been hard to achieve, but given more time released from routine work it may be made easier. Modelling of real problems can be enhanced by CAS of course. For example, traditional exam questions which asked a student to set up but not solve a DE to model a situation could now require that solution be obtained and outputs discussed for numerical validity – thus (say) asymptotic behaviour could be discussed when the solution is at hand.

The issue of different CAS systems raised in the introduction, has not been addressed here but can be important. Consider the simple problem of solving a cubic equation. A CAS procedure like Mathematica will generate exact solutions, often with such a convoluted form that the real solution(s) are not very visible directly. The Derive package in the TI-89 actually looks for simpler exact solutions but if the problem does not admit these conveniently, it reverts to giving numerical approximates directly when coefficients are numbers. This is quite a reasonable option for the user of course but it places the algebraic solver on these calculators on the same level as the numerical solvers of the traditional graphics calculators.

Far more mathematics including engineering mathematics is now data driven. Statistics are more important to use now because there are ready tools for number crunching and modelling. As the models can be automated, it becomes important to dwell on the meaning of the models, their range of use and limitations. In practice this may mean that a test question on regression should not be simply asking for a linear model to be found and maybe used in a given context. Far better to require students to decide if the model is suitable and how it can be used. They may be required to consider residuals, nonlinearity and other matters, which are not mere numerical outputs from a calculator. In this way we ensure that no calculators can take away conceptual understanding.

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MATHEMATICS FOR COMPUTER GAMES TECHNOLOGY

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ABSTRACT

Despite their entertainment focus, Computer Games are at the forefront of computer science. These days the successful computer games (programs) make strong calls on a number of other disciplines: psychology, mathematics, statistics and physics to name but a few.

Enrolment numbers in mathematics courses across Australian Universities are in decline, however we have found that students who have enrolled in our new degree, Bachelor of Computer Science (Games Technology), are not just tolerant, but are enthusiastic about the mathematics component of the degree. In this paper we describe the mathematical demands of games technology along with the development of the mathematical sub-component of the degree.

Our program, being a full strength computer science degree demands some standard mathematical components: discrete mathematics, linear algebra, numerical analysis, statistics and ordinary differential equations.

The strong emphasis on computer graphics programming needs a firm foundation in aspects of linear algebra; the virtual world of the game scene development is underpinned by the physics of movement. Acceleration, cornering, collisions, explosions and disintegration all require ODEs.

A subtle shift in the shape of a probability distribution can help to maintain game balance and player interest by giving the underdog an unseen helping hand. Logic, computational complexity and numerical analysis speak for themselves.

Mathematics educators have always known that relevance is a strong motivator of mathematics.
This has again been ably demonstrated by our first cohort of games technology students who want more mathematics.

Key Words: Computer Games; Computer Graphics; Mathematics Education.

1. Introduction

In 2001 the School of Information Technology at Charles Sturt University in Bathurst, Australia admitted the first cohort of students into its new degree, Bachelor of Computer Science (Games Technology). For the previous two years the School had been developing curricula for this course. The degree was to span four years, and was designed for an unusually bright cohort. We were aiming the curricula for students from the top 10% of their age cohort with proven mathematics and computing skills from their secondary schooling.

The Australian Computer Games industry was an enthusiastic supporter of this degree. Their help in the development of this program should be acknowledged. It would be the only degree level training available in Australia for their workforce. The degree program has a considerable component of work experience and agreements are in place with the industry to place the final year students within appropriate projects.

On a worldwide basis the Computer Games industry is larger than the Movie industry. It employs more people. This is not so surprising when you consider its scope. It encompasses hand held devices (Game Boy), the games consoles (Play Station, Sega, Microsoft's X-box), PC games, Internet multiplayer games, arcade games, gambling casino games (poker machines) and Internet gambling. The games themselves range from sophisticatedly intelligent strategy games, such as chess, to first person shooter games, to games of pure chance (gambling).

The divide between the Movie industry and the Computer Games industry is blurring at a rapid rate. Movie special effects with computer generated backgrounds or worlds are becoming common. Animated characters are no longer just the province of cartoons. Filmed video sequences played out with live actors are now being spliced into some computer games as scene setters.

Computer graphics also have serious uses. Medical imagery and other simulation programs are examples. Computer Games are at the leading edge of real-time computer graphics technology, but much of the mathematics associated with the generation of these images is fundamental. It is to be found within the mathematics courses that we already teach.

The 'Games Technology Development Committee' allocated only four subjects to mathematics, however some other subjects, such as computer graphics, do contain other elements of mathematics (A subject is about 48hrs of instruction over one ½year session.) Efficiency within the university dictated that we did not have a free hand to develop completely new subjects. There were some suitable mathematics subjects already in existence. Three existing subjects were chosen to fill the allocated space and one new subject had to be developed. They are:

- Discrete Mathematics
- Dynamics (New)
- Ordinary Differential Equations
- Life, Chaos and Virtual Worlds.

There is always some freedom to tweak the existing subjects to fit a new cohort of students. Some examples from computer games programming were added and small changes to the list of topics to be covered were made. You will also note that the new subject is also quite traditional.

Our campus does not have a physics department so we did not have a Dynamics subject on offer. However it will be developed with Computer Games focus.

Our teaching experience at this stage is limited to the Discrete Mathematics subject. The subject was taught to a class of some 74 students, of which 28 were the Games Technology students.

2. Discrete Mathematics

Discrete Mathematics covers a standard set of topics. These are:

- Logic, Sets and proof.
- Numbers, Combinatorics, Complex numbers.
- Probability.
- Vectors, Matrices and Solutions of Systems of Equations.
- Graphs.
- Recurrence and complexity.
- Boolean Algebra and Logic Circuits.

The subject was taught as a mathematics subject, with the usual emphasis on understanding. The development of the theory was not compromised. To cater for the new cohort of students, Probability was a new topic introduced to an already full curriculum, and examples drawn from Games programming were used to enhance the relevance of the study to these students.

The majority of the group, not just the Games students were motivated by these examples. These examples are from the students own field of interest and experience, real (perhaps I should say virtual) world examples of mathematics in use.

We shall have a look in a little detail at some of the ideas used.

Probability. Lecky-Thompson [2001] discusses two quick and dirty pseudo random number generators (my description). Both are probably not good enough for Poker Machines, and both will quickly generate a sequence of numbers, random enough for most game applications. Both sequences are repeatable given the same seed value. These random numbers are sampled from the uniform distribution.

How do we deal out a set of shuffled cards at the start of 'Free-cell' or 'Pairs'?

How do we generate random samples from other distributions?

How do we make a monster so that its behaviour is a little unpredictable and becomes a challenge to shoot?

Two-person games quickly loose their appeal if one player consistently outperforms the other. This is known as Game Balance. How can we change a probability distribution to bring the game back into balance and keep both competitors enthusiastic? A shift to the underlying probability distributions can also be used to increase the level of difficulty of a game as we enter new phase or level.

Matrices. (3D Graphics). The action of the First Person Shooter or the Car Racing Simulation takes place in a 3D virtual world. Objects and characters are built from a set of points in 3D. These points are stored in a data structure as columns of coordinates relative to a convenient local coordinate system. The objects are manipulated, (moved, rotated, scaled) to their desired shape and orientation then positioned in the world by a 'change of coordinates' to the world coordinate system.

Interestingly, a 4D homogeneous coordinate system is used. This has the distinct advantage over 3D of allowing translations to be achieved by a matrix multiplication. Associativity of matrix multiplication allows this entire placement procedure to be accomplished in one pass of matrix multiplication across the data structure.

Example. Homogeneous coordinates.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \begin{bmatrix} w_x \\ w_y \\ w_z \\ w \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{w_x}{w} \\ \frac{w_y}{w} \\ \frac{w_z}{w} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

3D ⇒ 4D **4D ⇒ 3D**

Transformation matrices.

Rotation about x-axis.

$$R = \begin{bmatrix} Cos\theta & -Sin\theta & 0 & 0 \\ Sin\theta & Cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling to desired shape and size.

$$S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation.

$$T = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

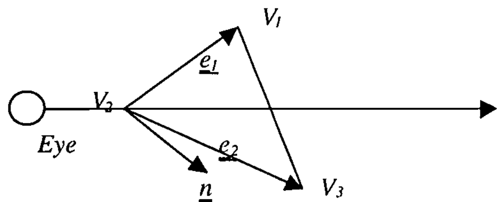
Note:

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix}$$

Transformation Matrices for rotations about the Y and Z axes are similarly built.

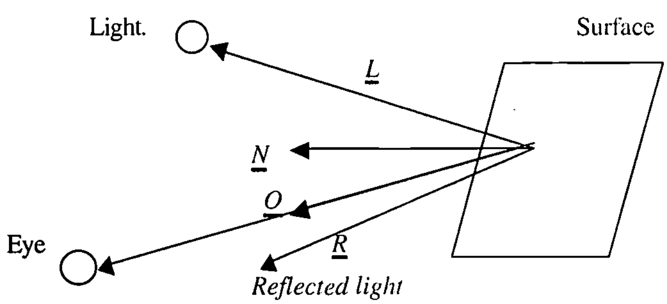
The usual composition of transformations rules apply, non-commutivity can be demonstrated easily and matrix inverses are needed to undo transformations and for change of coordinate systems. Put the movement in a loop and you have a stealth bomber flying in doing a barrel roll.

Vector products. (Hidden Surfaces). Once an object is positioned in the game space, we need to render its seen surface. Deciding which part of an object is seen, and which part is hidden, is achieved by a process that uses both vector products. We cover the set of points that define the object with a convex polygon mesh. Within this data structure the polygon vertices are always ordered in a clockwise pattern when viewed from the outside of the object. By taking cross products of two suitably defined edge vectors we generate a normal vector to the plane of the polygon. This normal will point out from the object's surface. This allows us to quite simply determine if this section of the surface is visible or hidden from view. If the dot product of the vector from the observer to V_2 with the normal a $\underline{n}$ is positive then this is a back surface and hidden from the observer. Hidden sections of the surface are deleted from the data set.



$$\underline{n} = \underline{e_1} \times \underline{e_2}$$
$$= (V_1 - V_2) \times (V_3 - V_2)$$

More vectors. (Light). Various lighting scenarios, ambient light, parallel light and point source light are used to give the Game scene a touch of realism. If either of the last two scenarios is used then the brightness of that part of an object is dependent on the component of reflected light reaching the observer.



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If $\underline{N}$ is a unit normal then we can show that $\underline{R} = 2(\underline{N}\bullet\underline{L})\underline{N} - \underline{L}$. The component of reflected light is $(\underline{O}.\underline{R})\underline{O}$ if $\underline{O}$ is a unit vector in the direction of the observer. Scaling factors on the length of this vector can then be used to generate the desired effect. Spot lights from the observer that can be turned on and off (a torch) are used to good effect in some game play.

Graphs. (Path Finding). Perez and Royer [2000] make the observation that finding the shortest path between two nodes on a graph is quite often needed when programming. It can be used for planning airline trips to generating the 'door to door' directions using GIS software in modern cars. In the game environment often the 'badies' are hunting you. Djkstra's algorithm finds the shortest path from any source node to all connected nodes in a directed graph with positive weights. Getting the ghosts to attack PacMan or the 'beast' to attack in Doom seems a little more exciting than the well trodden travelling salesman.

3. Dynamics

Some of the earlier computer games captured motion by holding pre-scripted animation sequences in storage. These sequences were fairly easy to detect as they all started and finished with the same neutral position. In the older fight games this position was the boxing stance from which the player had a choice to have his character punch, kick or jump. Storage requirements were later reduced by holding only the significant positions of the motion sequence and achieving the animation by dynamically interpolating between these positions. This is still quite limiting as only a small number of prerecorded options are available to the player.

Dynamic simulation (or physics based animation) can be used effectively to script the motion of objects. These can be used to great effect in Simulation Games like F1 Racer and Flight Sim.

For the sake of realism some modern car racing simulation games have gone as far as modelling weight changes on sprung suspensions components in cornering. [See Brian Beckman's articles on *The Physics of Racing*, Tutorials section of the Gamedev.net site.]

The Dynamics subject covers topics such as:

- Newton's Laws of motion.
- 1D particle dynamics within various types of force fields.
- 1D oscillating systems.
- Motion in 2D and 3D.
- Systems of particles, conservation laws and collisions.
- Rigid Body motion, Moment of Inertia and rotational motion.
- The Lagrangian and Hamiltonian formulations of dynamics.
- Rigid body motion, inertia tensor and Eurler's equations of motion.

There is still "work in progress" on this subject for now.

4. ODEs

In any simulation, the equations for the physical system must be set up in advance of any play. If the ODEs have an analytical solution then these are the stored equations and the motion can be calculated accurately and quickly. However if the ODEs do not lead to such a solution then they will need to be solved in real time by a numerical integration technique. The speed of solution, along with the level of desired accuracy will determine the type of numerical procedure employed.

The ODE subject currently covers:

- First and second order linear ODEs.
- Power series and Laplace transforms
- Special functions, Gamma, Bessel and Legendre polynomials.

It is the writer's view that the special functions section of the course should be replaced with a section that covers numerical integration.

5. Life, Chaos and Virtual Worlds

This subject was developed initially as an elective study for interested students (mainly Information Technology students). It was thought to be important enough in ideas and concepts that it covered to become a core subject within the Games degree.

Fractals are now a mainstream topic in computer graphics. They are used in creating exciting visual patterns of plants, mountains and surface textures. Students will visit Cantor sets, measurement, mapping, iterated function systems and a set of tools for describing chaotic attractors. It is shown that Chaos can arise from very simple differential equations of just three variables in a dscrete system. Students will learn how to cast an equation into a form, which is amenable for numeric solution. They will develop an appreciation of the sensitivity of such systems to initial conditions. In artificial life we get to a more difficult and less understood topic. Evolution and fitness landscapes are useful mathematical techniques. Emergent behaviour is a powerful idea but it still lacks a rigorous formulation. Finally, aside from the beautiful images that we can create with fractals, chaos and emergent behaviour, students will gain an appreciation of the unifying nature of mathematics and its powerful inner beauty.

$$\text{chaos} \Rightarrow \text{fractals} \Rightarrow \text{emergence} \Rightarrow \text{chaos.}$$

6. Conclusion

For a person of my age a certain level of proficiency with a pool queue is said to be a sure sign of a misspent youth. These days a misspent youth would be signalled by a similar level of proficiency with a computer game pad. We can take advantage of the student's intense interest in computer games because of the wealth of mathematics that is used in the development of these games. The development of the mathematics curricula for the Bachelor of Computer Science (Games Technology) degree shows that although much of the needed mathematics is quite fundamental in nature.

There are of course many areas of intersection between Computer Games and Mathematics that I have not touched upon, artificial intelligence paradigms such as neural nets and genetic algorithms are to be found in Game intelligence. There are also a number of mathematical skills that are developed in the secondary schools that are found in Games. Intersections of lines and planes and spheres are used in collision detection techniques. The breadth of mathematics used is in this area is immense.

It was not only the Games Technology students who were motivated by these examples. It was apparent that all students were interested by the examples. Vectors, Matrices, probability and graphs were seen to be useful mathematical constructs by the students.

Mathematics educators have always known that relevance is a strong motivator of mathematics, but relevance has to be **'- to the student'**. Games development was found to be an ideal vechile for the motivation of much of the mathematics that we teach.

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ENTREPRENEURIAL MATHEMATICS GRADUATES

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ABSTRACT

The university of Ulster is developing a strategy for the introduction of "entrepreneurial studies" into all of its programmes. The Learning Outcomes (LOs) expected of each programme must include the following:

On completion of this programme students will be able to:

-Demonstrate innovative thinking and creativity.
-Demonstrate knowledge of future trends in her or his subject area
-Identify the steps required to research a market for a business opportunity.
-Explain the impact of intellectual property rights with respect to new idea generation and product innovation.
-Describe the component parts of a business plan.
-Demonstrate familiarity with the range of organisational support available to assist with new enterprise development within UU and the local community.
-Demonstrate team building ability.
-Identify the steps required with respect to new company set up and incorporation.
-Identify the key sources of finance available for business start-up.
-Communicate new ideas effectively.
-Demonstrate familiarity with an e-learning environment.

The course team for the honours degree in Mathematics with Computing has devised a curriculum which seeks to develop these LOs in students. The curriculum innovations involved include the provision of two modules - "Mathematical Modelling" and "Statistics for Industry with Entrepreneurship". It is intended to offer these modules in Semester 4 of the course, i.e. the 2nd semester of Year 2. They will be taught in parallel along with a module of computer science.

The paper starts with a discussion of "entrepreneurship". It goes on to outline the gradual development of the course over a number of years, and how the required LOs came to be embedded in the course modules. It will indicate how we propose to teach the latest additions to the curriculum, starting in the 2nd semester of 2003/04.

Keywords: - enterprise, entrepreneurship, mathematics, key skills.

1. Introduction

There is a vast literature on "entrepreneurial education" although often it is found under a different name. We will cite just two papers, a few years old, because they give an extensive survey of entrepreneurship education and training programmes; these are the papers by Garavan and O'Cinneide (1994a and 1994b). Many of these programmes are in the context of professional development courses for people who already know that they want to at least try to be entrepreneurs, who have got an innovative idea that they wish to exploit and who now seek the specialist business knowledge required to do it. Such courses would be delivered by departments of business and management or specialist entrepreneurial units established for this purpose. There are also references to aspects of undergraduate teaching involving engineering students engaging in joint entrepreneurial studies with business studies students. We found no references to entrepreneurial mathematics students.

There are three terms, which are used frequently. These are "enterprise education", "entrepreneurship education" and "small business education". Sometimes the word "training" is used in addition to "education". We suggest that it is necessary to distinguish between these; often they are used almost interchangeably and they tend to have different meanings in different parts of the world, notably North America and the UK and Ireland. Garavan and O'Cinneide (1994a) point out that "the term 'entrepreneurship education' is commonly used in Canada and the USA, but it is less commonly used in Europe. The preferred term in the UK and Ireland is 'enterprise' and it is primarily focussed on the development of personal attributes. The term 'enterprise' does not necessarily embrace the small business project idea or the entrepreneur". This would be our use of the term "enterprise" and it has been used in this sense at the University of Ulster for many years, although things are now changing. Garavan and O'Cinneide go on to define "entrepreneurship" as "independent small business ownership", and they distinguish carefully between small businesses that have entrepreneurial owners and those that do not. Another term we will explore the meaning and usefulness of is "intrapreneurship", and we shall look at this in section 2 when we describe the development of the course thus far.

Some people suggest that entrepreneurs are born and not made. Others suggest the opposite. The evaluative evidence of courses which seek to teach entrepreneurial knowledge, skills and attitudes tends to support the latter point of view, and it is widely accepted that knowing about entrepreneurial skills is valuable in itself. It is generally recognised that an "entrepreneurial attitude" is essential to success. Garavan and O'Cinneide (1994a) write, "..this later topic of attitudes, the psycho-social forces of the individual and the cultural context, is of prime importance in influencing innovative and entrepreneurial behaviour patterns" and, as Cool Hand Luke says in a classic 1967 movie, "Boss, I now got my head right!" Hence the inculcation of "entrepreneurial attitude" is something a teaching intervention should strive for.

There is a strong belief that in the future innovative business ideas will come from small to medium businesses managed or owned by entrepreneurs. Since economic growth and national prosperity are goals that most governments wish for their respective countries, entrepreneurial education is encouraged and supported from the highest levels. In the University of Ulster, the move to include entrepreneurial studies in all courses came as an edict from Senate, under the leadership of the vice-chancellor. Additional funds were secured from government agencies to promote the idea through the hiring or seconding of staff with the necessary expertise. The staff appointed to this unit were not to deliver entrepreneurial teaching to each course, nor to devise a global, generic, "one size fits all" module for courses to adopt. They were first of all to win over staff who felt that entrepreneurial studies were a nonsense for their students, and then to act as a resource person cum advisor who would help academics to devise their own modules for their own courses. These could then be presented in the culture of the particular subject community. Many people did not need to be won over. In particular engineers of all hues, art and design lecturers and others whose graduates make and then sell artefacts, were all keen on the idea. But those who prepared graduates for service in the caring professions like nursing, for example, were up in arms, claiming that entrepreneurial studies were all about "bottom line" management, and this was inappropriate for their students. The mathematics and statistics group was not immediately enamoured by the idea. We felt we were already doing a good job in preparing enterprising graduates for employment and that the idea of a newly qualified mathematics graduate going out and setting up a small business was laughable. It was beyond our experience, our own world picture, and we could see no use for it, nor had we any confidence in our own ability to deliver the required teaching with authority.

But an edict is an edict and we joined in the ensuing debates with the specialist entrepreneurial advisor. We, and even our colleague in the caring professions, came to see the value of what was proposed for our students. We shall discuss this further in section 3 wherein we shall also describe the new developments to be introduced in the near future.

2. Previous development

Between ten and fifteen years ago the course in Mathematics with Computing (then called the honours degree in Mathematics, Statistics and Computing) was heavily influenced by the Enterprise in Higher Education Initiative which was a government agency funded project to encourage universities to produce "enterprising" graduates (TEED, 1989). The University of Ulster received its share of the funding and used it to facilitate staff development. A small number of staff with the necessary expertise were seconded to the Enterprise unit, and each faculty appointed one if its academics as its Enterprise Advisor. The main criterion for appointment was enthusiasm for the idea. Money trickled down to faculty staff to help pay for staff development and pilot schemes.

The EHE scheme allowed universities to develop their own definition of "enterprise" and UU chose to describe an enterprising graduate in terms of his or her personal transferable skills. Skills that have more recently been called Key Skills (Dearing 1997). These are the skills of communication, problem solving, independent learning and group or team work. At this stage there were no edicts, but a sufficient number of academics became enthused by the idea that the scheme worked. Later, the embedding of "enterprise competencies" or key skills became mandatory, with all new courses and all established courses at quinquennial review being required to demonstrate how key skills were taught.

Another influence on the development of our course was the Peer Tutoring Project, again a teaching initiative project funded by government agency. Innovative teaching schemes, which made use of peer learning and self and peer assessment were encouraged. Since learning from colleagues is a feature of industrial environments, and since we sought to develop employable graduates, these ideas also found their way into our course (Griffiths, Houston and Lazenbatt, 1995).

We chose to introduce key skills in a variety of ways and modules. Prominent among these were first year modules on "use of ICT in mathematics" and "mathematical modelling", a second module on modelling in Year 2, a whole year, Year 3 of the course, spent in a sandwich placement in industry, and a final year individual Project. In addition students undertook a diet of traditional modules in mathematics, statistics and computing.

In the innovative modules in years 1 and 2, group project work and peer learning were exploited to good effect. For example in the Year 1 module on modelling, students performed several tasks. Working in small groups, students would research a modelling application using suggested references. They would write lecture notes for their colleagues to use to learn the topic and they would give a seminar on the topic. This seminar programme occupied mainly the middle third of the semester. Students needed four weeks to do their research and preparation, and they also needed time at the end of the semester to complete other tasks and to prepare for the written examination which examined knowledge of all of the seminars. The other major task each group was asked to carry out was to do a project on a suggested title. This involved active modelling (as distinct from the study and analysis of existing models), and the writing of a research report. Groups also presented and defended their work at an end-of-term poster session.

During Year 2, besides taking the second modelling module, students would be taught how to write their CV and how to apply and present themselves for job interviews for the placement jobs. Many of these jobs were outside Northern Ireland, and it was either a great adventure or a great trial for them to go and live and work somewhere seemingly far away from home. [Virtually all students at UU live at home or go home every weekend with no more than a ninety-minute journey.]

Student placements were largely in statistical enquiry houses and government agencies, or in some other computing environment where their mathematical skills gave them an additional attribute. Usually they would work at a level suitable for undergraduates, performing tasks that mattered. They were full time employees and had to accept the personal discipline involved. Often they had the satisfaction of having their work praised and used by higher officers in the company. They were supervised by the company and visited by an academic tutor. They received training from the company and they kept a work diary. On return to university, they wrote a reflective placement report which described their experiences living and working in a new environment and which also described the company, its structure and its work.

Our graduates were all employed within three months of graduation and employers usually spoke highly of them when asked for an opinion. We felt we were doing a good job, and, when we looked at the list of Learning Outcomes required by the entrepreneurial studies edict, we felt that we were already meeting what seemed to us to be the most important, those that made them so employable, the key skills of communication, inventive problem solving, team work and independent learning. We felt that they had the right attitude to life, being motivated to go out and get a good job that was satisfying and rewarding. The other entrepreneurial LOs seemed to us to be about setting up as self employed, small business managers or owners, and none of our students was going to do that! Well certainly not at the age of 22 or so.

But an edict is an edict. "Enterprising" was no longer enough; "entrepreneurial studies" were required. Through discussion with the entrepreneurial advisor we came to realise that in a culture like mathematics, where virtually all graduates went into the employ of some company, we saw that it would be valuable to our graduates as employees to have a greater knowledge of the real-world side, the financial side of businesses, both small and large. It would be valuable to them to have an entrepreneurial attitude, even in employment where they might demonstrate "intrapreneurialship", that is seeking to be innovative and inventive in their employment. Garavan and O'Cinneide say that "entrepreneurs are characterised by innovative behaviour and employ strategic management practices, the main goals being profit and growth."

3. Entrepreneurial studies

The teaching programme outlined in this section is, today, merely ideas and plans. It will be the second semester of 2003/04 before we get the opportunity to put these ideas into practice, so we have some time to improve our own knowledge base and to prepare suitable learning materials for our students.

The revised course in Mathematics and Computing has one fewer module in mathematics and one more module in statistics. One reason for the change to the course was that since many of our graduates found employment in statistical enquiry businesses, we considered it highly desirable to include more training in the use of sophisticated statistical software packages like SAS.

To achieve this change, we compressed the two mathematical modelling modules, previously taught in years 1 and 2, into one module called simply "Mathematical Modelling". The new statistics module introduced is called "Statistics for Industry with Entrepreneurship". Both will be taught in semester 2 of year 2, after a substantial foundation in mathematical methods and statistical theory has been laid. It is at this time that students will be thinking purposefully about their placement year and we believe that is an appropriate time to teach some entrepreneurial studies.

After discussion with the entrepreneurial specialist advisor, it has been agreed that we should teach all the key skills elements of our course in the same way as before, namely in the context of teaching students to adopt the way of life of industrial mathematicians. Problem solving, group working and communication are all learnt in a mathematics environment. It is planned to do this mainly in the new Mathematical Modelling module. Similar teaching and learning methods as are employed at present will be used. There will be student seminars and student project work, but now involving more advanced mathematics and statistics than was possible when the module was taught in Year 1. Emphasis is placed on good writing and presentation skills and on harmonious group work. Some of the early semester tasks will be "course requirements", in that students must engage with them in order to complete the module. But they will be assessed only formatively, that is, a grade will not be given but extensive formative feedback - including praise where it is earned - will be provided. This is a new venture for us (i.e. not grading and counting every single piece of coursework), but it is a venture that reduces the overall assessment anxiety, and a venture that is being promoted by some educational developers like Gibbs (2001).

The specific entrepreneurial studies are included in the new statistics module. We agreed with the advisor that, for our students, it would be sufficient for them to have theoretical knowledge of small business start-up procedures. We will teach this mainly as "head knowledge" but we expect it to become alive to them during and after their placement. We will expect them, while on placement, to explore with their employing company, the "real-world" aspects of business life. A survey of students returning from placement indicated that several of them found "learning about the work environment and the workings of business" to be one of the best features of placement. Most of the LOs listed in the Abstract are included in this module. Teaching and learning resources are currently available, provided through e-learning packages prepared by colleagues in Business and Management and by the entrepreneurial advisors. It is to some extent, an "add-on" to our syllabus, but it is placed beside material that will be very pertinent to students on placement and in graduate employment, namely: - Questionnaire Design and Analysis, Quality Control, Use of Statistical Databases and Advanced Statistical Software (SAS). The entrepreneurial studies element will be assessed summatively through a case study plus essay, and will look particularly at researching the market for a business opportunity and at setting up a new company.

We can think of only one graduate from our course over the years who could be described as a true entrepreneur, who is now running his own small business and employing at least one other graduate from UU. This man came to us as a mature student in his later twenties, he finished top of his class, he completed a PhD in a mathematical aspect of computer science, and he worked for a few years in the employ of a large software company in Northern Ireland. With the recession in the software industry in recent years the company was downsizing and there was a threat of redundancy. This gave him the impetus to become self employed and to set up a small software company, specialising in a niche market. His profile is typical of many entrepreneurs - a highflying student with some years of employment experience and with an innovative idea to sell. Not all of our students will be able to emulate this graduate, but part of the rationale behind the entrepreneurial studies programme is that more students will have the necessary knowledge and may be inspired with the necessary innovative idea or ideas.

4. Conclusions

The most valuable asset a student possesses is himself or herself, but they do not always appreciate this when they leave high school. Many of the students at UU come from working class social backgrounds and may be the first generation in their families to go to university. Many would come from an environment where unemployment is the norm and social security benefits and casual work are the main income sources. Such students have to learn to value themselves, their talents and the opportunities higher education affords them. Many are frightened by the prospect of borrowing money to complete their education; they work long hours at pumping petrol and stacking supermarket shelves in order to stay out of debt, and thus they leave themselves with less time for study and normal student socialising. Being able to create a personal business plan for themselves might help students in this situation. Setting themselves goals and ambitions to achieve should help motivate them to learn and not to be afraid of borrowing money to invest in themselves and the possibilities of their own bright future. It might help them to get their "head right", to adopt the attitude needed for success.

In this paper we have described the situation at the University of Ulster regarding the university-wide teaching of entrepreneurial studies. Students are expected to take at least half a module's worth of this subject (about 8% of one year's total). It is desired that these studies be embedded in mainstream subject teaching, rather than being a simple add-on half module. The course in Mathematics and Computing, which we have described, seeks to achieve this. It is based on our previous good experience of embedding the teaching of key skills and it makes use of expertise and resources available elsewhere in the university. We shall report at ICME-3 how successful (or not) we will have been.

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GRAPHING CALCULATOR AS A TOOL FOR ENHANCING THE EFFICACY OF MATHEMATICS TEACHING

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ABSTRACT

New trends in mathematics teaching have emerged during recent years. These trends are connected with the current approach to school mathematics as a component of education in general, where mathematics is viewed as a tool for practical life. The rapid progress of technology is one of the aspects that have affected mathematics teaching at all levels, including the preparation of prospective teachers.

In the traditional teaching of mathematics the teacher passes complete information to the students and the students are passive recipients, while the integration of technology (computers, calculators, www resources) encourages and enables new approaches and procedures in mathematics teaching and learning - in particular a deeper investigation of problems, discovery of connections between phenomena etc.

Furthermore, technology can help to develop a better understanding of abstract mathematical concepts by their visualization or graphic representation; we can show relationships between objects and their properties. Such deeper understanding of concepts will in turn increase the ability of the students to acquire a better working knowledge of mathematics.

The article deals with the utilisation of graphing calculators in pre-service education of prospective mathematics teachers at the Faculty of Mathematics and Physics of Charles University in Prague. We use this type of technology in some of the subjects included in their study programs – specifically in "Didactics of mathematics" and "Methods of Problem Solving". According to our experience it is advantageous to use graphing calculators in these subjects, especially when we introduce new important mathematical concepts, such as function, because most of the properties of functions can be found from the graphs drawn on the calculator display.

KEYWORDS: Preparation of teachers, graphing calculators, visualization, concept development, problem solving, equivalent and non-equivalent transformations, geometric transformations, derivatives.

1. Introduction

During several last years graphing calculators found their way gradually into secondary school mathematics of many countries in the world including the Czech Republic. According to the series of researches in this field [1], [3], [6] the using graphing calculators in mathematics teaching and learning can help the students to improve their knowledge and skills in some domains as concept development, problem solving, computation skills etc. Using graphing calculators in mathematics education bring also new methods of work - especially the possibility of *exploration and modelling of mathematical problems*, *multiple representation of mathematical problems* (numerical, algebraic, graphic, algorithmic representation) and *graphic support of the results obtained by algebraic procedures*. There are many positive aspects of the usage of graphic calculator in mathematics education (if we use this aid by proper way) and therefore it is necessary to react to this fact in pre-service education of mathematics teachers too.

Faculty of Mathematics and Physics of Charles University in Prague has amongst other programs also the one for prospective mathematics teachers, especially for the high schools. The study program for mathematics teachers is at the Master level, usually five years in duration. In the last two years of their study the prospective teachers get acquainted with the use of technology in mathematics teaching (including graphic calculators) in such subject as "Didactics of Mathematics" and "Methods of problem solving". Owing to our experiences (we investigated the influence of this tool on mathematics teaching and learning in classroom practice during 1993 - 1996) we have advised our students to use graphic calculators especially when teaching the topics of secondary school mathematics connected with the concept of function. In doing so, we emphasize to use it not only to study the definition of function, its properties and graphs, differential and integral calculus, limits of sequences, but also the solve equations, inequalities, their systems, investigating mutual positions of lines and regular conic sections and to study geometric transformations. In this respect it is appropriate to use graphing calculators in the *concept development process* (via its visualisation on the display), for the *simplification of the solution of mathematics tasks* and for *problem solving*.

According to psychologists, it is desirable to create an adequate image of concept issues out of its visualisation and also to involve students in a concrete experience with it. The process of concept acquiring is active and it consists of several parts: there is exerted visual cognition at first, followed by the verbal description of the image gained during the discussion with the teacher and classmates and resulting at the end by the image fixed through the students' own activity. The concept development is affected by many factors (e.g. student's motivation, knowledge, topic of learning), but teaching methods belong to the most important. In the particular parts of this procedure it is possible to use the graphing calculator as a tool for enhancing of the efficacy of this process.

In the following part the ideas mentioned above will be illustrated by several examples from different parts of school mathematics in such a way as we have used in didactical part of teachers' preparation at Faculty of Mathematics and Physics. In "Didactics of Mathematics" and "Methods of problem solving" we have used the graphing calculator TI-83 most of the time, because this type of calculator is used most often in our secondary schools. We have tested the algebraic calculator TI-89 in these subjects too.

2. Concept development in algebra, geometry and differential calculus

The solution of equations belongs among the basic skills in ***algebra*** course in the secondary school. The students have usually learnt to solve equations and inequalities using equivalent transformations. There is the basic concept - *an equivalent transformation* (i.e. the change of the algebraic form of the equation in such a way that the original, as well as the resulting equations have the same sets of solutions) and the students must understand the difference between the equivalent and non-equivalent transformation. The typical non-equivalent transformations are multiplication by an expression with a variable and also squaring. Traditional way for learning, which operations are equivalent, is memorising but using graphing calculator we can do it easier via graphical representation on the display. We can compare the solution sets at particular steps of solution process graphically.

Example 2.1

Where is a mistake?

$$x = 3$$
$$x(x-2) = 3(x-2)$$
$$x^2 - 2x = 3x - 6$$
$$x^2 - 5x + 6 = 0$$
$$(x-3).(x-2) = 0$$
$$x = 3 \lor x = 2$$

Solution

The solution set {2, 3} is wrong because the written procedure contains non-equivalent transformation which is applied from the first line to the second one – multiplication by $(x-2)$. We draw the graphs which are located at the second line and we can see (Figure 1) that there exist two common points, i.e. two solutions, in comparison with the first line $x = 3$. It means this transformation is not equivalent; more exactly, the multiplication by an expression is not equivalent if the expression is equal to zero.

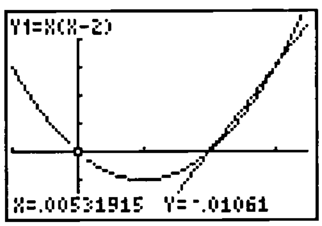

Figure 1

When solving equations or inequalities with radicals we usually use squaring and therefore we can obtain extraneous results that are not solutions of the original equation.

Example 2.2

Solve this equations $\sqrt{x+1} = x - 1$ for $x \in \mathbb{R}$.

Solution

At first we determine the condition for the radical ($x \geq -1$) to be defined and then we proceed by traditional method – we remove the radical by squaring:

$$\sqrt{x+1} = x - 1$$

$$x + 1 = (x - 1)^2$$
$$x + 1 = x^2 - 2x + 1$$
$$0 = x^2 - 3x$$
$$0 = x.(x - 3)$$
$$x = 0 \vee x = 3 \text{ (all solutions satisfy the condition for radical)}$$

When we substitute the results into the original equation we recognise that number 0 doesn't belong to the solution set. It is easy to show graphically that squaring transformation is not equivalent generally – we draw graphs from the first line (Figure 2) and from the second one (Figure 3) and we compare the number of common points.

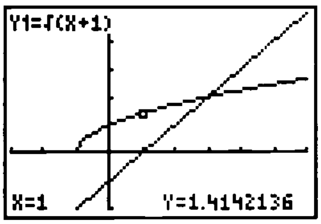

Figure 2

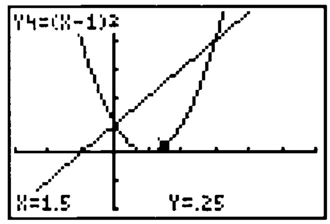

Figure 3

We can portray the concept of equivalent transformations by solving other examples while discussing whether or not squaring or multiplication is an equivalent transformation (e.g. find the solutions $\sqrt{x^2 + 2} = \sqrt{6 + 3x}$ or $\dfrac{6 - x}{x} = x + 4$). The graphing calculators let the students to concentrate on the introducing concept and not on the algebraic manipulation with expressions.

It is self-understood that we shall use graphing calculators in ***geometry*** for visualization of geometric objects in a plane (lines, circles and other regular conic sections) and investigate their mutual position and relationship. Furthermore, on the base of geometrical interpretation of algebra problems (if that is possible), we are able to develop the linkage between concepts in algebra and geometry. For example, when we solve the system of one linear and one quadratic equation we investigate the mutual position of line and regular conic section. It means, we can investigate the mutual position of two lines by comparing their slopes and y-intercepts, we can explore common points of conic section and line or the mutual position of two conic sections, depending on the type of equations (linear or quadratic) [7], [8]. This approach can assist in deeper conceptual comprehension that may influence the level of students' knowledge and skills in mathematics.

Another topic of geometry represents geometric transformations in a plane. The students learn to recognize distinct types of isometries (translations, axial symmetries, rotations) and also similarities (dilatation and shrinkage). The meaning of *isometry* (or the rigid motion) concept consists of the fact that isometry preserves distances and therefore it maps any geometric object to its image which is congruent with the original object. We can demonstrate it using graphic calculator effortlessly, because this tool (algebraic calculator) allow us to map any geometrical object to its image if we know the equations of the transformation.

Example 2.3

Find the image of the circle, the center of which has coordinates [-3, 3] and radius $r = 2$, in translation T: $(x, y) \rightarrow (x + 8, y)$. Compare the circle with its image.

Solution

We enter the parametric equations of the circle and equations of translation, we set up "thick curve" for the image and then we compare figures and radiuses (Figure 4, Figure 5) using the numeric functions of the calculator. Similarly we map lines, triangles etc. not only in translation but in other isometries too.

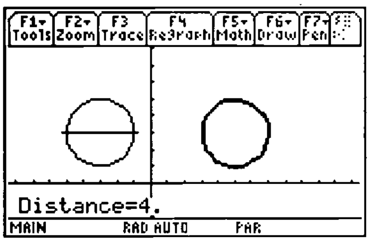

Figure 4

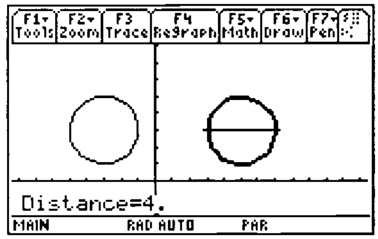

Figure 5

After that the students can confirm their observations about distances using the distance formula and coordinate geometry method.

The basic notion of ***differential calculus*** is the *derivative*. In mathematics teaching we usually utilize the graphical interpretation of the derivative to explain the students this important concept. Using graphing calculator it is easy to demonstrate this interpretation of the value of derivative at a given point as a slope of the tangent line to the graph of the function at the same point because the calculator is equipped by graphic command for drawing tangent line and expressing its equation (Figure 6).

In differential calculus the students learn to sketch the graphs of functions using the properties of the first and the second derivative, however, they sometimes forget that the first or the second derivatives are the functions too. We can draw the graph of function f together the graphs of its derivatives f', f''.

Example 2.4

Find the intervals in which the polynomial $f(x) = x^3 - 6x^2 + 9x$ is increasing or decreasing.

Solution

Let us write the function f and its derivatives in editor (Figure 7). At first we draw the graph of f and f' (thick line). To decide whether the function f is increasing or decreasing in some interval we have to determine where is the first derivative $f'(x)$ positive and where it is negative. It means to compute zeros of f'. We use the command "zero" and find the zeros $x = 1$ and $x = 3$ (Figure 8). The polynomial is increasing in $(-\infty, 1)$ and $(3, \infty)$, because there is $f'(x) > 0$; the function is decreasing in $(-1, 3)$ where $f'(x) < 0$. We can find the local maximum and minimum similarly. Using the graph of the second derivative of f we can find the intervals of concavity or convexity.

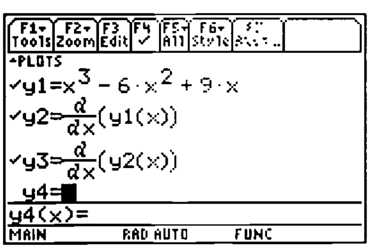

Figure 7

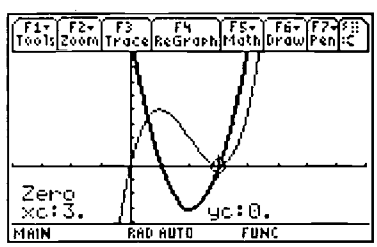

Figure 8

This procedure based on the graphic representation of abstract concepts is an invaluable instrument promoting the abstract idea to the knowledge and understanding of the topic.

3. Problem solving

The ability of using mathematics in practical life pertains to the important goals of mathematics teaching and learning. We have taught the students to solve the real world problems to show them the meaning of mathematics for their own life. However, this part belongs to the difficult ones. The solution process of the real world problems can be represented by the following schema [5]:

Real world		Model world
Problem	$\rightarrow$ *translate*	MP
		$\downarrow$ *solve, calculate*
Solution	$\leftarrow$ *translate*	MS

The first phase of the schema can be very laborious; transformation of the problem into the adequate mathematics model is the main phase in the process of the successful solution of the real problem. The students can use the graphing calculator not only in the second phase (calculation) but also in the third one. After finding the mathematics solution, the students need to verify that the results solve the real problem. The students are trained to recognise and interpret a "peculiar" solution (e.g. 2,51 pieces of bicycles or their negative number) but what about "nice" non-real solutions?

Example 3.1

In the warehouse the iron tubes are arranged into the layers in such a way that the tubes of higher layers fit in gaps of lower layers. We want to store 75 tubes into the layers with the lowest layer of 12 tubes. How many layers will we need?

Solution

We find the mathematics model - arithmetic sequence with the first term $a_1 = 12$, difference $d = -1$ and $s_n = 75$. We need to calculate the value of n, which means to solve the quadratic equation $n^2 - 25n + 150 = 0$. The roots of this equation are $n = 10$ and $n = 15$. Which of the results are solutions of real problem? We draw the graph of the arithmetic sequence with $a_n = 12 - (n - 1)$ on the display and we see that for $n = 15$ the number of tubes in 15-th line is negative -2 (Figure 9).

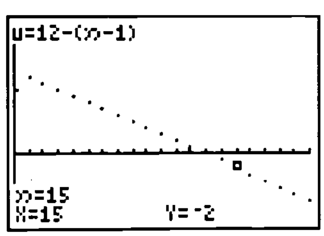

Figure 9

4. Conclusion

The procedures and examples mentioned in this contribution use the power of visualisation in learning process. The visualisation can help the student to understand and remember better the mathematical abstract concepts via their graphic representations, and graphic calculators can mediate this visualisation quickly and comfortably. Thus, graphing calculators represent the helpful tool for mathematics teaching and learning. However, the actual result depends on teachers themselves.

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USE OF INFORMAL COGNITION IN TEACHING MATHEMATICS

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ABSTRACT

The new curricula are based on the premise that the inclusion of mathematics of every day life in the teaching of mathematics is very important in order to make school mathematics meaningful. In Greece also, there is the same spirit in the new curriculum, without it being noted that the mathematics of every day life is not common for all students.

Through a comparative study, we have already conducted, in two culturally different groups of students we found that they carry in to their school different informal cognition. In particular, we have posed activities, based on conditions of every day life, to a group of gypsy students and to a non-gypsy group. The way these groups negotiated the activities made it obvious that the different cultural context dictates different strategies in problem solving. This leads to the conclusion that, formal education should take in consideration the backgrounds students have.

1. Introduction

Over the last few years there has been an ongoing interest in socio-cultural elements that are related to the teaching and learning mathematics. This turn of researchers and mathematics educators is a consequence on the one hand of the fact that cognitive approaches don't give working answers for school failure and on the other hand of the fact that current research shows that mathematics and social-cultural context are indissoluble related.

Stefano Pozzi (1998: 105) and his associates note that the cultural context is an important framework to think about mathematical activity since people think and act within these contexts. "New investigations tend to focus on how activities are shaped by the social practices and to examine how this shaping informs our understanding of mathematical behavior and learning".

Lave (Lave 1988) through her research distinguishes every day practices from school mathematics. She notices that in every day practices individuals use any available resource –based on common sense— in order to transform and solve problems, as there are no imposed strategies.

Freudenthal (1991:7) points out the very important role of common sense as the root of early mathematics development: "In the course of life, common sense generates common habits, in particular where arithmetic is concerned, algorithms and patterns of actions and thoughts, initially supported by paradigms, which in the long run are superseded by abstractions."

There is evidence from around the world that children develop mathematical knowledge through the every day activities in which they are involved. (Brenner 1998: 216) And as the activities depend on cultural context the children acquire the corresponding informal cognition.

Although this is something obvious very often the cognition the children acquire before –or out of—schooling and "which is usuful one for every day life and work lost during the first years of schooling........The former spontaneous abilities have been downgraded, repressed, and forgotten, while the learned ones have not been assimilated. Thus, early education instills a sense of failure and dependency".(D'Ambrosio 1985)

Curricula of the past that used to focus on typical and formalistic teaching were characterized by the underestimation of this kind of cognition. The new curricula focus on the inclusion of every-day mathematics in teaching school mathematics.

The usage of mathematics based on students' every day experiences --meaningful mathematics for all students-- is at the heart of the mathematics reform movement[i]. Likewise, in Greece, in the new curriculum the including of mathematics in classroom teaching and the development of what students have already learned is one of its objectives. Particularly, through problem solving, what is aimed at is "knowledge stabilization and application of what students have already learned through matters of their experience and through their environment". (Greek Curriculum)

The fact that the new curriculum mentions the value of using every-day mathematics is of course of great interest. Nevertheless, it is not taken into account the fact that mathematics of every day life are not the same for all students; students who come from different cultural groups other than the dominant one. So it is considered that low school aptitude and achievement of minority groups students is their own responsibility.

What it is presented here concerns a comparative study that has been conducted in groups of Romany and non-Romany students. Through this we can see the results we have if activities referred to are familiar and also the way of negotiation isn't strictly formal.

2. Context of the research and of the group

The study we present is based on research, which is conducted in the framework of a Ph.D. dissertation[ii]. A part of the first interesting findings that concerns the four operations is presented here.

The methodology that has been used is both ethnographic and educative. It is ethnographic in relation to the tools of data selected to find the answer to "what" connects cultural context and mathematics; it is educative in its purpose to make proposals that could improve mathematics education of the particular group and probably of minority groups, in general.

The main part of the fieldwork is a first grade class of exclusively Romany children. In the school there are also mixed background classes and pure non-Romany ones. During the past year we conducted observation, we posed activities as well as interviewing the students.

We extended the observation during the break to the school canteen where the students had to undertake transactions. The ease with which the Romany students conducted their dealings was remarkable and it is obviously a consequence of their way of life - of their cultural context.

Their different cultural context consist of the following elements:

-Semi-nomadic way of life with directs consequence on their schooling such as the time of starting school and the inconsistency in attendance.

-The socio-economic organization which is based on family and so children are involved in their families' business and through a horizontal way of teaching they become familiar with doing mental calculations.

-The fact of being a minority group which is related to their background and also to their limited expectation of education depending on their cultural fund.

Initially there were about 30 students, but after Christmas holiday there remained about ten. Only three of them had the corresponding age of their class --among them a girl. The rest were aged of ten to twelve. Some of the students were brothers with two and more years age difference.

The observations extended to the students' families and mostly to their businesses in order to examine the context in which the students develop. The parents of all the students deal with commerce. The majority of them are street greengrocers or sell household items on the street. One family apart from street commerce had got a small shop where we also observed young children in money dealings.

During the observation we were impressed by the fact that even children who were only three years old were dealing with money. Although they didn't know the value of coins they managed to carry out their purchases. Even children of five years old could distinguish between coins, mainly the ones they use more frequently.

Through the research we realized that Romany children are very much familiar with doing calculations—especially regarding money dealing. The following questions are posed:

1. What kind of informal cognition do students acquire through their every day practices?
2. Does this informal cognition dictate solving strategies in the class and out of it?
3. Could this cognition become a suitable didactical context, especially for these students, to teach mathematics and so to improve their aptitude?

3. Presentation of activities

Á. During the first period of the research in October, we conducted a test, mostly diagnostic. Among the activities there were two with money dealings context:

a. You have 5 hundred drachmas and you want to buy two cheese pies. If every cheese pie costs 2 hundred drachmas, would the money be enough?

b. Your father has given 1 thousand drachmas to your brother and to you five hundred drachmas, four hundred drachmas and two fifty drachmas coins. Has either of you got more money than the other? if so which of you?

At this time the number of students was about twenty-five. Twelve of them were selected as a sample representative in relation to age, gender and aptitude. The test was administered to each student separately at different moments and the students didn't collaborate.

With the exception of one girl who possibly got confused with the actual price of the cheese pie, all the rest answered correctly to both questions, although in some cases they could not justify their answers. In relation to the first question, the majority of them answered spontaneously and how much the change was. Almost all the anwers were of this kind:

"yes, I get 1 hundred drachmas change"

"yes, and I keep 1 hundred drachmas "

Some of the answers to the second question were:

" the same we get together, the same we get together"

"mine becomes one thousand"

"1 thousand, are all of these"

"9 hundred, and two fifty drachmas coin, 1000, the same"

"they will become the same, he gets as much as I get. ***We know them Miss****"*

We would like refer to an example of the way in which the students justify theirs answers:

"how did you find the answer;"

"I thought it up, in my mind Miss"

"please try to explain to us Anna!", (Anna was a girl of 7 years old.)

"my mother told me that 4 hundred and 5 hundred and 2 fifty drachmas coins give us 1 thousand"

A boy, of the same age, justified his answer with this way:

"I was looking for, I was looking for, I was looking for ...."

Since the results were fascinating the test was tried in a non-Romany first-grade class. The sample was one boy and one girl with the best aptitude, one boy and one girl of bad aptitude, according to their teacher's estimation.

There are quoted all of theirs answers to show the differences.

-*"No, I need 1 thousand"*, (to buy two cheese pies).

"My sister" (has got more money)

-*"Yes, the money is enough. I don't know how much change"*

"I get more than 1 thousand. I don't know how much"

-*"It is enough, I get 4 hundred change"*

"Fanis", he means his brother gets more money.

-*"No, the money is not enough"*

"To me"

B. Activity

In this class a group of 4 students –with ages ten to twelve- had separated themselves from the rest of the class and progressed at their own pace. The activity that is presented here is in a familiar context for them; even the name of the student and his father's occupation was true.

Basilis wanted to help his father to distribute apples in crates, which his father had got from the vegetable market. All the apples were 372 kg and every crate hold 20kg. How many crates does he need in order to put in all the apples?

Because of the limited of the space only some parts of the negotiation of the activity there are presented here. Students were encouraged to cooperate, without being obligated it.

(Apostolis was drawing lines on his desk: for every crate one line).

R: *please, tell me Apostolis what are you doing here?*

A: *10 crates Miss.*

R: *How many kilos do the ten crates hold?*

A: *20 kg every crate.*

R: *So....*

A: *Well, 20, 40, ...180, 200.*

R: *And how many are there?*

A: *372*

Cr: *I am thinking Miss....*

J: *(He continues) 220, 240, ...*

R: Y*ou Cris, what are you doing;*

Cr: *On this hand 72, the 60*

E: *What sixty; you mean sixty crates;*

Cr: *I don't mean crates, 3 crates.*

R: *How did you find it; (at the same time Apostolis and John continue to step by 20 up to 372).*

Cr: *I said 20 (he shows for every crate one finger) and 20 and 20, 60 and the rest are 12. I get for these (and shows the hand he imagines that he has the 300 kilos) 8 more, so I have 4 crates.*

....

Cr: *I get from the 300 the 8, 4 crates miss, 8 and 12 miss the rest are........ 302. No they are ....*

A: *250.*

Cr: *Wait! 292.*

R: *Bravo Cris. You had better write down the number so that you don't forget it.*

Cr: *I get from the 292, the 20, 5 crates, and the rest 272. Is it ok miss*

....

J: *Should I also do the same miss?*

....

Cr: *10 crates and the rest are 172. I get some more and they become 152. I am correct* (with self-confidence).

J: *look at him miss, he is doing them, he is doing them!!!* (with admiration).

Cr: *I get one more. I have 12 crates and the rest are 132. Now I get these 20 and the rest are 112, and I have 13 crates, all right?*

A: *All right.*

Cr: *Then, miss the rest are 112. Am I right; from the 100.... What I have done now, I am confused.*

E: *You are here at 112, you get the 12...*

Cr: *And I get 8 more from the 100, and now I have 92.*

A: *look at it! Look at it!*

Cr: *I put one more here (he means one little cube, he used for corresponding the crates)*

....

A: P*ut one more crate, the rest kilos are 12*

Cr: *Twelve*

....

Cr: 1,2, *19* (he counts the cubes)

E: *So, we needed 19 crates. What did you find?*

A: *19*

E: *Could you show as your one solution;*

A*: (Corresponding every line to 20 kilos) 20, 40, ...*

B: *100, 120, ...360, 380.*

E: *This last crate is going to become full?*

All together: *no*

Reviewing Chris solution we see that he used continuant subtraction. He got to subtract, from 372, 20 kilos at a time and so he found the number of the crates were needed.

It must be noted here that the students didn't know the algorithm of division, as they are students of first grade level. What they had been taught were simple operations with number up to 20.

After this, the test was tried in a fourth grade non-Romany class to see the differences in relation to children of the same age that had been taught the algorithm of division.

The results of the test were very important and very different from the results of the Romany students. Namely, none of them managed to find the correct solution. Only a few of them selected the correct operation and nobody of them used the algorithm correctly. The majority of them selected the operation of subtraction in order to solve the problem. Although they found illogical results they didn't question them. Many of the answers were: "he needs 352 crates", having selected the operation of subtraction. Some others doing multiplication: "he needs 7440 crates".

After these very disappointing results we conducted the test in a fifth grade class. Here six students of a total number of 18 selected the correct operation and also performed the algorithm correctly. Some of the rest selected the correct operation but made mistakes in calculation. About the half of the total number of students, used subtraction and multiplication. Of the remainder one of them firstly did multiplication and then to check, he did division. As he found the correct number (372) he was sure that his answer was correct.

4. Discussion

The way Romany students manipulated the activities makes it obvious that they had acquired concrete informal cognition through involvement in their parents' business. They have become conscious of the fact that: if you have to solve a problem which presents itself you should invent any suitable strategy to deal with it.

From the money dealings activity it is evident that money is a particularly familiar context to Romany students for teaching mathematics in primary school. Maybe it is useful to note the fact that Romany students approach decontextualized problems differently than in a money context. For example:

-5+3 =?

-......... (pause)

-You have 5 hundred drachma and your mother gives you 3 more.

-8 miss.

-

In the second activity we see that Romany students invented different strategies --subsequent subtraction, subsequent addition-- even the strategies they selected were based on the same store of cognition. As the context was a familiar one for them they faced it in the way they would do it in their life, out of school. They got over the fact that they hadn't been taught the algorithm of division, mobilizing what they had already learned; mostly the out of school cognition.

Chris acted in an absolutely 'natural' way. His strategy reflected a real situation. If he had been called to solve a problem like this he would probably have taken 20 kilos at a time and put them in the crates. In this negotiation he used addition, subtraction, multiplication and also correspondence[iii] in order to solve a standard division problem in terms of formal education.

The main feature of the other three students' strategy was the spontaneous selection of correspondence: for every crate (20 kilos) of apples they drew one line on their notebooks. This was also based on their parents' every day practices as observed during the research. Apart from that, the students used multiplication in the form of subsequent addition.

The results become more important if we compare them with those of the students of the dominant cultural group.

The results become more important if we compare them with those of the students of the dominant cultural group.

Firstly, in relation to the money dealing problem it is evident that the students of the first grade level didn't have any familiarity with money dealings as unlike the Romany students they don't use to have these kinds of dealings without someone else who has the responsibility.

More remarkable is the fact that the students of 4th and 5th grade (10-11 years old, that is the age of the Romany students) although they had been taught the operation of division and its algorithm didn't get the correct results. It is worth how far formal education leads the students to solve problems mechanically and not to care if the problems have any meaning for them. They didn't feel the need to reconsider of the results—to see if they were logical. They accepted even the number of 7440 crates in whicht to put 372 kilos of apples.

So we think that what arises here is the fact that these students who are presented as having low aptitude for school mathematics simply don't have the suitable didactical context for it. If we had tested the same groups of students in strictly typical form it is certain that we would have had different results.

5. Conclusions

The weakness of the educational system to be designed or at least to be adaptable for Romany students --and generally for students with cultural diversity-- is presented as incompatibility between typical education and Romany children. More than this, formal education ignores or has the contempt for the cognition children acquire through their everyday context.

The particular case is indicative of the particular cognition students have got as they live in a group with different cultural elements such as their involvement in families business. Also, an other point that differentiates them is the non-corresponding age to their grade.

If we accept that children learn more easily through problems of every day life—problems through their experience fund—it is necessary research to be conducted that tracks down which are the practices and the techniques that are used and what kind of informal cognition arises from them.

Then this information should be utilized in design the curricula and also to educate teachers. It is very important for future teachers to know the cultural differences of a group, the special practices and the cognition the children could carry to school being members of this group.

If the teacher is contemptuous of and rejects cognition that children carry from their home culture the consequence would be school failure by students by different cultural groups and their alienation instead of empowerment.

School also should compose the expectations[iv] these students have from education and the objects of the typical education. In this framework a crucial issue arises: how school could develop the students informal cognition and at the same time find suitable didactical ways to pass in the typical form.

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[i] As Brenner (Brenner, 1998a: 216) points out.

[ii] The title is "The Role of Cultural Context for Mathematics through the Study of an Ethnocultural Group".

[iii] During the process of solving the problem he used a cube for every crate (20 kilos) of apples.

[iv] To notice that they are related with students background.

THE USE OF THE JIGSAW IN HYPOTHESIS TESTING

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ABSTRACT

The Jigsaw is a cooperative learning technique in which the class is first divided into expert groups that are assigned different but related tasks. New "home" groups consisting of one member from each expert group are then formed. Each expert instructs the other members of this new group about what they have learned. In our application of the jigsaw, we use the expert groups to give students the opportunity to study one particular example in-depth. We use the home groups as a way to compare the different examples, and observe their similarities and differences. This gives the students a chance to create their own generalizations. We provide two examples of how the Jigsaw can be used in an introductory statistics class. The first example is for presenting different sampling techniques and a more advanced application is for introducing hypothesis testing. In the latter, we have found it effective to have expert groups use experiments to investigate a specific claim. Our experience indicates that these preliminary concrete activities will provide a smoother transition to understanding the formal and symbolic hypothesis testing framework.

Keywords: cooperative learning, hypothesis test, jigsaw, statistics

1. Introduction

It has been the experience of the authors and others (Rogers, et. al. 2001) that group activities can enhance the learning and understanding of many statistical concepts. Yet many times the students need some structure to guide their explorations. One technique that can be useful when there are a variety of similar tasks is known as the "jigsaw." This technique gets its name from the fact that individuals or groups each study a piece of a project and then the students put the pieces together to get a complete picture of the project. It is reminiscent of many spy stories in which the only way a code may be broken or a fortune recovered is if each individual in the plot contributes their portion of the puzzle.

In the cooperative learning setting, the jigsaw organizes students together in what have been called expert groups. Each expert group does a specific task and then new "home" groups are formed so that each home group includes at least one member from each expert group. Each individual expert is expected to convey their knowledge to the rest of the group members, that is, contribute their piece to the material to be mastered. The method has been primarily used in the elementary school classroom to teach social studies and reading. One of its strengths is that it creates a learning environment that "made it imperative that the children treat each other as resources" (Aronson, et. al., 1978).

The jigsaw is often used when the tasks are different but similar. It has been used in situations such as studying different positions of a certain issue and for sharing the workload of reading several related articles. In mathematics, it has been used to study the properties of complex numbers (Lucas, 2000) and for sharing the calculation tasks in standard statistical procedures (Perkins & Saris, 2001). In our own classes, we have found the technique useful when presenting the proofs of properties of an algebraic structure (for example, the proofs of the properties of logarithms or determinants.) The students often do not recognize the similarity in the structure of the proofs of these distinctive properties. However, when each group is assigned a specific proof and then they teach other students, they tend to more readily see the patterns among them. When used in this manner, perhaps the greatest benefit of employing the jigsaw is that the students notice *for themselves* the differences and similarities in these procedures. In fact, it was the authors' many frustrating experiences trying to point out to students such connections in the hypothesis testing of various parameters that led them to seek out an alternative pedagogy.

This paper details an implementation of the jigsaw to introduce new and somewhat complicated material. In what follows we suggest activities for both the expert groups and the home groups in each of the areas of sampling design and hypothesis testing.

2. Sampling Design

A good introduction to the use of the jigsaw early in the semester is in the study of sampling techniques. For the pieces of the jigsaw, the class is divided into groups and each of the groups studies one of the following sampling techniques: random, systematic, cluster, stratified and convenience sampling. Then they use that particular technique to collect data to estimate, for example, the average number of words per page in a dictionary. Similar activities have been used before to teach sampling techniques (Paranjpe & Shah, 2000), but not by the jigsaw method. New groups consisting of at least one member from each expert group are formed and each "expert" teaches their sampling method to the rest of the group. The puzzle is now complete. As a follow-up exercise to assess individual student's understanding of each of the procedures, the students are assigned to collect data to calculate the average number of advertisements in their favorite magazine using the different sampling methods. This stresses individual accountability and gives us feedback on whether or not the "experts" were successful in teaching other students.

We observed the following advantages in using the jigsaw in this setting. Because of the simplicity of the tasks, students are able to gain experience in studying a concept on their own, work with other students to plan and implement a task (all the work is outside class time), and analyze the results of the task. The students tend to be more focused and pay more attention to explanations given by other students.

3. Hypothesis Testing

In our experience, we have observed the following difficulties that students have with hypothesis testing. The obstacles occur at both the procedural and conceptual level. These observations are consistent with other studies (Hong, 1992; Albert, 2000).

(a) Inability to distinguish a test of hypothesis situation from other situations such as estimation or finding probabilities.
(b) Failure to recognize the population parameter to be tested and whether more than one population is involved.
(c) Difficulty specifying the null and alternative hypothesis and determining the rejection region.
(d) Confusing the sample and the population, resulting in a weak conceptual understanding of the role of sampling distributions in statistical inference.
(e) Difficulty interpreting their conclusion to reject or not to reject the null hypothesis in the context of the problem.
(f) Poor understanding of the reasoning behind the structure of hypothesis testing even if they can procedurally do all the textbook exercises.

In our opinion, the reasoning behind hypothesis testing is a natural process, one that is common to the students' experiences. This is why many instructors and textbooks use familiar examples such as cards, dice and coins to demonstrate the principles of statistical reasoning (Rossman, 1996; Maxwell, 1994; Eckert, 1994). We think that many of the above noted difficulties develop as the students try to translate their intuitive (and usually correct) notions of hypotheses testing to a formal statistical framework. In trying to learn the steps of hypothesis testing, as well as the terminology and the symbolism, the students tend to lose sight of the intuitive reasoning behind the process.

For this reason, we focus our pedagogy on the affirmation of the students' intuitive notions of hypothesis testing and then on the transference of this knowledge to the components of the formal framework of hypothesis testing. In order to facilitate this transition, we would like our students to first realize that the reasoning behind hypothesis testing is natural and within their range of experiences. Then we introduce the students to the formal procedures of a hypothesis test in a specific setting. Finally, we require our students to reflect on the *process* of hypothesis testing by studying several examples in a variety of accessible contexts.

One can focus lectures to address these concerns and we have observed that most students do see patterns in hypothesis testing, but that they tend to focus on the pattern in the formalism and symbolism rather than on the reasoning behind the formalism. As a result, they tend to view hypothesis testing as a collection of independent algorithms, one for each parameter. We feel that group work would allow the students more opportunity and time to grapple with the issues involved in making the transition to the formal framework.

However, it has been our experience that understanding one example in depth, even in a group setting, is not sufficient to reinforce the students' innate understanding of inference. In particular, we observed that students have trouble applying the knowledge that they gained in studying a specific example to similar situations. We believe that it would be helpful if the students had the chance to work through many similar, but related examples, so that they can identify common patterns, as well as differences. But of course, this is time-consuming.

Our implementation of the jigsaw, as described below, gives students this opportunity, in a relatively short amount of time. By studying a single problem in their expert groups, and then sharing each other's work in the students' home groups, the students are guided to recognize that they are procedurally doing the same kinds of tasks in different settings---thus focusing on the structure and not the technical details. Also, their understanding is reinforced when they communicate their knowledge to other members of the group.

4. Implementation

The class is divided into "expert" groups of three to five people each. Each group is given a worksheet with a story that involves testing a claim. In our statistics classes (typically classes of 25 – 35 students), there will be 6 to 8 groups and two "expert" groups working on identical claims. The claims we use vary in three aspects: the parameter to be tested (proportion or mean), the rejection region (one-tailed or two-tailed), and the conclusion of the test (reject the null hypothesis H_0 or not reject the null hypothesis H_0). A sample worksheet is included in the appendix, and examples of claims we have used are listed below:

1. Is a specific coin fair? (test for a proportion, two-tailed, do not reject H_0)
2. Is a specific coin more likely to show heads? (test for a proportion, one-tailed, reject H_0)
3. Are there about 56 M&M's in a 47.9-g bag? (test for a mean, two-tailed, do not reject H_0)
4. Is the average number of M&M's in a 47.9-g bag greater than 42? (test for a mean, one-tailed, reject H_0)

The numbering system used in the worksheet parallels the formal steps used in hypothesis testing and the wording of the questions in number 3 of worksheet #1 emphasizes the condition under which a hypothesis test is conducted, namely, that the null hypothesis is true.

After completing this part of the activity, the students read a summary of the hypothesis testing procedure. The terminology, symbolic representation and step-by-step procedure are introduced, except that there is no mention of the significance level nor is there a formal presentation of Type I and Type II errors. Then they translate their earlier observations into the formal framework (see sample worksheet #2 in the appendix).

To put the pieces of the jigsaw together, the class forms new "home" groups with at least one person from each expert group. Each "expert" teaches the rest of the group about their problem, procedure and results. One advantage of duplication of group assignments is that there are two experts for that problem in the home group, thereby reducing the possibility of having an ineffective "expert." To complete the picture of this statistical jigsaw puzzle, the students are asked to synthesize their results by noting the similarities and differences in their various tasks. Specifically, they are asked to address the following:

(a) Explain your problem and solution to the other members of the group.

(b) What similarities do you observe in all of your activities and results?
(c) What differences do you observe?

At the end of the group activities, the instructor leads the students in a class discussion to summarize their findings and address any misconceptions or missed conceptions. The amount of class time needed to complete these activities is about 1 ½ fifty-minute class periods.

5. Development of our Implementation

Our current implementation of the jigsaw is a result of two refinements of previous implementations, and is still undergoing revisions. In our initial attempt at adopting this strategy, we assumed (incorrectly) that the students' intuitive notions were well-developed and so we concentrated on the transition to the formal framework. At that time, the activities of the expert groups started with a worksheet similar to our current worksheet #2. In addition, the experiments were more complicated – some involving the difference of two population parameters – and there was no replication of the individual problems among the expert groups. Also, the types of problems were too varied for the students to readily see the common themes, and they focused on the experiments rather than on the structure of the tasks. The result was a situation that was more time-consuming and frustrating (for both students and instructor) than was anticipated. The exercises were completed with much instructor input to both the expert and home groups.

Despite the problems encountered, we were encouraged to persist with this approach, albeit with modifications. In the second round, we had the students first use the jigsaw technique to learn sampling designs so as to familiarize themselves with the technique in a relatively easy setting. Also, we simplified the experiments and focused the student's attention on the informal aspects and natural logic of hypothesis testing. This time, we felt we were more successful in developing the students' confidence in their natural problem-solving abilities. However, we were not satisfied in how the activities bridged the gap to the formal structure and terminology of statistical inference

Our combined experiences have led us to develop the particular version that is discussed in this paper and we will continue to modify the exercises and worksheets as needed. We have also developed computer simulation activities for two of the experiments to enhance our students' understanding of how a decision criterion is chosen to reject the null hypothesis. These explorations leads the students to the concepts of significance level, Type I and Type II errors.

6. Summary

As noted in the introduction, the jigsaw has most often been applied at the elementary school level and in the areas of reading and social science. In statistics, it is most often used to divide up a tedious calculation task, for example, having each expert group find the summary statistics for a particular sample to be used in an application of an ANOVA (Perkins and Saris, 2001). In this paper, we go beyond these usual ways of using the jigsaw. We use the technique in a statistics class to introduce a concept (hypothesis testing), by giving the students a chance to study different examples of this concept, in order to lead them to make their own generalizations. Their observations form the basis of further classroom discussion. We feel that in this way, a concept, procedure or formula would be more grounded in the students' experience.

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WORKSHEET #1: Introduction to Hypothesis Testing

Paul and Joshua are again squabbling over what TV show to watch. Their mom suggested that they flip a specific coin – if it comes up heads, Paul would choose the show; if it comes up tails, Joshua would decide. Both boys were hesitant to agree, thinking that the other one had an advantage. Mom says: " It's a shiny new coin – I am sure it is a fair coin."

1. **Hypotheses**
   This scenario involves two claims.
   (a) What is the mother's claim? (This claim is the prevailing view about any new coin.)
   (b) The boys have a different, or alternative viewpoint. What is their claim?

2. **Experiment Design**
   The boys ask their mother if they can prove to her that their claim is correct. Design an experiment that the boys can perform and write your procedure below. DO NOT conduct the experiment yet.

3. **Decision Criterion**
   The mother agrees to the boys' experiment but she insists that they must all agree beforehand on what criteria they will use to decide if the coin is fair. They discuss the following situations. Imagine that you are part of the discussion, and answer the questions below.
   (a) What results would you expect if the coin is fair?
   (b) What results would you expect if the coin is not fair?
   (c) If the coin is fair, is it possible to get 80% or more heads? Is it likely to get 80% or more heads?
   (d) If the coin is fair, is it possible to get 10% or fewer heads? Is it likely to get 10% or fewer heads?
   (e) If the coin is fair, is it possible to get 45% or fewer heads? Is it likely to get 45% or fewer heads?

   A **decision criterion** is a method to decide whether a claim is valid before conducting an experiment. An example of a decision criterion is the following:

   Paul and Joshua are correct in saying that the coin is not fair if the experiment shows 55% or more heads or 45% or fewer heads.

   In your group, develop a decision criterion for your experiment that is acceptable to all members of the group, and write it below.

   At this point, consult with your instructor before proceeding.

4. **Gathering the Evidence**
   Conduct the experiment and record your results below.

5. **Decision**
   Based on your results from #4 and your decision criterion from #3, is the boys' claim that the coin is unfair supported? Is the mother's claim that the coin is fair supported?

WORKSHEET #2: Terminology and Framework of a Statistical Test of Hypothesis

A formal statistical test of a hypothesis has several components.

1. Hypotheses. Every hypothesis testing situation has two competing hypotheses or claims. One is called the *null hypothesis* and is denoted H_0. This hypothesis represents the prevailing view or the status quo. Others may believe that the null hypothesis is not true. Their viewpoint is a competing, or *alternative hypothesis*, denoted by H_1. In the situation in the first worksheet, Paul and Joshua disagree with their mother's claim. Using this formal terminology for competing claims, state in ordinary language the null and alternative hypotheses for your experiment. Then restate these hypotheses in terms of the binomial parameter p, the proportion of heads in n tosses.

	Words	Symbols
H_0:		
H_1:		

2. Experiment Design. Now that you know the claim that you (acting as Paul and Joshua's representatives) want to establish, you need to collect data to support your alternative hypothesis. However, until you can support your alternative hypothesis with evidence, you must conduct your experiment under the assumption that the null hypothesis is true.

3. Rejection Region. The term *rejection region* refers to what was called the decision criterion in worksheet #1. This region is a range of what you believe to be unlikely values obtained from your experiment if the null hypothesis were indeed true. In other words, it is a range of values obtained from your experiment, which would convince most people to reject the null, or prevailing hypothesis (this is why it is called the rejection region), in favor of your claim H_1. In your experiment, the rejection region would be those values of $\hat{p}$ that you think would be unlikely if indeed the null hypothesis is true.

What is the rejection region for your experiment?

4. Test Statistic. The test statistic is the evidence obtained from an experiment that will be used to try and refute, or reject, the null hypothesis. Generally, you compute the sample statistic that corresponds to the population parameter used to state the null and alternative hypotheses. The population parameter of interest in your experiment is the proportion or percentage of heads when a coin is flipped. The corresponding sample statistic in your problem is thus the percentage of heads observed in the sample. What is the value of the test statistic $\hat{p}$ from your experiment?

5. Decision. Lastly, a decision is made based on whether or not the value of your test statistic falls in the rejection region given in #3. If it falls in this region, then you would reject the null hypothesis and support the alternative claim. If it does not, then you cannot reject the null hypothesis. In your test, will the null hypothesis be rejected? If so, state your conclusion in ordinary language. If the null hypothesis is not rejected, state this conclusion in ordinary language.

Whatever your decision, there is a chance that your conclusion is not correct. Explain how this can happen.

THE USE OF THE JIGSAW IN HYPOTHESIS TESTING

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ABSTRACT

The Jigsaw is a cooperative learning technique in which the class is first divided into expert groups that are assigned different but related tasks. New "home" groups consisting of one member from each expert group are then formed. Each expert instructs the other members of this new group about what they have learned. In our application of the jigsaw, we use the expert groups to give students the opportunity to study one particular example in-depth. We use the home groups as a way to compare the different examples, and observe their similarities and differences. This gives the students a chance to create their own generalizations. We provide two examples of how the Jigsaw can be used in an introductory statistics class. The first example is for presenting different sampling techniques and a more advanced application is for introducing hypothesis testing. In the latter, we have found it effective to have expert groups use experiments to investigate a specific claim. Our experience indicates that these preliminary concrete activities will provide a smoother transition to understanding the formal and symbolic hypothesis testing framework.

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This paper details an implementation of the jigsaw to introduce new and somewhat complicated material. In what follows we suggest activities for both the expert groups and the home groups in each of the areas of sampling design and hypothesis testing.

2. Sampling Design

A good introduction to the use of the jigsaw early in the semester is in the study of sampling techniques. For the pieces of the jigsaw, the class is divided into groups and each of the groups studies one of the following sampling techniques: random, systematic, cluster, stratified and convenience sampling. Then they use that particular technique to collect data to estimate, for example, the average number of words per page in a dictionary. Similar activities have been used before to teach sampling techniques (Paranjpe & Shah, 2000), but not by the jigsaw method. New groups consisting of at least one member from each expert group are formed and each "expert" teaches their sampling method to the rest of the group. The puzzle is now complete. As a follow-up exercise to assess individual student's understanding of each of the procedures, the students are assigned to collect data to calculate the average number of advertisements in their favorite magazine using the different sampling methods. This stresses individual accountability and gives us feedback on whether or not the "experts" were successful in teaching other students.

We observed the following advantages in using the jigsaw in this setting. Because of the simplicity of the tasks, students are able to gain experience in studying a concept on their own, work with other students to plan and implement a task (all the work is outside class time), and analyze the results of the task. The students tend to be more focused and pay more attention to explanations given by other students.

3. Hypothesis Testing

In our experience, we have observed the following difficulties that students have with hypothesis testing. The obstacles occur at both the procedural and conceptual level. These observations are consistent with other studies (Hong, 1992; Albert, 2000).

(a) Inability to distinguish a test of hypothesis situation from other situations such as estimation or finding probabilities.

(b) Failure to recognize the population parameter to be tested and whether more than one population is involved.

(c) Difficulty specifying the null and alternative hypothesis and determining the rejection region.

(d) Confusing the sample and the population, resulting in a weak conceptual understanding of the role of sampling distributions in statistical inference.

(e) Difficulty interpreting their conclusion to reject or not to reject the null hypothesis in the context of the problem.

(f) Poor understanding of the reasoning behind the structure of hypothesis testing even if they can procedurally do all the textbook exercises.

In our opinion, the reasoning behind hypothesis testing is a natural process, one that is common to the students' experiences. This is why many instructors and textbooks use familiar examples such as cards, dice and coins to demonstrate the principles of statistical reasoning (Rossman, 1996; Maxwell, 1994; Eckert, 1994). We think that many of the above noted difficulties develop as the students try to translate their intuitive (and usually correct) notions of hypotheses testing to a formal statistical framework. In trying to learn the steps of hypothesis testing, as well as the terminology and the symbolism, the students tend to lose sight of the intuitive reasoning behind the process.

For this reason, we focus our pedagogy on the affirmation of the students' intuitive notions of hypothesis testing and then on the transference of this knowledge to the components of the formal framework of hypothesis testing. In order to facilitate this transition, we would like our students to first realize that the reasoning behind hypothesis testing is natural and within their range of experiences. Then we introduce the students to the formal procedures of a hypothesis test in a specific setting. Finally, we require our students to reflect on the *process* of hypothesis testing by studying several examples in a variety of accessible contexts.

One can focus lectures to address these concerns and we have observed that most students do see patterns in hypothesis testing, but that they tend to focus on the pattern in the formalism and symbolism rather than on the reasoning behind the formalism. As a result, they tend to view hypothesis testing as a collection of independent algorithms, one for each parameter. We feel that group work would allow the students more opportunity and time to grapple with the issues involved in making the transition to the formal framework.

However, it has been our experience that understanding one example in depth, even in a group setting, is not sufficient to reinforce the students' innate understanding of inference. In particular, we observed that students have trouble applying the knowledge that they gained in studying a specific example to similar situations. We believe that it would be helpful if the students had the chance to work through many similar, but related examples, so that they can identify common patterns, as well as differences. But of course, this is time-consuming.

Our implementation of the jigsaw, as described below, gives students this opportunity, in a relatively short amount of time. By studying a single problem in their expert groups, and then sharing each other's work in the students' home groups, the students are guided to recognize that they are procedurally doing the same kinds of tasks in different settings---thus focusing on the structure and not the technical details. Also, their understanding is reinforced when they communicate their knowledge to other members of the group.

4. Implementation

The class is divided into "expert" groups of three to five people each. Each group is given a worksheet with a story that involves testing a claim. In our statistics classes (typically classes of 25 – 35 students), there will be 6 to 8 groups and two "expert" groups working on identical claims. The claims we use vary in three aspects: the parameter to be tested (proportion or mean), the rejection region (one-tailed or two-tailed), and the conclusion of the test (reject the null hypothesis H_0 or not reject the null hypothesis H_0). A sample worksheet is included in the appendix, and examples of claims we have used are listed below:

1. Is a specific coin fair? (test for a proportion, two-tailed, do not reject H_0)
2. Is a specific coin more likely to show heads? (test for a proportion, one-tailed, reject H_0)
3. Are there about 56 M&M's in a 47.9-g bag? (test for a mean, two-tailed, do not reject H_0)
4. Is the average number of M&M's in a 47.9-g bag greater than 42? (test for a mean, one-tailed, reject H_0)

The numbering system used in the worksheet parallels the formal steps used in hypothesis testing and the wording of the questions in number 3 of worksheet #1 emphasizes the condition under which a hypothesis test is conducted, namely, that the null hypothesis is true.

After completing this part of the activity, the students read a summary of the hypothesis testing procedure. The terminology, symbolic representation and step-by-step procedure are introduced, except that there is no mention of the significance level nor is there a formal presentation of Type I and Type II errors. Then they translate their earlier observations into the formal framework (see sample worksheet #2 in the appendix).

To put the pieces of the jigsaw together, the class forms new "home" groups with at least one person from each expert group. Each "expert" teaches the rest of the group about their problem, procedure and results. One advantage of duplication of group assignments is that there are two experts for that problem in the home group, thereby reducing the possibility of having an ineffective "expert." To complete the picture of this statistical jigsaw puzzle, the students are asked to synthesize their results by noting the similarities and differences in their various tasks. Specifically, they are asked to address the following:

(a) Explain your problem and solution to the other members of the group.

(b) What similarities do you observe in all of your activities and results?
(c) What differences do you observe?

At the end of the group activities, the instructor leads the students in a class discussion to summarize their findings and address any misconceptions or missed conceptions. The amount of class time needed to complete these activities is about 1 ½ fifty-minute class periods.

5. Development of our Implementation

Our current implementation of the jigsaw is a result of two refinements of previous implementations, and is still undergoing revisions. In our initial attempt at adopting this strategy, we assumed (incorrectly) that the students' intuitive notions were well-developed and so we concentrated on the transition to the formal framework. At that time, the activities of the expert groups started with a worksheet similar to our current worksheet #2. In addition, the experiments were more complicated – some involving the difference of two population parameters – and there was no replication of the individual problems among the expert groups. Also, the types of problems were too varied for the students to readily see the common themes, and they focused on the experiments rather than on the structure of the tasks. The result was a situation that was more time-consuming and frustrating (for both students and instructor) than was anticipated. The exercises were completed with much instructor input to both the expert and home groups.

Despite the problems encountered, we were encouraged to persist with this approach, albeit with modifications. In the second round, we had the students first use the jigsaw technique to learn sampling designs so as to familiarize themselves with the technique in a relatively easy setting. Also, we simplified the experiments and focused the student's attention on the informal aspects and natural logic of hypothesis testing. This time, we felt we were more successful in developing the students' confidence in their natural problem-solving abilities. However, we were not satisfied in how the activities bridged the gap to the formal structure and terminology of statistical inference

Our combined experiences have led us to develop the particular version that is discussed in this paper and we will continue to modify the exercises and worksheets as needed. We have also developed computer simulation activities for two of the experiments to enhance our students' understanding of how a decision criterion is chosen to reject the null hypothesis. These explorations leads the students to the concepts of significance level, Type I and Type II errors.

6. Summary

As noted in the introduction, the jigsaw has most often been applied at the elementary school level and in the areas of reading and social science. In statistics, it is most often used to divide up a tedious calculation task, for example, having each expert group find the summary statistics for a particular sample to be used in an application of an ANOVA (Perkins and Saris, 2001). In this paper, we go beyond these usual ways of using the jigsaw. We use the technique in a statistics class to introduce a concept (hypothesis testing), by giving the students a chance to study different examples of this concept, in order to lead them to make their own generalizations. Their observations form the basis of further classroom discussion. We feel that in this way, a concept, procedure or formula would be more grounded in the students' experience.

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WORKSHEET #1: Introduction to Hypothesis Testing

Paul and Joshua are again squabbling over what TV show to watch. Their mom suggested that they flip a specific coin – if it comes up heads, Paul would choose the show; if it comes up tails, Joshua would decide. Both boys were hesitant to agree, thinking that the other one had an advantage. Mom says: " It's a shiny new coin – I am sure it is a fair coin."

1. **Hypotheses**

   This scenario involves two claims.
   (a) What is the mother's claim? (This claim is the prevailing view about any new coin.)
   (b) The boys have a different, or alternative viewpoint. What is their claim?

2. **Experiment Design**

   The boys ask their mother if they can prove to her that their claim is correct. Design an experiment that the boys can perform and write your procedure below. DO NOT conduct the experiment yet.

3. **Decision Criterion**

   The mother agrees to the boys' experiment but she insists that they must all agree beforehand on what criteria they will use to decide if the coin is fair. They discuss the following situations. Imagine that you are part of the discussion, and answer the questions below.
   (a) What results would you expect if the coin is fair?
   (b) What results would you expect if the coin is not fair?
   (c) If the coin is fair, is it possible to get 80% or more heads? Is it likely to get 80% or more heads?
   (d) If the coin is fair, is it possible to get 10% or fewer heads? Is it likely to get 10% or fewer heads?
   (e) If the coin is fair, is it possible to get 45% or fewer heads? Is it likely to get 45% or fewer heads?

   A **decision criterion** is a method to decide whether a claim is valid before conducting an experiment. An example of a decision criterion is the following:

   Paul and Joshua are correct in saying that the coin is not fair if the experiment shows 55% or more heads or 45% or fewer heads.

   In your group, develop a decision criterion for your experiment that is acceptable to all members of the group, and write it below.

   At this point, consult with your instructor before proceeding.

4. **Gathering the Evidence**

   Conduct the experiment and record your results below.

5. **Decision**

   Based on your results from #4 and your decision criterion from #3, is the boys' claim that the coin is unfair supported? Is the mother's claim that the coin is fair supported?

WORKSHEET #2: Terminology and Framework of a Statistical Test of Hypothesis

A formal statistical test of a hypothesis has several components.

1. Hypotheses. Every hypothesis testing situation has two competing hypotheses or claims. One is called the *null hypothesis* and is denoted H_0. This hypothesis represents the prevailing view or the status quo. Others may believe that the null hypothesis is not true. Their viewpoint is a competing, or *alternative hypothesis*, denoted by H_1. In the situation in the first worksheet, Paul and Joshua disagree with their mother's claim. Using this formal terminology for competing claims, state in ordinary language the null and alternative hypotheses for your experiment. Then restate these hypotheses in terms of the binomial parameter p, the proportion of heads in n tosses.

	Words	Symbols
H_0:		
H_1:		

2. Experiment Design. Now that you know the claim that you (acting as Paul and Joshua's representatives) want to establish, you need to collect data to support your alternative hypothesis. However, until you can support your alternative hypothesis with evidence, you must conduct your experiment under the assumption that the null hypothesis is true.

3. Rejection Region. The term *rejection region* refers to what was called the decision criterion in worksheet #1. This region is a range of what you believe to be unlikely values obtained from your experiment if the null hypothesis were indeed true. In other words, it is a range of values obtained from your experiment, which would convince most people to reject the null, or prevailing hypothesis (this is why it is called the rejection region), in favor of your claim H_1. In your experiment, the rejection region would be those values of $\hat{p}$ that you think would be unlikely if indeed the null hypothesis is true.

What is the rejection region for your experiment?

4. Test Statistic. The test statistic is the evidence obtained from an experiment that will be used to try and refute, or reject, the null hypothesis. Generally, you compute the sample statistic that corresponds to the population parameter used to state the null and alternative hypotheses. The population parameter of interest in your experiment is the proportion or percentage of heads when a coin is flipped. The corresponding sample statistic in your problem is thus the percentage of heads observed in the sample. What is the value of the test statistic $\hat{p}$ from your experiment?

5. Decision. Lastly, a decision is made based on whether or not the value of your test statistic falls in the rejection region given in #3. If it falls in this region, then you would reject the null hypothesis and support the alternative claim. If it does not, then you cannot reject the null hypothesis. In your test, will the null hypothesis be rejected? If so, state your conclusion in ordinary language. If the null hypothesis is not rejected, state this conclusion in ordinary language.

Whatever your decision, there is a chance that your conclusion is not correct. Explain how this can happen.

PRINCIPIA PROGRAM: EXPERIENCES OF A COURSE WITH INTEGRATED CURRICULUM, TEAMWORK ENVIRONMENT AND TECHNOLOGY USED AS TOOL FOR LEARNING

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ABSTRACT

The ITESM's teaching model has evolved in the last few years. Nowadays, several abilities, attitudes and values (AAV's) are taken into account without forgetting the development of knowledge in students. These AAV's include teamwork, the use of technology as a tool for learning, self-learning, problem solving, among others.

Within this evolution process, several problems were identified in the former model used at ITESM to teach mathematics and engineering. These problems involved both teachers and students. For instance, there was poor knowledge retention in students, courses were too centered on algebra instead of developing mathematical reasoning and rules and algorithms were preferred to practical applications in the areas students are usually interested.

"Principia" is an engineering academic program which comes out from the idea of overcoming those difficulties. The main purpose of Principia is to develop a mathematical, physical and technological culture in students that will make them able to analyze and solve complex problems. This is achieved with the integration of different subjects in one unique program where the classroom and learning environment are considered.

"Principia" has been planned and implemented for the four first semesters of engineering. Some of the basic tools used in this program are problem based learning (PBL) and heavy use of computer technology. There are five fundamental principles in "Principia":

a) Integration of the curriculum for mathematics, physics, and computer sciences.
b) Collaborative learning.
c) Teamwork.
d) Emphasis on mathematical modeling.
e) Use of technology in the learning process.

With all these elements, "Principia" has evolved as an integrated program that considers objectives, knowledge, methodology and an evaluation system. In this paper, we share our experiences in "Principia" over three generations of students and some statistical and comparative results.

Keywords: curricula, innovation, technology.

1. Background

In the last years, a change in the ITESM's teaching model has been observed. Under these new ITESM teaching model, in addition to knowledge, development of some abilities, attitudes and values (AAV's) is being taken into account. In the past five years, the Math Department has developed different projects in which the following AAV's are being emphasized: use of technology in every math class and going in pursuit of students learning. Some problems in math teaching and learning in teachers and students have been identified thanks to these projects. "Principia" program comes out from the idea of overcoming these problems, looking forward to improve the teaching and to enlarge the learning spectrum of students from different areas of engineering. Besides, it considers the classroom and the environment where this learning process takes place and it introduces an educational strategy in mathematics, physics and computer science that leads to the development of the AAV's stated in the ITESM's mission. Some of the most important characteristics and methodology of "Principia" are included in this document.

2. "Principia" program

"Principia" program is a teaching-learning model of basic sciences aided by technology, which fosters the use of abilities for teamwork, self-learning, creativity, analysis and synthesis of information in engineering students, in agreement with the objectives of the ITESM mission. "Principia" is based on the following fundamental principles:

a) Integration of mathematics, physics, and computer science courses curricula .
b) Collaborative learning.
c) Teamwork.
d) Mathematical modeling as a fundamental tool for sciences and engineering.
e) Use of technology in the learning process.

The objectives and principles of "Principia" were stated as a result of research on the deficiencies of the teaching model in the area of mathematics and basic sciences in engineering. Having identified these deficiencies in teaching and evaluation of concepts learned, new alternatives experimented in the teaching environment are being researched. As a consequence, those elements and methodologies that have been successful in the development of AAV's were then selected. Design of physical spaces and technology that affect this process were also considered. Several activities which constitute operative and methodological design are used to follow these principles:

- Field of study
- Lectures
- Exercises solving
- Laboratory
- Presentations
- Subject Evaluation
- Problem based learning (PBL)
- Project oriented learning (POL)
- Learning based on technology (LBT)
- Exams with integrated curriculum (EIC)

The first six correspond to the classical activities in the classroom. The last four have been introduced in "Principia" taking up to 50% of the effective time of the program as basic elements in its structure.

3. Curricular integration

Curricular integration is not an isolated issue, different experiences have been carried out [1, 2, 3]. In "Principia", curricular integration is, of the five principles, the moving axis of activities, while the following four are the means to reach the objective. To achieve curricular integration in "Principia" entailed introducing additional activities that required more time. This had to be reduced as much as possible to achieve a balance with its former antecessor scheme. Therefore, PBL, POL, LBT and EIC allowed us to:

a) Consider the content of all integrated areas and long term objectives.

b) Take advantage of recurrent contents to achieve meaningful learning.

Table 1 shows the basic topics for each semester of "Principia", under the scheme of integrated curricula[1].

4. Collaborative environment and use of technology

1) The use of collaborative techniques and technology in learning and the classroom.

Table 2 shows some aspects and desired objectives. We must point out the fact that the design of space (classrooms equipped with complementary facilities) comes out in a natural way when considering the processes that occur in our activities. Each classroom is a room with movable divisions. It has 10 tables for teamwork that allow connection to Internet. Additionally there are working zones and library space.

The curriculum integration is based on PBL and POL methodologies. The first one allows progress in all areas, working on their specific goals. In the second we integrate all areas.

2) Problem Based Learning (PBL)

Collaborative learning among students is developed in the program in several activities:

a) Exercises solving, where students leave their basic team to form new heterogeneous teams to solve excercises of academic nature. The objective is to develop elemental level and to return to the original teams to share and to enrich the knowledge of their team members.

b) PBL that in its design integrate some of the organization of the exercises solving, to identify and to solve a more real and complex situation, normally with integrated curriculum. In "Principia", commonly this activity requires the use of technology for its development.

c) Development of projects is the open solution of a complex situation which involves the acquisition of additional formal knowledge. In these projects, future knowledge of the field of students is concerned.

[1] This curricula comes as a result of the integrated program. The traditional curricula is not as long.

Of these, PBL is the most recurrent and the most useful activity in "Principia" to develop its principles. Since beginning of PBL as a formal paradigm in medical education at Mc Master University [4, 5], several other universities have adopted this educational practice in various countries [6] and inclusively in some areas of Engineering [7, 8], as also in levels of basic education [9, 10], and high school [11, 12, 13, 14].

In the PBL approach, students are confronted with complex, usually multidisciplinary problems, which must be solved in teams. Problems should be sufficiently complex that students' prior knowledge and conceptual frameworks become insufficient to solve them. So, during the initial discussions the problems should trigger the questions that guide student's search for information and self-directed learning. Under these conditions, learning is guided by the students' questions.

Generally, the PBL curriculum is organized around general themes, instead of the discipline-based organization that characterizes the more traditional curricula. This kind of organization requires teams of teachers with different disciplinary backgrounds to prepare activities. Here, some general principles that guide most PBL educational practices may be summarized in didactic principles and professional orientation principles [15].

The didactic principles may be summed up as follows: First, the instructor may facilitate the process, but students must become responsible for their own learning. Secondly, knowledge and skills acquisition is a process that require students' active participation. Lecturing and other "transmission of knowledge" approaches are of little value under PBL. Third, students are oriented to cooperative work rather than to competition.

With regard to professional orientation: professional practice is seen from a holistic point of view. As was mentioned before, instead of the specialized disciplinary organization that characterizes traditional education, PBL arranges contents around multidisciplinary issues. Therefore, PBL aims to generate an integrated learning process. This integration is twofold. On the one hand, students should integrate knowledge from different domains. On the other hand, PBL should help students to integrate knowledge with skills and abilities.

As Douady[16] states, "For a teacher, 'teaching' refers to the creation of the conditions that will produce the acquirement of knowledge by students. For a student, 'learning' means to get involved on an intellectual activity where the final consequence is the availability of a knowledge in its double status of tool and object". This idea allows us to understand the complexity of an ideal teaching-learning process. Additional to knowledge, there are other elements that participate in the process. In this sense we can say that this process is multidimensional, knowledge being just one of the dimensions.

Beyond Polya's ideas [17], PBL takes us to the consideration of elements that are present either as the natural part of a mathematics problem or related to the solving processes involved. The common problem solving design elements normally include [18, 19, 20, 21]:

- Objectives
- Material
- Discussion outline
- Requirements
- Instrumentation
- Evaluation

Furthermore, some authors consider these aspects in the itself problem level. They mention that there is a second level that corresponds to the environment of the problem [22] and summarize the consideration of this level in four principles:

a) Goal of the activity can or cannot be accomplished by the students.

b) Problems can modify the mathematics comprehension of the student.

c) There are different ways to understand a problem.

d) There are different levels of comprehension in every theme and they are never reached the first time.

The consideration of using technology within a problem solving activity must at least take into account:

a) Technology used must not be near or superior in complexity to the problem.

b) Use of technology must be significant. It must be justified that the problem can't be solved without the use of this technology or at least, it must conform as a tool that enables the student to focus on concepts and mathematical comprehension.

These elements are normally considered as a basis for the creation of a common problem. Besides, we must consider some particular elements in any problem solving activity.

Based on the accomplishment of the above considerations and the basic principles of the program, added to the ITESM mission's objectives, the following dimensions for the design of a PBL activity are proposed:

- **Environment:** it refers to the real situations that may occur when the activity is taking place. These situations focus on the level of comprehension achieved or used by the student just as in the traditional scheme.
- **Curriculum:** the content on which the activity is based. The curriculum is the traditional basis of teaching, but a problem solving activity underlies other elements.
- **Frame of analysis:** It refers to the curricula of the integrated areas. Previous goals and future goals are taken into account to make the problem easier and to detect future necessities.
- **Use of Technology:** The technological elements (software, laboratory, etc.) that conform the activity. This dimension must establish an analysis of its significance and the role it has in such activity.
- **Development of formative objectives:** within the ITESM context, this dimension naturally caters to those AAV's stated in its mission.

Once the problem is determined, it may be endowed of these dimensions. Their lack may sometimes result in modification or disposal of the problem. The importance of creating a consistent network of problems with the above dimensions allows student to enforce the faith of the student regarding the goal of each activity.

The projects involve several of the elements and dimensions. They belong to a different level of knowledge and occasionally they have more similarity with open-ended problems. For this reason, we want to focus on PBL and EIC activities. All of them use technology (reason for considering it as a design dimension).

In a typical session of "Principia" three stages are observed. They are summarized in Table 3. The idea of these stages is to introduce students step by step until able to manage the solution by themselves. So, in some activities steps I & II may be omitted. These steps are always omitted in any EIC activity.

The creation of a network of problems under these considerations that establish a frame of analysis, allows evaluating the recurrence to previous and future subjects. In this way, the whole network is more important than the problem itself because it allows to give continuous sense to PBL activities within the course.

The use of PBL in mathematics and physics courses has not constituted a distortion to education for students currently in the program. The evaluation of the students in this program has been inside their comprehensive evaluation (including all courses). The consistency of results in proficiency of problem solving is strongly correlated with the global results obtained for each student. Effectiveness of these kind of activities is more influenced by teacher's preparation for leading an activity than by student preparation [23].

5. Studies about the effectiveness of the program

The department of institutional effectiveness of ITESM (DEI-RZS), has been the area which has evaluated the project since its beginning, with the help of the teachers who work on it. Since 1998, more than 14 studies about the effects in the learning of students who participate in Principia have been done. Studies have been both: qualitative and quantitative. They are very local but then they are expanded to a very global one. We show only some aspects of evaluation of effectiveness scheme.

a) Collaborative activities index

Several studies on the effectiveness of the program network problems have been performed. Figure 1 shows an index of consistency of each problem in an intermediate course of "Principia"; together with the result that is obtained by dividing the student evaluation in the activity by the global evaluation in the period, the standard deviation is obtained. So an index above 1.0 means that activity is easy, an index under 1.0 means the activity is complex, for the group. These charts allow us to determine corrections on the activities and adapt them each semester. This test is administered to a group of 60 students.

b) Students opinion about the program in the development of AAV's

About the evaluation that students made of the program, the following test was administered and had following objectives:

- Analyzing the effects of "Principia" on the development of AAV's.
- Comparing the effects of "Principia" with equivalent courses.

The characteristic elements of the test were:

- Leadership
- Analysis, synthesis
- Critical thinking
- Communication
- Teamwork
- Creativity
- Search for and management of information
- Entrepreneur spirit
- Quality and excellence
- Use of technology
- Work capacity
- Self-Learning
- Problem solving
- Learning
- Motivation

The student was asked to compare the level in which the course contributed to develop each ability, attitude or value of the above statements with the average of the other courses. A scale 0-10 was used, where 0 is less, 5 equal and 10 more. In previous research [24] one of the authors reported some preliminary data on students' self-perception of skill development. Students were formally assessed on oral and written communication. The test was administered to three groups of students as described below.

"Principia" group Students in "Principia" courses	*Witness 1 group Students in equivalent non "Principia" courses with "Principia" teachers*	*Witness 2 group Students in matching courses with non "Principia" teachers*
126 students	154 students	111 students

Figure 2 shows comparative results for each group and the dimension of the research in which the smallest and the highest difference with respect to the evaluation given to "Principia" was obtained (full description of test in [22]).

c) Evaluation based upon measurable observation through evaluations

The following test (Figure 3) compares three different groups in the same course (final course of the program). One of them corresponds to "Principia" program, another (traditional) to the way it was taught in 1995 and at last, to the way it is currently being taught under circumstances of the new educational model of ITESM (reengineered). Some aspects derived from the evaluations of the proposed activities in the course are compared. They show on an indirect way the evidence of the dimension compared (the evaluations are based on a 0-100 scale).

Actually, a new test was given and results are being processed, based on the criteria that compare, with a witness group, the development of two groups ("Principia" and reengineered) the capability to solve integrated problems. This study will finished in July 2002. The research is intended to measure the recurrent effect of the execution of collaborative activities, use of technology and those of "Principia".

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Semester	*Mathematics*	*Physics*	*Computer Science*
First	Single variable differential calculus. Vector functions and differential equations.	Mechanics.	Microsoft office and Mathematica.
Second	Single and multiple integral calculus. Vectorial fields.	Mechanics, elasticity, thermodynamics.	Matlab and C++.
Third	Multiple integrals and ordinary differential equations. Probability and statistics.	Electromagnetism and modern physics.	Numerical methods.
Fourth	Differential equations systems and modeling.	Study of mechanic and electric systems.	Simulation.

Table 1. The general curricula of Principia

Technology	*Activities*	*Objectives*
Matlab and Mathematica	Projects, practice and homework assignments.	To permit student applies physics, mathematics and computer knowledge to problems of higher complexity than ones studied in traditional courses.
Use of the internet and Learning Space	Lectures and assignments.	To ease the process of collecting information. Apply the technology in the process of learning-teaching. Link the student with the technology.
Laptop	Projects, homework and practice assignments.	To link student with the cutting edge technological elements.
Microsoft Office	Projects, presentations, homework and assignments.	To develop numerical and graphical strategies for problem solving and written and oral skills.
Equipped classroom.	The entire project.	Ease some learning process (work, visualization).

Table 2. Technology, AAV and objectives.

STAGES OF A PROBLEM RESOLUTION ACTIVITY			
	STAGE I: acquisition of knowledge	**STAGE II: Collaborative Learning.**	**STAGE III: Problem**
Instructions and rules	**Teacher:** does not give the information, but gives orientation and feedback to each team. **Student:** each team may access the necessary sources of information. .	**Teacher:** keeps the information and gives feedback on the performance and amount of participation of each expert. **Student:** can't interact with other teams. Allows each expert to talk in each section of the activity.	**Teacher:** keeps the information. Watches the time and gives advice on the objective to the team. **Student:** can't interact with other teams and allows to each member participates the same.
Action elements	• They define specialty fields. • They conform expert teams based on the ability of each student.	• An application activity is defined. It must allow the interaction and interchange of experiences from each student with his/her team members.	• A problem that involves the use of previous stages is proposed. • And of other contents within the analysis frame proposed.
Way to work	Each team is divided to form expert teams integrated by elements from different teams.	The base team gets together to solve an intermediate problem where each expert contributes to the team with individual knowledge.	The base team is oriented as a team to solve the problem.
Evaluation	Each expert team makes a presentation and is evaluated according to the activities specified in the outline.	The field evaluation (in what refers to efficiency and teamwork).	The evaluation centers in the report on site that the team prepares.

Table 3. Complete stages of a Problem solving activity

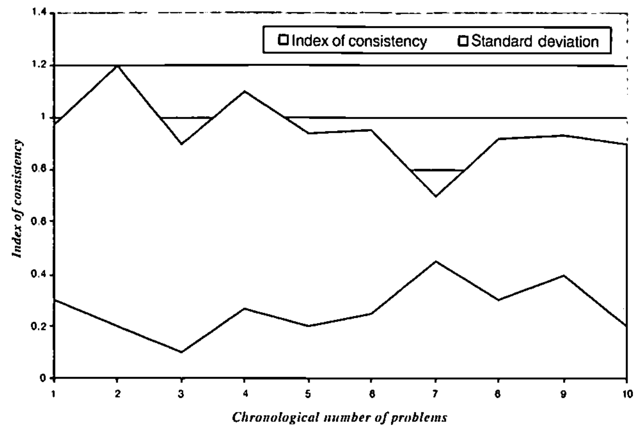


Figure 1. Index of consistency of a group of problems

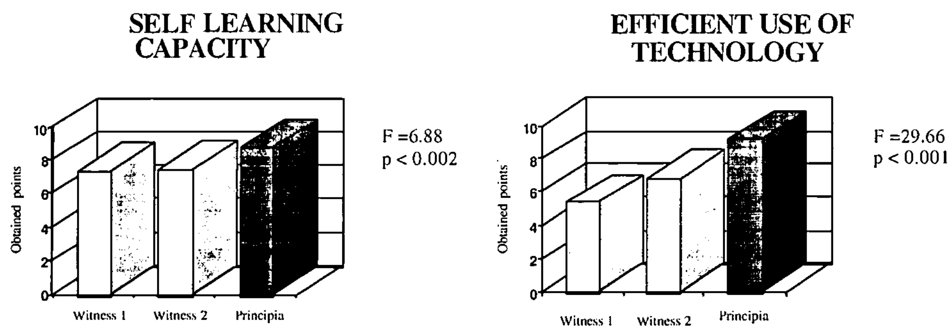
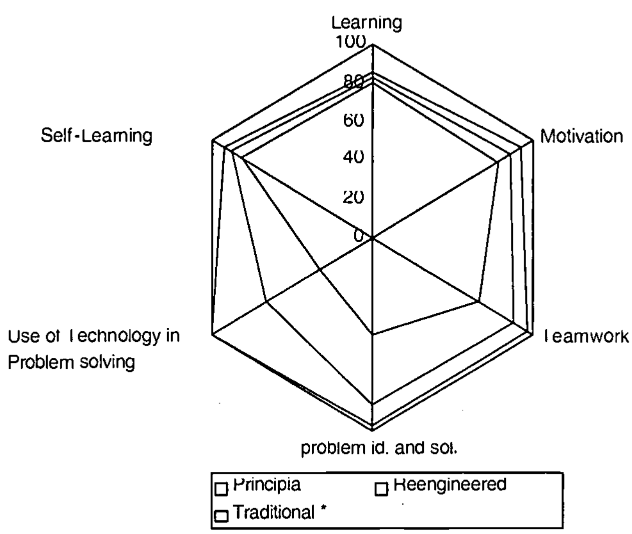


Figure 2. AAV with smallest and highest difference in evaluation with respect to "Principia" courses according to the opinion of students.



*Referring a previous research (1998)

Figure 3. Comparison of the different schemes of teaching-learning at ITESM, in relation with "Principia" program, based on student's evaluation. These students are taking one of the final courses of the program.

ASSESSMENT OF SOLVING TYPES OF PROBLEMS IN ALGEBRA

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ABSTRACT

A classroom experience in algebra which considered and interaction-bound view of the didactics of Mathematics regarding the categorization of types of mathematical problems was carried out. This experience aimed at relating the teaching of concepts and the resolution of algebraic problems previously validated and classified according to their nature into routine, non-routine problems; and according to their context into real, realistic, fantasy and purely mathematical problems. To that effect, a study of the learning of the unit about proportion variation for the first year high school based on the syllabus suggested by the Chilean Educational Reform was devised.

Learner-centered actions which gave way to group discussions and active interaction with the teacher were preferred in an attempt to reach the concept of proportion variations. The class work was carried out following a constructivist view of learning and was supported by class material specially prepared for such purpose.

Considering the students' actions in the classroom plus their interaction with the types of problems studied, it is possible to claim that the students could gain highly significant algebraic learning, demonstrating ability to recognize relations and transform the given data from a problem. By this means, they showed evidence of understanding the concept of proportion and its relations, knowledge and understanding of mathematical processes with accurate and fast calculations and an ability to reason in order to solve routine problems - preferably of purely mathematical, fantasy and realistic context - as encouraged by the Chilean Educational Reform.

Background information

All forms of learning imply the search for adequate knowledge or an effective skill. Knowledge about skills has increased in the past decades and this has contributed to the identification and development of cognitive skills such as problem solving.

In trying to find an appropriate heuristics to aid the solving of problems, Polya (1957) described a number of general strategies that could facilitate the procedure. Such concern increased later on as can be seen in articles about the same topic by Schoenfeld and Herman(1982), Mayer (1985), Sweller and Cooper (1985), Gick (1986), Minsky (1988), Schoenfeld and Herman, Mayer, Sweller and Cooper, Bifk, Minasky in Valenzuela, 1992).

Curricular tendencies for the teaching of Mathematics of the last decade have stressed 'the need to place capacities of higher level on a first position...'(Abrantes, 1996); that is, those capacities that are linked to the identification and resolution of problems, to critical thinking, and the use of metacognitive strategies.

Problems solving has been recognised as a main aspect of the learning process; hence, it has given rise to the formulation of operative strategies, to the classification of the different problems, and to the varied approaches given to the studies about solving types of problems. However, there is a need for further studies that suggest didactic strategies which allow an improvement in the teaching of mathematics, specifically of algebra, within the context of educational reforms (Díaz, Poblete, 2000).

At present, problem solving continues to be a topic of interest, especially from the perspective of the educational reforms that are being implemented in recent years. This consideration to problem solving tasks has been recognized by different educational reforms in Latin America. In Chile, it constitutes a fundamental element in the current teaching of mathematics at different levels, due to their relevance in everyday life application and usefulness (Díaz, Poblete, 1999). Such considerations of the conception of a problem have been expanded by means of a distinction between them that categorises them as routine and non-routine.

In our view, categorizing problems provides the conceptual basis for any didactic procedure in the school curriculum. To that effect, we have devised a classification that considers the nature and the context of the problem, and have devised the following categorization. Based on their nature, problems are categorized as Routine and Non-Routine problems; and based on their context they are categorized as Real, Realistic, Fantasy and Purely Mathematical problems (Díaz, Poblete, Fondecyt Project Number 1990558, 1999).

Routine Problems

Based on their context, we have classified problems as:

Real context problems: A context is real if it is effectively produced in reality and the student is involved in it.

Example: using a piece of thread, to measure the diameter and length of the circumference of three coins of different ze each. Find the ratio between the diameter and the length of each coin. What can you conclude from these ratios?

Realistic context problems: A context is realistic if it is likely to be really produced. It deals with a simulation of reality or of a part of reality.

Example: An industrial washing machine, working 8 continuous hours for 6 days has washed 1200 kilograms of clothes. How many kilograms of clothes will it wash in 20 days working for 10 hours daily?

Fantasy context problems: A context is considered a fantasy if it is the product of imagination not founded on reality.

Example: Two inhabitants from Krypton planet have been brought to Earth: Superman and Supergirl. In order for them not to be affected by Kryptonite, they need to drink an amount of liquid equivalent to one ninth of their weight. If Superman drank 21 litters of liquid in 3 days, how much liquid does he need to drink in a week?

Purely mathematical context problems: A context is purely mathematical if it refers exclusively to mathematical objects such as numbers, relations and arithmetic operations, geometry figures, etc.

Example: The sides of two squares have a ratio of 1:3. What is the ratio of their perimeters?

Non- Routine Problems:

Non- Routine Problems : these are those for which the student does not know an answer nor a previously established procedure or routine to find the answer.

Example: Think of two everyday life situations that are inversely proportional and determine the value of proportion constant in each case.

Note that non-routine problems can also be classified according to their context into real, realistic, fantasy and purely mathematical.

Development of the study

A didactic experience to articulate the mathematical concepts regarding a specific teaching unit and problem solving types for secondary school was devised.

The problems presented were algebra problems and the unit dealt with was that of proportion variations. The contents were structured as indicated by the Chilean Educational Reform for secondary schools. Primary education in Chile comprises 8 years and secondary school comprises 4 years. The qualitative research was mainly conducted as participating observations, and took place in May 2001, within a class of 40 students at a Science and Humanities Secondary School.

The whole process taking place within the class was described, analyzed and interpreted continuously by means of the interaction with the students. Three pairs of students were voluntarily chosen to be observed during the research process.

Active learner-centered actions rather than teacher-oriented actions were chosen as the methodology. Students worked with didactic materials designed by the researchers that included 55 tasks involving situations and problems based on the categorization devised.

Data were collected through observations in class. Individual interviews on some occasions and open-ended opinion questionnaires complemented such observations. These were passed at the beginning and at the end of the experience and aimed at finding out the meaning students give to their individual actions and to get valid and accurate conclusions about the study.

Students started work with the materials given from the second session, could do the tasks either individually or in groups and ask questions either to the teacher or the observer. All sessions were recorded, so it was possible to get information about the conversations among the six students observed, the kinds of questions raised about the text and the questions asked to both the teacher and the observer.

The teaching material was used as a facilitating element of the didactic procedure; that is, as a means to improve and complement the teaching and learning procedure of the study unit. The material included problems, situations and questions related to direct and inverse proportionality, proportionality constant, its relation with a quotient or a constant product, composed proportionality, graphs, charts with values, and algebra expressions. All of it based on types of problems classified depending on their nature as routine and non-routine; and depending on their context as real context, fantasy and purely mathematical.

The activities consisting of didactic situations based on types of problems allowed the introduction of the concept of proportion variations. Most of them were accompanied by drawings and charts to visually explain the relations involved. Thus, the situations presented through types of problems and associated to charts allowed the students to identify themselves with different contexts and use their own learning styles.

Some real context problems were solved in order to relate the equivalence of the ratios with the constant of proportionality in each case and introduce the concept of proportion. Some examples of this are the following: "Measure the sides of your desk. Measure the sides of your teacher's desk. Establish the ratio between them and decide if they make a proportion"; "Draw three squares of different sizes. Draw the diagonal in each and determine the ratio between the side of the square and the diagonal".

Some other tasks based on realistic context problems were performed in class. Examples of these are: "A farmer from the south of Chile, in Osorno, needs 750 kilos of pasture to feed 50 cows for 10 days. How many days will he be able to feed 40 cows with 800 kilos of pasture?"; "Two cities that are at a distance of 18 km one from the other appear 6 cm. apart on a map. What is the real distance between two cities that appear 21 cm. apart on that same map? Both of these problems are likely to occur in real life; they correspond to a simulation of reality or part of reality.

Similarly, fantastic context problems were also included. Some of them were: "A specimen from Saturn has been brought to Earth. It covers 21 meters in three jumps. What distance would it cover in 5 jumps?"; "If three cats eat three mice in three minutes each, how many cats are necessary to eat nine mice in nine minutes?." Both problems are just part of imagination.

The purely mathematical problems used are similar to those that normally appear in traditional coursebooks regarding proportion variations. Those have to do exclusively with mathematical objects: numbers, mathematical relations, geometry figures, etc. Students did not have major problems solving these problems.

Students appeared highly motivated with the methodology employed and were actively engaged in their groups trying to figure out the solving procedures.

Similarly, students solved non-routine problems such as: "Write down the mathematical formulation of the following variations: a) the N number of long distance calls between two cities is inversely proportional to the distance between both, b) The D distance, expressed in meters, covered by a vehicle in 15 minutes in inversely proportional to the average speed V, expressed as m/min." "Invent a problem that combines both types of proportionality." Faced with these problems, students discussed them for a few minutes and then tried solving them by means of numeric rather than by algebraic expressions.

In order to assess the students' previous knowledge about the topic, a pre- and meta-test were applied. This was useful to compare the level of achievement obtained by the students during the didactic experience and also as an assessment instrument for the unit.

Six protocols were developed during the research: they involved class observations, individual interviews and an opinion questionnaire.

Results

The application of the pre- and meta- test in this didactic experience based on routine and non-routine context problem solving has enlightened our understanding of the processes involved by means of a comparative study of individual performance.

Student 1 got 15.6% in the pre-test and throughout the development of the research increased his performance to reach 87.5% achievement. He developed a significant ability in solving routine problems of either realistic, fantasy or real context. He developed ability to identify relations and convert data from a given problem into other form, thus showing evidence of understanding the concept of proportionality and its relations.

Student 2 got 6.25% in the pre-test and by the end of the research had increased his performance to 75% achievement. He showed evidence of knowledge and understanding of the concept of proportionality, of mathematical procedures with fast and accurate calculations and ability to reason and solve routine problems, mainly those of purely mathematical context, fantasy and realistic context.

Student 3 got 6.25% in the pre-test and improved greatly throughout the development of the research to get 90.6% achievement in the final test. He showed greater skills in reading and interpreting routine problems of purely mathematical, realistic and fantasy context. He also developed the routine problem of real context with a certain degree of success, though minor errors led him to a wrong answer.

Student 4 was only able to solve one problem in the pre-test, getting 3.12% of achievement, improved his performance so much that he finally got 96.8% of achievement by the end of the experience, thus getting the highest achievement percentage of all students observed. The student developed the skills necessary to do mathematical reasoning, and showed evidence of knowledge and understanding of generalizations about asking and answering routine problems of all contexts. Similarly, he was the only one able to solve the non-routine problem presented in the test, thus indicating that he was able to apply prior knowledge to solve an uncommon type of problem, by making use of more complex mental processes since this problem required a higher kind of analysis.

Student 5 went from a 9.3% achievement in the pre-test to a 68.7% in the final test. He showed evidence of a discrete skill development regarding problems solving. He demonstrated greater ability doing routine problems of realistic and fantasy context and an adequate interpretation of the data given for such problems.

Finally, student 6 only did 3 problems in the pre-test, getting a 9.3% achievement

and increased it to 87.5% in the meta-test. His achievement improved greatly, specifically in his reading, interpreting, analyzing the data from a problem, and finally solving practically all routine problems successfully, except the one of real context. Certain data show that his answer was sensibly oriented yet a wrong interpretation misguided his answer.

Categorization of the research

Based on the data obtained, the class observation, literally transcribed at the end of each working session, plus the pre- and post- opinion questionnaire, all the information was classified in order to find the convergence. The idea was to obtain a corpus of data that allowed a more systematic analysis of each situation, that led to the formation of categories from the similarities in order to maintain internal homogeneity, or the differences related to external heterogeneity, trying to establish clear and coherent criteria for the classification and ordering of the information obtained. Similarities found for the three pairs of students observed are detaile d below.

Similarities

(1) About the problem solving

Regarding the pre-test:

They find it difficult, and argue that they cannot remember the contents studied in previous courses.

Their amount of knowledge was not enough to allow them to do the test.

Difficulties encountered mainly have to do with geometry.

Regarding the development:

They are more used to solving exercises than problems.

By the end of the study, students are able to discover applicability of algebra to everyday life situations with examples of realistic situations.

(2) About their knowledge

Initially, students do not recognize the applicability of algebra to their everyday life.

They appear to have increasing difficulty understanding metalanguage.

Students make no distinction between Mathematics and Algebra.

They are able to define correctly, and without major difficulty the concept of direct and inverse proportionality exemplifying each with a routine problem of realistic context.

They claim to have studied and learned contents referring to proportion variation and are able to expand their answers.

(3) About their teaching

Students can make a comparison and agree in identifying differences between the teaching of algebra in primary and secondary school.

They find the material entertaining and highlight the motivational feature of drawings.

Differences

(1) About the mathematical concepts

The difficulties encountered during the classroom sessions were concentrated on the solution of equations, the problems that involved geometric concepts, real context problems that required the measuring and solving of problems related with compound proportionality.

Students define and exemplify with difficulty what they understand by compound proportionality.

(2) *About the methodology*

For some students the methodology employed was adequate because it was entertaining and different to traditional classes, while for some others it was so because they could work in groups and share their ideas.

Students' views about the tasks performed are different: they are good and easy for some students, and very complicated for some others.

(3) *About their achievement*

In the application of the pre and meta-test students obtained achievement that ranged from 3.1% and 15.6% in the pre-test to 58.3% and 100% in the meta-test.

It is worth mentioning that only one student got an achievement score below 70% in the meta-test. This indicates that there was a significant increase in the development of certain skills and the use of knowledge related to problem solving in algebra.

Conclusions

This research into the algebra area considered a didactic conception of mathematics that related the teaching of general algebraic concepts and problem solving. The investigation was aided by teaching material based on types of problems categorized according to their nature and their context as a means to reach the concept of proportion variations in the students.

The students showed ability to recognize relations and transform data from a problem given in one way into another, by this means they got to the understanding of the concept of proportionality and its relations. They displayed knowledge and understanding of proportionality concept and its relations. They showed evidence of knowledge and understanding of proportionality situations, of mathematical processes with fast and accurate calculations and ability to reason about and solve routine problems, preferably of purely mathematical, fantasy and realistic context.

By the end of the study, a group of students managed to transfer their previous knowledge to a non-habitual problem, making use of more complex mental processes, since this one belongs to a category of higher analysis. The students showed difficulties with geometry and were more skillful with arithmetic than with algebra. The notions and approaches the students used in arithmetic previously and that still maintain can explain this difficulty. Working with algebra requires students' change of mind so that they move away from concrete numeric situations to more general situations, like the ones given by non-routine problems. Transition from what can be labeled as an informal representation and problem solving to a more formal one is complex and disorientating for most students who start studying algebra, since they continue to use the approach that worked for them in arithmetic.

In general, solving a problem of real, realistic or fantasy context requires the mathematization of the given situation; that is, it has to be translated into mathematical language. Since we are dealing with a problem, the mathematization process requires that the students search for the solution. If the student is able to 'mathematize' the situation in an automatic way without much effort, then he is not in the presence of a context problem but rather of a simple mathematization exercise. In everyday life there are concrete situations that can be made into problems. These situations can be given a mathematical formulation and can become isomorphic to those presented in the school curriculum, encouraging the students' constructive mental activity in the processes of knowledge acquisition and an effective development of the ability to deal with problem solving.

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STUDENT' ALGORITHMIC, FORMAL AND INTUITIVE KNOWLEDGE: THE CASE OF INEQUALITIES

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ABSTRACT

This paper presents a study that aimed at investigating Italian and Israeli high school students' solutions to algebraic inequalities. The findings of this study are analysed with reference to Fischbein' notions of intuitive and algorithmic knowledge. Students' intuitive ideas and their algorithmic models when solving algebraic inequalities are identified and discussed. Our claim is that students intuitively used the solutions of equations as a prototype for solving inequalities. They developed an equation-algorithmic model for solving inequalities. Students were intuitively drawing analogies to either correct or incorrect solutions of related equations, either by excluding only zero values when dividing both sides of an inequality by a not-necessarily positive value, or when dividing by an expression without the exclusion of zero values as well. The latter tended to use the balance model when solving both equations and inequalities, explaining that it is always permitted to "do the same thing on both sides" of either an equation or an inequality.

Student' algorithmic, formal and intuitive knowledge: The case of inequalities

In his analysis of students' mathematical performance, Fischbein (1993) related to the algorithmic, the formal and the intuitive components. Algorithmic knowledge is the ability to activate procedures in solving given tasks, and understand why these procedures "work". Formal knowledge refers to a wider perspective of the mathematical realm, what is accepted as valid and how to validate statements in mathematical context. Intuitive knowledge is described as an immediate self-evident cognition of which students are sure, feeling no need of validation.

The three components are usually inseparable, and Fischbein explained that "sometimes, the intuitive background manipulates and hinders the formal interpretation or the use of algorithmic procedures" (Fischbein, 1993. p. 14). He presented a number of algorithmic procedures, which he called algorithmic models, when referring, for instance, to methods of reduction in processes of simplifying algebraic or trigonometric expressions. For example, students' tendencies to treat $(a+b)^5$ as a^5+b^5 or log(x+t) as logx + logt were interpreted by Fischbein as evolving from the application of the distributive law, which he identified as a prototype for simplifying algebraic and trigonometric expressions (Fischbein, 1993; Fischbein & Barash, 1993). In this paper we use Fischbein's theory to analyze Italian and Israeli secondary-school students' knowledge, their intuitive ideas and their algorithmic models when solving algebraic inequalities.

We chose algebraic inequalities for several reasons. First, inequalities provide a complementary perspective to equations and are part of many mathematical topics including algebra, trigonometry, linear programming and the investigation of functions (e.g, Mahmood & Edwards, 1999). Accordingly, various document, such as the NCTM standards(2000) recommend that all students in Grades 9-12 should learn to represent situations that involve equations and inequalities, and that students should "understand the meaning of equivalent forms of expressions, equations, [and] inequalities .. and solve them with fluency" (ibid., p. 296). However, in both Italy and Israel, algebraic inequalities usually receive relatively little attention and are commonly presented in an algorithmic way, merely by discussing various algebraic manipulations. Moreover, in both countries, the researchers witnessed students' and teachers' frustration with the difficulties encountered when dealing with inequalities.

We believe that teaching algebraic inequalities, like any other mathematical topic, should take into consideration students' correct and incorrect ideas when solving related tasks. Therefore, an understanding of students' ways of thinking about inequalities is a necessary condition for making didactical decisions, and for implementing the recommendations made by the NCTM. As a first step for investigating students' common solutions to algebraic inequalities we decided to study the publications in the professional literature. We found that many related articles deal with suggestions for instructional approaches, usually with no research support, and considerably less attention has been paid to students' conceptions of inequalities (e.g., Bazzini, 2000; Linchevski & Sfard, 1991; Tsamir & Bazzini, 2001). The latter studies pointed, for instance, to students' difficulties in grasping the role of the sign, to their tendency to reject the R or Φ solutions, and to difficulties they encounter when using logical connectives.

The present study was designed in order to extend the existing body of knowledge regarding students' ways of thinking and their difficulties when solving various types of algebraic inequalities. In this paper we focus on the question: What intuitive ideas and what algorithmic models can be identified in Italian and Israeli secondary school students' solutions to algebraic inequalities?

Methodology

Participants

One-hundred-and-ninety two Italian and 210 Israeli high school students participated in this study. All participants were 16-17 year old who planned to take final mathematics examinations in high school. Success in these examinations is a condition for acceptance to academic institutions, such as universities.

Tools

Italian and Hebrew versions of a 15-task questionnaire were administered to the students. Here we focus on five tasks. Of these, three deal with dividing an inequality by a not-necessarily-positive factor, and the other two dealing with quadratic inequalities. The first three tasks are:

Task I: Examine the following claim: for any a in R, $a \cdot x < 5 \Longrightarrow x < 5/a$

Task II: Examine the following statement: for any $a \neq 0$ in R, $a \cdot x < 5 \Longrightarrow x < 5/a$

Task III: Solve the inequality: $(a-5) \cdot x > 2a-1$, x being the variable and 'a' being a parameter.

Research findings indicate that when solving rational inequalities, students frequently multiply both sides of the inequality by a negative number without changing the direction of the inequality (e.g. Tsamir & Almog, 2001). It was also reported that students encounter difficulties when solving mathematical tasks, presented in a way different from the way they are used to. For example, when having to deal with parametric equations and inequalities that are commonly not discussed in class (e.g., Furinghetti & Paola, 1994; Ilani, 1998). We took theses data into account when constructing tasks I, II, and III.

The second type of tasks included two "solve" tasks.

Task IV: Indicate which of the following is the truth set (the solution) of $5x^4 \leq 0$,

(a) $\{x: x>0\}$ (b) R (c) $\{x: x<-5\}$ (d) $\{x: 0<x<1/5\}$ (e) ϕ (f) $x=0$ (g) $\{x: x\leq 0\}$

Explain your choice

Task V: Indicate which of the following is the truth set (the solution) of $\frac{1}{4} \cdot x^2 \geq 0$,

(a) $\{x: x>0\}$ (b) R (c) $\{x: x \geq 4\}$ (d) $\{x: x \geq 2\}$ (e) ϕ (f) $\{x: x \geq 0\}$ (g) $\{x: x \leq 0\}$

Explain your choice

Tasks IV results in a single value and Task IV or in any real number. They and were presented in a manner similar to other tasks presented in Israeli and Italian classes. As such, we assumed that students would feel they can solve the tasks, but that a substantial number of them will reach incorrect solutions in accordance with findings reported in the literature regarding such tasks (e.g., Tsamir & Almog, 2001).

Procedure

The students were given approximately one hour, during mathematics lessons, to complete their written solutions. In order to get a better insight into the students' ways of thinking, forty-five students were individually interviewed. In the interviews we asked students to elaborate on their written solutions. Each interview lasted 30 to 45 minutes.

Results

No significant differences between Italian and Israeli students' solutions, thus, we present the data of all students.

Students' Answers to Task I

In their responses to Task I, about three-quarters of the students correctly judged the claim: "for any a in R, $a \cdot x < 5 \Longrightarrow x < 5/a$" as being false, and accompanied their correct judgement with an acceptable justification. While about 20% elaborated on the role of the sign of the value substituted for a in determining the direction of the ">", about 55% mentioned only the "zero case" as a counterexample to the given statement.

In their explanations, students who *related to the sign of a* did it either in a general, verbal manner, or by means of a specific counterexample. Those who provided a verbal explanation, wrote, for instance, "there are three cases: when $a>0$ the statement is correct, for $a = 0$ it is impossible to divide by zero, and when a is negative the conclusion is that $x<5/a$". Some students just mentioned that, "for $a>0$ this is a correct statement, but for $a\leq0$ it is not", and others explained more briefly that, "this statement is correct only for positive 'a's"; or "when a is negative the direction of the sign changes". A number of students provided specific *counterexamples*. They wrote, for instance, "if $a=(-1)$ the statement is not correct"; or "if $-5x<5$ the conclusion is that $x>(-1)$, instead of $x<(-1)$". A few students exhibited a good understanding of the role of their single counterexample in refuting the given statement, by adding, "I gave one example, but a single counterexample is sufficient for proving that the statement is false."

Most prevalent was the students' tendency to use *only the '$a=0$' case,* as a *counterexample* to refute the statement. They wrote, for instance, "this statement is false when a equals zero"; or "the statement is false, because of the case of $a=0$". Many added "division by zero is undefined, therefore the statement is not *always* correct."

In their oral interviews these students' typically commented,

Sophia: the statement here refers to any number. BUT, since it is false for $a=0$, the statement is not true for *any* number. It is, therefore, false.

The few students, who incorrectly judged the statement as "true" either explained "we divided both sides by the same thing" or provided an example, "$5x<5$, for example, means that $x<1$". In their oral interviews these students typically added a confirmation like,

Jonathan: It's OK to do the same thing on both sides. When doing the same operation on both sides, the equivalency is preserved.

Jonathan went on to talk about equations and when the interviewer commented on his shift to equations he said: "It's the same..."

Students' Answers to Task II

Only about 30% of the participants correctly responded that the claim "for any $a\neq0$ in R, $a \cdot x < 5 \Longrightarrow x<5/a$" is false and accompanied their response with a valid justification. All of them related to the role of the sign in their decision. They explained, for instance, "if a is negative then $x>5/a$"; or "the claim is correct only for positive a". Some students added specific examples, "It is false, because it holds only when a is positive. For example, if $a=(-2)$, then $-2 \cdot x<5 \Longrightarrow x>(-2.5)$". Students were usually satisfied with a single counterexample, occasionally explaining, "one counterexample is sufficient in order to show that the proposition is false".

Most prevalent (over 50%) was the incorrect response that the statement is true, accompanied by a comment explicitly based on the given that $a\neq0$. Students wrote, for instance, "It is correct because of the given condition that $a\neq0$." In the interviews, these students pointed to connections they made between equations and inequalities. For example, Daniel had used the example of $2x=6$ to explain his solution to the given inequality in the questionnaire. When the interviewer related to it, he said,

Daniel: It's the same [pause]. In a way inequalities are a certain type of equations. Just that equations are easier, so I simply use examples of equations when I have a difficult inequality.

Several students gave explanations, similar to the following one,

John: In equations and inequalities, dividing by zero is problematic. But if we solve an inequality by operating with identical numbers on both sides, it is not only permitted, it is actually *the way* to solve the given tasks.

Students' Answers to Task III

Only a little more than 10% of the participants provided a comprehensive analysis of the various (positive, zero and negative) options for 'a'. About 45% of the participants either wrote that $x>(2a-1)/(a-5)$ for $a\neq5$ (about 30%), or were satisfied with writing $x>(2a-1)/(a-5)$ without any limiting condition. Surprisingly, in their interviews, a substantial number of the students clearly mentioned drawing analogies to equations. Bettina, for instance, wrote in her solution that $x>(2a-1)/(a-5)$ for $a\neq5$, and she explained in her interview,

Bettina: I divided both sides by the same expression, but I had to make sure that it is a non-zero expression. So, I wrote that a cannot be 5, because then a-5 equals zero...

Interviewer: Are you sure that five is the only problematic value here?

Bettina: [confidently] sure. I have done that a million times when solving equations.

On the other hand, Anna who gave the $x>(2a-1)/(a-5)$ solution (without mentioning any limitation), also mentioned the use of equation-ideas in the oral interviews. Like others who provided this solution, she explained that it is allowed to "do the same thing on both sides". She related interchangeably to equations and to inequalities,

Interviewer: Is it OK to divide both sides by a-5?

Anna: Yes. I have done the same thing on both sides. If you do the same thing on both sides of an equation [pause], I mean an inequality [pause], actually both, you reach an equation or an inequality that has the same solution as the given one.

Interviewer: Always?

Anna: It is not only allowed, it is necessary to do that in order to solve the problem.

Students' Amswers to Task IV

About 60% of the participants correctly wrote that the solution for $\frac{1}{4}x^2\geq0$, is R. The common error made by the other participants was that the set of solutions is $\{x: x\geq0\}$. This conclusion was usually reached in the following algorithmic manner:

$$1/4x^2\geq0/\ \bullet4$$
$$x^2\geq0/\ \ ^{-}$$
$$x\geq0$$

In their interviews, these students usually elaborated on the way they had solved the inequality, and mentioned having in mind the way they usually solve equations. Kim, for instance, wrote a solution like the one written above, and in her interview she explained,

Kim: Here [pointing to the / • 4 that she wrote in the first line] I showed that I multiplied both sides by four, and here [pointing to the / ‾ written in the second line] I showed that I calculated the square root of both sides. I used on both sides the same operation with the same number, until I isolated x on the right side and thus reached $x \geq 0$, [pause] the solution.

Interviewer: Why is this the solution?

Kim: I reached it by means of permitted actions in each stage [pause].

Interviewer: [Questioning look] ???

Kim: Like in the case of $3x = 6$, I divide both sides by 3 so I get $x = 2$. I do the same thing on both sides. The same operation, I divide by the same number, three [pause] and therefore 2 is the solution.

Interviewer: You used an equation, while here we have an inequality...

Kim: It is the same thing.

Interviewer: [Questioning look] ???

Kim: There is no difference between the ways of solving equations and inequalities.

The connections that Kim made between equations and inequalities could be identified both in her intuitive choice of an equation to exemplify the solution of an inequality, and in her explicit saying that "it is the same thing".

Students' Reactions to Task V

Only about 50% of the participants correctly responded to this task, marking $x = 0$ as the solution of the inequality. A substantial number of students in both Israel (about 20%) and Italy (about 15%) incorrectly wrote that the set of solutions of $5x^4 \leq 0$ is $\{x : x \leq 0\}$, which was usually reached in the following algorithmic manner:

$$5x^4 \leq 0 \ / :5$$
$$x^4 \leq 0 \ / ^4\sqrt{}$$
$$x \leq 0$$

In their interviews of these students, most students related to connections they made between the solutions of equations and those of inequalities. Betty, for instance, said,

Betty: I divided both sides of the inequality [$5x^4 \leq 0$] by five and reached $x^4 \leq 0$ [pause]

...

Betty: I calculated the fourth root of both sides, and got $x \leq 0$.

Interviewer: Is it OK to calculate the fourth root of both sides of an inequality?

Betty: Sure. The fourth root is a root of an even order, so we can calculate it when the given expressions are not negative. This is exactly the case here. Neither $5x^4$ nor zero is negative, so it's OK to perform this calculation for which I got $x \leq 0$.

Interviewer: Are you sure?

Betty: Sure. These are all procedures I know very well from solving equations.

Betty performed a valid manipulation of "dividing both sides by 5", but incorrectly defended the conclusions derived from her "calculation of the fourth root" of both sides. She explained that her certainty in the correctness of her solution was rooted in her experience with such procedures for solving equations.

Final Comments

The aim of this study was to deepen the understanding of students' performance with inequalities, by identifying intuitive ideas and algorithmic models in Israeli and Italian secondary school students' solutions to algebraic inequalities. As mentioned before, according to Fischbein algorithmic models evolve when students' intuitive ideas manipulate their formal reasoning and/or their use of algorithmic procedures. The latter usually express formal overgeneralizations and/or rigid algorithms (e.g., Fischbein, 1993). These models are usually coercive, used with confidence and grasped as being self-evident, even though they frequently lead to erroneous solutions.

We found that equations serve as a prototype in the algorithmic model of solving inequalities. This algorithmic model had mainly the appearance of "doing the same operation with the same numbers on both sides is valid for any operation with any number", in solving equations and inequalities. Students tended to correctly multiply both sides by 4 in Task IV, and divide both sides by 5 in Task V. However, they also frequently applied this algorithmic model when incorrectly calculating the square root of both sides in Task IV, the 4th root in Task V, and when dividing both sides by a not necessarily positive number in Tasks I, II and III.

Participants who applied this algorithmic model actually overgeneralized the balance model when incorrectly solving both equations and inequalities. They assumed that "doing the same thing on both sides of an equation *always* leads to an equivalent equation, and consequently to the solution". This assumption which is not even always true for equations, is much more problematic in the case of inequalities. By drawing the equation-analogy to cases of inequalities, these students reached incorrect solutions. In Tasks I and II they judged the statements to be true, in Task III the erroneously wrote that $x>(2a-1)/(a-5)$ is the solution to the inequality $(a-5)\cdot x>(2a-1)$, in Task IV the concluded that $x \geq 0$and in Task V they wrote that $x \leq 0$. Consequently, most prevalent errors were rooted in this algorithmic model, which was clearly and explicitly referred to in the students' oral interviews.

A substantial number of participants applied a version of this algorithmic model. That is to say, they knew that when solving equations they should be careful not to divide by zero, and since they held the equation-model for solving inequalities, they imposed the same condition in the case of inequalities. This assumption which is true for equations, is problematic in the case of inequalities. Again, the equation-analogy derived from inequalities, led to incorrect solutions for inequalities. These students usually answered Task I correctly, providing the 'a=0' case as a counterexample to refute the given proposition. However, they frequently, incorrectly regarded the proposition of Task II as valid or suggested $x>(2a-1)/(a-5)$ $a\neq 5$ as the solution for Task III.

We have seen that students tend to apply the equation algorithmic model when solving inequalities. This was done by students who correctly solved and by those who incorrectly solved the related equations. This understanding of students' solutions should be considered when planning instruction. Fischbein recommended that when teaching, students' be made aware of their erroneous ways of thinking (e.g., Fischbein, 1987). How to promote students' awareness is another issue for research. One way to go about it is by presenting students with parametric inequalities, similar to the ones given here. These inequalities were found helpful in triggering students, who hold different equation-based models, to answer differently, and occasionally also incorrectly. The various solutions should be discussed in class, while shedding light on the mathematical similarities and differences between equations and inequalities, and on students' intuitive ideas and the resulting algorithmic model that they intuitively use. How to implement such instruction and their impact on students' performance should further be investigated.

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DESIGN AND IMPLEMENTATION OF AN INDUSTRIAL MATHEMATICS DEGREE COURSE

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ABSTRACT

The reform of the Italian academic organisation gave the to mathematics community the opportunity to create new degree courses where elements of informatics and computational mathematics are introduced together with the classical mathematical courses.

The aim is to give the students the necessary tools and methods to apply their mathematical knowledge to the solution of real problems arising from scientific and industrial applications; in the third paragraph we briefly describe the economic and industrial situation that characterises Bologna and its geographical area and some of the connections of mathematics with local industries and their demands. A three-year course of Informatics-Computational Mathematics started in Bologna at the beginning of the current academic year (2001-2002): all the teaching and research skills of the mathematicians of Bologna (in particular in the area of computational and applied mathematics) converge towards the local industrial demands and the needs for a new interdisciplinary research. In order to achieve a high level characterisation, it is necessary to create collaboration among the teachers in the definition of the courses programs and in their realisations, because the solution of real problems requires a unitary knowledge of different mathematical and informatics instruments. We report in particular the experience of the informatics and computational courses.

Keywords: informatics and computational mathematics, industrial mathematics, new degree courses.

1. The Italian Academic Organisation

From the beginning of the academic year 2001-2002 almost all the Italian universities adopted the new organisation disposals defined by the Italian government in compliance with the joint declarations of the European ministers for Higher Education (May 1998 and June 1999). They proposed the adoption of a system essentially based on two main cycles, undergraduate and graduate. Access to the second cycle shall require successful completion of first cycle studies, lasting a minimum of three years. The degree awarded after the first cycle shall also be relevant to the European labour market as an appropriate level of qualification. The second cycle should lead to the master and/or doctorate degree as in many European countries. (MIUR, 1999).

In order to follows these guidelines, the Italian Government enacted two decrees, one regarding the first cycle studies and the other regarding the second cycle studies where specific guidelines are given for the new organisation of the University courses. In these documents it is established, for example, that the new courses must be based on the system of credits and that one credit corresponds to 25 hours of work per student.

2. Applied mathematics at the University of Bologna

The Italian Universities have created new undergraduate three years degree courses that prepare the student to enter the job market with a high level of qualification. In the area of mathematics, many new mathematical courses started in the present academic year. These courses contain elements of financial mathematics, computational mathematics, informatics and applied mathematics. The aim of these courses is to apply the theoretical mathematical knowledge to the solution of real problems.

The research of applied and computational mathematics developed in Bologna ([2]) is nationally and internationally acknowledged. The interest and application in the study and solution of real problems characterise the Numerical Analysis group. An overview of the most recent results is collected in (D. Trigiante, 2000). We quote in particular:

- the works in the field of medicine for the reconstruction of tomographic and magnetic resonance images, in collaboration with the public health institutions of Bologna and Florence, and for the reconstruction of echocardiografic images in collaboration with a leading appliance manufacturing industry;
- the research on geometrical modelling, in collaboration with local shoes industries;
- the activities of VisLab (Visual Laboratory) in pattern recognition and involvement in European projects, in collaboration with ESA.

Many of these activities are also part of the MIUR, Italian Minister for the University and Research, research project "Inverse problems in medical imaging" 2001-2002 ([4]).

The strong point is represented by the competencies in parallel computation, acquired since the 80s, with the CINECA university consortium, one of the largest computing centres in Europe. It was a pioneer centre for parallel computation with a Cray I in 1985 and since then it has always offered parallel and distributed architectures with very high performance.

The pressure on academic researchers to help industry and launch new enterprises has changed the "ivory tower" attitude of the 80s and the most important universities established liaison offices with the industry to help the companies get a better insight in academic research.

This diffused culture and attitude towards the applications of mathematics gave rise to the following activities, promoted by the numerical mathematicians:

- the degree course of Computer Science, that started in 1987, and one of the most attractive in Italy;
- a master one-year graduate course in Industrial and Applied Mathematics where faculty members give lectures with an application flavour;
- cooperation activity between Departments focused on engineering applications;
- at the national level, a leading role in the activation of several applied projects of the Nation Research Council namely on parallel computing, information systems, mathematical applications.

In this context, a new mathematical three-year degree, named *Informatics-Computational Mathematics,* stemmed from the described experience and collaborations; it started in the Academic year 2001-2002 in Bologna. Its program contains new and interesting elements of informatics and computational mathematics. The computational mathematics studies aim to show how to efficiently use mathematical knowledge combined with the power of digital computation.

It is a relatively new discipline, expanding very fast with the extraordinary increase of information technologies and it is very attractive for society as an effective instrument to solve new scientific and industrial challenges.

The creation of pedagogically sound applications modules that show how mathematics is used to solve real world problems is an enormously challenging task. The task of introducing new materials will become simpler as an effective dialogue between mathematicians, users and problems providers is developed.

3. Fields of interest and connection with the local industrial demand

Emilia-Romagna region has 4 millions inhabitants, about 446.000 small and medium enterprises (SME) and is the 13^{th} - over 190 regions in Europe - for gross per capita product. Bologna is its capital city.

Emilia-Romagna produces 17% of the national scientific outcome according to OCSE reviews ([5]); there are four public Universities and a private one, and a few centres of the National Research Council and of the National Energy, Environment and new Technologies organisation.

The economic development model is characterised by a dense network of subcontracts and a local sector specialisation inside the industrial districts that had been studied world-wide for its characteristics of productivity, flexibility and specialisation. The districts traditionally deal in the textile, clothing, industrial, shoe, ceramics, motorcycles, agrimechanics, packaging, bio-medical and wood machinery sectors. In the last ten years, in accordance with the information and communication technologies (ICT) growth, there have been plans and initiatives for a multimedia virtual district and general improvements of the use of ICT in the innovation of processes and products for the traditional districts. At the same time several ICT regional companies have grown considerably and have been acknowledged on an international level.

There are many connections between the research environment and the regional SMEs represented by joint ventures, consultancy, spin-offs, research and demonstrational activities often in the European research framework.

The mathematical community has not always been an active partner in these initiatives, at times due to a poor demand (e.g. the widespread use of software packages, use to solve engineering problems, restrain the user from entering the algorithmic and modelling representation of its products) or because of the academic indifference to the local demand, many times inaccurate and not paying in the research arena.

The formulation of the SMEs' needs is now more accurate, the attitude of the renewed university is more inclined to co-operation with industry, the students, often computer-literate since their teens, desire to enter the job market with stronger determination.

What the University has to guarantee is the scientific level and quality of the co-operation: application tools and methods must be at a state of the art level, both in research and tools availability.

Moreover the research community follows the objectives of:

- pursuing the dissemination of the results in the industrial and service sectors;
- promoting the technological transfer towards the local economic bodies.

According to (NESTI, 1963) we use the term Research and Development (R&D) if there are remarkable indications of novelty and the reduction of scientific and technological uncertainty.

The innovation process is based on the following phases: design, R&D, equipment and knowledge acquisition, engineering, production, deployment, marketing .

The scientific and technological activities include: education, R&D, scientific services.

In figure 1 the interconnections between the phases and activities are graphically represented; the circle intersections have the following meaning:

- on the vertical axis there is the border between prototypes and pilot products;
- on the horizontal axis there are advancements in an area (e.g. software tools) or in the knowledge provisions for an application area (e.g. improvements to a simulation numerical technique).

We have a tradition of co-operation with some SMEs and we have collected (in our opinion at a high level) industrial R&D demands that are useful to structure the courses especially regarding their final steps when industrial seminars, stages, joint research projects are possible and desirable.

We provide a list of the current requests to provide an insight of our point of view, rather than to discuss the mathematics behind the proposals.

Modelling and analysis software for fluid flow and heat transfer in the processes of atomisation, water reduction and reshaping of the particles constituting the "casting slip" (mix of different powder clays and other materials) take place before the phase of **forming ceramic articles**. Nowadays, manufacturers of fluid flow-machines design their equipment relying on the knowledge of a small number of cases practically tested. The simulation software and the use of parallel computing will make possible to carry out a large numbers of simulation runs, studying different geometric configurations, with different sets of parameters, in order to approach a design solution close to optimal behaviour in all operating conditions.

The multibody simulation is a well-established technique to help in the study of new mechanical systems; it is called "virtual prototyping" methodology. Several general-purpose programs for cinematic and dynamic analysis are available; they are not design tools but rather analysis tools. Analysis programs can perform the simulation of a system once its geometric and inertial characteristics have been designed. However, in the design problem, the desired response of the system is known and the designer wants to find out the values of the design parameters that better satisfy the design requirements. To solve this problem the designer needs to know how changes of the parameters affect the systems behaviour. The **suspension design for shock absorber** manufacturers of two-wheeler vehicle is a difficult task because of the complexity of the mechanical system, the influence of the rider, the presence of a passenger that can increase the static load of the rear suspension.

There is a strong demand of tools for simulating the **percolation phenomenon** in a variety of applications like the design of new products in the coffee industry, the experimentation of elasticity properties of batches of tyres and the monitoring of chemical contamination of soils. Cellular automata, running on parallel platforms, are a promising software technology.

There is a need by the motor automotive industry to design better transient air and fuel film compensation algorithms capable of improving the performance of **automotive exhaust emission** control system by dynamically compensating for the transient response of the engine during thermal, speed and load transients. The reduction of engine emissions, during cold starts and dynamic operations conditions, is essential to ensure compliance with future regulations.

The **rapid prototyping** techniques to reproduce objects in the mechanical and furniture industry allow a generation of manufactured articles starting from a mathematical definition based on a three dimensional geometric description; selective laser sintering, fused deposition, solid ground curing and laminated object manufacturing are the most used techniques.

The **geometrical modelling** is a fundamental sector for household appliances industries, in the design of products. It is based on mathematical knowledge and graphical representation. Moreover, a growing sector in the software development is represented by CAD techniques that are the fundamental instrument for creating new shapes.

4. Objectives and organisation of the Course

The aim of the Informatics-Computational Mathematics Course is to give to the student solid base of mathematics together with an adequate competence in developing and applying the mathematical and informatics instruments necessary to solve real problems.

The professional and scientific profile of the graduate in this course should be that of an applied mathematician with high critical ability, a good knowledge of the mathematics used for the description of models typical of technological and industrial processes and the ability of applying it using informatics tools. If the student enters the job market, his/her profile answers the demand of the local industrial reality; if the student intends to continue his/her studies to the second level degree and in a doctorate/master, she or he has good basis to go into the research. The competencies in computational mathematics are required not only for an academic research, but also in high level scientific research developed in projects involving physics, engineering, biology, medicine. Many scientific challenges of our times need computational mathematics: let us think of the large and complex computations in biological research on human genome, where huge amounts of data must be accurately processed. Trace of it is the relevance that some widespread journals, such as *Scientific Computations*, give to these researches and to the role of numerical mathematics.

The premise we start from is that what would motivate someone to learn mathematics might not be only the intrinsic beauty of mathematics itself, but something quite different arising from concern and dedication to another subject altogether.

The studies are organised in compulsory and optional courses. The student gains the credits correspondent to a course by attending classes (8 hours per credit in the classroom, 9 hours in the laboratory) and by passing the exam. 180 credits are necessary to achieve the degree.

In the definition of the computational and informatics parts of the curriculum, the directions given in the *ACM Computing Curriculum 2001* (The Joint task force, 2000) have been deeply considered. It identifies a set of knowledge areas in the field of Computing and in each area has defined the body of knowledge. Among them, the area of Programming Fundamentals (PF), Algorithms and Complexity (AL), Programming Languages (PL), Computational Science (CN) and Graphics, Visualisation, Multimedia (GR) have been regarded as essential in the curriculum of the our degree Course.

In the Italian university organisation, these areas are labelled as Informatics (PF, AL, PL) and Numerical Analysis (CN, GR).

About 80 credits of the whole 180 are assigned to the characterizing courses of Informatics, Numerical Analysis and Statistics.

The first year courses started in October 2001; their plan is reported in table 1.

The Multimedia Laboratory is the first course of informatics attended by the students. The aim of the course is to introduce the student to future courses of Informatics and Numerical Calculus, thus sewing the seed for a new mathematical culture, not only confined to abstract spaces, but also applicable to real phenomena. In this context, the terms "informatics" and "computational" should become adjectives of the term "mathematics" in the everyday student language. We plan a laboratory-based on application-oriented instruction and thus diverging from traditional computer science approaches that are often model-based and research-oriented.

In the second semester, the course of Informatics I starts a systematic approach to algorithms and programming with the C language.

The course of Numerical Calculus I introduces the student to the concepts needed in the numerical problem solution: the floating point numbers, the conditioning of a problem and the stability of an algorithm and presents some examples of simple numerical methods. The aim is to give some instruments for problem-solution and mainly to develop a critical ability to the numerical approach to a problem. Really, it is not sufficient to find a resolution algorithm, because many factors affect the efficiency and precision of the method. Some practices are planned in collaboration with the "Informatics I" course; they rely on numerical topics but they make use of the programming language and instruments dealt with in Informatics.

In the following years, in addition to a deeper insight to the informatics (software engineering, net-centric computing, information management) and classical numerical analysis topics, some optional specific courses are proposed. We mention the courses of image processing, parallel computation, computational graphics and geometric modelling.

We are giving a "doing-centred" approach to the students' problem solving skills.

We think that industry increases the communication between the people in charge of hiring applied mathematics graduates and those in charge of educating and defining the curricula. Most of the jobs taken by just completing undergraduate level involve software design and development. But students are weak in a number of areas such as general communication skills, team development experience, user-oriented development practice, analysis of the design experience. Hence we stress the importance of work and experience in the computing laboratories.

With regard to the management of labs activities, we encourage the following behaviour that, we know and experienced, can lead to a virtuous circle:

1. students' aggregation and demand are to be encouraged;
2. students' community awareness are to be developed and strengthened;
3. students' motivation in the courses is to be sought and monitored;
4. students' help and suggestions for the services of the labs are to be considered and accepted;
5. information is to be widely distributed and discussed;

6. services are to be improved on the assumption that the users are involved in the operation and management of the labs.

For the success of 1 and 2 the factors of importance are: the positive attitude by the teachers and instructors, the motivation of the students, the quality of the computer systems and an appropriate credits award.

For the creation of 3 the success factors are: effective information updating, qualified technical assistance, organisation of the students into groups and evaluation of the groups' results.

For the issue of 4, the feeling of belonging to a scientific community strengthens positive behaviours and, as by-products, hackerism and stealing are hopefully eliminated.

Points 1 and 5 concern teachers, instructors as well as technicians and porters.

Point 6 stresses the vision of the labs activity as a core line in the education process.

5. The first semester experience

So far we have only experienced the Multimedia Laboratory course in the first semester of this academic year.

At the beginning of the course, the students compiled a questionnaire on their basis informatics knowledge; the questionnaire showed that about the 30% of the students (60 altogether) had never used a computer and of the remaining 70%, more than 50% had used a computer only for writing with a word processor.

Hence the program course contained some basic knowledge of informatics (the concept of algorithm, network and internet, the Web, multimedia data, data bases, hardware components, operating systems) together with examples and simulations, taken from the Web, of real applications based on mathematics in the area of image processing, astronomy, geometrical modelling, fluidodynamics. We consider a by-product of the course the capabilities of the first modules of the European Computer Driving License [8] that students learnt in the labs and through homework.

The students never imagined that mathematics was so important for these applications and they were fascinated by this fact. In the last ten hours we introduced the concept of a programming language, its main structures and its representation with some examples. Finally, we implemented some simple algorithms, in order to show how a program runs, the possible errors so as to point out the difference between the result obtained with the same method in the real arithmetic and in finite arithmetic on the computer.

We always give lessons with the help of multimedia instruments, both in the classroom and in the laboratory and the course material was distributed through the Web.

At the exam, not all the students showed a full understanding of the informatics concepts, but they all could write some C programs, and they had understood that this is the instrument to create numerical simulations.

6. Conclusions

We are experiencing the first year of a new three-year degree course of Informatics-Computational Mathematics. Our main effort is to prepare the students to enter the job market, with a high level of knowledge, with a particular attention to the demand of the local industry and, at the same time, we want to provide the students that intend to continue their studies with a good level of knowledge. We picked out the essential features for achieving a high scientific level: the definition of a good curriculum, the collaboration between the teachers and the work of the students in computing laboratories following some experienced behaviour. The curriculum should contain, together with basics and advanced classical mathematical courses of analysis, geometry and mathematical physics, basics and advanced courses of informatics and computational mathematics. As far as the informatics and computational courses are concerned, the teachers have planned a jointed action in order to practice on computers, sharing the different knowledge for the common objective of making the students aware that mathematics is an essential instrument for studying and solving many real problems.

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Course (semester)	Credits	Hours of lesson
Algebra I (I)	7	56
Geometry (I)	7	56
Analysis (I)	7	56
Mathematical Physics (I)	7	56
Multimedia Laboratory (Informatics) (I)	4	36
Geometry II (II)	6	48
Analysis II (II)	6	48
Informatics I(II)	6	48
Numerical Calculus (II)	6	48

Table 1 : the first year courses of the Informatics-Computational Mathematics degree plan.

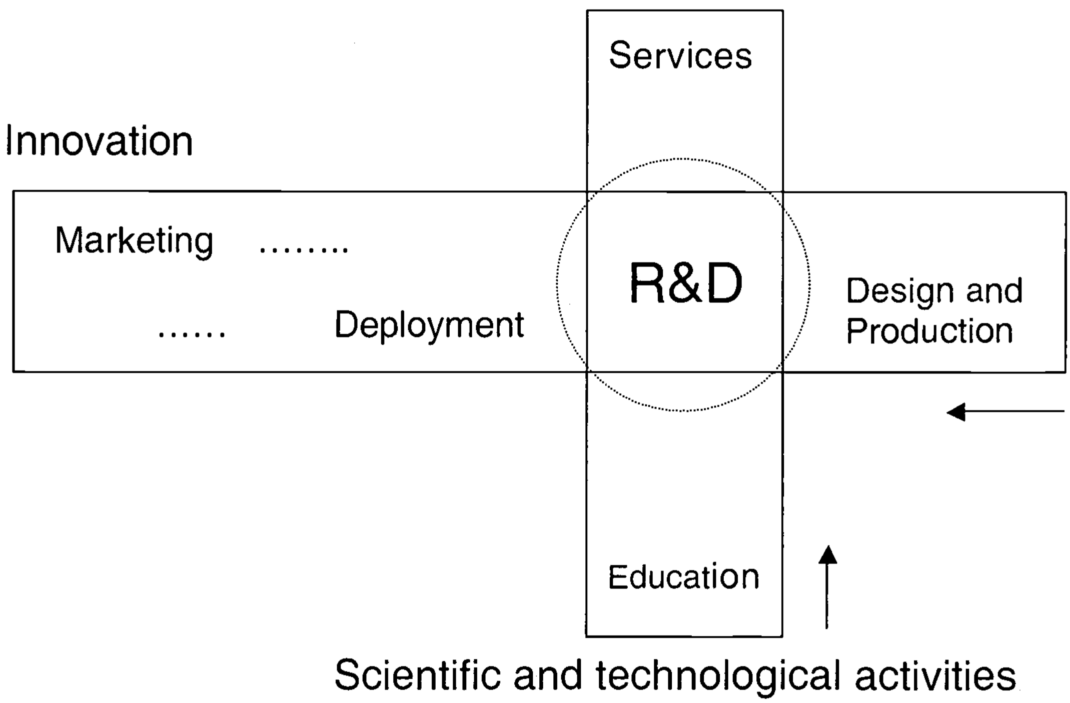

Figure1: Innovation and Scientific and technological activities.

PRACTICE MAKES PERFECT ON THE BLACKBOARD:
A Cultural Analysis of Mathematics Instructional Patterns in Taiwan

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ABSTRACT

Taiwanese students' math achievement ranked the 3rd among the 38 countries participating in the 1999 Third International Math and Science Study (TIMSS-R). This study aims to show how Taiwanese teachers teach math so as to produce such a remarkable achievement. Math teachers in the middle school in Taiwan were videotaped giving instruction on a math concept. The videotapes were reviewed and analyzed by using both the quantitative (Teacher Observation Schedule) and qualitative methods (observation notes). The results show that Taiwanese teachers focus more on demonstrating math procedures rather than on math concepts and tend to ask students to practice both on the blackboard and at their seats. Moreover, Taiwanese teachers are inclined to ask their students to offer the "right" answer to questions on the blackboard after individual practice at their seats. Cultural beliefs including "practice makes perfect," "one standard for all," and "motivation by wish for self-improvement" underlying the Taiwanese instructional patterns are discussed.

Key Words: Mathematics teaching, instructional pattern, Taiwan, cultural analysis

1. Introduction

According to the TIMSS 1999 Math Report (Mullis et al., 2000), there is a sharp difference in math achievement between students in the U.S. and students in several East Asian countries, amongst them Taiwan, where student math achievement ranked the 3rd among the 38 participating countries. This difference in achievement was discussed earlier in Stevenson and Stigler's (1992) work on the "learning gap" in student performance with reference to differences in teacher practice and parental rearing patterns between the West (American) and East (Chinese and Japanese).

Stigler and Hiebert (1999) now further contend that while teachers' general practices do make a difference to students' performance, it is the processes of teaching in the classroom that might really bring about the differences in students' learning. By analyzing TIMSS videotapes on math instruction given by teachers in Japan and the U.S., they have found a "teaching gap" between cultures. While American teachers focus more on procedural skills, Japanese teachers emphasize more on conceptual understanding. Thus, in an American math class, students spend most of their time acquiring isolated skills through repeated practice, whereas in a Japanese classroom, students devote as much time to solving challenging problems and discussing mathematical concepts as they do practicing skills. It can be concluded that teachers in different countries display markedly different teaching patterns, which result in very different approaches to how students learn math. Different teaching methods need to be understood in relation to the cultural beliefs and assumptions imbedded in different countries. Thus, teaching appears as a cultural activity. It would be interesting, then, to know how Taiwanese teachers teach math so as to produce such a remarkable achievement for their students in international math competition. The purpose of this study is to discover the instructional pattern shaped by teachers in Taiwan.

2. Research Design

Due to the prevailing reservations of Taiwanese teachers toward being videotaped in the classroom, the researchers had tried very hard to recruit teachers to participate in the study, and could finally obtain agreement from three math teachers in two middle schools in the Taipei area. This sample of three teachers showed a variation in age and gender. While the two male teachers were beginning teachers, the female teacher had more than 20 years of experience.

The data sources for this study were videotapes of the teachers' instruction and observation notes made by independent observers in the classroom. The researchers videotaped instruction on a math concept, lasting for 3-4 periods (hours), given by each teacher. The videotapes were reviewed and analyzed using both the quantitative and qualitative methods. A Teacher Observation Schedule, adapted from Stallings Observation System (Freiberg and Waxman, 1988; Stallings, 1986) was used to quantify the number of instructional activities and teacher-student interactions in the classroom. Three members of the research team coded the occurrences of different types of instructional activities and teacher-student interactions at the frequency of two times each five minutes. A high consistency was reached among the three raters with a .90 inter-rater reliability. Also, a close review of each segment of the videotapes was conducted to yield rich qualitative data. Extensive observation notes were taken on the instructional flow, student-teacher interaction and classroom atmosphere to discover the distinctive characteristics of the instructional patterns common to three math teachers.

3. Research Outcomes

From analysis of videotapes, we found that the instructional pattern common to the three teachers consisted of the following six steps: (1) review of previous materials, (2) presentation of the topic for the day, (3) presentation of definitions of terms and rules, (4) demonstration with examples, (5) practice, and (6) assignment of homework. At the beginning of a class, the teacher usually starts with a check of the homework assignment or gives a quiz to review material taught in the previous period. He/she usually calls on students to write up the procedures and solutions on the blackboard and then checks if the students give the right answers. When the teacher moves on to the new topic, the students "automatically" take out the math textbook and turn to the exact page from which the new topic begins. The teacher then presents the new terms and rules by either contrasting them with the previously established ones which can not apply to the new situation or by highlighting the "knack" of deriving correct answers to math problems in the new section being studied. At this stage, the teacher usually asks some closed questions to check if students get the point. The teacher provides little context relevant to the students' previous experiences and raises few questions to arouse students' interest or curiosity on the topic. Most students appear to show a "readiness to learn."

He/she then demonstrates with two to four problems with different degrees of difficulty to show how to apply the rules to get the answer. All of the three teachers tend to use the deductive, rather than the inductive, approach to math instruction. To check if students have learnt the rules and skills, the teacher calls on some students to practice problems from the textbook on the blackboard while other students do the same problems at their seats. The teacher usually calls upon students by drawing lots or based on a certain sequence so that each student has similar opportunity to practice on the blackboard. Usually several students are called upon at a time to solve problems of various types and/or degrees of difficulty. If students at the blackboard are stuck, the teacher usually helps them by giving out some hints. If the students get the wrong answers, the teacher will correct the mistakes and remind the whole class to be aware not to make the same errors. In the case of "hopeless" students, the teacher will solve the problem for the students. With the correct procedures and answers listed on the board, the teacher will then ask the whole class to check their own answers against the "standard" ones. This cycle of teacher demonstration and student practice at the board and in seats is usually repeated several times, occupying the major block of time in an instruction period. At the end of the class, the teacher usually gives homework either from the textbooks or from self-produced worksheets. He/she may also announce a quiz to be held in the next period on the topic just taught.

4. Discussion

4.1 Comparison with the American and Japanese instructional patterns

Comparing our findings with those of Stigler & Hiebert's (1999), it is found that math teachers in the U.S, Japan, and Taiwan all review previous materials, present the problems, and have students practice problems at their seats. It is the process of presenting the problem that reveals a great difference among the three countries. Similar to American teachers, Taiwanese teachers focus more on demonstrating procedures rather than on math concepts and ask students to practice procedural skills rather than understanding of the reasoning behind the procedures. However, American students tend to practice procedures at their seats, while Taiwanese students practice both on the blackboard and at their seats. Both Japanese and Taiwanese teachers present questions and call students to present their answers on the blackboard. However, Japanese teachers encourage students to offer alternative solutions to the questions after their group discussions, while Taiwanese teachers ask their students to come out with the "right" answer to the question after individual practice at their seats.

4.2 Cultural beliefs underlying the Taiwanese instructional patterns

Concurring with Stigler and Hiebert's (1999) argument that teaching is a cultural activity, it is of interest to consider the cultural beliefs underlying the distinctive Taiwanese math instructional pattern.

4.2.1 Practice makes perfect

The reason why practice plays such a major role in math instruction may lie in the deep-rooted conviction that "*shou neng sheng qiao* (practice makes perfect)." Many Chinese idioms express a concept that places "practice" in the pivotal role in human learning. It is believed that only through constant practice can a task of learning be perfected. Therefore, in the context of a math classroom, only through repeated practice on the problem can a student master the skills to solve the problem. Such a cultural belief can be seen as embodying a view of an "incremental" perception of intelligence that characterizes human intelligence as a malleable quality that can be increased through effort, in contrast with an "entity" theory that sees human intelligence as a fixed permanent entity that cannot be changed (Dweck, Chiu, and Hong, 1995; Hong, Chiu, and Dweck, 1995). From this perspective, one possible explanation for the reason why the Taiwanese teachers tend to ask their students to practice repeatedly is that they may believe that students' intelligence is malleable and can be increased through such constant practice.

4.2.2 Student practice on the blackboard

Further, for reason why Taiwanese teachers tend to call upon students to demonstrate on the chalkboard so frequently, several possible explanations can be offered: first, teachers use this method to check if most students understand what is taught. Faced with Taiwanese students who are characterized as "passive and reluctant" to speak out in class, the teachers may feel that they need to call upon individual students to see if they really understand. An experienced teacher may ask a student whose math achievement level is in the middle of the class to come to the board, to, in effect, see if half of the students could understand what is taught. Moreover, the teacher may use this "blackboard" method to save time and energy. When teachers call upon several students to practice a variety of math problems on the board at one time, they can see if students have mastered different types of skills or if they make the same or different types of errors. Within a relatively short period of time, command of a variety of skills can be demonstrated and errors can be corrected. This is an efficient way of instruction given time constraints. Third, the teacher may use this method to provide the "standard" answer for the whole class to check against their own answers. If students do not know how to solve the problems, they can copy down the standard procedures and answers. Lastly, the teacher may use this method to caution the whole class from making the same errors committed by individual students. When a student on the board is stuck or produces wrong answers, the teacher usually capitalizes on this opportunity to elaborate on the right procedure and caution against possible mistakes.

Two cultural beliefs underlying this work on the blackboard are, first, an orientation to conformity by establishment of a standard criterion for individuals to compare against; second, an inclination to self-improvement through constant comparison of self achievement with the standard criterion. Based on their work on personality traits in East Asian countries, Kitayama et al. (1997) and Heine et al. (1999; in press) report that individuals in these countries are encouraged to identify socially shared images of ideal and to compare the state of self with this ideal. Individuals seek and discover the externally set standards of excellence, critically assess themselves to determine what they are missing, and endeavor to eliminate the perceived deficit. This practice of self-improvement through constant self-comparison and self- criticism gradually leads the individual closer to the ideal criteria, and thus build his/her self-esteem in the process. As a country in the East Asian Confucian Circle, Taiwan shares the cultural beliefs of "common standard" and "self-improvement." These cultural beliefs penetrate into the actual classroom practices of the math instruction when the teacher asks the students to practice on the blackboard. The standard answer on the board serves as the "ideal criteria" for students to compare against their own. Through constant comparison, students can finally make improvement on their math performance.

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THE NEW MATHEMATICS MATRICULATION/ENTRANCE EXAM SYSTEM IN HUNGARY

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ABSTRACT

We are witnessing a long-term educational reform after the political changes in Hungary. The main elements of this reform - beside the question of educational management and finance - are the changes of the curriculum and the matriculation examination. Matriculation examination will have double function in the future that is, on the one hand, a final exam for secondary education and, on the other hand, an entrance examination for the tertiary level of education.

Within the frame of this examination reform we analysed the advantages and disadvantages of the present examination in the mirror of the expected social, educational policy and curriculum changes.

During our research and developmental work we considered the international trends and the applicable Hungarian traditions. We concentrated on the development of a new examination model and new types of tasks and items.

We had the opportunity to field test the new tasks and items and also to collect teacher's opinions and suggestions. After a careful analysis the experiences has been built in the new examination model.

In our presentation we would like to demonstrate the new crystallized examination model and some of the new examination tasks.

The main characteristics of this model:

- Two levels, the upper level has the selective function for the tertiary education.
- The examination has a centrally developed written and oral part.
- Among the tasks there are short answer questions and some complex mathematical problems with multiple questions.
- The evaluation of the written part is based on a detailed evaluation guide.

The model will be illustrated with concrete examination tasks and their solutions.

Key words: Mathematics, curriculum reforms, matriculation and entrance examination, examination model, requirements for the matriculation and entrance examination, Hungary.

1. The reform of the matriculation/entrance exam in view of the education reform

In Hungary, education had a centralized system for nearly 40 years. This was apparent from the unified, central curriculum, which was compulsory for all, and the lack of choice in textbooks. In the teacher training university and college research workshops research and innovation has been done from as far back as the beginning of the 80s, whose aim was to create new textbooks and textbook families that will better serve methodological directions. Other important steps in the direction of decentralization were the following: the running of schools was decentralized, and the role and responsibility of local councils and communities increased. At the same time, new schools appeared that were run by foundations or churches. These changes have, of course, brought along a renewal of the context and regulation of the education system. As a result of a long innovation process a new curriculum was born in 1995, which instead of the old, strictly specified subjects was based on "cultural domains". It made a summary of its requirements for two-year periods, and it did not fill the whole number of lessons but gave way to - and in fact expected - additions to the curriculum on the basis of local needs. (NAT, 1995) This meant more freedom but also more responsibility for teachers – and it was welcomed by many, but was too fast a change for the majority. The preparation of local curricula meant such new tasks that teachers found it difficult to cope with them - and they made their changes with very mixed quality levels. Therefore, after the change of government in 1998 the new education ministry overruled the introduction of the core curriculum and created a new type of frame curriculum, one that gives more freedom to teachers than the old centralized one but which also has stricter regulations than the core curriculum. (Kerettantervek, 2000a; Kerettantervek, 2000b) For example, this new frame curriculum went back to the old subject system and to a yearly definition of requirements.

In the process of the education reform the main change from the point of view of the graduation/entrance exam system is that whereas in the past the core of the exam was determined by the contextual elements of the curriculum, there has now appeared - as new elements of the exam reform, based on the curriculum changes - a detailed description of requirements and a more strictly structured exam description. These changes will also serve the new needs that society creates, which will in turn increase the reliability of the exam results and ensure equity and comparability. This is not only a Hungarian but an international trend too. (Galbraith, 1993; Niss, 1993; Wain, 1994; Gipps and Murphy, 1996, Mátrai, 2001)

2. Description of the Mathematics matriculation/entrance exam

The current Maths graduation exam can be, in short, summarized as follows.

Students can choose between two ways of taking their exam according to their plans for further studies.

1. A school exam can be taken by students who do not want to continue their studies or would like to apply to a higher education institute that does not require them to take a Maths entrance exam. These exams are based on the material covered by the minimum compulsory number of lessons. Such an exam has two versions, which are linked to the two types of secondary schools.

2. *A joint matriculation/entrance exam* must be taken by students who would like to continue their studies in a higher education institute that requires a entrance exam in Maths. With regard to their material context, these exams are not different from the school exam, though the questions are more complicated and require a higher level of Maths problem-solving skill.

The most important characteristics of the current Maths exams are summarized in Table 1:

Table 1. *Summarizing the possibilities of the current Maths Graduate exam*

Exam attributes	**Centrally designed, but locally taken exam- school exam**		**Centrally designed, externally taken exam – joint matriculation/entrance exam**
Version	For secondary grammar schools	For vocational secondary schools	For students taking a Maths entrance exam
The set up of the exam	6 open-ended problems and the verification of 1 known theorem.	5 open-ended problems, 1 definition and the verification of 1 known theorem	8 open-ended problems
Duration	180 minutes	180 minutes	240 minutes
Scores	Maximum 80	Maximum 80	Maximum 100
Evaluator	Secondary school teacher	Secondary school teacher	External (+) secondary school teacher
The source of the assigned problems	Chosen from known problems (Gimes, 1992)	Chosen from known problems (Gimes, 1992)	Unknown problems

The chart refers to the way of evaluation and the method of problem assignment. The fundamental difference between the two ways of evaluation is that while the school exam paper is checked by the secondary school teacher of the student, the joint exam is evaluated by two independent teachers - for two different reasons. On one hand, the secondary school teacher will decide the grade that a student will obtain as his/her graduate exam result; on the other hand, the external evaluator assigned by the given higher education institute gives the result that the success of an entrance exam will depend upon. The assignment of the problems for the school exam is based on a collection of problems that has a 20-year history and which contains over 4000 problems that have remained basically unchanged during this time and which are announced on the day of the exam via the media. (Gimes, 1992) The set-up of the test has also stayed unchanged over the years. The design of the joint exam is undertaken by a professional board and contains problems that are especially designed for the exam every year. (The taking of this type of exam is helped by the publishing of test papers from previous years.) We examined the advantages and disadvantages of the current matriculation exam as part of the research/innovation process pertaining to the new matriculation exam. On one hand, our research covered the analysis/evaluation of the design method and evaluation instructions going with matriculation exam test papers coming from previous years. (Tompa, 1999.) On the other hand, we analysed and re-evaluated randomly chosen actual written test papers and their corrections and evaluations as done by teachers.

From an analysis of the documents and a comparison of the results achieved by students in concurrent years it was evident that the exams of each year came with a different level of difficulty. (Tompa, 2001) As a result of this, the Maths grades of the various years are unable to serve as a reliable basis for an evaluation of students' actual knowledge. This research also showed that exams set up according to these principles do not fulfil the criteria of objectivity and equality and comparability; in other words, owing to a lack of sufficient evaluation instructions there is room for subjective evaluations. As a result of a teacher's strictness or leniency the results going with individual classes can easily become up- or downgraded – so that an equal value being given to different results cannot be guaranteed. (Frisbie, 1988; Gipps and Murphy, 1996)

Teachers' opinions given during the course of the creation of the new requirements and the testing of the new type of graduate exam show that Hungarian Maths teachers in general rejected the type of exam containing closed-ended test questions i.e. which would be the best way to ensure objective evaluations. When analysing exam models coming from other countries, such elements are more common in exams that serve as higher education entrance exams. (Mátrai, 2001) Thus, we concluded that, basically, our new exam model also favours open-ended test questions. We simply cannot ignore the great amount of rejection involved here and choose closed-ended (e.g. multiple-choice) test questions to out-rule the possibilities of subjectivity (Osterlind, 1998). This view - which most teachers share - is also in line with the Maths exam philosophy of the exam-designing workgroups.

3. The development of the joint Maths matriculation-entrance exam

Before we give further details about the new elements of the Maths matriculation-entrance exam, we would briefly like to summarize those educational policy decisions that have an effect on the whole of the graduate-entrance exams.

The new matriculation exam is unified - which means that it measures students' knowledge under the same regulations, with the same test papers and evaluation mechanisms both in the framework of regular and adult education, and both in secondary grammar and vocational schools.

The other important difference is the introduction of two levels relating to all subjects, i.e. students can choose between a lower and a higher level of graduate exam; this latter will also serve as an entrance exam. (This in the past was only possible in the case of a few subjects.)

The Maths exam design process is similar to that of the other subjects. The development was preceded by a research period that made an analysis and comparison of Hungarian traditions and international trends. (Lukács, 1997; Mátrai, 2001) The development has been carried out by a diversely selected workgroup (among its members one can find experienced grammar and vocational school teachers, higher education experts, curriculum and evaluation experts, and textbook writers). Every document created by the workgroup (exam requirements, exam model, exam descriptions, sample test papers, evaluation guidelines etc.) has to succeed in a multiple professional evaluation, which means (among other things) professional proofreading, tutorial and higher educational opinion polling, and the collection and use of the points of view of professional pedagogical organizations. There are, of course, in these design-groups people whose main task is to make up these new types of test questions and the detailed answers. The creation of the new exam model was also preceded by testing some of its versions in schools.

The legal document for the exam contains the requirements in detail and a description of the exam for both levels.

4. The introduction of the joint Maths matriculation-entrance exam under development

In our present study we only have the chance to introduce the high level exam.

The requirement system consists of the following content elements, which is given further detail in the exam document, thus describing the contextual and underlying differences between the two levels.

4.1 Mathematical content of the requirements

1. Methods of Mathematical thinking, sets, logics, combinatorics, graphs
    - 1.1 Sets
        - 1.1.1 Operations on sets
        - 1.1.2 Cardinality, sub-sets
    - 1.2 Mathematical logic
        - 1.2.1 Concepts, theorems, proof and verification in Math
    - 1.3 Combinatorics
    - 1.4 Graphs
2. Arithmetic, algebra, number theory
    - 2.1 Basic operations
    - 2.2 Set of natural numbers, basic knowledge of number theory
        - 2.2.1 Divisibility
        - 2.2.2 Number Systems
    - 2.3 Rational and irrational numbers
    - 2.4 Real numbers
    - 2.5 Powers, roots, logarithm
    - 2.6 Formulas ("letter equations")
        - 2.6.1 Notable identities
    - 2.7 Proportionality
        - 2.7.1 Percentages
    - 2.8 Equations, equation systems, inequalities, inequality systems
        - 2.8.1 Algebraic equations, equation systems (linear, quadratic and higher order, square-root)
        - 2.8.2 Non-algebraic equations (absolute values, exponential, logarithmic, trigonometric)
        - 2.8.3 Inequalities, inequality systems
    - 2.9 Means, inequalities
3. Relations, functions, the elements of calculus
    - 3.1 The concept of functions
    - 3.2 One-variable real functions
        - 3.2.1 Graphs of functions, transformation of functions
        - 3.2.2 Characteristics of functions
    - 3.3 Series
        - 3.3.1 Number series, geometrical series
        - 3.3.2 Infinite geometrical series
        - 3.3.3 Compound interest, allowances
    - 3.4 The elements of calculus – One-variable real functions
        - 3.4.1 Limit, continuity
        - 3.4.2 Differential calculus
        - 3.4.3 Integration
4. Geometry, coordinate geometry, trigonometry

4.1 Elementary geometry
    4.1.1 Elements of solid geometry
    4.1.2 Sets of points defined by the concept of distances
4.2 Geometric transformations
    4.2.1 Congruency (on the plane, in the space)
    4.2.2 Similarity transformation
    4.2.3 Other transformation (orthogonal projection)
4.3 Geometrical shapes (Plane shapes – solid figures)
    4.3.1 Plane shapes (triangles, quadrilaterals, polygons, circle)
    4.3.2 Solid figures
4.4 Vectors (two dimensional, three dimensional)
4.5 Trigonometry
4.6 Coordinate geometry
    4.6.1 Points, vectors
    4.6.2 Line
    4.6.3 Circle
    4.6.4 Parabola
4.7 Circumference, Area,
4.8 Surface, volume

5. Probability, statistics
    5.1 Descriptive statistics
        5.1.1 Data collection, systematisation of the data, data representation, visualization, diagrams
        5.1.2 The characteristics of the mass of data, measures of central tendency and dispersion statistical indicators
    5.2 Probability and the elements of inductive statistics (point-estimation)
        5.2.1 Characteristics of stochastic phenomena, probability
        5.2.2 Estimate of the relative frequency of a sample by the parameters of a population (Lukács, 2001a)

4.2. The structure of the joint matriculation-entrance exam

The high level Maths exam consists of a 240-minute written test and a 20-minute oral exam. Students can use a calculator and a Collection of Formulas and Functions both for the written and the oral parts. The parameters of these will have to be redefined every year.

Written exam

Content structure

The test thematically covers the 5 main topic groups of the requirement system.
When designing the exam paper, the following proportions will have to serve as guidelines:

Methods of Mathematical thinking, sets, logics, combinatorics, graphs	25%
Arithmetic, algebra, number theory	20%
Relations, functions, the elements of calculus	20%
Geometry, coordinate geometry, trigonometry	20%
Probability, statistics	15%

These proportions, of course, are only guidelines, as a considerable number of test questions could belong to more than one thematic group, being built on a complex circle of knowledge; also due to the arbitrary parts of the exam these proportions could vary with each and every student depending on their choice of test questions. The first thematic group includes the parts of all those problems that require a translation of the text into the language of Mathematics or the creation of mathematical models.

40% of the test problems are situation-based, problems in connection with the everyday life, which will require the application of simple mathematical modelling.

The attributes of the test paper

The exam paper consists of 3 different parts that need to be attended to continually. Students have a maximum of 240 minutes to complete it, which time can be freely used. The maximum number of points that can be achieved is *115.*

Part I consists of 4 questions. These can be regarded as easier problems based on the requirements of this high level exam; in general, they can be solved with the knowledge of the lower level requirements. (There is no free choice of questions in this section.) The questions might contain more than one sub-question. The *maximum score is 50 points.*

Part II consists of 4 questions, all worth 15 points. The candidate has to solve three of the four and only these three can be taken into account. The questions are, in general, based on the knowledge of one or two thematic groups. The *maximum score* for part II is *45 points.*

Part III contains one complex question that combines several sub-questions, ones that are based on several thematic groups and which require practical problem solving and mathematical modelling. The correct solution to this problem is worth *20 points.*

Evaluation

The guidelines for the evaluation contains a detailed solution to the test questions and its possible versions as well as the different sub-points that can be given in the various steps of the solution process.

Oral exam

Content structure

The oral exam is an external exam. The proportion of contents in the central list of series of questions reflects the proportion in the description of the written exam.

Attributes

Each series of questions is chosen from a specific thematic group. Every series of questions requires a student:

- to give a definition,
- to verify a theorem,
- to solve a problem,
- and to give an example for the application of the given thematic group within or outside Mathematics.

As the difficulty of the various theorems can vary, the equal level of the oral exam can be granted by a balance being given to the complexity and difficulty of the chosen questions.

Evaluation

The maximum score in the oral exam is 35 points.

The elements of evaluation:

1. Theoretical question and the problem-solving	25 points
2. The example demonstrating the application	5 points

3. The ability to work independently, to demonstrate logical problem solving, use of the terminology and the ability of Mathematical communication 5 points

5. Some results gained from the development of the tests and the evaluation of the documents

The earlier-mentioned trail exams had about 250-300 participants on every occasion on both levels. The students represented the two school types in equal proportions. From these trail exams, we gained information partly regarding the difficulty of the test questions and partly about how well the different tasks are capable of measuring the mathematical knowledge, skills and abilities as laid out in the requirements. During this experiment both teachers and students were asked to give their opinions about the types of tasks and the whole structure of the exam; and teachers were also questioned about how useful they found the evaluation guidelines given to them.

Students liked the new types of practical questions; however, some tasks, especially the ones that required a higher level of theoretical knowledge, were not carried out to an acceptable standard. (Lukács-Vancsó, 2001) The majority of teachers did not like the idea of free choice among the questions, fearing that their students would be put under even more stress when having to make such a decision in an exam situation. Yet the actual results show a different picture, in that students welcomed this new opportunity and used it well. Nevertheless, the experiment proved that this decision situation requires sufficient time to be allocated to it - and this will have to be taken into account when setting up the exam model.

Teachers had their reservations about the new type of practical questions, which could be summarized as follows:

- The new contents that appeared in the exam requirements and the actual questions are at the moment quite frustrating for some teachers. This is especially true with the theory of probability and statistics, which they will have to teach without actually studying or will have to do differently from the way *they* were taught. (This problem, of course can be solved by the further training of teachers.)
- Hungarian Maths teaching in general was always more theory-centred, and many teachers would not like to change that for reasons of conviction.
- Some teachers experienced that some of the new types of questions are more favourable to students who are less hardworking but have the necessary intelligence and creativity.
- In this modelling, several of the situational questions require the sort of communicational (comprehension) skills that have so far not been emphasized in Hungarian Maths teaching; thus, some teachers would find it a little problematic to test these types of question in an exam.
- Some teachers feel that Mathematics will suffer if this new exam drops the reproductive verification of mathematical theorems.

On the whole, however, the majority of teachers understand and accept the need for a change. This is shown in the following data. With regard to the higher level of exam, teachers gave the following responses:

- 92% of them agree with the set up of the detailed requirements
- for 90% of them the requirements of the framework curriculum are in line with exam requirements
- 74% of them agree with the introduction of free choice in the written exam

- 74% of them think that this type of written exam is suitable for the reliable measuring of a student's performance
- 67% of them think that this type of oral exam is suitable for the reliable measuring of a student's performance (Lukács, 2001b)

School will receive the new exam document at the beginning of the 2002/2003 school year in order to enable teachers to prepare their students for the new exam first taken in 2005.

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Annex 1: An example of the test papers

Part I

1. In a 70-membered sports delegation the average age of men is 37, of the women it is 23, and of the whole group it is 28. How many men and women were in the group?

2. The radius of the Earth is 6380 km, and the radius of the Sun is about 110 times this.

a) How many square metres is the surface of the Earth?
b) How many cubic metres is the volume of the Sun?
Give the results in a normal form.
c) The shadow of a ball standing on the ground reaches as far as 42.5 cm from its touching point. At the same time, the shadow of a 1 meter-tall child standing next to it is 2 metres.
How large is the diameter of the ball?

3. József smokes one packet of cigarettes a day. The price of a box of cigarettes went up from 210 Ft to 250 Ft.

a) How many percent is the price rise?
b) If József's net monthly income is 80 000 Fts, how many percent of his monthly income did he spend on cigarettes after the price rise? (Take into account 30-day months.)
c) To protect his health and pressed by the recent price rise, József has decided to stop smoking. He will put the price of the 250 Ft. cigarettes in a bank at the beginning of every month. The bank will reinvest the interest, i.e. on the last day of every month they will add it to the actual amount on his account and this increased amount will continue to produce interest. The monthly interest rate is 2%. How big will the amount be that József can receive at the end of the 12^{th} month?

4.

a) Every year several thousand people apply for pilot training. They have to undergo 3 tests:
A — a vision test,
B — an allergy test, and a
C — a height-endurance test.

One year there were 2000 applicants.
After the tests we have the following data:
570 of them failed the vision test,
798 people had some kind of allergy-related problems,
65 could not endure heights,
120 people had both vision and allergy problems,
32 could not endure heights and had a vision problem,
42 had an allergy and could not endure heights,
25 of them failed all three tests.

- How many applicants passed all three tests?
- How many applicants had only allergy-related problems?
- How many applicants had exactly two problems?

b) You can get to and from any of five different airports. The airline runs 2 flights from the first, the second and the third airport, one flight from the fourth, and three from the fifth. Draw a network based on the above information.

Part II

From the next four question (5-8) you will have to choose *three* to solve.

5. What is more likely? If a regular dice thrown up six times will produce at least one six, or if a regular coin thrown up 10 times will produce at least 5 heads?

6. Before light bulbs were invented the windows of factories were designed to enable as much light as possible to get into them. Some factories used the so-called "Noorman window". These consisted of a rectangle and a semicircle, the semicircle joined the rectangle on one of its sides and its diameter was as long as this side of the rectangle.
If the circumference of the window is constant, how wide and how long should the rectangle be to let the largest amount possible of light through it?

7. 65% of Hungarian health officers are women. On one training course there were 100 health officers present. Give the interval that will include the number of women health visitors present with 90% certainty.

8. Solve the following equation on the set of real numbers:

$$\left|5-x^2\right|-y=0$$

$$x^2-6\cdot\sqrt{x^2+y^2}+y^2=-5$$

Part III

9. A radio tower is sending signals to an engine while it is moving along a line. Placed in a Cartesian coordinate system the radio tower is on the R(1;0) point. The equation of the t line is: $2x + y = 30$, where all data is given in km.

   a) Represent the situation assuming a coordinate system where the units on both axes are the same.

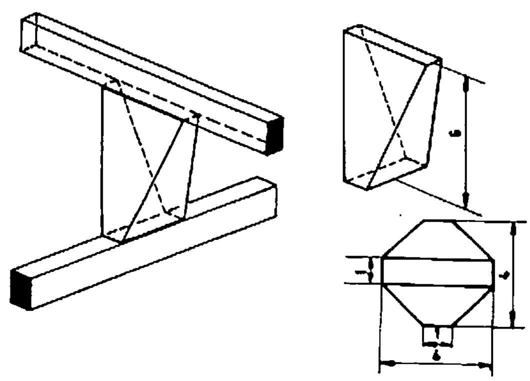
   b) The engine gets its strongest signal in point C, so C is the point of line t that is closest to R. Define the coordinates of point C.

   c) When the engine is more than 28 km away from the tower, it does not receive the signals anymore. Define the two end points of the section where the radio signals can still be received.

   d) A further two equally strong radio towers will be set up in such a way that their signals can be received in the greatest possible area. Where should we place these two towers – taking into account the above parameters – so that one of the two towers could be received on the longest continuous line along t? Give their coordinates.

   e) When building the towers the following iron units are used as supporting elements, whose height is 6 dm. Their other parameters can be read from their pictures in dm. How much will corrosion protection cost considering that the application of 1 square metre costs 700 Ft?

   f) How big will the weight of one unit be if the density of iron is 7800kg/m^3?



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Annex 2: An example of the evaluation guidelines going with one question.

3rd question

a)		
$250 : 210 = 1{,}19$		
The price rise is 19%.	1 point	
b)		
After the price rise		
$30 \cdot 250 = 7500$		
$7500 : 80\,000 = 0{,}09375$	1 point	
At present, he is spending 9.4% of his wages on them.	1 point	
c)		
1 month's saving: $A = 30 \cdot 250 = 7500$.	2 points	*If the student only counts with 1 day, he won't get these 2 points.*
Monthly interest rate: 2%, therefore $q = 1{,}02$		
At the end of the 1st month: Aq At the endof the 2nd month: Aq^2+Aq . .		*If the yearly amount is individually calculated correctly, he will receive these 4 points. If he makes a mistake while doing it, he gets 1 point for every 3 good amounts.*
At the end of the 12th month: $Aq^{12}+Aq^{11}+Aq^{10}+\ldots+Aq^2+Aq =$	2 points	
$= Aq(1+q+q^2+\ldots+q^{11}) = Aq \cdot \dfrac{q^{12}-1}{q-1} =$	2 points	*For the formula of the geometrical series.*
$= 7500 \cdot 1{,}02 \cdot \dfrac{1{,}02^{12}-1}{1{,}02-1} = 102\,602$ Ft	1 point	*If he only writes down the last formula, he will still receive 4 points.*
At the end of the 12th month he could receive 102 602 Ft.		*If he takes the formula for the annuity from the Collection of Formulas, he will only get the amount he can receive at the beginning of the 12th month - and for this he can get 5 points.*
Total:	**10 points**	

MATHEMATICS TEACHERS INITIAL TRAINING AND COLLABORATIVE WORK

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ABSTRACT

For several years, Communication Technology has been intensively used by the trainers in mathematics of the training college I.U.F.M. of the North of France. It concerns use of e-mail for maths dissertation papers, of WEB resources for the history of mathematics, an on-line bibliographic data base, a data- base of dissertation papers and web sites created by groups of trainers: C.R.E.A.M, Mathadoc, LILIMATH, FUNCTIONS, GEOWEB, several of them have received national prizes. These groups use a groupware for their work.

In 2001-2002, a new program for the training in new technologies began linked with dissertation papers. The trainers are themselves engaged in the creation of resources and animation of networks, and they work in a collaborative way for preparing and managing the training. They have set up, for trainees, working modalities that are mostly induced by the collaborative way trainers have been working.

After briefly presenting the French educational and teacher training system, this communication will give essential points of this program and will analyze the first year of training.

Keywords: initial training , cooperative and collaborative work, distributed cognition, network.

The first part of this paper gives a presentation of the teacher training colleges in France, called I.U.F.M. (Instituts Universitaires de Formation des Maîtres). It is based on texts written by Pierre Louis, Director of the I.U.F.M. of the Nord-Pas-de-Calais in France and by Eliane Cousquer, head of the LAMIA. In the second part, we present and analyse a teacher training organization in witch we have experimented collaborative work.

Education in France and teacher training at an I.U.F.M.

Organization of teaching in France

One can consider that in France, teaching is organized in three levels:

· primary teaching (1st level) concerns children from 2 to 11, compulsory education beginning at the age of 6. The teachers, called "school professors", have comprehensive skills;

· secondary teaching (2nd level) concerns children and teenagers from the age of 11 to 18. The teachers, "professors of secondary teaching ", have the responsibility for one, sometimes two disciplinary domains (e.g. history and geography, English and French literature);

· higher education is for young people having successfully finished their secondary curriculum. Academic establishments have also a research mission. Most teachers are also researchers and are specialists in one discipline.

What is the I.U.F.M. ?

I.U.F.M. is an academic institute for teacher training. Its first role is that of the vocational training of new teachers, along with training in the discipline the future teacher will teach. Vocational training begins during the first year with an awareness of the teacher's profession and is strongly strengthened in the second year. Most professional training is given during the second year. The main aim of the training is to establish links between necessary information for teaching (didactics of a discipline, child and teenager psychology, sociology and philosophy of education, and so on), practical competence to be acquired (such as management of a class, communication, teamwork), and the real experience of responsibility for a class. The competitive examination, following the training, takes into account three domains:

· general professional and disciplinary training,

· professional dissertation paper,

· practice in responsibility of a class.

The professional dissertation paper is a trainee's personal work on his real experience of teaching. He has to theorize his experience in a written report and an oral presentation in front of a board of examiners.

A strengthened mission : continuing education for secondary school teachers

Since their creation, the I.U.F.M. have been recognized for their participation in continuing education of teachers. By giving this mission to the institutes, the political leaders want:

· to establish a continuity between trainee-teacher education and continuing education, each training becoming a time of global improvement of knowledge along the career,

· to strengthen the academic character of the training in order to maintain teachers at a high level of knowledge,

· to train together all the teachers, of primary and secondary schools, of general or technical or professional schools; the point being to strengthen collaboration between the different parts of the educational system.

Research and pedagogical innovation in the I.U.F.M.

From their birth, the institutes inherited pedagogical research, of an applied type, based on experiments in classes. This activity is particularly emphasized in the general frame of training. Additionally, the I.U.F.M. want to develop research having a real academic status and have created teams with other institutes or universities, led by confirmed researchers.

The teacher trainers of the I.U.F.M.

One of the strong peculiarities of the institutes consists in the variety of their trainers. Primary school teachers, secondary schools teachers, teachers of higher education, heads of establishments, inspectors … trainers come from all degrees of teaching and all parts of the educational system. In order to maintain the quality and the level of their trainers and to recruit new ones, the institutes set up a training for trainers, often linked with research. It takes different forms: lectures, thematic work groups, production teams, workshops …

Creation of the LAMIA

A laboratory for multimedia creation within the I.U.F.M.: what for ?

New tools (hypertexts, hypermedia, animations, virtual reality ...) were developed for other aims than training, in particular for the accessibility on the Internet of large quantities of data. They present, however, a great interest in training, since they can allow the learner to be more active and to take more initiatives. Re ection on the uses of these techniques for training is still at an early stage, but one should bear in mind that some basic ideas can be drawn from research on practices:

· Systems based on knowledge require considerable elaboration. To be really useful, they must be integrated into the training and not placed side by side. We must focus on an active use of hypermedia by trainees to solve problems or to carry out personal work given by their trainers.

· If creation is sometimes due to a minority of teachers, full integration requires interest and implication from a lot of them. This involvement can not be obtained at once. Nothing will be done without the teachers' full cooperation.

The objectives

· To give creative teachers an institutional frame which facilitates and enriches their creative work;

· to lead activities of research and development in the uses of multimedia tools and of information and communication technologies;

· to analyse the possible effects of new technologies on teaching practices, on learning strategies and processes;

· to contribute to the creation of new tools, ready to be used when a need shows, as well as the analysis of the uses and of the practices for implementing them;

· to animate a workshop which has the specific task of validating the capitalization of reflection and production. The workshop has been dedicated to the contributions of cognitive psychology. These last years, it focused on collaborative work.

Research activity at LAMIA

This activity is centered on production because we want to experiment different training organization and environments in real situations. It is the case with the training in collaborative work we wanted to set up. The laboratory financially encourages teams of creation. It organizes for them a network modality of working. LAMIA plays the role of catalyst between the various teams and supports their collaborative work.

LAMIA productions in mathematics

· C.R.E.A.M.: center of pedagogical resources for trainees in mathematics,

· A6-3: *The Electronic Schoolbag* of the secondary school teacher is a downloadable software programme ; it provides an important data base on the curriculum concerning the last ten years in France, with a set of lessons and exercises for four class levels (from the first form to the fourth form). The teacher can develop it and modify it in order to create his own database,

· LILIMATH: discovery workshops for the teacher to use in classes; LiliMath received in 1998 the first prize in a national competition (Cervod) for computer supported learning tools,

· FUNCTIONS: learning of functions in secondary technological schools. This software programme allows an individual follow-up of pupil work and received the fourth prize in the same competition,

· GEOWEB: a web site presenting secondary schools pupils creations. They make up and write up geometry problems and organize access to the concepts needed to solve them. It received the 2001 "Prize for educational innovation".

Another network is financially encouraged by the LAMIA: GÉOMÉTRIX: this software helps to structure writing of demonstrations by secondary school pupils; it is written in prolog programming language and uses artificial intelligence techniques; it received the second prize in the Cervod competition.

Continuation

Networks created by LAMIA for development teams have subsequently become involved in teacher training. A project was launched in 2001-2002 by the head of the LAMIA, also in charge of professional dissertation papers in mathematics. It consists of the creation of the *Math Ring* (workgroup of teacher-creators in mathematics), and in the design of training in new technologies for second year mathematics trainees.

Collaborative work between trainers

Although the *Math Ring* is under the supervision of the head of the LAMIA who organizes the agenda, the scientific program and invites the speakers, the following training principle are applied:

· emergent collaboration: the trainers are registered in the *Math Ring* group, where they make up the content of the training they want to provide, as well as the content of the training from which they want to benefit,

· no prior strong structuration of activities nor role assignment, learning by training: each participant is successively trainee and trainer according to his competences. Little by little responsibilities show up, for example, the final making of a CD-ROM of resources,

· full size experiment: it involves all the trainers (between 40 and 50) participating in the training in new technologies for the discipline.

Tools and technical aspects

Members of LAMIA don't only have training skills, they also have technical abilities needed to implement the collaborative training organization. The environment gradually evolved from email to a complex platform with the following functionalities:

· Email: the exchange of emails allows planning of meetings and exchange of documents.

· LAMIA Web Server: this server is used for submitting large documents. It also allows each document to be put in a proper learning context with links to appropriate information.

· The *Virtual Campus*: it was our main tool during 2000-2001. However it quickly showed its limits. It didn't support more than 50 registrations and his double checking system handicapped communication.

· A Yahoo group: in 2001-2002, our entire organization migrated to *Yahoo groups*. *Yahoo groups* is a free and rather comprehensive collaborative environment. Each group has a zone for file deposits, a diary and a mailing list. The impossibility to create sub groups was finally felt as something positive, because it prevented the setting up of a hierarchy between groups.

· The GANESHA platform (PHP MYSQL): the evolution of computing languages, and free disposal of resources written in these new languages (e.g. PHP and MYSQL) allows the current development of our own collaborative working system. Members of LAMIA are currently working to adapt a GANESHA platform

The *Math Ring* functioning

Trainers meet once a month. At each meeting, one of them presents his work of creation. He clarifies the educational problems he wants to solve and how his method can favor the teaching of mathematics. The other trainers test their colleague's work as pupils and can express their critics. Researchers are also invited to these meetings and their presentation is often followed by a contradictory debate about the motivation of their work and their actual and original contribution to teaching of mathematics.

Additionally, all the trainers are registered in the *Math Ring* group. Summaries of presentations and web links and documents provided during meetings are posted in the Yahoo group in order to continue discussions through the mailing list.

The aim of the group is not only trainer training but also the design of mathematics training in new technologies. As we pointed above, during this process no specific role was assigned among trainers of the group, nor was any prior instruction given. The only structuring element was the common and urgent objective to be achieved. As a matter of fact, some individuals showed leading and organizing skills and took responsibilities as well.

As planned, the group production was the content of the training. But the group, during the process, decided to produce a CD-Rom of various resources to be given to trainees; the showing up of this need was induced by the collaborative organization of trainers work. Another consequence of this collaborative organization was that trainers resolved to have trainees work in a collaborative way.

Training in new technologies for prospective teachers

The staff of the I.U.F.M. decided to experiment a new disciplinary program to integrate new technologies in mathematical training. The head of the LAMIA chose six young and very creative trainers. Their task was to train for twelve hours without a fixed program. They had to show the trainee their own pedagogical use of technology and how to guide a network. Collaboration between the trainers and analysis of practices will allow a more specific training program to be set up in future years.

For who and by who is this training made?

This training is organized for 90 future teachers of mathematics who are going to teach in September 2002. All the trainers are teachers who actually use new technologies in their professional practice. Some of them also create computer supported learning tools and use them in their classes; thus, they have investigated the ergonomic and pedagogical aspect of the tools they have created. It can often be said that the products they developed are a computer supported solution of an educational obstacle they met with their class.

In what way is this training different, in content, from previous training ?

Previous training in new technologies was essentially cross-disciplinary, consisting mainly of office automation and communication by the internet. The content of the above described training is, on the contrary, essentially disciplinary. It also favors free software packages or resources and singles out software created by teachers to answer an educational problem, rather then non-specialized commercial software package.

Principle of this training

We intended to apply the same principles as those that prevailed in the organization of the trainers group.

· emergent collaboration: trainees are included in distant computer supported working groups. After several weeks, messages circulate in the network and documents are put at the disposal of the group. These interactions actually constitute the content of the training we want to provide;

· no prior roles assigned to trainees when included into the group;

· full size experiments: all the 90 mathematics trainees of the North of France (Académie du Nord Pas de Calais) were involved in this year's experiment.

A main obstacle to overcome: passivity

As former pupils and students, trainees have been used to be rather passive learners. Thus, encounting passivity among trainees is an inherent difficulty for our collaborative way of working. Our concern is that trainee's participation in exchanging is the basis of the training, consequently passive trainees just don't benefit from the training at all. Besides, in distant training, trainees can easily drop out, by not opening emails and not attending meetings any more.

Passivity is not that problematic when the entire course is given with the trainees physically present. The trainee is then taken up in the dynamics of the group in spite of himself. Even if he does not participate, he attends the group session and can reconnect with its work at any time. We identified two main factors to overcome the passivity obstacle in distant training:

· the holding of a minimum number of regular meetings: the whole training can not be made using distant collaborative software,

· a real pedagogical project: although collaborative work is the most important characteristic of the training according to trainers, it is introduced to trainees as the means to prepare and to improve the efficiency of the next meeting's work. It's also the mode set for working on the dissertation paper they will have to return later in the year. As we said previously, the trainers group also had a project, which was to set up the contents of the training.

Conclusion

The collaborative work is set up at several levels in the teacher training institute (I.U.F.M.): between members of the LAMIA, between the teams of production led by the LAMIA, between members of the trainers network, between the various teaching groups of trainees and their members. Whatever the level where it is applied, the collaborative work arises from the same philosophy, that is emergent collaboration (no frame of activity fixed in advance), no prior role assignment and full size experiments. As far as our experiment is concerned, interests of such an organization are:

· favoring learning among peers;

· allowing the realization of a common project;

· stimulating creativity in training design;

· optimizing team competency: using complementary skills, status and functions, enhancing each individual and team competence. The competences concerned are various: designing computer supported learning tools in mathematics, pedagogical uses of computer supported learning tools in mathematics, designing and managing teacher training as well as working in team.

We would like to point out some important characteristics of this organization:

· permeability of sub-networks: one individual can be part of several networks,

· mobility and context-dependency of the network design; the network cannot be drawn, for its representation would depend, among other things, on the person under focuse, the moment, the activity this person is involved in at the concerned time etc.,

· although the organization is set up under an institutional frame, it seems to remain even if the concerned people move out of the institution. For example, it is already the case for former members of production teams. We hope that it will also be the case for trainees next year, when they will be teachers.

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-C.R.E.A.M	http://www.lille.iufm.fr/labo/cream/entree.html
-A6-3	http://www.lille.iufm.fr/labo/6A3/index.html
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MATHEMATICAL TECHNOLOGY TRANSFER - INDUSTRIAL APPLICATIONS AND EDUCATIONAL PROGRAMMES IN MATHEMATICS

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ABSTRACT

Mathematical technology is a term referring to the interdisciplinary area combining applied mathematics, engineering and computer science. Computational technology has made sophisticated mathematical methods viable for practical applications. There is a window of opportunity for mutually beneficial two-way knowledge transfer between academia and industry. This also means a challenge for the university education. The modern and dynamic view of mathematics should be reflected in educational practices. New kinds of expertise are called for.

The area of applications for mathematical technology is wide and diverse. Models are used to

-replace or enhance experiments or laboratory trials.
-create virtual and/or visualized images of objects and systems
-forecast system behaviour and analyse what-if situations
-optimise certain values of design parameters
-analyse risk factors and failure mechanisms
-create imaginary materials and artificial conditions
-gain understanding of intricate mechanisms and phenomena
-perform intelligent analyses on measurement data
-manage and control large information systems, networks, data-bases.

The education should bring the flavour of this fascinating art to the classroom. We should shape the image of an emerging profession, industrial mathematician, computational engineer or symbonumeric analyst? The education should convey the vision about mathematics at work, to display the diversity of application areas, to demonstrate the practical benefits. A number of groups worldwide are working towards fresh solutions in applied mathematics education. The goal is to combine mathematical knowledge with modelling skills, project work and a touch with real world applications. Possible tools for improvement include

-revision of curricula
-educational software environments
-problem seminars and project work
-teachers (re)training

We point to the challenge of mathematics education to find a way to communicate to the students the end-user perspective of mathematical knowledge. In this paper we describe the growing sectors of real life applications, industrial processes and R&D-questions where mathematical methods have a significant role. These examples are meant to emphasize the nature of mathematics as a versatile environment of problem solving. We discuss the educational challenge, curriculum development, the contents and viewpoints that could be used in undergraduate education and teachers training. The relevant didactic point is the search for and presentation of illustrative and interesting case examples on a right level of abstraction and technical sophistication.

1. Computational Technology

The development of technology has modified in many ways the expectations facing the mathematics education and practices of applied research. Today's industry is typically high tech production. Sophisticated methods are involved at all levels. Computationally intensive methods are also used in ordinary production chains, from timber industry and brick factories to bakeries and laundry machines. The increased supply of computing power has made it possible to implement and apply computational methods. Mathematics is emerging as a vital component of R&D and an essential development factor. The increasing demand and sphere of applications and the evolving computational possibilities have created what may be called *mathematical technology* or *industrial mathematics*. Terms like *computational modelling* or *mathematical simulation* are also used to describe this active contact zone between technology, computing and mathematics.

Mathematics as a resource for development Modeling means an imitation of a real system or process. The model is assumed to represent the structure and the laws governing the time evolution of the system or phenomenon that it was set out to mimic. Once we are able to produce a satisfactory model, we have a powerful tool to study the behavior and hence to understand the nature of the system. The models can be used to

-gain understanding of intricate mechanisms by testing assumptions about the systems nature.

-carry out structural analysis tasks

-replace or enhance experiments or laboratory trials.

-evaluate the systems performance capabilities

-forecast system behaviour and analyse what-if situations, to evaluate the effects of modifications, consequences of changes to systems parameters.

-perform sensitivity analyses and study the system behaviour at exceptional circumstances.

-analyse risk factors and failure mechanisms

-create virtual and/or visualized images of objects and systems in design processes

-create imaginary materials and artificial conditions prior to the possible synthesis or construction.

-optimise certain values of design parameters or the whole shape of a system component.

-perform intelligent analyses on the measurement data which is produced by the process monitoring, experimenting etc.

-manage and control large information systems, networks, data-bases

The model can describe situations that are impossible to be realized as physical models or are too extreme for making observations (one can't repeat the Big Bang or observe at close distance the explosion of a mine, but one can numerically simulate both).

2. Increasing Sphere of Applications

Economics and management. The daily functioning of our modern society is based on numerous large-scale systems. Examples are transportation, communication, energy distribution and community service systems. The planning, monitoring and management of these systems offers a lot of opportunities for mathematical approach. System models, methods of operations analysis, simulation etc. can be used to gain understanding on the behaviour of these mechanisms.

Corporate management uses methods in which mathematical knowledge is embedded in different levels. Econometric models are used especially at the banking sector to describe the macro level changes and mechanisms in the national economy. Risk analysis, game theory, decision analysis etc are used to back up strategic decisions, to design a balanced financial strategy, to optimise a stock portfolio. The mathematics of the financial derivatives (options, securities) has been a sector of rapid mathematical development in recent years.

Traffic and transportation. Roads, railway networks and air traffic contain many challenges for modelling. In railway industry one is interested on the mechanical models about the rail-wheel contact (Fig 8), explaining the phenomena of wear, slippage, braking functions etc. The train itself is a dynamical system with a lot of vibrations and other phenomena. Analysis of traffic flow, scheduling, congestion effects, planning of timetables, derivation of operational characteristics etc. (Fig 6) need sophisticated models. In air traffic guidance systems and the flight control of an aircraft represent sophisticated mathematical control theory.

Maritime industry. The maritime and offshore industries use advanced mathematical methods in the design of ships and mechanical analysis of offshore structures. An example is the dynamical behaviour of floating structures under wave force effects and wind conditions. Individual technical tasks like the optimal design of an anchor cable or the laying of communication cables at sea-bottom lead to interesting mathematical problems. One particular challenge is the modelling of the sea and the wave conditions itself for the sake of simulation purposes.

Space technology. Modelling of the mechanical properties of the manmade structures in the spatial orbit lead to advanced mathematical questions. An example could be the stability study of a large extremely light antenna structures in the weak gravity field. Each individual space mission represents a massive task for dynamical modelling and optimal control.

Product design and geometry. The modern toolbox of analytic and numerical method has made mathematics a real power tool for design engineers, production engineers, architects etc. One can bypass costly trial and error prototyping phases by resorting to symbolic analysis and numerical models. Mathematics is a natural tool to handle geometrical shapes (Fig 9), like the surfaces of car bodies and in the visualization techniques in CAD and virtual prototyping. In fact entertainment industry is one of the great clients for mathematical software nowadays. Visualization and animation is the basis of computer games and the vivid special effects in movies etc. These tricks are made possible by mathematical models.

Performance analysis, manufacturing systems, reliability. The major source of economic added value in using mathematical methods comes from the possibility of simulate devices, mechanisms, systems including complex large scale systems prior to their physical existence. A whole new system - like an elevator system in a high rise building, a microelectronic circuit containing millions of elements, or a high tech manufacturing system – can be designed and tested for its performance and reliability.

Chemical reactions and processes. Chemical processes are being modelled on various scales. In the study of molecular level phenomena mathematical models are used to describe the spatial structures and dynamical properties of individual molecules, to understand the chemical bonding mechanisms etc. The chemical reactions are modelled using probabilistic and combinatorial methods, the reaction kinetics take the form of differential equations etc. An example is the biochemical response in the design of a laboratory test (Fig 1). Chemical factories use large models to monitor the full-scale production process (Fig 3). The increasingly important area of environmental monitoring benefits from models that describe and explain biochemical processes.

Materials behaviour. Materials science is one of the really active fields where the mathematics based methods have proved their necessity and power. The aim is to understand the microlevel molecular and subatomic effects, subtle engineering of special compounds etc. The behaviour of non-typical materials (Fig 7) or new materials like semiconductors, polymer crystals, composite materials, piezoelectric materials, optically active compounds, optical fibres (Fig 5) etc. create a multitude of research questions, some of which can be approached with mathematical models. The models can further be used to design and control the manufacturing processes.

Metal industry The whole production chain of metals starting from mining industry, enrichment processes, furnace, casting, hot rolling, sheet forming, profiling etc. contains a lot of challenge to mathematical models. Quite modern and sophisticated methods are employed, like optimal control theory, free boundary problems, optimisation methods and advanced probabilistic methods. There are delicate questions like modelling of the material deformation during manufacturing processes, the phase change phenomena in the heat treatment of steel (Fig 11) and the study on the fatigue mechanisms (Fig 10).

Food and brewing industry Mathematics has to do with butter packages, lollipop ice-cream, beer cans and freezing of meat balls (Fig 2). The food and brewing industry means biochemical processes, mechanical handling of special sorts of fluids and raw materials. These less typical constituents lead into non-trivial mathematical questions. The control of microbial processes is quite crucial and adds to the complexity. Some of the questions deal with simple aesthetics, like the problem of proper filling of lollipop moulds in an automatic production chain.

Flow phenomena The ability to model sophisticated phenomena, including non-linear effects, the possibility to solve the equations with advanced numerical methods, combined with the latest visualization tools have created a luxury environment for mathematical engineering. The computational simulation can be used to support the design of systems from tooth paste tubes, regional heating networks and aircraft fuselage design to ink-bubble printers and making the fascinating flow phenomena visually observable. One of the important fields of application is diffusion phenomena, like the spreading of pollutants in air, soil, rivers etc.

Semiconductor industry The tiny devices are so small that it takes a microscope to see the details. The modelling of the single transistor has generated a lot of research. The industry wants accurate device models describing the performance characteristics of a chip prior to its production. To find the optimal architecture for an integrated circuit demands heavy calculations. The procedure of etching or electron beam lithography that is used during the manufacturing of the integrated circuit leads to interesting problems for mathematical modelling.

Systems design and control The design engineers and systems engineers have always been active users of mathematics in their profession. The possibility to set up realistic large-scale system models (Fig 3) and the development of modern control theory have made the computational platform a powerful tool with new dimensions.

Measurement technology, signals and image analysis The computer and the advanced technologies for measurement, monitoring devices, camera, microphones etc. produce a flood of digital information. The processing, transfer and analysis of multivariate digital process data (Fig 4) has created a need for a considerable amount of mathematical theory and new techniques. The area of signal processing is one of the hot areas for applied mathematics. Examples of advanced measurement technologies are mathematical imaging applications (Fig 12).

Experiments and data analysis The ample output of process data means a demand of mathematics. Intelligent methods are needed for the utilization of experimental data. The process control and monitoring systems, the sampling procedures etc have to be designed carefully. The quality inspection at different parts of the production chain and the testing procedures for the finished products all involve the questions for intelligent techniques for the handling of data. An example is the area of accelerated testing of mechanical components (Fig 10). There has been a speedy development of methods for data-analysis and the novel techniques for processing data.

3. Types of Models and Mathematical Projects

Mathematical models represent many different forms and types. *Continuous* models deal with quantities (like time, distance, force, electric potential) that vary smoothly over space and time. The models typically take the form of a set of algebraic or differential equations, integral equations, PDEs. Discrete models deal with quantities that vary in a stepwise manner, they take values from a discrete set. Examples of discrete models are recurrence relations, difference equations, Markov chain, digital coding and signals, autoregressive – moving average time series models (ARMA), graphs, integer LP-models.

A model which is based on the understanding of the internal mechanisms (physics, chemistry, biology, economics etc) is called a *mechanistic* models. When a mechanical model and analytic solution is not available we may resort to *simulation* model. *Empirical* models and *model fitting* are terms that describe the efforts to deduce the model equations from measurement data.

Deterministic models describe the phenomenon by predicting the actual values of the dependent variables. Known input values lead to unique output values. *Stochastic* models incorporate different random effects into the model structure and they are aimed to describe random behaviour and predict the probability distribution of the output values.

Models may represent different scales, conceptual levels and the model may contain inner structures, *partial models* or *submodels*. *Macroscopic* model tries to catch the big picture, *microscopic model* zooms at more minutiae details. Take for example the dynamics of weather phenomena where different version of models are needed to describe the formation of rain drops or local air pressure variations, to explain the creation of tornadoes or to understand the greenhouse effect.

Models can also be categorized due to the purpose for which they were created. Models may be designed to understand the mechanism of change, a *transition* like growth, decay, saturation or switching from one state to another. Other models are aimed at describing permanence, *equilibrium* and balanced condition. An example of this sort would be a set of equations describing the operating status of a chemical reactor.

Often the models are used for the purposes for control. *Optimisation models* are geared to a specific purpose, to help to find the best operating conditions, to find an optimal design for a product, etc. *Control models* are the devices for control engineering, process control and different mechanisms of guidance. Examples of this sort are the models for steering the operation of power stations and the guidance systems for air traffic.

4. Educational challenge

As the previous material shows, the computer age has generated a need and a window of opportunity for a new kind of expertise. This field could be called industrial mathematics, mathematical technology, computational engineering. This presents a challenge to the educational programmes and curriculum development. Some universities already offer specialized MS-programs oriented towards the professional use of mathematics. There are excellent programs that equip the students with the skills that are needed in the mathematical projects in the R&D-sections in industry. In general there is still a lot of room for improvement. Some mathematics departments have stayed too long in the pasture of isolated abstract mathematics and failed to face the challenge coming from the changing world.

A good educational package would contain a selection of mathematics, computing skills and basic knowledge of physics, engineering or other professional sector. The job title in industry is very seldom that of a mathematician. It can be a researcher, a research engineer, systems specialist, development manager. Industrial mathematics is teamwork. Success stories are born when a group of specialists can join their expertise and visions together in a synergic manner. The team-work makes communications skills a necessary matter. It would be very important to train oneself to work in a project team, where the interpersonal communication is continuously present. To become a good applied mathematician one should be curious about other areas as well, to be interested and learn basic facts from a few neighbouring areas outside mathematics.

To tackle the fascinating tasks and challenges, development questions in modern industry, the student need a solid and sufficiently broad theoretical education and operational skills in the methods of applied mathematics. However, the most important single skill is the experience in modelling projects. The lectures, books and laboratory exercises are necessary, but the actual maturing into an expert can only be achieved by "treating real patients".

From the point of view of successful transfer of mathematical knowledge to client disciplines a crucial and current educational challenge is the theme of mathematical modelling. Many departments have introduced modelling courses in the curriculum in recent years. The active development is reflected in a boom of literature. A variety of books of different flavour are available on the subject. A course in modelling may contain study of case examples, reading texts and solving exercises from literature. The actual challenge and fascination is the students' exposure to open problems, addressing questions arising from real context. The real world questions may be found from the student's own fields of activity, hobbies, summer jobs, from the profession of their parents etc. Reading newspapers and professional magazines with a mathematically curious eye may produce an idea for a modelling exercise. A good modelling course should

(a) contain an interesting collection of case examples which is able to stir students' curiosity
(b) give an indication of the diversity of model types and purposes
(b) show the development from simple models to more sophisticated ones.
(c) stress the interdisciplinary nature, teamwork aspect, communication skills
(d) tell about the open nature of the problems and non-existence of "right" solutions
(e) bring home the understanding of practical benefits, the usage of the model
(f) tie together mathematical ideas from different earlier courses

The modelling courses have been run in different forms. Traditional lecture course with weekly exercise session is a possibility. It would be important to implement group work mode and PC-lab activities in the course. The most rewarding form of activity might be projects and weekly session where the student report and discuss about their work and progress on the problems. A very successful form and educational innovation is a modelling week, and intensive problem-solving workshop that has been implemented in Europe and US since late 80ties.

The supply of good classroom examples and case studies from different application areas is a key factor for the development of attractive and inspiring educational modules in applied mathematics. Especially in the courses on mathematical modelling we would need a flow of fresh problems to maintain an intellectual urge. It would be important to have ongoing contacts to different special sectors, professions, diverse pockets of innovative processes.

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Illustrative examples

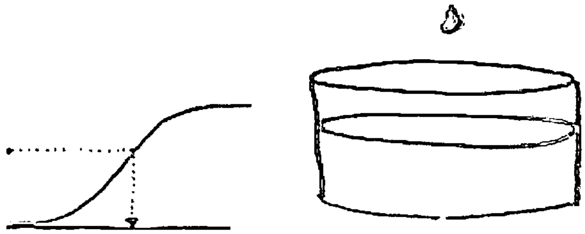

Fig 1. Design of a blood test for clinical use

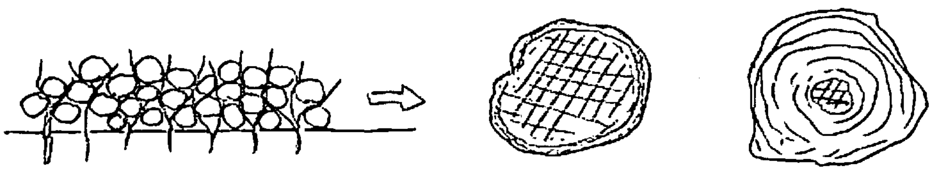

Fig 2. Quality control for the freezing of meat balls

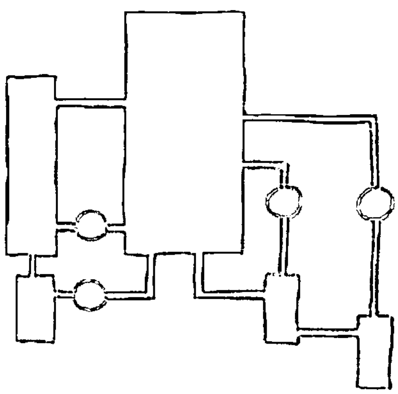

Fig 3. Chemical process modelling

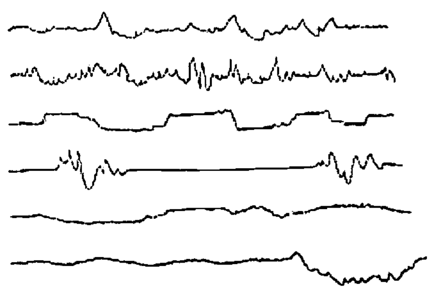

Fig 4. Process monitoring and diagnostics

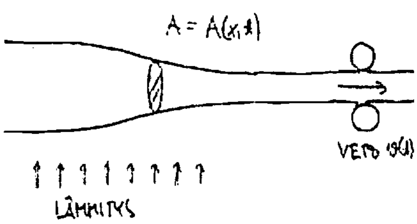

Fig 5. Tapering process for optic fiber

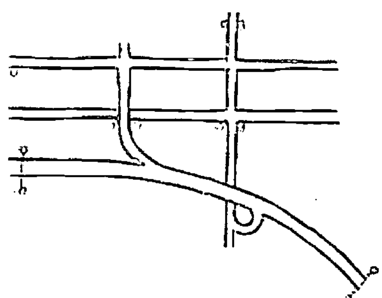

Fig 6. Dynamic traffic guidance

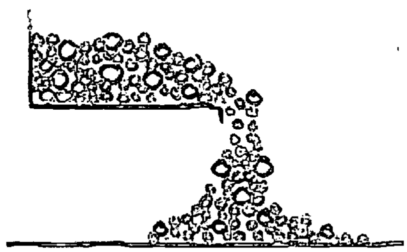

Fig 7. Models for granular materials

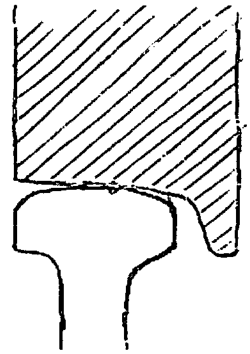

Fig 8. Modelling the rail-wheel contact

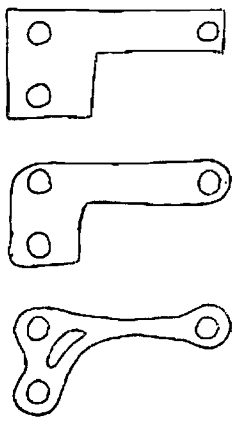

Fig 9. Optimal shape design

Fig 10. Analysis and design of accelerated fatigue testing

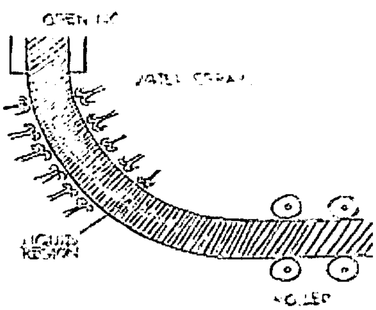

Fig 11. Continuous casting of steel

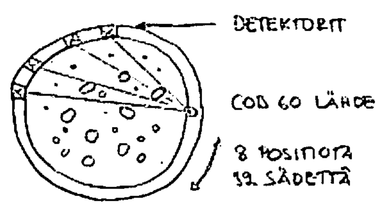

Fig 12. On-line process tomography

"COMPOUNDING RATIOS'
A Musical Approach on Mathematics Education

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ABSTRACT

In this study I shall consider educational aspects of the development of ratio and proportion, focusing on the arithmetization undergone by these concepts in the light of the relations between mathematics and music. Since such relations, even if confined to the context of ratio and proportion, are fairly wide-reaching and also that the process of arithmetization is quite complex, we shall concentrate mainly on the instructional aspects of a structural peculiarity presented in such a fascinating dynamics. This peculiarity is the so-called *compounding ratios*, a curious feature present in the structure of ratio since the Classical Period whose irregular transformation into the operator multiplication is quite representative of the importance of theoretical music in the arithmetization of ratios. As a consequence we shall also point out features of the differences between *identity* and *proportion*, which are capable of being didactically explored with a mathematic-musical approach.

The reason for choosing music for the present approach is not only historical, but more specifically didactic insofar as the subtle semantic differences between *compounding* and *multiplication* and also between *identity* and *proportion* are clearer if one thinks of ratios as musical intervals when looking at such constructs. Grattan-Guinness argues that the well-known difficulties in teaching fractions can be alleviated by converting the latter into ratios, and thus using a musical approach. These considerations corroborate the need to explore didactically specific contexts in which differences between given constructs manifest themselves more clearly.

In order to fulfil the aforementioned aim we shall first of all introduce some historical aspects of ratio in mathematical-musical contexts as well as of the corresponding structure in which compounding makes sense, and then follow these with examples of the practice of compounding on the monochord and by the didactic-epistemological aspects that underlie such a practice.

1. Historic considerations: compounding ratios or musical intervals?

There are several themes on the relation between mathematics and music or even between ratios and musical intervals, which can be explored in mathematics education. We will concentrate here on an intriguing characteristic of the structure originally associated with the concept of ratio, namely compounding ratios, which we could call an operator, although it never attained the status of a technical term in mathematics (Sylla, 1984, p.19). Such an operator occurred tacitly in contexts involving ratios since the Classical period up to the 17th century, being eventually superseded by multiplication.

The structural change is from conceptions of operations -- *compounding ratios* -- strongly tied to contiguous musical intervals to theories that admit the composition of general ratios -- multiplication -- with an essentially arithmetic character, for example, the idea that a ratio is equal to a number. The point is how to approach in classroom dynamics an epistemological change such as this, which occurred in the course of the development of ratio, in such a way that one succeeds in creating an ordinary situation in which such a difference manifests itself more clearly than it does in purely arithmetical domains.

When one considers that this transitory structure with which ratios were very partially and irregularly equipped over a long period of their history is derived from musical contexts and also that *compounding* makes no sense out of musical contexts, it is quite reasonable to take music as the scenario for approaching such differences, since here the previous structure attached to ratio stands out. But before moving on to the instructional aspects of such a topic, we will have to delve into compounding ratios in more detail.

Some indicators of the different theories attached to the concept of ratio are found in connection with issues such as Euclid's restriction on the operation of *composition with ratios* implied in definitions 9 and 10, Book V as well as in proposition 23, Book VI (Heath, 1956, p.248). Such operations consisted of compounding ratios of the type a:b with b:c to produce a:c, which then allows the repetition of this process with c:d and so on.

This operation, which had strong musical affinities, required in general that given a sequence of ratios to be compounded the second term of a ratio should equal the first term of the next ratio. Quite apart from the interest, which it holds for the historian of science, this ontological difference deserves attention in educational contexts. We will try to propose now how to explore in didactic-pedagogical contexts these two completely different understandings of ratio, one geometric-musical where ratio has no semantic proximity with number and the other, where ratio is semantically a number, capable of being multiplied in the same way as numbers are multiplied. In order to emphasize such an important epistemological change present in the history of ratio, we will make use of musical contexts.

2. Practicing mathematics/music: compounding ratios/intervals on the monochord

Compounding on the monochord is a case in point. *Compounding* in Euclid's sense must definitely *not* be put in the same category as *multiplication* although the former presents structural similarities with the latter. Both differences and similarities between *compounding* and *multiplication* concerned with musical and arithmetical fields respectively can be better felt and grasped with the help of an enriched reconstruction in learning/teaching context of the monochord's experiment. Such reconstruction can encourage students with promising tendencies in music to get interested in mathematics and vice-versa. Such crossing capacity not only stimulates the relationship between both *areas* and the related *skills* but also demands mathematics *skills* in musical *contexts* and musical *skills* in mathematical *contexts* through an simple arrangement involving elementary concepts.

Concerning the pertinent part of workshop in mathematics/music carried out in São Paulo, monochords were first handed out to the participants who were initiated into the perception of basic musical concepts, such as musical interval, necessary for the following performance. Once the students discovered by means of the monochord the ratios 1:2, 2.3 and 3:4 underlying the basic Greek consonances octave, fifth and fourth, respectively, one can set problems like:

- Let L be the length, which produces a determined pitch in the monochord. What is the length necessary to produce a pitch obtained raising the original one by an octave and a fifth, following by the lowering of two fourths? Listen to the resulting pitch in the monochord and compare that with the pitch obtained on the piano. Comment.

- Let *do* be the pitch corresponding to the length L. Which is the pitch provided by the length 32L/27? Indicate in terms of superposition of fourths, fifths and octaves, the successive steps to reach that result. In raising a fourth from the given pitch, what are the pitch and length obtained? Listen to resulting pitch in the monochord comparing it with the pitch obtained on the piano.

Such problems in particular, presented in a workshop with children between 11 and 14 years old in *Estação Ciência* -- a museum for dissemination of science, culture and technology within the University of São Paulo --, for instance, demanded simultaneously musical and mathematical aptitudes and/or at least could awaken curiosity of students who were at first interested exclusively in either mathematics or music. Depending on where each student's greatest potential lies, students solve these kind of problems either by finding the interval and checking the compounding ratios which provide it or by finding the combination of ratios that when compounded provide the requested interval, and checking the interval.

Such problems provide one with the opportunity not only to experience, perhaps even unconsciously, the compounding of ratios but also to simulate operations with ratios in Greek and medieval musical contexts, inasmuch as the students have as basic operational elements the perfect consonances, that is, the discrete ratios 1:2, 2:3 and 3:4, which in this context have no categorical relation with numbers in principle, but are merely instruments for comparison.

In order to illustrate my points, it may be worthwhile to describe some of the reactions that occur in solving these problems. I will take as an example a workshop for students of the '8th serie' -- around 14 years old -- carried out at 'Escola de Aplicação' in São Paulo. Because of size limitations, I will confine my discussion to some approaches to the first problem as well as some questions, which were raised as a consequence. In this case, the solutions passed basically from a geometric approach to an arithmetic one.

First of all, the students were familiarized in the workshops with intervals and compounding of musical intervals/ratios in the monochord. This experience enabled them to compound contiguous intervals or mathematical ratios where the endpoint of the second magnitude of the first ratio coincided with the first magnitude of the second ratio -- ratios of the type a:b with b:c -- which is what they saw in the monochord during the familiarization. The classroom was then divided into groups comprising students of different tendencies in order not only to make possible different kinds of interpretations of the problems but also to provide an appreciation of the diversified potential of each group since all problems would eventually claim the use of at least music and mathematics skills.

Initially, they were asked to solve problem one using a ruler with only four divisions and a compass. After visualizing how compounding operated in the monochord, students evinced basically two tendencies in solving the problem: one tendency was to make the calculation by always transferring the ratios to the string and dividing the string into as many parts as the denominator and then taking the number of parts that were in the numerator -- in the case of 2:3 two parts of the strings previously divided in 3 parts -- which is clearly compounding in the classic sense. Other students tried to find the resulting note -- in the case a la -- but tried to check such a result by compounding the ratios 1:2, 2:3 and decompounding the ratios 3:4 two times, as in the first case. In order to perform this operation they availed themselves of the operation, used in the first step-by-step demonstrations, of the basic consonance -- octave, 1:2; fifth, 2:3; and fourth, 3:4.. In general, they found the part of the string which when sounded resulted in the note la without knowing precisely to which ratio or note such a point or pitch corresponded.

In this first stage, no arithmetical interpretations resulted. They did the procedure as in the demonstration of the consonances, in which we used rule and compass to build similar triangles in order to divide a segment in 2,3 and 4 parts. The following question emerged:

-- Do we get the same result if I change the order of the procedure?

They figured it out from a musical point of view, an approach that makes the answer fairly intuitive, since compounding is nothing but the 'addition' and 'subtraction' of intervals. Such an interpretation makes the commutativity of this operation more intuitive. It shows also to some extent how the musical context could facilitate the 'feeling' of the meaning of such a property in the structure of ratio.

The situation provided also a suitable context for moving on to the following question:

-- How could we compound musical intervals when we know only the lengths of the strings whose ratio provide each interval? Again without metric ruler.

In this case some students tried to adapt by trial and error the first term of the second ratio to the second term of the first by taking ratios equivalent to the second term expressed as multiples of its two original magnitudes. A musical solution also emerged. For this, they tried to hear the intervals defined by each pair of strings by singing their compounding and sometimes keeping the partial result in a keyboard in order to keep the tuning. They confirmed the result doing it musically sometimes step by step, at other times at the end of the operation, based on the initial musical auditive experience with intervals and consonances. They could do it almost automatically, subsequently verifying the length of the string that corresponded to the discovered pitch. To accomplish such an operation they must always find the 'musical' fourth proportion insofar as in each step they have a reference ratio and the first factor of a second ratio that provided the lower note over which the reference interval should be translated.

Others students even tried a mixed solution by guessing through hearing the probable ratio from which they could give a good guess as to the factor by which it was necessary multiply both factors of the second ratio. In all cases the students often make use of a proportional pair of strings which are naturally not equal but that have some property, which makes them similar in some way to the first pair. This feeling of similarity realizable by hearing is one important point that pervaded many different situations in these workshops and both emphasized and eventually eased the differentiation between proportionally and equality, a feeling which disappeared when they later faced the problem with an arithmetical approach using a metric ruler. The advantage of the musical approach in comparison with the geometrical one consists in the fact that the former provides the feeling, based on a perceptive skill, that both pairs of magnitudes are not equal but that at the same time they have a common attribute, which is musically the interval defined by them. In the face of such similar ratios/intervals, some comments like the following were heard:

- They are not equal but one is 'as if' it were the other.

The rationalization of such a feeling was refined when not only harmonic but also melodic versions of the same ratio were provided. Then some comments like the following one appeared:

- The notes 'walked' or 'climbed' with the same step:

They are probably doing albeit not necessarily consciously a musical or logarithmical approach.

In order to provide a similar visual perception by geometry, on the other hand, the four magnitudes should be laid in a particular configuration -- not necessary in music --, which was also approached -- as the following shows -- in order to strengthen such a differentiation.

In such a dynamics, the following question came out.

-- Could we calculate it only once?

Then similarity was introduced so that one could build precisely the proportional second ratio in such a way that its first term had the same measure of the second term of the first ratio, emphasizing a geometric/musical connotation to proportionality.

Still without metric rule, it was possible to pose the following question:

-- Could we calculate the compounding of all ratios applying it at the end to the monochord?

One possibility was to do it analogically to the geometric procedure using now whole numbers, which involves the knowledge that a:b :: ma:mb -- proposition 18 of Book VII of The Elements -- going on working technically just with integers. In such a dynamics the following question came out:

-- What do we do to compound a:b with c:d when there is no integer m so that mc = a?

When we dealt only with geometrical magnitudes this question did not arise, since one can always adapt one magnitude to another but that is not the case with whole numbers to be adapted to each other using integer multiples.

In this case, one must multiply the numerator and denominator of both ratios, resulting as factors c and b respectively which make the original compounding proportional to (ac:bc).(bc:bd) :: ac:bd. Based to some extent on the trial and error experience done before with geometrical magnitudes, they tried now to do something analogical with integers represented geometrically which resulted eventually in the use of the Minimal Common Multiple between b and c.

The compounding of all ratios was curiously very easily done with intervals, that is, from a determinate interval with a certain low pitch, they could build the correspondent equivalent interval -- proportional ratio -- from hearing and feeling the same 'growth' of interval.

The comments and questions mentioned above concerning the solution of the first problem reflect to some extent the dynamics of this workshop. The example mentioned above tried to reflect partially how the workshops could provide a suitable environment to experience this arithmetic sense of ratios, by introducing this approach before turning to the metric ruler.

The problem was repeated allowing the use of metric rule and gradually ratios and compounding were equated to decimal numbers and multiplication respectively, thus diminishing the emphasis in the differentiation between identity and proportionality.

It was possible to realize that the problem became even more interesting insofar as one could restrict the available tools for the solutions: compass, non-metric ruler, metric ruler, instruments - which provide different meanings to ratio and proportion, and could get the student to operate at times with compounding, and at other times with multiplication. Such an enriched arrangement proves useful not only for illustrating the importance of ratio as a medium for comparison but also and most importantly for providing a context for practicing the differentiation between both compounding and multiplication as well as between proportionally and identity within a meaningful practical situation.

3. Didactic-epistemological aspects

Besides the difference between *compounding* and *multiplication*, there are deeper differences within the arithmetization of ratios that become transparent through the aforementioned arrangement, such as that between *identity* and *proportion*. In Euclid, the idea of equality of ratios is not as natural as that of numbers or magnitudes. Such a way of establishing relations between ratios gains greater meaning when we consider that on the monochord, for instance, *do - sol* and *la - mi* are the same intervals - in this case, a fifth - but they are not equal, inasmuch as the latter is a sixth above the former, or even that do-sol 'is as' la-mi. The *identity* is normally a philosophically difficult concept to be worked out in learning/teaching dynamics. Stressing the distinction between identity and proportion in mathematical/musical contexts, where such differences become clearer when they are visible and 'audible', can ease such difficulty.

The problems and the device mentioned above also encourage the perception of such a difference insofar as the students can hear the intervals provided by proportional ratios like 9:12 and 12:16 - both are fourths, that is, the same intervals, but they are not equal - which are proportional but definitely *not identical*. This elucidates by the use of mathematics and music the differences and similarities between both concepts which also contribute to the better understanding of the identifications of *ratio* and *fraction* and of *proportion* and *equality*. It opens several possibilities for exploration of such concepts in both contexts. For instance, they can find the forth proportional and deduce what is the associated pitch or reciprocally, given an interval, they can figure out the note which will produce the same interval given a determinate lower pitch: both situations deal with proportional magnitudes in mathematical and musical contexts simultaneously. The students must not necessarily be aware of the epistemological procedure underlying such dynamics. What is actually important is that they experience such a situation and thus establish a reference with which they can bridge and anchor the comprehension of future situations involving these concepts. In the same way, the experience will enable them to detach concepts associated with fixed areas and interweave them in a more general context.

The aforementioned arrangement in teaching/learning as well as the long history of ratio and proportions show that, within the rich semantic field associated with these concepts, ratio was a natural vehicle for human beings to use in comparing different contexts through proportions, that is, analogies. In this sense, the proposition that 3:2 corresponds to a fifth, as well as that one that the aforementioned intervals of fourths are proportional mean that these two concepts pertaining to mathematical and/or musical fields are capable of being compared to one another by means of the ratio of numbers and the interval between notes through proportions. In this sense, it is possible to experience that the geometrical/musical proposition A:B::C:D is semantically distinct from yet structurally similar to the arithmetical proposition $A \div B = C \div D$, as well as that the corresponding cases in which ratios are not proportional and fractions are not equal.

Reciprocally, by means of the device of the monochord, ratio and proportions are viewed as instruments for evaluating the degree of similarities between different contexts. Such a device can also help the comprehension of the categorical distinction between ratio and proportion—sometimes misunderstood—inasmuch as ratio is clearly viewed as a *definition* involving two magnitudes of the same kind whereas proportion functions in all the aforementioned situations either as a *logical proposition* to which one may attribute a valuation or as a tool to make a proposition true. In the case, such a difference is experienced through the question about the plausibility of the equality between two intervals or of the proportion between two ratios. The differences between these two mathematical entities are less ambiguous when understood in this way than when viewed in purely arithmetical contexts.

4. Conclusion

The present musical approach widens our comprehension of ratio and proportion in mathematics not only because of its historical-cultural contextualization and the interdisciplinary aspect which underlies it, but also, and most importantly, because of the role that analogical thought plays in the construction of meaning, in this case, that of ratio and proportion. If we wanted to extend Kieren's argument (Kieren, 1976, p.102) about rational numbers to ratios, we could claim that to understand the ideas of *ratios*, one must have adequate experience with their many interpretations. The aforementioned device not only provides a fertile ground for the understanding of the subtle differences and structural similarities underlying the diversity of interpretations associated with ratio and proportions but also contributes to constructing and to experiencing in a broader way their associated meanings.

In a general sense, discovering common schemes and archetypes is an efficient way of constructing concepts that concern in principle different areas. An analogy or metaphor used in a sensible and discerning way may re-configure a student's thought in a problematic situation of learning, enabling a better understanding of matters that escape immediate intuition, or that seem too abstract to him/her, such as the many interpretations associated with ratio and proportions as well as with the wide variety of structures historically associated with them.

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THE BLK SCHOOL PROJECT IN BADEN-WUERTTEMBERG

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ABSTRACT

The project to be described is entitled "Promoting classroom culture in mathematics" and is the contribution of the state of Baden-Wuerttemberg to a programme of the Bund-Länder-Kommission (BLK), a commission of the German Federal Government and the governments of the Federal states. The title of the programme is "Furthering the efficiency of mathematics and science teaching". This four-year programme for the lower secondary level is intended as a meaningful response to the less than satisfactory German TIMSS results. The approach of this project focuses on changing the teaching style, the major objective being to develop a holistic concept for mathematics teaching, integrating comprehension, active participation and long-term productive learning. A report will be given on initial experience obtained with the project in Baden-Wuerttemberg.

1. TIMSS and the BLK Project

The not very flattering results of Germany in TIMSS, the Third International Mathematics and Science Study, turned out to have the effect of a catalyst inducing a nation-wide debate about educational goals and the content of mathematics teaching. The most perceptible signal of response is the programme "Furthering the efficiency of mathematics and science teaching", initiated by the Bund-Länder-Kommission (BLK), the Federation-Laender Commission for Educational Planning and Research Promotion. This four-year programme for the lower secondary level was started with the school year of 1998/99. In preparation of the BLK programme (BLK 1997), a report was made pointing out the basic assumptions and principles of future educational policy as well as the various problems encountered in mathematics and science teaching at school. Based on this report, the Germany-wide school experiment with accompanying measures was put into place. A major approach adopted for this experiment was that schools and teachers were not to be confronted with ready-made teaching concepts. Instead, 11 so-called modules were developed which offer promising starting points for the teachers who are free to develop their own approaches for promoting mathematics and science teaching.

The Federal states of Germany participate in this BLK-programme with altogether 30 experiments. Six schools each forming a so-called school set participate in an experiment. The Federal states have chosen the modules they wished to apply in the participating schools, and have organised their respective experiments in their own manner.

2. The school project in Baden-Wuerttemberg

The project "Promoting Classroom Culture in Mathematics" is the contribution of Baden-Wuerttemberg to the BLK programme (Blum&Neubrand 1998, Henn 1999). There are three different school sets: one representing six "Hauptschulen" (lower level education), one representing six "Realschulen" (intermediate level education), and one representing six "Gymnasien" (higher level education). The schools are working together closely and there is a lively exchange of experience between the school sets. For our model schools, we have focused on the following four out of the 11 modules:

Module 1: Developing a problem culture in mathematics and science teaching. Emphasis is on open problems, appealing to all students. Individual problem solving abilities are challenged. Problems are given in varied contexts, to permit development of various, qualitatively different solutions and to provide systematic and productive exercises.

Module 3: Learning from mistakes. Psychological and pedagogical theories describing the conditions which foster learning from mistakes form the basis and are applied in practical teaching in the class room. "Mistake-friendly" lessons can improve the students' mental activities.

Module 5: Experiencing the growth of competence: cumulative learning. Possibilities to make a vertical net and prepare the ground for cumulative learning are explored. Long-term orientation and guidance in building learning history should help students to accumulate a bigger "learning possession".

Module 10: Assessment: Comprehension and feedback on growing competence. The development of challenging examination problems and examination types suitable to measure comprehension and the versatile application of knowledge are highly important to the task of improving teaching.

The modules are applied in all year groups of the lower secondary level. The approach underlying our project aims not so much at changing the mathematical content, but rather focuses on a change of teaching style. The aim is to develop a holistic design of teaching which leads to understanding, active involvement and long-term fruitful learning. Central deficiencies of the current teaching style are short-term learning for the next test, restrictions resulting from a rigidly guided "questioning-developing" teaching style, and a strong bias towards calculation in mathematics teaching, whereby calculations are used without insight and understanding.

The central question is not "what is to be learned", but "how should the learning process proceed", "how can mathematical literacy be promoted", and also "how can learning processes be measured". The willingness to question and to rethink current teaching, to change one's own reception and to realize opportunities brought about by new practice and teaching methods are important. We do not intend to reject everything from the past for being bad, or to follow new fashionable slogans such as "tasks as open as possible" or "application at all cost". Our objective rather is to bring about a reasonable shift of emphasis, balancing the importance of instruction (by the teacher) and construction (by the students themselves), teaching and discovery, convergent routine problems and divergent open problems, different modes of testing and achievement measurement.

Of course this reorientation has to develop in a natural way in the course of the years spent at school from elementary to upper secondary level. Our approach to the BLK-project was highly influenced by the concepts of the Dortmund project *mathe 2000* of the group around E.Ch. Wittmann and G.N. Mueller (Mueller et al. 1997). Inherent in the concepts is the concentration on fundamental ideas of mathematics and a long-term development following the spiral principle, across the years spent at school. An aspect of major importance is student-centred mathematics teaching, which means taking seriously the answers, ideas and products of the students. This means a conscious change in teachers' as well as students' attitudes, especially towards attributing more importance to learning processes than to results. The special merit of a "good teacher" is not his "good explanations", but the ability to promote thinking processes and active discovery learning in a productive learning environment.

Two main aspects which are not self-evident, at least in teaching at upper secondary level, have emerged: to take children and their products seriously, and to construct productive learning environments.

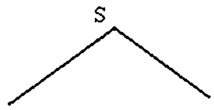
3. Taking children seriously

In "How children compute", a book worth reading, the authors plead that teachers should think and argue with children, listen to them, take their products seriously, not ignore their mistakes, but rather discuss them productively (Selter&Spiegel 1997). We usually expect that children think in the same way as we do, the mathematicians (whereby we often have remarkable blinkers ...). However, as was clearly pointed out by Selter and Spiegel, children use to calculate in a different way, which may differ from the way we do it, the way we assume, the way other children do it, and then again from the way they have used themselves before in dealing with the "same" problem.

Consequentely it is not enough to question children, but rather to take their questions seriously, discuss and try to understand them. Thus teachers are able to recognize and understand their students' cognitive structures. *One* way is to gather all solutions without any comment on the blackbord in a first step and then to discuss them. Often learners then point out the mistakes they made themselves. The following examples illustrate this approach.

Example 1: Geometry (grade 6):

The problem is (relating to Fig. 1): *Complete the drawing in such a way that there are two adjacent* *supplementary angles.*



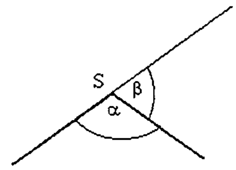


Fig. 1 **Fig. 2**

In a normal lesson, students would be guided to a solution by making one line at the side of the angle longer (cf. Fig. 2). Here, the teacher's students were able to accomplish the task by themselves, without any hints and guidance. Single solutions were then presented at the blackboard. Questions were only allowed after the drawing was finished. Then the student had to explain his or her solution, mistakes had to be realized. Solutions turned out to be much more general than the narrow schoolbook solution. Often there where several pairs of adjacent supplementary angles. Fig. 3 shows some of the solutions:

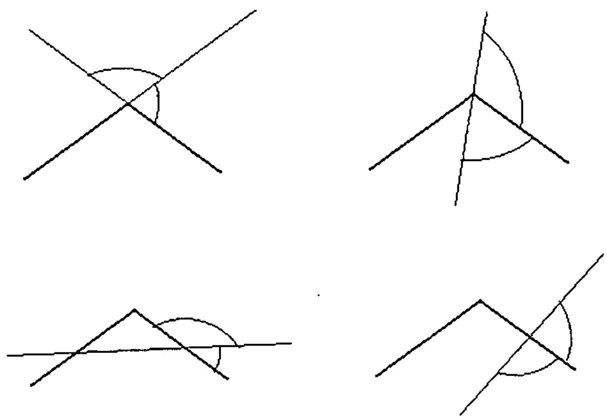


Fig. 3

Interestingly enough, the schoolbook solution of Fig. 2 was not mentioned (and not forced upon the students by the teacher). Taking the children seriously created a productive working atmosphere. "I could easily see the disappointment of students whose solution was already presented by others. A positive disappointment", reports the teacher.

Example 2: Expressions (grade 7):

In a grade 7 project class the three solides given in Fig. 4 were presented.

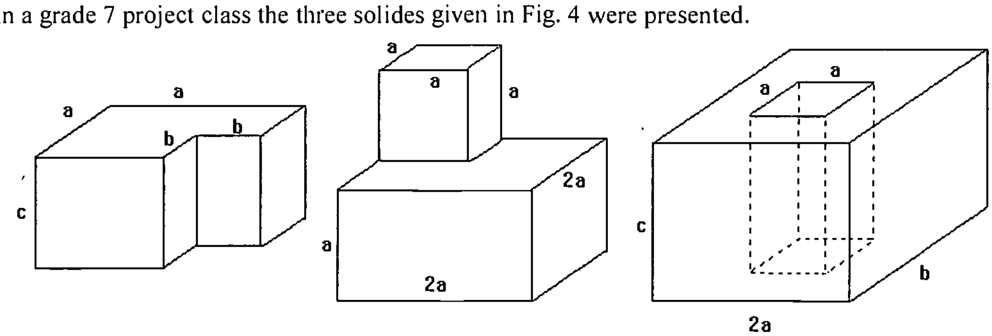


Fig. 4

The class agreed to tackle the task to determine surface areas and volumes of the solids individually or in groups. Students started to work, results were written (without comment) onto the blackboard. After some time there were up to eight different expressions for the sought quantities. Now a discussion started about who had found the correct solution, obviously the teacher did not intervene. Some expressions, for example including a term a^6, could be sorted out and found incorrect with respect to the unit. For the remaining terms it was not obvious whether they were correct or incorrect. The students tried intensively to compare the expressions, which gave an excellent motivation for finding strategies for the manipulation of expressions. To calculate with the children was of great value also for the teacher: "I was very content about how the lesson developed. I gained insight into the cognitive structure of students".

4. Productive practise

E.Ch. Wittmann describes the didactics of mathematics as a *design science* which delevops and researches "productive learning evironments". Problems with rich content are worked on holistically. This is presented exemplarily in both *Handbooks of Productive Arithmetic Practise* for the four elementary school years (Wittmann&Mueller 1990/1992). The single learning sections create meaningful relations and propose problems of different degrees of difficulties, leading to a natural differentiation. In contrast to the usual step-by-step teaching not all obstacles are removed. Students gain experience in using "common sense" and are challenged to think about problems on their own, to judge their own considerations and to test whether they make sense.

One example for a productive learning environment is the following sequence of problems on number walls, developed by a collegue for his grade 5 class:

Example 3: For the empty number wall given in Fig. 5 the following questions were asked:

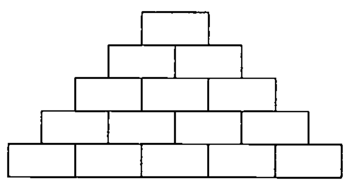


Fig. 5

Can you build number walls in each of the following cases?

- *In the first line write down five numbers you like.*
- *Write down only four numbers in line one.*
- *There are only odd numbers in line two.*
- *At the top is a number close to 500.*
- *At the top is exactly 500.*
- *Can you find several walls with 500 at the top?*
- *Is it possible that there are only numbers divisible by three in line three?*
- *Can you write down five numbers in any space and still complete the number wall?*

Number walls are a tried and tested exercise format which is introduced in elementary education. The sum of two adjacent spaces will appear in the space above. This exercise format has been successfully used also for other number spaces and operations, as well as for variables. More complex questions can be posed easily. For example, write down the unit fractions $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ from top to bottom on the left side of the wall. Then all other places can be filled unambigiously. What do you observe?

Standard problems, too, can often be improved in a "productive way", by taking away a restriction, by opening up a too rigorous problem statement, or by posing the problem rather vaguely. Some examples from our project:

Example 4: In the standard problem

Arrange the following numbers according to their size

$$-1;\ \frac{2}{3};\ 3.\overline{9};\ -\frac{143}{6};\ -184.76;\ \frac{14}{7};\ 8.23;\ -5\frac{2}{9}.$$

the words "according to their size" were omitted in a grade 7 project class. This resulted at first in a creative restlessness among the children who wondered what might be meant by "arrange". They were used to managing convergent, unambiguous questions. Only after the teacher pointed out that there may be more than one solution (without, of course, mentioning them), the children started to work and arranged the numbers according to positive/negative, number space, or size.

Example 5: *Choose four fractions. Use them to make expressions as large as possible (as near as possible to 1, ...).*

Example 6: *Calculate some powers that you like.*

It is typical of such problems that pupils work intensively and offer many ideas, but naturally make mistakes, too. But they realize and correct their mistakes. It is time well spent because children are involved quite differently. Emotional AND cognitive aspects are addressed.

Often only standard knowledge and skills are applied to standarized question types. If practical teaching continues in such a way, thinking and computing are separated. A problem setting should be looked at from different perspectives. Inverse problems often appear to be easier at first sight, but then lay bare missing basic concepts. Here some tried and tested examples from our project:

Example 7: *State two different fractions between $\frac{6}{17}$ and $\frac{7}{17}$ or explain why there are none.*

Example 8: *Find all solutions or explain why there are none:*

$$6\times(20 - ?) = 144;\quad 7\times(12 + ?)=100.$$

Example 9: Better than the convergent question *"7 + 5 = ?"* which asks for the synthesis of twelve is the divergent question "*Which is the most beautiful twelve?*" asking for the analysis of twelve and resulting in many correct and important answers, i.e. 12 = 11 + 1 or 12 = 1 + 2 + 3 + 3 + 2 + 1.

These illustrative examples show that only by choosing another formulation in standard problems new aims can be addressed to further creative ideas, to differentiate the possiblities, to order, and to classify.

7. Assessment: Measuring and feedback of gain in competence

It is important to differentiate between learning and assessment situations. Understandably so, students try to avoid failure in assessment situations. Nevertheless, problems that are open and ask for own decisions have to be included in tests. In our experience students did not see these problems as something new because firstly they were used to such problems from their lessons and secondly these questions were included sensitively in test situations. In particular, one cannot force creativity in relatively little time and under stress. However, some test problems from our project classes (grades 5) show what is possible.

Example 10: *Armin buys a rubber and two pencils from Beate. The rubber costs 2 €, and each pencil 1 €. Beate asks for 6 €, and Armin protests. Then Beate writes down an expression and explains her calculation to Armin. Then Armin understands. He tells Beate that the term is correct but that she has broken a rule in her calculation. Armin pays the correct price. What is the expression Beate wrote down and what rule has she used incorrectly?*

Example 11: *Form expressions with the numbers 24, 9, 8 and 5 and calculate them. For at least three of the expressions the results should be between 0 and 10. For at least three of them the results should be between 100 and 110.*

Naturally, with this type of questions, basic arithmetic knowledge is checked, but in addition, algebraic competence on expressions is necessary. The important computation rules are used independently rather than merely checked out of context.

Example 12: Supplement:

a) $7 \times (50 + [\ ]) = 350 + 28$; *b)* $7 \times 14 + [\ ] = [\ ] \times (14 + 6)$.

For a) the solution $7\times (50+4) = 350+28$ is unique. For b) mostly, as expected, the solution $7\times14 + 7\times6 = 7\times (14 + 6)$ was given triggered by the distribution law. In reality, the equation $7\times14 + a = b\times (14 + 6)$ has the positive integer solution $a = 2\times (10b-49)$ for every positive integer $b \geq 5$. In a few cases some of those solutions where found by trying out different numbers, which, of course, is a creative, original achievement.

8. Experience resulting from the project

In posing more open problems one has to take starting difficulties into account. Students tend to ask for a recipe, and are unsure in the beginning. They often are afraid of failure and do not start at all. But their attitude changes after some time. "During the school year unsureness was lessening, and with growing confidence the students would develop more problem solutions on their own and accepted that there is more than one way to reach the solution", reported two of the colleagues involved. One could observe a growing familiarity with more complex, more open problem statements. "The development of a wide range of solutions was only possible when I myself as a teacher retreated in the decisive moment – left the problem completely to the class and did not break down the problem into bite-sized pieces by questioning-answering techniques until they were convinced that the problem could be solved only with a linear equation – is that not what often goes completely wrong in mathematics teaching?"

The increase in creative and heuristic abilities is difficult to measure. But in the feedback we received it was reported that problems were increasingly dealt with as a matter of routine, with perserverance instead of resignation. All involved were convinced that students gained metaknowledge rather than a collection of easily assessible but quickly forgotten information. The deliberate change in the teacher's rôle (to stay back in working and solution phases, to challenge and to accept solutions, to encourage alternatives) was not restricted to project classes only!

9. The WUM inservice teacher education

The experience gained up to now in the 18 project schools resulted in the development of a new regional inservice teacher education named "*W*eiterentwicklung der *U*nterrichtskultur im Fach *M*athematik" for all school types (abbreviated to WUM, that is to say, further development of the teaching culture in mathematics). Teachers themselves request that an inservice teacher education team comes to their school. In courses lasting for one whole and three half days, generation of problem awareness is the first item on the agenda, followed by a number of short presentations introducing the new methods (productive practise, opening and variation of problems, open-ended approach, non-routine examination problems). The main task of the teachers then is independent preparation of teaching material for their own lessons. The tremendous demand shows the great interest of our teachers.

10. Conclusion

Obviously, our experience accumulated in the BLK project for about three years now is not yet reliable enough to serve as a source for deriving reliable final conclusions. Certainly however the teaching climate and the active participation of most learners have improved decidedly. We have reason to believe that through open, more challenging work and problem style improved basic concepts will be developed.

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THREE EXPERIMENTS IN TEACHING UNDERGRADUATE STUDENTS IN MATHEMATICS

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ABSTRACT

The aim of this presentation is to describe the pleasures and the problems in teaching undergraduate students. We look at three experiments. One experiment looks at the methodology of teaching a large class of 250 students using the overhead projection method. The advantages/disadvantages of this method from the viewpoints of students, teachers and the administration are discussed. The other two experiments look at teaching a class size of about 40 students in an examination free set up, in a more interactive way. The feed back of the students about the several aspects of these methods are discussed.

Experiment 1: Teaching a large class

Introduction

This is the experiment carried out at Indian Institute of Technology Bombay (IITB) from 1997-2000.

At IITB, about 450-500 students are admitted every year to various engineering programs. All the first year students are given core–courses, one each in Mathematics, Physics and Chemistry. Till 1997 these students were divided in 4 divisions, each division assigned to a teacher – one of them being the instructor-in-charge, for coordination of the course. The method of teaching was the traditional `blackboard-chalk method'. In 1997, for various reasons, it was decided to undertake an experiment of teaching large classes (of the size of 250 or so) with the help of `modern technology'.

Methodology

The methodology proposed to conduct the course was the following.

In view of the large class strength, the traditional blackboard-chalk method of delivering instructions has to be replaced. It was proposed (in fact that is what was finally implemented) to use overhead projection of instructions. Also to have uniformity (across different divisions) it was felt desirable that the same material be used in all the divisions. Moreover, since the place of instruction has to be dimly lit (to make the overhead projection effective), it was felt that the student would find it difficult to take notes during the lecture. Thus a `concise' set of notes needs to be prepared for the students.

To implement this, a team of two instructors (one for each division of 250 students) was selected about 3-months before the start of the course. They achieved the required preparations, see [2], and the course was conducted in 1997. The experiment was repeated in 1998 and 1999. In all these experiments, conducting a class meant explaining (to 250 students seated in a dimly lit hall) mathematics from a set of notes projected on a screen.

I will list below some of the advantages and disadvantages of this method of instructions (see also [1]).

Advantages

(i) From administration point of view

- Large number of students can be taught with lesser faculty.

(i) From the teachers point of view

- Teachers have sufficient time to plan, discuss and prepare the course in advance.
- During the lecture the teacher has more time to explain, since he does not have to write.
- It is cleaner (no rubbing of messy black-board again and again).

(iii) From students point of view

- They have more time to listen and understand the concepts, as they do not have to take notes.

Disadvantages

(i) From administrations point of view

- None

(ii) From teachers point of view

- There is more rigidity in the lecture as the contents are already documented. There is no spontaneity.

- Teachers own style is constrained. There is not much scope for innovation.
- There is no interaction. Mostly it is one-way traffic.
- Difficult to manage the class because of its size and the classroom environment.

(iii) From students point of view

- Lectures tend to go at a faster rate as compared to the traditional method. So student gets less time to assimilate the concepts.
- The charm of seeing the contents being developed is lost. In the traditional method, there is a sense of contents being developed then and there.
- The availability of notes, even if concise, gives student a false sense of security. They tend to be less attentive.
- The good students do not get an opportunity to interact with the teacher.
- The classroom environment makes them feel sleepy.

The positive outcome of this experiment was that some teachers have started using overhead projections partially to supplement their traditional classroom teaching.

Experiment 2: Workshops in Mathematics

In 1993, the mathematics faculty at IITB felt that efforts should be made to attract good students for the M.Sc. programs. I proposed to the department the concept of Workshop in Mathematics. Since 1994, it has become a yearly activity at the department. The Department of Science and Technology, Government of India funded the last four workshops.

Objectives

"Experience shows that it is unwise to expect much mathematical background in the case of a student entering college. Many dread Mathematics. They should be assured that mathematics is not so difficult, and it will prove interesting if carefully studied.

American Mathematical Monthly, 40 (1993)

"Do not satisfy your vanity by teaching great things. Awake their curiosity. It is enough to open their minds, do no overload them. Put there just a spark. If there is some inflammable stuff, it will catch fire."

Anatole France.

The broad objectives of the workshop are to encourage final year graduate (BA/BSc) students for higher studies in mathematics. And this can be best achieved as follows:

- Workshop should be held in an examination free environment.
- Topics for the workshop should neither be too hard nor be disjoint from their course curriculum.
- Lectures should analyze not only what is in the topic, but also try to answer whys and how's of the topic.
- Efforts should be made to instill confidence for problem solving and to encourage independent thinking.

Methodology of the Workshop

Posters, announcing the workshop and inviting applications are sent to colleges in Mumbai, Pune, Ratnagiri, Kolhapur, Nasik, Dhule, and some other nearby cities. On personal level also teachers are contacted and asked to recommend 2/3 students. On the average about 120 applications are received. Based on the marks obtained by the student and the teacher's recommendation, 40 students are selected for the workshop.

The topics of the workshop are selected keeping in mind the course curriculum of the students. Efforts are made to make the lectures inter-active and discussion oriented. Students are encouraged to discover and develop independent thinking via problem sessions. Expository lectures are arranged to given an over-all view of some topics. A carrier-guidance-lecture on the avenues for higher studies in mathematics at IIT Bombay is also organized. Lecture notes are prepared for the topics to be taught and are distributed to the students.

Students' feedback on the 6th workshop conducted in the year 2000

At the end of each workshop, the students are asked to give their feedback in a form. A summary of the feedback received for the workshop conducted in the year 2000 is as follows:

(1) How did you find the lecture contents of the topics?

Topic → Nature ↓	Set Theory	Probability Theory	Linear Algebra	Analysis
Heavy	28%	5%	19%	45%
Medium	8%	45%	47%	25%
Light	8%	---	5%	3%
Totally New	--	--	--	2%
Not new but useful	56%	50%	29%	25%

(2) How useful were the problem sessions?

Do you think more time should be devoted to them?

Very useful	Useful	No Response	More time
64%	25%	11%	80%

(3) How were the non-academic facilities?

Very good	Good	OK	No Response
61%	20%	2%	17%

(4a) Do you think such workshops are useful? Yes **-100%.**

(4b) Do you think such workshop should be held in the future also? Yes - **100%.**

OUT OF THE PARTICIPANTS EACH YEAR ABOUT EIGHT TO TEN PARTICIPANTS GET SELECTED AND JOIN M.Sc. PROGRAMMES AT DEPARTMENT OF MATHEMATICS, IIT BOMBAY.

Before drawing any conclusions from the above experiment, I would like to present to you another similar experiment.

Experiment 3: Mathematics Training & Talent Search Program (MTTS)

Introduction

During the "Discussion Meeting on Harmonic Analysis" held at Indian Institute of Science, Bangalore (India) in 1992, a session was devoted to discuss the academic preparation of the students who come for Ph.D. programs in Mathematics in various Universities and Institutions in the country. In order to improve the level of Ph.D. aspirants it was felt that a training program should be started (starting at the B.Sc. level itself) which should expose bright young minds to the excitement of doing mathematics. The National Board for Higher Mathematics (NBHM) of India was approached with the proposal and it agreed to fund the pgrogram. The first program was held in the summer of 1993. This program is being conducted every summer since 1993 under the directorship of Prof. S. Kumaresan, Department of Mathematics, University of Mumbai, India and funded by NBHM.

Methodology:

The program consists of 3 levels: 2 for undergraduate students and one for postgraduate students. The program is advertised in leading national newspapers and applications are invited for participation. On the average about 1500 applications are received out of which about 120 (140 for each level) participants are chosen. The daily program consists of 3 hours of lectures in the morning, 2 hours of problems sessions in the evening on basic topics: Algebra, Analysis, Geometry, Topology, Number Theory, Probability Theory. Contact hours for each topic during the program (of 4 weeks) is approximately equal to that of a one-semester course. Some teachers are also invited to the workshop.

Objectives:

- To teach mathematics in an interactive way rather than the usual passive presentation. To promote active learning, the teachers usually ask questions and try to develop the theory based on the answers and typical examples. At every level, the participants are encouraged to explore, guess and formulate definitions and results.
- To promote independent thinking in mathematics.
- To provide a platform for the talented students so that they can interact with their peers and experts in the field. This serves two purposes: (i) the participants come to know where they stand academically and what they have to do to bring out their full potential and (ii) they establish a rapport with other participants and teachers which help them shape their career in mathematics.
- The precise and linear exposition of a typical textbook often leads students to believe that mathematics is a dry, rigid and unchanging subject. The program aims at dispelling such beliefs and tries to exhibit to them the vibrant nature and the essential unity of mathematics.

The program is highly appreciated by the participants and teachers. Many of the participants have gone for higher studies and write back appreciating the training they had received. Some are now teachers at colleges and feel that the training at MTTS is enabling them to do a better justice to their jobs.

Conclusions

As is clear from the advantages and disadvantages listed in the first experiment, neither the teachers nor the students found the experiment worth continuing. Both found it too monotonous and devoid of any human interaction. The success of the second and third experiment lies mainly in the facts that in both these programs there is lot of interaction not only between the teacher and the students also between the students themselves. The interaction not only helps students to understand the subject better, it is also useful for the teachers. It helps them to know the stumbling blocks in the process of understanding of the students and to device new ways/methods of presenting the subject. Students are not under pressure to perform (for an examination) or to compete with each other. They get a chance to ally their simplest doubts. They find there is spontaneity, concepts being developed in front of their eyes rather than just being displayed. Teacher also feels happy when he sees a glint of satisfaction in the eyes of the students. For him there is a sense of achievement. All this is because there is active (interactive) teaching. There is a human touch and that makes all the difference. I feel, whatever technology we bring into our teaching, it should only be to assist the teacher to make the human touch more effective and not replace it.

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A QUASI-QUALITATIVE APPROACH TO LIMITS

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ABSTRACT

The classical definition of limit of a function involving 'epsilon and delta' is not readily understood by students studying calculus for first time. Though teaching/learning calculus from Non-standard models of number system and infinitesimals is relatively easier , it is not widely practised. Under these circumstances increased use of Landau symbols is suggested. This will promote a greater qualitative understanding of limits and the rate of growth of functions.

1. Introduction

Every teacher of basic calculus experiences some difficulty in communicating the concept of limit of a function / sequence to students learning the subject for the first time. This is not surprising, for the precise formulation of the concept of limit of a function or sequence eluded the best of mathematical minds for centuries. More specifically, Cauchy's treatment of the concept of limit in his *Analyse algebrique* (1821) and subsequent treatment of calculus in his *Lecons sur le calcul infinitesimal* (1823 and 1829) was considered an enormous advance over the exposition of Newton and Leibnitz. However, Cauchy's definition of limit (and continuous function) was discontinued after 1880 when Heine and Weierstrass formulated the modern ε - δ definition. Cauchy was not always rigorous and he did not distinguish between continuity and uniform continuity. His discussion of power series reveals that Cauchy treated pointwise and uniform convergence of the functional series without distinction. Later in 1872 Weierstrass shocked the mathematical world with an example of a nowhere differentiable everywhere continuous function when his predecessors and contemporaries thought otherwise. In the light of these historical developments it is clear that the concepts of limit, continuity and differentiability of functions and their interrelationship are intrinsically abstruse. It is but natural that students find these concepts hard to comprehend. For an enlightening discussion of the difficulties involved in providing formal approaches to these concepts, Artigue [1] may be referred.

2. Diverse (equivalent) definitions of limit and continuity

The well-known Weierstrassian 'ε - δ definition' of continuity of a function at a point is only one of the several options for formulating this idea precisely. A slight variation, based on the concept of a neighbourhood of a point (real number) makes the definition qualitative (and topological). In terms of neighbourhoods the continuity of f at x_0 can be restated as follows :

for each neighbourhood V of $f(x_0)$ there is a neighbourhood U of x_0 such that $f(U) \subseteq V$.

Based on the concept of convergence of real sequences, continuity of a function f at x_0 can be viewed as a property of regularity that requires the convergence of $f(x_n)$ to $f(x_0)$ whenever the sequence (x_n) converges to x_0.

Around 1960, Abraham Robinson validated the use of infinitesimals in calculus by means of his Non-standard Analysis. Since then, there have been numerous attempts to simplify Nonstandard Analysis and make it more accessible to undergraduate and high school students. The purpose of such attempts is to retain the intuitive approach of Newton and Leibnitz based on infinitesimals without compromising on rigor. At the same time these soften Robinson's original metanumerical foundations of the real number system. Notable among such contribution are those due to Kinsler, Schwarzenberger and Tall. Such a non-standard formulation leads to simpler algebraic treatment of problems of calculus.

3. A quasi-qualitative approach

Concepts of nets and filters suffice to investigate questions of convergence in a general topological context. However, from a pedagogical point of view, it is impracticable to introduce these abstract concepts at the undergraduate level. Besides, Non-standard methods of calculus, despite their merits are not widely practised. However, the' $\varepsilon - \delta$ approach' is still in vogue at the undergraduate level. Nevertheless it is worthwhile to expose students to qualitative methods of studying limits, even if the concept of limit is defined after Weierstrass using epsilons and deltas. In the sequel, a procedure embodying such a quasi-qualitative approach is outlined. It is quasi-qualitative as it is based on classical 'ε - δ definition' of a limit. This is exemplified by the systematic use of the three Landau symbols defined below.

Definition 3.1

Let (x_n) and (y_n) be sequences of real numbers, where $y_n > 0$.

(i) If there is a constant K such that $|x_n| \leq K y_n$ for all n one writes $x_n = O(y_n)$;

(ii) If $\lim_{n\to\infty} \frac{x_n}{y_n} = 0$, one writes $x_n = o(y_n)$;

(iii) If $\lim_{n\to\infty} \frac{x_n}{y_n} = 1$, one writes $x_n \sim y_n$

The symbols o , O and $\sim$ above are usually called Landau symbols as the German mathematician Landau (1877 – 1938) was the first to use the o, O notations systematically. But Landau himself attributes this notation to Paul Bachman (1837-1920). P.Du Bois-Reymond (1831-89) had earlier used a notation that included the symbol $\sim$ defined above for comparing the rates of growth of two increasing functions tending to infinity.

For functions defined in a neighbourhood of zero, one has the following

Definition 3.2 Let f,g be two real-valued functions defined in a neighbourhood of zero and suppose g is non-zero in that neighbourhood.

(i) If $|f(x)| \leq K g(x)$ for some $K > 0$ for all x with $|x| < \delta$, then one writes $f = O(g)$ as $x \to 0$;

(ii) If $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$, then one writes $f = o(g)$ as $x \to 0$;

(iii) If $\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$, then one writes $f \sim g$ as $x \to 0$;

These growth conditions can also be considered when the independent variable x tends to infinity. Landau symbols have been discussed in the text-books of Burkill and Burkill [2] and Hardy [3] and are used extensively in analytic number theory. Early training in the use of these symbols helps students acquire of qualitative understanding of the relative growth of functions. Table 1 displays some well-known limits and their quasi-qualitative versions (in terms of the Landau symbols) :

Students can be encouraged to reformulate problems on limits quasi-qualitatively using Landau symbols, in the spirit of the above table of limits.

Some basic properties of Landau symbols are presented below in the form of a theorem (see Hardy [3]).

Theorem 3.1 Let f and g be real-valued functions defined in a neighbourhood of zero. Then as $x \to 0$,

(a) $O(f) + O(g) = O(f+g)$;

(b) $\mathcal{O}(f)\,\mathcal{O}(g) = \mathcal{O}(fg)$;

(c) $\mathcal{O}(f)\,o(g) = o(fg)$;

(d) If $h \sim g$, then $h + o(f) \sim f$.

The above result is true even when f, g and h are considered real sequences. Students must be cautioned to note that $o(1) = \mathcal{O}(1)$ will not, in general, imply $\mathcal{O}(1) = o(1)$. For instance, as $x \to 0$, $\sin x = \mathcal{O}(\cos x)$, while $\cos x \neq o(\sin x)$, though $\sin x = o(1)$ and $\cos x = \mathcal{O}(1)$.

These symbols can be profitably employed to define differentiability of functions, as in the following

Definition 3.3 Let $f : D \subseteq \mathbf{R} \to \mathbf{R}$ be a function and x_0 an interior point of D. f is said to be differentiable at x_0 if there is a number $f'(x_0)$ such that

$$f(x_0 + h) = f(x_0) + hf'(x_0) + o(h) \text{ as } h \to 0 \tag{I}$$

Formula (I), essentially due to Weierstrass, is often called first order Taylor's formula and can be readily extended to real-valued functions of n-variables and to vector-valued functions of n - variables. For $f : D \subseteq \mathbf{R}^n \to \mathbf{R}$ and $x_0 = (x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)})$ an interior point of D and $h = (h_1, h_2, \ldots, h_n)$ (I) can be modified as

$$f(x_0+h) = f(x_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(x_0) + o(\|h\|) \text{ as } h \to 0 \tag{II}$$

Here $\frac{\partial f}{\partial x_i}$ are first-order partial- derivatives of f and $\|h\| = \sqrt{\sum_{i=1}^{n} h_i^2}$.

Use of first order Taylor's formula and Landau symbols leads to a quick proof of the chain rule (see Rudin [4]). It also clarifies the ideas underlying the proof of L' Hospital's rule . As a sample we have

Theorem 3.2 Let $f, g : D \subseteq \mathbf{R} \to \mathbf{R}$, where x_0 is an interior point of D. Suppose f, g, f' and g' are defined at x_0 and $f(x_0) = g(x_0) = f'(x_0) = g'(x_0) = 0$. If $g''(x_0) \neq 0$, then

$\lim_{h\to 0} \frac{f(x_0+h)}{g(x_0+h)} = \frac{f''(x_0)}{g''(x_0)}$, h being a real number sufficiently small in absolute value.

<u>Proof</u> : Using (I), as $h \to 0$

$$\frac{f(x_0+h)}{g(x_0+h)} = \frac{f(x_0+\frac{h}{2}) + \frac{h}{2} f'(x_0+\frac{h}{2}) + o(\frac{h}{2})}{g(x_0+\frac{h}{2}) + \frac{h}{2} g'(x_0+\frac{h}{2}) + o(\frac{h}{2})} \tag{III}$$

$$= \frac{f(x_0) + f'(x_0)\frac{h}{2} + o(\frac{h}{2}) + \frac{h}{2}(f'(x_0) + \frac{h}{2} f''(x_0) + o(\frac{h}{2}))}{g(x_0) + g'(x_0)\frac{h}{2} + o(\frac{h}{2}) + \frac{h}{2}(g'(x_0) + \frac{h}{2} g''(x_0) + o(\frac{h}{2}))}$$

$$= \frac{\frac{h^2}{4} f''(x_0) + o(\frac{h}{2}) + \frac{h}{2} o(\frac{h}{2})}{\frac{h^2}{4} g''(x_0) + o(\frac{h}{2}) + \frac{h}{2} o(\frac{h}{2})} \quad \text{as } f'(x_0) = g'(x_0) = 0 = f(x_0) = g(x_0)$$

As $g''(x_0) \neq 0$, Proceeding to the limit in (III) as $h \to 0$ we get

$$\lim_{x\to 0}\frac{f(x_0+h)}{g(x_0+h)}=\frac{f''(x_0)}{g''(x_0)}.$$

Theorem 3.2 can be readily formulated for the case when higher order derivatives of f and g also vanish at x_0. The proof is a direct application of first order Taylor's formula without recourse to Taylor's mean-value theorem and the use of Landau symbols makes the proof direct and transparent.

4. Conclusion

The use of Landau symbols affords a qualitative approach to many problems involving limits and derivatives. It also serves to mitigate the punctilious use of epsilons and deltas. Clearly an increased use of Landau symbols in a basic calculus course will improve the learner's understanding of the concepts of limit and derivative.

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Table 1

Some well-known limits	Their formulations with Landau symbols
1. $\lim_{x\to 0}\frac{\text{Sin } x}{x}=1$	$\text{Sin } x \sim x$ as $x \to 0$
2. $\lim_{x\to 0}\frac{\text{Cos } x}{1-\frac{x^2}{2}}=1$	$\text{Cos } x \sim 1-\frac{x^2}{2}$ as $x \to 0$
3. $\lim_{x\to\infty}\frac{x^n}{e^x}=0$, $n \varepsilon \mathbf{N}$	$x^n = o(e^x)$ as $x \to \infty$, $n \varepsilon \mathbf{N}$
4. $\lim_{x\to\infty}\frac{\log x}{x}=0$	$\log x = o(x)$ as $x \to \infty$
5. $\lim_{x\to\infty} e^{-x}=0$	$e^{-x} = o(1)$ as $x \to \infty$
6. $\lvert \text{Cos } x + \text{Sin } x \rvert \le 2$, $x \varepsilon \mathbf{R}^+$	$\text{Cos } x + \text{Sin } x = O(1)$ as $x \to \infty$

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